Canonical Transformation

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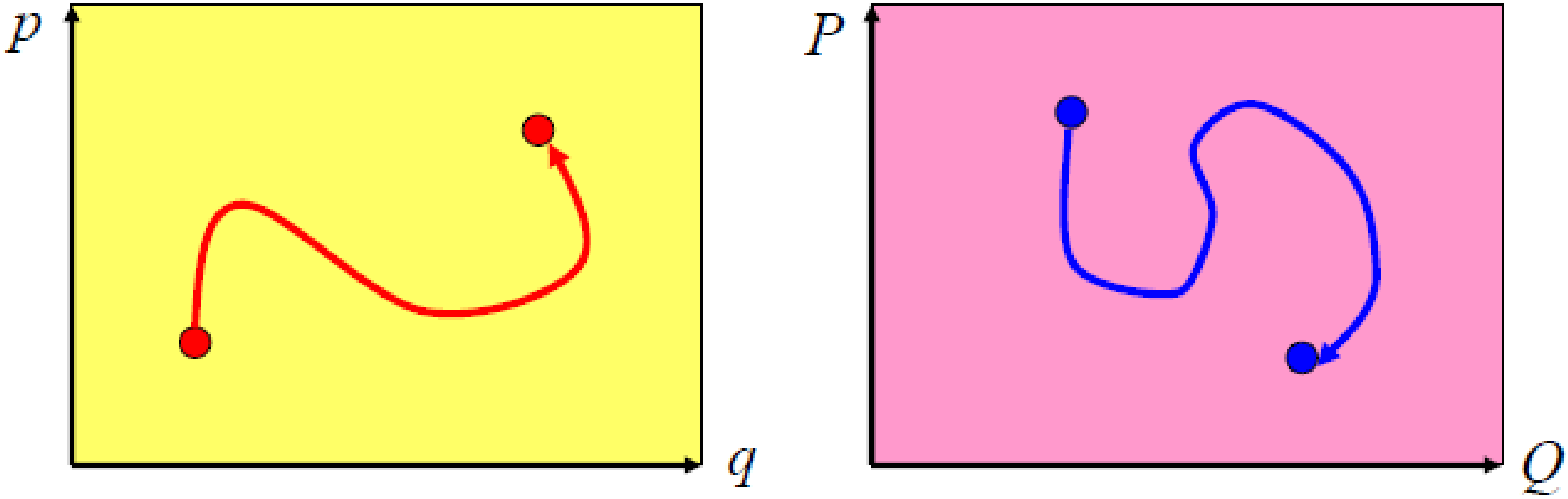
Two Points of View

- Canonical Transformation allows one system to be described by multiple sets of coordinates/momenta
  - Same physical system is expressed in different phase spaces

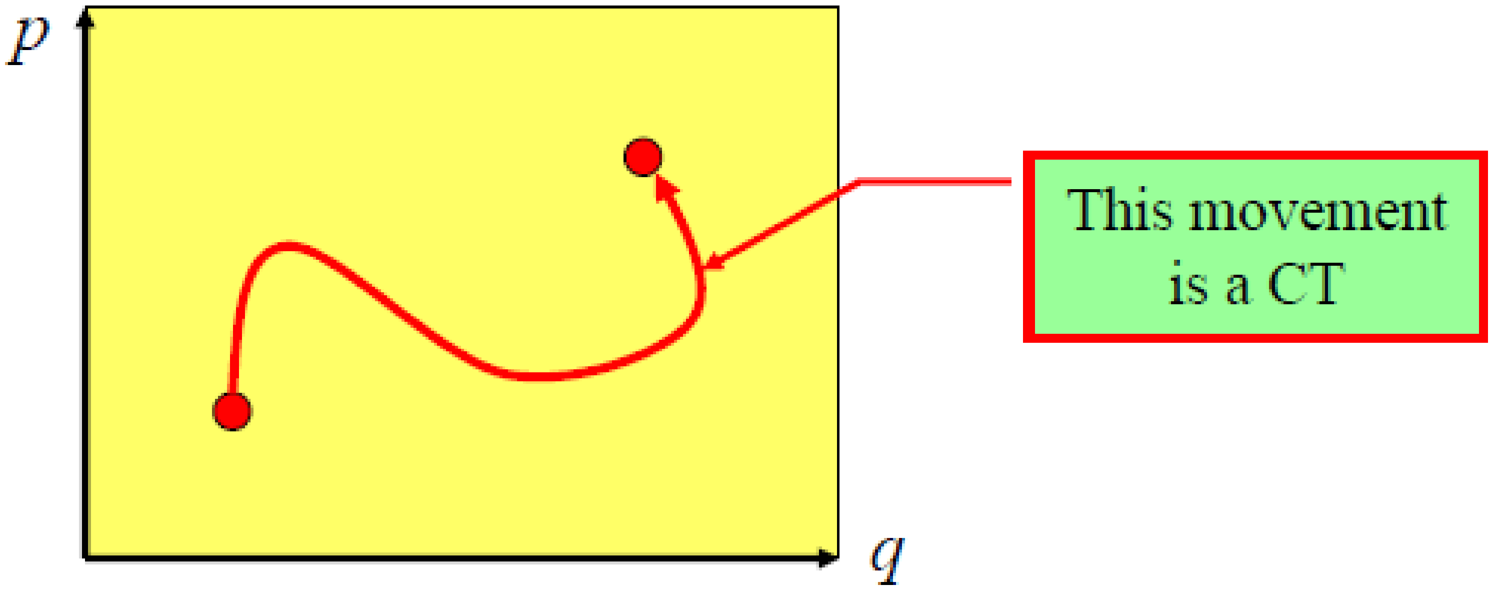


- This is the static view – The system itself is unaffected

Is there a dynamic view?

Dynamics View of CT

- A system evolves with time $q(t_0), p(t_0) \Rightarrow q(t), p(t)$
  - At any moment, q and p satisfy Hamilton's equations
  - The time-evolution must be a Canonical Transformation!



- Static View = Coordinate system is changing
- Dynamic View = Physical system is moving

Canonical Transformation

- Goal: To find transformations

$$Q_i = Q_i(q_1, \ldots, q_n, p_1, \ldots, p_n, t) \qquad P_i = P_i(q_1, \ldots, q_n, p_1, \ldots, p_n, t)$$

that satisfy Hamilton's equation of motion

$$\dot{q}_i = \frac{\partial H}{dp_i} \qquad \dot{p}_i = -\frac{\partial H}{dq_i} \quad \Longrightarrow \quad \dot{Q}_i = \frac{\partial K}{dP_i} \qquad \dot{P}_i = -\frac{\partial K}{dQ_i}$$

  - K is the transformed Hamiltonian $K = K(Q, P, t)$

- Hamilton's principle requires

$$\delta \int_{t_1}^{t_2} (p_i \dot{q}_i - H(q, p, t))\, dt = 0 \quad \text{and} \quad \delta \int_{t_1}^{t_2} \left(P_i \dot{Q}_i - K(Q, P, t)\right) dt = 0$$

General Transformation

$$\delta\int_{t_1}^{t_2}\left(p_i\dot{q}_i - H(q,p,t)\right)dt = 0$$ and $$\delta\int_{t_1}^{t_2}\left(P_i\dot{Q}_i - K(Q,P,t)\right)dt = 0$$

- Two types of transformations are possible
  - $P_i\dot{Q}_i - K = \lambda(p_i\dot{q}_i - H)$ ⇐ Scale transformation
  - $P_i\dot{Q}_i - K + \dfrac{dF}{dt} = p_i\dot{q}_i - H$ ⇐ Canonical transformation
  - Both satisfy Hamilton's principle
- Combined, we find

$$P_i\dot{Q}_i - K + \frac{dF}{dt} = \lambda(p_i\dot{q}_i - H)$$ ⇐ Extended Canonical transformation

Scale Transformation

- We can always change the scale of (or unit we use to measure) coordinates and momenta

  $P_i = \nu p_i$ $Q_i = \mu q_i$

- To satisfy Hamilton's principle, we can define

  $K(P,Q,t) = \mu\nu H(p,q,t)$

  $\Rightarrow$ $P_i\dot{Q}_i - K = \mu\nu(p_i\dot{q}_i - H)$ ⇐ Scale transformation

- This is trivial

- We now concentrate on – Canonical transformations

Canonical Transformation

$$P_i\dot{Q}_i - K + \frac{dF}{dt} = p_i\dot{q}_i - H$$

- Hamilton's principle

$$\delta\int_{t_1}^{t_2}\left(P_i\dot{Q}_i - K\right)dt = \delta\int_{t_1}^{t_2}\left(p_i\dot{q}_i - H - \frac{dF}{dt}\right)dt = -\delta\left[F\right]_{t_1}^{t_2} = 0$$

