Classical Mechanics
(MP-301)

Physics:
The most basic of all sciences!

- Physics: The "king" of all sciences!
- Physics = The study of structure of matter and energy and their interactions.
- Its Applications

Physics: General Discussion

- *The Goal of Physics* (& all of science): To quantitatively and qualitatively **describe the "world around us".**
- Physics *IS NOT* merely a collection of facts & formulas!
- Physics *IS* a creative activity!
- Physics $\longrightarrow$ Observation $\longrightarrow$ Explanation.
- Requires *IMAGINATION!!*

Physics & Its Relation to Other Fields

- *The "Parent" of all Sciences!*
- The foundation for and is connected to *ALL* branches of *science and engineering*.
- Also useful in everyday life and in *MANY* professions
  - Chemistry
  - Life Sciences (Medicine also!!)
  - Architecture
  - Engineering
  - Various technological fields

The Nature of Science

- Physics is an *EXPERIMENTAL* science!
- Experiments & Observations:
  - Important first steps toward scientific theory.
  - It requires imagination to tell what is important
- Theories:
  - Created to *explain* experiments & observations. Will also make predictions
- Experiments & Observations:
  - Will tell if predictions are accurate.
- *No theory* can be absolutely verified
  - But a theory *CAN be proven false!!!*

Theory

- *Quantitative* (mathematical) *description* of experimental observations.
- Not just ***WHAT*** is observed but ***WHY*** it is observed as it is and ***HOW*** it works the way it does.
- **Tests of theories:**
  - Experimental observations:
    More experiments, more observation.
  - Predictions:
    Made before observations & experiments.

Model, Theory, Law

- **Model:** An analogy of a physical phenomenon to something we are familiar with.
- **Theory:** More detailed than a model. Puts the model into mathematical language.
- **Law:** Concise & **general** statement about **how nature behaves.** Must be verified by many, many experiments! Only a few laws.
  - Not comparable to laws of government!

- How does a new theory get accepted?
- Predictions agree better with data than old theory
- Explains a greater range of phenomena than old theory
- Example:
  - Aristotle believed that objects would return to a state of rest once put in motion.
  - Galileo realized that an object put in motion would stay in motion until some force stopped it.

The Structure of Physics

SUMMARY: THE STRUCTURE OF PHYSICS

	Low Speed $v << c$	High Speed $v < \sim c$
Large size >> atomic size	Classical Mechanics (Newton, Hamilton, Lagrange)	Special Relativity (Einstein)
Small size < ~ atomic size	Quantum Mechanics (Schrodinger, Heisenberg)	Relativistic Quantum Mechanics (Dirac)

Atomic Physics

Molecular Physics

Solid State Physics

Nuclear & Particle Physics

Quantum Field Theory (Feynman, Schwinger)

Quantum Electrodynamics (Photons, Weak Nuclear Force)

Quantum Chromodynamics (Gluons, Quarks, Leptons Strong Nuclear Force)

Mechanics

- The science of *HOW* objects move (behave) under *given forces*.
- (Usually) Does not deal with the *sources* of forces. Answers the question: “Given the forces, how do objects move”?

Mechanics: "Classical" Mechanics

"Classical" Physics:

"Classical" ≡ ≈ Before the 20th Century

The foundation of pure & applied macroscopic physics & engineering!

- *Newton's Laws* + Boltzmann's Statistical Mechanics (& Thermodynamics):
  ≈ Describe most of macroscopic world!
- However, at high speeds ($v \sim c$) we need

Special Relativity: (Early 20th Century: 1905)

- Also, for small sizes (atomic & smaller) we need

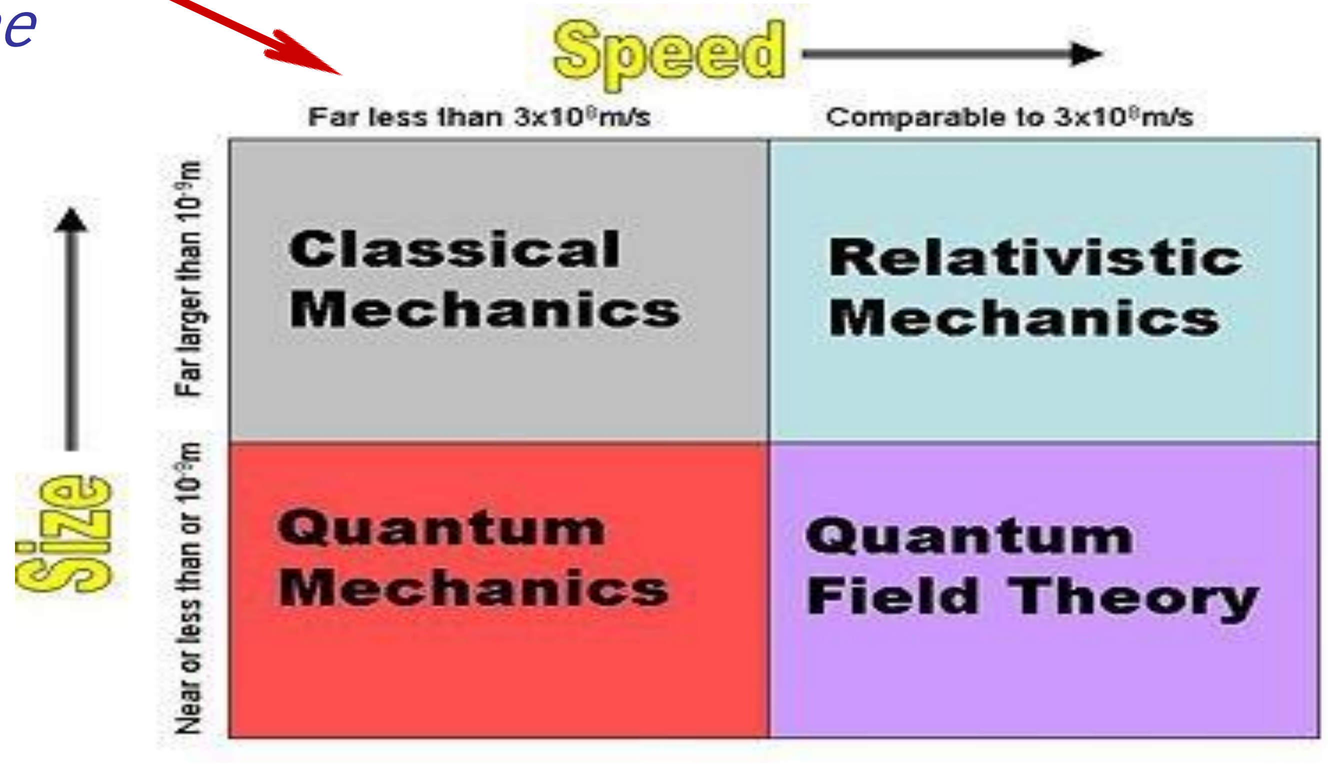
Quantum Mechanics: (1900 through ~ 1930)

"Classical" Mechanics: (17th & 18th Centuries) Still useful today!

"Classical" Mechanics

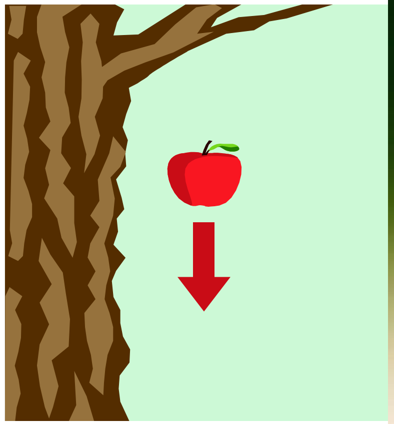
The mechanics in this course is *limited to macroscopic objects* moving at *speeds* v *much, much smaller than the speed of light* $c = 3 \times 10^8$ m/s. As long as $v \ll c$, our discussion will be valid.

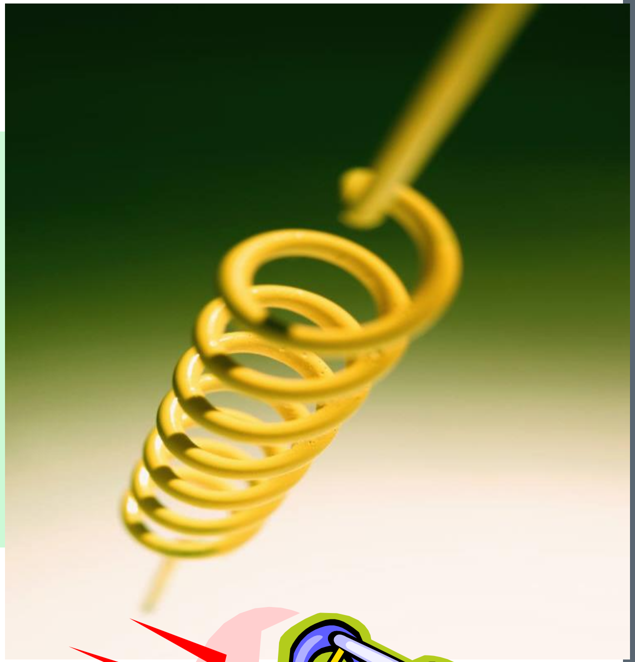
So, we will work exclusively in the gray region in the figure.

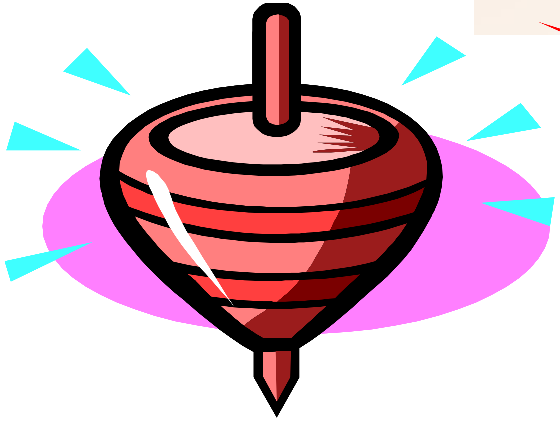


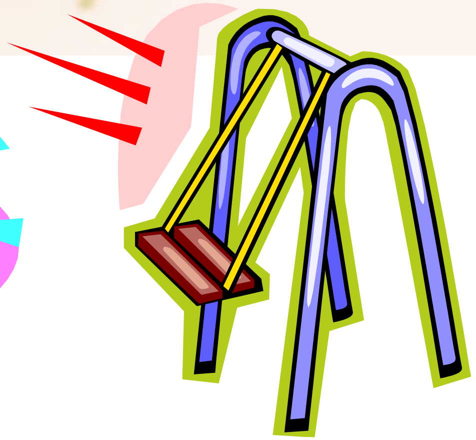
Classical Mechanics

- Kinematics – how objects move
  - Translational motion
  - Rotational motion
  - Vibrational motion
- Dynamics – Forces and why objects move as they do
  - Statics – special case - forces cause no motion









Classical Mechanics

- First began with Galileo (1584-1642), whose experiments with falling bodies (and bodies rolling on an incline) led to Newton's 1st Law.
- Newton (1642-1727) then developed his 3 laws of motion, together with his universal law of gravitation.
- Two additional, highly mathematical frameworks were developed by the French mathematician Lagrange (1736-1813) and the Irish mathematician Hamilton (1805-1865).
- Together, these three alternative frameworks by Newton, Lagrange, and Hamilton make up what is generally called Classical Mechanics.
- They are distinct from the other great forms of non-classical mechanics, Relativistic Mechanics and Quantum Mechanics, but both of these borrow heavily from Classical Mechanics.

This Course

- **The first part of this course will be a review in terms of the basic ideas of mechanics, which you should have already thoroughly learned in your introductory physics courses.**
- PRINCIPLE OF VIRTUAL WORK, Constraints, Generalised Coordinates, Velocities And Momenta, D' Alembert's Principle
- LAGRANGE'S FORMULATION, Variational technique for many independent, variables, Euler Lgrangian differential equation, Application of Lagrange's equation of motion.
- HAMILTON'S EQUATIONS; Hamilton's canonical equation of, motion, Physical significance of H, Advantage of Hamilton approach,
- POISSON BRACKETS, Invariance of Poisson bracket with respect to canonical transformation, Equation of motion in Poisson bracket form.
- TWO-BODY CENTRAL FORCE PROBLEM, Reduced mass, Planet orbits, Kepler Laws, Virial theorem.
- COLLISIONS AND SCATTERING, Centre of mass and lab frames, Scattering cross section
- RIGID BODY DYNAMICS, Euler equations, Euler angles, The inertia tenser, Motion of the symmetric top.
- MOTION IN NON-INERTIAL FRAMES, Coriolis Force
- HARMONIC OSCILLATOR; NONLINEAR OSCILLATOR; INTRODUCTION TO CHAOS

Space and Time

- We live in a three-dimensional world, and for the purpose of this course we can consider space and time to be a fixed framework against which we can make measurements of moving bodies.

- Each point P in space can be labeled with a distance and direction from some arbitrarily chosen origin O. Expressed in terms of unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$

$$\mathbf{r} = x\ \hat{\mathbf{x}} + y\ \hat{\mathbf{y}} + z\ \hat{\mathbf{z}}$$

- It is equivalent to write the vector as an ordered triplet of values $\mathbf{r} = (x,\ y,\ z)$

- We can also write components of vectors using subscripts

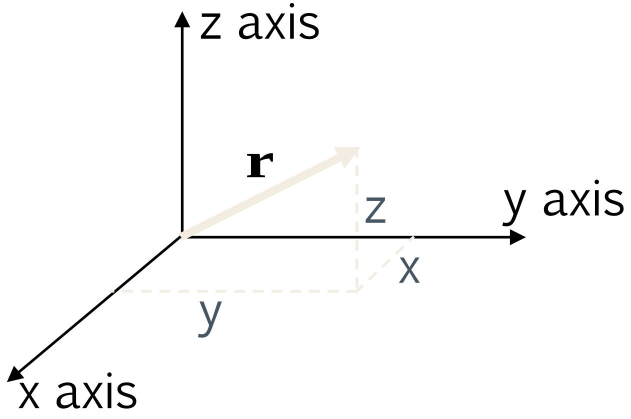
$$\mathbf{v} = (v_x,\ v_y,\ v_z) \qquad \mathbf{a} = (a_x,\ a_y,\ a_z)$$

Ways of writing vector notation

$$\mathbf{F} = m\mathbf{a}$$

$$\vec{F} = m\vec{a}$$

$$\underline{F} = m\underline{a}$$



Other Vector Notations

- You will be used to unit vector notation **i**, **j**, **k**, but we will follow the text and use the $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ notation.
- At times, it is more convenient to use notation that makes it easier to use summation notation, so we introduce the equivalents:

$$r_1 = x, \quad r_2 = y, \quad r_3 = z$$

$$\mathbf{e}_1 = \hat{\mathbf{x}}, \quad \mathbf{e}_2 = \hat{\mathbf{y}}, \quad \mathbf{e}_3 = \hat{\mathbf{z}}$$

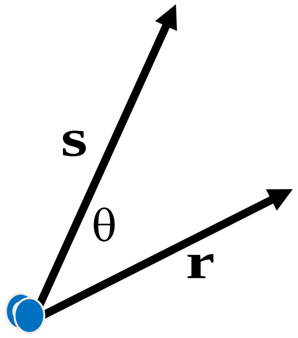
  which allows us to write

$$\mathbf{r} = r_1\mathbf{e}_1 + r_2\mathbf{e}_2 + r_3\mathbf{e}_3 = \sum_{i=1}^{3} r_i\mathbf{e}_i$$

- In the above example, this form has no real advantage, but in other cases we will meet, this form is much simpler to use. The point is that we may choose any convenient notation, and you should become tolerant of different, but consistent forms of notation.

Vector Operations

- Sum of vectors $\mathbf{r} = (r_1, r_2, r_3);\ \mathbf{s} = (s_1, s_2, s_3)\quad \mathbf{r} + \mathbf{s} = (r_1 + s_1, r_2 + s_2, r_3 + s_3)$
- Vector times scalar $c\mathbf{r} = (cr_1, cr_2, cr_3)$
- Scalar product, or dot product



$$\mathbf{r} \cdot \mathbf{s} = rs\cos\theta$$

$$= r_1 s_1 + r_2 s_2 + r_3 s_3 = \sum_{n=1}^{3} r_n s_n$$

$$\mathbf{p} = \mathbf{r} \times \mathbf{s}; \qquad |\mathbf{r} \times \mathbf{s}| = rs\sin\theta$$

- Vector product, or cross product

$$p_x = r_y s_z - r_z s_y$$

$$p_y = r_z s_x - r_x s_z$$

$$p_z = r_x s_y - r_y s_x$$

$$\mathbf{r} \times \mathbf{s} = \det\begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ r_x & r_y & r_z \\ s_x & s_y & s_z \end{bmatrix} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ r_x & r_y & r_z \\ s_x & s_y & s_z \end{vmatrix}$$

Differentiation of Vectors

- This course makes heavy use of Calculus (and differential equations, and other forms of advanced mathematics). In general, we will refresh your memory about the techniques you will need as they come up, but we will do so from a Physics perspective—only paying lip-service to the underlying mathematical proofs.
- What we need now is a simple form of something called Vector Calculus. As long as you remember that vectors are just triplets of numbers, and vector equations can be thought of as three separate equations, you will be fine.
- For now, consider only the derivative of the position vector $\mathbf{r}(t)$, which you should know gives the velocity $\mathbf{v}(t) = d\mathbf{r}(t)/dt$. Likewise, the derivative of the velocity (the second derivative of the position) gives the acceleration: $\mathbf{a}(t) = d\mathbf{v}(t)/dt = d^2\mathbf{r}(t)/dt^2$. Formally:

for scalars: $$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \qquad \text{where} \qquad \Delta x = x(t+\Delta t) - x(t)$$

for vectors: $$\frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} \qquad \text{where} \qquad \Delta \mathbf{r} = \mathbf{r}(t+\Delta t) - \mathbf{r}(t)$$

Differentiation of Vectors

- By the usual rules of differentiation, the derivative of a sum of vectors is

$$\frac{d}{dt}(\mathbf{r}+\mathbf{s}) = \frac{d\mathbf{r}}{dt} + \frac{d\mathbf{s}}{dt}$$

- derivative of a scalar times a vector follows the usual product rule

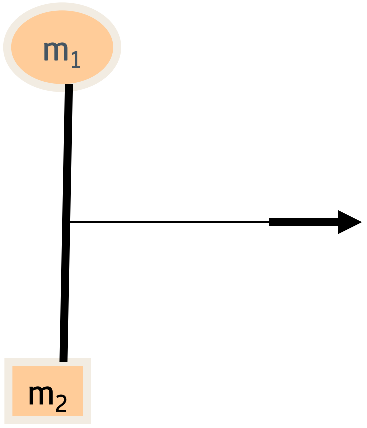
$$\frac{d}{dt}(f\mathbf{r}) = f\,\frac{d\mathbf{r}}{dt} + \frac{df}{dt}\mathbf{r}$$

- Also note: $\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$ so $\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\hat{\mathbf{x}} + \frac{dy}{dt}\hat{\mathbf{y}} + \frac{dz}{dt}\hat{\mathbf{z}}$ $\quad \mathbf{v} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}$

  implies that the unit vectors are constant (i.e. $\frac{d\hat{\mathbf{x}}}{dt} = \frac{d\hat{\mathbf{y}}}{dt} = \frac{d\hat{\mathbf{z}}}{dt} = 0$).

Mass and Force

- **What is the difference between mass and weight?**
- Mass has to do with inertial force (*ma*).
- Weight has to do with gravitational force (*mg*).
- In the first case, the mass is "resistance to changes in motion" while in the second case it is a rather mysterious "attractive property" of matter
- **Inertial balance:**



Allows measurement of inertial mass without mixing in gravitational force.

Point Mass (Particle)

- For now, we want to focus on the concept of a point mass, or particle. This is an approximation, which is worthwhile to look at carefully. It basically refers to a body that can move through space but has NO internal degrees of freedom (rotation, flexure, vibrations).

- Later we will talk about bodies as collections of particles, or a continuous distribution of mass, and in considering such bodies the laws of motion are considerably more complicated.

- Despite this being an approximation, the approximation is still useful in many cases, such as for elementary particles (protons, neutrons, electrons), or even planets and stars (sometimes).

Newton's Three Laws

› Law of Inertia

- In the absence of forces, a particle moves with constant velocity $\mathbf{v}$.
- (An object in motion tends to remain in motion, an object at rest tends to remain at rest.)

› Force Law

- For any particle of mass m, the net force $\mathbf{F}$ on the particle is always equal to the mass m times the particle's acceleration: $\mathbf{F} = m\mathbf{a}$.

› Conservation of Momentum Law

- If particle 1 exerts a force $\mathbf{F}_{21}$ on particle 2, then particle 2 always exerts a reaction force $\mathbf{F}_{12}$ on particle 1 given by $\mathbf{F}_{12} = -\mathbf{F}_{21}$.
- (For every action there is an equal and opposite reaction.)

Aside: Dot Notation

- Dot Notation:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}} = \frac{d^2\mathbf{r}}{dt^2} = \ddot{\mathbf{r}}$$

- We will be using this dot notation extensively. It means differentiation with respect to time, t (only!).
- You may have seen "prime" notation, but if the differentiation is not with respect to time, it is NOT equivalent to y-dot.

$$y' = \frac{dy}{dx} \neq \dot{y}$$

- y-dot means dy/dt only.

Equivalence of First Two Laws

- The Law of Inertia and the Force Law can be stated in equivalent ways.
- Obviously, if $\mathbf{F} = m\mathbf{a}$, then in the absence of forces $\mathbf{F} = m\mathbf{a} = 0$

$$\frac{d\mathbf{v}}{dt} = 0 \Rightarrow \mathbf{v} = \mathbf{v}_o$$

- Thus, the velocity is constant (objects in motion tend to remain in motion) and could be zero (objects at rest tend to remain at rest).
- The second law can be rewritten in terms of momentum:

$$\mathbf{p} = m\mathbf{v}$$

- In Classical Mechanics, the mass of a particle is constant, hence

$$\dot{\mathbf{p}} = m\dot{\mathbf{v}} = m\mathbf{a}$$

- So we can write $\mathbf{F} = \dot{\mathbf{p}}$.
- In words, forces cause a change in momentum, and conversely any change in momentum implies that a force is acting on the particle.

The Equation of Motion

- Newton's Second Law is the basis for much of Classical Mechanics, and the equation $\mathbf{F} = m\mathbf{a}$ has another name—the equation of motion.
- The typical use of the equation of motion is to write

$$m\mathbf{a} = \sum \text{Forces}$$

  where the right hand side lists all of the forces acting on the particle.
- In this text, an even more usual way to write it is:

$$m\ddot{\mathbf{r}} = \sum \text{Forces}$$

  which is perhaps an easier way to understand why it is called the equation of motion. This relates the position of the particle vs. time to the forces acting on it, and obviously if we know the position at all times we have an equation of motion for the particle.

Differential Equations

- Most of you should have had a course in differential equations by now, or should be taking the course concurrently.
- A differential equation is an equation involving derivatives, in this case derivatives of the particle position $\mathbf{r}(t)$.
- Consider the one-dimensional equation for the position $x(t)$ of a particle under a constant force:

$$\ddot{x}(t) = \frac{F_0}{m}$$

- This equation involves the second derivative (with respect to time) of the position, so to get the position we simply integrate twice:

$$\dot{x}(t) = \int \ddot{x}(t)dt = v_0 + \frac{F_0}{m}t$$

$$x(t) = \int \dot{x}(t)dt = x_0 + v_0 t + \frac{F_0}{2m}t^2$$

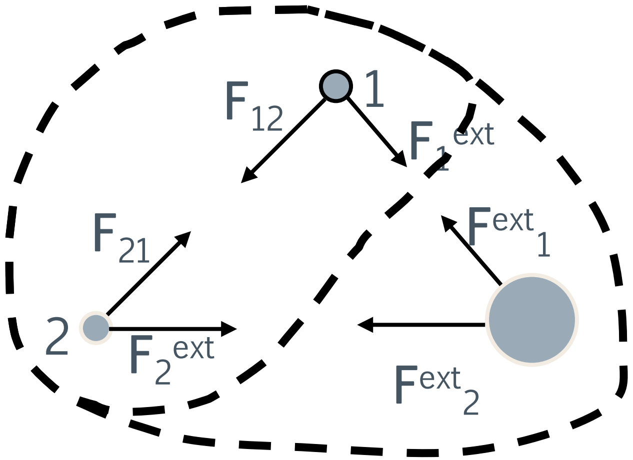
- This was so easy we did not actually need to know the theory of differential equations, but we will meet with lots of more complicated equations where the DiffEQ theory is needed, and will be introduced as needed.

