The Central Force Problem

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Central Forces

Introduction

- Interested in the **"2 body"** problem!

  Start out generally, but eventually restrict to motion of **2 bodies** interacting through a **central force**.

- ***Central Force*** ≡ Force between 2 bodies which is directed along the line between them.
- ***Important*** physical problem! Solvable ***exactly***!
  - Planetary motion & Kepler's Laws.
  - Nuclear forces
  - Atomic physics (H atom). Needs quantum version!

Reduction to Equivalent 1-Body Problem

- General 3d, 2 body problem. 2 masses $\mathbf{m_1}$ & $\mathbf{m_2}$: Need **6 coordinates:** For example, components of 2 position vectors $\vec{\mathbf{r}}_1$ & $\vec{\mathbf{r}}_2$ (arbitrary origin).
- **Assume** only forces are due to an interaction potential **U**. At first, **U** = any function of the vector between 2 particles, $\vec{\mathbf{r}} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$, of their relative velocity $\dot{\vec{\mathbf{r}}} = \dot{\vec{\mathbf{r}}}_1 - \dot{\vec{\mathbf{r}}}_2$, & possibly of higher derivatives of $\vec{\mathbf{r}} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$: $\mathbf{U} = \mathbf{U}(\vec{\mathbf{r}},\dot{\vec{\mathbf{r}}},..)$
  - Very soon, will restrict to central forces!

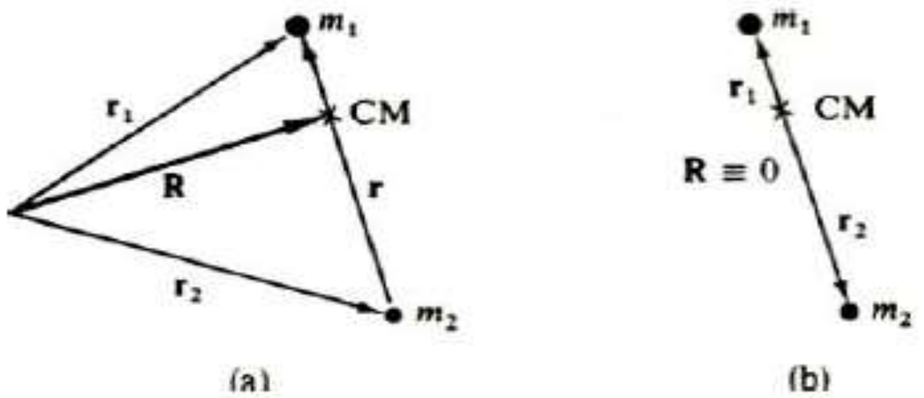
Lagrangian: $L = (\frac{1}{2})m_1|\dot{\vec{r}}_1|^2 + (\frac{1}{2})m_2|\dot{\vec{r}}_2|^2 - U(\vec{r},\dot{\vec{r}}, ..)$

- Instead of 6 components of 2 vectors $\mathbf{r}_1$ & $\mathbf{r}_2$, usually *transform* to (6 components of) Center of Mass (CM) & Relative Coordinates.
- ***Center of Mass Coordinate:*** $(M \equiv (m_1+m_2))$

$$\vec{R} \equiv (m_1\vec{r}_1 + m_2\vec{r}_2)/(M)$$

- ***Relative Coordinate:***

$$\vec{r} \equiv \vec{r}_1 - \vec{r}_2$$



- *Define:* **Reduced Mass:** $\mu \equiv (m_1m_2)/(m_1+m_2)$

Useful relation: $\mu^{-1} \equiv (m_1)^{-1} + (m_2)^{-1}$

- Algebra $\Rightarrow$ Inverse coordinate relations:

$$\vec{r}_1 = \vec{R} + (\mu/m_1)\vec{r};\ \vec{r}_2 = \vec{R} - (\mu/m_2)\vec{r}$$

Lagrangian: $L = (\frac{1}{2})m_1|\dot{\vec{r}}_1|^2 + (\frac{1}{2})m_2|\dot{\vec{r}}_2|^2 - U(\vec{r},\dot{\vec{r}}, ..)$ (1)

- **Velocities related** by

$$\dot{\vec{r}}_1 = \dot{\vec{R}} + (\mu/m_1)\dot{\vec{r}}; \quad \dot{\vec{r}}_2 = \dot{\vec{R}} - (\mu/m_2)\dot{\vec{r}} \quad (2)$$

- Combining (1) & (2) + algebra gives Lagrangian in terms of $\vec{R},\vec{r},\dot{\vec{r}}$: $L = (\frac{1}{2})M|\dot{\vec{R}}|^2 + (\frac{1}{2})\mu|\dot{\vec{r}}|^2 - U$

Or: $L = L_{CM} + L_{rel}$. Where: $L_{CM} \equiv (\frac{1}{2})M|\dot{\vec{R}}|^2$

$$L_{rel} \equiv (\tfrac{1}{2})\,\mu\,|\dot{\vec{r}}|^2 - U$$

$\Rightarrow$ **Motion separates into 2 parts:**

1. **CM motion,** governed by $L_{CM} \equiv (\frac{1}{2})M|\dot{\vec{R}}|^2$
2. **Relative motion,** governed by

$$L_{rel} \equiv (\tfrac{1}{2})\mu|\dot{\vec{r}}|^2 - U(\vec{r},\dot{\vec{r}},..)$$

CM & Relative Motion

- Lagrangian for **2 body problem**: $L = L_{CM} + L_{rel}$

$$L_{CM} \equiv (\tfrac{1}{2})M|\dot{\vec{R}}|^2 \; ; \; L_{rel} \equiv (\tfrac{1}{2})\mu|\dot{\vec{r}}|^2 - U(\vec{r},\dot{\vec{r}},..)$$

$\Rightarrow$ Motion separates into **2 parts:**

1. Lagrange's Eqtns for 3 components of CM coordinate vector $\vec{\mathbf{R}}$ clearly gives eqtns of motion independent of $\vec{\mathbf{r}}$.

2. Lagrange Eqtns for 3 components of relative coordinate vector $\vec{\mathbf{r}}$ clearly gives eqtns of motion independent of $\vec{\mathbf{R}}$.

- By transforming from $(\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2)$ to $(\vec{\mathbf{R}},\vec{\mathbf{r}})$:

The 2 body problem has been separated into 2 one body problems!

- Lagrangian for **2 body problem**

$$L = L_{CM} + L_{rel}$$

$\Rightarrow$ Have transformed the 2 body problem

into ***2 one body problems!***

1. **Motion of the CM**, governed by

$$L_{CM} \equiv (\tfrac{1}{2})M|\dot{\vec{R}}|^2$$

2. **Relative Motion**, governed by

$$L_{rel} \equiv (\tfrac{1}{2})\mu|\dot{\vec{r}}|^2 - U(\vec{r},\dot{\vec{r}},..)$$

- **Motion of CM** is governed by $L_{CM} \equiv (\frac{1}{2})M|\dot{\vec{R}}|^2$
  - Assuming no external forces.
- $\vec{R} = (X,Y,Z)$ $\Rightarrow$ 3 Lagrange Eqtns; each like:

$$(d/dt)(\partial[L_{CM}]/\partial\dot{X}) - (\partial[L_{CM}]/\partial X) = 0$$

$$(\partial[L_{CM}]/\partial X) = 0 \Rightarrow (d/dt)(\partial[L_{CM}]/\partial\dot{X}) = 0$$

$\Rightarrow$ $\ddot{X} = 0$, *CM acts like a free particle!*

- Solution: $\dot{X} = V_{x0}$ = constant
  - Determined by initial conditions!

$\Rightarrow$ $X(t) = X_0 + V_{x0}t$, *exactly like a free particle!*

- Same eqtns for **Y, Z**:

$\Rightarrow$ $\vec{R}(t) = \vec{R}_0 + \vec{V}_0 t$, *exactly like a free particle!*

CM Motion is identical to trivial motion of a free particle.

Uniform translation of CM. Trivial & uninteresting!

- **2 body** Lagrangian: $L = L_{CM} + L_{rel}$

$\Rightarrow$ 2 body problem is transformed to 2 one body problems!

1. Motion of the CM, governed by $L_{CM} \equiv (\frac{1}{2})M|\dot{\vec{R}}|^2$

Trivial free particle-like motion!

2. Relative Motion, governed by

$$L_{rel} \equiv (\tfrac{1}{2})\mu|\dot{\vec{r}}|^2 - U(\vec{r},\dot{\vec{r}},..)$$

$\Rightarrow$ 2 body problem is transformed to 2 one body problems,

one of which is trivial!

All interesting physics is in relative motion part!

$\Rightarrow$ ***Focus on it exclusively!***

Relative Motion

- **Relative Motion** is governed by

$$L_{rel} \equiv (\tfrac{1}{2})\mu|\dot{\vec{r}}|^2 - U(\vec{r},\dot{\vec{r}},..)$$

  - Assuming no external forces.
  - Henceforth: $L_{rel} \equiv L$ (Drop subscript)
  - For convenience, take **origin of coordinates at CM**:

$$\Rightarrow \quad \vec{R} = 0$$

$$\vec{r}_1 = (\mu/m_1)\vec{r}; \qquad \vec{r}_2 = -\,(\mu/m_2)\vec{r}$$

$$\mu \equiv (m_1 m_2)/(m_1+m_2)$$

$$(\mu)^{-1} \equiv (m_1)^{-1} + (m_2)^{-1}$$

- **The 2 body, central force problem** has been formally reduced to an

<u>EQUIVALENT ONE BODY PROBLEM</u>

in which the motion of a "particle" of mass $\boldsymbol{\mu}$ in $\mathbf{U}(\vec{\mathbf{r}},\dot{\vec{\mathbf{r}}},\ldots)$ is what is to be determined!

  – Superimpose the uniform, free particle-like translation of CM onto the relative motion solution!

  – If desired, if get $\vec{\mathbf{r}}(\mathbf{t})$, can get $\vec{\mathbf{r}}_1(\mathbf{t})$ & $\vec{\mathbf{r}}_2(\mathbf{t})$ from above. **Usually, the relative motion (orbits) only is wanted & we stop at $\vec{\mathbf{r}}(\mathbf{t})$.**

Eqtns of Motion & 1st Integrals

- **System:** "Particle" of mass $\mathbf{\mu}$ ($\mathbf{\mu} \rightarrow \mathbf{m}$ in what follows) moving in a force field described by potential $\mathbf{U(\vec{r},\dot{\vec{r}},..)}$.
- Now, restrict to **conservative Central Forces:**

$$\mathbf{U \rightarrow V} \qquad \text{where } \mathbf{V = V(r)}$$

  – **Note:** $\mathbf{V(r)}$ depends only on $\mathbf{r} = |\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2|$ = distance of particle from force center. No orientation dependence. $\Rightarrow$ **System has spherical symmetry**

  $\Rightarrow$ Rotation about any fixed axis can't affect eqtns of motion.

  $\Rightarrow$ **Expect the angle representing such a rotation to be cyclic &** ***the corresponding generalized momentum (angular momentum) to be conserved*****.**

Angular Momentum

- By the discussion in Ch. 2: **Spherical symmetry**

$\Rightarrow$ ***The Angular Momentum of the system is conserved:***

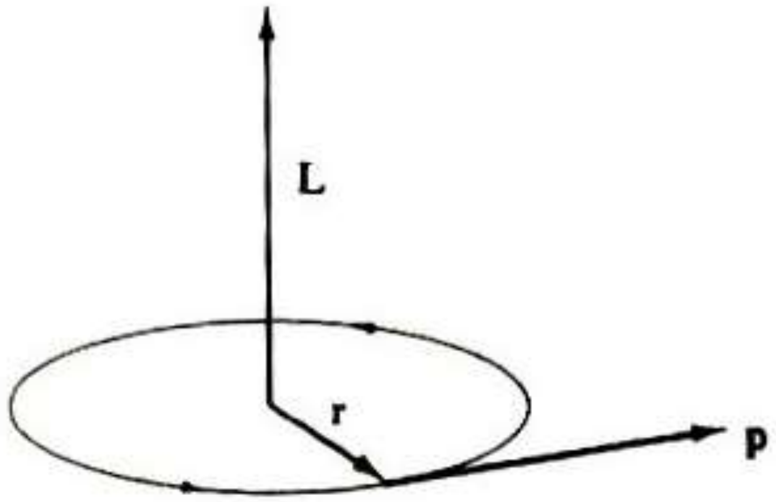
$$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}} = \text{constant (magnitude \& direction!)}$$

Angular momentum conservation!

$\Rightarrow \vec{\mathbf{r}}$ & $\vec{\mathbf{p}}$ (& thus the particle motion!) always **lie in a plane** $\perp \vec{\mathbf{L}}$, which is fixed in space.

Figure:

(See text discussion for $\vec{\mathbf{L}} = \mathbf{0}$)



$\Rightarrow$ ***The problem is effectively reduced from 3d to 2d***

(particle motion in a plane)!

Motion in a Plane

- **Describe 3d motion in spherical coordinates** (Goldstein notation!): $(\mathbf{r},\theta,\psi)$. θ = angle in the plane (plane polar coordinates). ψ = azimuthal angle.
- $\vec{L}$ ***is fixed,*** as we saw. $\Rightarrow$ **The motion is in a plane.** Effectively reducing the 3d problem to a 2d one!
- Choose the polar (z) axis along $\vec{L}$.

$\Rightarrow$ $\psi = (\tfrac{1}{2})\pi$ ***& drops out of the problem.***

- **Conservation of angular momentum** $\vec{L}$

$\Rightarrow$ *3 independent constants of the motion*

(1st integrals of the motion): Effectively we've used 2 of these to limit the motion to a plane. The third ($|\vec{L}|$ = constant) will be used to complete the solution to the problem.

Summary So Far

- Started with **6d, 2 body problem.** Reduced it to **2, 3d 1 body problems,** one (CM motion) of which is trivial. **Angular momentum conservation** reduces 2nd 3d problem (relative motion) **from 3d to 2d** (motion in a plane)!
- **Lagrangian** ($\mu \rightarrow$ **m** , conservative, central forces):

$$L = (½)m|\dot{\vec{r}}|^2 - V(r)$$

- **Motion in a plane**

$\Rightarrow$ Choose plane polar coordinates to do the problem:

$$\Rightarrow \qquad L = (½)m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

$$L = (½)m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

- The Lagrangian is cyclic in θ

$\Rightarrow$ **The generalized momentum p_θ is conserved:**

$$p_\theta \equiv (\partial L/\partial\dot{\theta}) = mr^2\dot{\theta}$$

Lagrange's Eqtn: $(d/dt)[(\partial L/\partial\dot{\theta})] - (\partial L/\partial\theta) = 0$

$\Rightarrow$ $\dot{p}_\theta = 0$, $\quad p_\theta = \text{constant} = mr^2\dot{\theta}$

- **Physics:** $p_\theta = mr^2\dot{\theta}$ = angular momentum about an axis $\perp$ the plane of motion. ***Conservation of angular momentum,*** as we already said!
- The problem symmetry has allowed us to integrate one eqtn of motion. $p_\theta \equiv$ a **"1st Integral"** of motion. Convenient to define: $\ell \equiv p_\theta \equiv mr^2\dot{\theta}$ = constant.

$$L = (½)m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

- In terms of $\ell \equiv mr^2\dot{\theta}$ = constant, the Lagrangian is:

$$L = (½)m\dot{r}^2 + [\ell^2/(2mr^2)] - V(r)$$

- **Symmetry** & the resulting conservation of angular momentum has reduced the effective 2d problem (2 degrees of freedom) to an effective **1d problem!**

1 degree of freedom, one generalized coordinate r!

- Now: Set up & **solve the problem** using the above Lagrangian. Also, follow authors & do with **energy conservation.** However, first, **a side issue.**

Kepler's 2nd Law

- Const. angular momentum $\ell \equiv \mathbf{m r^2 \dot{\theta}}$
- Note that ℓ could be < 0 or > 0.
- **Geometric interpretation:** $\ell =$ const: See figure:

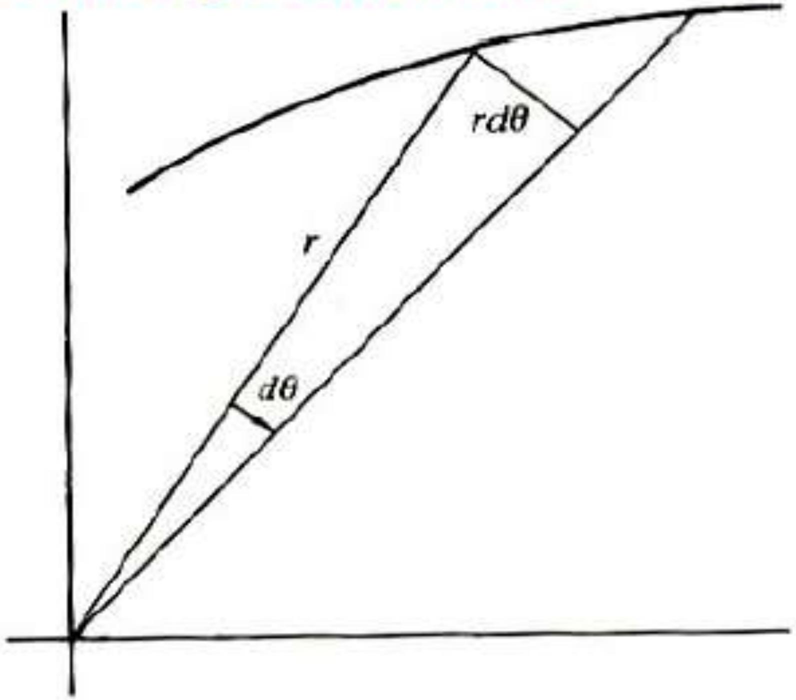


FIGURE 3.2 The area swept out by the radius vector in a time dt.

- In describing the path $\mathbf{r(t)}$, in time $\mathbf{dt}$, the radius vector sweeps out an area: $\mathbf{dA = (\tfrac{1}{2}) r^2 d\theta}$

- In **dt**, radius vector sweeps out area $\mathbf{dA = (½)r^2 d\theta}$
  - Define ***Areal Velocity*** $\equiv$ **(dA/dt)**

$\Rightarrow$ $\mathbf{(dA/dt) = (½)r^2(d\theta/dt) = dA = (½)r^2\dot{\theta}}$ (1)

But $\ell \equiv \mathbf{mr^2\,\dot{\theta}}$ = constant

$\Rightarrow$ $\mathbf{\dot{\theta} = (\ell/mr^2)}$ (2)

- Combine (1) & (2):

$\Rightarrow$ $\mathbf{(dA/dt) = (½)(\ell/m)}$ = constant!

$\Rightarrow$ Areal velocity is constant in time!

$\equiv$ ***the Radius vector from the origin sweeps out equal areas in equal times*** $\equiv$ **Kepler's 2nd Law**

- First derived by empirically by Kepler for planetary motion.

General result for central forces!

Not limited to the gravitational force law ($\mathbf{r^{-2}}$).

Lagrange's Eqtn for r

- In terms of $\ell \equiv \mathbf{m}r^2\dot{\theta}$ = const, the **Lagrangian** is:

$$L = (½)m\dot{r}^2 + [\ell^2/(2mr^2)] - V(r)$$

- **Lagrange's Eqtn** for **r**:

$$(d/dt)[(\partial L/\partial \dot{r})] - (\partial L/\partial r) = 0$$

$$\Rightarrow \quad m\ddot{r} - [\ell^2/(mr^3)] = -(\partial V/\partial r) \equiv f(r)$$

($\mathbf{f(r)} \equiv$ force along **r**)

Rather than solve this directly, its easier to use **Energy Conservation.** Come back to this later.

Energy

- **Note:** Linear momentum is conserved also:
  - Linear momentum of CM.

  $\Rightarrow$ Uninteresting free particle motion
- **Total mechanical energy is also conserved** since the central force is conservative:

$$\mathbf{E = T + V = constant}$$

$$\mathbf{E = (\tfrac{1}{2})m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)}$$

- Recall, **angular momentum** is:

$$\ell \equiv \mathbf{mr^2\dot{\theta}} = \text{const}$$

$$\Rightarrow \quad \dot{\theta} = [\ell/(mr^2)]$$

$$\Rightarrow \quad \mathbf{E = (\tfrac{1}{2})m\dot{r}^2 + (\tfrac{1}{2})[\ell^2/(mr^2)] + V(r)} = \text{const}$$

Another **"1st integral"** of the motion

$r(t)$ & $\theta(t)$

$$E = (\tfrac{1}{2})m\dot{r}^2 + [\ell^2/(2mr^2)] + V(r) = \text{const}$$

- **Energy Conservation** allows us to get solutions to the eqtns of motion in terms of **r(t)** & **θ(t)** and **r(θ)** or **θ(r)** ≡ **The orbit of the particle!**
  - Eqtn of motion to get **r(t)**: One degree of freedom

⇒ Very similar to a 1 d problem!

- Solve for $\dot{r} = (dr/dt)$:

$$\dot{r} = \pm (\{2/m\}[E - V(r)] - [\ell^2/(m^2r^2)])^{\frac{1}{2}}$$

  - **Note:** This gives $\dot{r}(r)$, the phase diagram for the relative coordinate & velocity. Can qualitatively analyze (**r** part of) motion using it, just as in 1d.
- Solve for **dt** & formally integrate to get **t(r).** In principle, invert to get **r(t).**

$$\dot{r} = \pm (\{2/m\}[E - V(r)] - [\ell^2/(m^2r^2)])^{1/2}$$

- Solve for **dt** & **formally integrate** to get **t(r):**

$$t(r) = \pm \int dr(\{2/m\}[E - V(r)] - [\ell^2/(m^2r^2)])^{-1/2}$$

  – Limits $r_0 \rightarrow r$, r_0 determined by initial condition
  – Note the square root in denominator!

- Get **θ(t)** in terms of **r(t)** using **conservation of angular momentum** again: $\ell \equiv mr^2\dot{\theta}$ = const

$$\Rightarrow \quad (d\theta/dt) = [\ell/(mr^2)]$$

$$\Rightarrow \quad \theta(t) = (\ell/m)\int(dt[r(t)]^{-2}) + \theta_0$$

  – Limits $0 \rightarrow t$

  θ_0 determined by initial condition

- ***Formally***, **the 2 body Central Force problem has been reduced to the evaluation of 2 integrals:**

(Given **V(r)** can do them, in principle.)

$$\mathbf{t(r) = \pm \int dr(\{2/m\}[E - V(r)] - [\ell^2/(m^2r^2)])^{-1/2}}$$

  – Limits $\mathbf{r_0 \rightarrow r}$, $\mathbf{r_0}$ determined by initial condition

$$\mathbf{\theta(t) = (\ell/m)\int(dt[r(t)]^{-2}) + \theta_0}$$

  – Limits $\mathbf{0 \rightarrow t}$, $\mathbf{\theta_0}$ determined by initial condition

- To solve the problem, need 4 integration constants:

$$\mathbf{E, \ell, r_0, \theta_0}$$

Orbits

- Often, we are much more interested in the path in the **r, θ** plane: **r(θ)** or **θ(r)** ≡ ***The orbit.***
- Note that (chain rule):

$$(d\theta/dr) = (d\theta/dt)(dt/dr) = (d\theta/dt)/(dr/dt)$$

Or: $$(d\theta/dr) = (\dot{\theta}/\dot{r})$$

Also, $\ell \equiv mr^2\dot{\theta} = \text{const} \Rightarrow \dot{\theta} = [\ell/(mr^2)]$

Use $$r = \pm (\{2/m\}[E - V(r)] - [\ell^2/(m^2r^2)])^{1/2}$$

$$\Rightarrow (d\theta/dr) = \pm [\ell/(mr^2)](\{2/m\}[E - V(r)] - [\ell^2/(m^2r^2)])^{-1/2}$$

Or:

$$(d\theta/dr) = \pm (\ell/r^2)(2m)^{-1/2}[E - V(r) - \{\ell^2/(2mr^2)\}]^{-1/2}$$

- **Integrating this will give θ(r) .**

- ***Formally:***

$$(d\theta/dr) = \pm (\ell/r^2)(2m)^{-1/2}[E - V(r) - \{\ell^2/(2mr^2)\}]^{-1/2}$$

- Integrating this gives a formal eqtn for the orbit:

$$\theta(r) = \pm \int (\ell/r^2)(2m)^{-1/2}[E - V(r) - \{\ell^2/(2mr^2)\}]^{-1/2}\, dr$$

- Once the central force is specified, we know **V(r)** & can, in principle, do the integral & get the orbit **θ(r),** or, (if this can be inverted!) **r(θ)**.

⇒ This is quite remarkable! Assuming only a central force law & nothing else:

We have reduced the original 6 d problem of 2 particles to a 2 d problem with only 1 degree of freedom. The solution for the orbit can be obtained simply by doing the above (1d) integral!

Equivalent “1d” Problem

- *Formally,* the 2 body Central Force problem has been **reduced to evaluation of 2 integrals**, which will give **r(t)** & **θ(t)** : (Given **V(r)** can do them, in principle.)

$$t(r) = \pm\int dr(\{2/m\}[E - V(r)] - [\ell^2/(m^2r^2)])^{-1/2} \quad (1)$$

  - Limits $r_0 \rightarrow r$, r_0 determined by initial conditions
  - Invert this to get **r(t)** & use that in **θ(t)** (below)

$$\theta(t) = (\ell/m) \int (dt/[r^2(t)]) + \theta_0 \qquad (2)$$

  - Limits $0 \rightarrow t$, θ_0 determined by initial condition
- Need **4 integration constants:** E, ℓ, r_0, θ_0
- Most cases: **(1)**, **(2)** can’t be done except numerically
- Before looking at cases where they can be done: Discuss the ***PHYSICS*** of motion obtained from conservation theorems.

- Assume the system has **known energy** **E** & **angular momentum** ℓ ($\equiv mr^2\dot{\theta}$).
  - Find the magnitude & direction of velocity **v** in terms of **r**:
- ***Conservation of Mechanical Energy:***

$\Rightarrow$ $$E = (½)mv^2 + V(r) = \text{const} \qquad (1)$$

Or: $$E = (½)m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = \text{const} \qquad (2)$$

v^2 = square of total (2d) velocity: $v^2 \equiv \dot{r}^2 + r^2\dot{\theta}^2$ **(3)**

(1) $\Rightarrow$ **Magnitude** of **v**:

$$v = \pm (\{2/m\}[E - V(r)])^{½} \qquad (4)$$

(2) $\Rightarrow$ $$\dot{r} = \pm (\{2/m\}[E - V(r)] - [\ell^2/(m^2r^2)])^{½} \qquad (5)$$

Combining **(3), (4), (5)** gives the **direction** of **v**

- Alternatively, $\ell = mr^2\dot{\theta}$ = const, gives $\dot{\theta}$. Combined with (5) gives both magnitude & direction of **v**.