  - Satisfied if $\delta p = \delta q = \delta P = \delta Q = 0$ at t_1 and t_2

- F can be any function of p_i, q_i, P_i, Q_i and t
  - It defines a canonical transformation
  - Call it the generating function of the transformation

or generator

Simple Example

$$P_i\dot{Q}_i - K + \frac{dF}{dt} = p_i\dot{q}_i - H$$

- Try a generating function: $F = q_i P_i - Q_i P_i$
  - Canonical transformation generated by F is

$$P_i\dot{Q}_i - K + \frac{dF}{dt} = -K + (q_i - Q_i)\dot{P}_i + P_i\dot{q}_i = p_i\dot{q}_i - H$$

$\Rightarrow$ $Q_i = q_i$ $P_i = p_i$ ⇐ Identity transformation

$K = H$

  - OK, that was too simple
    - Let's push this one step further…

Continue…

$$P_i\dot{Q}_i - K + \frac{dF}{dt} = p_i\dot{q}_i - H$$

- Let's try this one: $F = f_i(q_1,\ldots,q_n,t)P_i - Q_iP_i$
  - f_i are arbitrary functions of $q_1 \ldots q_n$ and t

$$P_i\dot{Q}_i - K + \frac{dF}{dt} = -K + (f_i - Q_i)\dot{P}_i + P_i\frac{\partial f_i}{\partial q_j}\dot{q}_j + \frac{\partial f_i}{\partial t}P_i = p_i\dot{q}_i - H$$

$\Rightarrow$ $Q_i = f_i(q_1,\ldots,q_n,t)$

All "point transformations" of generalized coordinates are covered

$$p_i = \frac{\partial f_j}{\partial q_i}P_j$$

Must invert these n equations to get P_i

$$K = H + \frac{\partial f_i}{\partial t}P_i$$

  - We can do all what we could do before

Arbitrarity

- Generating function F → a canonical transformation
  - Opposite mapping is not unique
    - There are many possible Fs for each transformation
  - e.g. add an arbitrary function of time $g(t)$ to F

$$P_i\dot{Q}_i - K + \frac{dF}{dt} \rightarrow P_i\dot{Q}_i - K + \frac{dF}{dt} + \boxed{\frac{dg(t)}{dt}}$$

Does not affect the action integral

$$\Rightarrow K \rightarrow K + \frac{dg(t)}{dt}$$

Just modifies the Hamiltonian without affecting physics

- F is arbitrary up to any function of time only
  - So is the Hamiltonian

Finding the Generator

- Let's look for a generating function
  - Suppose $K(Q,P,t)=H(q,p,t)$ for simplicity

$$\Rightarrow \frac{dF}{dt}=p_i\dot{q}_i-P_i\dot{Q}_i$$

  - Easiest way to satisfy this would be

$$F=F(q,Q) \qquad \frac{\partial F}{\partial q_i}=p_i \qquad \frac{\partial F}{\partial Q_i}=-P_i$$

- Trivial example: $F(q,Q)=q_iQ_i$

$$\Rightarrow p_i=Q_i \qquad P_i=-q_i$$

In the Hamiltonian formalism, you can freely swap the coordinates and the momenta

Type -1 Generator

$$P_i\dot{Q}_i - K + \frac{dF}{dt} = p_i\dot{q}_i - H$$

- $F = F(q,Q)$ is not very general
  - It does not allow *t*-dependent transformation
  - Fix this by extending to $F = F_1(q,Q,t)$ ← Call it Type-1

$$p_i = \frac{\partial F_1(q,Q,t)}{\partial q_i} \qquad P_i = -\frac{\partial F_1(q,Q,t)}{\partial Q_i}$$

  - This affects the Hamiltonian

$$\frac{dF}{dt} = \frac{\partial F_1}{\partial q_i}\dot{q}_i + \frac{\partial F_1}{\partial Q_i}\dot{Q}_i + \frac{\partial F_1}{\partial t} = p_i\dot{q}_i - P_i\dot{Q}_i + K - H$$

$$\Rightarrow \quad K = H + \frac{\partial F_1}{\partial t}$$

Harmonic Oscillator

- Consider a 1-dimensional harmonic oscillator

$$H(q,p) = \frac{p^2}{2m} + \frac{kq^2}{2} = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right) \qquad \omega^2 \equiv \frac{k}{m}$$

  - Sum of squares → Can we make them sine and cosine?
  - Suppose $p = f(P)\cos Q \quad q = \dfrac{f(P)}{m\omega}\sin Q$

$$\Rightarrow K = H = \frac{\{f(P)\}^2}{2m} \Leftarrow Q \text{ is cyclic} \rightarrow P \text{ is constant}$$

  - Trick is to find $f(P)$ so that the transformation is canonical
    - How?

Continue…

- Let's try a Type-1 generator

$$F_1(q,Q,t) \qquad p = \frac{\partial F_1}{\partial q} \qquad P = -\frac{\partial F_1}{\partial Q}$$

  - Express p as a function of q and Q

$$p = f(P)\cos Q \qquad q = \frac{f(P)}{m\omega}\sin Q \;\Rightarrow\; p = m\omega q \cot Q$$

  - Integrate with q $\Rightarrow$ $F_1 = \frac{m\omega q^2}{2}\cot Q$

$$\Rightarrow\; P = -\frac{\partial F_1}{\partial Q} = \frac{m\omega q^2}{2\sin^2 Q}$$

We are getting somewhere

Continue…

$$p = \frac{\partial F_1}{\partial q} = m\omega q \cot Q \qquad P = -\frac{\partial F_1}{\partial Q} = \frac{m\omega q^2}{2\sin^2 Q}$$

- We need to turn $H(q, p)$ into $K(Q, P)$
- Solve the above equations for q and p

$$q = \sqrt{\frac{2P}{m\omega}} \sin Q \qquad p = \sqrt{2Pm\omega} \cos Q$$

- Now work out the Hamiltonian

$$K = H = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right) = \omega P$$

  - Things don't get much simpler than this…

Continue…

$$K = \omega P = E$$

- Solving the problem is trivial

$$P = \text{const} = \frac{E}{\omega} \qquad \dot{Q} = \frac{\partial K}{\partial P} = \omega \qquad Q = \omega t + \alpha$$

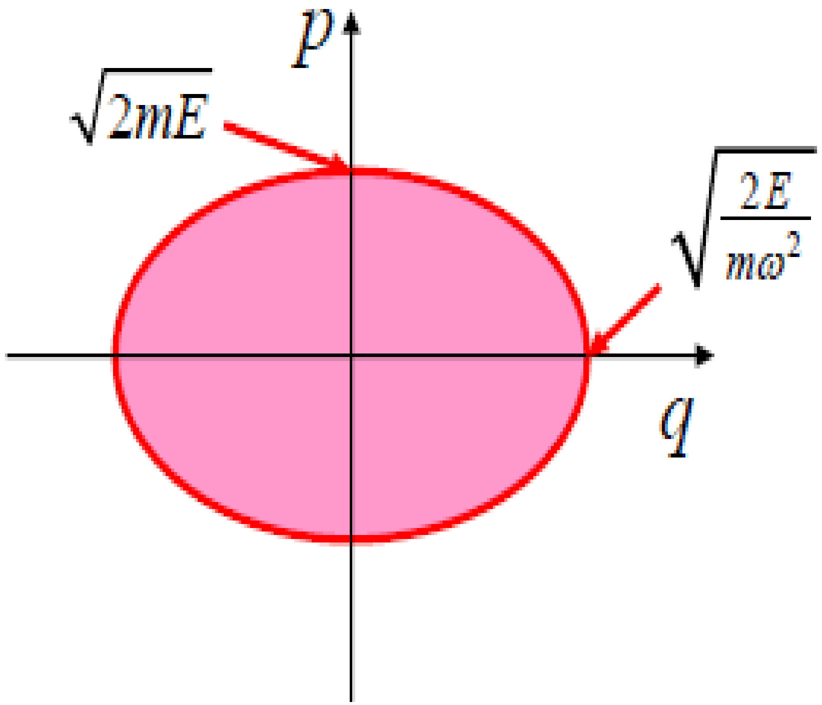
Finally ⇒

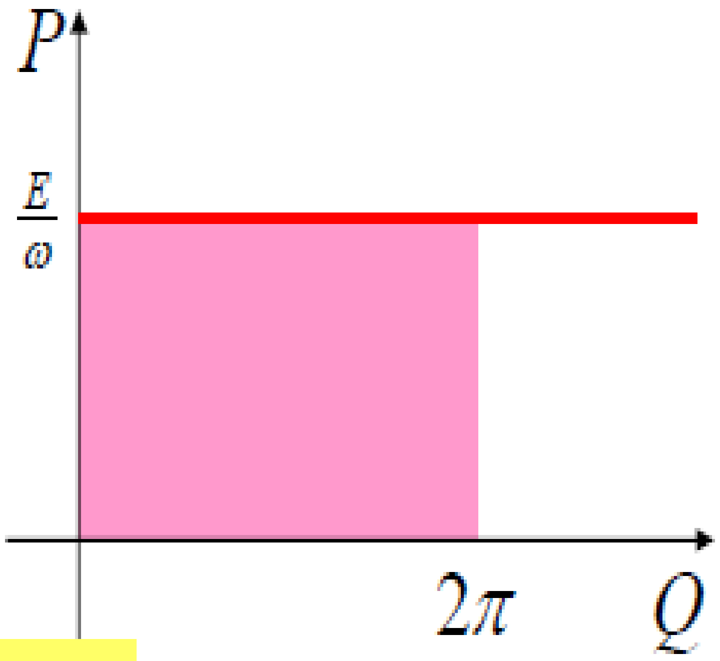
$$p = \sqrt{2Pm\omega}\cos Q = \sqrt{2mE}\cos(\omega t + \alpha)$$