Third Law and Conservation of Momentum

- Newton's first two laws refer to forces acting on a single particle. The Third Law, by contrast, explicitly refers to two particles interacting—the particle being accelerated, and the particle doing the forcing.
- Introduce notation $\mathbf{F}_{21}$ (F-on-by) to represent the force on particle 2 by particle 1. Then

Newton's Third Law

If particle 1 exerts a force F_{21} on particle 2, then particle 2 always exerts a reaction force F_{12} on particle 1 given by $F_{12} = - F_{21}$.



$$\dot{\mathbf{p}}_1 = \mathbf{F}_1 = \mathbf{F}_{12}; \quad \dot{\mathbf{p}}_2 = \mathbf{F}_2 = \mathbf{F}_{21};$$
$$\dot{\mathbf{P}} = \mathbf{F}_1 + \mathbf{F}_2 = \mathbf{F}_{12} + \mathbf{F}_{21} = 0$$

$$\dot{\mathbf{p}}_1 = \mathbf{F}_1 = \mathbf{F}_1^{\text{ext}} + \mathbf{F}_{12}; \quad \dot{\mathbf{p}}_2 = \mathbf{F}_2 = \mathbf{F}_2^{\text{ext}} + \mathbf{F}_{21};$$
$$\dot{\mathbf{P}} = \mathbf{F}_1 + \mathbf{F}_2 = \mathbf{F}_1^{\text{ext}} + \mathbf{F}_{12} + \mathbf{F}_2^{\text{ext}} + \mathbf{F}_{21} = \mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}} = \mathbf{F}^{\text{ext}}$$

$$\dot{\mathbf{p}}_1 = \mathbf{F}_1 = \mathbf{F}_1^{\text{ext}} + \mathbf{F}_{12}; \quad \dot{\mathbf{p}}_2 = \mathbf{F}_2 = \mathbf{F}_2^{\text{ext}} + \mathbf{F}_{21};$$
$$\dot{\mathbf{p}}_{ext} = \mathbf{F}_{ext} = \mathbf{F}^{\text{ext}}{}_1 + \mathbf{F}^{\text{ext}}{}_2; \quad \dot{\mathbf{P}} = 0$$

Multi-Particle Systems

- It should be fairly obvious how to extend this to systems of N particles, where N can be any number, including truly huge numbers like 10^{23}.
- Let α or β designate one of the particles. Both α and β can take any value $1, 2, \ldots, N$. The net force on particle α is then

$$\mathbf{F}_\alpha = \sum_{\beta \neq \alpha} \mathbf{F}_{\alpha\beta} + \mathbf{F}_\alpha^{\text{ext}} = \dot{\mathbf{p}}_\alpha$$

where the sum runs over all particles except α itself (a particle does not exert a force on itself).

- The total force on the system of particles is just the sum of all of the $\dot{\mathbf{p}}_\alpha$

$$\dot{\mathbf{P}} = \sum_\alpha \dot{\mathbf{p}}_\alpha$$

$$\dot{\mathbf{P}} = \sum_\alpha \sum_{\beta \neq \alpha} \mathbf{F}_{\alpha\beta} + \sum_\alpha \mathbf{F}_\alpha^{\text{ext}} = \sum_\alpha \mathbf{F}_\alpha^{\text{ext}} = \mathbf{F}^{\text{ext}}$$

- Each term $\mathbf{F}_{\alpha\beta}$ can be paired with $\mathbf{F}_{\beta\alpha}$:

$$\sum_\alpha \sum_{\beta \neq \alpha} \mathbf{F}_{\alpha\beta} = \sum_\alpha \sum_{\beta > \alpha} \left(\mathbf{F}_{\alpha\beta} + \mathbf{F}_{\beta\alpha} \right) = 0$$

Law of Conservation of Linear Momentum

When the total external force on a system is zero, the total momentum of the system remains constant

Law of Conservation of Momentum (collisions):

The ***total (vector) momentum before a collision =***
the ***total (vector) momentum after a collision.***

$$\Rightarrow \qquad p_{total} = p_A + p_B = (p_A)' + (p_B)' = \text{constant}$$

Or: $$\Delta p_{total} = \Delta p_A + \Delta p_B = 0$$

$p_A = m_A v_A$, $p_B = m_B v_B$, Initial momenta

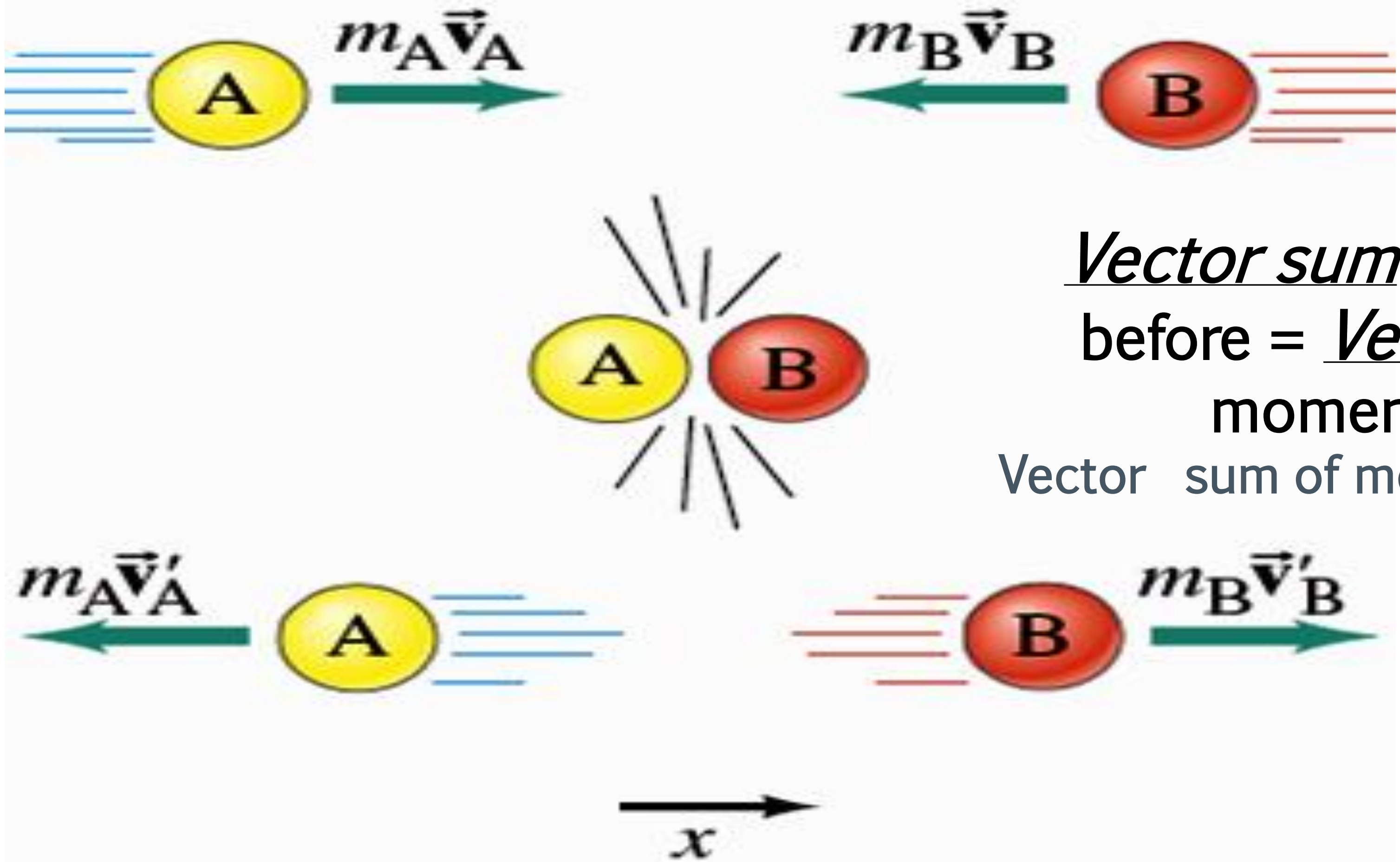
$(p_A)' = m_A (v_A)'$, $(p_B)' = m_B (v_B)'$, Final momenta

$$\Rightarrow \quad m_A v_A + m_B v_B = m_A (v_A)' + m_B (v_B)'$$

Example: 2 billiard balls collide (zero external force)

$$m_A \vec{\mathbf{v}}_A + m_B \vec{\mathbf{v}}_B = m_A \vec{\mathbf{v}}'_A + m_B \vec{\mathbf{v}}'_B$$

The vector sum of the momenta is a constant!

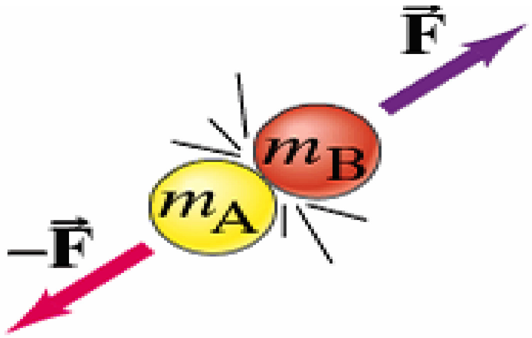


Vector sum of momenta before = *Vector sum* of momenta after!

Vector sum of momenta = constant!

Another brief *Proof,* using Newton's 2^{nd} & 3^{rd} Laws

Two masses, m_A & m_B in collision:
Internal forces: $F_{AB} = - F_{BA}$ by Newton's 3^{rd} Law



Newton's 2^{nd} Law:

The force on A, due to B, for a small time Δt:

$$F_{AB} = \Delta p_A/\Delta t = m_A[(v_A)' - v_A]/\Delta t$$

The force on B, due to A, for the same small Δt:

$$F_{BA} = \Delta p_B/\Delta t = m_B[(v_B)' - v_B]/\Delta t$$

Newton's 3^{rd} Law: $F_{AB} = - F_{BA} = F$

$$\Rightarrow \quad m_A[(v_A)' - v_A]/\Delta t = - m_B[(v_B)' - v_B]/\Delta t$$

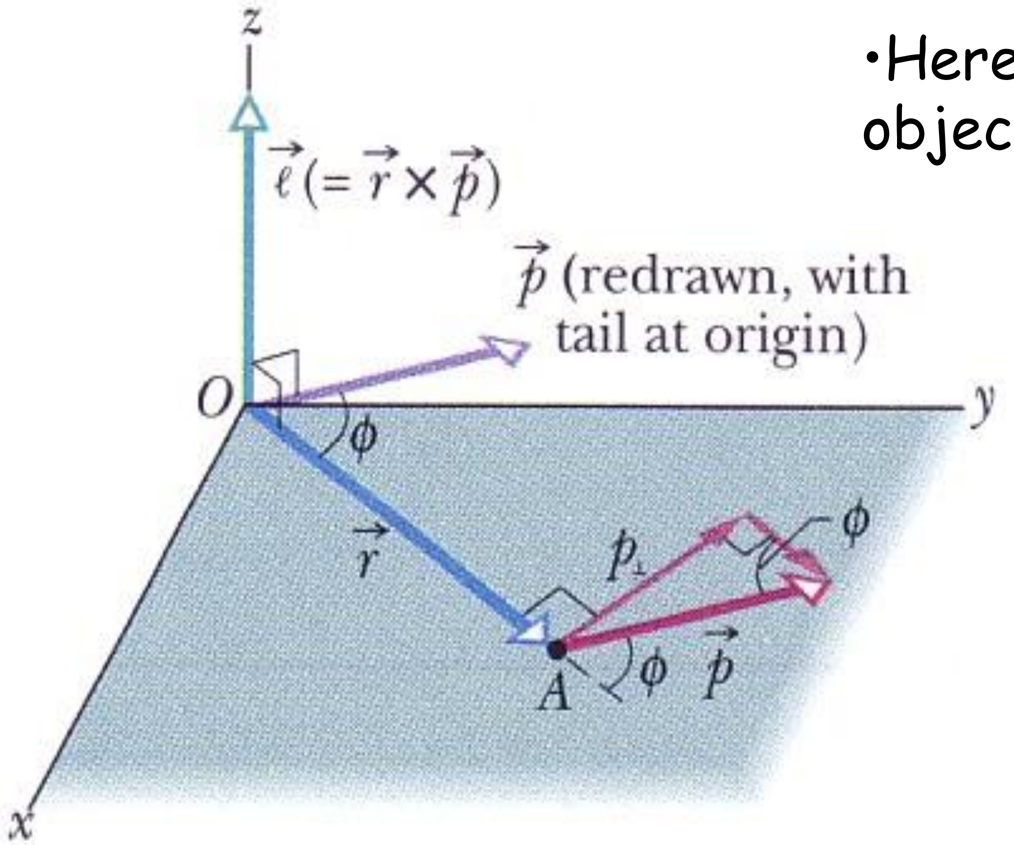
or: $m_A v_A + m_B v_B = m_A(v_A)' + m_B(v_B)' \Rightarrow$ *Proven!*

So, **for Collisions:** $m_A v_A + m_B v_B = m_A(v_A)' + m_B(v_B)'$

Torque and angular momentum

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad \text{(definition)}$$

Angular momentum $\vec{l}$ is defined as: $\vec{l} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$



- Here, p is the linear momentum mv of the object.

$$l = mvr\sin\phi$$

$$= rp_\perp = rmv_\perp$$

$$= r_\perp p = r_\perp mv$$

Newton's second law in angular form

No surprise:

$$\vec{F}_{net} = \frac{d\vec{p}}{dt}$$

Linear form

$$\vec{\tau}_{net} = \frac{d\vec{l}}{dt}$$

angular form

The vector sum of all the torques acting on a particle is equal to the time rate of change of the angular momentum.

For a system of many particles, the total angular momentum is:

$$\vec{L} = \vec{l}_1 + \vec{l}_2 + \vec{l}_3 + \cdots\cdots + \vec{l}_n = \sum_{i=1}^{n} \vec{l}_i$$

$$\frac{d\vec{L}}{dt} = \sum_{i=1}^{n} \frac{d\vec{l}_i}{dt} = \sum_{i=1}^{n} \vec{\tau}_{net,i} = \vec{\tau}_{net}$$

The net external torque acting on a system of particles is equal to the time rate of change of the system's total angular momentum.

Conservation of angular momentum

It follows from Newton's second law that:

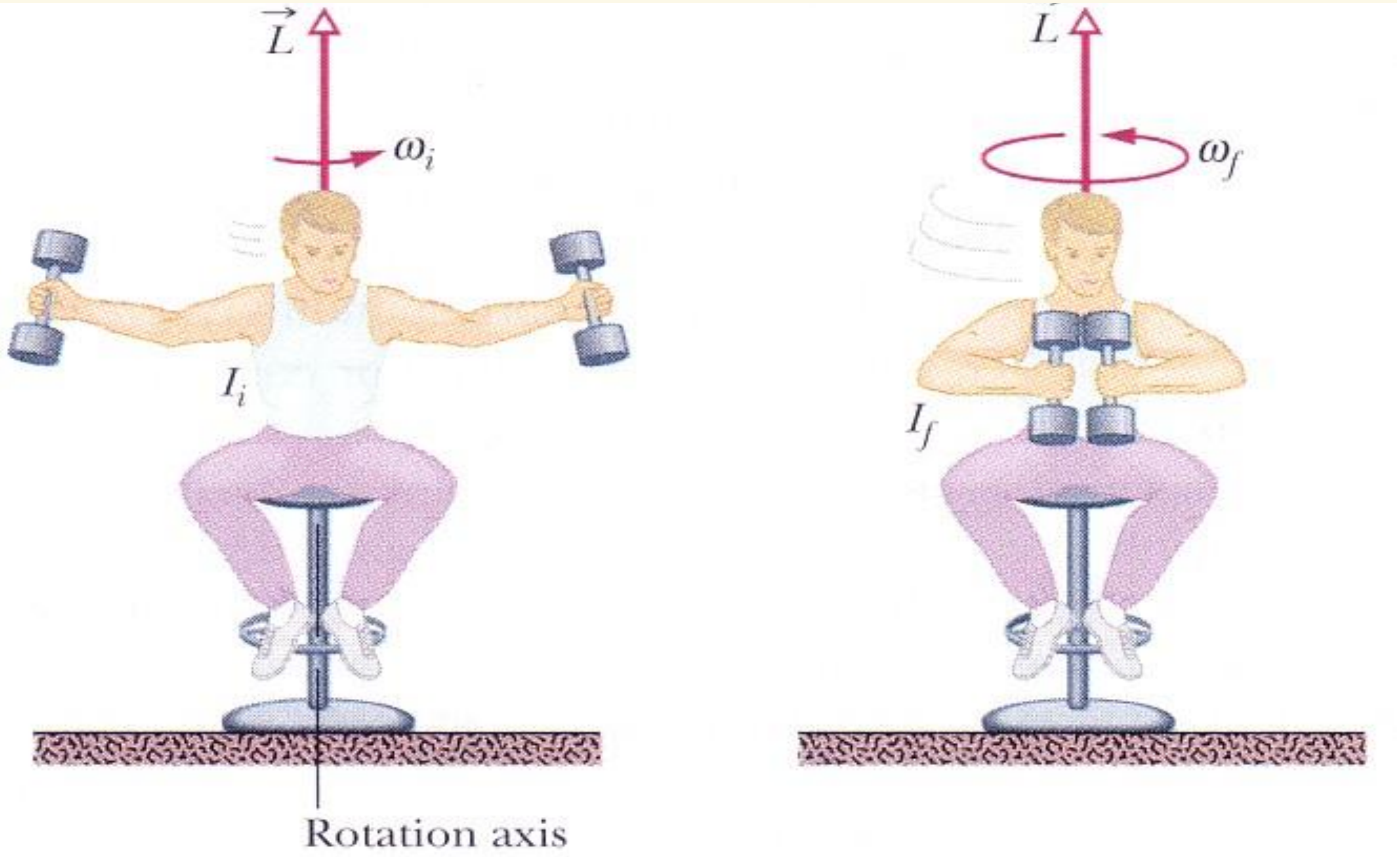
If the net external torque acting on a system is zero, the angular momentum of the system remains constant, no matter what changes take place within the system.

$$\vec{L} = \text{a constant}$$

$$\vec{L}_i = \vec{L}_f$$

$$I_i \omega_i = I_f \omega_f$$

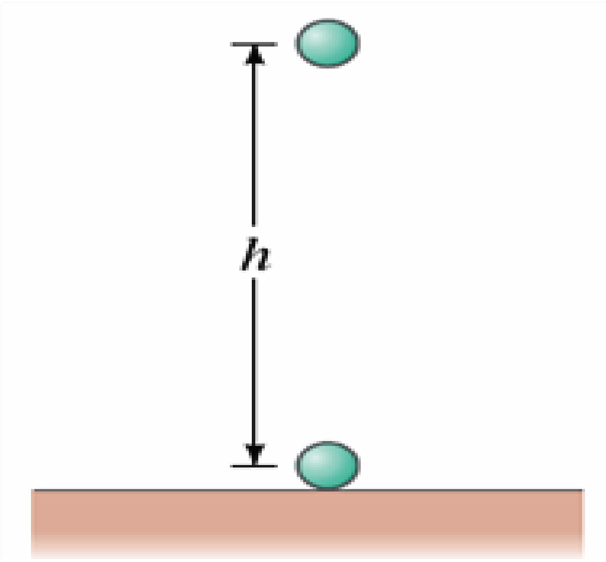
$$\frac{\omega_f}{\omega_i} = \frac{I_i}{I_f}$$

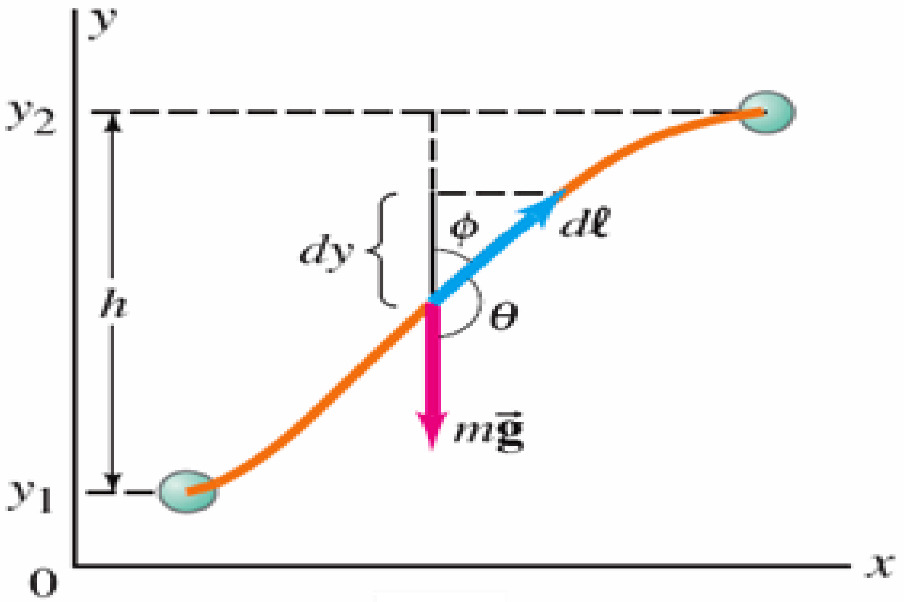


Conservative & Non-conservative Forces

Definition: A force is **conservative** if & only if *the work done by that force on an object moving from one point to another depends ONLY on the initial & final positions of the object, & is independent of the particular path taken.*

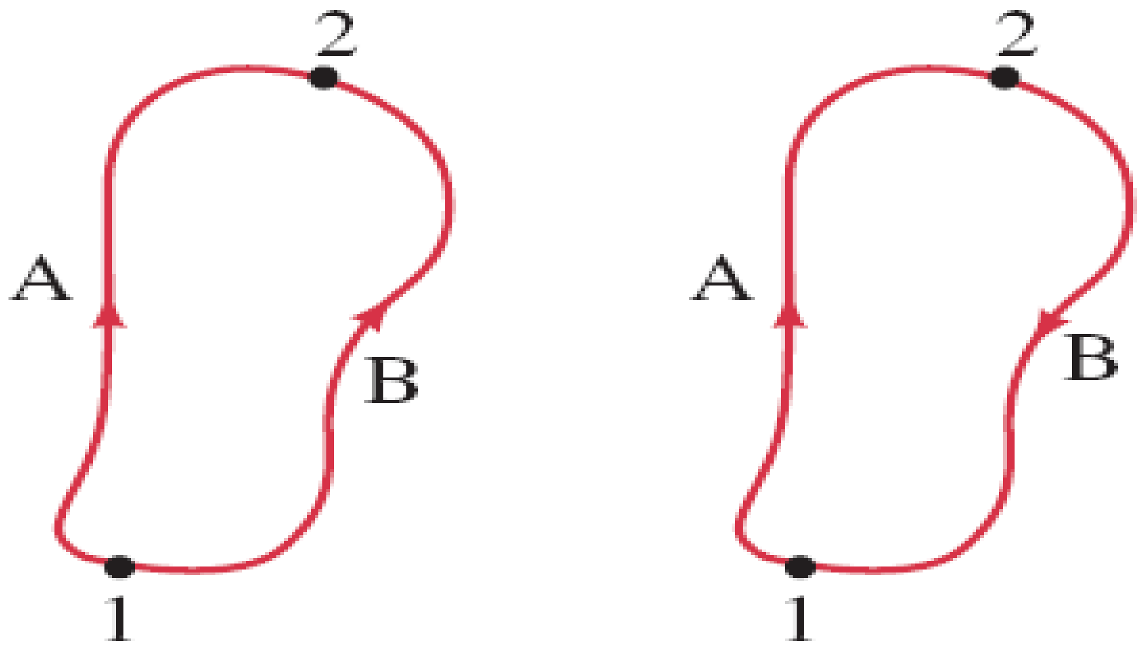
Example: gravity.



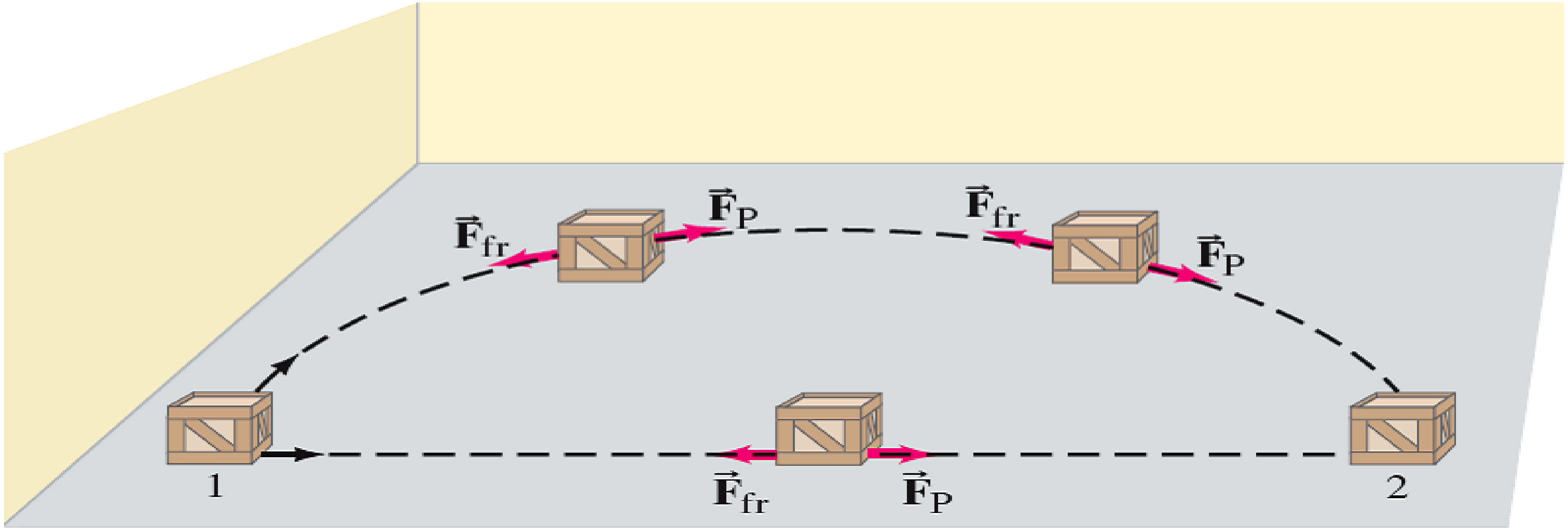


Conservative Force: Another definition:

A force is conservative if the net work done by the force on an object moving around any closed path is zero.