- **Lagrangian :** $L = (\frac{1}{2})m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$
- In terms of $\ell \equiv mr^2\dot{\theta}$ = const, this is:

$$L = (\tfrac{1}{2})m\dot{r}^2 + [\ell^2/(2mr^2)] - V(r)$$

- Lagrange Eqtn for **r**: $(d/dt)[(\partial L/\partial\dot{r})] - (\partial L/\partial r) = 0$

$\Rightarrow$ $$m\ddot{r} - [\ell^2/(mr^3)] = -(\partial V/\partial r) \equiv f(r)$$

$(f(r) \equiv$ force along $r)$

Or: $$m\ddot{r} = f(r) + [\ell^2/(mr^3)] \qquad (1)$$

- **(1)** involves only **r** & $\dot{\mathbf{r}}$. $\Rightarrow$ **Same Eqtn** of motion (Newton's 2nd Law) as for a **fictitious** (or effective) 1d **(r)** problem of mass **m** subject to a force:

$$f'(r) = f(r) + [\ell^2/(mr^3)]$$

Centrifugal "Force" & Potential

- Effective 1d **(r)** problem: **m** subject to a force:

$$f'(r) = f(r) + [\ell^2/(mr^3)]$$

- *PHYSICS:* Using $\ell \equiv mr^2\dot{\theta}$:

$$[\ell^2/(mr^3)] \equiv mr\dot{\theta}^2 \equiv m(v_\theta)^2/r \equiv \text{“Centrifugal Force”}$$

  - Return to this in a minute.

- Equivalently, energy:

$$E = (\tfrac{1}{2})m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = (\tfrac{1}{2})m\dot{r}^2 + (\tfrac{1}{2})[\ell^2/(mr^2)] + V(r) = \text{const}$$

- **Same energy Eqtn as** for a **fictitious** (or effective) 1d **(r)** problem of mass **m** subject to a potential:

$$V'(r) = V(r) + (\tfrac{1}{2})[\ell^2/(mr^2)]$$

  - Easy to show that $f'(r) = -(\partial V'/\partial r)$
  - Can clearly write $E = (\tfrac{1}{2})m\dot{r}^2 + V'(r) = \text{const}$

Comments on Centrifugal "Force" & Potential:

- Consider: $\mathbf{E} = (\tfrac{1}{2})m\dot{r}^2 + (\tfrac{1}{2})[\ell^2/(mr^2)] + V(r)$
- ***Physics*** of $[\ell^2/(2mr^2)]$. Conservation of angular momentum: $\ell = mr^2\dot{\theta} \Rightarrow [\ell^2/(2mr^2)] \equiv (\tfrac{1}{2})mr^2\dot{\theta}^2$

  $\equiv$ *Angular part of kinetic energy* of mass **m**.
- Because of the form $[\ell^2/(2mr^2)]$, this contribution to the energy depends only on **r**: ***When analyzing the* r *part of the motion***, can treat this as **an additional part of the *potential energy.***

$\Rightarrow$ It's often convenient to call it another potential energy term $\equiv$ ***"Centrifugal" Potential Energy***

- $[\ell^2/(2mr^2)] \equiv$ ***"Centrifugal" PE*** $\equiv V_c(r)$
  - As just discussed, this is really the angular part of the ***Kinetic Energy***!

$\Rightarrow$ "Force" associated with $V_c(r)$:

$$f_c(r) \equiv -(\partial V_c/\partial r) = [\ell^2/(mr^3)]$$

Or, using $\ell = mr^2\dot{\theta}$:

$$f_c(r) = [\ell^2/(mr^3)] = mr\theta^2 \equiv m(v_\theta)^2/r$$

$\equiv$ ***"Centrifugal Force"***

- $\mathbf{f_c(r) = [\ell^2/(mr^3)]} \equiv$ *"Centrifugal Force"*

- $\mathbf{f_c(r)}$ = ***Fictitious*** "force" arising due to fact that the reference frame of the relative coordinate **r** (of "particle" of mass **m**) is ***not* an inertial frame!**

  – *NOT (!!)* a force in the Newtonian sense! A part of the "**ma**" of Newton's 2nd Law, rewritten to appear on the "**F**" side.

  Direction of $\mathbf{f_c}$: **Outward from the force center!**

- Particle moving in a circular arc: Force *in an Inertial Frame* is directed **INWARD TOWARDS THE CIRCLE CENTER**

  $\equiv$ *Centripetal Force*

Effective Potential

- For both qualitative & quantitative analysis of the ***RADIAL*** motion for "particle" of mass **m** in a central potential **V(r)**, $\mathbf{V_c(r) = [\ell^2/(2mr^2)]}$ acts as an additional potential & we can treat it as such!
  - But recall that **physically,** it comes from the ***Kinetic Energy*** of the particle!

$\Rightarrow$ As already said, lump **V(r)** & $\mathbf{V_c(r)}$ together into an ***Effective Potential*** $\equiv$

$$\mathbf{V'(r) \equiv V(r) + V_c(r)}$$
$$\mathbf{\equiv V(r) + [\ell^2/(2mr^2)]}$$

- ***Effective Potential*** $\equiv$

$$V'(r) \equiv V(r) + V_c(r) \equiv V(r) + [\ell^2/(2mr^2)]$$

- Consider now:

$$E = (\tfrac{1}{2})m\dot{r}^2 + (\tfrac{1}{2})[\ell^2/(mr^2)] + V(r) = (\tfrac{1}{2})m\dot{r}^2 + V'(r) = \text{const}$$

$$\Rightarrow \quad \dot{r} = \pm (\{2/m\}[E - V'(r)])^{1/2} \qquad (1)$$

- Given **U(r)**, can use **(1)** to **qualitatively analyze the *RADIAL* motion** for the "particle". Get turning points, oscillations, etc. Gives $\dot{\mathbf{r}}$ *vs.* **r** phase diagram.
  - Similar to analysis of 1 d motion where one analyzes particle motion for various **E** using

$$E = (\tfrac{1}{2})m\dot{x}^2 + V(x) = \text{const}$$

- Important **special case: Inverse square law central force:** $f(r) = -(k/r^2) \Rightarrow V(r) = -(k/r)$
  - Taking $V(r \to \infty) \to 0$
  - $k > 0$: Attractive force. $k < 0$: Repulsive force.
  - Gravity: $k = GmM$. Always attractive!
  - Coulomb (SI Units): $k = (q_1q_2)/(4\pi\varepsilon_0)$. Could be attractive or repulsive!
- For **inverse square law force,** effective potential is: $V'(r) \equiv V(r) + [\ell^2/(2mr^2)]$

$$= -(k/r) + [\ell^2/(2mr^2)]$$

V´(r) for Attractive r^{-2} Forces

- **Qualitatively analyze motion** for different energies **E** in effective potential for inverse square law force. (Figure): **V´(r) = -(k/r) + [ℓ²⁄(2mr²)]**

  $$E = (½)m\dot{r}^2 + V'(r) \Rightarrow E - V'(r) = (½)m\dot{r}^2 \geq 0$$

$\Rightarrow \dot{r} = 0$ at turning points ($E = V'(r)$)

NOTE: This analysis is for the **r** ***part of the motion only***. To get the particle orbit **r(θ)**, must superimpose **θ** motion on this!

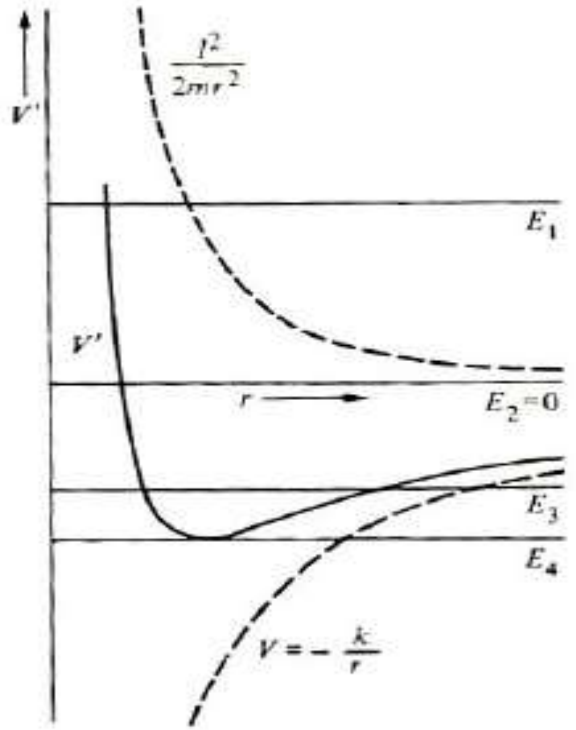


FIGURE 3.3 The equivalent one-dimensional potential for attractive inverse-square law of force.

- Motion of particle with **energy $E_1 > 0$** (figure):

$E_1 - V'(r) = (½)m\dot{r}^2 \geq 0$

$\Rightarrow \dot{r} = 0$ at turning point r_1 $(E_1 = V'(r_1))$

- r_1 = min distance of approach
- No max **r**:

$\Rightarrow$ **Unbounded orbit!**

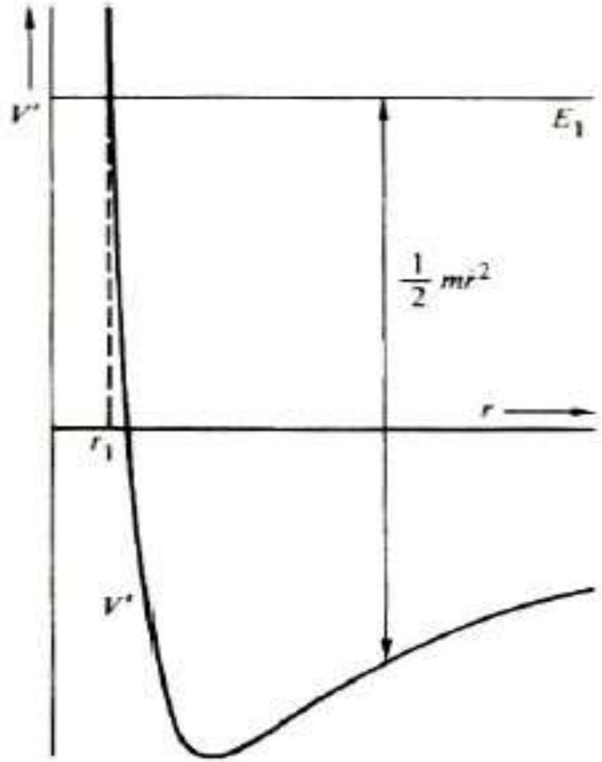


FIGURE 3.4 Unbounded motion at positive energies for inverse-square law of force.

- Particle from $r \to \infty$ comes in towards **r=0**. At $r = r_1$, it "strikes" the "repulsive centrifugal barrier", is repelled (turns around) & travels back out towards $r \to \infty$. It speeds up until $r = r_0$ = min of $V'(r)$. Then, slows down as it approaches r_1. After it turns around, it speeds up to r_0 & then slows down to $r \to \infty$.

- Motion of particle with **energy $E_1 > 0$** (continued):

$$E_1 - V'(r) = (\tfrac{1}{2})m\dot{r}^2 \geq 0$$

- Also:

$$E_1 - V(r) = (\tfrac{1}{2})mv^2 \geq 0$$

$$\Rightarrow V(r) - V'(r) =$$

$$(\tfrac{1}{2})mv^2 - (\tfrac{1}{2})m\dot{r}^2 =$$

$$(\tfrac{1}{2})mr^2\dot{\theta}^2 = [\ell^2/(2mr^2)]$$

$$= V_c(r)$$

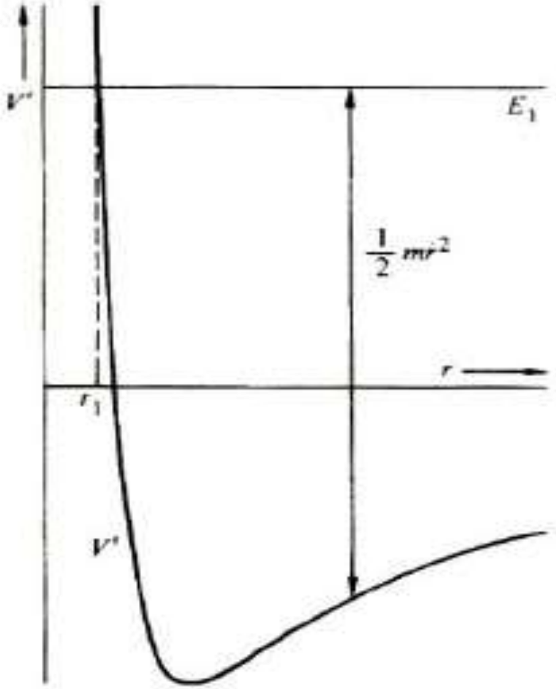


FIGURE 3.4 Unbounded motion at positive energies for inverse-square law of force.

$\Rightarrow$ From this analysis, can, for any **r**, get the magnitude of the velocity **v** + **its r & θ components**.

$\Rightarrow$ Can use this info to get an ***approximate*** **picture of the particle orbit r(θ)**.

- Motion of particle with **energy $E_1 > 0$** (continued):

  $\mathbf{E_1 - V'(r) = (½)m\dot{r}^2 \geq 0.}$ Qualitative **orbit** $\mathbf{r(\theta)}$.

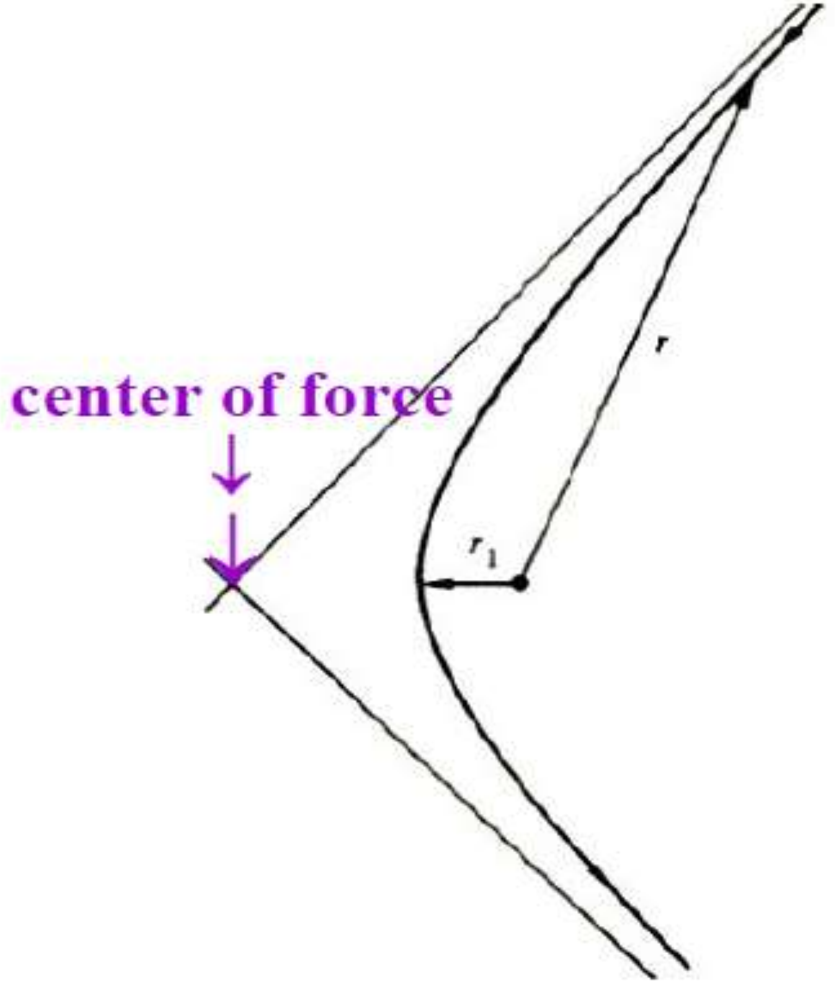


FIGURE 3.5 The orbit for E_1 corresponding to unbounded motion.

- Motion of particle with **energy $E_2 = 0$:**

  $$E_2 - V'(r) = (½)m\dot{r}^2 \geq 0. \quad \Rightarrow -V'(r) = (½)m\dot{r}^2 \geq 0$$

  Qualitative motion is ~ the same as for $\mathbf{E_1}$, except **the turning point is at r_0** (figure):

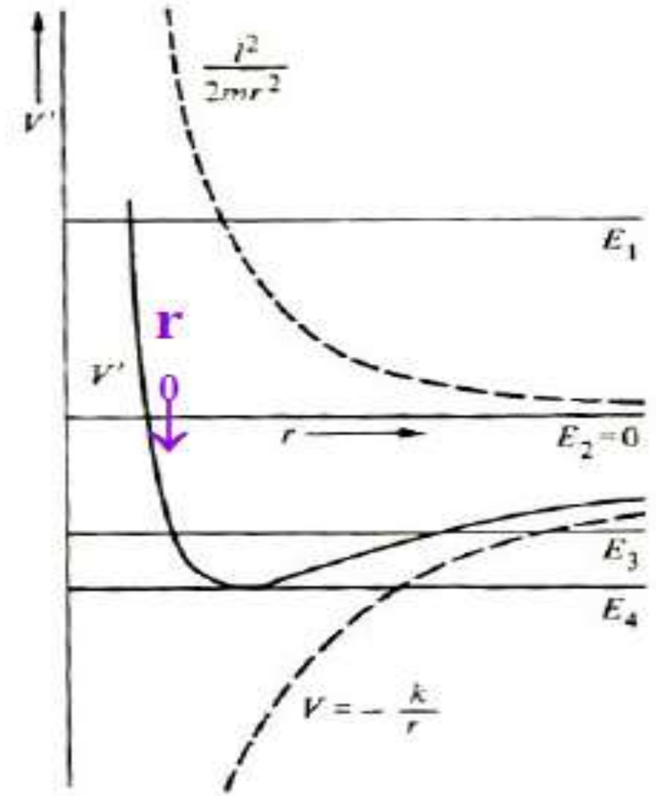


FIGURE 3.3 The equivalent one-dimensional potential for attractive inverse-square law of force.

- Motion of particle with **energy $E_3 < 0$:**

$E_3 - V'(r) = (½)m\dot{r}^2 \geq 0$. Qualitative motion:

"oscillatory" in r

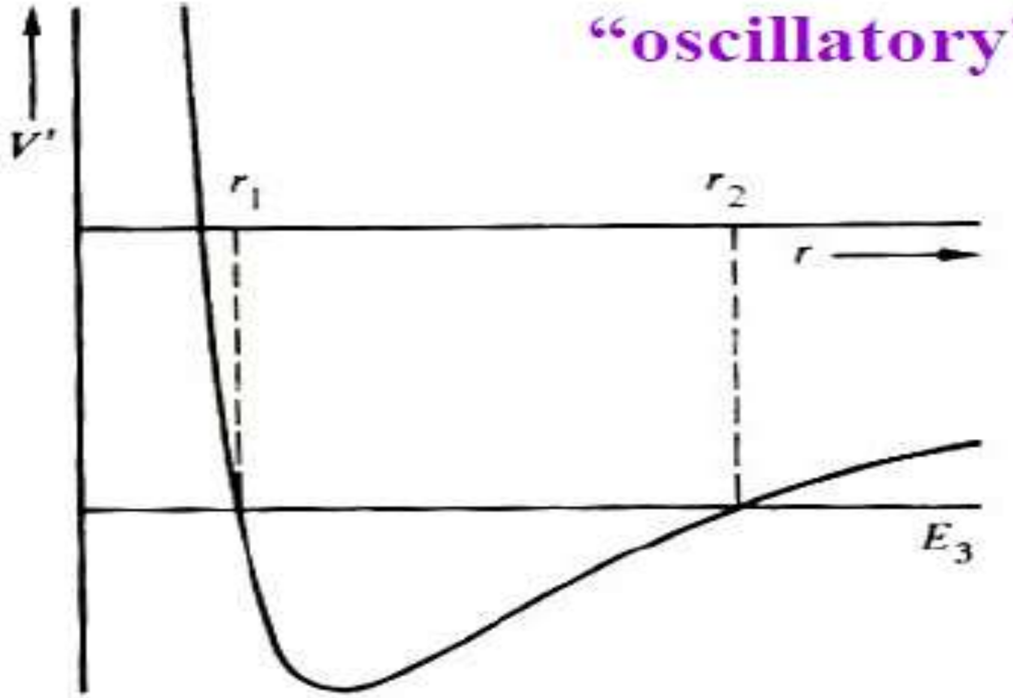


FIGURE 3.6 The equivalent one-dimensional potential for inverse-square law of force, illustrating bounded motion at negative energies.

2 turning points, min & max **r**: (**r_1** & **r_2**). Turning points given by solutions to **$E_3 = V'(r)$**. **Orbit is bounded.** **r_1** & **r_2** = "apsidal' distances.

- Motion of particle with **energy $E_3 < 0$** (continued):

  $E_3 - V'(r) = (\frac{1}{2})m\dot{r}^2 \geq 0$. **Qualitative orbit $r(\theta)$**.

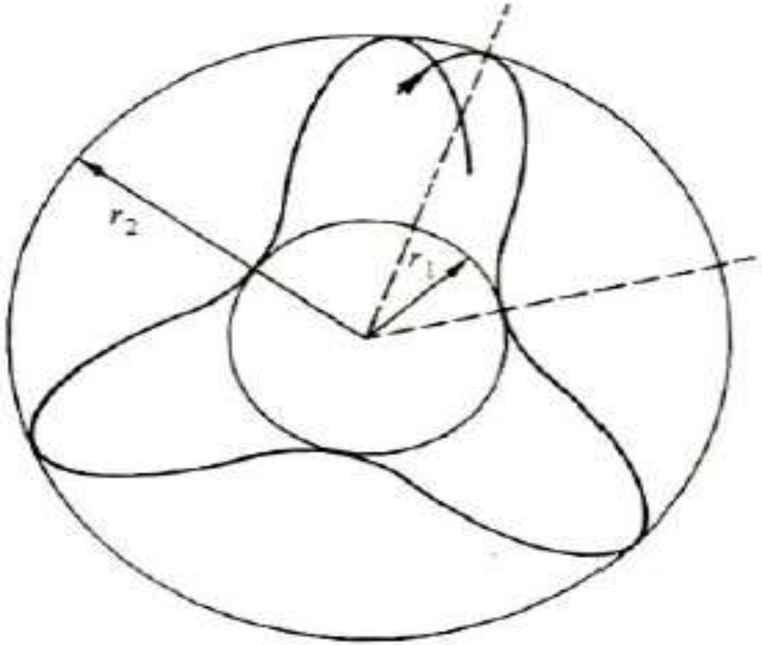


FIGURE 3.7 The nature of the orbits for bounded motion

Turning points, $\mathbf{r_1}$ & $\mathbf{r_2}$. **Orbit is bounded; but not necessarily closed! Bounded:** Contained in the plane between the 2 circles of radii $\mathbf{r_1}$ & $\mathbf{r_2}$. Only **closed** if eventually comes back to itself & retraces the same path over. More on this later.

- Motion of particle with **energy $E_4 < 0$: $E_4 - V'(r) = 0$** **($\dot{r} = 0$) $\Rightarrow$ r = r_1** (min **r** of **V´(r)**) = constant $\Rightarrow$ **Circular orbit** (& bounded, of course!) **r(θ) = r_1**!

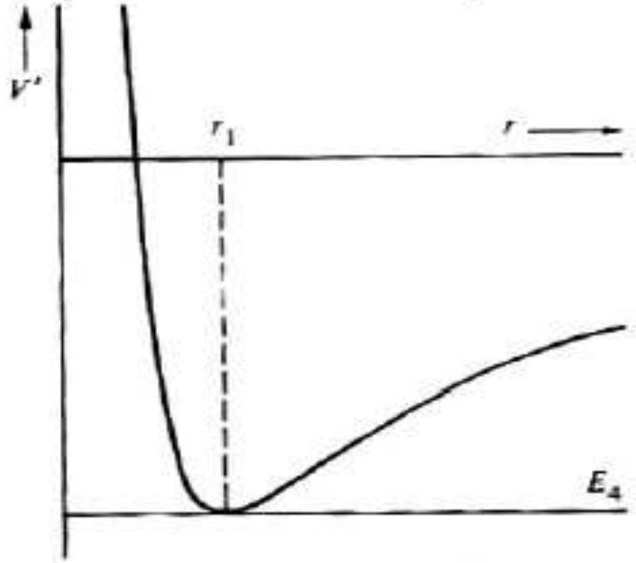


FIGURE 3.8 The equivalent one-dimensional potential of inverse-square law of force, illustrating the condition for circular orbits.

Effective potential: $V'(r) = V(r) + (\frac{1}{2})[\ell^2/(mr^2)]$

Effective force: $f'(r) = f(r) + [\ell^2/(mr^3)] = -(\partial V'/\partial r)$.

At **r = r_1** (min of **V´(r)**) $\Rightarrow$ **$f'(r) = -(\partial V'/\partial r) = 0$** or **$f(r) = -[\ell^2/(mr^3)] = -mr\dot{\theta}^2$.** Appl. force = centripetal force

- **Energy $E < E_4$?** $\Rightarrow \mathbf{E - V'(r) = (\frac{1}{2})m\dot{r}^2 < 0}$

$\Rightarrow$ Unphysical! Requires $\dot{\mathbf{r}}$ = imaginary.

- All discussion has been for **one value of angular momentum** ℓ. Clearly changing ℓ changes $\mathbf{V'(r)}$ quantitatively, but not qualitatively (except for $\ell = \mathbf{0}$ for which the centrifugal barrier goes away.)

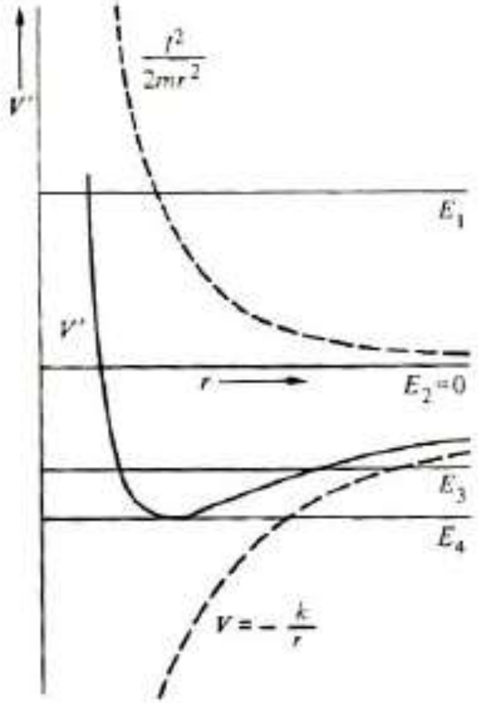


FIGURE 3.3 The equivalent one-dimensional potential for attractive inverse-square law of force.