$$q = \sqrt{\frac{2P}{m\omega}}\sin Q = \sqrt{\frac{2E}{m\omega^2}}\sin(\omega t + \alpha)$$

Phase Space

- Oscillator moves in the *p*-*q* and *P*-*Q* phase spaces





  - One cycle draws the same area $\dfrac{2\pi E}{\omega}$ in both spaces

Other types of generator

- Type-1 generator $F = F_1(q, Q, t)$ is still not so general
  - Just try to find a generator for $Q_i = q_i$ $P_i = p_i$
- We need generating functions of different set of independent variables
  - In fact, we may have 4 basic types of them
    $F_1(q, Q, t)$ $F_2(q, P, t)$ $F_3(p, Q, t)$ $F_4(p, P, t)$
- We can derive them using the now-familiar rule
  - i.e. we can add any dF/dt inside the action integral

Continue…

- In the last lecture, I used $F=-q_i p_i$ to convert

$$\delta\int_{t_1}^{t_2}\left(p_i\dot{q}_i - H(q,p,t)\right)dt = 0 \quad \Rightarrow \quad \delta\int_{t_1}^{t_2}\left(-\dot{p}_i q_i - H(q,p,t)\right)dt = 0$$

  - Switch the definition of canonical transformations

$$P_i\dot{Q}_i - K + \frac{dF}{dt} = p_i\dot{q}_i - H \quad \Rightarrow \quad -\dot{P}_i Q_i - K + \frac{dF}{dt} = p_i\dot{q}_i - H$$

$$\Rightarrow \quad \frac{dF}{dt} = p_i\dot{q}_i + Q_i\dot{P}_i + K - H$$

  - To satisfy this

$$F = F_2(q,P,t) \qquad \frac{\partial F_2}{\partial q_i} = p_i \qquad \frac{\partial F_2}{\partial P_i} = Q_i \qquad K = H + \frac{\partial F_2}{\partial t}$$

Continue…

- If we go back to the original definition of generating function $P_i\dot{Q}_i - K + \dfrac{dF}{dt} = p_i\dot{q}_i - H$

$$F = F_2(q,P,t) - Q_iP_i \qquad \frac{\partial F_2}{\partial q_i} = p_i \qquad \frac{\partial F_2}{\partial P_i} = Q_i \qquad K = H + \frac{\partial F_2}{\partial t}$$

  - Trivial case: $F_2 = q_iP_i$

$\Rightarrow$ $p_i = P_i$ $\quad Q_i = q_i$ $\Leftarrow$ Identity transformation

- We push the same idea to define the other 2 types

Four Basics Generator

Generator	Derivatives	Trivial Case
$F_1(q,Q,t)$	$p_i = \frac{\partial F_1}{\partial q_i} \quad P_i = -\frac{\partial F_1}{\partial Q_i}$	$F_1 = q_i Q_i \quad Q_i = p_i \quad P_i = -q_i$
$F_2(q,P,t) - Q_i P_i$	$p_i = \frac{\partial F_2}{\partial q_i} \quad Q_i = \frac{\partial F_2}{\partial P_i}$	$F_2 = q_i P_i \quad Q_i = q_i \quad P_i = p_i$
$F_3(p,Q,t) + q_i p_i$	$q_i = -\frac{\partial F_3}{\partial p_i} \quad P_i = -\frac{\partial F_3}{\partial Q_i}$	$F_3 = p_i Q_i \quad Q_i = -q_i \quad P_i = -p_i$
$F_4(p,P,t) + q_i p_i - Q_i P_i$	$q_i = -\frac{\partial F_4}{\partial p_i} \quad Q_i = \frac{\partial F_4}{\partial P_i}$	$F_4 = p_i P_i \quad Q_i = p_i \quad P_i = -q_i$

Continue…

- The 4 types of generators are almost equivalent
  - It may look as if F_1 is special, but it isn't

$$P_i\dot{Q}_i - K + \frac{dF_1}{dt} = p_i\dot{q}_i - H$$

$$-\dot{P}_iQ_i - K + \frac{dF_2}{dt} = p_i\dot{q}_i - H$$

$$P_i\dot{Q}_i - K + \frac{dF_3}{dt} = -\dot{p}_iq_i - H$$

$$-\dot{P}_iQ_i - K + \frac{dF_4}{dt} = -\dot{p}_iq_i - H$$

There is no reason to consider any of these 4 definitions to be more fundamental than the others

We arbitrarily chose the first form (which happens to be the Lagrangian form) to write the generating functions in the table

Continue…

- Some canonical transformations cannot be generated by all 4 types
  - e.g. identity transf. is generated only by F_2 or F_3
- This does not present a fundamental problem
  - One can always swap coordinate and momentum
    $Q_i = p_i \quad P_i = -q_i$
  - One can always change sign by scale transformation
    $Q_i = \pm q_i \quad P_i = \pm p_i$
- These transformations make the 4 types practically equivalent

Example

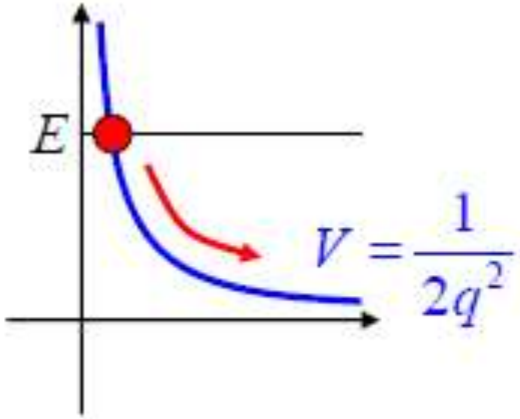
- 1-dim system with $H = \frac{p^2}{2} + \frac{1}{2q^2}$
  - Try $P = pq$
  - Let's use Type-2

    $F = F_2(q, P, t)$ $\quad \frac{\partial F_2}{\partial q} = p \quad \frac{\partial F_2}{\partial P} = Q$

  - Step 1: Express p with q and P ➡ $p = \frac{P}{q}$
  - Step 2: Integrate with q to get

    $F_2 = P \log q$ ⇦ assuming $q > 0$

  - Step 3: Differentiate to get $Q = \log q$ ➡ $q = e^Q$
  - Now we have a canonical transformation



Continue…

$$F_2 = P\log q \quad q = e^Q \quad p = \frac{P}{q} = Pe^{-Q}$$

- Now rewrite the Hamiltonian

$$H = \frac{p^2}{2} + \frac{1}{2q^2} = \frac{P^2+1}{2}e^{-2Q} = E$$

constant

- Equation of motion: $\dot{P} = (P^2+1)e^{-2Q} = 2E$

$$\Rightarrow P = 2Et + C$$

$$\Rightarrow q = e^Q = \sqrt{\frac{P^2+1}{2E}} = \sqrt{2Et^2 + 2Ct + \frac{C^2+1}{2E}}$$

Summary

- Canonical transformations

$$P_i\dot{Q}_i - K + \frac{dF}{dt} = p_i\dot{q}_i - H$$

  - Hamiltonian formalism is invariant under canonical + scale transformations
  - Generating functions define canonical transformations
  - Four basic types of generating functions

    $F_1(q,Q,t)$ $F_2(q,P,t)$ $F_3(p,Q,t)$ $F_4(p,P,t)$

    - They are all practically equivalent
- Used it to simplify a harmonic oscillator
  - Invariance of phase space area