If **friction** is present, the work done depends not only on the starting & ending points, but also on the path taken. ***Friction is a Nonconservative Force!***



Friction is a Nonconservative Force.
The work done by friction depends on the path!

Potential Energy

A mass can have a *Potential Energy* due to its environment

Potential Energy (U) ≡

The energy associated with the position or configuration of a mass.

Examples of potential energy:

- A wound-up spring
- A stretched elastic band
- An object at some height above the ground

TABLE 8–1 Conservative and Nonconservative Forces

Conservative Forces	Nonconservative Forces
Gravitational	Friction
Elastic	Air resistance
Electric	Tension in cord
	Motor or rocket propulsion
	Push or pull by a person

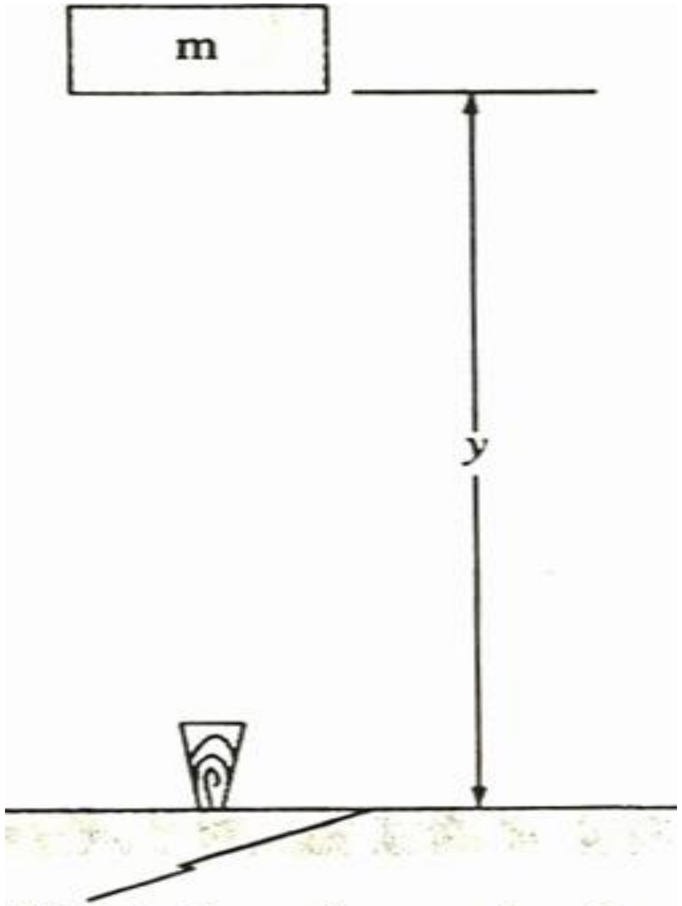
Potential Energy:

Can only be defined for

Conservative Forces!

- ***Potential Energy*** (U) ≡ Energy associated with the position or configuration of a mass.

  Potential work done!



***Gravitational Potential Energy*:**

$$U_{grav} \equiv mgy$$

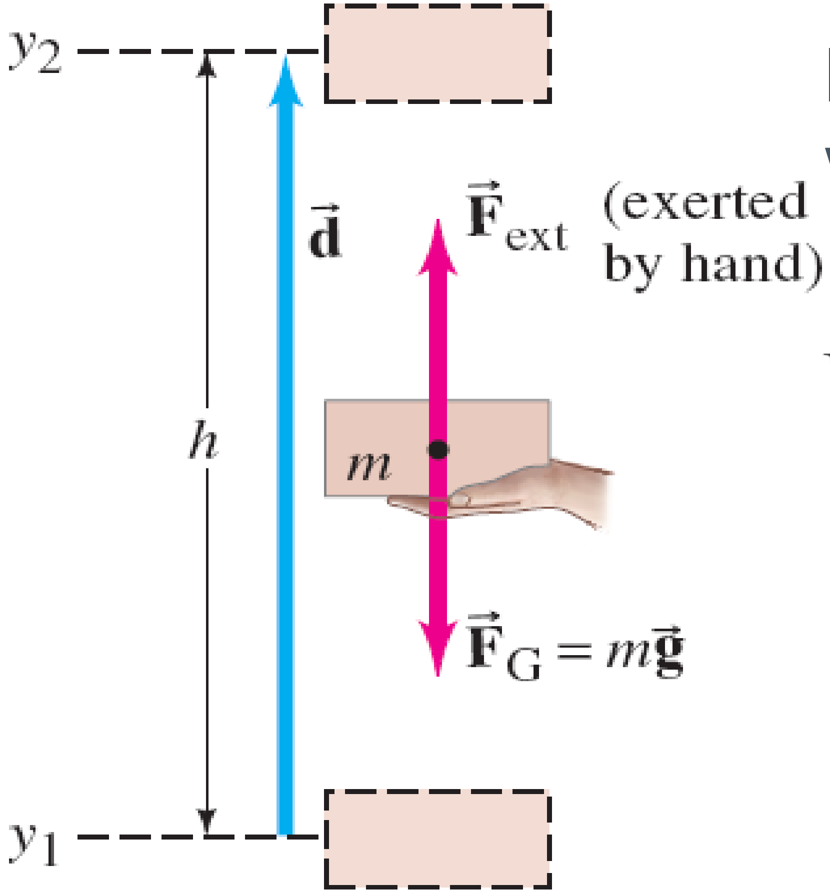
y = distance above Earth

m has the ***potential*** to do work

mgy when it falls

(W = Fy, F = mg)

Gravitational Potential Energy



In raising a mass **m** to a height **h**, the work done by the external force is

$$W_{ext} = \vec{\mathbf{F}}_{ext} \cdot \vec{\mathbf{d}} = mgh\cos 0^\circ$$

$$= mgh = mg(y_2 - y_1)$$

So we *Define the Gravitational Potential Energy* at height **y** above some reference point as

$$U_{grav} = mgy$$

- Consider a problem in which the height of a mass above the Earth changes from y_1 to y_2:

- The ***Change in Gravitational Potential Energy*** is:

$$\Delta U_{grav} = mg(y_2 - y_1)$$

- The work done on the mass by gravity is: $W = \Delta U_{grav}$

y = distance above Earth

Where we choose $y = 0$ is *arbitrary*, since we take the difference in 2 y's in calculating ΔU_{grav}

Of course, this potential energy will be converted to kinetic energy if the object is dropped.

Potential energy is a property of a system as a whole, not just of the object (because it depends on external forces).

If $U_{grav} = mgy$, from where do we measure y?

Doesn't matter, but we need to be consistent about this choice!

This is because only ***changes*** in potential energy can be measured.

Work & Energy

› Particle is acted on by a total external force **F**. *Work* done *ON* particle in moving it from position **1** to position **2** in space is defined as line integral

(**ds** = differential path length, assume mass **m** = constant)

$$W_{12} \equiv \int \mathbf{F} \bullet d\mathbf{s} \qquad \text{(limits: from 1 to 2)}$$

› Newton's 2nd Law (& chain rule of differentiation): $\mathbf{F} \bullet d\mathbf{s} = (d\mathbf{p}/dt) \bullet (d\mathbf{r}/dt)\, dt$

$$= m(d\mathbf{v}/dt) \bullet \mathbf{v}\, dt = (\tfrac{1}{2})m[d(\mathbf{v} \bullet \mathbf{v})/dt]\, dt$$

$$= (\tfrac{1}{2})m(dv^2/dt)\, dt$$

Work-Energy Principle

$\Rightarrow \quad W_{12} = \int \mathbf{F} \bullet d\mathbf{s} = (½)m\int[d(v^2)/dt]\, dt$

$= (½)m\int d(v^2)$ (limits: from 1 to 2)

Or: $W_{12} = (½)m[(v_2)^2 - (v_1)^2]$

- Kinetic Energy of Particle: $T \equiv (½)mv^2$

$\Rightarrow \quad W_{12} = T_2 - T_1 = \Delta T$

Total Work done = Change in kinetic energy

(Work-Energy Principle or Work-Energy Theorem)

Conservative Forces

- Special Case: Force **F** is such that the work W_{12} *does not depend on path* between points 1 & 2:

  F and the system are then ≡ *Conservative*.

- Alternative definition of conservative: Particle goes from point 1 to point 2 & back to point 1 (different paths, total path is closed). Work done is

$$W_{12} + W_{21} = \oint \mathbf{F} \bullet d\mathbf{s} = 0$$

Work done in closed path is zero

Because path independence means $W_{12} = -W_{21}$

Conservative Forces $\Rightarrow$ Potential Energy

- Consider $\mathbf{W_{12} = \int F \bullet ds}$ (limits: from 1 to 2)
- Conservative force **F** $\Rightarrow$ $\mathbf{W_{12}}$ is path independent.
  - Clearly, friction & similar forces are not conservative!
- *For conservative forces*, define a **Potential Energy** function **V(r)**. By definition:

$$W_{12} = \int F \bullet ds = V_1 - V_2 = -\Delta V$$

Depends only on end points **1** & **2**

For conservative forces the total work done =

- (the change in potential energy)

Potential Energy Function

- For conservative forces:

$$W_{12} = \int \mathbf{F} \bullet d\mathbf{s} = V_1 - V_2 = -\Delta V$$

- Vector calculus theorem. W_{12} path independent

$\Rightarrow$ F = gradient of some scalar function. That is this is satisfied if & only if the force has the form:

$$\mathbf{F} = -\nabla V(\mathbf{r}) \quad \text{(minus sign by convention)}$$

For conservative forces, the force is the negative gradient of the potential energy (or potential).

› For conservative forces: $\mathbf{F} = -\nabla V(\mathbf{r})$.

⇒ Can write: $\mathbf{F} \bullet d\mathbf{s} = -\nabla V(\mathbf{r}) \bullet d\mathbf{s} = -dV$

⇒ $\mathbf{F} = -(\partial V/\partial s)$

› Physical (experimental) quantity is $\mathbf{F} = -\nabla V(\mathbf{r})$

⇒ **The zero of V(r) is arbitrary**

(since **F** is a derivative of **V(r)**!)

Energy Conservation

- For conservative forces only we had:

$$W_{12} = \int \mathbf{F} \bullet d\mathbf{s} = V_1 - V_2 \quad \text{(independent of path)}$$

- In general, we had (Work-Energy Principle):

$$W_{12} = T_2 - T_1$$

- Combining ⇒ *For conservative forces:*

$$V_1 - V_2 = T_2 - T_1 \quad \text{or} \quad \Delta T + \Delta V = 0$$

or

$$T_1 + V_1 = T_2 + V_2$$

or

$$E = T + V = \text{constant}$$

$E = T + V \equiv$ Total Mechanical Energy

(or just Total Energy)

- For conservative forces:

$$\Delta T + \Delta V = 0$$

or $$T_1 + V_1 = T_2 + V_2$$

or $$E = T + V = \text{constant (conserved)}$$

Energy Conservation Theorem for a Particle:

If only conservative forces are acting on a particle, then the total mechanical energy of the particle, $E = T + V$, *is conserved.*

- Consider a **special case** where **F** is a function of both position **r** & time **t**: **F = F(r,t)**
- Further, suppose we can define a function **V(r,t)** such that: **F = - (∂V/∂s)**

⇒ Work done on particle in differential distance **ds** is still **F•ds = - (∂V/∂s)ds**

However, in this case, *cannot* write **F•ds = -dV** since **V** is a function of *both time & space.* May still define a total mechanical energy **E = T + V**. However, **E** *is no longer conserved*! **E = E(t)**!!

(Conserved ⇒ **dE/dt = 0**)

Mechanics of a System of Particles

Mechanics of a System of Particles

- Generalization to *many* (N) particle system:
  - Distinguish External & Internal Forces.

  - **Newton's 2nd Law** (eqtn. of motion), particle *i*:

$$\sum_j F_{ji} + F_i^{(e)} = (dp_i/dt) = \dot{p}_i$$

$F_i^{(e)} \equiv$ Total external force on the i^{th} particle.

$F_{ji} \equiv$ Total (internal) force on the i^{th} particle due to the j^{th} particle.

- $F_{jj} = 0$ of course!!

$$\sum_j F_{ji} + F_i^{(e)} = (dp_i/dt) = \dot{p}_i \qquad (1)$$

- *Assumption:* Internal forces F_{ji} obey **Newton's 3rd Law:** $F_{ji} = - F_{ij}$
  - ≡ The *"Weak" Law of Action and Reaction*
  - – Original form of the 3rd Law, but is not satisfied by all forces!

- Sum (1) over all particles in the system:

$$\sum_{i,j(\neq i)} F_{ji} + \sum_i F_i^{(e)} = \sum_i (dp_i/dt)$$

$$= d(\sum_i m_i v_i)/dt = d^2(\sum_i m_i r_i)/dt^2$$

Newton's 2nd Law for Many Particle Systems

- Rewrite as:

$$d^2(\sum_i m_i r_i)/dt^2 = \sum_i F_i^{(e)} + \sum_{i,j(\neq i)} F_{ji} \qquad (2)$$

$$\sum_i F_i^{(e)} \equiv \text{ total external force on system } \equiv F^{(e)}$$

$\sum_{i,j(\neq i)} F_{ji} \equiv 0.$ By **Newton's 3rd Law:**

$F_{ji} = - F_{ij} \Rightarrow F_{ji} + F_{ij} = 0$ (*cancel pairwise* !)

- So, (2) becomes ($\mathbf{r}_i \equiv$ position vector of $\mathbf{m}_i$):

$$d^2(\sum_i m_i r_i)/dt^2 = F^{(e)} \qquad (3)$$

$\Rightarrow$ Only *external* forces enter Newton's 2nd Law to get the equation of motion of a many particle system!!

$$d^2(\sum_i m_i r_i)/dt^2 = F^{(e)} \qquad (3)$$

- Modify **(3)** by defining **R** ≡ mass weighted average of position vectors $\mathbf{r}_i$.

$$\mathbf{R} \equiv (\sum_i m_i \mathbf{r}_i)/(\sum_i m_i) \equiv (\sum_i m_i \mathbf{r}_i)/M$$

$M \equiv \sum_i m_i$ (total mass of particles in system)

R ≡ *Center of mass* of the system (schematic in Figure)

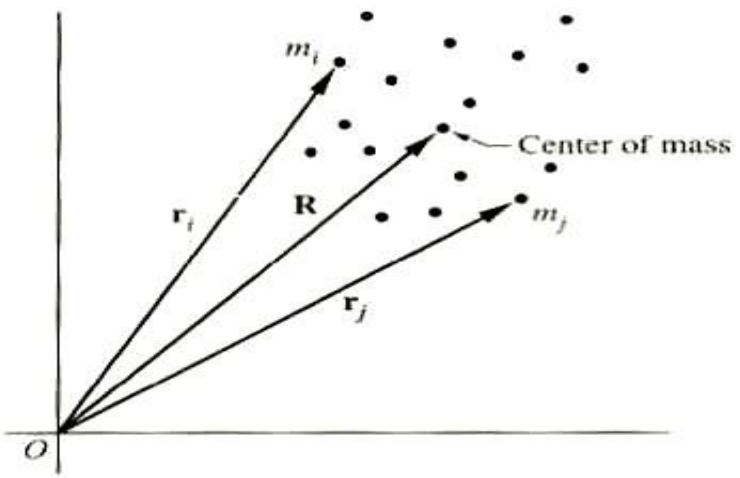


FIGURE 1.1 The center of mass of a system of particles.

⇒ (3) becomes:

$$M(d^2\mathbf{R}/dt^2) = M\mathbf{A} = M(d\mathbf{V}/dt) = (d\mathbf{P}/dt) = \mathbf{F}^{(e)} \quad (4)$$

Just like the eqtn of motion for mass M at position R under the force $F^{(e)}$!

$$M(d^2R/dt^2) = F^{(e)} \qquad (4)$$

⇒ Newton's 2nd Law for a many particle system: *The Center of Mass moves as if the total external force were acting on the entire mass of the system concentrated at the Center of Mass!*

– **Corollary:** Purely *internal* forces (assuming they obey Newton's 3rd Law) have no effect on the motion of the Center of Mass (CM).

Momentum Conservation

- $MR = (\sum_i m_i r_i)$. Consider: Time derivative (const M):

$M(dR/dt) = MV = \sum_i m_i[(dr_i)/dt] \equiv \sum_i m_i v_i \equiv \sum_i \dot{p}_i \equiv \dot{P}$

(total momentum = momentum of CM)

$\Rightarrow$ Using the definition of P, Newton's 2nd Law is:

$$(dP/dt) = F^{(e)} \qquad (4')$$

- Suppose $F^{(e)} = 0$: $\Rightarrow (dP/dt) = \dot{P} = 0$

$\Rightarrow$ P = constant (conserved)

Conservation Theorem for the Linear Momentum of a System of Particles:

If the total external force, $F^{(e)}$, *is zero, the total linear momentum,* P, *is conserved.*

Angular Momentum

- Angular momentum **L** of a many particle system (sum of angular momenta of each particle): $\mathbf{L} \equiv \sum_i[\mathbf{r}_i \times \mathbf{p}_i]$
- Time derivative: $\mathbf{L} = (d\mathbf{L}/dt) = \sum_i d[\mathbf{r}_i \times \mathbf{p}_i]/dt$

$$= \sum_i[(d\mathbf{r}_i/dt) \times \mathbf{p}_i] + \sum_i[\mathbf{r}_i \times (d\mathbf{p}_i/dt)]$$

$$(d\mathbf{r}_i/dt) \times \mathbf{p}_i = \mathbf{v}_i \times (m_i\mathbf{v}_i) = 0$$

$$\Rightarrow \quad (d\mathbf{L}/dt) = \sum_i[\mathbf{r}_i \times (d\mathbf{p}_i/dt)]$$

- Newton's 2nd Law: $(d\mathbf{p}_i/dt) = \mathbf{F}_i^{(e)} + \sum_{j(\neq i)} \mathbf{F}_{ji}$

$\mathbf{F}_i^{(e)} \equiv$ Total external force on the i^{th} particle

$\sum_{j(\neq i)} \mathbf{F}_{ji} \equiv$ Total internal force on the i^{th} particle due to interactions with all other particles (j) in the system

$$\Rightarrow \quad (d\mathbf{L}/dt) = \sum_i[\mathbf{r}_i \times \mathbf{F}_i^{(e)}] + \sum_{i,j(\neq i)} [\mathbf{r}_i \times \mathbf{F}_{ji}]$$

$$(dL/dt) = \sum_i [\mathbf{r}_i \times \mathbf{F}_i^{(e)}] + \sum_{i,j(\neq i)} [\mathbf{r}_i \times \mathbf{F}_{ji}] \qquad (1)$$

› Consider the 2nd sum & look at *each particle pair* (i,j). Each term $\mathbf{r}_i \times \mathbf{F}_{ji}$ has a corresponding term $\mathbf{r}_j \times \mathbf{F}_{ij}$. Take together & use Newton's 3rd Law:

$$\Rightarrow [\mathbf{r}_i \times \mathbf{F}_{ji} + \mathbf{r}_j \times \mathbf{F}_{ij}] = [\mathbf{r}_i \times \mathbf{F}_{ji} + \mathbf{r}_j \times (-\mathbf{F}_{ji})] = [(\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ji}]$$

$(\mathbf{r}_i - \mathbf{r}_j) \equiv \mathbf{r}_{ij}$ = vector from particle j to particle i. (Figure)

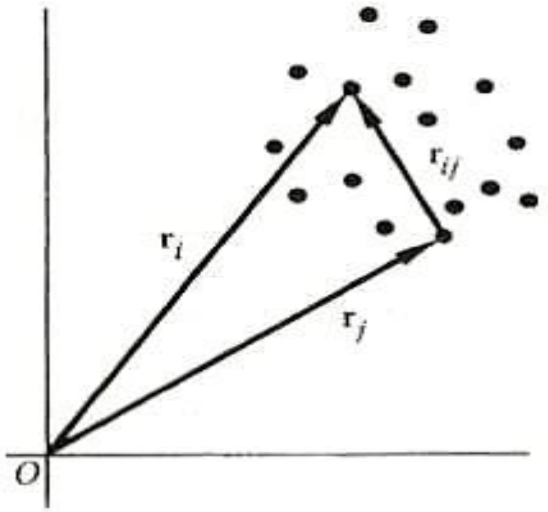


FIGURE 1.2 The vector r_{ij} between the ith and jth particles.

$$(dL/dt) = \sum_i [r_i \times F_i^{(e)}] + (½)\sum_{i,j(\neq i)}[r_{ij} \times F_{ji}] \qquad (1)$$

↑

To Prevent Double Counting!

- ***Assumption:*** Internal forces are ***Central Forces***: **Directed the along lines joining the particle pairs**

  (≡ ***The "Strong" Law of Action and Reaction***)

⇒ $r_{ij} \,||\, F_{ji}$ for each (i,j) & $[r_{ij} \times F_{ji}] = 0$ for each (i,j)!

⇒ 2nd term in (1) is $(½)\sum_{i,j(\neq i)} [r_{ij} \times F_{ji}] = 0$

$$\Rightarrow \qquad (dL/dt) = \sum_i [r_i \times F_i^{(e)}] \qquad (2)$$

- Total external torque on particle *i*:

$$N_i^{(e)} \equiv r_i \times F_i^{(e)}$$

- (2) becomes:

$$(dL/dt) = N^{(e)}$$

(2′)

$$N^{(e)} \equiv \sum_i [r_i \times F_i^{(e)}] = \sum_i N_i^{(e)}$$

= Total external torque on the system

$$(dL/dt) = N^{(e)} \qquad (2)$$

$\Rightarrow$ Newton's 2nd Law (rotational motion) for a many particle system: *The time derivative of the total angular momentum is equal to the total external torque.*

- Suppose $N^{(e)} = 0$: $\Rightarrow$ $(dL/dt) = \dot{L} = 0$

$$\Rightarrow \quad L = \text{constant} \qquad (\text{conserved})$$

Conservation Theorem for the Total Angular Momentum of a Many Particle System:

If the total external torque, $N^{(e)}$, *is zero, then* $(dL/dt) = 0$ *and the total angular momentum,* L, *is conserved.*

- $(dL/dt) = N^{(e)}$. A vector equation! Holds component by component. $\Rightarrow$ **Angular momentum conservation holds component by component.** For example, if $N_z^{(e)} = 0$, L_z is conserved.

- **Linear & Angular Momentum Conservation Laws:**
  - Conservation of Linear Momentum holds if internal forces obey *the "Weak" Law of Action and Reaction*: Only Newton's 3rd Law $F_{ji} = - F_{ij}$ is required to hold!
  - Conservation of Angular Momentum holds if internal forces obey *the "Strong" Law of Action and Reaction*: Newton's 3rd Law $F_{ji} = - F_{ij}$ holds, *PLUS* the forces **must be Central Forces**, so that $r_{ij} \parallel F_{ji}$ for each (i,j)!

    Valid for many common forces (gravity, electrostatic). Not valid for some (magnetic forces, etc.).

Center of Mass & Relative Coordinates

- More on angular momentum. Search for an **analogous relation** to what we had for linear momentum:

$$P = M(dR/dt) = MV$$

Want: Total momentum = Momentum of CM = Same as if entire mass of system were at CM.

- Start with total the angular momentum: $L \equiv \sum_i [r_i \times p_i]$
- $R \equiv$ CM coordinate (origin O). For particle *i* define:

$r_i' \equiv r_i - R$ = relative coordinate vector from CM to particle *i* (Figure)

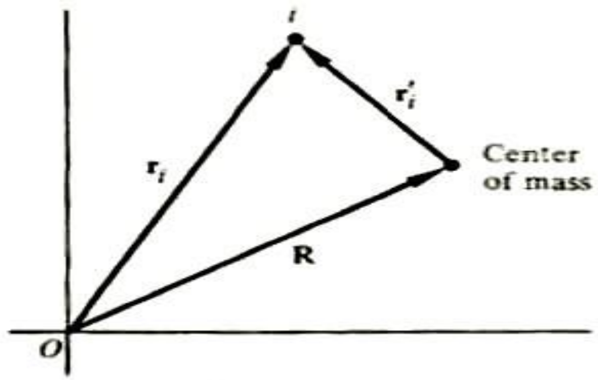


FIGURE 1.3 The vectors involved in the shift of reference point for the angular momentum.