$\Rightarrow$ **Orbit types will be the same for similar energies.**

$V'(r)$ for Attractive r^{-2} Forces

- Will analyze orbits in detail later. Will find:
- Energy $\mathbf{E_1 > 0}$: **Hyperbolic Orbit**
- Energy $\mathbf{E_2 = 0}$: **Parabolic Orbit**
- Energy $\mathbf{E_3 < 0}$: **Elliptic Orbit**
- Energy $\mathbf{E_4 = [V'(r)]_{min}}$: **Circular Orbit**

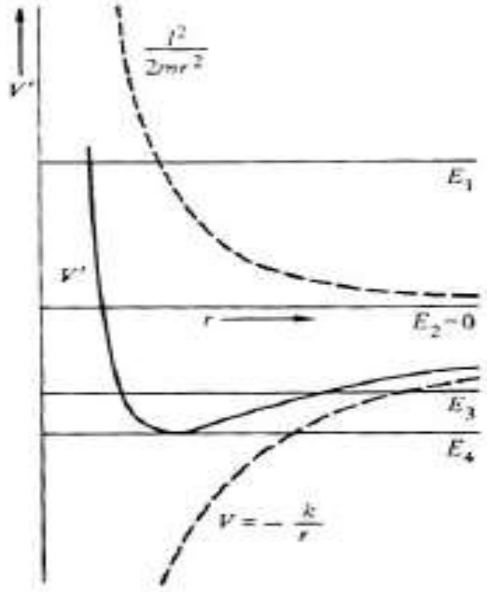


FIGURE 3.3 The equivalent one-dimensional potential for attractive inverse-square law of force.

Other Attractive Forces

- For **other types of Forces**: Orbits aren't so simple.

- For any ***attractive*** **V(r)** still have the same qualitative division into open, bounded, & circular orbits if:

1. V(r) falls off *slower* than $\mathbf{r}^{-2}$ as $\mathbf{r} \to \infty$

Ensures that $\mathbf{V(r)} > (\frac{1}{2})[\ell^2/(\mathbf{mr}^2)]$ as $\mathbf{r} \to \infty$

$\Rightarrow$ **V(r)** dominates the Centrifugal Potential at large **r**.

2. V(r) $\to \infty$ ***slower* than $\mathbf{r}^{-2}$** as $\mathbf{r} \to 0$

Ensures that $\mathbf{V(r)} < (\frac{1}{2})[\ell^2/(\mathbf{mr}^2)]$ as $\mathbf{r} \to 0$

$\Rightarrow$ The centrifugal Potential dominates **V(r)** at small **r**.

- If the *attractive* potential **V(r)** doesn't satisfy these conditions, the qualitative nature of the orbits will be **altered from our discussion.**
- However, we can still use same method to examine the orbits.
- Example: $\mathbf{V(r) = -(a/r^3)}$ $\quad$ ($\mathbf{a}$ = constant)

$\Rightarrow$ Force: $\mathbf{f(r) = -\,(\partial V/\partial r) = -(3a/r^4)}$.

$V'(r)$ for Attractive r^{-4} Forces

- **Example:** $V(r) = -(a/r^3); \Rightarrow f(r) = -(3a/r^4)$.
  (Fig): Eff. potential: $V'(r) = -(a/r^3) + (½)[\ell^2/(mr^2)]$
- Energy **E**, 2 motion types, depending on **r**:
- $r < r_1$, ***bounded orbit.***

$r < r_1$ always. Particle passes through center of force ($r = 0$).

- $r > r_2$, ***unbounded orbit***.

$r > r_2$ always. Particle can never get to the center force ($r = 0$).

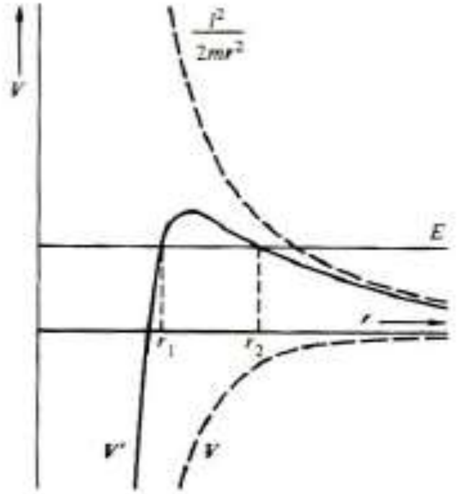


FIGURE 3.9 The equivalent one-dimensional potential for an attractive inverse-fourth law of force.

- $r_1 < r < r_2$. Not possible physically, since would require $E - V'(r) = (½)m\dot{r}^2 < 0 \Rightarrow$ Unphysical! $\Rightarrow \dot{r}$ imaginary!

$V'(r)$: Isotropic Simple Harmonic Oscillator

- **Example:** Isotropic Simple Harmonic Oscillator:

$$f(r) = -kr,\ V(r) = (\tfrac{1}{2})kr^2$$

Effective potential: $V'(r) = (\tfrac{1}{2})kr^2 + (\tfrac{1}{2})[\ell^2/(mr^2)]$

- $\ell = 0 \Rightarrow V'(r) = V(r) = (\tfrac{1}{2})kr^2$ (figure):

Any **E >0**: Motion is straight line in "**r**" direction. **Simple harmonic**. Passes through **r = 0**. Turning point at $\mathbf{r_1}$ = motion amplitude.

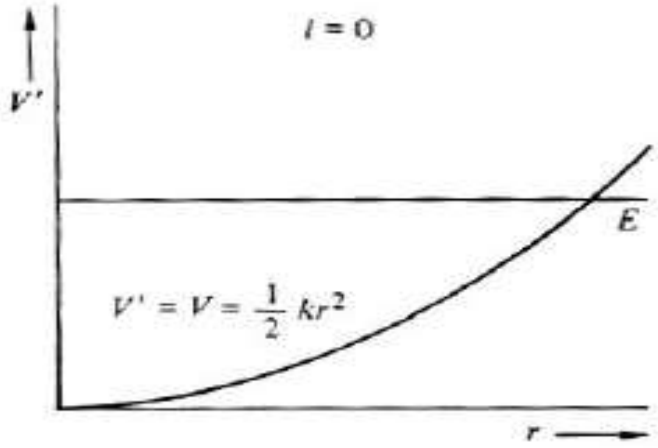


FIGURE 3.10 Effective potential for zero angular momentum.

$\mathbf{E - V(r) = (\tfrac{1}{2})m\dot{r}^2 > 0} \Rightarrow$ Speeds up as heads towards **r = 0**, slows down as heads away from **r = 0**. Stops at $\mathbf{r_1}$, turns around.

- Isotropic Simple Harmonic Oscillator:

  $\mathbf{f(r) = -\,kr,\ V'(r) = (½)kr^2 + (½)[\ell^2/(mr^2)]}$

- $\ell \neq \mathbf{0} \Rightarrow$ (fig):

All **E**: **Bounded orbit.** Turning points $\mathbf{r_1}$ & $\mathbf{r_2}$.

$\mathbf{E - V'(r)} = \mathbf{(½)\, m\dot{r}^2 > 0}$

Does not pass through $\mathbf{r = 0}$

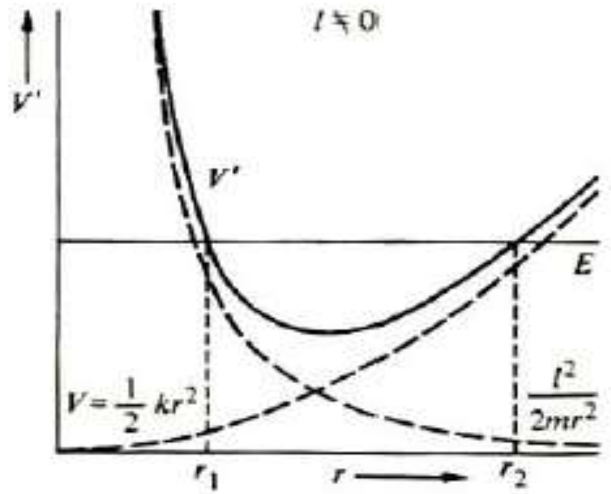


FIGURE 3.11 The equivalent one-dimensional potential for a linear restoring force.

$\Rightarrow$ Oscillates in **r** between $\mathbf{r_1}$ & $\mathbf{r_2}$. **Motion in plane** $\mathbf{(r(\theta))}$ **is elliptic.** Proof: Take **x** & **y** components of force: $\mathbf{f_x = -kx}$, $\mathbf{f_y = -ky}$. $\mathbf{r(\theta)}$ = Superposition of 2, 1d SHO's, same frequency, moving at right angles to each other

Kepler Problem: r^{-2} Force Law

- **Inverse square law force:**

$$F(r) = -(k/r^2);\ V(r) = -(k/r)$$

  – The ***most important*** special case of Central Force motion!

- **Special case:** **Motion of planets** (& other objects) about Sun. (Also, of course, motion of Moon & artificial satellites about Earth!)

Force = Newton's Universal Law of Gravitation

$\Rightarrow$ **k = GmM**; **m =** planet mass, **M =** Sun mass

(or **m =** Moon mass, **M =** Earth mass)

› Relative coordinate problem was solved using reduced mass **μ**: $\boldsymbol{\mu^{-1} \equiv m^{-1} + M^{-1} = (m^{-1})[1 + mM^{-1}]}$

SOME PROPERTIES OF THE PRINCIPAL OBJECTS IN THE SOLAR SYSTEM

Name	Semimajor axis of orbit (in astronomical units[a])	Period (yr)	Eccentricity	Mass (in units of the mass of the Earth[b])
Sun	—	—	—	333,480
Mercury	0.3871	0.2408	0.2056	0.0553
Venus	0.7233	0.6152	0.0068	0.8150
Earth	1.0000	1.0000	0.0167	1.000
Eros (asteroid)	1.4583	1.7610	0.2230	2×10^{-9} (?)
Mars	1.5237	1.8809	0.0934	0.1074
Ceres (asteroid)	*c*	4.6035	0.0765	1/8000 (?)
Jupiter	5.2028	*c*	0.0483	317.89
Saturn	9.5388	29.456	0.0560	*c*
Uranus	19.191	84.07	0.0461	14.56
Neptune	30.061	164.81	0.0100	17.15
Pluto	39.529	248.53	0.2484	0.002
Halley (comet)	18	76	0.967	$\sim10^{-10}$

[a] One astronomical unit (A.U.) is the length of the semimajor axis of the Earth's orbit. One A.U. $\cong 1.495 \times 10^{11}$ m $\cong 93 \times 10^6$ miles.

[b] The mass of the Earth is approximately 5.976×10^{24} kg.

$$\mu^{-1} \equiv m^{-1} + M^{-1} = (m^{-1})[1+ mM^{-1}]$$

› From Table: ***For all planets***

$$m << M \Rightarrow \mu^{-1} \cong m^{-1} \text{ or } \mu \cong m$$

– Similarly, ≈ true **for Moon and Earth**

› Definitely true for artificial satellites & Earth!

– **Corrections:** $\mu = (m)[1+ mM^{-1}]^{-1}$

$$\mu \cong m[1 - mM^{-1} + mM^{-2} - \dots]$$

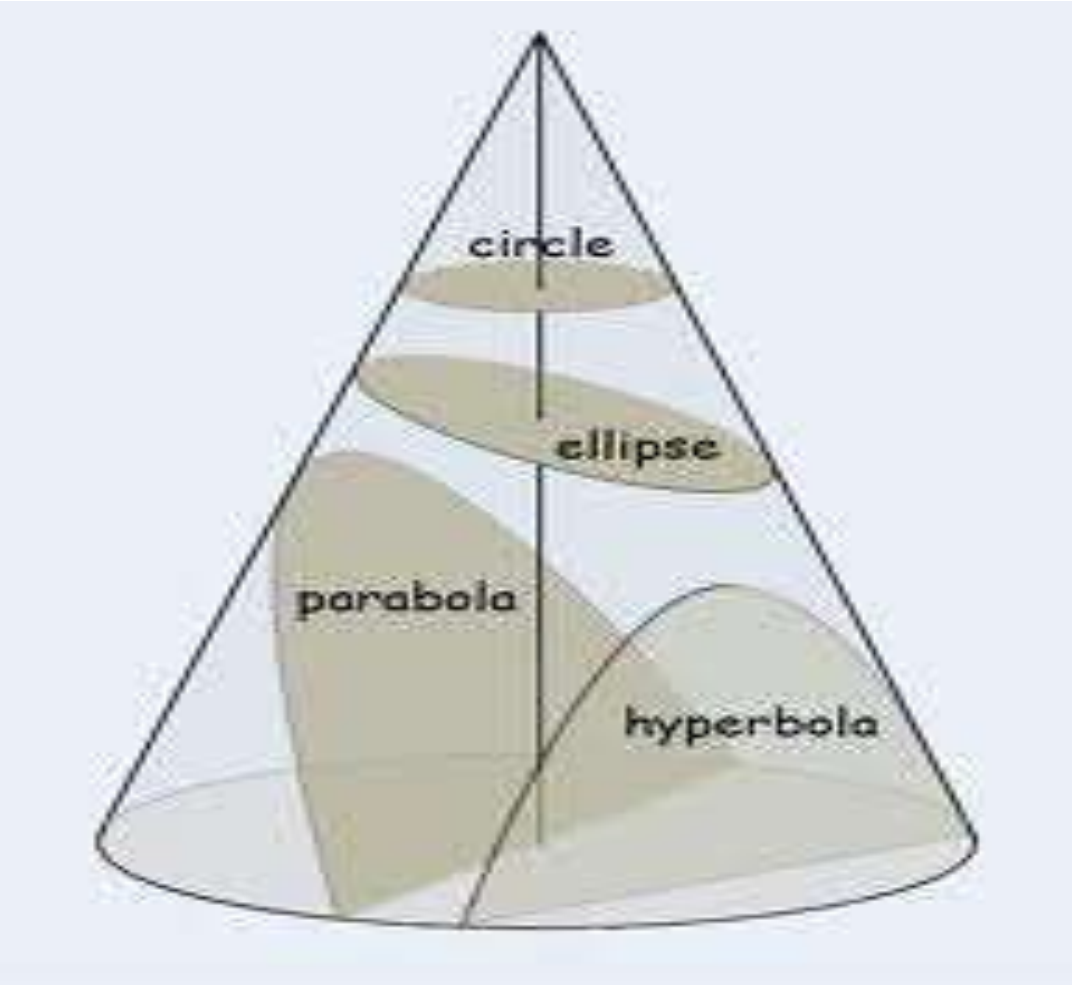
⇒ In what follows, **μ** is replaced by **m** (as it has been for most of the discussion so far)

› Note also (useful for numerical calculations):

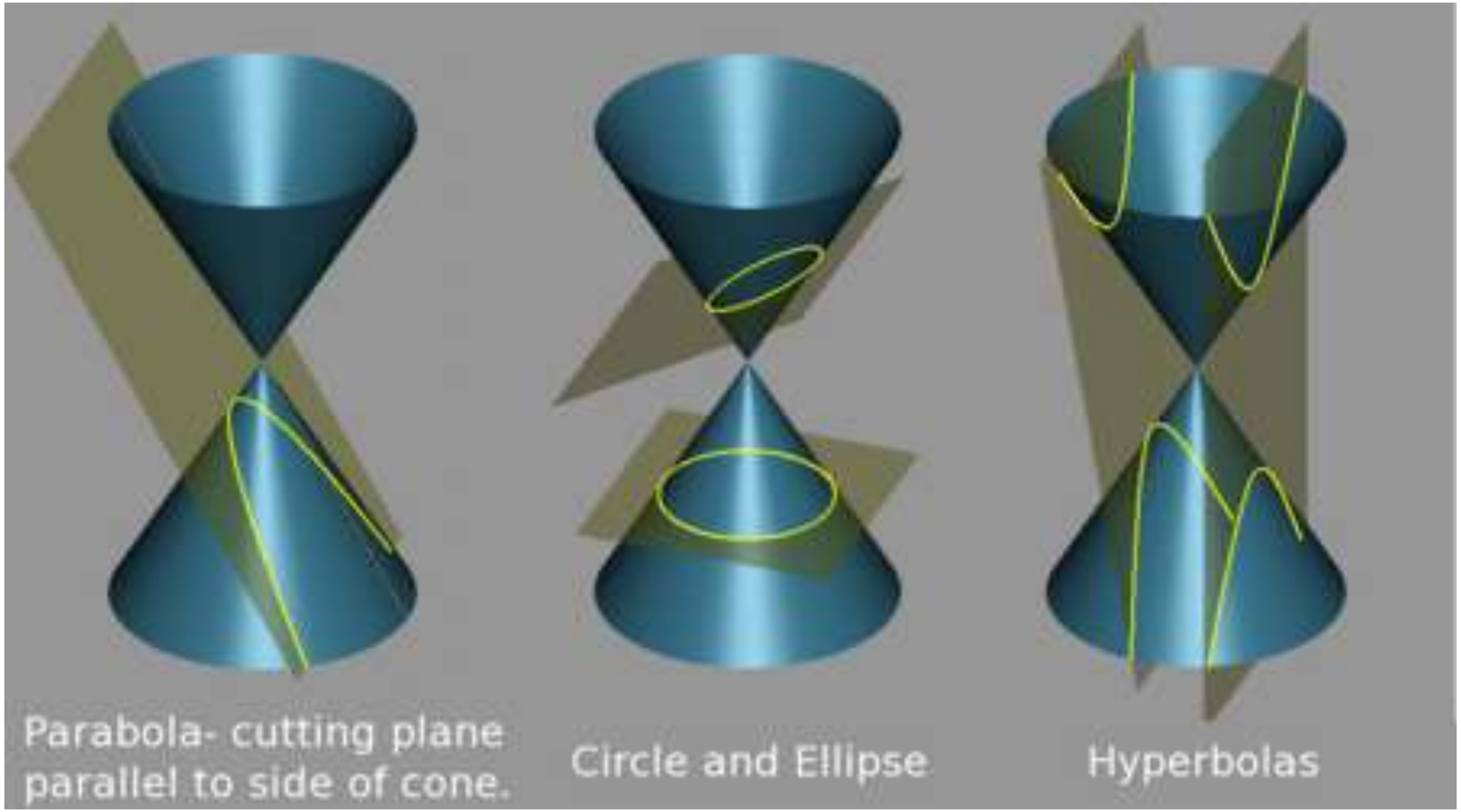
$$\Rightarrow \quad (k/\mu) \cong (k/m) \cong GM$$

CONIC SECTIONS

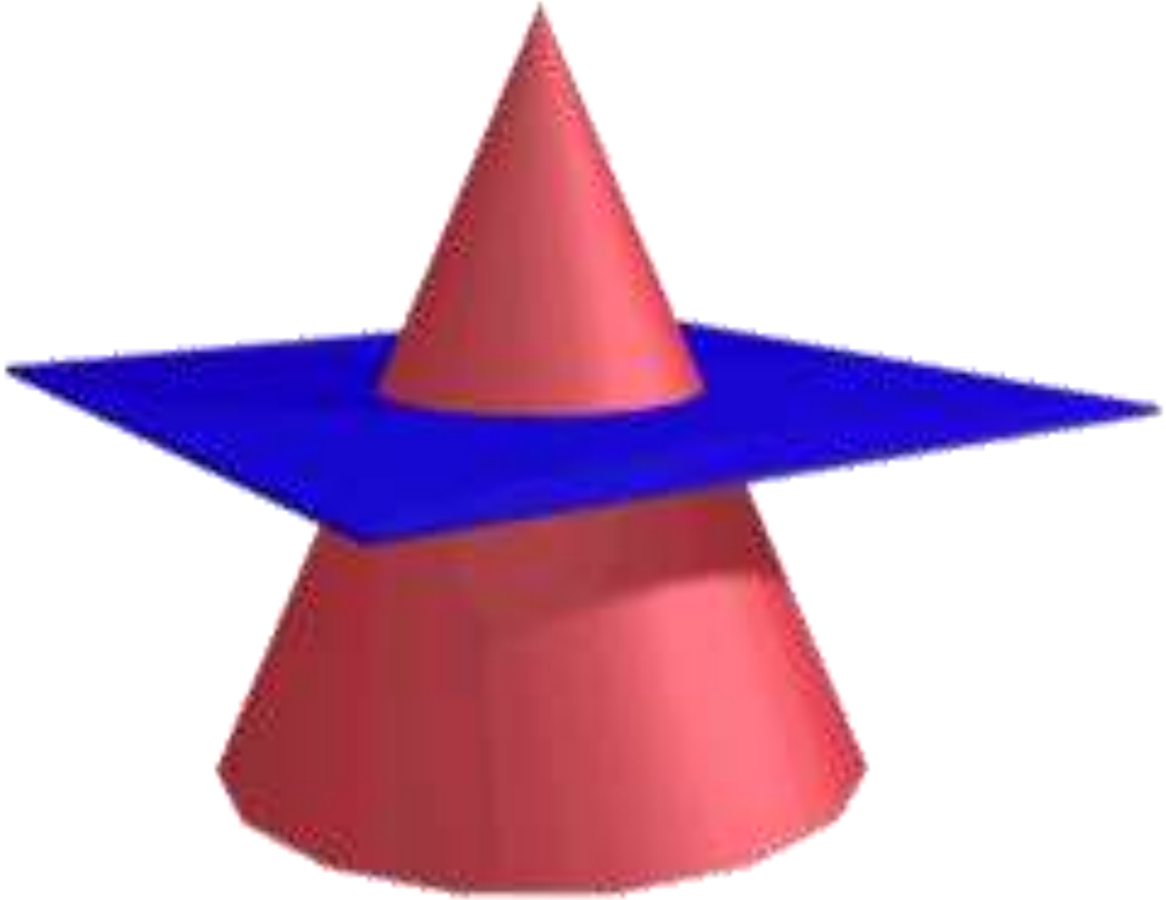
Conic Section: Any figure that can be formed by slicing a double cone with a plane



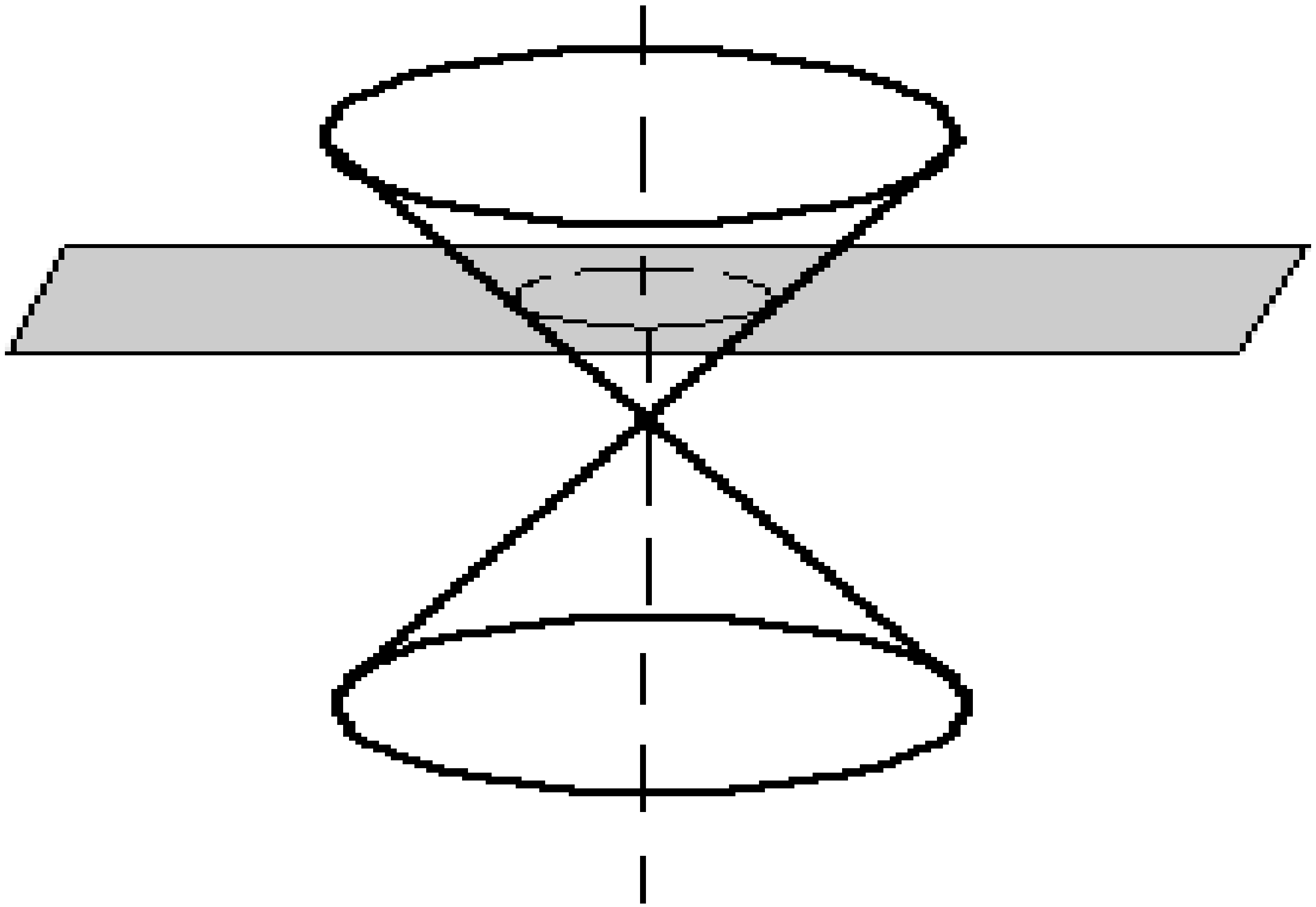
OTHER VIEW OF CONIC SECTIONS



THE CIRCLE



CONIC SECTION – THE CIRCLE



Equation for a Circle

- Standard Form: $x^2 + y^2 = r^2$
- You can determine the equation for a circle by using the distance formula then applying the standard form equation.
- Or you can use the standard form.
- Most of the time we will assume the center is (0,0). If it is otherwise, it will be stated.
- It might look like: $(x-h)^2 + (y - k)^2 = r^2$