Continue…

- Dig deeper into Canonical Transformations
  - Infinitesimal Canonical Transformation
    - Very small changes in q and p
    - Define generator G for an ICT
  - Direct Conditions for Canonical Transformation
    - Necessary-and-sufficient conditions for any CT
  - Poisson Bracket
    - Invariant of any Canonical Transformation
    - Connect to Infinitesimal Canonical Transformation

Infinitesimal CT

- Consider a CT in which q, p are changed by small (infinitesimal) amounts

$$Q_i = q_i + \delta q_i \qquad P_i = p_i + \delta p_i$$

Infinitesimal Canonical Transformation (ICT)

  - ICT is close to identity transf.
  - Generating function should be $F_2(q,P,t) = q_i P_i + \varepsilon G(q,P,t)$

Identity CT generator

Small

Look at the generator table

$$p_i = \frac{\partial F_2}{\partial q_i} = P_i + \varepsilon \frac{\partial G}{\partial q_i} \qquad Q_i = \frac{\partial F_2}{\partial P_i} = q_i + \varepsilon \frac{\partial G}{\partial P_i}$$

Since ε is infinitesimal

$$\delta q_i = \varepsilon \frac{\partial G}{\partial P_i} \approx \varepsilon \frac{\partial G}{\partial p_i} \qquad \delta p_i = -\varepsilon \frac{\partial G}{\partial q_i} \approx -\varepsilon \frac{\partial G}{\partial Q_i}$$

Generator of ICT

- An ICT is generated by $F_2(q,P,t) = q_i P_i + \varepsilon G(q,P,t)$

$$Q_i = q_i + \varepsilon \frac{\partial G}{\partial P_i} \qquad P_i = p_i - \varepsilon \frac{\partial G}{\partial q_i}$$

  - G is called (inaccurately) the generator of the ICT
  - Since the CT is infinitesimal, G may be expressed in terms of q or Q, p or P, interchangeably
  - For example: $G = G(q,p,t)$ $\quad Q_i = q_i + \varepsilon \frac{\partial G}{\partial p_i} \quad P_i = p_i - \varepsilon \frac{\partial G}{\partial q_i}$

Hamiltonian

- Consider $G = H(q, p, t)$

$$\Rightarrow \quad \delta q_i = \varepsilon \frac{\partial H}{\partial p_i} = \varepsilon \dot{q}_i \qquad \delta p_i = -\varepsilon \frac{\partial H}{\partial q_i} = \varepsilon \dot{p}_i$$

  - What does ε look like? → Infinitesimal time δt

$$\delta q_i = \dot{q}_i \delta t \qquad \delta p_i = \dot{p}_i \delta t$$

- Hamiltonian is the generator of infinitesimal time transformation
  - In QM, you learn that Hamiltonian is the operator that represents advance of time

Direct Condition

- Consider a restricted Canonical Transformation
  - Generator has no t dependence

$$\frac{\partial F}{\partial t} = 0 \Rightarrow K(Q,P) = H(q,p)$$

Hamiltonian is unchanged

  - Q and P depends only on q and p

$$Q_i = Q_i(q,p) \qquad P_i = P_i(q,p)$$

$$\dot{Q}_i = \frac{\partial Q_i}{\partial q_j}\dot{q}_j + \frac{\partial Q_i}{\partial p_j}\dot{p}_j = \frac{\partial Q_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial Q_i}{\partial p_j}\frac{\partial H}{\partial q_j}$$

$$\dot{P}_i = \frac{\partial P_i}{\partial q_j}\dot{q}_j + \frac{\partial P_i}{\partial p_j}\dot{p}_j = \frac{\partial P_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial P_i}{\partial p_j}\frac{\partial H}{\partial q_j}$$

Hamilton's equations

Continue…

- On the other hand, Hamilton's eqns say

$$\dot{Q}_i = \frac{\partial H}{\partial P_i} = \frac{\partial H}{\partial q_j}\frac{\partial q_j}{\partial P_i} + \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial P_i} \Longleftrightarrow \dot{Q}_i = \frac{\partial Q_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial Q_i}{\partial p_j}\frac{\partial H}{\partial q_j}$$

$$\dot{P}_i = -\frac{\partial H}{\partial Q_i} = -\frac{\partial H}{\partial q_j}\frac{\partial q_j}{\partial Q_i} - \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial Q_i} \Longleftrightarrow \dot{P}_i = \frac{\partial P_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial P_i}{\partial p_j}\frac{\partial H}{\partial q_j}$$

Direct Conditions for a Canonical Transformation

$$\left(\frac{\partial Q_i}{\partial q_j}\right)_{q,p} = \left(\frac{\partial p_j}{\partial P_i}\right)_{Q,P} \qquad \left(\frac{\partial Q_i}{\partial p_j}\right)_{q,p} = -\left(\frac{\partial q_j}{\partial P_i}\right)_{Q,P}$$

$$\left(\frac{\partial P_i}{\partial q_j}\right)_{q,p} = -\left(\frac{\partial p_j}{\partial Q_i}\right)_{Q,P} \qquad \left(\frac{\partial P_i}{\partial p_j}\right)_{q,p} = \left(\frac{\partial q_j}{\partial Q_i}\right)_{Q,P}$$

Continue…

$$\left(\frac{\partial Q_i}{\partial q_j}\right)_{q,p} = \left(\frac{\partial p_j}{\partial P_i}\right)_{Q,P} \qquad \left(\frac{\partial Q_i}{\partial p_j}\right)_{q,p} = -\left(\frac{\partial q_j}{\partial P_i}\right)_{Q,P}$$

$$\left(\frac{\partial P_i}{\partial q_j}\right)_{q,p} = -\left(\frac{\partial p_j}{\partial Q_i}\right)_{Q,P} \qquad \left(\frac{\partial P_i}{\partial p_j}\right)_{q,p} = \left(\frac{\partial q_j}{\partial Q_i}\right)_{Q,P}$$

- Direct Conditions are necessary and sufficient for a time-independent transformation to be canonical
  - You can use them to test a CT
- In fact, this applies to all Canonical Transformations

Continue…

$$\delta q_i = \varepsilon \frac{\partial G}{\partial P_i} \approx \varepsilon \frac{\partial G}{\partial p_i}$$

$$\delta p_i = -\varepsilon \frac{\partial G}{\partial q_i} \approx -\varepsilon \frac{\partial G}{\partial Q_i}$$

- Does an ICT satisfy the DCs?

$$\frac{\partial Q_i}{\partial q_j} = \frac{\partial (q_i + \delta q_i)}{\partial q_j} = \delta_{ij} + \varepsilon \frac{\partial^2 G}{\partial P_i \partial q_j}$$

$$\frac{\partial p_j}{\partial P_i} = \frac{\partial (P_j - \delta p_j)}{\partial P_i} = \delta_{ij} + \varepsilon \frac{\partial^2 G}{\partial P_i \partial q_j}$$

$$\frac{\partial Q_i}{\partial p_j} = \frac{\partial (q_i + \delta q_i)}{\partial p_j} = \varepsilon \frac{\partial^2 G}{\partial P_i \partial p_j}$$

$$\frac{\partial q_j}{\partial P_i} = \frac{\partial (Q_j - \delta q_j)}{\partial P_i} = -\varepsilon \frac{\partial^2 G}{\partial P_i \partial p_j}$$

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$$\frac{\partial P_i}{\partial p_j} = \frac{\partial (p_i + \delta p_i)}{\partial p_j} = \delta_{ij} - \varepsilon \frac{\partial^2 G}{\partial Q_i \partial p_j}$$

$$\frac{\partial q_j}{\partial Q_i} = \frac{\partial (Q_j - \delta q_j)}{\partial Q_i} = \delta_{ij} - \varepsilon \frac{\partial^2 G}{\partial Q_i \partial p_j}$$

Yes!