- $r_i = r_i' + R$

Time derivative: $(dr_i/dt) = (dr_i'/dt) + (dR/dt)$ or:

$v_i = v_i' + V$, $V \equiv$ CM velocity relative to O

$v_i' \equiv$ velocity of particle *i* relative to CM. Also:

$p_i \equiv m_i v_i \equiv$ momentum of particle *i* relative to O

- Put this into angular momentum:

$$L = \sum_i [r_i \times p_i] = \sum_i [(r_i' + R) \times m_i(v_i' + V)]$$

Manipulation: (using $m_i v_i' = d(m_i r_i')/dt$)

$$L = R \times \sum_i (m_i) V + \sum_i [r_i' \times (m_i v_i')] +$$

$$\sum_i (m_i r_i') \times V + R \times d[\sum_i (m_i r_i')]/dt$$

- **Note:** $\sum_i (m_i r_i')$ defines the CM coordinate with respect to the CM & is thus zero!! $\sum_i (m_i r_i') \equiv 0$!

$\Rightarrow$ *The last 2 terms are zero!*

$\Rightarrow \qquad L = R \times \sum_i(m_i)V + \sum_i[r_i' \times (m_i v_i')] \qquad (1)$

- Note that $\sum_i(m_i) \equiv M$ = total mass & also

  $m_i v_i' \equiv p_i'$ = momentum of particle *i* relative to the CM

$\Rightarrow \quad L = R\times(MV)+ \sum_i[r_i' \times p_i'] = R\times P + \sum_i[r_i'\times p_i'] \qquad (2)$

The total angular momentum about point O = the angular momentum of the motion of the CM + the angular momentum of motion about the CM

- (2) $\Rightarrow$ In general, **L** depends on the origin **O**, through the vector **R**. Only if the CM is at rest with respect to **O**, will the first term in (2) vanish. Then & only then will **L** be independent of the point of reference. Then & only then will **L** = angular momentum about the CM

Work & Energy

- The **work done by all forces** in changing the system from configuration 1 to configuration 2:

$$W_{12} \equiv \sum_i \int F_i \bullet ds_i \qquad \text{(limits: from 1 to 2)} \qquad (1)$$

As before: $F_i = F_i^{(e)} + \sum_j F_{ji}$

$$\Rightarrow \quad W_{12} = \sum_i \int F_i^{(e)} \bullet ds_i + \sum_{i,j(\neq i)} \int F_{ji} \bullet ds_i \qquad (2)$$

- Work with **(1)** first:
  - Newton's 2nd Law $\Rightarrow$ $F_i = m_i(dv_i/dt)$. Also: $ds_i = v_i dt$

    $F_i \bullet ds_i = m_i(dv_i/dt) \bullet ds_i = m_i(dv_i/dt) \bullet v_i dt$

    $= m_i v_i dv_i = d[(½)m_i(v_i)^2]$

$$\Rightarrow \quad W_{12} = \sum_i \int d[(½)m_i(v_i)^2] \equiv T_2 - T_1$$

where $T \equiv (½)\sum_i m_i(v_i)^2$ = **Total System Kinetic Energy**

Work-Energy Principle

- $W_{12} = T_2 - T_1 = \Delta T$

  The total Work done = The change in kinetic energy

  (*Work-Energy Principle or Work-Energy Theorem*)

- Total Kinetic Energy: $T \equiv (\frac{1}{2})\sum_i m_i(v_i)^2$
  - Another useful form: Use transformation to CM & relative coordinates: $v_i = V + v_i'$, $V \equiv$ CM velocity relative to O, $v_i' \equiv$ velocity of particle *i* relative to CM.

  $\Rightarrow$ $T \equiv (\frac{1}{2})\sum_i m_i(V + v_i')\bullet(V + v_i')$

  $T = (\frac{1}{2})(\sum_i m_i)V^2 + (\frac{1}{2})\sum_i m_i(v_i')^2 + V\bullet\sum_i m_i v_i'$

  Last term: $V\bullet d(\sum_i m_i r_i')/dt$. From the angular momentum discussion: $\sum_i m_i r_i' = 0 \Rightarrow$ The last term is zero!

$\Rightarrow$ Total KE: $T = (\frac{1}{2})MV^2 + (\frac{1}{2})\sum_i m_i(v_i')^2$

Total KE

$$T = (\tfrac{1}{2})MV^2 + (\tfrac{1}{2})\sum_i m_i(v_i')^2$$

- *The total Kinetic Energy of a many particle system is equal to the Kinetic Energy __of__ the CM plus the Kinetic Energy of motion __about__ the CM.*

Work & PE

› 2 forms for work:

$W_{12} = \sum_i \int \mathbf{F}_i \bullet d\mathbf{s}_i = T_2 - T_1 = \Delta T$ (just showed!) (1)

$W_{12} = \sum_i \int \mathbf{F}_i^{(e)} \bullet d\mathbf{s}_i + \sum_{i,j(\neq i)} \int \mathbf{F}_{ji} \bullet d\mathbf{s}_i$ (2)

Use (2) with *Conservative Force Assumptions:*

1. **External Forces:** $\Rightarrow$ Potential functions $V_i(\mathbf{r}_i)$ exist such that (for each particle *i*): $\mathbf{F}_i^{(e)} = -\nabla_i V_i(\mathbf{r}_i)$
2. **Internal Forces:** $\Rightarrow$ Potential functions V_{ij} exist such that (for each particle pair *i,j*): $\mathbf{F}_{ij} = -\nabla_i V_{ij}$
3. **Strong Law of Action-Reaction:** $\Rightarrow$ Potential functions $V_{ij}(r_{ij})$ are functions only of distance

$r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$ between *i* & *j* & the forces lie along line joining them (Central Forces!): $V_{ij} = V_{ij}(r_{ij})$

$\Rightarrow \quad \mathbf{F}_{ij} = -\nabla_i V_{ij} = +\nabla_j V_{ij} = -\mathbf{F}_{ji} = (\mathbf{r}_i - \mathbf{r}_j) f(r_{ij})$

$\leftarrow$ f is a scalar function!

› Conservative external forces:

$\Rightarrow \quad \sum_i \int \mathbf{F}_i^{(e)} \bullet d\mathbf{s}_i = - \sum_i \int \nabla_i V_i \bullet d\mathbf{s}_i = - \sum_i (V_i)_2 + \sum_i (V_i)_1$ Or:

$\sum_i \int \mathbf{F}_i^{(e)} \bullet d\mathbf{s}_i = (V^{(e)})_1 - (V^{(e)})_2$

Where: $V^{(e)} \equiv \sum_i V_i$ = total PE associated with external forces.

› Conservative internal forces: Write (sum over pairs)

$$\Rightarrow \quad \sum_{i,j(\neq i)} \int \mathbf{F}_{ji} \bullet d\mathbf{s}_i = (1/2) \sum_{i,j(\neq i)} \int [\mathbf{F}_{ji} \bullet d\mathbf{s}_i + \mathbf{F}_{ij} \bullet d\mathbf{s}_j]$$

$$= - (1/2) \sum_{i,j(\neq i)} \int [\nabla_i V_{ij} \bullet d\mathbf{s}_i + \nabla_j V_{ij} \bullet d\mathbf{s}_j]$$

Note: $\nabla_i V_{ij} = - \nabla_j V_{ij} = \nabla_{ij} V_{ij}$ ($\nabla_{ij} \equiv$ grad with respect to $\mathbf{r}_{ij}$)

Also: $d\mathbf{s}_i - d\mathbf{s}_j = d\mathbf{r}_{ij}$

$$\Rightarrow \quad \sum_{i,j(\neq i)} \int \mathbf{F}_{ji} \bullet d\mathbf{s}_i = - (1/2) \sum_{i,j(\neq i)} \int \nabla_{ij} V_{ij} \bullet d\mathbf{r}_{ij}$$

do integral!!

$$= - (1/2) \sum_{i,j(\neq i)} (V_{ij})_2 + (1/2) \sum_{i,j(\neq i)} (V_{ij})_1$$

- Conservative internal (Central!) forces:

  $\sum_{i,j(\neq i)} \int F_{ji} \bullet ds_i = -(½)\sum_{i,j(\neq i)}(V_{ij})_2 + (½)\sum_{i,j(\neq i)}(V_{ij})_1$

  or: $\sum_{i,j(\neq i)} \int F_{ji} \bullet ds_i = (V^{(I)})_1 - (V^{(I)})_2$

  Where: $V^{(I)} \equiv (½)\sum_{i,j(\neq i)} V_{ij}$ = Total PE associated with internal forces.

- *For conservative external forces & conservative, central internal forces, it is possible to define a potential energy function for the system:*

$$V \equiv V^{(e)} + V^{(I)} \equiv \sum_i V_i + (½)\sum_{i,j(\neq i)} V_{ij}$$

Conservation of Mechanical Energy

› *For conservative external forces & conservative, central internal forces:*

– The total work done in a process is:

$$W_{12} = V_1 - V_2 = -\Delta V$$

with $V \equiv V^{(e)} + V^{(I)} \equiv \sum_i V_i + (\tfrac{1}{2})\sum_{i,j(\neq i)} V_{ij}$

– In general

$$W_{12} = T_2 - T_1 = \Delta T$$

Combining $\Rightarrow V_1 - V_2 = T_2 - T_1$ or $\Delta T + \Delta V = 0$

or $$T_1 + V_1 = T_2 + V_2$$

or $$E = T + V = \text{constant}$$

$E = T + V \equiv$ Total Mechanical Energy

(or just Total Energy)

Energy Conservation

$$\Delta T + \Delta V = 0$$

or $$T_1 + V_1 = T_2 + V_2$$

or $$E = T + V = \text{constant (conserved)}$$

Energy Conservation Theorem for a Many Particle System:

If only conservative external forces & conservative, central internal forces are acting on a system, then the total mechanical energy of the system,

$E = T + V$, *is conserved.*

- Consider the potential energy:

$$V \equiv V^{(e)} + V^{(I)} \equiv \sum_i V_i + (½)\sum_{i,j(\neq i)} V_{ij}$$

- 2nd term $V^{(I)} \equiv (½)\sum_{i,j(\neq i)} V_{ij}$ ≡ Internal Potential Energy of the System. This is generally non-zero & might vary with time.
  - Special Case: *Rigid Body:* System of particles in which distances $\mathbf{r}_{ij}$ are fixed (do not vary with time).

  $\Rightarrow$ $d\mathbf{r}_{ij}$ are all $\perp$ $\mathbf{r}_{ij}$ & thus to internal forces $\mathbf{F}_{ij}$

  $\Rightarrow$ $\mathbf{F}_{ij}$ do no work. $\Rightarrow$ $V^{(I)}$ = **constant**

  Since V is arbitrary to within an additive constant, we can ignore $V^{(I)}$ for rigid bodies only.

Constraints

Constraints

- Discussion up to now ⇒ **All mechanics is reduced to solving a set of simultaneous, coupled, 2nd order differential eqtns** which come from Newton's 2nd Law applied to each mass individually:

  $$(dp_i/dt) = m_i(d^2r_i/dt^2) = F_i^{(e)} + \sum_j F_{ji}$$

  ⇒ Given forces & initial conditions, problem is reduced to pure math!

- Oversimplification!! Many systems have *CONSTRAINTS* which limit their motion.
  - Example: Rigid Body. Constraints keep $\mathbf{r}_{ij}$ = constant.
  - Example: Particle motion on surface of sphere.

Types of Constraints

- In general, constraints are expressed as a mathematical relation or relations between particle coordinates & possibly the time.
  - Eqtns of constraint are relations like:

$$f(r_1,r_2,r_3,\ldots r_N,t) = 0$$

- Constraints which may be expressed as above:

$\equiv$ ***Holonomic Constraints.***

- Example of **Holonomic Constraint:** Rigid body. Constraints on coordinates are of the form:

$$(r_i - r_j)^2 - (c_{ij})^2 = 0$$

c_{ij} = some constant

- Constraints not expressible as $f(r_i,t) = 0$

  $\equiv$ *Non-Holonomic Constraints*

- Example of **Non-Holonomic Constraint**: Particle confined to surface of rigid sphere, radius **a**: $r^2 - a^2 \geq 0$
- Time dependent constraints:

  $\equiv$ *Rhenomic or Rhenomous Constraints.*

- If constraint eqtns don't explicitly contain time: $\equiv$ *Fixed or Scleronomic or Scleronomous Constraints.*

- Difficulties constraints introduce in problems:

1. Coordinates $\mathbf{r}_i$ are no longer all independent. Connected by constraint eqtns.

2. To apply Newton's 2nd Law, need ***TOTAL*** force acting on each particle. Forces of constraint aren't always known or easily calculated.

$\Rightarrow$ **With constraints, it's often difficult to *directly* apply Newton's 2nd Law.**

Put another way: **Forces of constraint are often among the unknowns of the problem!**

Generalized Coordinates

- To handle the 1st difficulty (with holonomic constraints), introduce *Generalized Coordinates*.
  - Alternatives to usual Cartesian coordinates.
- System (3d) N particles & no constraints.

  $\Rightarrow$ 3N **degrees of freedom**

  (3N independent coordinates)
- With k holonomic constraints, each expressed by eqtn of form:

  $f_m(r_1,r_2,r_3,\ldots r_N,t) = 0 \quad (m = 1, 2, \ldots\ k)$

  $\Rightarrow$ 3N - k **degrees of freedom**

  (3N - k independent coordinates)

- General mechanical system with **s = 3N - k** degrees of freedom (**3N - k** independent coordinates).
- Introduce **s = 3N - k** independent ***Generalized Coordinates*** to describe system:

Notation: $q_1, q_2, \ldots$ Or: q_ℓ (ℓ = 1,2,… s)

- In principle, can always find relations between generalized coordinates & Cartesian (vector) coordinates of form: $\mathbf{r}_i = \mathbf{r}_i(q_1,q_2,q_3,..,t)$ (i = 1,2,3,…N)
  - These are ***transformation eqtns*** from the set of coordinates ($\mathbf{r}_i$) to the set (q_ℓ) . They are parametric representations of ($\mathbf{r}_i$)
  - In principle, can combine with **k** constraint eqtns to obtain inverse relations $q_\ell = q_\ell(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,..t)$ (ℓ = 1,2,… s)

- *Generalized Coordinates* ≡ Any set of **s** quantities which *completely specifies* the state of the system (for a system with **s** degrees of freedom).
- These **s** generalized coords need not be Cartesian! Can choose *any set of s coordinates* which completely describes state of motion of system. **Depending on problem:**
  - Could have **s** curvilinear (spherical, cylindrical, ..) coords
  - Could choose *mixture* of rectangular coords (**m** = # rectangular coords) & curvilinear (**s** - **m** = # curvilinear coords)
  - The **s** generalized coords needn't have units of length! Could be dimensionless or have (almost) *any units*.

- Generalized coords, $\mathbf{q}_\ell$ will (often) not divide into groups of 3 that can be associated with vectors.
  - Example: Particle on sphere surface: A convenient choice of

    $\mathbf{q}_\ell$ = latitude & longitude.
  - Example: Double pendulum:

    A convenient choice of

    $\mathbf{q}_\ell = \boldsymbol{\theta}_1$ & $\boldsymbol{\theta}_2$ (Figure) →

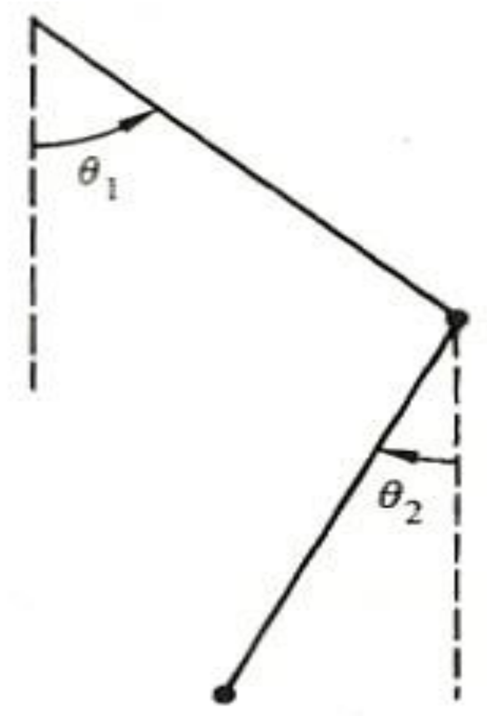


FIGURE 1.4 Double pendulum.

- Sometimes, it's convenient & useful to use **Generalized Coords** (non-Cartesian) even in systems with no constraints.
  - **Example:** Central force field problems:
    $V = V(r)$, it makes sense to use spherical coords!

- Generalized coords need not be orthogonal coordinates & need not be position coordinates.

- **Non-Holonomic constraint:**

⇒ Eqtns expressing constraint can't be used to eliminate dependent coordinates.

- **Example:** Object rolling without slipping on a rough surface.

Coordinates needed to describe motion: Angular coords to specify body orientation + coords to describe location of point of contact of body & surface. Constraint of rolling ⇒ Connects 2 coord sets: They aren't independent. BUT, # coords cannot be reduced by the constraint, because cannot express rolling condition as eqtn between coords! Instead, (can show) rolling constraint is condition on the *velocities*: a differential eqtn which can be integrated only after solution to problem is known!

Example: Rolling Constraint

› Disk, radius **a**, constrained to be vertical, rolling on the horizontal (**xy**) plane. Figure:

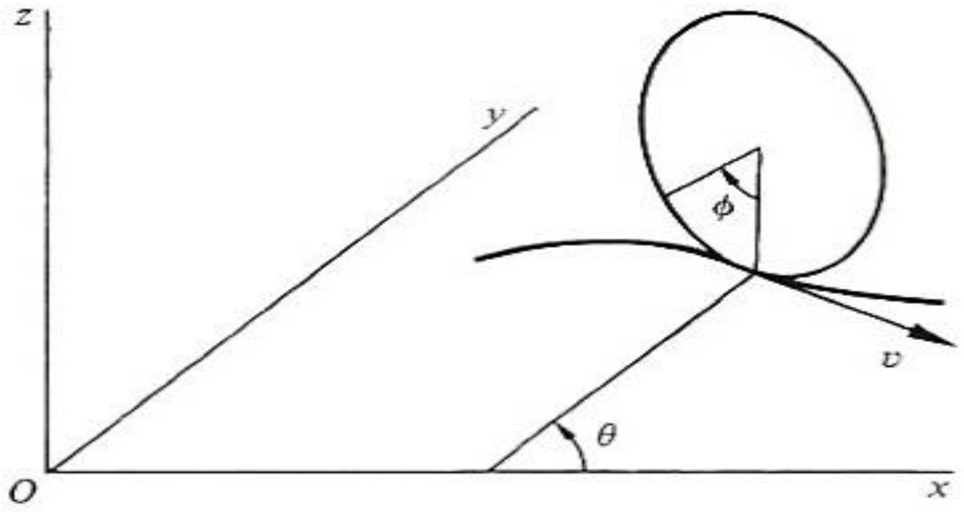


FIGURE 1.5 Vertical disk rolling on a horizontal plane.

› Generalized coords: **x, y** of point of contact of disk with plane + **θ** = angle between disk axis & **x**-axis + **ϕ** = angle of rotation about disk axis

- **Constraint:** Velocity **v** of disk center is related to angular velocity **($d\phi/dt$)** of disk rotation:

$$v = a(d\phi/dt) \qquad (1)$$

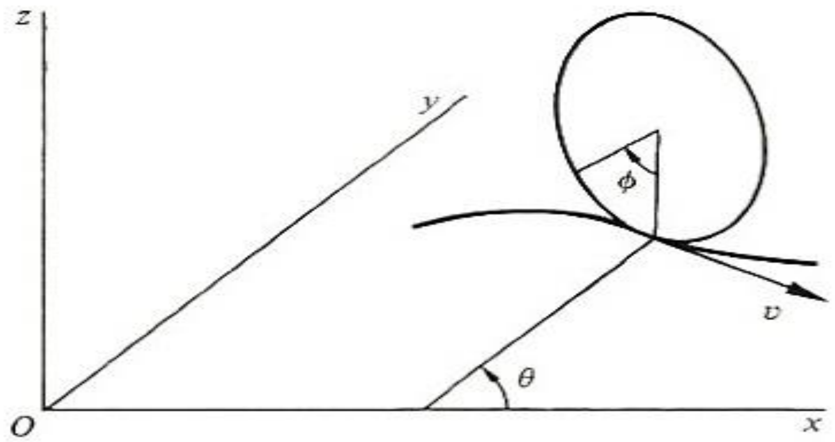


FIGURE 1.5 Vertical disk rolling on a horizontal plane.

Also Cartesian components of **v**:

$$v_x = (dx/dt) = v\sin\theta,\ v_y = (dy/dt) = -v\cos\theta \qquad (2)$$

Combine **(1)** & **(2)** (multiplying through by dt):

$$\Rightarrow \qquad dx - a\sin\theta\ d\phi = 0 \quad dy + a\cos\theta\ d\phi = 0$$

Neither can be integrated without solving the problem! That is, a function $f(x,y,\theta,\phi) = 0$ cannot be found. Physical argument that ϕ must be indep of x,y,θ: See pp. 15 & 16

- Non-Holonomic constraints can also involve higher order derivatives or inequalities.
- Holonomic constraints are preferred, since easiest to deal with. No general method to treat problems with Non-Holonomic constraints. Treat on case-by-case basis.
- In special cases of Non-Holonomic constraints, when constraint is expressed in differential form (as in example), can use method of Lagrange multipliers along with Lagrange's eqtns (later).
- Authors argue, except for some macroscopic physics textbook examples, most problems of practical interest to physicists are microscopic & the constraints are holonomic or do not actually enter the problem.

› Difficulties constraints introduce:

1. Coordinates $\mathbf{r}_i$ are no longer all independent. Connected by constraint eqtns.
   - Have now thoroughly discussed this problem!

2. To apply Newton's 2nd Law, need the ***TOTAL*** force acting on each particle. Forces of constraint are not always known or easily calculated.

$\Rightarrow$ With constraints, it's often difficult to ***directly*** apply Newton's 2nd Law.

Put another way: Forces of constraint are often among the unknowns of the problem! To address this, long ago, people reformulated mechanics. Lagrangian & Hamiltonian formulations. No direct reference to forces of constraint.

D'Alembert's Principle & Lagrange's Equations

- *Virtual* (infinitesimal) *displacement* ≡ Change in the system configuration as result of an arbitrary infinitesimal change of coordinates $\delta \mathbf{r}_i$, *consistent with the forces & constraints imposed on the system at a given time t.*
- *"Virtual"* distinguishes it from an actual displacement $d\mathbf{r}_i$, occurring in small time interval **dt** (during which forces & constraints may change)

- Consider the system at *equilibrium*: The total force on each particle is $F_i = 0$. *Virtual work* done by F_i in displacement δr_i:

$$\delta W_i = F_i \bullet \delta r_i = 0. \text{ Sum over } i\text{:}$$

$$\Rightarrow \quad \delta W = \sum_i F_i \bullet \delta r_i = 0.$$

- Decompose F_i into **applied force** $F_i^{(a)}$ & **constraint force** f_i:
  $F_i = F_i^{(a)} + f_i$

$$\Rightarrow \quad \delta W = \sum_i (F_i^{(a)} + f_i) \bullet \delta r_i \equiv \delta W^{(a)} + \delta W^{(c)} = 0$$

- *Special case* (often true, see text discussion): Systems for which the net virtual work due to constraint forces is zero:
  $\sum_i f_i \bullet \delta r_i \equiv \delta W^{(c)} = 0$

Principle of Virtual Work

⇒ *Condition for system equilibrium:* Virtual work due to APPLIED forces vanishes:

$$\delta W^{(a)} = \sum_i F_i^{(a)} \bullet \delta r_i = 0 \qquad (1)$$

≡ *Principle of Virtual Work*

- **Note:** In general coefficients of δr_i , $F_i^{(a)} \neq 0$ even though $\sum_i F_i^{(a)} \bullet \delta r_i = 0$ because δr_i are not independent, but connected by constraints.
  - In order to have coefficients of $\delta r_i = 0$, must transform Principle of Virtual Work into a form involving virtual displacements of generalized coordinates $q_{\square}$, which are independent. (1) is good since it does not involve constraint forces f_i . But so far, only statics. Want to treat dynamics!

D'Alembert's Principle

- Dynamics: Start with **Newton's 2nd Law** for particle *i*:

  $F_i = (dp_i/dt)$ Or: $F_i - (dp_i/dt) = 0$

$\Rightarrow$ Can view system particles as in "equilibrium" under a force

= actual force + "reversed effective force" $= -(dp/dt)$

- **Virtual work** done is

$$\delta W = \sum_i [F_i - (dp_i/dt)] \bullet \delta r_i = 0$$

- Again decompose F_i: $F_i = F_i^{(a)} + f_i$

$$\Rightarrow \quad \delta W = \sum_i [F_i^{(a)} - (dp_i/dt) + f_i] \bullet \delta r_i = 0$$

- Again restrict consideration to ***special case***: Systems for which the net virtual work due to constraint forces is zero: $\sum_i f_i \bullet \delta r_i \equiv \delta W^{(c)} = 0$

$$\Rightarrow \ \delta W = \sum_i [F_i - (dp_i/dt)] \bullet \delta r_i = 0 \qquad (2)$$

$\equiv$ ***D'Alembert's Principle***

– Dropped the superscript **(a)**!