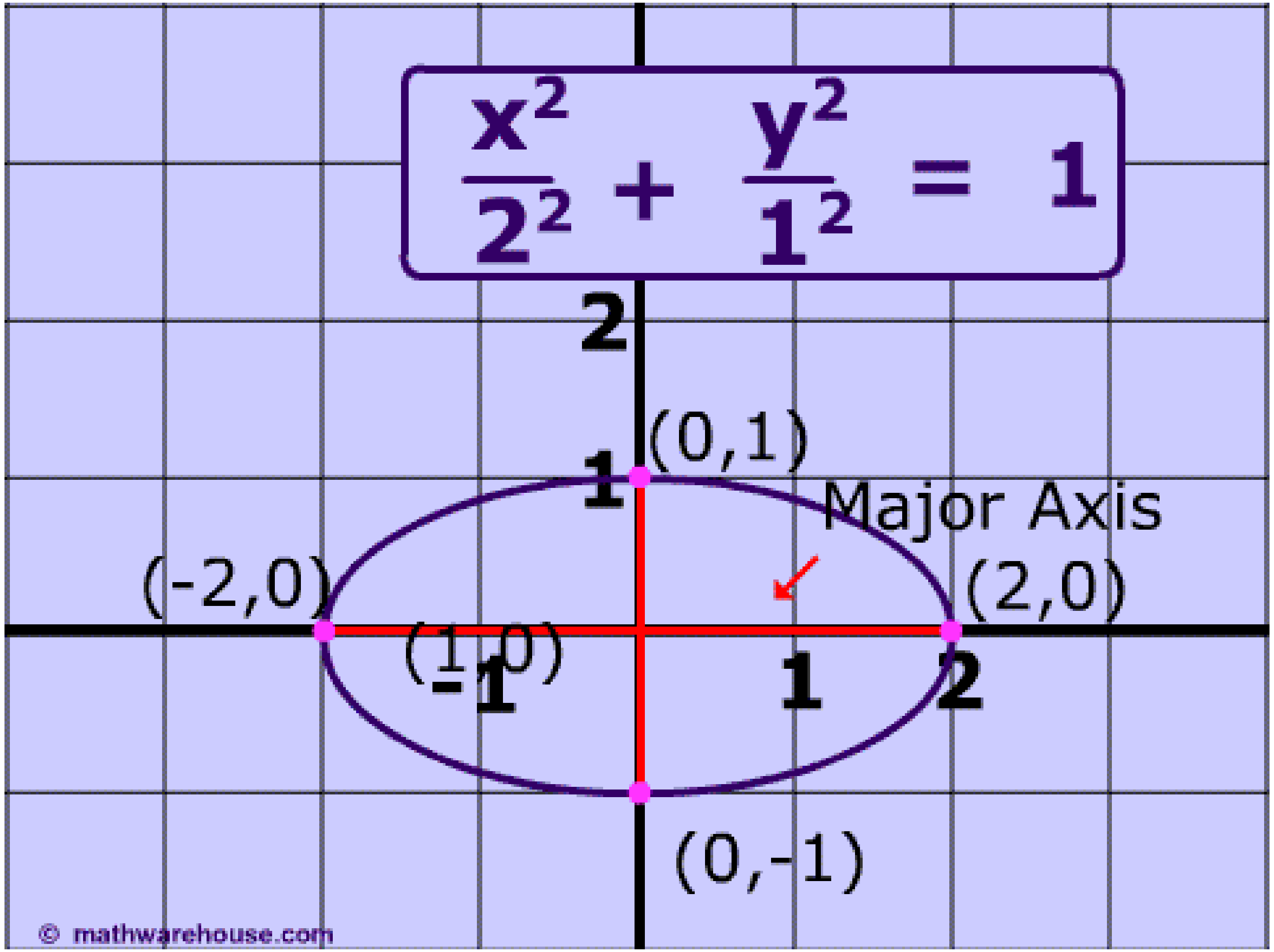
ELLIPSES

- Ellipse – A set of points in A plane such that the sum of the distance from two foci to any point on the ellipse is constant
- focus (foci - plural) – one of two fixed points within in an ellipse such that the sum of the distances from the points to any other point on the ellipse is constant

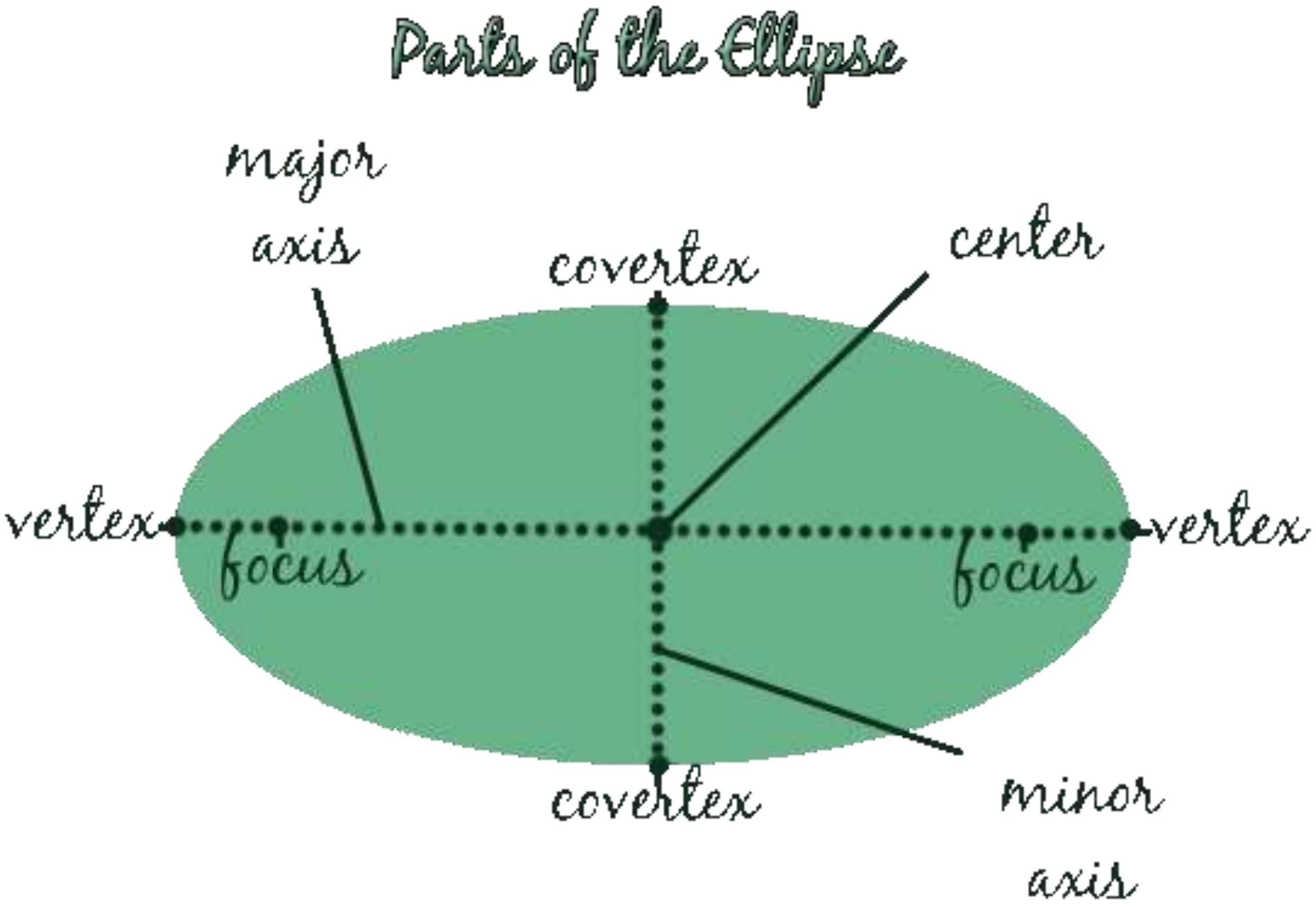
Vocabulary for Ellipses

- Vertices – for an ellipse, the y and x intercepts are the vertices
- Major axis – for an ellipse, the longer axis of symmetry, the axis that contains the foci
- Minor axis – for an ellipse, the shorter axis of symmetry
- Center – for an ellipse, the intersection of the major and minor arcs

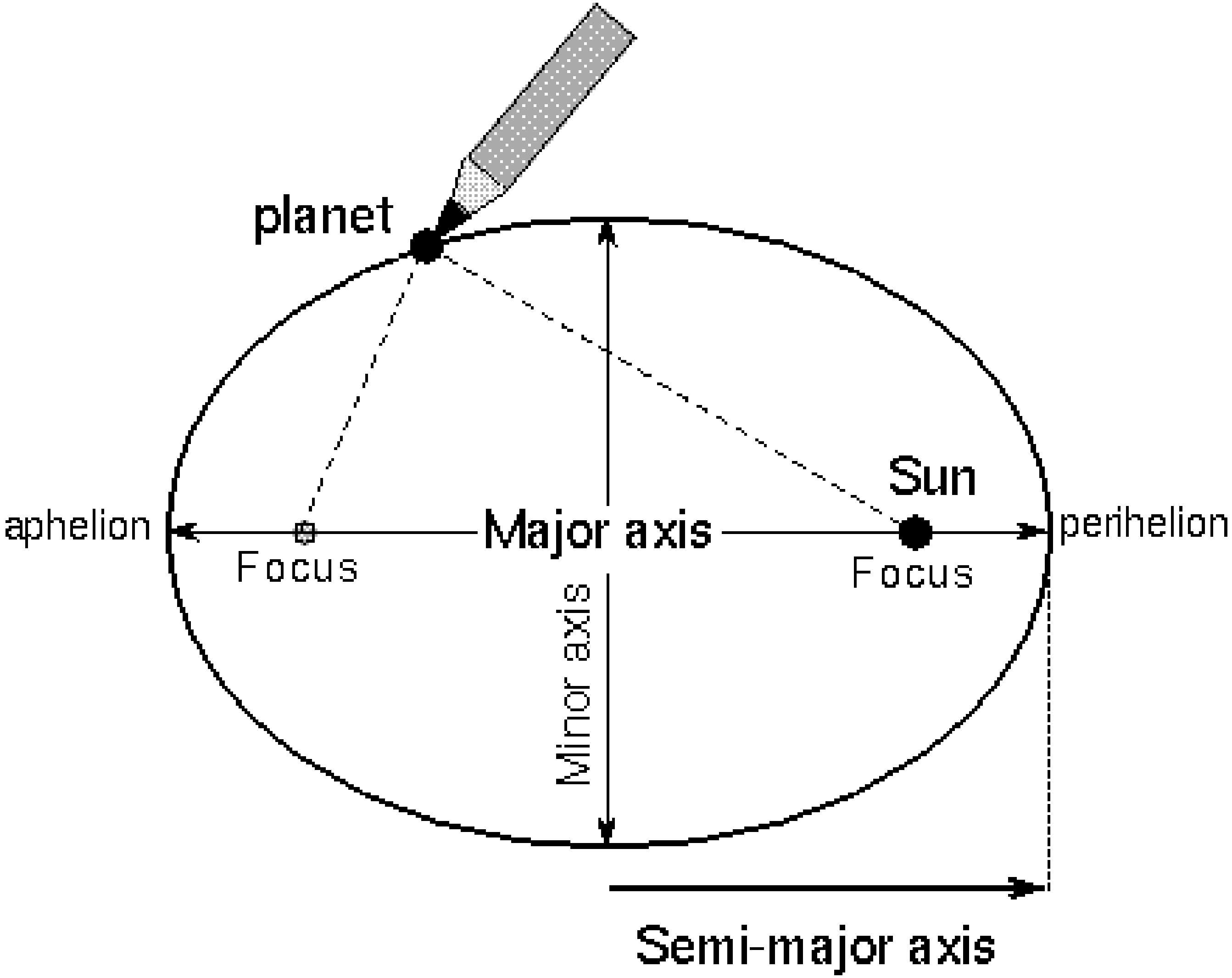
Equation for an Ellipse



Parts of an Ellipse



EXAMPLES



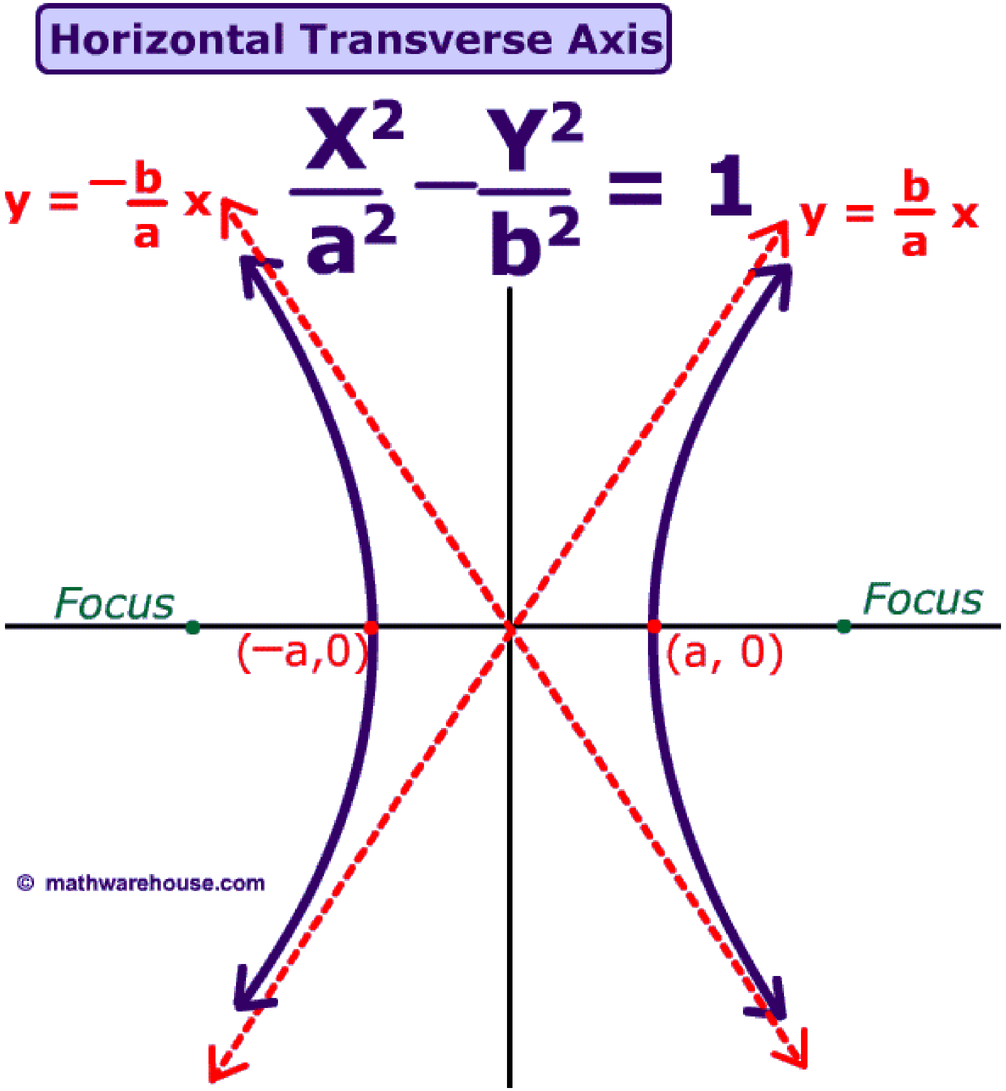
Perihelion – point *closest* to the sun in such an orbit

Aphelion – point *farthest* from the sun in such an orbit

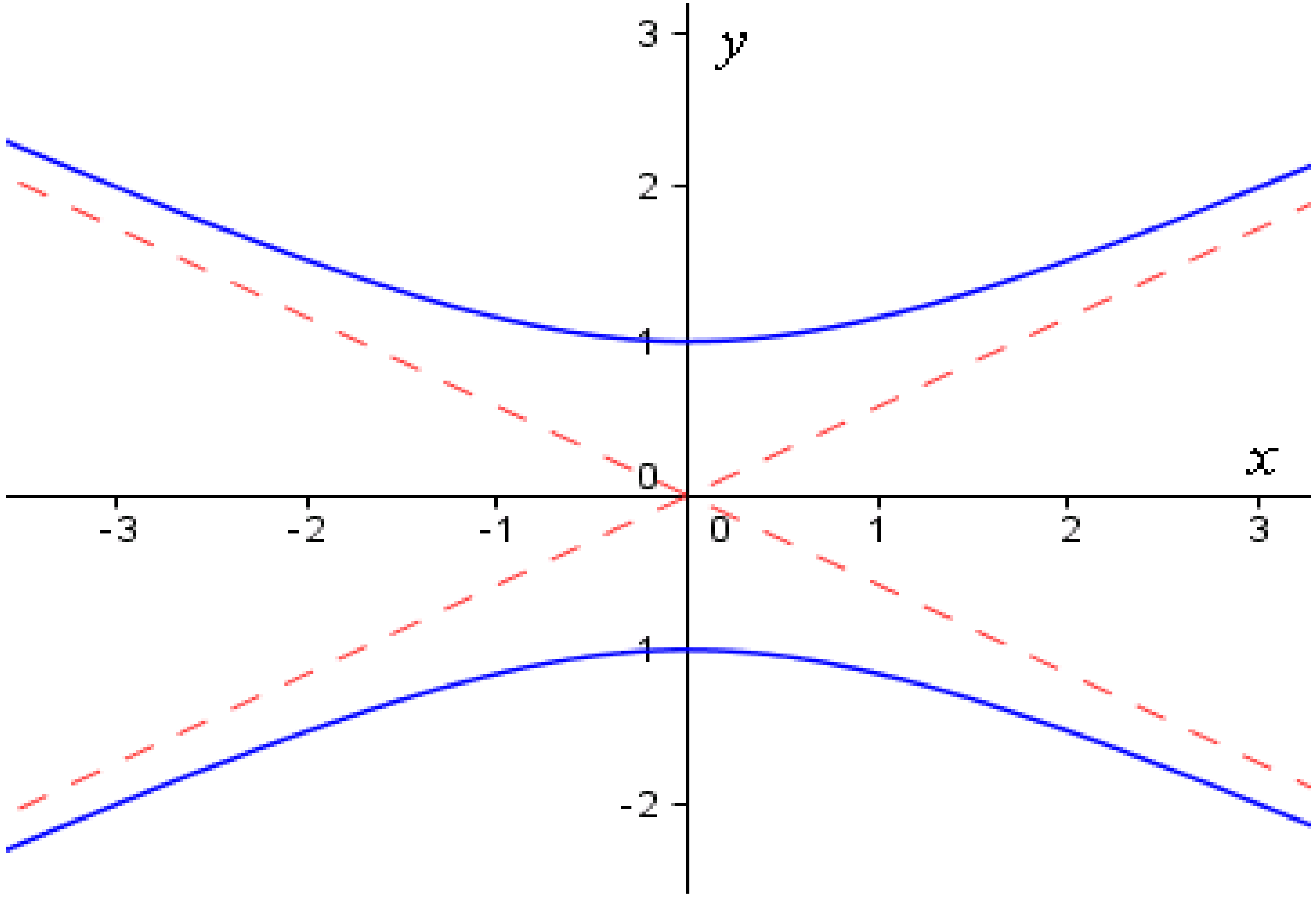
HYPERBOLAS

- Hypoberla – a set of points such that the difference of the distances from two fixed points to any point on the hyperbola is constant
- Vertices – x or y intercepts of a hyperbola
- Asymptote – a straight line that a curve approaches but never reaches

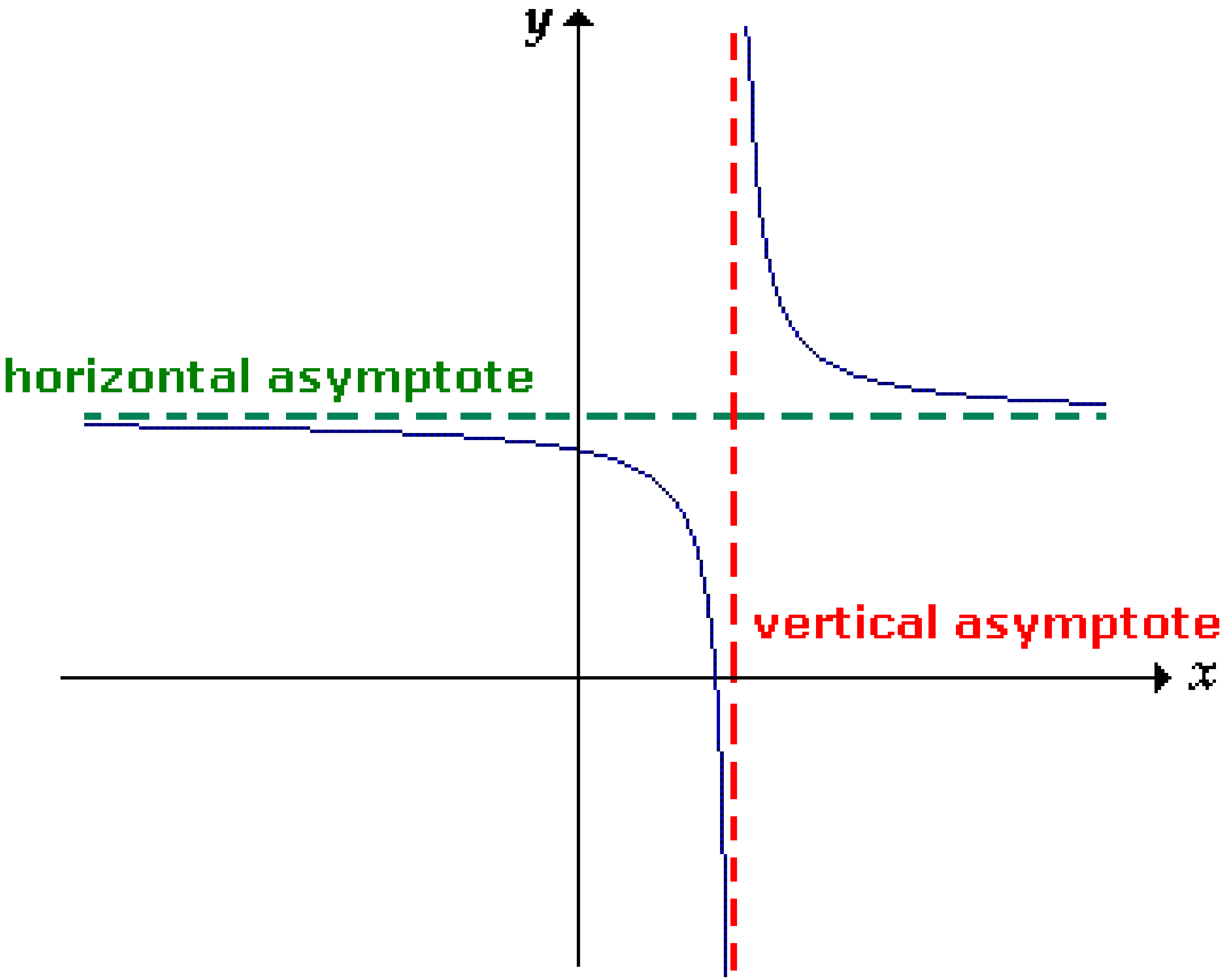
WHAT DOES IT LOOK LIKE?AND WHAT IT ITS FORMULA?



ASYMPTOTES (IN RED)



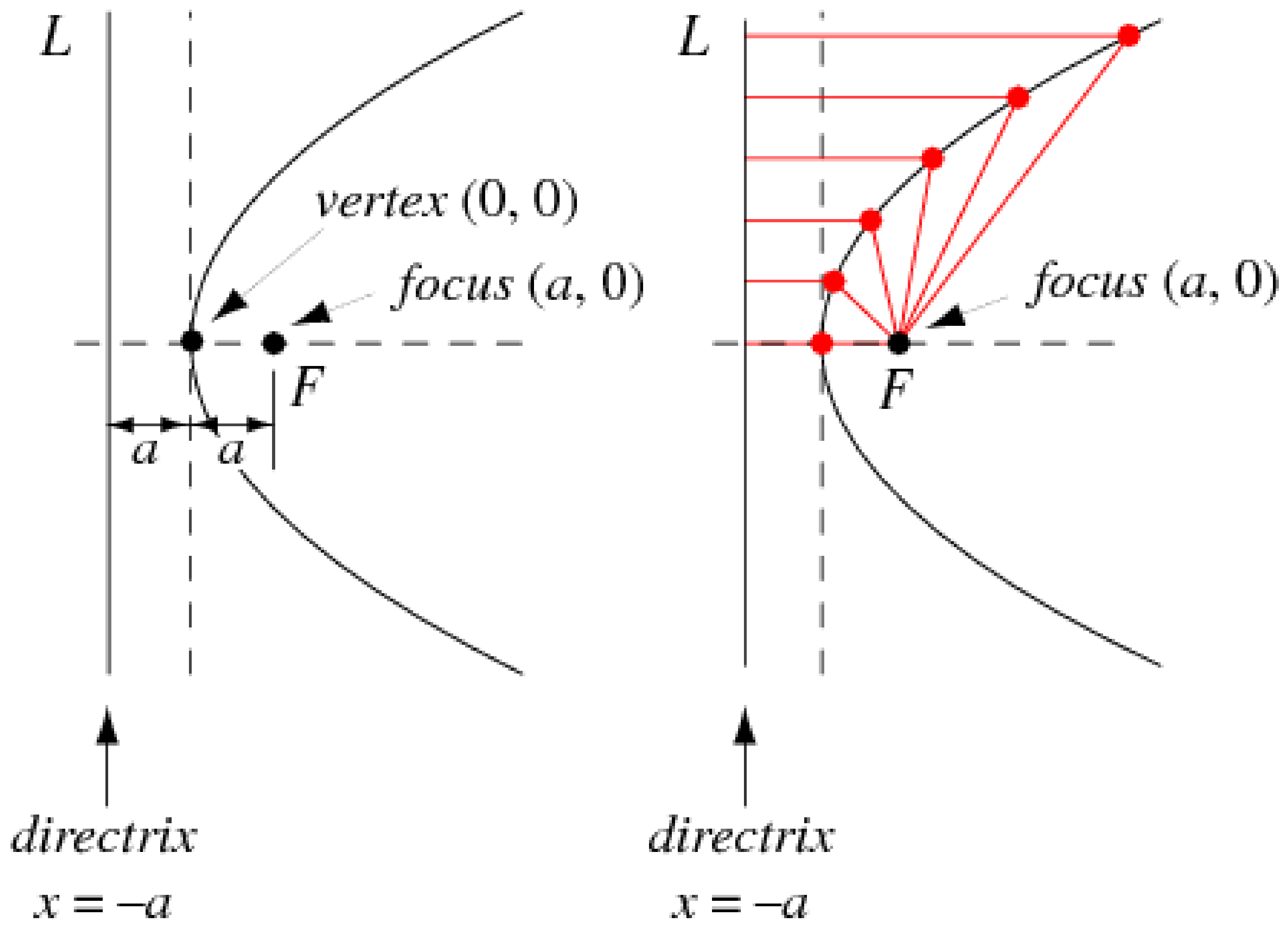
ASYMPTOTES



PARABOLAS

- **Parabolas – a set of points in a plane that are equidistant from a focus and a fixed line – the directrix**
- **Directrix – the fixed straight line that together with the point known as the focus serves to define a parabola.**

WHAT DOES IT LOOK LIKE?