Continue…

- Two successive CTs make a CT

$$P_i\dot{Q}_i - K + \frac{dF_1}{dt} = p_i\dot{q}_i - H \quad + \quad Y_i\dot{X}_i - M + \frac{dF_2}{dt} = P_i\dot{Q}_i - K$$

$$\Rightarrow Y_i\dot{X}_i - M + \frac{d(F_1 + F_2)}{dt} = p_i\dot{q}_i - K$$

True for unrestricted CTs

- Direct Conditions can also be "chained", e.g.,

$$\left(\frac{\partial Q_i}{\partial q_j}\right)_{q,p} = \left(\frac{\partial p_j}{\partial P_i}\right)_{Q,P} \quad + \quad \left(\frac{\partial X_i}{\partial Q_j}\right)_{Q,P} = \left(\frac{\partial P_j}{\partial Y_i}\right)_{X,Y}$$

$$\Rightarrow \left(\frac{\partial X_i}{\partial q_j}\right)_{q,p} = \left(\frac{\partial p_j}{\partial Y_i}\right)_{X,Y}$$

Easy to prove

Unrestricted CT

- Now we consider a general, time-dependent CT

$$Q_i = Q_i(q,p,t) \qquad P_i = P_i(q,p,t) \qquad K = H + \frac{\partial F}{\partial t}$$

  - Let's do it in two steps

$$q,p \Rightarrow Q(q,p,t_0), P(q,p,t_0) \Rightarrow Q(q,p,t), P(q,p,t)$$

Time-independent CT

Time-only CT

Fixed time

- First step is t-independent → Satisfies the DCs
  - We must show that the second step satisfies the DCs

Continue…

- Concentrate on a time-only CT $Q(t_0), P(t_0) \Rightarrow Q(t), P(t)$
  - Break $t - t_0$ into pieces of infinitesimal time dt

    $Q(t_0), P(t_0) \Rightarrow Q(t_0 + dt), P(t_0 + dt) \Rightarrow\Rightarrow\Rightarrow\Rightarrow Q(t), P(t)$
  - Each step is an ICT → Satisfies Direct Conditions
  - "Integrating" gives us what we needed

All Canonical Transformations satisfies the Direct Conditions, and vice versa

  - The proof worked because a time-only CT is a continuous transformation, parameterized by t

Poisson Bracket

- For u and v expressed in terms of q and p

$$[u,v]_{q,p} \equiv \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}$$

Poisson Bracket

  - This weird construction has many useful features
  - If you know QM, this is analogous to the commutator

$$\frac{1}{i\hbar}[u,v] \equiv \frac{1}{i\hbar}(uv - vu)$$ for two operators u and v

- Let's start with a few basic rules

Continue…

$$[u,v]_{q,p} \equiv \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}$$

- For quantities u, v, w and constants a, b

$$[u,u]=0 \qquad [u,v]=-[v,u]$$

All easy to prove

$$[au+bv,w]=a[u,w]+b[v,w]$$

$$[uv,w]=[u,w]v+u[v,w]$$

Jacobi's Identity

$$[u,[v,w]]+[v,[w,u]]+[w,[u,v]]=0$$

This one is worth trying.
See Goldstein if you are lost

Continue…

- Consider PBs of q and p themselves

$$[q_j, q_k] = \frac{\partial q_j}{\partial q_i}\frac{\partial q_k}{\partial p_i} - \frac{\partial q_j}{\partial p_i}\frac{\partial q_k}{\partial q_i} = 0 \qquad [p_j, p_k] = 0$$

$$[q_j, p_k] = \frac{\partial q_j}{\partial q_i}\frac{\partial p_k}{\partial p_i} - \frac{\partial q_j}{\partial p_i}\frac{\partial p_k}{\partial q_i} = \delta_{jk} \qquad [p_j, q_k] = -\delta_{jk}$$

  - Called the Fundamental Poisson Brackets
- Now we consider a Canonical Transformation

$$q, p \rightarrow Q, P$$

  - What happens to the Fundamental PB?

Fundamentals of PB and CT

$$[Q_j, Q_k]_{q,p} = \frac{\partial Q_j}{\partial q_i}\frac{\partial Q_k}{\partial p_i} - \frac{\partial Q_j}{\partial p_i}\frac{\partial Q_k}{\partial q_i} = -\frac{\partial Q_j}{\partial q_i}\frac{\partial q_i}{\partial P_k} - \frac{\partial Q_j}{\partial p_i}\frac{\partial p_i}{\partial P_k} = -\frac{\partial Q_j}{\partial P_k} = 0$$

$$[P_j, P_k]_{q,p} = \frac{\partial P_j}{\partial q_i}\frac{\partial P_k}{\partial p_i} - \frac{\partial P_j}{\partial p_i}\frac{\partial P_k}{\partial q_i} = \frac{\partial P_j}{\partial q_i}\frac{\partial q_i}{\partial Q_k} + \frac{\partial P_j}{\partial p_i}\frac{\partial p_i}{\partial Q_k} = \frac{\partial P_j}{\partial Q_k} = 0$$

$$[Q_j, P_k]_{q,p} = \frac{\partial Q_j}{\partial q_i}\frac{\partial P_k}{\partial p_i} - \frac{\partial Q_j}{\partial p_i}\frac{\partial P_k}{\partial q_i} = \frac{\partial Q_j}{\partial q_i}\frac{\partial q_i}{\partial Q_k} + \frac{\partial Q_j}{\partial p_i}\frac{\partial p_i}{\partial Q_k} = \frac{\partial Q_j}{\partial Q_k} = \delta_{jk}$$

$$[P_j, Q_k]_{q,p} = -[Q_k, P_j] = -\delta_{jk}$$

Used Direct Conditions here

- Fundamental Poisson Brackets are invariant under CT

Poisson Bracket & CT

- What happens to a Poisson Bracket under CT?
  - For a time-independent CT

$$
\begin{aligned}
[u,v]_{Q,P} &\equiv \frac{\partial u}{\partial Q_i}\frac{\partial v}{\partial P_i} - \frac{\partial u}{\partial P_i}\frac{\partial v}{\partial Q_i} \\
&= \left(\frac{\partial u}{\partial q_j}\frac{\partial q_j}{\partial Q_i} + \frac{\partial u}{\partial p_j}\frac{\partial p_j}{\partial Q_i}\right)\left(\frac{\partial v}{\partial q_k}\frac{\partial q_k}{\partial P_i} + \frac{\partial v}{\partial p_k}\frac{\partial p_k}{\partial P_i}\right) - \left(\frac{\partial u}{\partial q_j}\frac{\partial q_j}{\partial P_i} + \frac{\partial u}{\partial p_j}\frac{\partial p_j}{\partial P_i}\right)\left(\frac{\partial v}{\partial q_k}\frac{\partial q_k}{\partial Q_i} + \frac{\partial v}{\partial p_k}\frac{\partial p_k}{\partial Q_i}\right) \\
&= \frac{\partial u}{\partial q_j}\frac{\partial v}{\partial q_k}[q_j, q_k]_{Q,P} + \frac{\partial u}{\partial q_j}\frac{\partial v}{\partial p_k}[q_j, p_k]_{Q,P} + \frac{\partial u}{\partial p_j}\frac{\partial v}{\partial q_k}[p_j, q_k]_{Q,P} + \frac{\partial u}{\partial p_j}\frac{\partial v}{\partial p_k}[p_j, p_k]_{Q,P} \\
&= \frac{\partial u}{\partial q_j}\frac{\partial v}{\partial p_k}\delta_{jk} - \frac{\partial u}{\partial p_j}\frac{\partial v}{\partial q_k}\delta_{jk} \\
&= [u,v]_{q,p}
\end{aligned}
$$

Poisson Brackets are invariant under CT

Invariance of Poisson Bracket

- Poisson Brackets are canonical invariants
  - True for any Canonical Transformations
    - Goldstein shows this using "simplectic" approach
- We don't have to specify q, p in each PB

$$\boxed{[u,v]_{q,p}} \Rightarrow \boxed{[u,v]} \quad \text{good enough}$$

ICT and Poisson Bracket

- Infinitesimal CT can be expressed neatly with a PB
  - For a generator G, $Q_i = q_i + \varepsilon \frac{\partial G}{\partial p_i}$ $P_i = p_i - \varepsilon \frac{\partial G}{\partial q_i}$
  - On the other hand

$$\varepsilon[q_i, G] = \varepsilon\left(\frac{\partial q_i}{\partial q_j}\frac{\partial G}{\partial p_j} - \frac{\partial q_i}{\partial p_j}\frac{\partial G}{\partial q_j}\right) = \varepsilon\frac{\partial G}{\partial p_i} = \delta q_i$$

$$\varepsilon[p_i, G] = \varepsilon\left(\frac{\partial p_i}{\partial q_j}\frac{\partial G}{\partial p_j} - \frac{\partial p_i}{\partial p_j}\frac{\partial G}{\partial q_j}\right) = -\varepsilon\frac{\partial G}{\partial q_i} = \delta p_i$$

  - We can generalize further…

Continue…

- For an arbitrary function $u(q,p,t)$, the ICT does

$$
\begin{aligned}
u \xrightarrow{ICT} u+\delta u &= u+\frac{\partial u}{\partial q_i}\delta q_i+\frac{\partial u}{\partial p_i}\delta p_i+\frac{\partial u}{\partial t}\delta t \\
&= u+\frac{\partial u}{\partial q_i}\varepsilon\frac{\partial G}{\partial p_i}-\frac{\partial u}{\partial p_i}\varepsilon\frac{\partial G}{\partial q_i}+\frac{\partial u}{\partial t}\delta t \\
&= u+\varepsilon[u,G]+\frac{\partial u}{\partial t}\delta t
\end{aligned}
$$

  - That is $\delta u=\varepsilon[u,G]+\frac{\partial u}{\partial t}\delta t$

Infinitesimal Time Transf.