- Transform **(2)** to an expression involving virtual displacements of $\mathbf{q}_\ell$ (which, for holonomic constraints, are indep of each other). Then, by linear independence, the coefficients of the $\boldsymbol{\delta}\mathbf{q}_\ell = \mathbf{0}$

$$\delta W = \sum_i [F_i - (dp_i/dt)] \bullet \delta r_i = 0 \qquad (2)$$

- Much manipulation follows! Only highlights here!
- Transformation eqtns:

$$r_i = r_i(q_1, q_2, q_3, .., t) \qquad (i = 1,2,3,...n)$$

- Chain rule of differentiation (velocities):

$$v_i \equiv (dr_i/dt) = \sum_k (\partial r_i/\partial q_k)(dq_k/dt) + (\partial r_i/\partial t) \qquad (a)$$

- Virtual displacements $\delta \mathbf{r}_i$ are connected to virtual displacements $\delta \mathbf{q}_\ell$: $\quad \delta r_i = \sum_j (\partial r_i/\partial q_j)\delta q_j \qquad$ (b)

Generalized Forces

- 1^{st} term of **(2)** (Combined with **(b)**):

  $$\sum_i \mathbf{F}_i \bullet \delta \mathbf{r}_i = \sum_{i,j} \mathbf{F}_i \bullet (\partial \mathbf{r}_i/\partial q_j)\delta q_j \equiv \sum_j Q_j \delta q_j \qquad \mathbf{(c)}$$

  Define *Generalized Force* (corresponding to Generalized Coordinate q_j): $Q_j \equiv \sum_i \mathbf{F}_i \bullet (\partial \mathbf{r}_i/\partial q_j)$

  – Generalized Coordinates q_j need not have units of length!

  $\Rightarrow$ Corresponding **Generalized Forces** Q_j need not have units of force!

  – For example: If q_j is an angle, corresponding Q_j will be a torque!

- 2^{nd} term of **(2)** (using **(b)** again):

  $\sum_i (dp_i/dt)\bullet\delta r_i = \sum_i [m_i (d^2r_i/dt^2)\bullet\delta r_i\ ] =$

  $\sum_{i,j} [m_i (d^2r_i/dt^2)\bullet(\partial r_i/\partial q_j)\delta q_j]$ **(d)**

- **Manipulate** with **(d)**: $\sum_i [m_i (d^2r_i/dt^2)\bullet(\partial r_i/\partial q_j)] =$

  $\sum_i [d\{m_i(dr_i/dt)\bullet(\partial r_i/\partial q_j)\}/dt] - \sum_i [m_i(dr_i/dt)\bullet d\{(\partial r_i/\partial q_j)\}/dt]$

  Also: $d\{(\partial r_i/\partial q_j)\}/dt = \partial\{dr_i/dt\}/\partial q_j \equiv (\partial v_i/\partial q_j)$

  Use **(a)**: $(\partial v_i/\partial q_j) = \sum_k (\partial^2 r_i/\partial q_j \partial q_k)(dq_k/dt) + (\partial^2 r_i/\partial q_j \partial t)$

  From **(a)**: $(\partial v_i/\partial \dot{q}_j) = (\partial r_i/\partial q_j)$

  So: $\sum_i [m_i (d^2r_i/dt^2)\bullet(\partial r_i/\partial q_j)]$

  $= \sum_i [d\{m_i v_i\bullet(\partial v_i/\partial q_j)\}/dt] - \sum_i [m_i v_i\bullet(\partial v_i/\partial q_j)]$

More manipulation $\Rightarrow$ (2) is: $\sum_i[F_i-(dp_i/dt)]\bullet\delta r_i = 0$

$$\sum_j\{d[\partial(\sum_i (½)m_i(v_i)^2)/\partial\dot{q}_j]/dt - \partial(\sum_i(½)m_i(v_i)^2)/\partial q_j - Q_j\}\delta q_j = 0$$

- System kinetic energy is: $T \equiv (½)\sum_i m_i(v_i)^2$

$\Rightarrow$ ***D'Alembert's Principle*** becomes

$$\sum_j\{(d[\partial T/\partial\dot{q}_j]/dt) - (\partial T/\partial q_j) - Q_j\}\delta q_j = 0 \qquad (3)$$

– Note: If q_j are Cartesian coords, $(\partial T/\partial q_j) = 0$

$\Rightarrow$ In generalized coords, $(\partial T/\partial q_j)$ comes from the curvature of the q_j. (**Example:** Polar coords, $(\partial T/\partial\theta)$ becomes the centripetal acceleration).

- So far, no restriction on constraints except that they do no work under virtual displacement. q_j are any set.

Special case: ***Holonomic Constraints*** $\Rightarrow$ It's possible to find sets of q_j for which each δq_j is independent.

$\Rightarrow$ **Each term in (3) is separately 0!**

- Holonomic constraints $\Rightarrow$ *D'Alembert's Principle*:

$$(d[\partial T/\partial \dot{q}_j]/dt) - (\partial T/\partial q_j) = Q_j \qquad (4)$$

$(j = 1,2,3, \dots\ n)$

- Special case: *A Potential Exists* $\Rightarrow$ $F_i = -\nabla_i V$
  – Needn't be conservative! V could be a function of t!

$\Rightarrow$ Generalized forces have the form

$$Q_j \equiv \sum_i F_i \bullet (\partial r_i/\partial q_j) = -\sum_i \nabla_i V \bullet (\partial r_i/\partial q_j) \equiv -(\partial V/\partial q_j)$$

- Put this in (4): $(d[\partial T/\partial \dot{q}_j]/dt) - (\partial[T-V]/\partial q_j) = 0$
- So far, V doesn't depend on the velocities $\dot{q}_j$

$$\Rightarrow \qquad (d/dt)[\partial(T-V)/\partial \dot{q}_j] - \partial(T-V)/\partial q_j = 0 \qquad (4')$$

Lagrange's Equations

› **Define:** ***The Lagrangian*** L of the system:

$$L \equiv T - V$$

$\Rightarrow$ Can write ***D'Alembert's Principle*** as:

$$(d/dt)[(\partial L/\partial \dot{q}_j)] - (\partial L/\partial q_j) = 0 \qquad (5)$$

$(j = 1,2,3, \ldots\ n)$

(5) $\equiv$ ***Lagrange's Equations***

Lagrange's Equations

- ***Lagrangian:*** $\mathcal{L} \equiv T - V$
- ***Lagrange's Eqtns:***

$$(d/dt)[(\partial \mathcal{L}/\partial \dot{q}_j)] - (\partial \mathcal{L}/\partial q_j) = 0 \qquad (j = 1,2,3, \dots n)$$

- **Note:** $\mathcal{L}$ is <u>not unique</u>, but is arbitrary to within the addition of a derivative **(dF/dt)**. **F = F(q,t)** is ***any*** differentiable function of **q**'s & **t**.
- That is, if we define a new Lagrangian $\mathcal{L}'$

$$\mathcal{L}' = \mathcal{L} + (dF/dt)$$

It is easy to show that $\mathcal{L}'$ satisfies ***the same*** Lagrange's Eqtns (above).

Velocity-Dependent Potentials & the Dissipation Function

- **Non-conservative** forces? It's still possible, in a ***Special Case***, to use Lagrange's Eqtns unchanged, provided a **Generalized or Velocity-Dependent Potential** $U = U(q_j,\dot{q}_j)$ exists, where the generalized forces Q_j are obtained as:

$$Q_j \equiv -(\partial U/\partial q_j) + (d/dt)[(\partial U/\partial \dot{q}_j)]$$

- The **Lagrangian is now:** $L \equiv T - U$ & Lagrange's Eqtns are still:

$$(d/dt)[(\partial L/\partial \dot{q}_j)] - (\partial L/\partial q_j) = 0 \quad (j = 1,2,3, \ldots\ n)$$

- <u>A very important application:</u> Electromagnetic forces on moving charges.

Electromagnetic Force Problem

- Particle, mass **m**, charge **q** moving with velocity **v** in combined electric (**E**) & magnetic (**B**) fields.
- Lorentz Force (SI units!):

$$F = q[E + (v \times B)] \qquad (1)$$

- *E&M results that you should know!*

E = E(x,y,z,t) & B = B(x,y,z,t) are derivable from a scalar potential $\phi = \phi(x,y,z,t)$ and a vector potential A = A(x,y,z,t) as:

$$E \equiv -\nabla\phi - (\partial A/\partial t) \qquad (2)$$

$$B \equiv \nabla \times A \qquad (3)$$

- **Can obtain the Lorentz Force (1) from the velocity dependent potential:** $\mathbf{U} \equiv \mathbf{q\phi - qA\bullet v}$

$$\mathbf{F = -\nabla U}$$

  - Proof: **Exercise for student!** Use **(1),(2),(3)** together.

- Lagrangian is: $L \equiv \mathbf{T - U = (½)mv^2 - q\phi + qA\bullet v}$

- Use Cartesian coords. Lagrange Eqtn for coord **x** (noting $\mathbf{v}^2 = (\dot{\mathbf{x}})^2 + (\dot{\mathbf{y}})^2 + (\dot{\mathbf{z}})^2$ & $\mathbf{v} = \dot{\mathbf{x}}\,\hat{\mathbf{i}} + \dot{\mathbf{y}}\,\hat{\mathbf{j}} + \dot{\mathbf{z}}\,\hat{\mathbf{k}}$)

$$\mathbf{(d/dt)[(\partial \mathit{L}/\partial \dot{x})] - (\partial \mathit{L}/\partial x) = 0}$$

$$\Rightarrow \mathbf{m\ddot{x} = q[\dot{x}(\partial A_x/\partial x) + \dot{y}(\partial A_y/\partial x) + \dot{z}(\partial A_z/\partial x)]}$$

$$\mathbf{- q[(\partial\phi/\partial x) + (dA_x/dt)]} \qquad \mathbf{(a)}$$

Note that: $\mathbf{(dA_x/dt) = v\bullet\nabla A_x + (\partial A_x/\partial t)}$

$\Rightarrow \qquad m\ddot{x} = -\,q(\partial\phi/\partial x) - q(\partial A_x/\partial t)$

$+\, q[\dot{y}\{(\partial A_y/\partial x) - (\partial A_x/\partial y)\} + \dot{z}\{(\partial A_z/\partial x) - (\partial A_x/\partial z)\}]$

- Using **(2)** & **(3)** this becomes:

$$m\ddot{x} = q[E_x + \dot{y}B_z - \dot{z}B_y]$$

Or: $\quad m\ddot{x} = q[E_x + (v \times B)_x] = F_x$ (Proven!)

- If **some forces in the problem are conservative & some are not:** $\Rightarrow$ Have potential **V** for conservative ones & thus have the Lagrangian $L \equiv$ **T - V** for these. For non-conservative ones, still have generalized forces:

$$Q_j \equiv \sum_i F_i \bullet (\partial r_i/\partial q_j)$$

Non-Conservative Forces

- $L \equiv$ T - V for conservative forces.
- **Generalized forces:** $Q_j \equiv \sum_i F_i \bullet (\partial r_i/\partial q_j)$ for non-conservative forces.
- Follow derivation of Lagrange Eqtns & get:

$$(d/dt)[(\partial L/\partial \dot{q}_j)] - (\partial L/\partial q_j) = Q_j \quad (j = 1,2,3, ..n)$$

- **Friction:** A common non-conservative force.
- **Friction** (or air resistance): A common ***model:*** Components are proportional to some power of **v** (often the 1st power): $F_{fx} = -k_x v_x$ (k_x = const)

Frictional Forces

- ***Model* for Friction** (or air resistance): $\mathbf{F_{fx} = -k_x v_x}$
- Can Include such forces in Lagrangian formalism by introducing ***Rayleigh's Dissipation Function $\mathcal{F}$***

$$\mathcal{F} \equiv (\tfrac{1}{2})\sum_i [k_x(v_{ix})^2 + k_y(v_{iy})^2 + k_z(v_{iz})^2]$$

- Obtain components of the frictional force by:

$$F_{fxi} \equiv -(\partial\mathcal{F}/\partial v_{ix}), \text{ etc. Or, } F_f = -\nabla_v \mathcal{F}$$

- **Physical Interpretation** of $\mathcal{F}$: Work done by system *against* friction: $\mathbf{dW_f = -F_f \bullet dr = -F_f \bullet v\, dt}$

$$= -[k_x(v_{ix})^2 + k_y(v_{iy})^2 + k_z(v_{iz})^2]\, dt = -2\mathcal{F}\, dt$$

$\Rightarrow$ **Rate of energy dissipation due to friction:**

$$\mathbf{(dW_f/dt) = -2\mathcal{F}}$$

- ***Rayleigh's Dissipation Function*** $\mathcal{F}$

$$\mathcal{F} \equiv (½)\sum_i[k_x(v_{ix})^2 + k_y(v_{iy})^2 + k_z(v_{iz})^2]$$

- **Frictional force:** $\mathbf{F_{fi} = - \nabla_{vi} \mathcal{F}}$
- Corresponding **generalized force:**

$$Q_j \equiv \sum_i F_{fi} \bullet(\partial r_i/\partial q_j) = - \sum_i \nabla_{vi} \mathcal{F} \bullet(\partial r_i/\partial q_j)$$

Note that: $(\partial r_i/\partial q_j) = (\partial \dot{r}_i/\partial \dot{q}_j)$

$$Q_j = - \sum_i \nabla_{vi} \mathcal{F} \bullet(\partial \dot{r}_i/\partial \dot{q}_j) = - (\partial \mathcal{F}/\partial \dot{q}_j)$$

- Lagrange's Eqtns, with frictional (dissipative) forces:

$$(d/dt)[(\partial L/\partial \dot{q}_j)] - (\partial L/\partial q_j) = Q_j$$

Or

$$(d/dt)[(\partial L/\partial \dot{q}_j)] - (\partial L/\partial q_j) + (\partial \mathcal{F}/\partial \dot{q}_j) = 0$$

$$(j = 1,2,3, ..n)$$

Simple Applications of the Lagrangian Formulation

- **Lagrangian formulation: 2 *scalar functions*, T & V**
- **Newtonian formulation: *MANY* vector forces** & accelerations. (***Advantage of Lagrangian over Newtonian!***)
- **"Recipe"** for application of the Lagrangian method:
  - Choose appropriate generalized coordinates
  - Write **T** & **V** in terms of these coordinates
  - Form the Lagrangian $L = T - V$
  - Apply: ***Lagrange's Eqtns:***

    $$(d/dt)[(\partial L/\partial \dot{q}_j)] - (\partial L/\partial q_j) = 0 \qquad (j = 1,2,3, \ldots n)$$

  - Equivalently ***D'Alembert's Principle***:

    $$(d/dt)[\partial T/\partial \dot{q}_j] - (\partial T/\partial q_j) = Q_j \qquad (j = 1,2,3, \ldots n)$$

- Sometimes, **T** & **V** are easily obtained in generalized coordinates $\mathbf{q_j}$ & velocities $\dot{\mathbf{q}}_\mathbf{j}$ & sometimes not. If not, **write in Cartesian coordinates & transform to generalized coordinates.** Use transformation eqtns:

$$\mathbf{r_i = r_i(q_1,q_2,q_3,..,t) \qquad (i = 1,2,3,\ldots n)}$$

$$\Rightarrow \quad \mathbf{v_i \equiv (dr_i/dt) = \textstyle\sum_j(\partial r_i/\partial q_j)(dq_j/dt) + (\partial r_i/\partial t)}$$

$$\Rightarrow \quad \mathbf{T \equiv (\tfrac{1}{2})\textstyle\sum_i m_i(v_i)^2}$$

$$\mathbf{= (\tfrac{1}{2})\textstyle\sum_i m_i[\sum_j(\partial r_i/\partial q_j)(dq_j/dt) + (\partial r_i/\partial t)]^2}$$

- Squaring gives: $\mathbf{T = M_0 + \sum_j M_j\dot{q}_j + \sum_j M_{jk}\dot{q}_j\dot{q}_k}$

$$\mathbf{M_0 \equiv (\tfrac{1}{2})\textstyle\sum_i m_i(\partial r_i/\partial t)^2,\ \ M_j \equiv \sum_i m_i(\partial r_i/\partial t)\bullet(\partial r_i/\partial q_j)}$$

$$\mathbf{M_{jk} \equiv \textstyle\sum_i m_i(\partial r_i/\partial q_j)\bullet(\partial r_i/\partial q_k)}$$

- ***Always:*** $$\mathbf{T = M_0 + \sum_j M_j \dot{q}_j + \sum_j M_{jk} \dot{q}_j \dot{q}_k}$$

$$\text{Or: } \mathbf{T_0 + T_1 + T_2}$$

$\mathbf{T_0 \equiv M_0}$ independent of generalized velocities

$\mathbf{T_1 \equiv \sum_j M_j \dot{q}_j}$ linear in generalized velocities

$\mathbf{T_2 \equiv \sum_j M_{jk} \dot{q}_j \dot{q}_k}$ quadratic in generalized velocities

NOTE: From previous eqtns, if $\mathbf{(\partial r_i/\partial t) = 0}$ (if transformation eqtns do not contain time explicitly), then

$$\mathbf{T_0 = T_1 = 0} \qquad \Rightarrow \qquad \mathbf{T = T_2}$$

$\Rightarrow$ ***If the transformation eqtns from Cartesian to generalized coords do not contain the time explicitly, the kinetic energy is a homogeneous, quadratic function of the generalized velocities.***

Examples

- Simple examples (for some, the Lagrangian method is "overkill"):

  1. A single particle in space (subject to force **F**):

  a. Cartesian coords

  b. Plane polar coords.

  2. The Atwood's machine

  3. Time dependent constraint: A bead sliding on rotating wire

Particle in Space (Cartesian Coords)

- The Lagrangian method is "overkill" for this problem!
- Mass **m**, force **F**: Generalized coordinates $\mathbf{q_j}$ are Cartesian coordinates **x, y, z**! $\mathbf{q_1 = x}$, etc. Generalized forces $\mathbf{Q_j}$ are Cartesian components of force $\mathbf{Q_1 = F_x}$, etc.
- Kinetic energy: $\mathbf{T = (½)m[(\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2]}$
- Lagrange eqtns which contain generalized forces (***D'Alembert's Principle***):

$$\mathbf{(d[\partial T/\partial \dot{q}_j]/dt) - (\partial T/\partial q_j) = Q_j} \qquad \mathbf{(j = 1,2,3} \text{ or } x,y,z\mathbf{)}$$

- $T = (½)m[(\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2]$

$$(d[\partial T/\partial \dot{q}_j]/dt) - (\partial T/\partial q_j) = Q_j$$

(j = 1,2,3 or x,y,z)

$$(\partial T/\partial x) = (\partial T/\partial y) = (\partial T/\partial z) = 0$$

$$(\partial T/\partial \dot{x}) = m\dot{x},\ (\partial T/\partial \dot{y}) = m\dot{y},\ (\partial T/\partial \dot{z}) = m\dot{z}$$

$$\Rightarrow\ d(m\dot{x})/dt = m\ddot{x} = F_x\ ;\ d(m\dot{y})/dt = m\ddot{y} = F_y$$

$$d(m\dot{z})/dt = m\ddot{z} = F_z$$

Identical results (of course!) to Newton's 2nd Law.

Particle in Plane (Plane Polar Coords)

- **Plane Polar Coordinates:**

$$q_1 = r,\ q_2 = \theta$$

- **Transformation eqtns:**

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$\Rightarrow \quad \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$

$$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

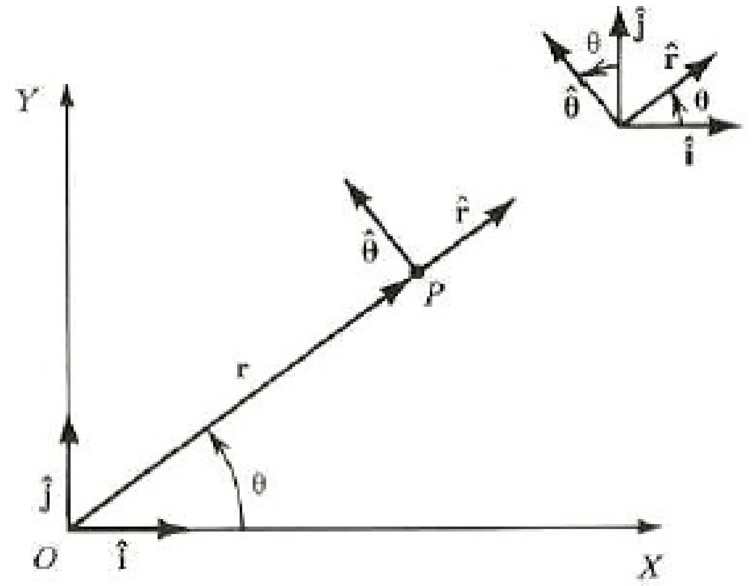


Figure 6.9 Unit vectors $\hat{r}$ and $\hat{\theta}$ in plane polar coordinates.

$\Rightarrow$ **Kinetic energy:**

$$T = (½)m[(\dot{x})^2 + (\dot{y})^2] = (½)m[(\dot{r})^2 + (r\dot{\theta})^2]$$

Lagrange: $(d[\partial T/\partial \dot{q}_j]/dt) - (\partial T/\partial q_j) = Q_j \quad (j = 1,2 \text{ or } r, \theta)$

Generalized forces: $Q_j \equiv \sum_i F_i \bullet (\partial \vec{r}_i/\partial q_j)$

$$\Rightarrow \quad Q_1 = Q_r = \vec{F}\bullet(\partial\vec{r}/\partial r) = \vec{F}\bullet\hat{r} = F_r$$

$$Q_2 = Q_\theta = \vec{F}\bullet(\partial\vec{r}/\partial\theta) = \vec{F}\bullet r\hat{\theta} = rF_\theta$$

$T = (½)m[(\dot{r})^2 + (r\dot{\theta})^2]$ **Forces:** $Q_r = F_r$, $Q_\theta = rF_\theta$

Lagrange: $(d[\partial T/\partial \dot{q}_j]/dt) - (\partial T/\partial q_j) = Q_j$ $(j = r, \theta)$

– *Physical interpretation:* $Q_r = F_r$ = radial force component.

$Q_r = F_r$ = radial component of force.

$Q_\theta = rF_\theta$ = torque about axis ⊥ plane through origin

- r: $(\partial T/\partial r) = mr(\dot{\theta})^2$; $(\partial T/\partial \dot{r}) = m\dot{r}$; $(d[\partial T/\partial \dot{r}]/dt) = mr$

$$\Rightarrow \quad m\ddot{r} - mr(\dot{\theta})^2 = F_r \qquad (1)$$

– *Physical interpretation:* $- mr(\dot{\theta})^2$ = centripetal force

- θ: $(\partial T/\partial \theta) = 0$; $(\partial T/\partial \dot{\theta}) = mr^2\dot{\theta}$; (Note: $L = mr^2\dot{\theta}$)

$(d[\partial T/\partial \dot{\theta}]/dt) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = (dL/dt) = N$

$$\Rightarrow \quad mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = rF_\theta \qquad (2)$$

– *Physical interpretation:* $mr^2\dot{\theta} = L$ = **angular momentum** about axis through origin ⇒ $(2) \equiv (dL/dt) = N = rF_\theta$

Atwood's Machine

- M_1 & M_2 connected over a massless, frictionless pulley by a massless, extensionless string, length ℓ.

  Gravity acts, of course!

  $\Rightarrow$ ***Conservative system, holonomic, scleronomous constraints***

- 1 indep. coord. (1 deg. of freedom). Position **x** of M_1.

  Constraint keeps const. length ℓ.

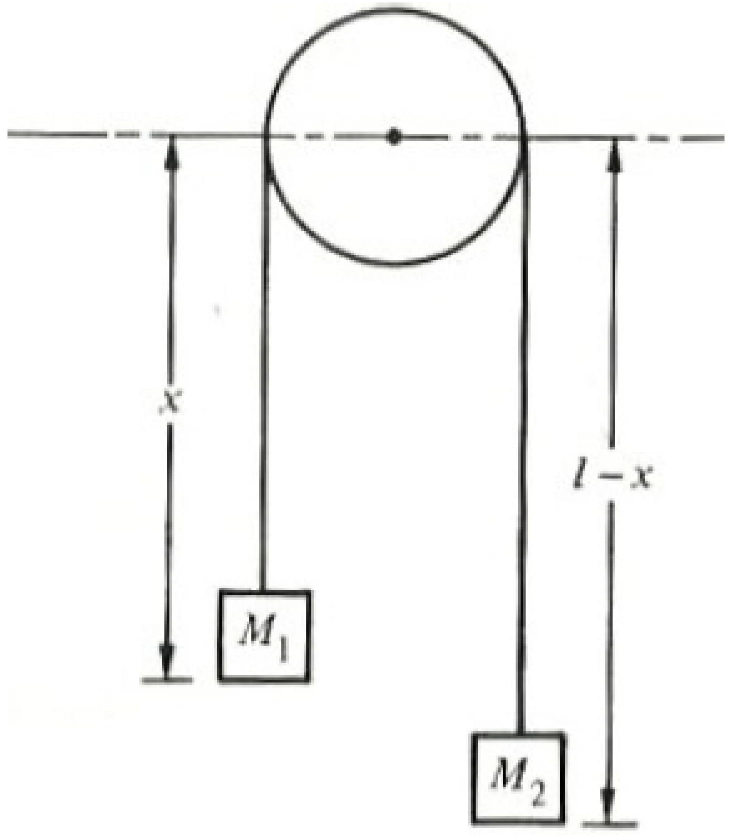


FIGURE 1.7 Atwood's machine.