ECCENTRICITY

- Eccentricity – a ratio of the distance from the focus and the distance from the directrix.
- Each shape has its own eccentricity: circle, parabolas, hyperbolas, and ellipses.
- Circle: $e = 0$
- Ellipse: $e = 0 < e < 1$
- Parabola: $e = 1$
- Hyperbola: $e > 1$

Definition: Eccentricity of an Ellipse

The **eccentricity** of an ellipse is

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}$$

where *a* is the semimajor axis, *b* is the semiminor axis, and *c* is the distance from the center of the ellipse to either focus.

What is the range of possible "*e*" values for an ellipse?

What happens when "*e*" is zero?

$$0 \leq e < 1$$

→ A CIRCLE!!!

Practice Problems

The Earth’s orbit has a semimajor axis $a \approx 149.598\,\text{Gm}$ and an eccentricity of $e \approx 0.0167$.

Calculate and interpret *b* and *c*.

$$c = ea \approx 0.0167 \times 149.598 = 2.4982866$$

$$b = \sqrt{a^2 - c^2} \approx \sqrt{149.598^2 - 2.4982866^2} \approx 149.577$$

Seminor Axis

The semiminor axis is only 0.014% shorter than the semimajor axis…

Practice Problems

The Earth's orbit has a semimajor axis $a \approx 149.598\text{Gm}$ and an eccentricity of $e \approx 0.0167$.

Calculate and interpret *b* and *c*.

$$c = ea \approx 0.0167 \times 149.598 = 2.4982866$$

$$b = \sqrt{a^2 - c^2} \approx \sqrt{149.598^2 - 2.4982866^2} \approx 149.577$$

Aphelion of Earth: $a + c \approx 152.096\text{Gm}$

Perihelion of Earth: $a - c \approx 147.100\text{Gm}$

The Earth's orbit is nearly a perfect circle, but the eccentricity as a percentage is 1.67%; this measures how far *off-center* the Sun is…

Planetary Motion

- General result for ***Orbit*** **θ(r)** was:

$$\theta(r) = \pm \int (\ell/r^2)(2m)^{-\frac{1}{2}}[E - V(r) - \{\ell^2/(2mr^2)\}]^{-\frac{1}{2}}\, dr + \theta'$$

  - θ' = integration constant

- Put **V(r) = -(k/r)** into this:

$$\theta(r) = \pm \int (\ell/r^2)(2m)^{-\frac{1}{2}}[E + (k/r) - \{\ell^2/(2mr^2)\}]^{-\frac{1}{2}}\, dr + \theta'$$

- Integrate by first changing variables: Let **u ≡ (1/r):**

$$\theta(u) = \ell(2m)^{-\frac{1}{2}}\int du\, [E + k\,u - \{\ell^2/(2m)\}u^2]^{-\frac{1}{2}} + \theta'$$

- Tabulated. Result is: (**r = 1/u**)

$$\theta(r) = \cos^{-1}[G(r)] + \theta'$$

$$G(r) \equiv [(\alpha/r) - 1]/e \; ; \; \alpha \equiv [\ell^2/(mk)]$$

$$e \equiv [\,1 + \{2E\ell^2/(mk^2)\}]^{\frac{1}{2}}$$

- **Orbit for inverse square law force:**

$$\cos(\theta - \theta') = [(\alpha/r) - 1]/e \qquad (1)$$

$$\alpha \equiv [\ell^2/(mk)]; \quad e \equiv [1 + \{2E\ell^2/(mk^2)\}]^{1/2}$$

- Rewrite **(1)** as:

$$(\alpha/r) = 1 + e\cos(\theta - \theta') \qquad (2)$$

- **(2)** ≡ ***CONIC SECTION*** (analytic geometry!)
- Orbit properties:

e ≡ *Eccentricity*

2α ≡ *Latus Rectum*

Conic Sections

$\Rightarrow$ A very important result!

<u>All</u> orbits for inverse r-squared forces

(attractive or repulsive) ***are conic sections***

$$(\alpha/r) = 1 + e\cos(\theta - \theta')$$

with

Eccentricity $\equiv e = [\,1 + \{2E\ell^2/(mk^2)\}]^{1/2}$

and

Latus Rectum $\equiv 2\alpha = [2\ell^2/(mk)]$

Conic Sections

- **Conic sections:** Curves formed by the intersection of a plane and a cone.
- **A conic section:** A curve formed by the loci of points (in a plane) where the ratio of the distance from a fixed point (***the focus***) to a fixed line (***the directorix***) is a constant.
- **Conic Section**

$$(\alpha/r) = 1 + e\cos(\theta - \theta')$$

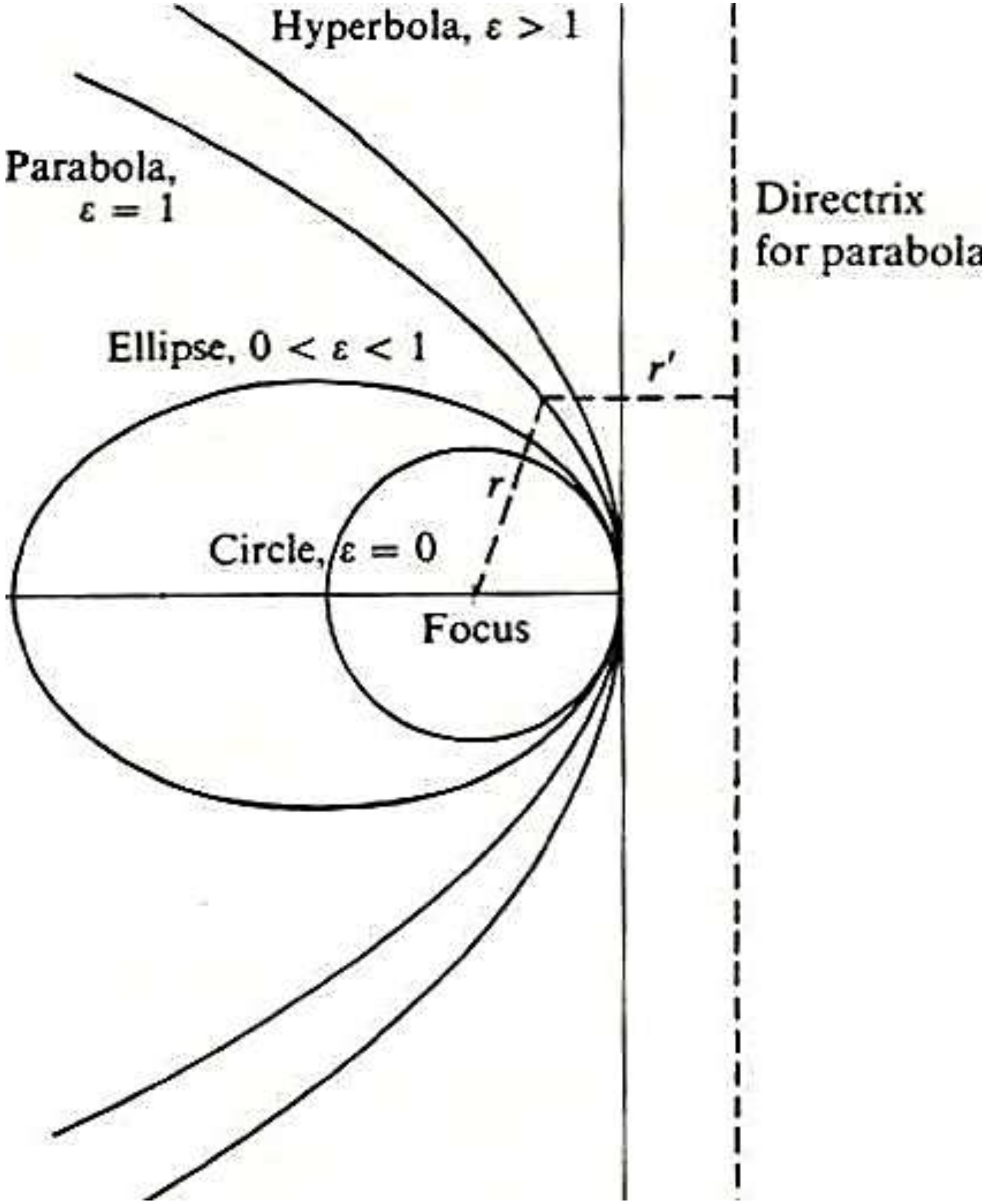
- The specific type of curve depends on eccentricity **e.** For objects in orbit, this, in turn, depends on the energy **E** and the angular momentum ℓ.

- **Conic Section**

$(\alpha/r) = 1 + e\cos(\theta - \theta')$

- Type of curve depends on eccentricity **e.**

  In Figure, **$\varepsilon \equiv e$**



Conic Section Orbits

› In the following discussion, we need 2 properties of the effective (1d, **r**-dependent) potential, which (as we've seen) **governs the orbit behavior** for a fixed energy **E** & angular momentum ℓ. For **V(r) = -(k/r)** this is:

$$V'(r) = -(k/r) + [\ell^2/\{2m(r)^2\}]$$

1. It is easily shown that **the r = r_0 where V´(r) has a minimum is: $r_0 = [\ell^2/(2mk)]$**. (We've seen in our general discussion that this is the radius of a circular orbit.)

2. Its also easily shown that **the value of V´at r_0 is:**

$$V'(r_0) = -(mk^2)/(2\ell^2) \equiv (V')_{min} \equiv E_{circular}$$

- We've shown that all orbits for inverse **r**-squared forces (attractive or repulsive) are **conic sections**

$$(\alpha/r) = 1 + e\cos(\theta - \theta')$$

- – As we just saw, ***the shape*** of curve (orbit) depends on the eccentricity $\mathbf{e} \equiv [1 + \{2E\ell^2/(mk^2)\}]^{1/2}$
- – Clearly this depends on energy **E**, & angular momentum ℓ**!**
- – Note: $(V')_{min} \equiv -(mk^2)/(2\ell^2)$

$e > 1 \Rightarrow E > 0 \Rightarrow$ ***Hyperbola***

$e = 1 \Rightarrow E = 0 \Rightarrow$ ***Parabola***

$0 < e < 1 \Rightarrow (V')_{min} < E < 0 \Rightarrow$ ***Ellipse***

$e = 0 \Rightarrow E = (V')_{min} \Rightarrow$ ***Circle***

$e =$ ***imaginary*** $\Rightarrow E < (V')_{min} \Rightarrow$ ***Not Allowed!***

- **Terminology for conic section orbits:**

Integration const $\Rightarrow$ $\mathbf{r = r_{min}}$ when $\boldsymbol{\theta = \theta'}$

$\mathbf{r_{min}} \equiv$ ***Pericenter*** ; $\mathbf{r_{max}} \equiv$ ***Apocenter***

Any radial turning point $\equiv$ ***Apside***

Orbit about sun: $\mathbf{r_{min}} \equiv$ ***Perihelion***

$\mathbf{r_{max}} \equiv$ ***Aphelion***

Orbit about earth: $\mathbf{r_{min}} \equiv$ ***Perigee***

$\mathbf{r_{max}} \equiv$ ***Apogee***

- **Conic Section:** $(\alpha/r) = 1 + e\cos(\theta - \theta')$

  $e \equiv [1 + \{2E\ell^2/(mk^2)\}]^{1/2} \quad \alpha \equiv [\ell^2/(mk)]$

- $e > 1 \Rightarrow E > 0 \Rightarrow$ ***Hyperbola***

  Occurs for the ***repulsive* Coulomb** force: See scattering discussion,

$0 < e < 1 \Rightarrow V_{min} < E < 0 \Rightarrow$ ***Ellipse***

$(V_{min} \equiv -(mk^2)/(2\ell^2))$ Occurs for the ***attractive* Coulomb** force & the **Gravitational** force:

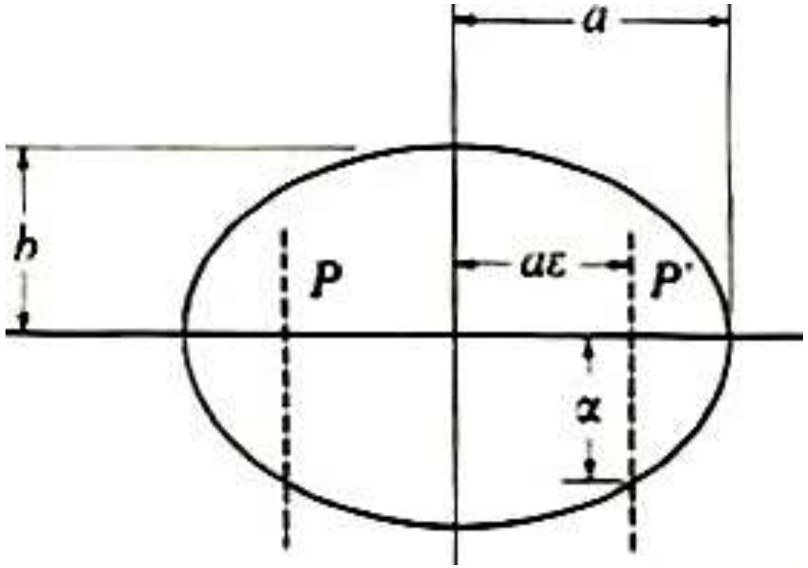
The Orbits of all of the planets (& several other solar system objects) are ellipses with the Sun at one focus. (Again, see table).

Most planets, $e \ll 1$ (see table) $\Rightarrow$ Their orbit is almost circular!

Planetary Orbits

- **Planetary orbits in terms of ellipse geometry.**

  In the figure, **ε ≡ e**



- Compute **major & minor axes** (**2a** & **2b**) as in text.

  Get (recall **k = GmM**):

  $\mathbf{a \equiv (\alpha)/[1 - e^2] = (k)/(2|E|)}$

  (depends only on energy **E**)

  $\mathbf{b \equiv (\alpha)/[1 - e^2]^{1/2} = (\ell)/(2m|E|)^{1/2} \equiv a[1 - e^2]^{1/2} \equiv (\alpha a)^{1/2}}$

  (Depends on both energy **E** & angular momentum ℓ)

- **Apsidal distances** $\mathbf{r_{min}}$ & $\mathbf{r_{max}}$ (or $\mathbf{r_1}$ & $\mathbf{r_2}$):

  $\mathbf{r_{min} = a(1- e) = (\alpha)/(1 + e), \; r_{max} = a(1+ e) = (\alpha)/(1 - e)}$

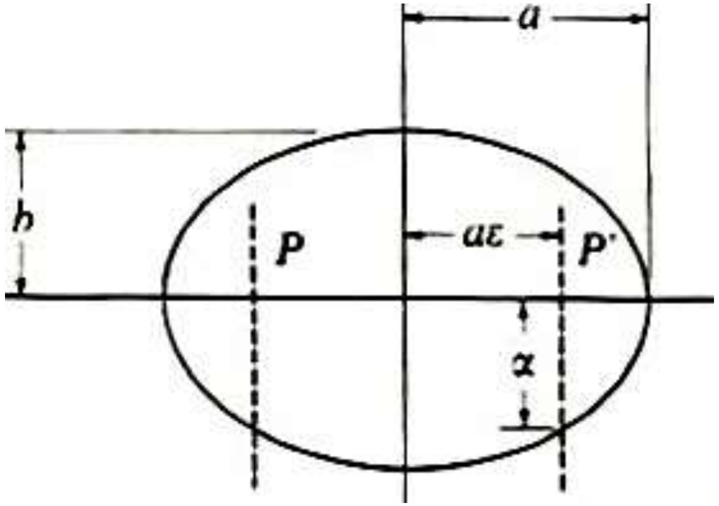
⇒ **Orbit eqtn** is: $\mathbf{r = a(1- e^2)/[1 + e\cos(\theta - \theta')]}$

- **Planetary orbits = ellipses, sun at one focus:** Fig:
- For a general central force,

  we had **Kepler's 2nd Law:**

  (Constant areal velocity!):

  $(dA/dt) = (\ell)/(2m)$ $=$ const



  Use to compute **orbit period:**

  $\Rightarrow$ **$dt = (2m)/(\ell)\, dA$**

  Period = time to sweep out ellipse area:

  $\tau = \int dt = [(2m)/(\ell)] \int dA = [(2m)/(\ell)]A$

- **Period of elliptical orbit:**

$$\tau = [(2m)/(\ell)]\,A \qquad (A = \text{ellipse area}) \qquad (1)$$

- Analytic geometry: Area of ellipse:

$$A \equiv \pi ab \qquad (2)$$

- In terms of **k**, **E** & ℓ**,** we just had:

$$a = (k)/(2|E|);\ b = (\ell)/(2mE)^{1/2} \qquad (3)$$

$$(1), (2), (3) \Rightarrow \tau = \pi k(m/2)^{1/2}|E|^{-(3/2)}$$

- Alternatively: $b = (\alpha a)^{1/2}$; $\alpha \equiv [\ell^2/(mk)]$

$$\Rightarrow \qquad \tau^2 = [(4\pi^2 m)/(k)]\,a^3$$

The square of the period is proportional to cube of semimajor axis of the elliptic orbit

$\equiv$ ***Kepler's Third Law***

› Kepler's Third Law

$$\Rightarrow \qquad \tau^2 = [(4\pi^2 m)/(k)]\, a^3$$

The square of period is proportional to the cube of the semimajor axis of the elliptic orbit

› ***Note:*** Actually, **m** $\rightarrow$ **μ**. The reduced mass **μ** actually enters! As derived empirically by Kepler: Kepler's 3rd Law states that this is true with the same proportionality constant for all planets. This ignores the difference between the reduced mass **μ** & the mass **m** of the planet: $\mu = (m)[1 + mM^{-1}]^{-1}$

$$\mu \cong m[1 - (m/M) + (m/M)^2 - \ldots]$$

Note: $k = GmM$; $\mu \cong m \quad (m \ll M)$

$$\Rightarrow \qquad (\mu/k) \cong 1/(GM)$$

$$\Rightarrow \qquad \tau^2 = [(4\pi^2)/(GM)]a^3 \quad (m \ll M)$$

So Kepler was only approximately correct!

Kepler's Laws

› ***Kepler's First Law:***

The planets move in elliptic orbits with the Sun at one focus.

– Kepler proved empirically. Newton proved this from Universal Law of Gravitation & calculus.

› ***Kepler's Second Law:***

The area per unit time swept out by a radius vector from sun to a planet is constant. (Constant areal velocity).

$$(dA/dt) = (\ell)/(2m) = \text{constant}$$

– Kepler proved empirically. We've proven in general for any central force.

› ***Kepler's Third Law:*** $\tau^2 = [(4\pi^2 m)/(k)]\, a^3$

The square of a planet's period is proportional to cube of semimajor axis of the planet's elliptic orbit.

Example

- Halley's Comet, which passed around the sun early in 1986, moves in a highly elliptical orbit: Eccentricity **e = 0.967**; period **τ = 76 years.** Calculate its minimum and maximum distances from the sun.

- Use the formulas just derived & find:

$$\mathbf{r_{min} = 8.8 \times 10^{10}\ m}$$

(Inside Venus's orbit & almost to Mercury's orbit)

$$\mathbf{r_{max} = 5.27 \times 10^{12}\ m}$$

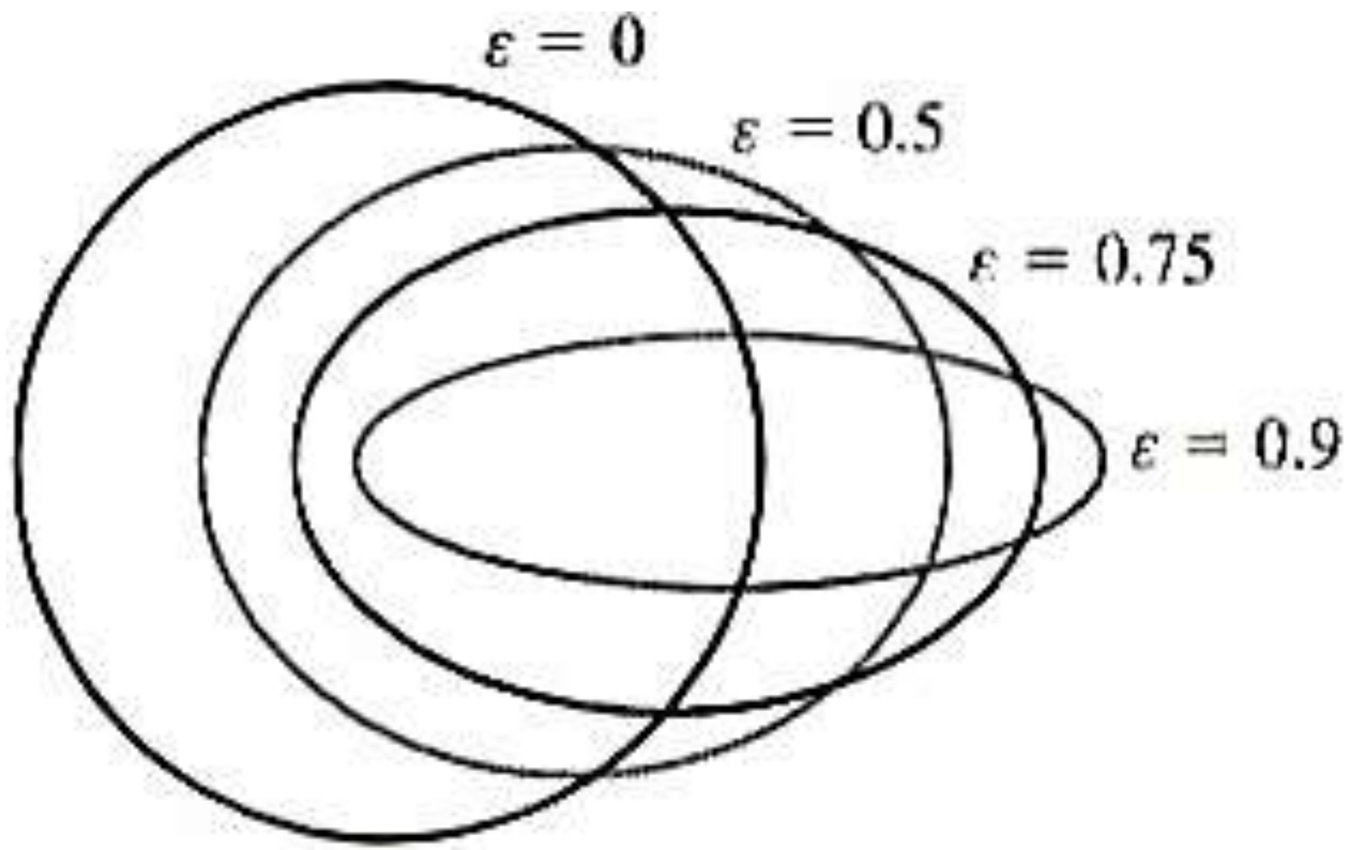
(Outside Neptune's orbit & near to Pluto's orbit)

› **Elliptical orbits:** Same semimajor axis **a = (k)/(2|E|)**

⇒ Same energy **E** & mass **m**, different eccentricities

$$e = [1+\{2E\ell^2/(mk^2)\}]^{1/2} \quad \text{(\& semiminor axes)}$$

$$b = (\ell)/(2m|E|)^{1/2} \Rightarrow \text{Different angular momenta } \ell$$



› **Orbit properties:** $\mathbf{r_1, r_2} \equiv$ apsidal distances, $\mathbf{p_r}$, $\mathbf{p_\theta} \equiv$ angular momenta, $\boldsymbol{\theta_1}, \boldsymbol{\theta_2} \equiv$ angular velocities at the apsidal distances, with respect to circular orbit, radius **a**. In Table, $\boldsymbol{\varepsilon} \equiv \mathbf{e}$

TABLE 3.1 Ellipse Properties

Ellipticity	$\frac{p_\theta}{l_0}$	$\frac{p_r a}{l_0}$ for $r = a$	$\frac{r_1}{a}$	$\frac{r_1}{a}$	$\frac{\dot\theta_1}{\dot\theta_0}$	$\frac{\dot\theta_2}{\dot\theta_0}$	$\frac{v_{\theta_1}}{v_0}$	$\frac{v_{\theta_2}}{v_0}$
ε	$\sqrt{1-\varepsilon^2}$	ε	$1-\varepsilon$	$1+\varepsilon$	$\sqrt{\frac{1-\varepsilon}{(1+\varepsilon)^3}}$	$\sqrt{\frac{1+\varepsilon}{(1-\varepsilon)^3}}$	$\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{3/2}$	$\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{3/2}$
0	1	0	1	1	1	1	1	1
0.1	0.995	0.1	0.9	1.1	0.822	1.23	0.740	1.35
0.25	0.968	0.25	0.75	1.25	0.620	1.72	0.465	2.15
0.5	0.867	0.5	0.5	1.5	0.384	3.46	0.192	5.20
0.75	0.661	0.75	0.25	1.75	0.216	10.58	0.054	18.5
0.9	0.435	0.9	0.1	1.9	0.121	43.6	0.012	82.8

› Velocity along particle path $\equiv \vec{\mathbf{v}} = \mathbf{v_r \hat{r} + v_\theta \hat{\theta}}$

$$\mathbf{v_r \equiv (p_r/m) = \dot{r},\ v_\theta \equiv r\dot{\theta} = [p_\theta/(mr)]}$$

› Orbit phase space properties:

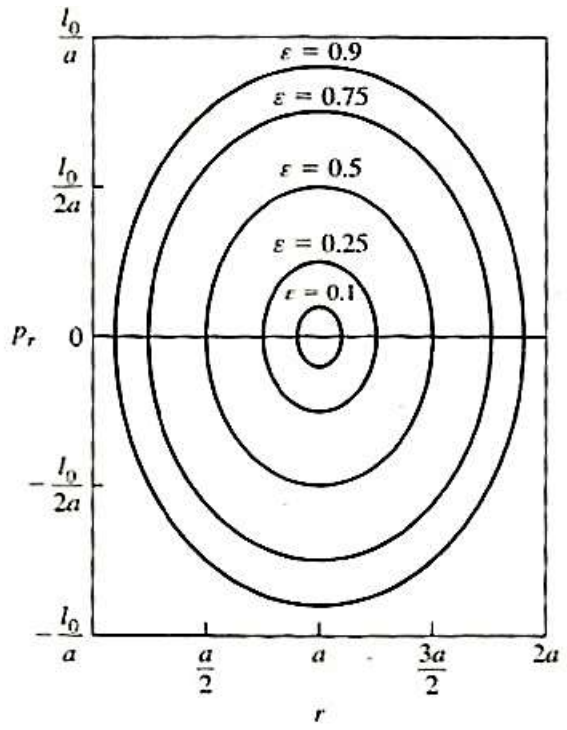


FIGURE 3.15 Phase-space plot for three ellipses in $r p_r$ space.

p_r *vs.* r

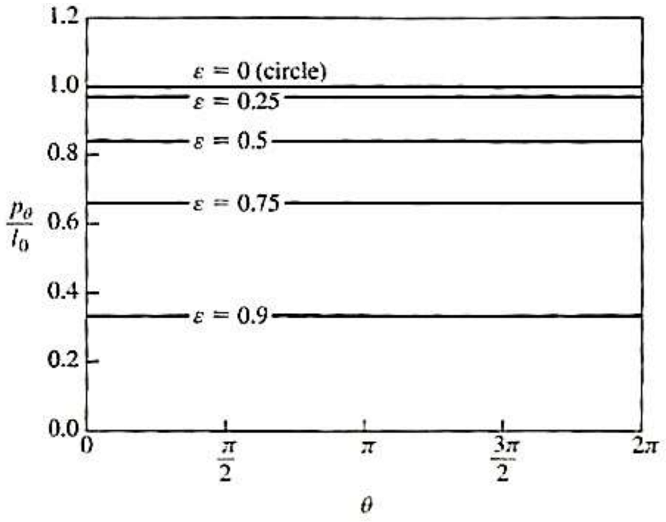


FIGURE 3.16 Phase-space plot for three ellipses in θp_θ space.

p_θ *vs.* θ

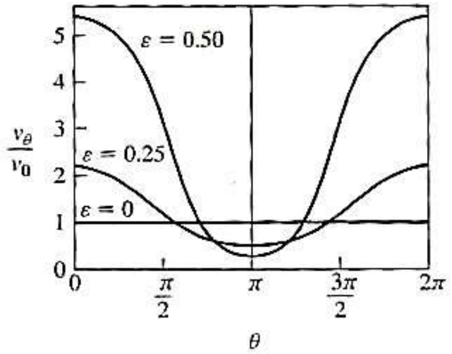


FIGURE 3.17 Velocity versus angle plot for three ellipses.

v_θ *vs.* θ

The Virial Theorem

- Skim discussion. Read details on your own!
- Many particle system. Positions $\vec{\mathbf{r}}_i$, momenta $\vec{\mathbf{p}}_i$. Bounded. ***Define*** $\mathbf{G} \equiv \sum_i \vec{\mathbf{r}}_i \bullet \vec{\mathbf{p}}_i$
- **Time derivative** of **G**:

$$(dG/dt) = \sum_i (\vec{r}_i \bullet \vec{p}_i + \vec{r}_i \bullet \vec{p}_i) \qquad (1)$$

- **Time average: (dG/dt)** in interval $\boldsymbol{\tau}$:

$$\langle (dG/dt) \rangle \equiv \tau^{-1} \int (dG/dt)dt \qquad (2)$$

(limits $0 < t < \tau$)

$$\langle (dG/dt) \rangle = [G(\tau) - G(0)]/\tau \qquad (3)$$

- **Periodic motion** $\Rightarrow G(\tau) = G(0)$:

$$(3) \Rightarrow \langle (dG/dt) \rangle = 0$$

$$\langle(dG/dt)\rangle = [G(\tau) - G(0)]/\tau \qquad (3)$$

- If motion ***isn't periodic,*** still make $\langle(dG/dt)\rangle = \langle\dot{G}\rangle$ as small as we want if τ is very large. $\Rightarrow$ For a periodic system or for a non-periodic system with large τ can (in principle) **make** $\langle\dot{G}\rangle = 0$
- When $\langle\dot{G}\rangle = 0$, (long time average) **(1) & (2)** combine:

$$\langle\textstyle\sum_i(\vec{p}_i\bullet\dot{\vec{r}}_i)\rangle = -\langle\textstyle\sum_i(\dot{\vec{p}}_i\bullet\vec{r}_i)\rangle \qquad (4)$$

- Left side of **(4)**: $\vec{p}_i\bullet\dot{\vec{r}}_i = 2T_i$

or $\langle\sum_i(\vec{p}_i\bullet\dot{\vec{r}}_i)\rangle = \langle 2\sum_i T_i\rangle = 2\langle T\rangle$ **(5)**

T_i = KE of particle **i; T** = total KE of system

- **Newton's 2nd Law:** $\Rightarrow \dot{\vec{p}}_i = \vec{F}_i$ = force on particle **i**

$\Rightarrow$ Right side of **(4)** : $\langle\sum_i(\vec{p}_i\bullet\vec{r}_i)\rangle = \langle\sum_i(\vec{F}_i\bullet\vec{r}_i)\rangle$ **(6)**

Combine **(5) & (6):**

$$\Rightarrow \qquad \langle T\rangle = -\,(\tfrac{1}{2})\langle \textstyle\sum_i(\vec{F}_i \bullet \vec{r}_i)\rangle \qquad (7)$$

$$-\,(\tfrac{1}{2})\langle \textstyle\sum_i(\vec{F}_i \bullet \vec{r}_i)\rangle \equiv$$ ***The Virial (of Clausius)***

$\equiv$ ***<u>The Virial Theorem:</u>***

The time average kinetic energy of a system is equal to its virial.

- Application to Stat Mech (ideal gas):

$\langle T\rangle = -(\frac{1}{2})\langle\sum_i(\vec{F}_i\bullet\vec{r}_i)\rangle \equiv$ ***The Virial Theorem:***

- **Application to classical dynamics:**
- For a conservative system in which a PE can be defined: $F_i \equiv -\nabla V_i \;\Rightarrow\; \langle T\rangle = (\frac{1}{2})\langle\sum_i(\nabla V_i\bullet r_i)\rangle$
- Special case: ***Central Force***, which (for each particle **i**):

$|\mathbf{F}| \propto \mathbf{r}^{\mathbf{n}}$, **n** *any power* (**r** = distance between particles) $\Rightarrow$

$$V = kr^{n+1}$$

$$\Rightarrow \qquad \nabla V\bullet r = (dV/dr)r = k(n+1)r^{n+1}$$

or:

$$\nabla V\bullet r = (n+1)V$$

$\Rightarrow$ **Virial Theorem gives:**

$$\langle T\rangle = (\tfrac{1}{2})(n+1)\langle V\rangle \qquad (8)$$

(Central forces ONLY!)

- ***Virial Theorem, Central Forces:***

$$(F(r) = kr^n\ ,\ V(r) = kr^{n+1})$$

$$\langle T \rangle = (½)(n+1)\langle V \rangle \qquad (8)$$

- **Case 1:** Gravitational (or electrostatic!) Potential:

$$n = -2 \quad \Rightarrow \quad \langle T \rangle = -(½)\langle V \rangle$$

- **Case 2:** Isotropic Simple Harmonic Oscillator Potential:

$$n = +1 \quad \Rightarrow \quad \langle T \rangle = \langle V \rangle$$

- **Case 3:** $n = -1 \quad \Rightarrow \quad \langle T \rangle = 0$

- **Case 4:** **n** $\neq$ integer (real power **x**):

$$n = x \quad \Rightarrow \quad \langle T \rangle = (½)(x+1)\langle V \rangle$$

Orbit Equation

- We had:

$$(d\theta/dr) = \pm (\ell/r^2)(2m)^{-½}[E - V(r) - \{\ell^2/(2mr^2)\}]^{-½}$$

- Integrating this gives:

$$\theta(r) = \pm \int(\ell/r^2)(2m)^{-½}[E - V(r) - \{\ell^2/(2mr^2)\}]^{-½}\, dr$$

- **Once the central force is specified, we know V(r) & we can, in principle, do the integral & get the orbit θ(r), or,** (if this can be inverted!) **r(θ)**. Quite remarkable! Assuming only a central force law & nothing else

  We have reduced the original 6d problem of 2 particles to a 2d problem with only 1 degree of freedom. The solution can be obtained simply by doing the above (1d) integral! (Not necessarily a closed form function, but integral can always be done. Usually numerically.)

- *General Eqtn for orbit* (*any* Central Potential **V(r)**) is:

$$(2m)^{1/2}\theta(r) = \pm \int (\ell/r^2)dr/D(r) \qquad (1)$$

$$D(r) \equiv [E - V(r) - \{\ell^2/(2mr^2)\}]^{1/2}$$

- In general, **(1)** must be evaluated numerically. True even for most **power law forces:**

$$f(r) = kr^n \, ; \, V(r) = kr^{n+1}$$

- For a few integer & fractional values of **n**, can express **(1)** in terms of certain elliptic integrals.

Orbits

$$(2m)^{1/2}\theta(r) = \pm \int (\ell/r^2)dr/D(r) \qquad (1)$$

$$D(r) \equiv [E - V(r) - \{\ell^2/(2mr^2)\}]^{1/2}$$

- **Can prove** that *only* for **n = 1, -2,** and **-3** can **(1)** be integrated to give **trigonometric functions.** Also, results for **n=5,3,0,-4,-5,-7** can be expressed as **elliptic integrals.** Interesting academically & mathematically, but most of these are uninteresting **physically!**

n = 1, f(r) = kr: Isotropic, 3d simple harmonic oscillator.

n = -2, f(r) = kr^{-2}: Inverse square law force:

Gravitation, Coulomb, …will treat in detail this chapter!

– Other cases: **Homework Problems!**

- **Recap:** Have solved problem for **r(t)** & orbit **θ(r)** or **r(θ)** using **conservation laws exclusively:**
  - Combined **conservation of angular momentum with conservation of energy** into a single result which gives the orbit **θ(r)** in terms of a single integral.
- Useful to take another (equivalent, of course!) approach which will result in a ***differential eqtn* for the orbit!**

- Go back to Lagrangian for relative coordinate (before using conservation of angular momentum):

$$L \equiv (½)m\,|\dot{\vec{r}}|^2 - V(r)$$

Or: $$L = (½)m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

$$L = (\tfrac{1}{2})m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

- Lagrange eqtn for **r** (again):

$$(\partial L/\partial r) - (d/dt)(\partial L/\partial \dot{r}) = 0$$

$$(\partial L/\partial r) = mr\dot{\theta}^2 - (\partial V/\partial r)$$

$$(d/dt)(\partial L/\partial \dot{r}) = m\ddot{r};$$

$$-(\partial V/\partial r) = f(r)$$

⇒ **Differential Eqtn of motion** for "particle" of mass **m** subject to Central Force **f(r)** :

$$m(\ddot{r} - r\dot{\theta}^2) = f(r) \qquad (2)$$

As we've already seen: Newton's 2nd Law in plane polar coordinates!

- **Diff Eqtn of motion:** $m(\ddot{r} - r\dot{\theta}^2) = f(r)$ **(2)**

(2) is a 2^{nd} order differential equation.

- Most convenient to solve by making a change of variables:
- Let $\mathbf{u} \equiv (1/r)$ $\Rightarrow$ $(du/dr) = -(1/r^2)$ $(\equiv -u^2)$
- More interested in orbit $\theta(r)$ or $r(\theta)$ than in $r(t)$.

  Manipulation (repeated use of chain rule):

  $(du/d\theta) = (du/dr)(dr/d\theta) = -(1/r^2)(dr/d\theta)$

$= -(1/r^2)(dr/dt)(dt/d\theta) = -(1/r^2)(dr/dt)/(d\theta/dt) = -(1/r^2)(\dot{r}/\dot{\theta})$

$$\Rightarrow \qquad (du/d\theta) = -(1/r^2)(\dot{r}/\dot{\theta}) \qquad (3)$$

- **Conservation of angular momentum:**

$$\ell \equiv mr^2\dot{\theta} = \text{const} \Rightarrow \dot{\theta} = [\ell/(mr^2)] \qquad (4)$$

- Combine (3) & (4): $\Rightarrow (du/d\theta) = -(m/\ell)\dot{r}$ (5)
- Similar manipulation for: $(d^2u/d\theta^2) = (d/d\theta)[-(m/\ell)\dot{r}]$
  $= -(m/\ell)(dt/d\theta)(d\dot{r}/dt) = -(m/\ell)(\ddot{r}/\dot{\theta})$

Again substitute $\dot{\theta} = [\ell/(mr^2)]$ $(d^2u/d\theta^2) = -(m^2/\ell^2)(r^2\ddot{r})$ (6)

- Solving (6) for $\mathbf{r}$ (& using $u^2 = (1/r^2)$) :

$$\Rightarrow \quad \ddot{r} = -(\ell^2/m^2)u^2(d^2u/d\theta^2) \qquad (7)$$

$$\ell \equiv mr^2\dot{\theta} = \text{const};\ \dot{\theta} = [\ell/(mr^2)]\ (\text{but } u = (1/r))$$

$$\Rightarrow \quad r\dot{\theta}^2 = (\ell^2/m^2)\,u^3 \qquad (8)$$

- Eqtn of motion:

$$m(\ddot{r} - r\dot{\theta}^2) = f(r) \qquad (2)$$

- Putting (7) & (8) into (2) gives (on simplifying):

$$(d^2u/d\theta^2) + u = -(m/\ell^2)u^{-2}f(1/u) \qquad (9)$$

- *Differential eqtn* which gives the orbit is (**u = (1/r)**)

$$(d^2u/d\theta^2) + u = -(m/\ell^2)u^{-2}\,f(1/u) \qquad (9)$$

In terms of potential **V(r) = V(1/u)** this is:

$$(d^2u/d\theta^2) + u = -(m/\ell^2)[dV(1/u)/du] \qquad (9')$$

Alternatively, could write:

$$(d^2[1/r]/d\theta^2) + (1/r) = -(m\ell^2)r^2\,f(r) \qquad (9'')$$

- *Note:* Because of right hand side, **(9), (9′), (9″)** are **nonlinear differential equations** in general.
  - **Exception:** When $\mathbf{f(r) \propto r^{-2}}$ (Inverse Square Law), for which right side = constant.
- **(9), (9′), (9″)** could, in principle be used to solve for orbit **r(θ)** or **θ(r)** given the force law **f(r).**
  - The result, of course will be same as if integral version of **θ(r)** is evaluated. Can show integral for **θ(r)** is solution to **(9), (9′), (9″)**

- ***Differential eqtn*** which gives the orbit (With **u = (1/r)**)

$$(d^2u/d\theta^2) + u = -(m/\ell^2)u^{-2} f(1/u) \qquad (9)$$

$$(d^2u/d\theta^2) + u = -(m/\ell^2)[dV(1/u)/du] \qquad (9')$$

$$(d^2[1/r]/d\theta^2) + (1/r) = -(m/\ell^2)r^2 f(r) \qquad (9'')$$

- Usually, rather than solve these for the orbit, given **f(r),** we usually use the integral formulation.
- However, **where these are most useful is for**

The Inverse Problem:

≡ ***Given a known orbit r(θ) or θ(r), determine the force law f(r).***

Examples

1: Find the force law for a central force field that allows a particle to move in a **logarithmic spiral orbit** given by $\mathbf{r} = \mathbf{k}e^{\alpha\theta}$, where **k** and α are constants.

2: Find **r(t)** and **θ(t)** for the same case.

3: What is the total energy of the orbit for the same case?

Solution on Board!

i) use $\frac{d^2u}{d\theta^2} + u = -\frac{m}{\ell^2} u^{-2} F\left(\frac{1}{u}\right)$

$r = k e^{\alpha\theta}$

$u = \frac{1}{k} e^{-\alpha\theta}$

$\frac{du}{d\theta} = -\alpha u$

$\frac{d^2u}{d\theta^2} = -\alpha \frac{du}{d\theta} = \alpha^2 u$

$\frac{d^2u}{d\theta^2} + u = (1+\alpha^2) u$

$e^{2\alpha\theta} = \frac{2\alpha\ell}{mk^2} t + e^{2\alpha\theta_0}$

$\theta(t) = \frac{1}{2\alpha} \ln\left[\frac{2\alpha\ell t}{mk^2} + e^{2\alpha\theta_0}\right]$

$r = k e^{\alpha\theta}$

$r^2 = k^2 e^{2\alpha\theta} = k^2\left[\frac{2\alpha\ell t}{mk^2} + e^{2\alpha\theta_0}\right]$

$r = k\left[\quad\right]^{\frac{1}{2}}$

Solution on Board!

$$F\left(\tfrac{1}{u}\right) = -\frac{\ell^2}{m} u^2 (1+\alpha^2) u$$

$$= -\frac{\ell^2}{m}\frac{(1+\alpha^2)}{r^3}$$

② $r(t)$ & $\theta(t)$? $r = k e^{\alpha\theta}$

$$\ell = m r^2 \dot{\theta}, \quad \dot{\theta} = \frac{\ell}{m r^2} = \frac{\ell}{m}\frac{1}{k^2 e^{2\alpha\theta}}$$

$$\dot{\theta} = \frac{d\theta}{dt}$$

$$\int_{\theta_0}^{\theta} e^{2\alpha\theta}\, d\theta = \frac{\ell}{m k^2}\int_0^t dt$$

$$\frac{e^{2\alpha\theta} - e^{2\alpha\theta_0}}{2\alpha} = \frac{\ell}{m k^2} t$$

③ $E = ?$ $\cancel{E = T + V}$

$$\cancel{T = \tfrac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)}$$

$$E = \tfrac{1}{2} m \dot{r}^2 + \frac{\ell^2}{2 m r^2} + V(r)$$

$$V(r) = -\int_\infty^r F\, dr = \frac{\ell^2 (1+\alpha^2)}{m}\int \frac{dr}{r^3}$$

$$= \frac{-\ell^2(1+\alpha^2)}{2m}\frac{1}{r^2}$$

$$\dot{r} = \frac{dr}{d\theta}\dot{\theta} = \frac{dr}{d\theta}\frac{\ell}{m r^2} = \frac{\alpha \ell}{m r}$$

$$E = 0$$

- ***Differential eqtn*** which gives the orbit (With $\mathbf{u = (1/r)}$)

$$(d^2u/d\theta^2) + u = -(m/\ell^2)u^{-2} f(1/u) \qquad (9)$$

$$(d^2u/d\theta^2) + u = -(m/\ell^2)[dV(1/u)/du] \qquad (9')$$

Can use these results (text, p 87) to prove that **the orbit is symmetric about 2 adjacent turning points.** Stated another way: Can prove a theorem that **the orbit is invariant under reflection about the apsidal vectors $\mathbf{r}_{min}$ & $\mathbf{r}_{max}$.** Figure:

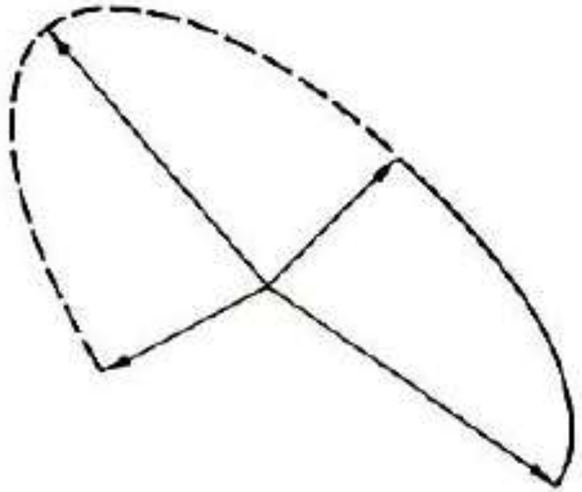


FIGURE 3.12 Extension of the orbit by reflection of a portion about the apsidal vectors

- **The integral form of the orbit eqtn** was:

$$\theta(r) = \pm\int(\ell/r^2)(2m)^{-1/2}[E - V(r) - \{\ell^2/(2mr^2)\}]^{-1/2}\,dr + \text{constant}$$

- It's sometimes useful to write in terms of $\mathbf{u = (1/r)}$

$$\theta(u) = \theta_0 - \int du[(2mE)/\ell^2 - (2mV(1/u)/\ell^2 - u^2]^{-1/2}$$

- For **power law potentials:** $\mathbf{V = ar^{n+1}}$

$$\theta(u) = \theta_0 - \int du[(2mE)/\ell^2 - (2mau^{-n-1})/\ell^2 - u^2]^{-1/2}$$

- **Open & closed orbits (*qualitative*).** Look (briefly) again at the **radial velocity** vs. **r**:

$$\dot{r}(r) = \pm ([2/m][E - V(r)] - [\ell^2/(m^2r^2)])^{1/2}$$

Or, **conservation of energy:**

$$E = (1/2)m\dot{r}^2 + [\ell^2/(2mr^2)] + V(r) = \text{const}$$

- As we already said in our qualitative analysis, the ***Radial turning points*** $\equiv$ Points where $\dot{r} = 0$ (Where the particle stops! Apsidal distances!)
  - Look at either of these eqtns & find, at turning points:

$$E - V(r) - [\ell^2/(2mr^2)] = E - V'(r) = 0 \qquad (1)$$

$$\mathbf{E - V(r) - [\ell^2/(2mr^2)] = 0} \qquad \mathbf{(1)}$$

- Solutions **r** to **(1)** = **r** where $\dot{\mathbf{r}} = \mathbf{0}$

$\Rightarrow$ **Turning points = Apsidal distances**

- Often (even usually), **(1)** has 2 roots (max & min **r**)

$$\equiv \mathbf{r_{max}} \,\&\, \mathbf{r_{min}}. \quad \Rightarrow \quad \mathbf{r_{max} \geq r \geq r_{min}}$$

Radial motion is *oscillatory* between $\mathbf{r_{max}}$ & $\mathbf{r_{min}}$.

- Some combinations of **E, V(r)**, ℓ give **(1)** **only one root,** but the orbit is still bounded then:

$\Rightarrow$ In this case, $\dot{\mathbf{r}} = \mathbf{0}$, for all **t**

$\Rightarrow$ **r** = constant

$\Rightarrow$ **The orbit r(θ) is *circular*.**

Closed & Open Orbits

- **Periodic motion in V(r)** $\Rightarrow$ The orbit **r(θ)** is *closed*.

$\Rightarrow$ After a finite number of oscillations of **r** between $\mathbf{r}_{\mathbf{max}}$ & $\mathbf{r}_{\mathbf{min}}$, the motion repeats itself exactly! If the orbit does not close on itself after a finite number of oscillations between $\mathbf{r}_{\mathbf{max}}$ & $\mathbf{r}_{\mathbf{min}}$, the orbit **r(θ)** is *open*. See figures

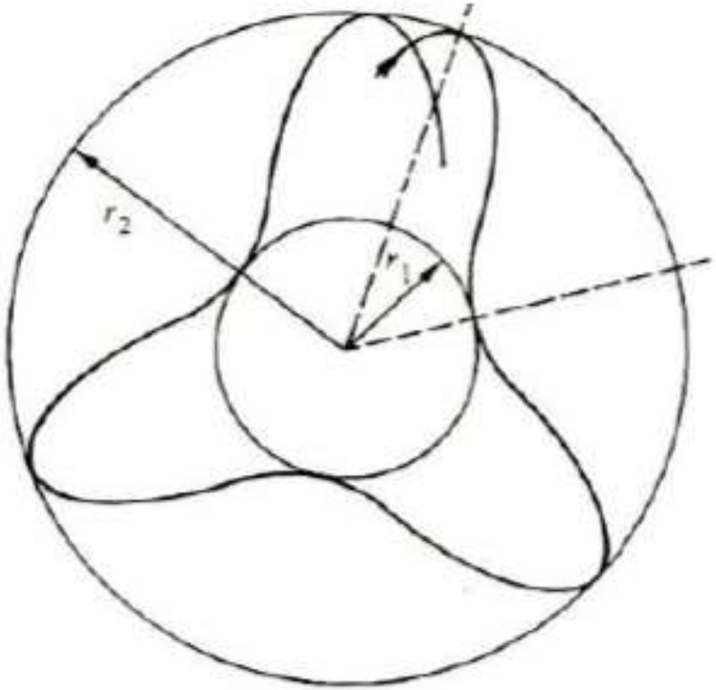


FIGURE 3.7 The nature of the orbits for bounded motion.

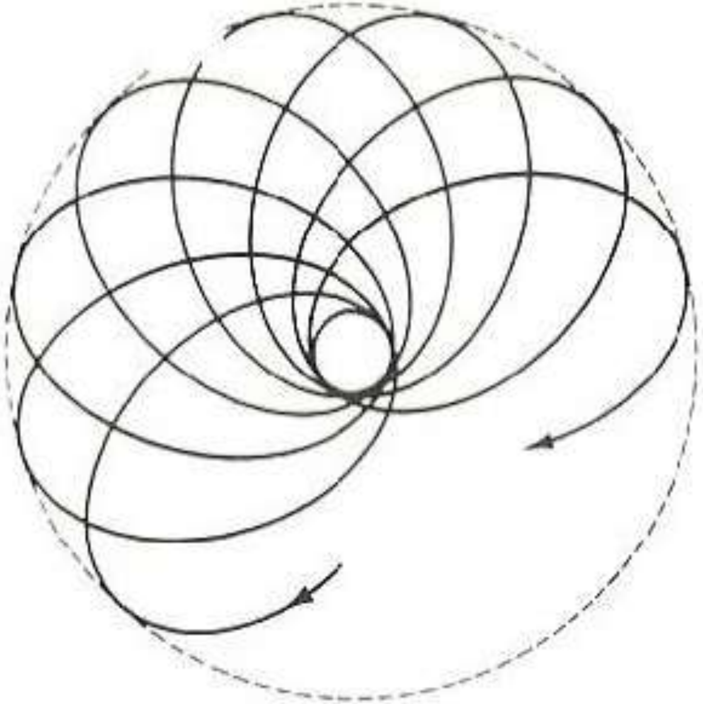


FIGURE 8-4

- Use **the eqtn for the orbit** $\boldsymbol{\theta(r)}$ to find the change in $\boldsymbol{\theta}$ due to one complete oscillation of $\mathbf{r}$ from $\mathbf{r_{min}}$ to $\mathbf{r_{max}}$ & back to $\mathbf{r_{min}}$:
  - The angular change is $\mathbf{2} \times$ the change in going once from $\mathbf{r_{min}}$ to $\mathbf{r_{max}}$:

$$\Rightarrow \quad \Delta\theta \equiv 2\int(\ell/r^2)(2m)^{-\frac{1}{2}}[E - V(r) - \{\ell^2/(2mr^2)\}]^{-\frac{1}{2}}\, dr$$

(limits $\mathbf{r_{min}} \le \mathbf{r} \le \mathbf{r_{max}}$)

$$\Delta\theta = 2\int(\ell/r^2)(2m)^{-\frac{1}{2}}[E - V(r) - \{\ell^2/(2mr^2)\}]^{-\frac{1}{2}}dr$$

(limits $r_{min} \leq r \leq r_{max}$)

- Results in periodic motion & a **closed orbit** ***only*** **if** $\Delta\theta$ = a rational fraction of 2π: $\Delta\theta \equiv 2\pi(a/b)$, ($a, b$ = integers)
- **If the orbit is closed,** after **b** periods, the radius vector of the particle will have made **a** complete revolutions & the particle will be back at its original position.
- Can show: **If the potential is a power law in r**: $V(r) = k\,r^{n+1}$ a closed ***NON-CIRCULAR*** path can occur ***ONLY*** for

  $n = -2$ (inverse square law force, gravity, electrostatics) &

  $n = +1$ (3 d isotropic, simple harmonic oscillator).