- Hamiltonian generates infinitesimal time transf.
  - Applying the Poisson Bracket rule

$$\delta u = \delta t[u, H] + \frac{\partial u}{\partial t}\delta t \Rightarrow \frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t}$$

  - Have you seen this in QM?
- If u is a constant of motion, $[u, H] + \frac{\partial u}{\partial t} = 0$
  - That is, $\boxed{[H, u] = \frac{\partial u}{\partial t} \Leftrightarrow u \text{ is a constant of motion}}$

Continue…

- If u does not depend explicitly on time,

$$\frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t} = [u, H]$$

  - Try this on q and p

$$\dot{q}_i = [q_i, H] = \frac{\partial q_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial q_i}{\partial p_j}\frac{\partial H}{\partial q_j} = \frac{\partial H}{\partial p_i}$$

$$\dot{p}_i = [p_i, H] = \frac{\partial p_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial p_i}{\partial p_j}\frac{\partial H}{\partial q_j} = -\frac{\partial H}{\partial q_i}$$

Hamilton's equations!

Summary

- Direct Conditions
  - Necessary and sufficient for Canonical Transf.

$$\left(\frac{\partial Q_i}{\partial q_j}\right)_{q,p} = \left(\frac{\partial p_j}{\partial P_i}\right)_{Q,P} \qquad \left(\frac{\partial Q_i}{\partial p_j}\right)_{q,p} = -\left(\frac{\partial q_j}{\partial P_i}\right)_{Q,P}$$

$$\left(\frac{\partial P_i}{\partial q_j}\right)_{q,p} = -\left(\frac{\partial p_j}{\partial Q_i}\right)_{Q,P} \qquad \left(\frac{\partial P_i}{\partial p_j}\right)_{q,p} = \left(\frac{\partial q_j}{\partial Q_i}\right)_{Q,P}$$

- Infinitesimal CT
- Poisson Bracket

$$[u,v] \equiv \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}$$

  - Canonical invariant
  - Fundamental PB $[q_i, q_j] = [p_i, p_j] = 0$ $\quad [q_i, p_j] = -[p_i, q_j] = \delta_{ij}$
  - ICT expressed by $\delta u = \varepsilon[u, G] + \dfrac{\partial u}{\partial t}\delta t$
  - Infinitesimal time transf. generated by Hamiltonian

Infinitesimal Time CT

- Infinitesimal CT $q(t), p(t) \Rightarrow q(t+dt), p(t+dt)$
  - We know that the generator = Hamiltonian

$$du = dt[u,H] + \frac{\partial u}{\partial t}dt \Rightarrow \dot{q} = [q,H] \quad \dot{p} = [p,H]$$

Hamiltonian is the generator of the system's motion with time

  - Integrating it with time should give us the "finite" CT that turns the initial conditions $q(t_0), p(t_0)$ into the configuration $q(t), p(t)$ of the system at arbitrary time
    - That's a new definition of "solving" the problem

Static Vs Dynamics

- Two ways of looking at the same thing
  - System is moving in a fixed phase space
    - Hamilton's equations → Integrate to get $q(t)$, $p(t)$
  - System is fixed and the phase space is transforming
    - ICT given by the PB → Integrate to get CT for finite t
- Equations are identical
  - You'll find yourself integrating exactly the same equations

Did we gain anything?

Conservation

- Consider an ICT generated by G $\delta u = \varepsilon[u, G] + \frac{\partial u}{\partial t}\delta t$
  - Suppose G is conserved and has no explicit t-dependence $\Rightarrow$ $[G, H] = 0$
  - How is H (without t-dependence) changed by the ICT?

$$\delta H = \varepsilon[H, G] + \frac{\partial H}{\partial t}\delta t = 0$$

If an ICT does not affect Hamiltonian, its generator is conserved

- A transformation that does not affect H
  → Symmetry of the system
  → Generator of the transformation is conserved

Continue…

- Simplest example:
  - What is the ICT generated by momentum p_i?

$$\delta q_j = \varepsilon[q_j, p_i] = \varepsilon\delta_{ij} \qquad \delta p_j = \varepsilon[p_j, p_i] = 0$$

    - That's a shift in q_i by ε → spatial translation
  - If Hamiltonian is unchanged by such shift, then $[H, p_i] = 0$
    → Momentum p_i is conserved
- This is not restricted to linear momentum

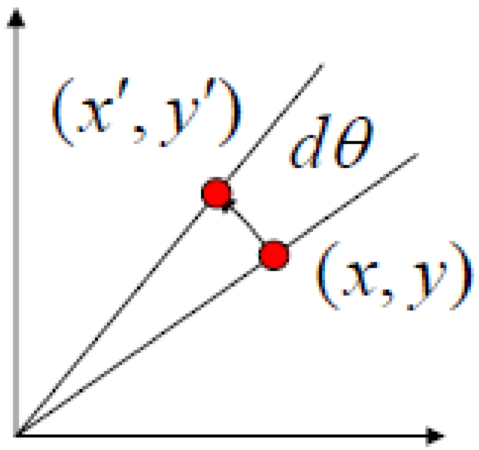
Hamiltonian is unchanged by a shift of a coordinate q → The generator of the ICT is the conjugate momentum p → $[H, p] = 0$

p is conserved

Angular Momentum

- Let's consider a specific case: Angular momentum
  - Pick *x-y-z* system with *z* being the axis of rotation
    - *n* particles' positions given by (x_i, y_i, z_i)
  - Rotate all particles CCW around *z* axis by $d\theta$

$x_i' = x_i - y_i d\theta \qquad y_i' = y_i + x_i d\theta$



  - Momenta are rotated as well

$p_{ix}' = p_{ix} - p_{iy} d\theta \qquad p_{iy}' = p_{iy} + p_{ix} d\theta$

  - Generator is $G = x_i p_{iy} - y_i p_{ix}$

$$d\theta[x_i, G] = d\theta \frac{\partial G}{\partial p_{ix}} = -y_i d\theta \qquad d\theta[p_{ix}, G] = -d\theta \frac{\partial G}{\partial x_i} = -p_{iy} d\theta$$

etc.

Continue…

- The generator $G = x_i p_{iy} - y_i p_{ix}$ is obviously $L_z = (\mathbf{r}_i \times \mathbf{p}_i)_z$
  - i.e. the *z*-component of the total momentum
  - Generator for rotation about an axis given by a unit vector **n** should be

$$G = \mathbf{L} \cdot \mathbf{n}$$

- We now know generators of 3 important ICTs
  - Hamiltonian generates displacement in time
  - Linear momentum generates displacement in space
  - Angular momentum generates rotation in space

Integrating ICT

- I said we can "integrate" ICT to get finite CT
  - How do we integrate $\delta u = \varepsilon[u, G]$?
- First, let's rewrite it as $du = d\alpha[u, G]$ ➡ $\dfrac{du}{d\alpha} = [u, G]$
  - We want the solution $u(\alpha)$ as a function of α, with the initial condition $u(0) = u_0$
  - Taylor expand $u(\alpha)$ from $\alpha = 0$

$$u(\alpha) = u_0 + \alpha \left.\frac{du}{d\alpha}\right|_0 + \frac{\alpha^2}{2!} \left.\frac{d^2 u}{d\alpha^2}\right|_0 + \frac{\alpha^3}{3!} \left.\frac{d^3 u}{d\alpha^3}\right|_0 + \cdots$$

This is $[u, G]_0$

What can I do with these?

Continue…

- Since $\frac{du}{d\alpha} = [u, G]$ is true for any u, we can say $\frac{d}{d\alpha} = [\,, G]$
  - Now apply this operator repeatedly

$$\frac{d^2u}{d\alpha^2} = \frac{d}{d\alpha}[u, G] = [[u, G], G] \Rightarrow \frac{d^ju}{d\alpha^j} = [\cdots[[u, G], G], \cdots, G]$$

  - Going back to the Taylor expansion,

$$u(\alpha) = u_0 + \alpha \left.\frac{du}{d\alpha}\right|_0 + \frac{\alpha^2}{2!} \left.\frac{d^2u}{d\alpha^2}\right|_0 + \frac{\alpha^3}{3!} \left.\frac{d^3u}{d\alpha^3}\right|_0 + \cdots$$

$$= u_0 + \alpha[u, G]_0 + \frac{\alpha^2}{2!}[[u, G], G]_0 + \frac{\alpha^3}{3!}[[[u, G], G], G]_0 + \cdots$$

  - Now we have a formal solution – But does it work?