- **PE:** $V = -M_1gx - M_2g(\ell - x)$
- **KE:** $T = (\tfrac{1}{2})(M_1 + M_2)(\dot{x})^2$
- **Lagrangian:** $L = T - V = (\tfrac{1}{2})(M_1+M_2)(\dot{x})^2 - M_1gx - M_2g(\ell - x)$

$$L = (½)(M_1+M_2)(\dot{x})^2 - M_1 g x - M_2 g(\ell - x)$$

- **Lagrange:** $(d/dt)[(\partial L/\partial \dot{x})] - (\partial L/\partial x) = 0$

$$(\partial L/\partial x) = (M_2 - M_1)g \; ; \; (\partial L/\partial \dot{x}) = (M_1+M_2)\dot{x}$$

$\Rightarrow$ $$(M_1+M_2)\ddot{x} = (M_2 - M_1)g$$

Or: $$\ddot{x} = [(M_2 - M_1)/(M_1+M_2)]\, g$$

Same as obtained in freshman physics!

- **Force of constraint = tension. Compute using Lagrange multiplier method (later!).**

Bead Sliding on Uniformly Rotating Wire in Free Space

- Straight wire, rotating about a fixed axis ⊥ wire, with constant angular velocity of rotation **ω**.
  - **Time dependent constraint!**
- **Generalized Coords:** Plane polar:

$\Rightarrow$ $\mathbf{x = r\cos\theta}$, $\mathbf{y = r\sin\theta}$, but $\boldsymbol{\theta = \omega t}$, $\boldsymbol{\dot{\theta} = \omega}$ = const

- Use plane polar results:

$$\mathbf{T = (\tfrac{1}{2})m[(\dot{r})^2 + (r\dot{\theta})^2] = (\tfrac{1}{2})m[(\dot{r})^2 + (r\omega)^2]}$$

- Free space $\Rightarrow$ $\mathbf{V = 0.}$ $\quad$ $\boldsymbol{L} \mathbf{= T - V = T}$

Lagrange's Eqtn: $\mathbf{(d/dt)[(\partial L/\partial \dot{r})] - (\partial L/\partial r) = 0}$

$\Rightarrow$ $\mathbf{m\ddot{r} - mr\omega^2 = 0}$ $\qquad$ $\Rightarrow$ $\mathbf{r = r_0\, e^{\omega t}}$

Bead moves exponentially outward.

- Use **(x,y)** coordinate system in figure to find **T, V**, & ***L*** for a **simple pendulum** (length ℓ , bob mass **m**), moving in **xy** plane. Write transformation eqtns from **(x,y)** system to coordinate **θ.** Find the eqtn of motion.

$$\mathbf{T = (\tfrac{1}{2})m[(x)^2 + (y)^2],\ V = mgy}$$

$$\Rightarrow \quad \boldsymbol{L} = \mathbf{(\tfrac{1}{2})m[(x)^2 + (y)^2] - mgy}$$

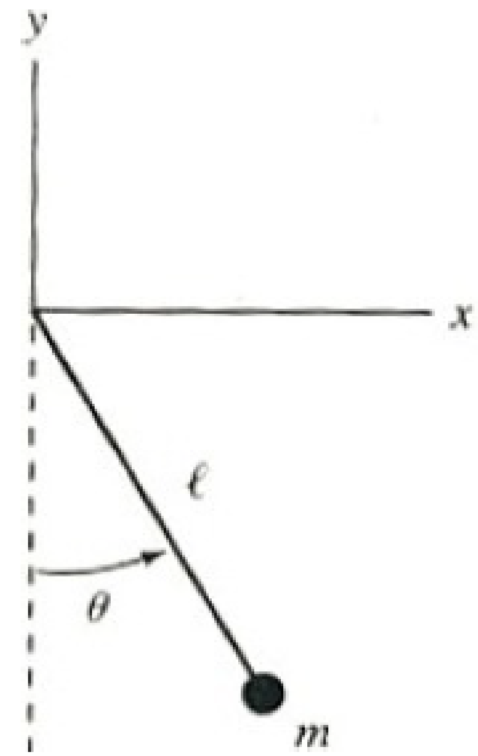
$$\mathbf{x = \ell \sin\theta,\ \ y = - \ell \cos\theta}$$

$$\mathbf{x = \ell\, \dot{\theta} \cos\theta,\ \ y = \ell\, \dot{\theta} \sin\theta}$$

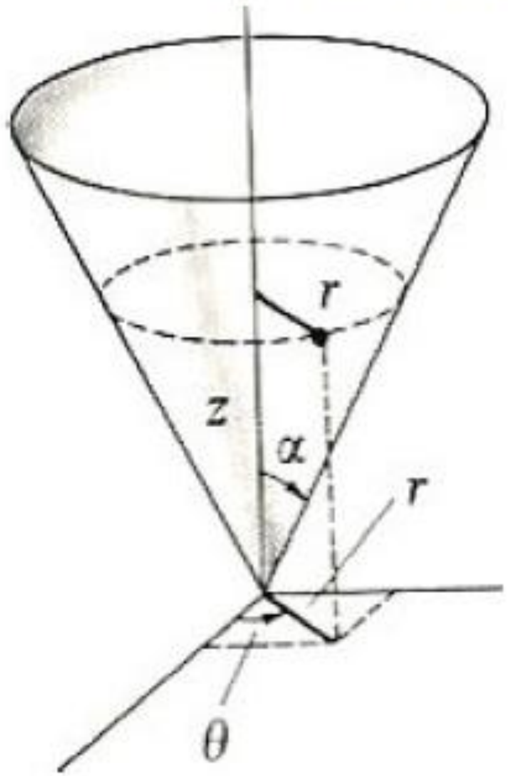
$$\boldsymbol{L} = \mathbf{(\tfrac{1}{2})m(\ell\dot{\theta})^2 + mg\, \ell \cos\theta}$$

$$\mathbf{(d/dt)[(\partial \boldsymbol{L}/\partial\dot{\theta})] - (\partial \boldsymbol{L}/\partial\theta) = 0}$$

$$\Rightarrow \quad \mathbf{\ddot{\theta} + (g/\ell) \sin\theta = 0}$$



- Particle, mass **m**, constrained to move on the inside surface of a smooth cone of half angle $\boldsymbol{\alpha}$ (Fig.). Subject to gravity. Determine a set of generalized coordinates & determine the constraints. Find the eqtns of motion.



Solution: Let the axis of the cone correspond to the z-axis and let the apex of the cone be located at the origin. Since the problem possesses cylindrical symmetry, we choose r, θ, and z as the generalized coordinates. We have, however, the equation of constraint

$$z = r\cot\alpha \tag{7.26}$$

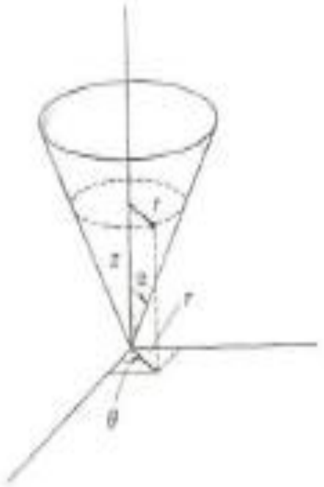


FIGURE 7-2

so there are only two degrees of freedom for the system, and therefore only two proper generalized coordinates. We may use Equation 7.26 to eliminate either the coordinate z or r; we choose to do the former. Then the square of the velocity is

$$\begin{aligned} v^2 &= \dot r^2 + r^2\dot\theta^2 + \dot z^2 \\ &= \dot r^2 + r^2\dot\theta^2 + \dot r^2\cot^2\alpha \\ &= \dot r^2\csc^2\alpha + r^2\dot\theta^2 \end{aligned} \tag{7.27}$$

The potential energy (if we choose $V = 0$ at $z = 0$) is

$$V = mgz = mgr\cot\alpha$$

so the Langrangian is

$$L = \tfrac{1}{2}m(\dot r^2\csc^2\alpha + r^2\dot\theta^2) - mgr\cot\alpha \tag{7.28}$$

We note first that L does not explicitly contain θ. Therefore $\partial L/\partial\theta = 0$, and the Lagrange equation for the coordinate θ is

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} = 0$$

Hence

$$\frac{\partial L}{\partial\dot\theta} = mr^2\dot\theta = \text{constant} \tag{7.29}$$

ut $mr^2\dot\theta = mr^2\omega$ is just the angular momentum about the z-axis. Therefore, Equa- on 7.29 expresses the conservation of angular momentum about the axis of sym- etry of the system.

The Lagrange equation for r is

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial\dot r} = 0$$

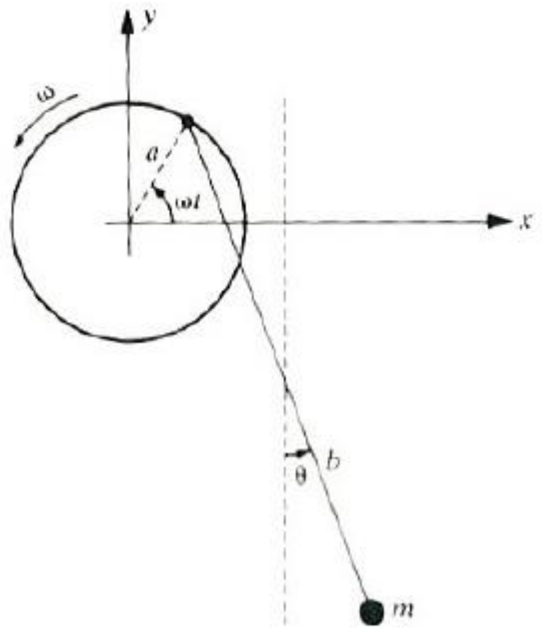
Calculating the derivatives, we find

$$\ddot r - r\dot\theta^2\sin^2\alpha + g\sin\alpha\cos\alpha$$

which is the equation of motion for the coordinate r.

We shall return to this example in Section 8.10 more detail.

- The point of support of a simple pendulum (length **b**) moves on massless rim (radius **a**) rotating with const angular velocity **ω**. Obtain expressions for the Cartesian components of velocity & acceleration of **m**. Obtain the angular acceleration for the angle **θ** shown in the figure.



Solution!

Solution: We choose the origin of our coordinate system to be at the center of the rotating rim. The Cartesian components of mass m become

$$\left.\begin{aligned} x &= a\cos\omega t + b\sin\theta \\ y &= a\sin\omega t - b\cos\theta \end{aligned}\right\} \qquad (7.32)$$

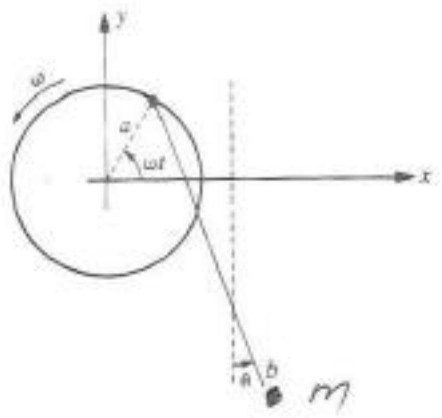
The velocities are

$$\left.\begin{aligned} \dot{x} &= -a\omega\sin\omega t + b\dot{\theta}\cos\theta \\ \dot{y} &= a\omega\cos\omega t + b\dot{\theta}\sin\theta \end{aligned}\right\} \qquad (7.33)$$

Taking the time derivative once again gives the acceleration:

$$\ddot{x} = -a\omega^2\cos\omega t + b(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)$$

$$\ddot{y} = -a\omega^2\sin\omega t + b(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)$$



It should now be clear that the single generalized coordinate is θ. The kin potential energies are

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$U = mgy$$

where $U = 0$ at $y = 0$. The Lagrangian is

$$L = T - U = \frac{m}{2}[a^2\omega^2 + b^2\dot{\theta}^2 + 2b\dot{\theta}a\omega\sin(\theta - \omega t)]$$

$$-mg(a\sin\omega t - b\cos\theta)$$

The derivatives for the Lagrange equation of motion for θ are

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = mb^2\ddot{\theta} + mba\omega(\dot{\theta} - \omega)\cos(\theta - \omega t)$$

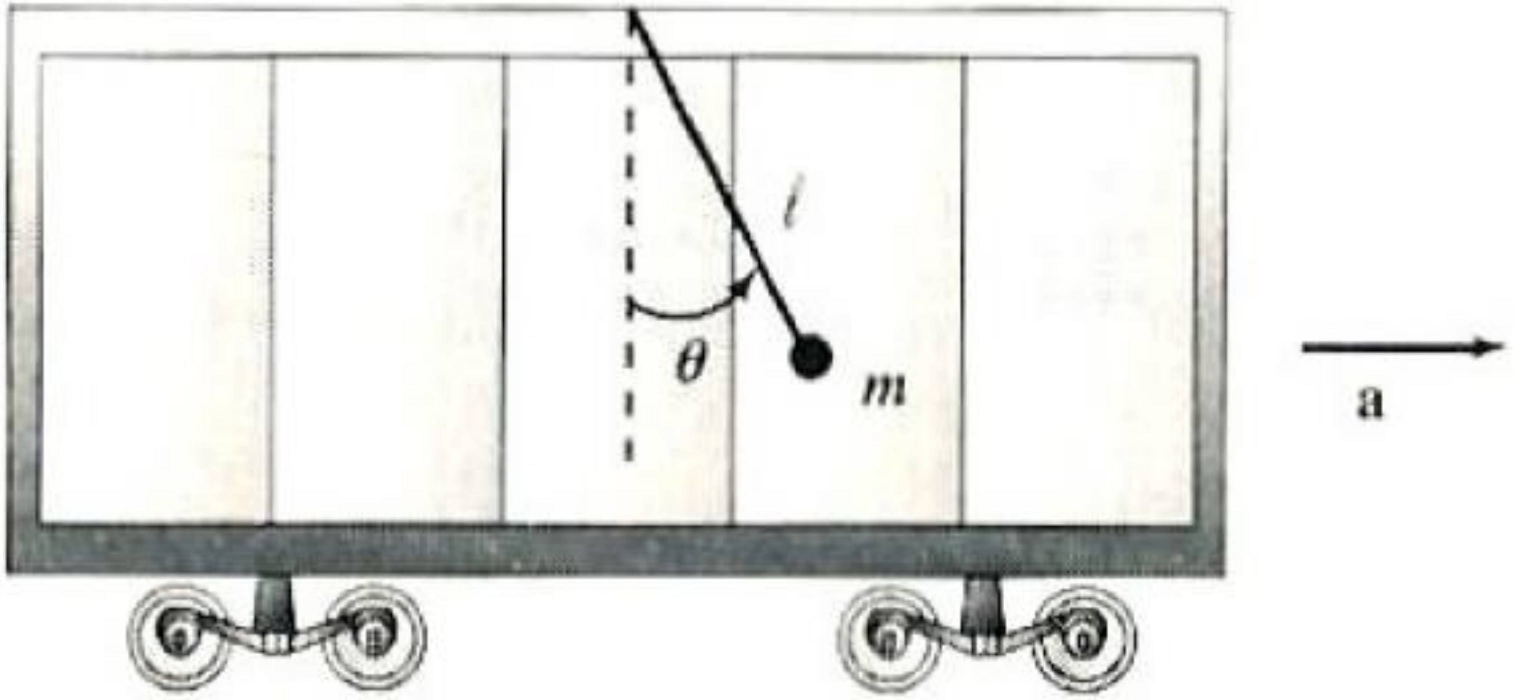
$$\frac{\partial L}{\partial\theta} = mb\dot{\theta}a\omega\cos(\theta - \omega t) - mgb\sin\theta$$

which results in the equation of motion (after solving for $\ddot{\theta}$)

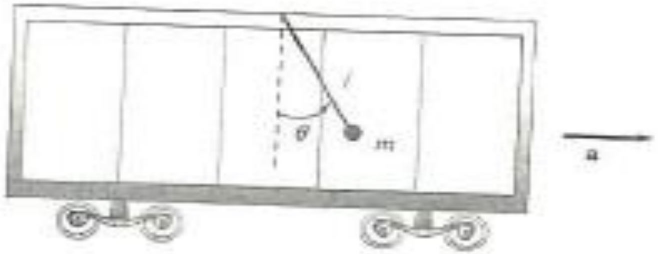
$$\ddot{\theta} = \frac{\omega^2 a}{b}\cos(\theta - \omega t) - \frac{g}{b}\sin\theta$$

Notice that this result reduces to the well-known equation of motion for pendulum if $\omega = 0$

- Find the eqtn of motion for a simple pendulum placed in a railroad car that has a const **x**-directed acceleration **a**.



Solution!



Solution: A schematic diagram is shown in Figure 7-4a for the pendulum of length ℓ, mass m, and displacement angle θ. We choose a fixed cartesian coordinate system with $x = 0$ and $\dot{x} = v_0$ at $t = 0$. The position and velocity of m become

$$x = v_0 t + \tfrac{1}{2} a t^2 + \ell \sin\theta$$

$$y = -\ell\cos\theta$$

$$\dot{x} = v_0 + at + \ell\dot{\theta}\cos\theta$$

$$\dot{y} = \ell\dot{\theta}\sin\theta$$

The kinetic and potential energies are

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) \qquad V = -mg\ell\cos\theta$$

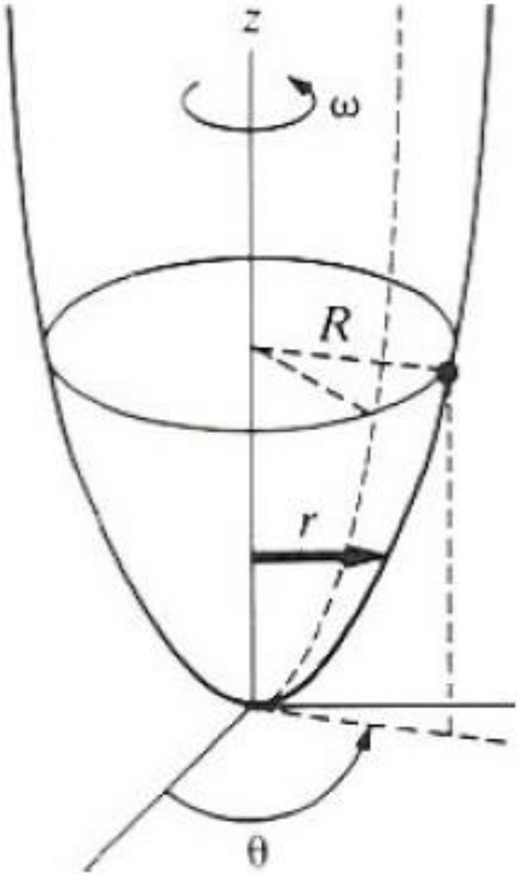
and the Lagrangian is

$$L = T - V = \tfrac{1}{2}m(v_0 + at + \ell\dot{\theta}\cos\theta)^2 + \tfrac{1}{2}m(\ell\dot{\theta}\sin\theta)^2 + mg\ell\cos\theta$$

The angle θ is the only generalized coordinate, and after taking the derivatives for Lagrange's equations and suitable collection of terms, the equation of motion becomes (Problem 7-2)

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta - \frac{a}{\ell}\cos\theta$$

- A bead slides along a smooth wire bent in the shape of a parabola, $\mathbf{z = cr^2}$ (Fig.) The bead rotates in a circle, radius **R,** when the wire is rotating about its vertical symmetry axis with angular velocity $\boldsymbol{\omega}$. Find the constant **c**.



Solution!

Solution: Because the problem has cylindrical symmetry, we choose r, θ, and z as the generalized coordinates. The kinetic energy of the bead is

$$T = \frac{m}{2}[\dot{r}^2 + \dot{z}^2 + (r\dot{\theta})^2] \qquad (7.43)$$

If we choose $U = 0$ at $z = 0$, the potential energy term is

$$U = mgz$$

But r, z, and θ are not independent. The equation of constraint for the par...

$$z = cr^2 \qquad (7.45)$$

$$\dot{z} = 2c\dot{r}r \qquad (7.46)$$

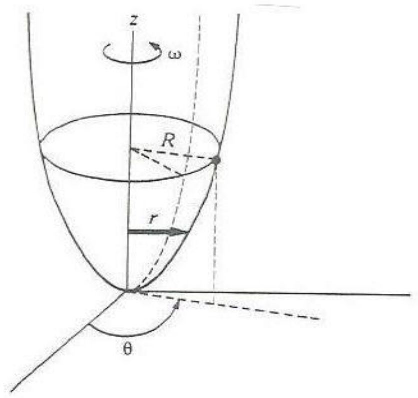


FIGURE 7-5

...ve an explicit time dependence of the angular rotation

$$\theta = \omega t$$

$$\dot{\theta} = \omega \qquad (7.47)$$

...v construct the Lagrangian as being dependent only on r, because there ... θ dependence.

$$L = T - U$$

$$= \frac{m}{2}(\dot{r}^2 + 4c^2r^2\dot{r}^2 + r^2\omega^2) - mgcr^2 \qquad (7.48)$$

...v construct the Lagrangian as being dependent only on r, because there ... θ dependence.

$$L = T - U$$

$$= \frac{m}{2}(\dot{r}^2 + 4c^2r^2\dot{r}^2 + r^2\omega^2) - mgcr^2 \qquad (7.48)$$

...n stated that the bead moved in a circle of radius R. The reader might ... at this point to let $r = R =$ const. and $\dot{r} = 0$. It would be a mistake ...ow in the Lagrangian. First, we should find the equation of motion for ... r and then let $r = R$ as a condition of the particular motion. This ... the particular value of c needed for $r = R$.

$$\frac{\partial L}{\partial \dot{r}} = \frac{m}{2}(2\dot{r} + 8c^2r^2\dot{r})$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{m}{2}(2\ddot{r} + 16c^2r\dot{r}^2 + 8c^2r^2\ddot{r})$$

$$\frac{\partial L}{\partial r} = m(4c^2r\dot{r}^2 + r\omega^2 - 2gcr)$$

...equation of motion becomes

$$\ddot{r}(1 + 4c^2r^2) + \dot{r}^2(4c^2r) + r(2gc - \omega^2) = 0$$

which is a complicated result. If, however, the bead rotates with ... then $\dot{r} = \ddot{r} = 0$, and Equation 7.49 becomes

$$R(2gc - \omega^2) = 0$$

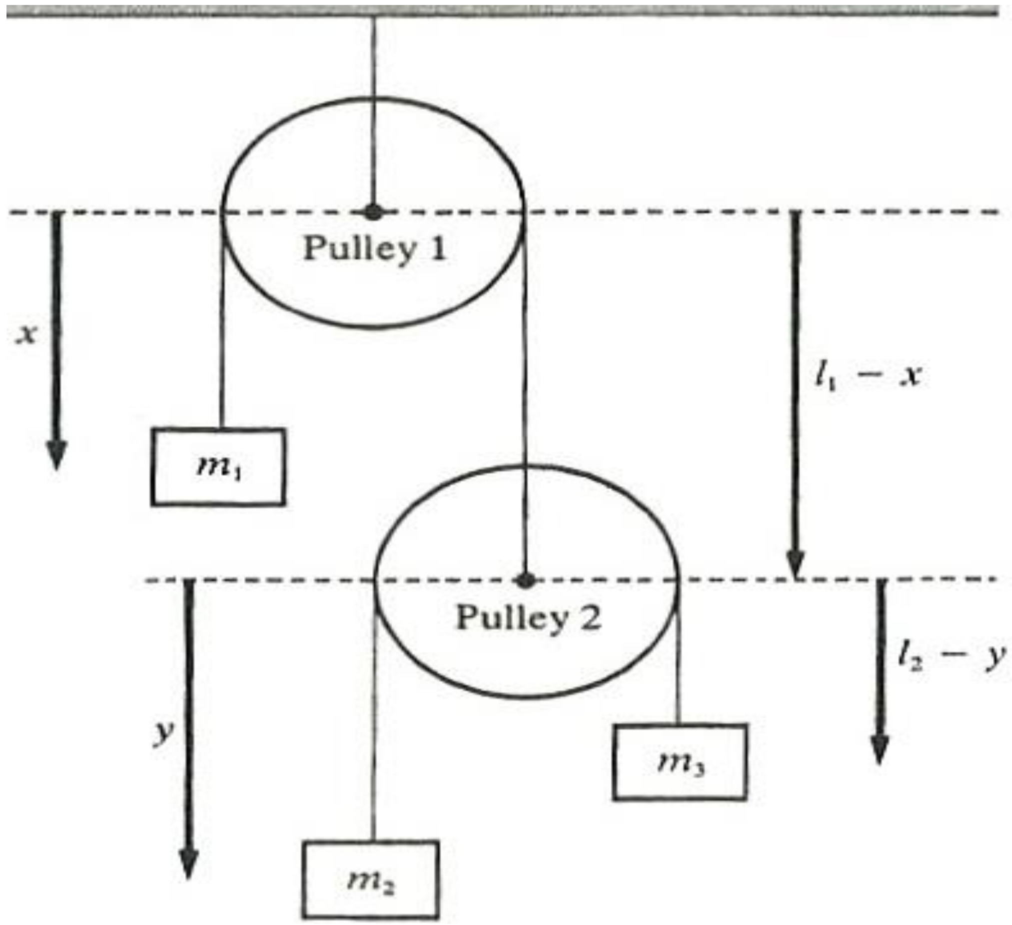
and

$$c = \frac{\omega^2}{2g}$$

is the result we wanted.