  – **Footnote:** Some fractional values of **n** also lead to closed orbits. Not interesting from the **PHYSICS** viewpoint.

Central Force Field Scattering

- Application of Central Forces outside of astronomy: **Scattering of particles**.
- Atomic scale scattering: Need QM of course!
- **Description of scattering processes:**
  - Independent of CM or QM.
- 1 body formulation = Scattering of particles by a

  Center of Force.
  - Original 2 body problem = Scattering of "particle" with the reduced mass **μ** from a center of force
  - Here, we go right away to the 1 body formulation, ***while bearing in mind that it really came from the 2 body problem***.

- Consider a **uniform beam** of particles (of any kind) of equal mass and energy incident on a center of force (Central force **f(r)**).
  - ***Assume*** that **f(r)** falls off to zero at large **r**.
- Incident beam is characterized by an intensity (flux density) **I** ≡ # particles crossing a unit area (⊥ beam) per unit time (= # particles per **m³** per **s**)
  - As a particle approaches the center of force, it is either attracted or repelled & thus it's orbit will be changed (deviate from the initial straight line path).

  ⇒ Direction of final motion is not the same as incident motion. ⇒ ≡ **Particle is Scattered**

- For **repulsive scattering** (what we mainly look at here) the situation is as shown in the figure:

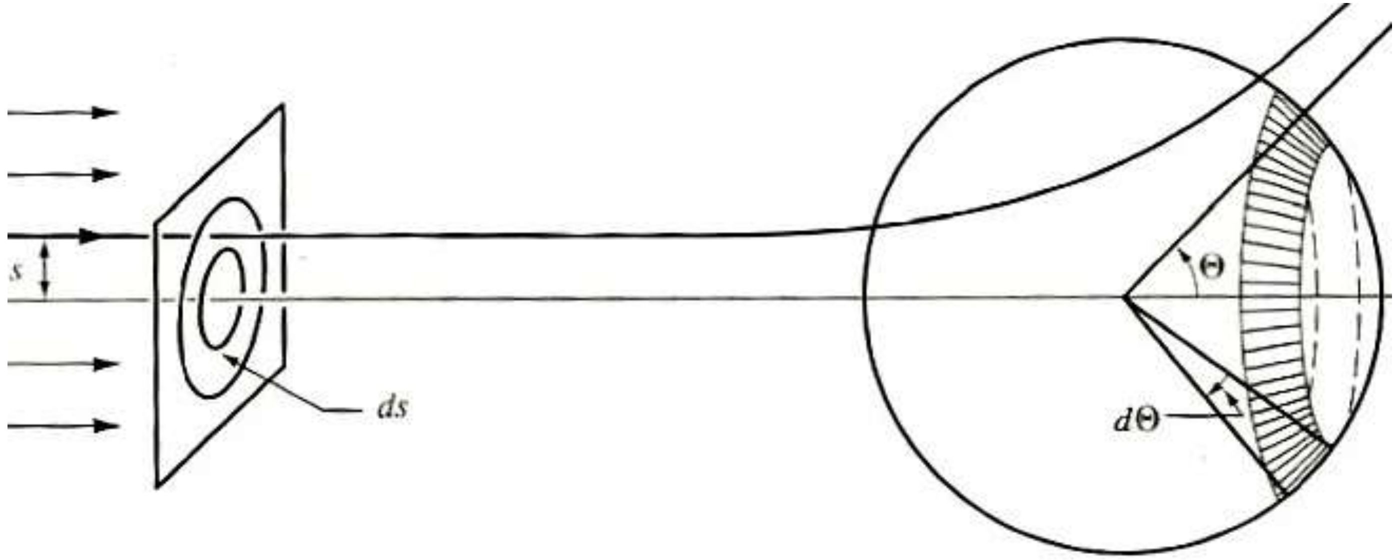


FIGURE 3.19 Scattering of an incident beam of particles by a center of force.

- ***Define:*** **Cross Section for Scattering** in a given direction (into a given solid angle $\mathbf{d\Omega}$):

$$\boldsymbol{\sigma(\Omega)d\Omega \equiv (N_s/I)}.$$ With $\mathbf{I}$ = incident intensity

$\mathbf{N_s}$ = # particles/time scattered into solid angle $\mathbf{d\Omega}$

Scattering Cross Section:

$\sigma(\Omega)d\Omega \equiv (N_s/I)$

I = incident intensity

N_s = # particles/time

scattered into angle **$d\Omega$**

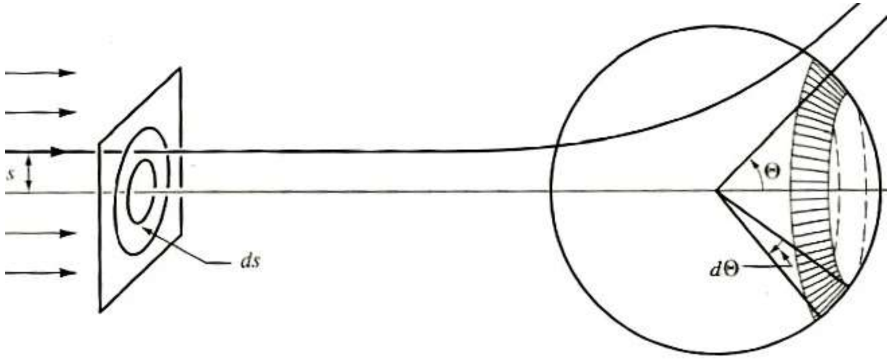


FIGURE 3.19 Scattering of an incident beam of particles by a center of force.

- In general, the solid angle **Ω** depends on the spherical angles **Θ**, **Φ**. However, for central forces, there must be **symmetry** about the axis of the incident beam

$\Rightarrow$ **$\sigma(\Omega)$** ($\equiv$ **$\sigma(\Theta)$**) is independent of azimuthal angle **Φ**

$\Rightarrow d\Omega \equiv 2\pi \sin\Theta d\Theta$, $\sigma(\Omega)d\Omega \equiv 2\pi \sin\Theta d\Theta$,

Θ $\equiv$ Angle between incident & scattered beams, as in the figure.

σ $\equiv$ "***cross section***". It has units of area

Also called the differential cross section.

› As in all Central Force problems, for a given particle, the orbit, & thus the amount of scattering,

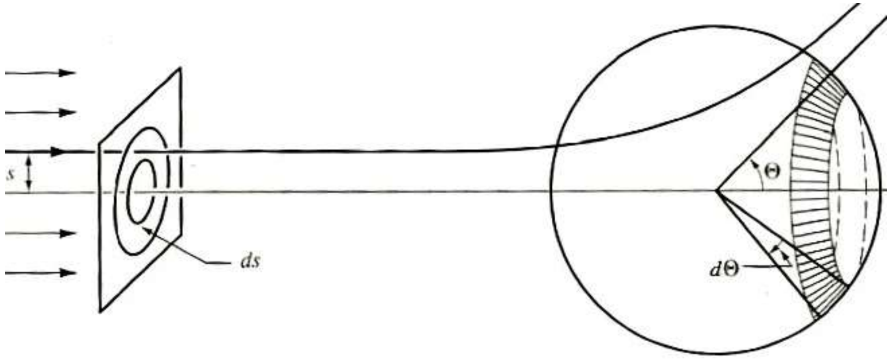


FIGURE 3.19 Scattering of an incident beam of particles by a center of force.

is determined by the energy **E** & the angular momentum ℓ

› ***Define***: **Impact parameter, s**, & express the angular momentum ℓ in terms of **E** & **s**.

› Impact parameter **s** ≡ the ⊥ distance between the center of force & the incident beam velocity (fig).

› ***GOAL:*** **Given** the energy **E**, the impact parameter **s**, & the force **f(r)**, what is the cross section **σ(Θ)**?

- **Beam**, intensity **I**.
  Particles, mass **m**,
  incident speed
  (at $\mathbf{r} \to \infty$) = $\mathbf{v_0}$.

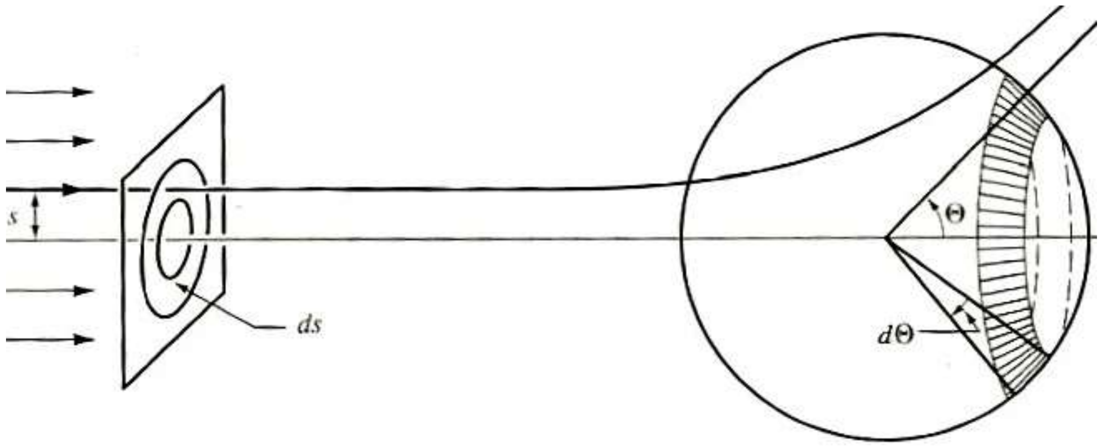


FIGURE 3.19 Scattering of an incident beam of particles by a center of force.

- **Energy conservation:**

$$E = T + V = (½)mv^2 + V(r) = (½)m(v_0)^2 + V(r \to \infty)$$

- Assume $V(r \to \infty) = 0 \;\Rightarrow\; E = (½)m(v_0)^2$

$$\Rightarrow\; v_0 = (2E/m)^{½}$$

- **Angular momentum:** $\ell \equiv mv_0s \equiv s(2mE)^{½}$

- **Angular momentum** $\ell \equiv mv_0 s \equiv s(2mE)^{1/2}$

  Incident speed v_0.

- $N_s \equiv$ # particles scattered into solid angle $d\Omega$ between Θ & $\Theta + d\Theta$. **Cross section definition**

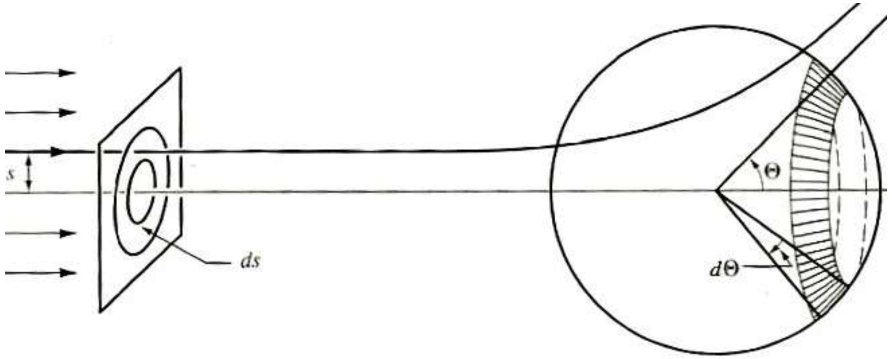


FIGURE 3.19 Scattering of an incident beam of particles by a center of force.

$$\Rightarrow \qquad N_s \equiv I\sigma(\Theta)d\Theta = 2\pi I\sigma(\Theta)\sin\Theta d\Theta$$

- $N_i \equiv$ # incident particles with impact parameter between **s** & **s + ds** . $N_i = 2\pi I s ds$

- **Conservation of particle number**

$$\Rightarrow N_s = N_i \ \text{ or: } \quad 2\pi I\sigma(\Theta)\sin\Theta|d\Theta| = 2\pi Is|ds|$$

2πI cancels out! (Use absolute values because **N**'s are always **> 0,** but **ds** & **dΘ** can have any sign.)

$\sigma(\Theta)\sin\Theta|d\Theta| = s|ds| \quad (1)$

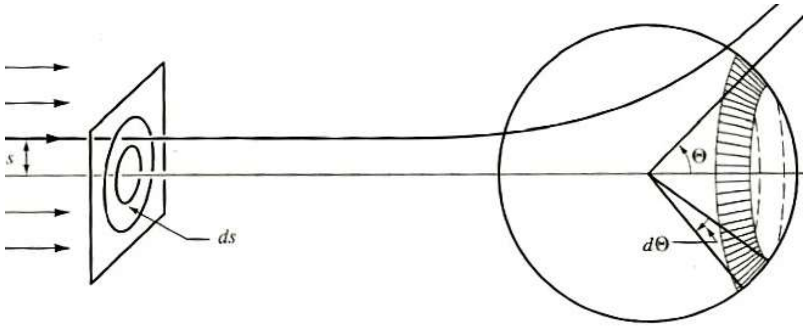


FIGURE 3.19 Scattering of an incident beam of particles by a center of force.

- **s** = a function of energy

  E & scattering angle **Θ:**

  $s = s(\Theta,E)$

$(1) \quad \Rightarrow \sigma(\Theta) = (s/\sin\Theta)\,(|ds|/|d\Theta|) \qquad (2)$

- To compute **σ(Θ)** we clearly need **s = s(Θ,E)**
- **Alternatively,** could use **Θ** = **Θ(s,E)** & rewrite **(2)** as:
  $\sigma(\Theta) = (s/\sin\Theta)/[(|d\Theta|/|ds|)] \qquad (2')$
- Get **Θ** = **Θ(s,E)** from the orbit eqtn. For general central force (**θ** is the angle which describes the orbit **r = r(θ); θ ≠ Θ**)

$$\theta(r) = \int(\ell/r^2)(2m)^{-1/2}[E - V(r) - \{\ell^2/(2mr^2)\}]^{-1/2}dr$$

- Orbit eqtn. **General central force:**

$$\theta(r) = \int(\ell/r^2)(2m)^{-\frac{1}{2}}[E - V(r) - \{\ell^2/(2mr^2)\}]^{-\frac{1}{2}}dr \qquad (3)$$

- Consider purely **repulsive scattering**. See figure:

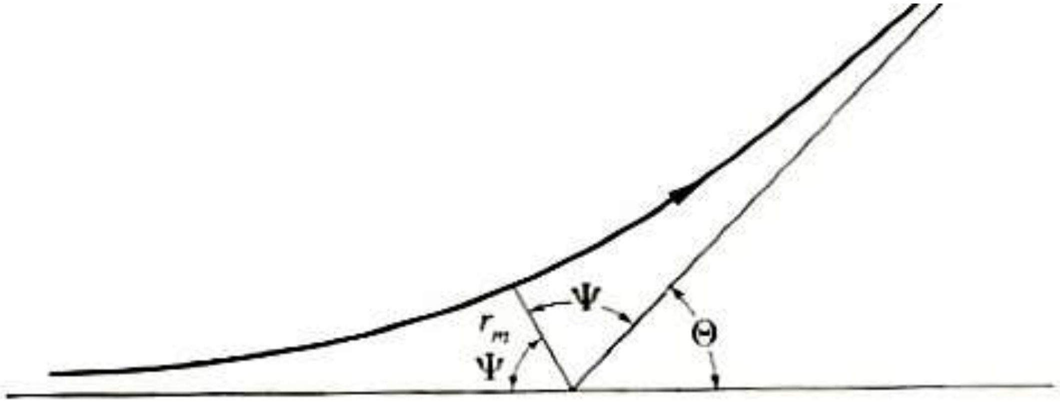


FIGURE 3.20 Relation of orbit parameters and scattering angle in an example of repulsive scattering.

- **Closest approach distance** $\equiv r_m$. Orbit must be symmetric about $r_m \Rightarrow$ Sufficient to look at angle (see figure): Scattering angle $\Theta \equiv \pi - 2\Psi$. Also, orbit angle

$$\theta = \pi - \Psi \textit{ in the special case } r = r_m$$

$\Rightarrow$ After some **manipulation** can write **(3)** as:

$$\Psi = \int (dr/r^2)[(2mE)/(\ell^2) - (2mV(r))/(\ell^2) - 1/(r^2)]^{-\frac{1}{2}} \quad (4)$$

› Integrate from r_m to $r \to \infty$

› Angular momentum in terms of impact parameter

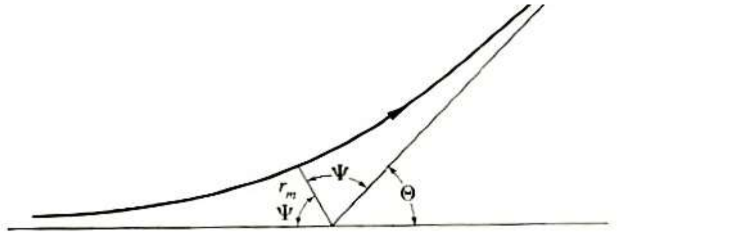


FIGURE 3.20 Relation of orbit parameters and scattering angle in an example of repulsive scattering.

s & energy **E**: $\ell \equiv mv_0 s \equiv s(2mE)^{\frac{1}{2}}$. Put this into **(4)**

& get for scattering angle **Θ** (after **manipulation**):

$$\Theta(s) = \pi - 2\int dr(s/r)[r^2\{1 - V(r)/E\} - s^2]^{-\frac{1}{2}} \qquad (4')$$

Changing integration variables to **u = 1/r**:

$$\Theta(s) = \pi - 2\int s\,du\,[1 - V(r)/E - s^2u^2]^{-\frac{1}{2}} \qquad (4'')$$

› Integrate from $u = 0$ to $u = u_m = 1/r_m$

- ***Summary:*** Scattering by a ***general central force***:
- **Scattering angle Θ** = **Θ(s,E)** (**s =** impact parameter, **E =** energy):

  $$\Theta(s) = \pi - 2\int s\,du\,[1 - V(r)/E - s^2u^2]^{-\frac{1}{2}} \quad (4'')$$

  Integrate from $u = 0$ to $u = u_m = 1/r_m$
- **Scattering cross section:**

  $$\sigma(\Theta) = (s/\sin\Theta)\,(|ds|/|d\Theta|) \qquad (2)$$

- **"<u>Recipe</u>":** To solve a scattering problem:
  1. Given force **f(r),** compute potential **V(r).**
  2. Compute **Θ(s)** using **(4´´).**
  3. Compute **σ(Θ)** using **(2).**
  - **Goldstein tells you this is rarely done to get σ(Θ)!**

Coulomb Scattering

- ***Special case:* Scattering by r^{-2} repulsive forces**:
  - **For this case,** as well as for others where the orbit eqtn

    r = r(θ) is known analytically, instead of applying **(4´´)** directly to get **Θ(s)** & then computing **σ(Θ)** using **(2)**, we make use of the known expression for **r = r(θ)** to get **Θ(s)** & then use **(2)** to get **σ(Θ).**

- Repulsive scattering by $\mathbf{r^{-2}}$ repulsive forces =

  Coulomb scattering by like charges

  - Charge **+Ze** scatters from the center of force, charge **Z´e**

$$\Rightarrow \qquad f(r) \equiv (ZZ'e^2)/(r^2) \equiv -k/r^2$$

  - Gaussian E&M units! Not SI! For SI, multiply by $(1/4\pi\varepsilon_0)$!

  $\Rightarrow$ For **r(θ)**, in the previous formalism for $\mathbf{r^{-2}}$ attractive forces, make the replacement $\mathbf{k \to -ZZ'e^2}$

- **Like charges**: $\mathbf{f(r) \equiv (ZZ'e^2)/(r^2)}$

  $\Rightarrow \mathbf{k \to - ZZ'e^2}$ in orbit eqtn $\mathbf{r = r(\theta)}$

- We've seen: Orbit eqtn for $\mathbf{r^{-2}}$ force is a conic section:

$$[\alpha/r(\theta)] = 1 + \varepsilon\cos(\theta - \theta') \qquad (1)$$

With: **Eccentricity** $\equiv \boldsymbol{\varepsilon} = [\,1 + \{2E\ell^2/(mk^2)\}]^{1/2}$ &

$\mathbf{2\alpha = [2\ell^2/(mk)]}$. Eccentricity = $\boldsymbol{\varepsilon}$ to avoid confusion with electric charge **e**. $\mathbf{E > 0 \Rightarrow \varepsilon > 1 \Rightarrow}$ **Orbit is a hyperbola**, by previous discussion.

- Choose the integration const $\boldsymbol{\theta' = \pi}$ so that $\mathbf{r_{min}}$ is at $\boldsymbol{\theta = 0}$
- Make the changes in notation noted:

$$\Rightarrow \quad [1/r(\theta)] = [(mZZ'e^2)/(\ell^2)](\varepsilon\cos\theta - 1) \qquad (2)$$

$$f(r) = (ZZ'e^2)/(r^2)$$

$$[1/r(\theta)] = [(mZZ'e^2)/(\ell^2)](\varepsilon\cos\theta - 1) \qquad (2)$$

- **Hyperbolic orbit.**
- With change of notation, eccentricity is

$$\varepsilon = [\, 1 + \{2E\ell^2/(mZ^2Z'^2e^4)\}]^{1/2}$$

- Using the relation between angular momentum, energy, & impact parameter, $\ell^2 = 2mEs^2$ this is:

$$\varepsilon = [\, 1 + (2Es)^2/(ZZ'e^2)^2\,]^{1/2}$$

$$[1/r(\theta,s)] = [(mZZ'e^2)/(\ell^2)](\varepsilon\cos\theta - 1) \qquad (2)$$

$$\varepsilon = [\, 1 + (2Es)^2/(ZZ'e^2)^2 \,]^{1/2} \qquad (3)$$

- From **(2)** get **θ(r,s)**. Then, use relations between orbit angle **θ** scattering angle **Θ**, & auxillary angle **Ψ** in the scattering problem, to get **Θ = Θ(s)** & thus **the scattering cross section.**

$$\Theta = \pi - 2\Psi$$

Focus of Hyperbola →

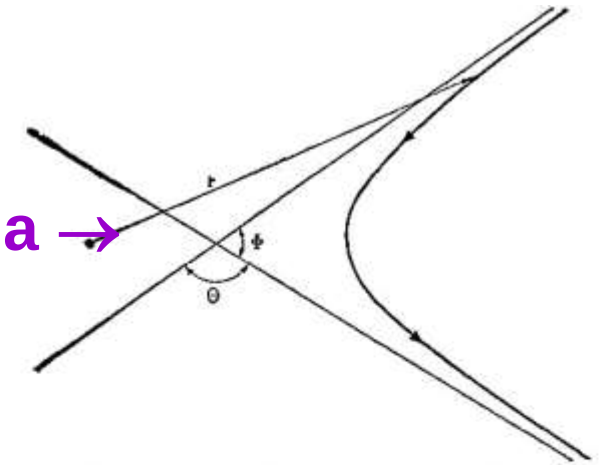


FIG. 3-15. Orbit for repulsive coulomb scattering, illustrating the connection between the angle between the asymptotes and the scattering angle.

- **Ψ** = direction of incoming

  asymptote. Determined by

  r$\to \infty$ in **(2)** $\Rightarrow$ **cosΨ = (1/ε)**.

- In terms of **Θ** this is: **sin(½Θ) = (1/ε)**. **(4)**

$$\varepsilon = [\, 1 + (2Es)^2/(ZZ'e^2)^2 \,]^{1/2} \qquad (3)$$

$$\sin(\tfrac{1}{2}\Theta) = (1/\varepsilon) \qquad (4)$$

- Manipulate with **(3)** & **(4)** using trig identities, etc.

  (4) & trig identities $\Rightarrow \varepsilon^2 - 1 = \cot^2(\tfrac{1}{2}\Theta)$ **(5)**

  Put **(3)** into the right side of **(5)** & take the square root:

$\Rightarrow$ $\cot(\tfrac{1}{2}\Theta) = (2Es)/(ZZ'e^2)$ **(6)** Typo in text! Factor of e!

- Solve **(6)** for the impact parameter $s = s(\Theta,E)$

$$\Rightarrow \quad s = s(\Theta,E) = (ZZ'e^2)/(2E)\cot(\tfrac{1}{2}\Theta) \qquad (7)$$

(7), the impact parameter as function of **Θ** & **E** for Coulomb scattering is an **important result!**

$$s = s(\Theta,E) = (ZZ'e^2)/(2E)\cot(\tfrac{1}{2}\Theta) \qquad (7)$$

- Now, use **(7)** to compute the **Differential Scattering Cross Section for Coulomb Scattering.**
- We had: $\sigma(\Theta) = (s/\sin\Theta)\,(|ds|/|d\Theta|) \qquad (8)$

  (7) & **(8)** (after using trig identities):

$$\Rightarrow \sigma(\Theta) = (\tfrac{1}{4})[(ZZ'e^2)/(2E)]^2\csc^4(\tfrac{1}{2}\Theta) \qquad (9)$$

(9) ≡ ***The Rutherford Scattering Cross Section***

- Get the same results in a (non-relativistic) QM derivation!

Total Cross Section

- What we've discussed up to now is the ***Differential Cross Section:*** $\boldsymbol{\sigma(\Omega)}$.
- For Central Forces, this is a function of $\boldsymbol{\Theta}$ only:

$$\sigma(\Omega)d\Omega \equiv 2\pi\sigma(\Theta)\sin\Theta d\Theta$$

- **Note:** To emphasize the differential nature of $\boldsymbol{\sigma(\Omega)}$, in some texts it is denoted: $\boldsymbol{\sigma(\Omega) = (d\sigma/d\Omega)}$

- Often, it is useful to consider the

Total Cross Section: $\sigma_T \equiv \int\sigma(\Omega)d\Omega$.