Rotation CT

- Let's integrate the ICT for rotation around z
  - Let me forget the particle index i $G = xp_y - yp_x$
  - Parameter α is θ in this case
  - Let's see how x changes with θ

$$x(\theta) = x_0 + \theta[x, G]_0 + \frac{\theta^2}{2!}[[x, G], G]_0 + \frac{\theta^3}{3!}[[[x, G], G], G]_0 + \cdots$$

  - Evaluate the Poisson Brackets

$[x, G] = -y$ $[[x, G], G] = -x$ $[[[x, G], G], G] = y$

$[[[[x, G], G], G], G] = x$ Repeats after this

  - Where does this lead us?

Continue…

$$x(\theta) = x_0 + \theta[x,G]_0 + \frac{\theta^2}{2!}[[x,G],G]_0 + \frac{\theta^3}{3!}[[[x,G],G],G]_0 + \cdots$$

$$= x_0 - \theta y_0 - \frac{\theta^2}{2!}x_0 + \frac{\theta^3}{3!}y_0 + \frac{\theta^4}{4!}x_0 - \cdots$$

$$= x_0\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) - y_0\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$= x_0 \cos\theta - y_0 \sin\theta$$

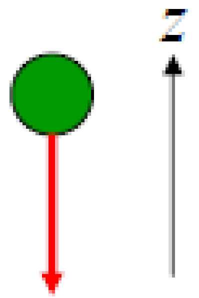
- Similarly

$$y(\theta) = y_0 + \theta[y,G]_0 + \frac{\theta^2}{2!}[[y,G],G]_0 + \frac{\theta^3}{3!}[[[y,G],G],G]_0 + \cdots$$

$$= y_0 \cos\theta + x_0 \sin\theta$$

Free fall

- An object is falling under gravity
  - Hamiltonian is $H = \frac{p^2}{2m} + mgz$
  - Integrate the time ICT



$$z(t) = z_0 + t[z,H]_0 + \frac{t^2}{2!}[[z,H],H]_0 + \frac{t^3}{3!}[[[z,H],H],H]_0 + \cdots$$

$$[z,H] = \frac{p}{m} \qquad [[z,H],H] = -g \qquad [[[z,H],H],H] = 0$$

$$\Rightarrow \quad z(t) = z_0 + \frac{p_0}{m}t - \frac{g}{2}t^2$$

It's easier than it looked

Infinitesimal Rotation

- ICT for rotation is generated by $G = \mathbf{L} \cdot \mathbf{n}$
  - We've studied infinitesimal rotation in Lecture 8
  - Infinitesimal rotation of $d\theta$ about $\mathbf{n}$ moves a vector $\mathbf{r}$ as

$$d\mathbf{r} = \mathbf{n} d\theta \times \mathbf{r}$$

  - Compare the two expressions

$$d\mathbf{r} = d\theta [\mathbf{r}, \mathbf{L} \cdot \mathbf{n}] = \mathbf{n} d\theta \times \mathbf{r} \Rightarrow [\mathbf{r}, \mathbf{L} \cdot \mathbf{n}] = \mathbf{n} \times \mathbf{r}$$

- Equation $[\mathbf{r}, \mathbf{L} \cdot \mathbf{n}] = \mathbf{n} \times \mathbf{r}$ holds for any $\mathbf{r}$ that rotates together with the system
  - Several useful rules can be derived from this

Scalar Product

$$[\mathbf{r}, \mathbf{L}\cdot\mathbf{n}] = \mathbf{n}\times\mathbf{r}$$

- Consider a scalar product $\mathbf{a}\cdot\mathbf{b}$ of two vectors
  - Try to rotate it $$\begin{aligned}[\mathbf{a}\cdot\mathbf{b}, \mathbf{L}\cdot\mathbf{n}] &= \mathbf{a}\cdot[\mathbf{b}, \mathbf{L}\cdot\mathbf{n}] + \mathbf{b}\cdot[\mathbf{a}, \mathbf{L}\cdot\mathbf{n}] \\ &= \mathbf{a}\cdot(\mathbf{n}\times\mathbf{b}) + \mathbf{b}\cdot(\mathbf{n}\times\mathbf{a}) \\ &= \mathbf{a}\cdot(\mathbf{n}\times\mathbf{b}) + \mathbf{a}\cdot(\mathbf{b}\times\mathbf{n}) \\ &= 0\end{aligned}$$
  - Obvious: scalar product doesn't change by rotation
  - Also obvious: length of any vector is conserved

Angular Momentum

Try with $\mathbf{L}$ itself ➡ $[\mathbf{L}, \mathbf{L}\cdot\mathbf{n}] = \mathbf{n}\times\mathbf{L}$

- x-y-z components are

$$[L_x, L_x] = 0 \qquad [L_x, L_y] = L_z \qquad [L_x, L_z] = -L_y$$

$$[L_y, L_x] = -L_z \qquad [L_y, L_y] = 0 \qquad [L_y, L_z] = L_x$$

$$[L_z, L_x] = L_y \qquad [L_z, L_y] = -L_x \qquad [L_z, L_z] = 0$$

$$\Rightarrow [L_i, L_j] = \varepsilon_{ijk} L_k$$

- These relationships are well-known in QM

Continue…

- Imagine two conserved quantities A and B

$$[A,H]=[B,H]=0$$

  - How does $[A,B]$ change with time?

$$\underbrace{[[A,B],H]=-[[B,H],A]-[[H,A],B]}_{\text{Jacobi's identity}}=0$$

  - Poisson bracket of two conserved quantities is conserved

- Now consider $[L_i,L_j]=\varepsilon_{ijk}L_k$
  - If 2 components of **L** are conserved, the 3rd component must → Total vector **L** is conserved

Continue…

- Remember the Fundamental Poisson Brackets?

$$[q_i, q_j] = [p_i, p_j] = 0 \qquad [q_i, p_j] = -[p_i, q_j] = \delta_{ij}$$

PB of two canonical momenta is 0

  - Now we know $[L_i, L_j] = \varepsilon_{ijk} L_k$
  - Poisson brackets between L_x, L_y, L_z are non-zero

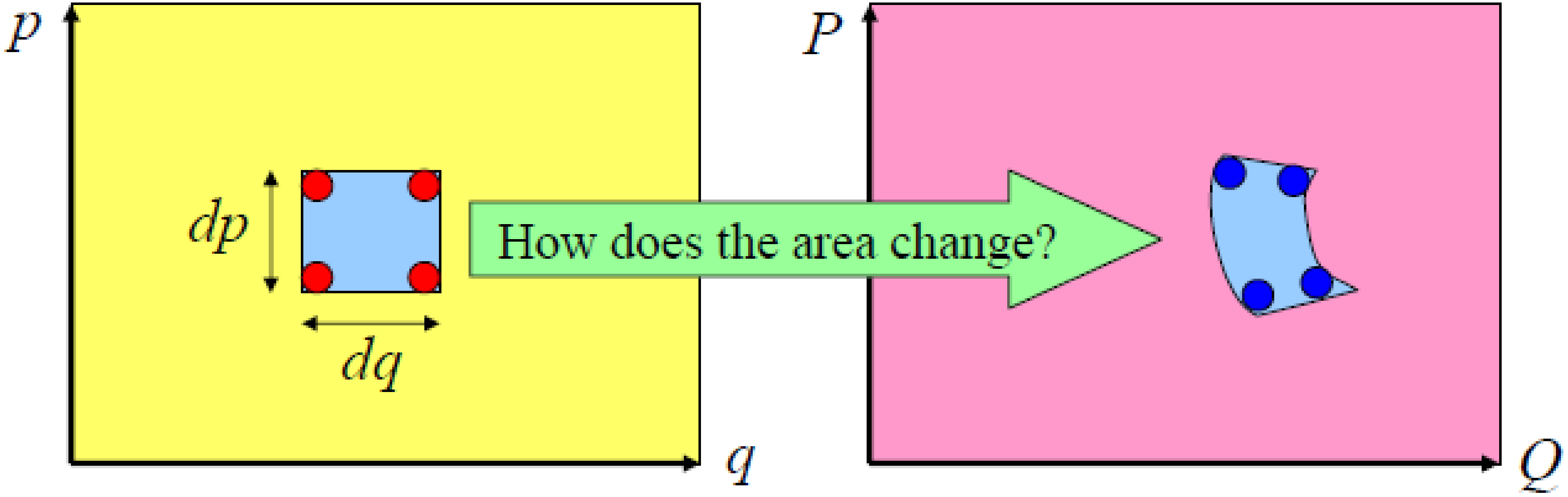
Only 1 of the 3 components of the angular momentum can be a canonical momentum

  - On the other hand, $[L^2, L_i] = 0$, so $|L|$ may be a canonical momentum
  - QM: You may measure $|L|$ and, e.g., L_z simultaneously, but not L_x and L_y, etc.