(7.49)

› Consider the double pulley system shown. Use the coordinates indicated & determine the eqtns of motion.

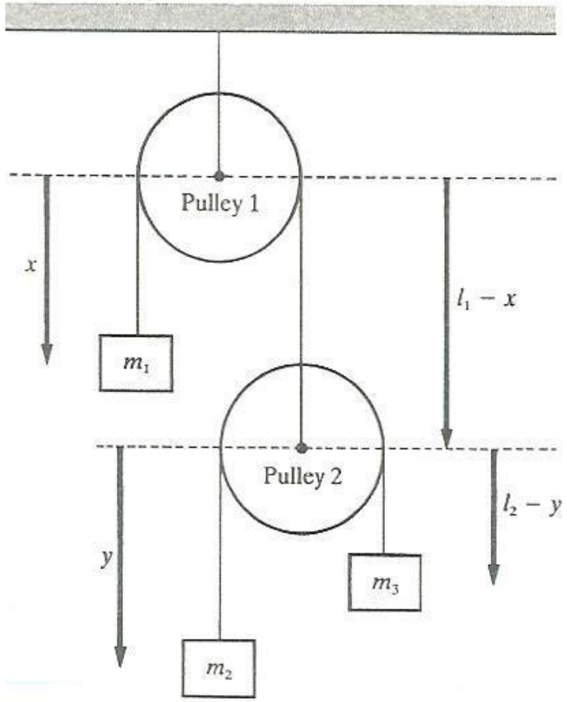


Solution!

Solution: Consider the pulleys to be massless, and let l_1 and l_2 be the lengths of rope hanging freely from each of the two pulleys. The distances x and y are measured from the center of the two pulleys.

m_1:

$$v_1 = \dot{x} \tag{7.51}$$



m_2:

$$v_2 = \frac{d}{dt}(l_1 - x + y) = -\dot{x} + \dot{y} \tag{7.52}$$

m_3:

$$v_3 = \frac{d}{dt}(l_1 - x + l_2 - y) = -\dot{x} - \dot{y} \tag{7.53}$$

$$T = \tfrac{1}{2}m_1 v_1^2 + \tfrac{1}{2}m_2 v_2^2 + \tfrac{1}{2}m_3 v_3^2$$

$$= \tfrac{1}{2}m_1\dot{x}^2 + \tfrac{1}{2}m_2(\dot{y} - \dot{x})^2 + \tfrac{1}{2}m_3(-\dot{x} - \dot{y})^2 \tag{7.54}$$

Let the potential energy $U = 0$ at $x = 0$.

$$U = U_1 + U_2 + U_3$$

$$= -m_1 g x - m_2 g(l_1 - x + y) - m_3 g(l_1 - x + l_2 - y) \tag{7.55}$$

Because T and U have been determined, the equations of motion can be obtained using Equation 7.18. The results are

$$m_1\ddot{x} + m_2(\ddot{x} - \ddot{y}) + m_3(\ddot{x} + \ddot{y}) = (m_1 - m_2 - m_3)g \tag{7.56}$$

$$-m_2(\ddot{x} - \ddot{y}) + m_3(\ddot{x} + \ddot{y}) = (m_2 - m_3)g \tag{7.57}$$

Equations 7.56 and 7.57 can be solved for $\ddot{x}$ and $\ddot{y}$.

Hamilton's Principle

- Our derivation of Lagrange's Eqtns from D'Alembert's Principle: Used Virtual Work - *A Differential Principle.* (A *LOCAL* principle).
- Here: An alternate derivation from **Hamilton's Principle**: *An Integral (or Variational) Principle* (A *GLOBAL* principle). More general than D'Alembert's Principle.
  - Based on techniques from the Calculus of Variations.
  - Brief discussion of derivation & of Calculus of Variations. More details: See the text!

- **System**: **n** generalized coordinates $\mathbf{q_1,q_2,q_3,..q_n}$.
  - At time $\mathbf{t_1}$: These all have some value.
  - At a later time $\mathbf{t_2}$: They have changed according to the eqtns of motion & all have some other value.
- **System Configuration:** A point in **n**-dimensional space ("*Configuration Space*"), with $\mathbf{q_i}$ as **n** coordinate "axes".
  - At time $\mathbf{t_1}$: Configuration of System is represented by a point in this space.
  - At a later time $\mathbf{t_2}$: Configuration of System has changed & that point has moved (according to eqtns of motion) in this space.
  - Time dependence of System Configuration: The point representing this in Configuration Space traces out a path.

- **Monogenic Systems** ≡ All Generalized Forces (except constraint forces) are derivable from a **Generalized Scalar Potential** that ***may*** be a function of generalized coordinates, generalized velocities, & time:

$$\mathbf{U(q_i,\dot{q}_i,t):\ \ Q_i \equiv -(\partial U/\partial q_i) + (d/dt)[(\partial U/\partial \dot{q}_i)]}$$

  – If **U** depends only on $\mathbf{q_i}$ (& not on $\mathbf{\dot{q}_i}$ & **t**), **U** = **V** & the system is conservative.

- Monogenic systems, **Hamilton's Principle**:

***The motion of the system** (in configuration space)* ***from time t_1 to time t_2 is such that the line integral** (the action or action integral)*

$$\mathbf{I} = \int L\, \mathbf{dt} \qquad (\text{limits } \mathbf{t_1} < \mathbf{t} < \mathbf{t_2})$$

has a stationary value for the actual path of motion.

$$L \equiv \mathbf{T} - \mathbf{V} = \text{Lagrangian of the system}$$

$$L = \mathbf{T} - \mathbf{U}, \ (\text{if the potential depends on } \dot{\mathbf{q}}_i \ \& \ \mathbf{t})$$

Hamilton's Principle (HP)

$$\mathbf{I} = \int L\, dt \qquad (\text{limits } t_1 < t < t_2, L = T - V)$$

- **Stationary value** $\Rightarrow$ **I** is an extremum (maximum or minimum, ***almost always*** a minimum).
- In other words: Out of all possible paths by which the system point could travel in configuration space from $\mathbf{t_1}$ to $\mathbf{t_2}$, it will ***ACTUALLY*** travel along path for which **I** is an extremum (usually a minimum).

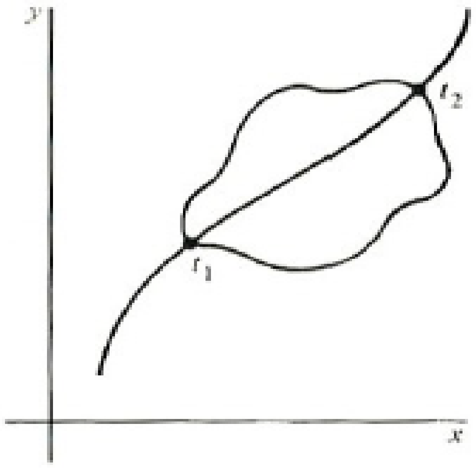


FIGURE 2.1 Path of the system point in configuration space.

$$\mathbf{I} = \int L\, \mathbf{dt} \qquad \text{(limits } \mathbf{t_1 < t < t_2},\ L = \mathbf{T - V}\text{)}$$

- In the terminology & notation from the **calculus of variations**: **HP** $\Rightarrow$ the motion is such that ***the variation of*** **I** (fixed $\mathbf{t_1}$ & $\mathbf{t_2}$) ***is zero***:

$$\delta\int L\, \mathbf{dt} = \mathbf{0} \qquad \text{(limits } \mathbf{t_1 < t < t_2}\text{)} \quad \mathbf{(1)}$$

$\boldsymbol{\delta} \equiv$ Arbitrary variation (calculus of variations).

$\boldsymbol{\delta}$ plays a role in the calculus of variations that the derivative plays in calculus.

- **Holonomic constraints** $\Rightarrow$ **(1)** is both a necessary & a sufficient condition for Lagrange's Eqtns.
  - That is, we can derive **(1)** from Lagrange's Eqtns.
  - However this text & (most texts) do it the other way around & derive Lagrange's Eqtns from **(1)**.
  - Advantage: ***Valid in any system of generalized coords.!!***

- History, philosophy, & general discussion, which is worth briefly mentioning (not in Goldstein!).
- Historically, to overcome some practical difficulties of Newton's mechanics (e.g. needing all forces & not knowing the forces of constraint)

$\Rightarrow$ Alternate procedures were developed

Hamilton's Principle

$\Rightarrow$ ***Lagrangian Dynamics***

$\Rightarrow$ ***Hamiltonian Dynamics***

$\Rightarrow$ **Also Others!**

- **<u>All</u>** such procedures obtain results *100% equivalent* to *Newton's 2nd Law:* **F = dp/dt**

⇒ *Alternate procedures are NOT new theories!*
But **reformulations of Newtonian Mechanics** in different math language.

- **Hamilton's Principle (HP):** Applicable outside particle mechanics! For example to fields in E&M.

- **HP:** Based on experiment!

- **HP: Philosophical Discussion**

HP: ⇒ No new physical theories, just new formulations of old theories

HP: Can be used to ***unify*** several theories: Mechanics, E&M, Optics, …

HP: ***Very elegant & far reaching!***

HP: "More fundamental" than Newton's Laws!

HP: Given as a (single, simple) postulate.

HP & Lagrange Eqtns apply (as we've seen) to non-conservative systems.

- **HP:** One of many **"Minimal" Principles:**

  (Or variational principles)

  – Assume Nature always minimizes certain quantities when a physical process takes place

  – Common in the history of physics

- **History:** List of (some) other minimal principles:

  – **Hero, 200 BC:** Optics: ***Hero's Principle of Least Distance:*** A light ray traveling from one point to another by reflection from a plane mirror, always takes shortest path. By geometric construction:

$\Rightarrow$ **Law of Reflection.** $\boldsymbol{\theta_i = \theta_r}$

Says nothing about the Law of Refraction!

- **"Minimal" Principles:**
  - **Fermat, 1657:** Optics: ***Fermat's Principle of Least Time:*** A light ray travels in a medium from one point to another by a path that takes the least time.

    $\Rightarrow$ **Law of Reflection:** $\theta_i = \theta_r$

    $\Rightarrow$ **Law of Refraction:** "Snell's Law"
  - **Maupertuis, 1747:** Mechanics: ***Maupertuis's Principle of Least Action:*** Dynamical motion takes place with minimum action:
    - **Action** $\equiv$ (Distance) $\times$ (Momentum) = (Energy)$\times$ (Time)
    - Based on ***Theological*** Grounds!!! (???)
    - Lagrange: Put on firm math foundation.
    - Principle of Least Action $\Rightarrow$ ***HP***

Hamilton's Principle

(As originally stated 1834-35)

– Of all possible paths along which a dynamical system may move from one point to another, in a given time interval (consistent with the constraints), the *actual path* followed is one which minimizes the time integral of the difference in the KE & the PE. That is, the one which makes the variation of the following integral vanish:

$$\delta\int[\mathbf{T - V}]\mathbf{dt} = \delta\int \mathit{L}\mathbf{dt} = \mathbf{0} \qquad (\text{limits } \mathbf{t_1} < \mathbf{t} < \mathbf{t_2})$$

- Consider the following problem in the **xy** plane:

The Basic Calculus of Variations Problem:

Determine the function **y(x)** for which the integral

$$\mathbf{J \equiv \int f[y(x),y'(x);x]dx} \quad \text{(fixed limits } \mathbf{x_1 < x < x_2}\text{)}$$

is an ***extremum*** (max or min)

$\mathbf{y'(x) \equiv dy/dx}$ (Note: The text calls this $\mathbf{\dot{y}(x)}$!)

- Semicolon in **f** separates independent variable **x** from dependent variable **y(x)** & its derivative **y′(x)**
- **f** ≡ **A GIVEN *functional*.** **Functional** ≡ Quantity **f[y(x),y′(x);x]** which depends on the ***functional form*** of the dependent variable y(**x**). "A function of a function".

- ***Basic problem restated:*** Given $\mathbf{f[y(x),y'(x);x]}$, find (for fixed $\mathbf{x_1}$, $\mathbf{x_2}$) the function(s) $\mathbf{y(x)}$ which minimize (or maximize) $\mathbf{J \equiv \int f[y(x),y'(x);x]dx}$ (limits $x_1 < x < x_2$)

$\Rightarrow$ Vary $\mathbf{y(x)}$ until an extremum (max or min; ***usually min!***) of $\mathbf{J}$ is found. Stated another way, vary $\mathbf{y(x)}$ so that the variation of $\mathbf{J}$ is zero or

$$\delta J = \delta\int f[y(x),y'(x);x]dx = 0$$

Suppose the function $\mathbf{y = y(x)}$ gives $\mathbf{J}$ a min value:

$\Rightarrow$ Every "***neighboring function***", no matter how close to $\mathbf{y(x)}$, must make $\mathbf{J}$ increase!

› *Solution to basic problem :* The text proves that to minimize (or maximize)

$$J \equiv \int f[y(x),y'(x);x]dx \qquad (\text{limits } x_1 < x < x_2)$$

or $$\delta J = \delta\int f[y(x),y'(x);x]dx = 0$$

$\Rightarrow$ The functional f must satisfy:

$$(\partial f/\partial y) - (d[\partial f/\partial y']/dx) = 0$$

$\equiv$ *Euler's Equation*

- Euler, 1744. Applied to mechanics

  $\equiv$ *Euler - Lagrange Equation*

- Various pure math applications,
- Read on your own!

- 1^{st}, extension of calculus of variations results to **Functions with Several Dependent Variables**
- Derived **Euler Eqtn** = Solution to problem of finding path such that $\mathbf{J = \int f\, dx}$ is an extremum or $\mathbf{\delta J = 0}$. Assumed one dependent variable $\mathbf{y(x)}$.
- In mechanics, we often have problems with many dependent variables: $\mathbf{y_1(x), y_2(x), y_3(x), \ldots}$
- In general, have a functional like:

$$\mathbf{f = f[y_1(x), y_1'(x), y_2(x), y_2'(x), \ldots; x]}$$

$$\mathbf{y_i'(x) \equiv dy_i(x)/dx}$$

- *Abbreviate* as $\mathbf{f = f[y_i(x), y_i'(x); x]}$, $\mathbf{i = 1,2, \ldots, n}$

- Functional: $f = f[y_i(x),y_i'(x);x]$, $i = 1,2, \ldots,n$
- **Calculus of variations problem:** Simultaneously find the "**n** paths" $y_i(x)$, $i = 1,2, \ldots,n$, which minimize (or maximize) the integral:

$$J \equiv \int f[y_i(x),y_i'(x);x]dx$$

$(i = 1,2, \ldots,n$, fixed limits $x_1 < x < x_2)$

Or for which $\delta J = 0$

- Follow the derivation for one independent variable & get:

$$(\partial f/\partial y_i) - (d[\partial f/\partial y_i']dx) = 0 \qquad (i = 1,2, \ldots,n)$$

$\equiv$ ***Euler's Equations***

(Several dependent variables)

- **Summary:** Forcing $\mathbf{J} \equiv \int \mathbf{f[y_i(x),y_i'(x);x]dx}$

  $(\mathbf{i = 1,2, \ldots,n}$, fixed limits $\mathbf{x_1 < x < x_2})$

  To have an extremum (or forcing

  $\mathbf{\delta J = \delta\int f[y_i(x),y_i'(x);x]dx = 0}$) requires **f** to satisfy:

  $$(\partial f/\partial y_i) - (d[\partial f/\partial y_i']dx) = 0 \qquad (i = 1,2, \ldots,n)$$

  $\equiv$ ***Euler's Equations***

- **HP** $\Rightarrow$ The system motion is such that $\mathbf{I} = \int \boldsymbol{L}\ \mathbf{dt}$ is an extremum (fixed $\mathbf{t_1}$ & $\mathbf{t_2}$)

  $\Rightarrow$ The variation of this integral **I** is zero:

  $$\delta\int L\, dt = 0 \qquad (\text{limits } t_1 < t < t_2)$$

- **HP $\Rightarrow$ Identical to abstract calculus of variations problem of with replacements:**

$$J \rightarrow \int L\, dt; \quad \delta J \rightarrow \delta \int L\, dt$$

$$x \rightarrow t\ ;\ y_i(x) \rightarrow q_i(t)$$

$$y_i'(x) \rightarrow dq_i(t)/dt = q_i(t)$$

$$f[y_i(x), y_i'(x); x] \rightarrow L(q_i, q_i; t)$$

$\Rightarrow$ **The Lagrangian L satisfies Euler's eqtns with these replacements!**

$\Rightarrow$ Combining **HP** with Euler's eqtns gives:

$$(d/dt)[(\partial L/\partial \dot{q}_j)] - (\partial L/\partial q_j) = 0 \quad (j = 1,2,3, \ldots n)$$

- **Summary: HP gives *Lagrange's Eqtns:***

$$(d/dt)[(\partial L/\partial \dot{q}_j)] - (\partial L/\partial q_j) = 0$$

(j = 1,2,3, … n)

- Stated another way, ***Lagrange's Eqtns ARE Euler's eqtns*** in the special case where the abstract functional **f** is the Lagrangian **L!**

⇒ They are sometimes called the

Euler-Lagrange Eqtns.

HP for Non-Holonomic Systems

- Can formally *extend* HP to include some types of **non-holonomic systems**.
  - Derivation of Lagrange's Eqtns: Holonomic constraint requirement does not appear until last step. (When δq_i are considered independent).

- Holonomic constraints: **m** constraint eqtns:

$$f_\alpha(q_1,q_2,\ldots q_n,t) = 0 \qquad (\alpha = 1,..m)$$

- Extension of **HP** to "**<u>semi-holonomic</u>**" systems for which constraint eqtns can be written:

$$\mathbf{f_\alpha(q_1,q_2,\ldots q_n;\dot{q}_1,\dot{q}_2,\ldots,\dot{q}_n) = 0} \qquad (\alpha = 1,..m)$$

  – Alternately, differential form:

$$\mathbf{\sum_k a_{\alpha k} dq_k + a_{\alpha t} dt = 0}$$

  – Latter version is non-holonomic ***unless it can be integrated*** (unless it is an exact differential, in which case it is holonomic):

$$\mathbf{a_{\alpha k} = (\partial f_\alpha/\partial q_k) \qquad a_{\alpha t} = (\partial f_\alpha/\partial t)}$$

- Consider "**semi-holonomic**" systems with constraints:

$$\mathbf{f}_\alpha(\mathbf{q}_1,\mathbf{q}_2,\ldots\mathbf{q}_n;\dot{\mathbf{q}}_1,\dot{\mathbf{q}}_2,\ldots\dot{\mathbf{q}}_n) = 0 \qquad (\alpha = 1,..m) \qquad (1)$$

- Multiply (1) by some **unknown function**

$$\lambda_\alpha = \lambda_\alpha(\mathbf{q}_1,\mathbf{q}_2,\ldots\mathbf{q}_n;\mathbf{t}), \;\; \text{\& sum over all } \alpha$$

$$\Rightarrow \qquad \textstyle\sum_\alpha \lambda_\alpha \mathbf{f}_\alpha = 0 \qquad (2)$$

$\lambda_\alpha \equiv$ ***Lagrange's undetermined multipliers***

- **HP** $\Rightarrow$ $\delta\int L\,\mathbf{dt} = 0$ (limits $\mathbf{t}_1 < \mathbf{t} < \mathbf{t}_2$)

Combining with (2) gives:

$$\delta\int(L + \textstyle\sum_\alpha \lambda_\alpha \mathbf{f}_\alpha) = 0 \qquad (3)$$

(limits $\mathbf{t}_1 < \mathbf{t} < \mathbf{t}_2$)

- **HP** $\Rightarrow \delta\int(L + \sum_{\alpha}\lambda_{\alpha}f_{\alpha}) = 0$ (limits $t_1 < t < t_2$) **(3)**

  $\lambda_{\alpha} \equiv$ ***Lagrange's undetermined multipliers***

- Follow the derivation from before of Lagrange's Eqtns from **HP** & get:

$$(d/dt)[(\partial L/\partial \dot{q}_k)] - (\partial L/\partial q_k) = Q_k \quad (4)$$

$$(k = 1,2,3,\ \ldots\ n)$$

With generalized constraint force Q_k is given as:

$$Q_k = \sum_{\alpha}\{\lambda_{\alpha}[(\partial f_{\alpha}/\partial q_k) - (d/dt)(\partial f_{\alpha}/\partial \dot{q}_k)] - (d\lambda_{\alpha}/dt)(\partial f_{\alpha}/\partial \dot{q}_k)\} \quad (5)$$

(4) & (5): $\equiv$ **Lagrange's Eqtns *with undetermined multipliers***

- ***Recipe for Lagrange formalism with semi-holonomic constraints:***

1. For each constraint, introduce a multiplier $\lambda_\alpha = \lambda_\alpha(t)$

2. Apply Eqtns **(4)** & **(5)**:

$$(d/dt)[(\partial L/\partial \dot{q}_k)] - (\partial L/\partial q_k) = Q_k \qquad (4)$$

$$(j = 1,2,3, \ldots n)$$

$$Q_k = \sum_\alpha \{\lambda_\alpha[(\partial f_\alpha/\partial q_k) - (d/dt)(\partial f_\alpha/\partial \dot{q}_k)]$$

$$- (d\lambda_\alpha/dt)(\partial f_\alpha/\partial \dot{q}_k)\} \qquad (5)$$

3. Combine with **m** constraint eqtns: $f_\alpha(q_k;\dot{q}_k) = 0$

Results ⇒ **Eqtns of motion** for each q_k **PLUS** eqtns which give the unknown multipliers λ_α

Physical Interpretation of λ_α

- $Q_k = \sum_\alpha\{\lambda_\alpha[(\partial f_\alpha/\partial q_k) - (d/dt)(\partial f_\alpha/\partial \dot{q}_k)]$
$$- (d\lambda_\alpha/dt)(\partial f_\alpha/\partial \dot{q}_k)\} \qquad (5)$$
- See text, $Q_k \equiv$ ***Generalized forces of constraint.***

$\Rightarrow$ From this formalism get eqtns of motion for q_k
PLUS generalized constraint forces.

- Holonomic constraint eqtn of the form

$$f_\alpha(q_1,q_2,\ldots q_n) = 0 \quad (\alpha = 1,..m)$$

is a special case of a semi-holonomic constraint eqtn:

$$f_\alpha(q_1,q_2,\ldots q_n;q_1,q_2,\ldots q_n) = 0 \qquad (\alpha = 1,..m)$$

$\Rightarrow$ Can *also* use this formalism with holonomic constraints. **Useful if want to find constraint forces.**

Example 1

- **Particle with Lagrangian:**

$$L = (½)m[(\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2] - V(x,y,z)$$

Subject to constraint: $f(\dot{x},\dot{y},y) = \dot{x}\,\dot{y} + ky = 0 \qquad (1)$

- Eqtns of motion:

$$m\ddot{x} + \lambda\ddot{y} + \dot{\lambda}\dot{y} + (\partial V/\partial x) = 0 \qquad (2)$$

$$m\ddot{y} + \lambda\ddot{x} - k\lambda + \dot{\lambda}\dot{x} + (\partial V/\partial y) = 0 \qquad (3)$$

$$mz + (\partial V/\partial z) = 0 \qquad (4)$$

(1), (2), (3), (4): 4 coupled (differential) eqtns, 4 unknowns: **x(t), y(t), z(t), λ(t)**

- **(1), (2), (3), (4): 4 coupled (differential) eqtns**, 4 unknowns: **$x(t)$, $y(t)$, $z(t)$, $\lambda(t)$**

- Once solved (very messy math!), compute **generalized constraint forces:**

$$Q_k = \lambda[(\partial f/\partial q_k) - (d/dt)(\partial f/\partial \dot{q}_k)] - (d\lambda/dt)(\partial f/\partial \dot{q}_k)$$

- Constraint eqtn: $f(\dot{x},\dot{y},y) = \dot{x}\,\dot{y} + ky = 0$

$$\Rightarrow\ Q_x = -\lambda\ddot{y} - \dot{\lambda}\dot{y};\ Q_y = \lambda[k - \ddot{x}] - \dot{\lambda}\dot{x};\ Q_z = 0$$

Example 2

- **Hoop rolling down inclined plane** (fig.). Find eqtns of motion, force of constraint, angular acceleration

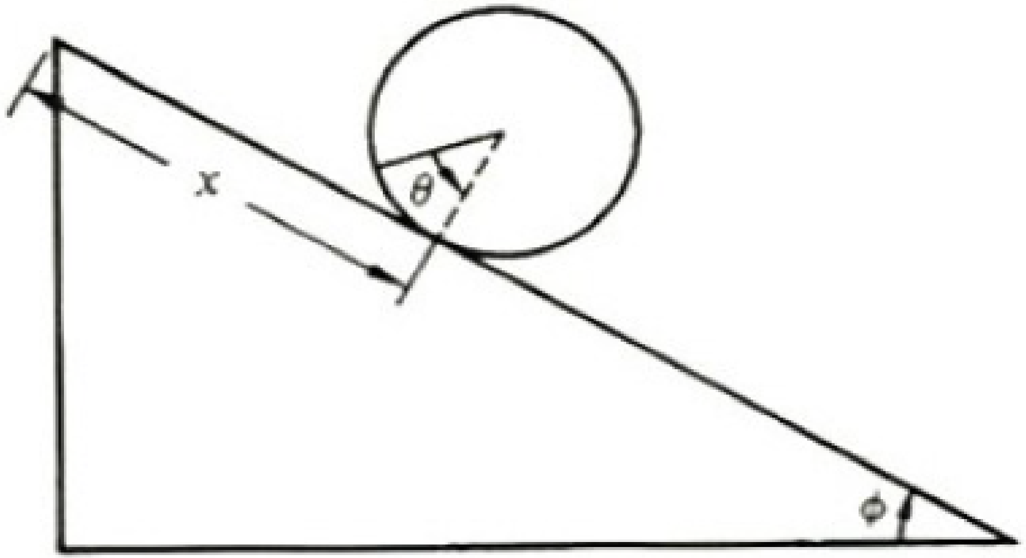


FIGURE 2.5 A hoop rolling down an inclined plane.