- For Central Forces, we have:

$$\sigma_T = 2\pi\int\sigma(\Theta)\sin\Theta d\Theta$$

⇒ For **repulsive Coulomb Scattering:**

$$\sigma_T = 2\pi\int\sigma(\Theta)\sin\Theta d\Theta \qquad (1)$$

where, **σ(Θ)** is given by the **Rutherford formula:**

$$\sigma(\Theta) = (\tfrac{1}{4})[(ZZ'e^2)/(2E)]^2\csc^4(\tfrac{1}{2}\Theta) \qquad (2)$$

- Putting **(2)** into **(1)** & integrating gives $\sigma_T \to \infty$ **!**
  - Repulsive Coulomb Scattering: σ_T ***diverges!***
- ***PHYSICS Reason*** for the divergence: By definition, σ_T = total # particles scattered (in all directions/unit time/intensity). Range of Coulomb force goes to $r \to \infty$. Also, very small deflections (**Θ** near **0** in integrand of **(1)**) occur for very large $s = s(\Theta,E) = (ZZ'e^2)/(2E)\cot(\tfrac{1}{2}\Theta)$. Only if the force "cuts off" or → **0** beyond a certain distance will σ_T be finite. **Actually happens in *real Coulomb scattering* due to screening effects**.

- **Impact parameter for Rutherford scattering:**

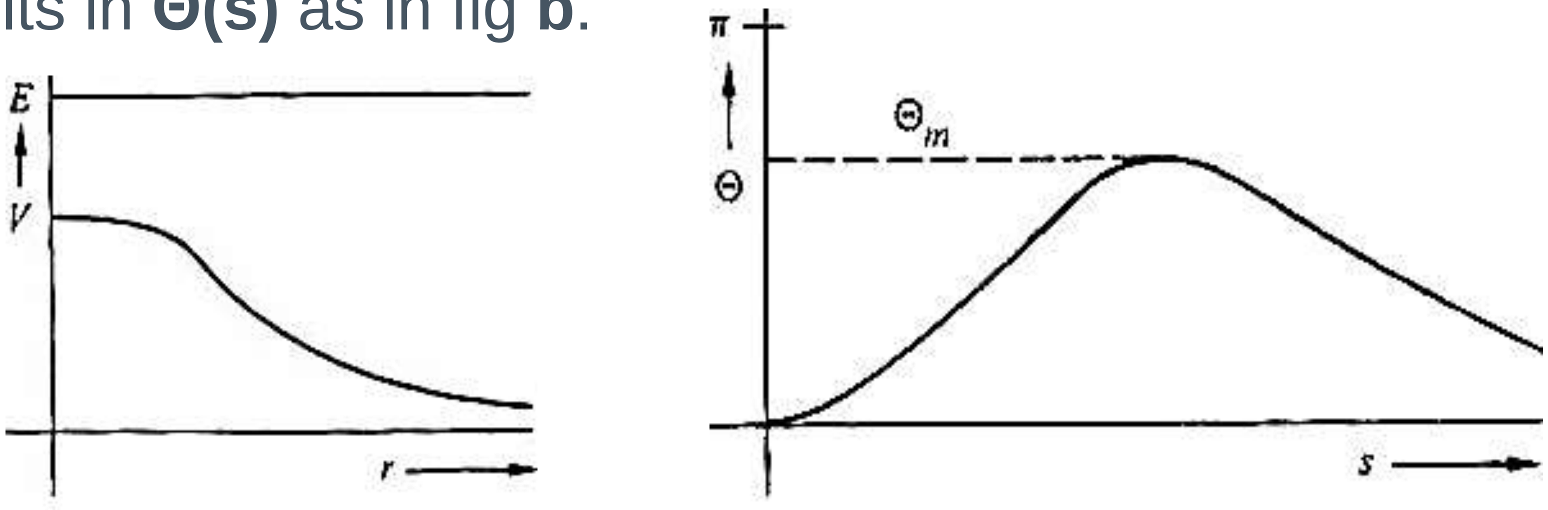
$$s = s(\Theta,E) = (ZZ'e^2)/(2E)\cot(\tfrac{1}{2}\Theta).$$

$\Rightarrow$ **$\Theta = \Theta(s,E)$** = a smooth, monotonic function of **s**.

At **$\Theta = 0$, $s \to \infty$; $\Theta = \pi$, $s = 0$**

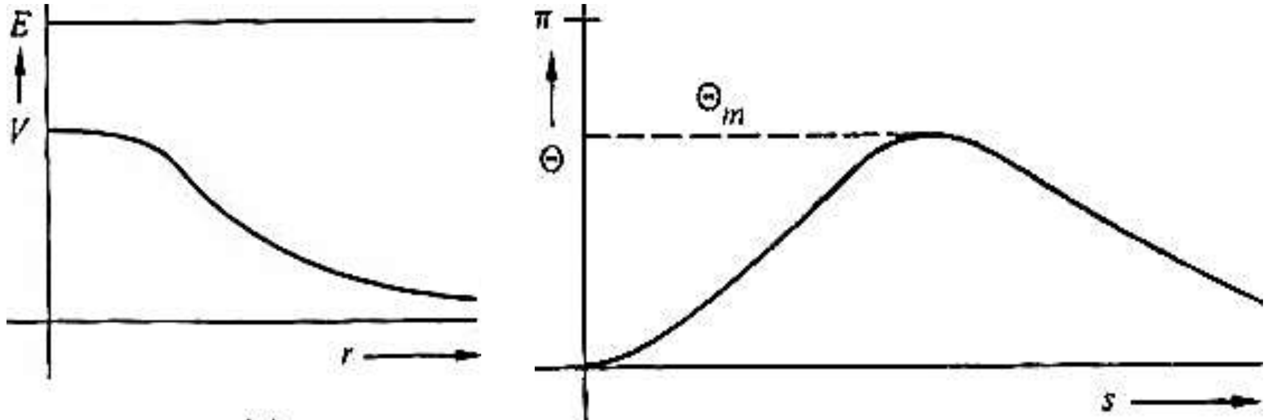
- **Other Central Potentials:** Can get other types of behavior for **$\Theta(s,E)$.** Some require some modification of the cross section prescription: **$\sigma(\Theta) = (s/\sin\Theta)(|ds|/|d\Theta|)$**
- **Example:** Repulsive potential & energy as in fig **a**.

Results in **$\Theta(s)$** as in fig **b**.



(a) (b)

› **Example:**



(a) (b)

› **From figures:**

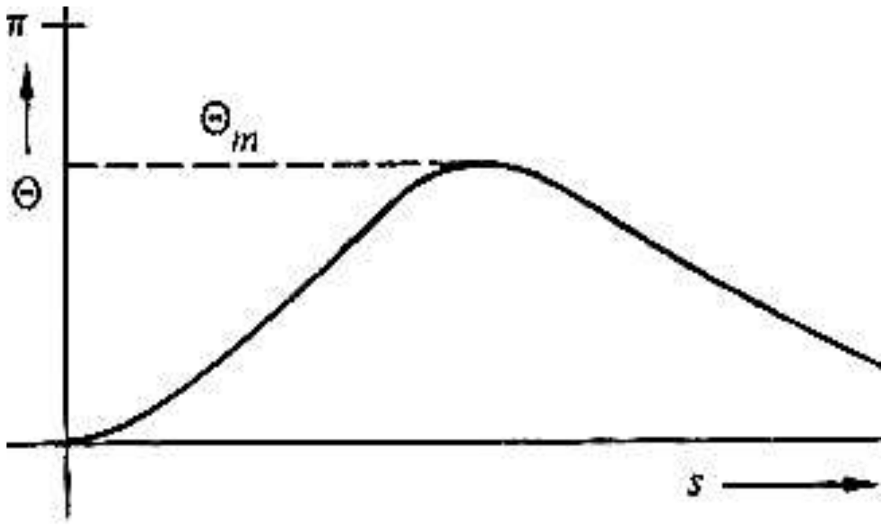
– Very large **s**: Particle always remains at large **r** from force center.

– Very small **s**, near **s = 0**, particle travels in a straight line into the force center **r = 0**. If **E > V**$_{max}$, it will travel through force center with very little scattering (zero for **s** exactly = **0**) $\Rightarrow$ For both limits, $\boldsymbol{\Theta = \Theta(s) \to 0}$

$\Rightarrow$ **Θ(s)** has a maximum $\equiv$ $\boldsymbol{\Theta_m}$ to the function as in fig b.

$\Rightarrow$ **Θ(s)** = double valued function! 2 different **s**'s give the same scattering angle **Θ.**

› **Example:**



(b)

› **Θ(s)** = **double valued function.**

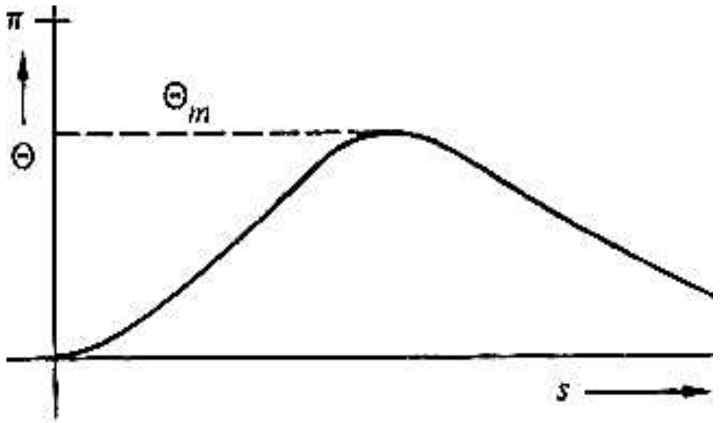
⇒ Must modify the cross section formula from

$$\sigma(\Theta) = (s/\sin\Theta)\,(|ds|/|d\Theta|).$$

To (for $\Theta \neq \Theta_m$): $\sigma(\Theta) = \sum_i (s_i/\sin\Theta)\,(|ds/d\Theta|_i)$

Subscript **i = 1,2,**: the 2 values of **s** which give the same **Θ**

› Example:



(b)

$$\sigma(\Theta) = \sum_i (s_i/\sin\Theta)\,(|ds/d\Theta|_i) \;\backslash,\; i = 1,2,$$

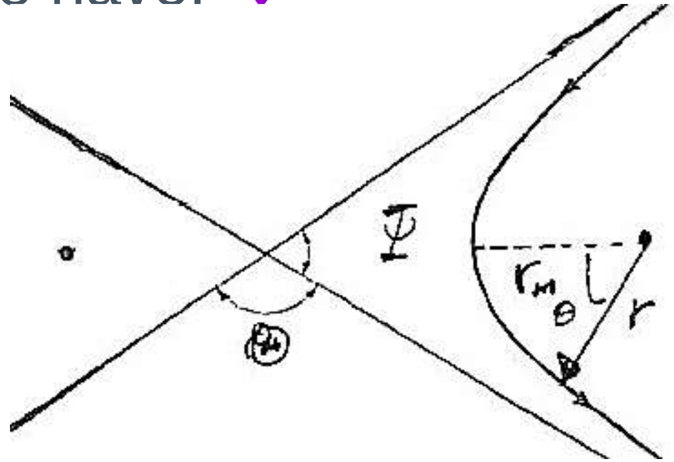
- Look at **σ(Θ)** for **Θ = Θ_m**: Since **Θ_m** = maximum of **Θ(s)**, **(dΘ/ds)** ≡ **0** at that angle & **(|ds/dΘ|)** in the cross section formula → ∞ ⇒ **σ(Θ)** → ∞.
- Note! If **Θ > Θ_m**, **σ(Θ) = 0** since **Θ_m** = maximum allowed **Θ** for scattering to occur.
  - Infinite rise of **σ(Θ)** followed by abrupt disappearance!
  - Similar to an optics phenomenon ≡ "rainbow scattering"

Attractive Scattering

- Up to now: Repulsive scattering. **Changes for Attractive Scattering?** Several complications:
- **Obvious:** Attraction pulls the particle towards force center rather than pushes it away.
- For $\mathbf{r}^{-2}$ scattering, $\mathbf{E > 0} \;\Rightarrow \boldsymbol{\varepsilon > 1} \;\Rightarrow$ Orbit is still a hyperbola.

However, instead of: ↓

We have: ↓



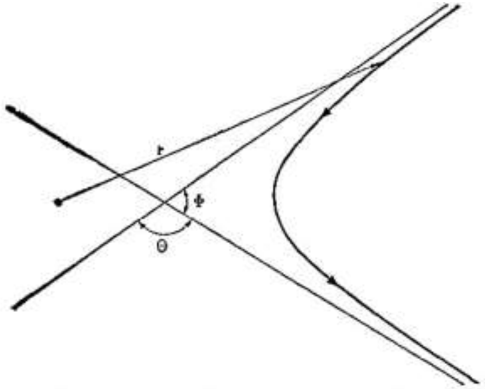
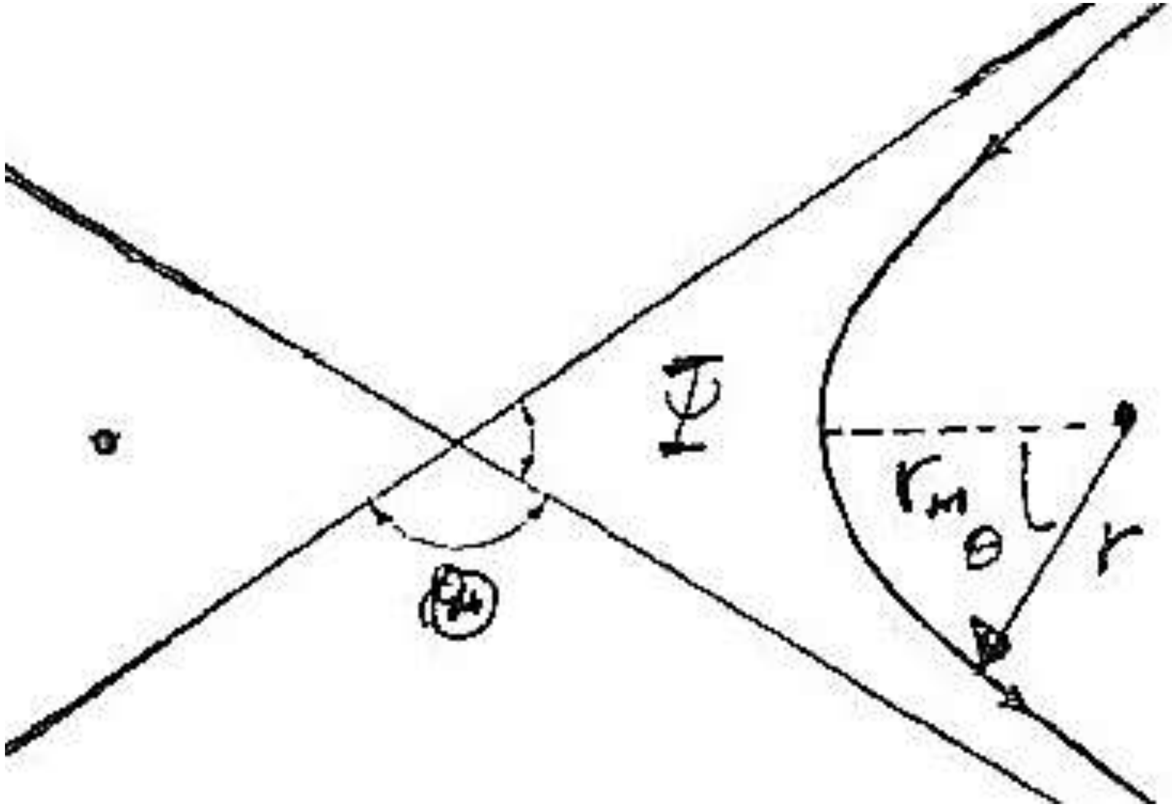


FIG. 3-15. Orbit for repulsive coulomb scattering, illustrating the connection between the angle between the asymptotes and the scattering angle.

The center of force is at the other focus of the hyperbola!

› **Attractive scattering** (r^{-2} force): **Hyperbolic orbit**.



$\Rightarrow$ Can have $\Psi > (\frac{1}{2})\pi \Rightarrow$ Can have $\Theta = \pi - 2\Psi < 0$.

Not a problem since $|\Theta|$ enters the calculation of $\sigma(\Theta)$

- **General, attractive Central Force:** In general:

$$\Theta(s) = \pi - 2\int dr(s/r)[r^2\{1- V(r)/E\} - s^2]^{-½}$$

Depending on attractive **V(r), s, & E**, can have **$\Theta(s) > 2\pi$**

⇒ It is possible for the scattered particle to circle the force center for one complete revolution *OR MORE* before moving off to $r \rightarrow \infty$!

- Consider ***qualitatively*** how this might happen. Effective potential **$V'(r) = V(r) + (½)[\ell^2/(mr^2)]$.** Plot for different values of **s** (equivalently, at several values of ℓ **= s(2mE)^{½}**)
- **Qualitative** discussion now!

› **V´(r)** for different values of **s:**

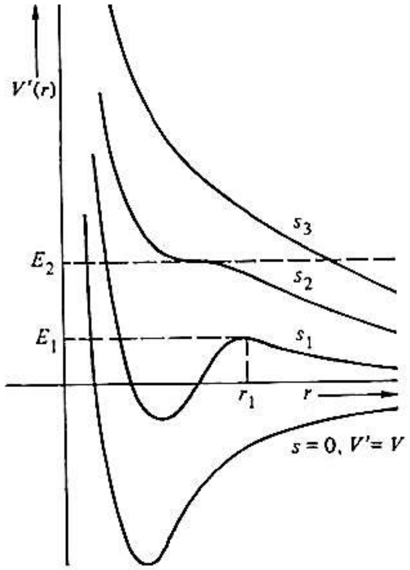


FIGURE 3.22 A combined attractive and repulsive scattering potential, and the corresponding equivalent one-dimensional potential at several values of the impact parameter s.

The **s =ℓ = 0** curve corresponds to **V´= V** (true potential) (looks ≈ a molecular potential) For **s ≠ 0 (& >0) & E > 0** the centrifugal barrier **(½)[ℓ²⁄(mr²)]** dominates at small r ***and*** at large r

⇒ **V´(r)** **has a bump, as shown.**

- Particle with impact parameter $\mathbf{s_1}$ & energy $\mathbf{E_1}$ at max of bump in $\mathbf{V'(r)}$: Conservation of energy

  $\Rightarrow \mathbf{E - V'(r_1) = (½)m\dot{r}^2 = 0}$

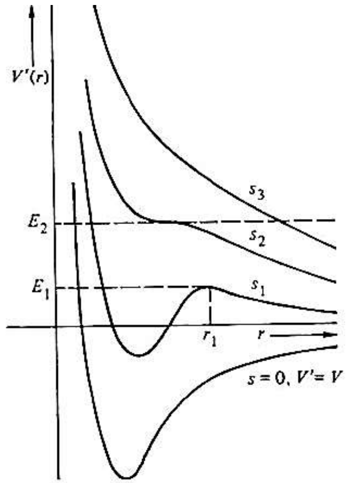


FIGURE 3.22 A combined attractive and repulsive scattering potential, and the corresponding equivalent one-dimensional potential at several values of the impact parameter s.

$\Rightarrow$ When the incoming particle reaches $\mathbf{r_1}$, $\mathbf{\dot{r} = 0}$

- **Previous discussion:** These are conditions for an ***unstable*** **circular orbit** at $\mathbf{r = r_1}$ $\Rightarrow$ In the absence of perturbations, the incoming particle is "captured" by the force center & goes into a circular orbit at $\mathbf{r = r_1}$ ***forever***! For $\mathbf{s = s_1}$ but $\mathbf{E > \approx E_1}$, no true circular orbit, but for very small $\mathbf{r - r_1} \Rightarrow$ the particle spends a **long time** at **r** near $\mathbf{r_1}$. It may ***orbit or spiral around*** the force center more than once before moving on inward towards it, or perhaps moving back on out towards $\mathbf{r \to \infty}$!

- Particle, impact parameter

  s_1, energy **$E > \approx E_1$**, **r** near

  the max of bump in **V´(r)**.

  <u>***Unstable***</u> **circular orbit** at

  $r = r_1$ $\Rightarrow$ No circular orbit for

  very small **$r - r_1$** $\Rightarrow$ The particle may ***orbit or spiral***

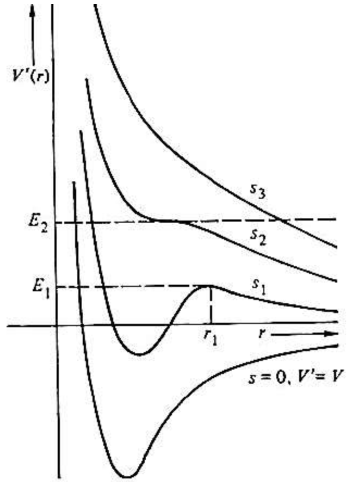


FIGURE 3.22 A combined attractive and repulsive scattering potential, and the corresponding equivalent one-dimensional potential at several values of the impact parameter s.

  around the center. The angular dependence of the motion is given by cons. of angular momentum: $\ell = \mathbf{m r^2 \dot{\theta}}$ = const $\Rightarrow$ for **$r \approx r_1$, $\dot{\theta} = [\ell/m(r_1)^2]$**. Use ℓ **= s(2mE)$^{1/2}$**

  $\Rightarrow$ **$\dot{\theta} = [s_1/(r_1)^2](2E/m)^{1/2}$**.

$\Rightarrow$ In the time for the particle to get through the region of the bump, the angular velocity may carry the particle through angles > **2π** $\equiv$ <u>***Orbiting or spiraling scattering***</u>.

- As **$s > s_1$**, the bump in **V´(r)** flattens out. At some **$s = s_2$** : **V´** has an inflection point at energy **E_2**. For **$E > E_2$** **no longer have orbiting.** But the combined effects of **V(r)** & the barrier **(½)[ℓ^2/(mr^2)]** can still lead to a scattering angle **Θ = 0** for some **s.**

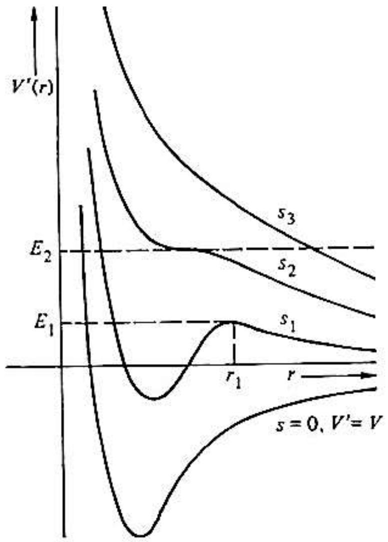


FIGURE 3.22 A combined attractive and repulsive scattering potential, and the corresponding equivalent one-dimensional potential at several values of the impact parameter s.

- Large **E** & small **s**: The scattering is dominated by the **(½)[ℓ^2/(mr^2)]** part & thus **σ(Θ)** is qualitatively similar to the Rutherford results.

- Just saw: For a **general Central Force**, can have the scattering angle $\boldsymbol{\Theta(s) > \pi}$. But: the **observed angle** is always $\boldsymbol{0 < \Theta(s) < \pi}$! So: A change of notation!

$\Rightarrow$ Introduce the ***deflection angle*** $\boldsymbol{\Phi} \equiv$ angle ***calculated*** from previous the formulas for $\boldsymbol{\Theta(s)}$:

$$\Phi \equiv \pi - 2\int dr(s/r)[r^2\{1- V(r)/E\} - s^2]^{-½}$$

Use the symbol $\boldsymbol{\Theta}$ for the ***observed scattering angle***

- Have the relation: $\boldsymbol{\Theta = \pm\Phi - 2m\pi}$ ($\mathbf{m}$ = integer $\mathbf{> 0}$)
  – Sign ($\pm$) & value of **m** is chosen so the observed angle $\boldsymbol{0 < \Theta < \pi}$. $\Rightarrow$ Sum in $\boldsymbol{\sigma(\Theta) = \sum_i(s_i/\sin\Theta)\,(|ds/d\Theta|_i)}$ covers all values of $\boldsymbol{\Phi}$ leading to the same $\boldsymbol{\Theta}$.

$$\Phi \equiv \pi - 2\int dr(s/r)[r^2\{1- V(r)/E\} - s^2]^{-½}$$

$\Theta \equiv$ *observed angle* $= \pm\Phi - 2m\pi$ (m > 0)

- **Φ** vs. **s**: For **$E = E_1$**

  & **$E = E_2$** →

  in **$V'(r)$** curves in ↓

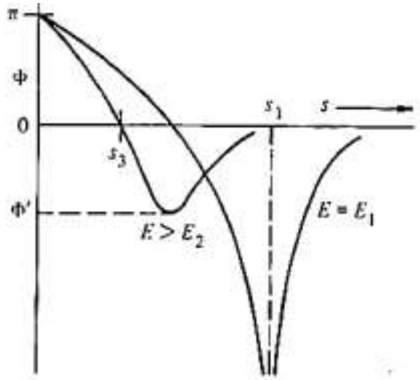


FIGURE 3.23 Curves of deflection angle Φ versus *s*, for the potential of Fig. 3.22 at two different energies.

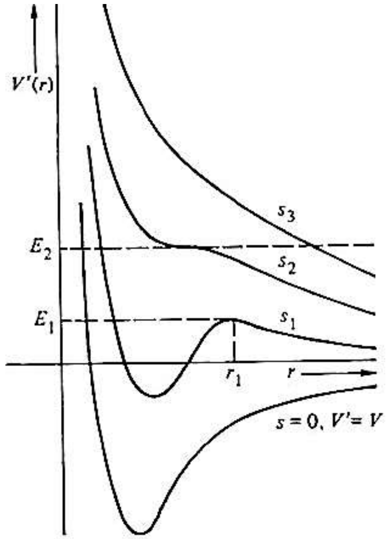


FIGURE 3.22 A combined attractive and repulsive scattering potential, and the corresponding equivalent one-dimensional potential at several values of the impact parameter *s*.

$E = E_1$: We've seen the orbiting.

Shows up as singularity the in **Φ** vs. **s** curve.

$E = E_2 > E_1$: No orbiting. "Rainbow effect" at **$\Theta = -\Phi'$** (min of curve). **$s = s_3$: $\Theta = \Phi = 0 \Rightarrow$**

$\sigma(\Theta) = (s/\sin\Theta)(|ds|/|d\Theta|) \to \infty$

Also can happen for **$\Theta = \pi$** $\equiv$ "glory scattering"

Thank You