Phase Volume

- Static view: CT moves a point in one phase space to a point in another phase space
- Dynamic view: CT moves a point in one phase space to another point in the same space

- If you consider a set of points, CT moves a volume to anther volume, e.g.



Continue…

- Easy to calculate the Jacobian for 1-dimension

$$dQdP = |\mathbf{M}|\, dqdp \quad \text{where} \quad \mathbf{M} = \begin{bmatrix} \partial Q/\partial q & \partial Q/\partial p \\ \partial P/\partial q & \partial P/\partial p \end{bmatrix}$$

$$|\mathbf{M}| = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} = [Q, P] = 1 \quad \Rightarrow \quad dQdP = dqdp$$

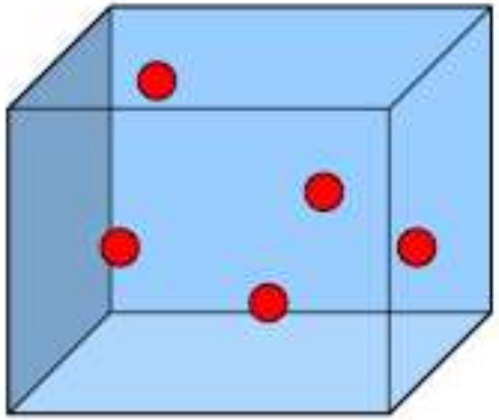
  - i.e., volume in 1-dim. phase space is invariant

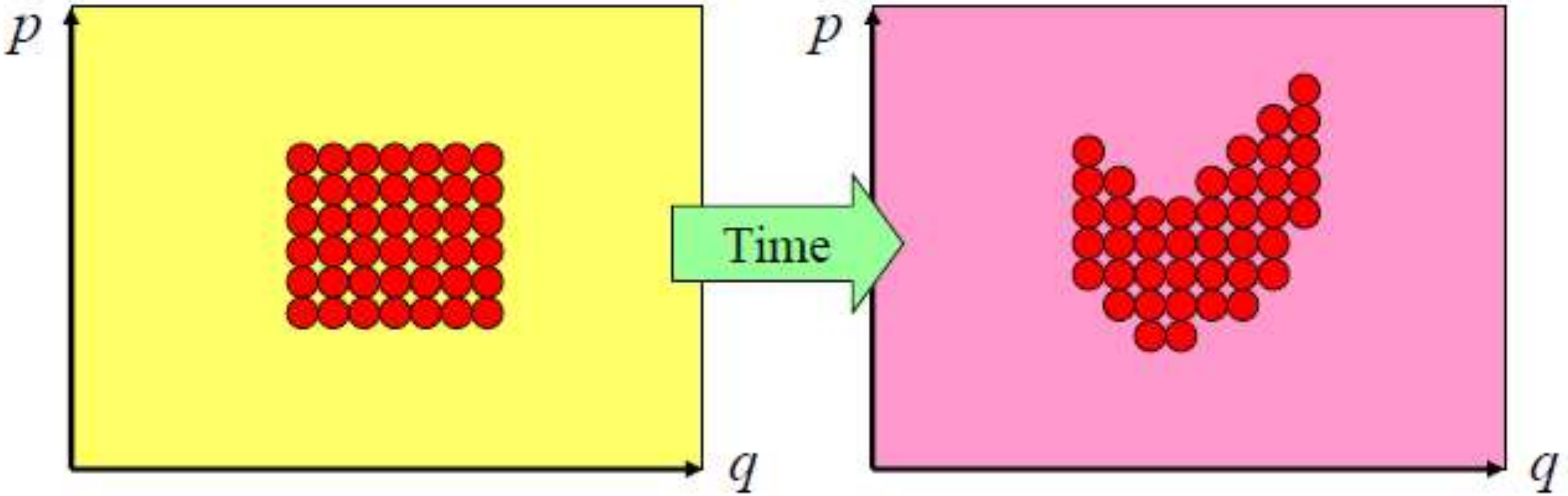
- This is true for *n*-dimensions
  - Goldstein proves it using *simplectic* approach

Volume in Phase Space is a Canonical Invariant

Dynamic View

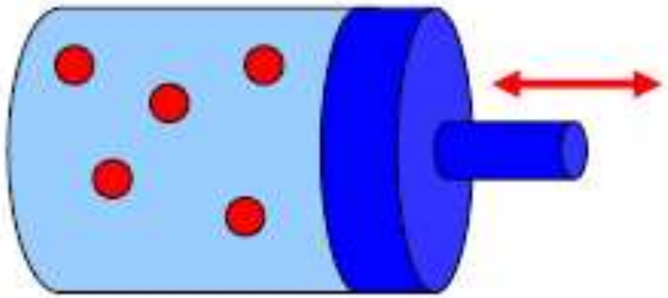
- Consider many particles moving independently
  - e.g., ideal gas molecules in a box
  - They obey the same EoM independently
  - Can be represented by multiple points in one phase space
  - They move with time → CT



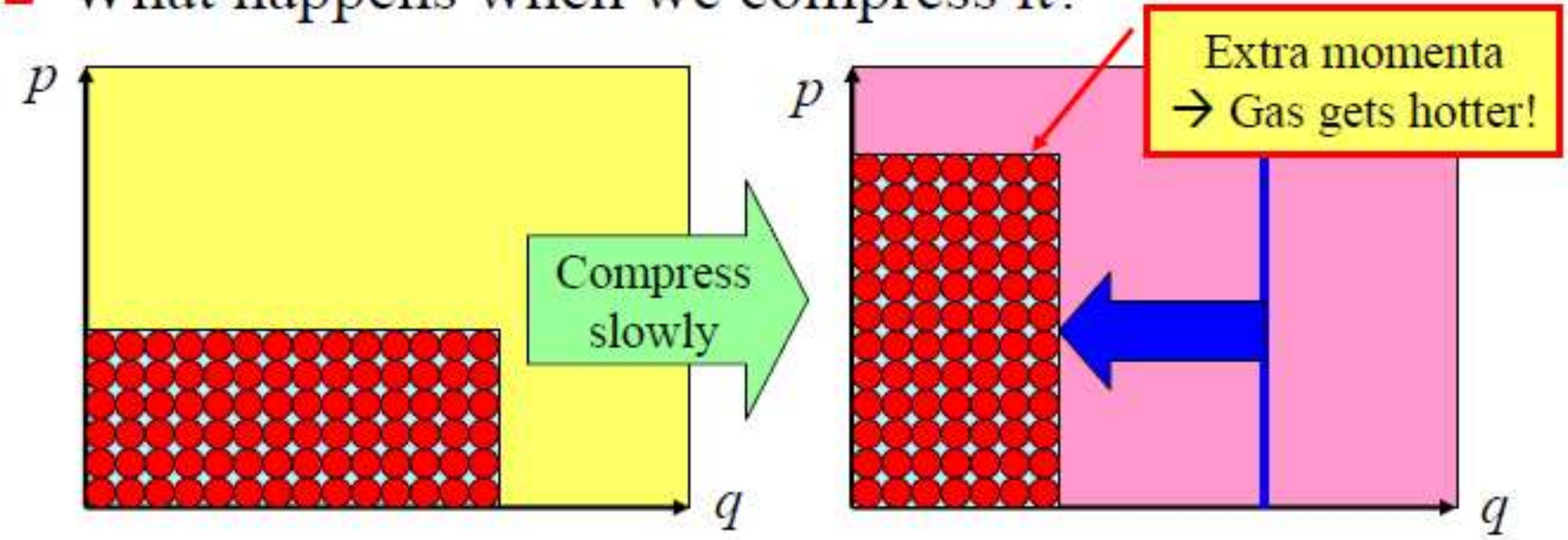


Ideal Gas Dynamics

- Imagine ideal gas in a cylinder with movable piston
  - Each molecule has its own position and momentum → They fill up a certain volume in the phase space



- What happens when we compress it?



Liouville's Theorem

- The phase volume occupied by a group of particles (*ensemble* in stat. mech.) is conserved
  - Thus the density in phase space remains constant with time
  - Known as Liouville's theorem
  - Theoretical basis of the 2nd law of thermodynamics
- This holds true when there are large enough number of particles so that the distribution may be considered continuous

Continue…

Thank You