- Radius **r**. Mass **M**. Plane length ℓ. Generalized coords: **x, θ**. Rolling constraint eqtn: **f(θ,x) = rθ - x = 0**. Differential form: **rdθ - dx =0**

- Constraint: $r d\theta - dx = 0$
  - Or, $r\theta - x = f(\theta, x) = 0$

  Note: $(\partial f/\partial\theta) = r$; $(\partial f/\partial\theta) = -1$

  $(\partial f/\partial\dot{\theta}) = 0$; $(\partial f/\partial\dot{x}) = 0$

  $$T = (½)M(\dot{x})^2 + (½)Mr^2(\dot{\theta})^2$$

  Moment of inertia of hoop = $I = Mr^2$

  $$V = Mg(\ell - x)\sin\phi$$

  $$\Rightarrow L = T - V = (½)M(\dot{x})^2 + (½)Mr^2(\dot{\theta})^2 - Mg(\ell - x)\sin\phi$$

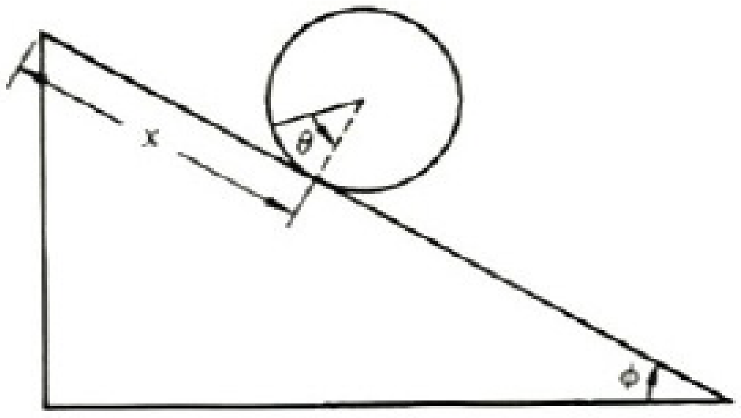


FIGURE 2.5 A hoop rolling down an inclined plane.

- **One Lagrange multiplier λ**
- Lagrange's Eqtns: $(k = r, \theta)$

  $$(d/dt)[(\partial L/\partial\dot{q}_k)] - (\partial L/\partial q_k) = Q_k$$

  $$Q_k = \lambda[(\partial f/\partial q_k) - (d/dt)(\partial f/\partial\dot{q}_k)] - (d\lambda/dt)(\partial f/\partial\dot{q}_k)$$

- $r\theta - x = 0 = f(\theta,x)$ (1)
  - $(\partial f/\partial\theta) = r;\ (\partial f/\partial\dot{\theta}) = 0,\ (\partial f/\partial x) = -1,\ (\partial f/\partial\dot{x}) = 0$

$\Rightarrow L = T - V = (½)M(\dot{x})^2 + (½)Mr^2(\dot{\theta})^2 - Mg(\ell - x)\sin\phi$

- $(d/dt)[(\partial L/\partial\dot{x})] - (\partial L/\partial x) = Q_x$

  $Q_x = \lambda[(\partial f/\partial x) - (d/dt)(\partial f/\partial\dot{x})] - (d\lambda/dt)(\partial f/\partial\dot{x})$

$\Rightarrow$ $M\ddot{x} - Mg\sin\phi + \lambda = 0$ (2)

- $(d/dt)[(\partial L/\partial\dot{\theta})] - (\partial L/\partial\theta) = Q_\theta$

  $Q_\theta = \lambda[(\partial f/\partial\theta) - (d/dt)(\partial f/\partial\dot{\theta})] - (d\lambda/dt)(\partial f/\partial\dot{\theta})$

$\Rightarrow$ $Mr^2\ddot{\theta} - \lambda r = 0$ (3)

- **(1), (2), (3)** together: **3 eqtns, 3 unknowns**:

  x(t), θ(t), λ(t)

- $r\theta - x = 0$ (1)
  - $(\partial f/\partial\theta) = r$; $(\partial f/\partial\dot{\theta}) = 0$, $(\partial f/\partial x) = -1$, $(\partial f/\partial\dot{x}) = 0$

$\Rightarrow$ $M\ddot{x} - Mg\sin\phi + \lambda = 0$ (2)

$\Rightarrow$ $Mr^2\ddot{\theta} - \lambda r = 0$ (3)

- (1), (2), (3): 3 eqtns, 3 unknowns: $x(t)$, $\theta(t)$, $\lambda(t)$.

$\Rightarrow$ **Linear acceleration** down the plane: $\ddot{x} = (\frac{1}{2})g\sin\phi$

$\Rightarrow$ **Angular acceleration** of hoop: $\ddot{\theta} = (\frac{1}{2})(g\sin\phi)/r = (\ddot{x}/r)$

$\Rightarrow$ **Lagrange multiplier:** $\lambda = (\frac{1}{2})Mg\sin\phi =$

"**Normal force**" between hoop & plane

Lagrange multiplier: $\lambda = (\frac{1}{2})Mg\sin\phi$

- **Generalized forces of constraint:**

$$Q_x = \lambda[(\partial f/\partial x) - (d/dt)(\partial f/\partial \dot{x})] - (d\lambda/dt)(\partial f/\partial \dot{x})$$

$$= -\lambda = -(\tfrac{1}{2})Mg\sin\phi$$

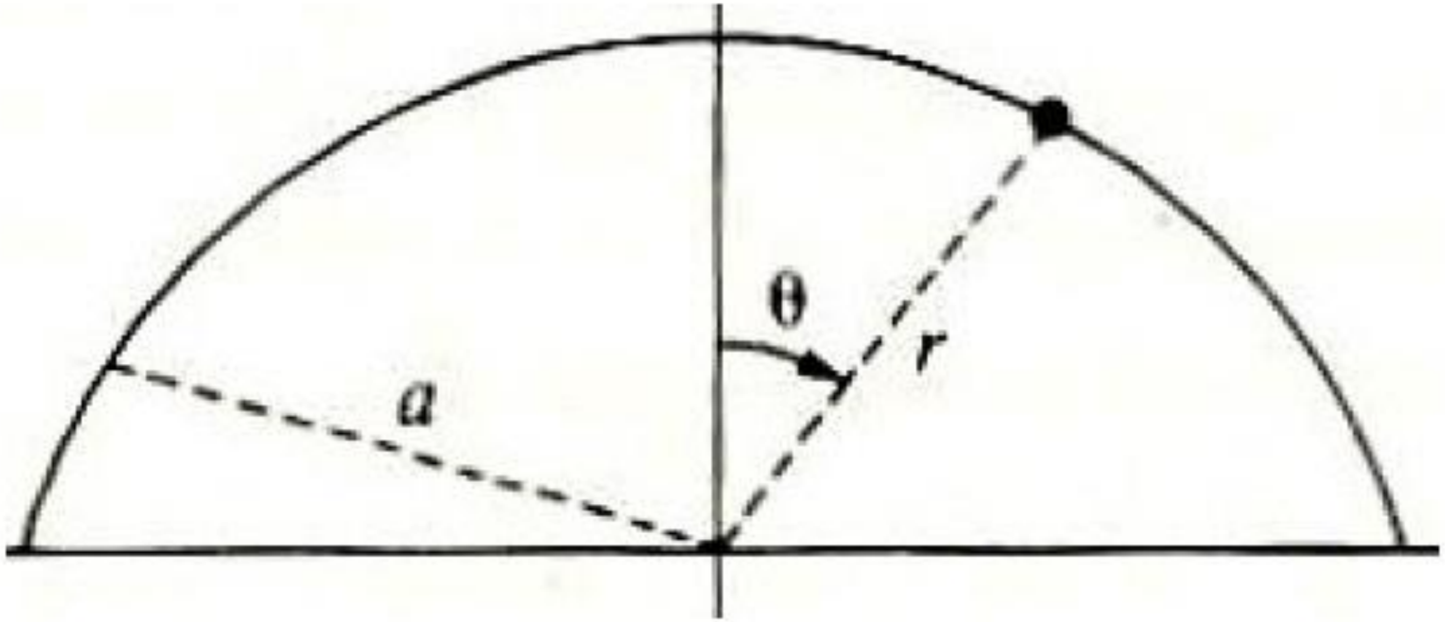
= **"Normal force"** between the hoop & the plane

$$Q_{\square} = \lambda[(\partial f/\partial \theta) - (d/dt)(\partial f/\partial \dot{\theta})] - (d\lambda/dt)(\partial f/\partial \dot{\theta})$$

$$= \lambda r = -(\tfrac{1}{2})Mg\sin\phi$$

= **Torque about hoop axis** produced by the "normal force" between the hoop & the plane

- A mass **m** starts from rest on top of a frictionless hemisphere, radius **a** (fig.). Find: Eqtns of motion, force of constraint & angle at which **m** leaves the hemisphere.



On the blackboard!

Solution, page 1!!

Solution: See Figure 7-7. Because we are considering the possibility of the particle leaving the hemisphere, we choose the generalized coordinates to be r and θ. The constraint equation is

$$f(r, \theta) = r - a = 0 \tag{7.80}$$

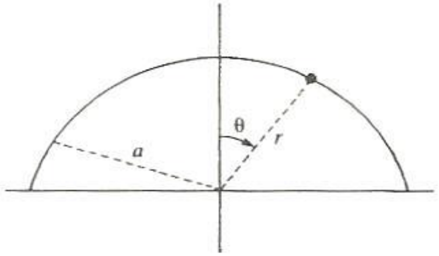


FIGURE 7-7

The Lagrangian is determined from the kinetic and potential energies:

$$T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$U = mgr\cos\theta$$

$$L = T - U$$

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta \tag{7.81}$$

where the potential energy is zero at the bottom of the hemisphere. The Lagrange equations, Equation 7.65, are

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} + \lambda\frac{\partial f}{\partial r} = 0 \tag{7.82}$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \lambda\frac{\partial f}{\partial \theta} = 0 \tag{7.83}$$

Performing the differentiations on Equation 7.80 gives

$$\frac{\partial f}{\partial r} = 1, \qquad \frac{\partial f}{\partial \theta} = 0 \tag{7.84}$$

Equations 7.82 and 7.83 become

$$mr\dot{\theta}^2 - mg\cos\theta - m\ddot{r} + \lambda = 0 \tag{7.85}$$

$$mgr\sin\theta - mr^2\ddot{\theta} - 2mr\dot{r}\dot{\theta} = 0 \tag{7.86}$$

Next, we apply the constraint $r = a$ to these equations of motion:

$$r = a, \qquad \dot{r} = 0 = \ddot{r}$$

Equations 7.85 and 7.86 then become

$$ma\,\dot{\theta}^2 - mg\cos\theta + \lambda = 0 \tag{7.87}$$

$$mga\sin\theta - ma^2\ddot{\theta} = 0 \tag{7.88}$$

Solution, page 2!!

From Equation 7.88, we have

$$\ddot{\theta} = \frac{g}{a} \sin \theta \tag{7.89}$$

We can integrate Equation 7.89 to determine $\dot{\theta}^2$.

$$\ddot{\theta} = \frac{d}{dt}\frac{d\theta}{dt} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{d\dot{\theta}}{d\theta} \tag{7.90}$$

We integrate Equation 7.89,

$$\int \dot{\theta}\, d\dot{\theta} = \frac{g}{a}\int \sin \theta \, d\theta \tag{7.91}$$

which results in

$$\frac{\dot{\theta}^2}{2} = \frac{-g}{a} \cos \theta + \frac{g}{a} \tag{7.92}$$

where the integration constant is g/a, because $\dot{\theta} = 0$ at $t = 0$ when $\theta = 0$. Substituting $\dot{\theta}^2$ from Equation 7.92 into Equation 7.87 gives, after solving for λ,

$$\lambda = mg(3 \cos \theta - 2) \tag{7.93}$$

which is the force of constraint. The particle falls off the hemisphere at angle θ_0 when $\lambda = 0$.

$$\lambda = 0 = mg(3 \cos \theta_0 - 2) \tag{7.94}$$

$$\theta_0 = \cos^{-1}\left(\frac{2}{3}\right) \tag{7.95}$$

As a quick check, notice that the constraint force is $\lambda = mg$ at $\theta = 0$ when the particle is perched on top of the hemisphere.

Advantages of a Variational Principle Formulation

- HP $\Rightarrow$ $\delta\int \mathcal{L}dt = 0$ (limits $t_1 < t < t_2$). An example of a *variational principle.*
- Most useful when a coordinate system-independent Lagrangian $L = T - V$ can be set up.
- HP: "Elegant". **Contains all of mechanics of holonomic systems in which forces are derivable from potentials.**
- HP: Involves only physical quantities (T, V) which can be generally defined without reference to a specific set of generalized coords.

$\Rightarrow$ *A formulation of mechanics which is independent of the choice of coordinate system!*

- **HP** $\Rightarrow$ $\boldsymbol{\delta\int L dt = 0}$ (limits $t_1 < t < t_2$).
- From this, we can see (again) that the **Lagrangian *L* is arbitrary to within the derivative (dF/dt) of an arbitrary function F = F(q,t).**
  - If we form $L' = L + (dF/dt)$ & do the integral, $\int L'dt$, we get $\int L dt + F(q,t_2) - F(q,t_1)$. By the definition of δ, the variation at t_1 & t_2 is zero $\Rightarrow$ $\delta\int L'dt$ will not depend on the end points.
- Another advantage to **HP** : **Can extend Lagrangian formalism to systems outside of classical dynamics:**
  - Elastic continuum field theory
  - Electromagnetic field theory
  - QM theory of elementary particles
  - Circuit theory!

Lagrange Applied to Circuit Theory

- **System: LR Circuit** (Fig.) Battery, voltage V, in series with inductor **L** & resistor **R** (which will give dissipation). Dynamical variable = charge **q**.

PE = **V** = **q**V

KE = **T** = $(\frac{1}{2})L(\dot{q})^2$

Lagrangian:

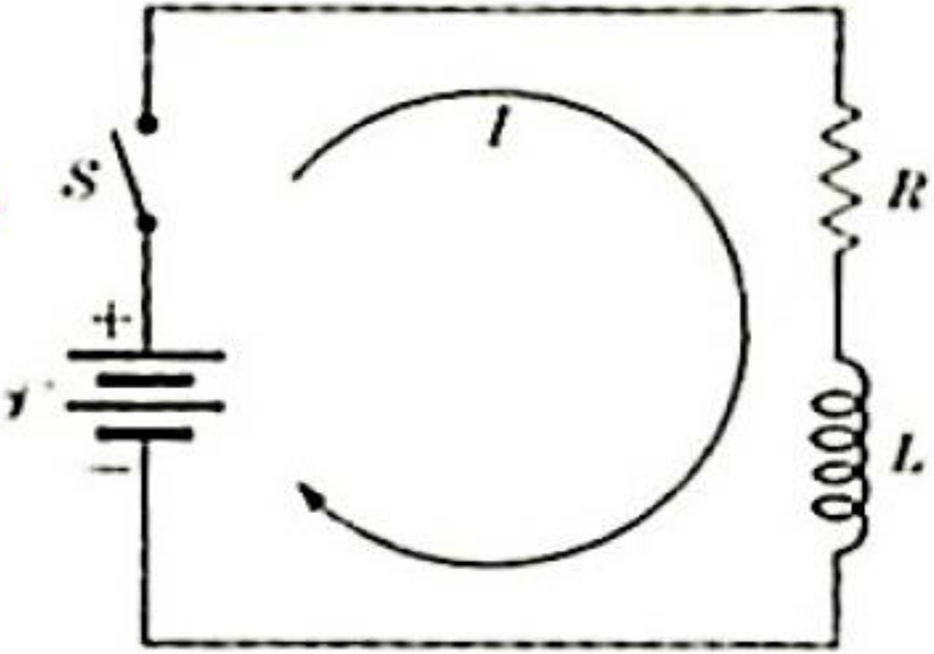
$$\mathcal{L} = T - V$$

Dissipation Function:

(last chapter!)

$$\mathcal{F} = (\tfrac{1}{2})R(\dot{q})^2 = (\tfrac{1}{2})R(I)^2$$

switch →



Lagrange's Eqtn (with dissipation):

$$(d/dt)[(\partial\mathcal{L}/\partial\dot{q})] - (\partial\mathcal{L}/\partial q) + (\partial\mathcal{F}/\partial\dot{q}) = 0$$

Lagrange Applied to RL circuit

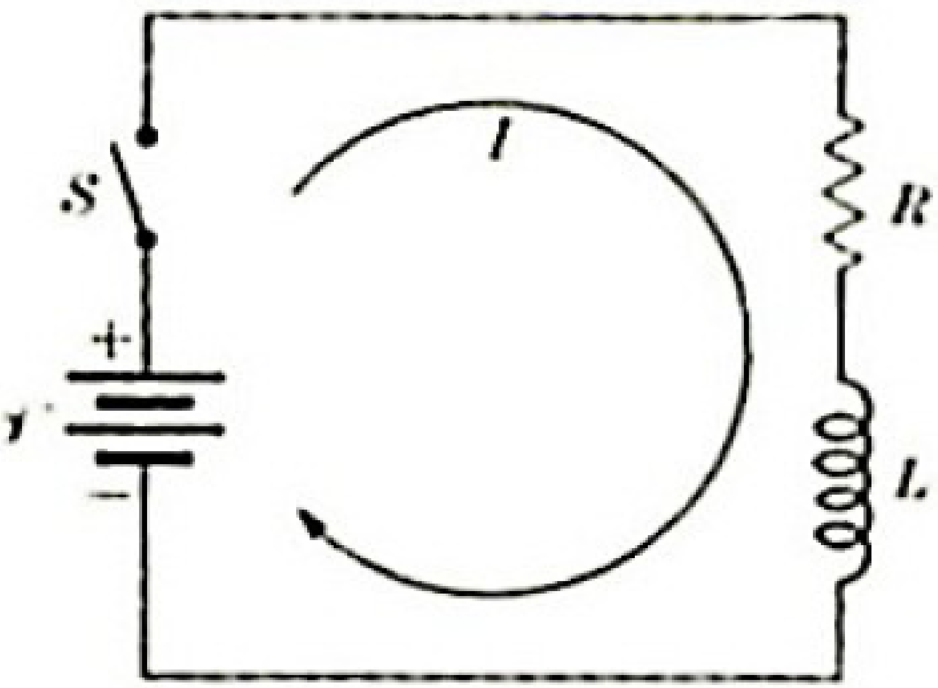
- **Lagrange's Eqtn (with dissipation):**

$$(d/dt)[(\partial L/\partial \dot{q})] - (\partial L/\partial q) + (\partial \mathcal{F}/\partial \dot{q}) = 0$$

$$\Rightarrow V = L\ddot{q} + R\dot{q}$$

$$I = \dot{q} = (dq/dt)$$

$$\Rightarrow V = L\dot{I} + RI$$



Solution, for switch closed at $t = 0$ is:

$$I = (V/R)[1 - e^{(-Rt/L)}]$$

Steady state ($t \to \infty$):

$$I = I_0 = (V/R)$$

Mechanical Analogue to RL circuit

- **Mechanical analogue:**

  Sphere, radius **a**, (effective) mass **m´**, falling in a const density viscous fluid, viscosity η under gravity.

  $\mathbf{m'} \equiv \mathbf{m} - \mathbf{m_f}$, $\mathbf{m} \equiv$ actual mass, $\mathbf{m_f} \equiv$ mass of displaced fluid (buoyant force acting upward: Archimedes' principle)

- $V = m'gy$, $T = (\frac{1}{2})m'v^2$, $L = T - V$ $\quad (v = \dot{y})$

  Dissipation Function: $\quad \mathcal{F} = 3\pi\eta a v^2$

  Comes from Stokes' Law of frictional drag force:

  $\mathbf{F_f} = 6\pi\eta a v$ and (Ch. 1 result that) $\mathbf{F_f} = -\nabla_v \mathcal{F}$

  Lagrange's Eqtn (with dissipation):

  $$(d/dt)[(\partial L/\partial \dot{y})] - (\partial L/\partial y) + (\partial \mathcal{F}/\partial \dot{y}) = 0$$

- $\mathbf{V = m'gy,\ T = (\frac{1}{2})m'v^2,}\ \mathbf{\mathit{L} = T - V} \quad \mathbf{(v = \dot{y})}$

Dissipation Function: $\mathcal{F} = \mathbf{3\pi\eta a v^2}$

Comes from Stokes' Law frictional drag force:

$\mathbf{F_f = 6\pi\eta a v}$ and (Ch. 1 result that) $\mathbf{F_f = -\nabla_v}\mathcal{F}$

Lagrange's Eqtn (with dissipation):

$$\mathbf{(d/dt)[(\partial \mathit{L}/\partial y)] - (\partial \mathit{L}/\partial y) + (\partial \mathcal{F}/\partial y) = 0}$$

$$\Rightarrow \qquad \mathbf{m'g = m'y + 6\pi\eta a\dot{y}}$$

Solution, for $\mathbf{v = \dot{y}}$ starting from rest at $\mathbf{t = 0}$:

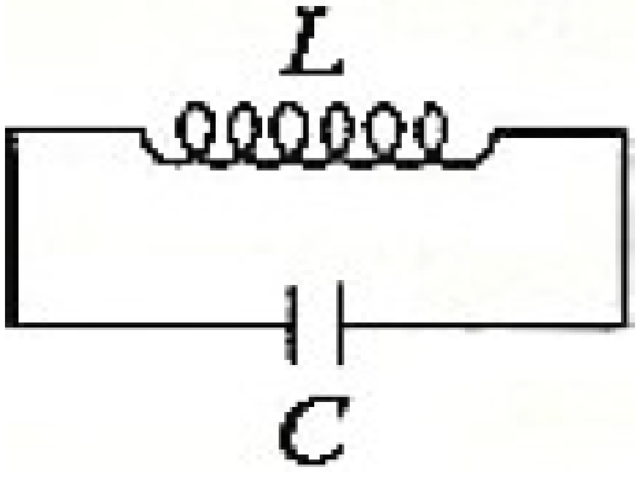
$\mathbf{v = v_0\ [1 - e^{(-t/\tau)}].\ \tau \equiv m'\ (6\pi\eta a)^{-1}} \equiv$ Time it takes sphere to reach $\mathbf{e^{-1}}$ of its terminal speed $\mathbf{v_0}$. Steady state

$(t \rightarrow \infty)$: $\mathbf{v = v_0 = (m'g)(6\pi\eta a)^{-1} = g\tau}$ = terminal speed.

Lagrange Applied to Circuit Theory

- **System: LC Circuit** (Fig.) Inductor **L** & capacitor **C** in series. Dynamical variable = charge **q**.

Capacitor acts a PE source:



PE = $(\frac{1}{2})q^2C^{-1}$, KE = $T = (\frac{1}{2})L(\dot{q})^2$

Lagrangian: $L = T - V$

(No dissipation!)

Lagrange's Eqtn:

$$(d/dt)[(\partial L/\partial \dot{q})] - (\partial L/\partial q) = 0 \Rightarrow L\ddot{q} + qC^{-1} = 0$$

Solution (for $q = q_0$ at $t = 0$):

$$q = q_0 \cos(\omega_0 t),\ \omega_0 = (LC)^{-(\frac{1}{2})}$$

ω_0 ≡ natural or resonant frequency of circuit

Mechanical Analogue to LC Circuit

- **Mechanical analogue:**

  Simple harmonic oscillator (no damping) mass **m**, spring constant **k**.

- $V = (\frac{1}{2})kx^2$, $T = (\frac{1}{2})mv^2$, $L = T - V$ $\quad (v = \dot{x})$

  Lagrange's Eqtn:

$$(d/dt)[(\partial L/\partial \dot{x})] - (\partial L/\partial x) = 0$$

$$\Rightarrow m\ddot{x} + kx = 0$$

Solution (for $x = x_0$ at $t = 0$):

$$x = x_0 \cos(\omega_0 t),\ \omega_0 = (k/m)^{\frac{1}{2}}$$

$\omega_0 \equiv$ natural or resonant frequency of circuit

- Circuit theory examples give analogies:

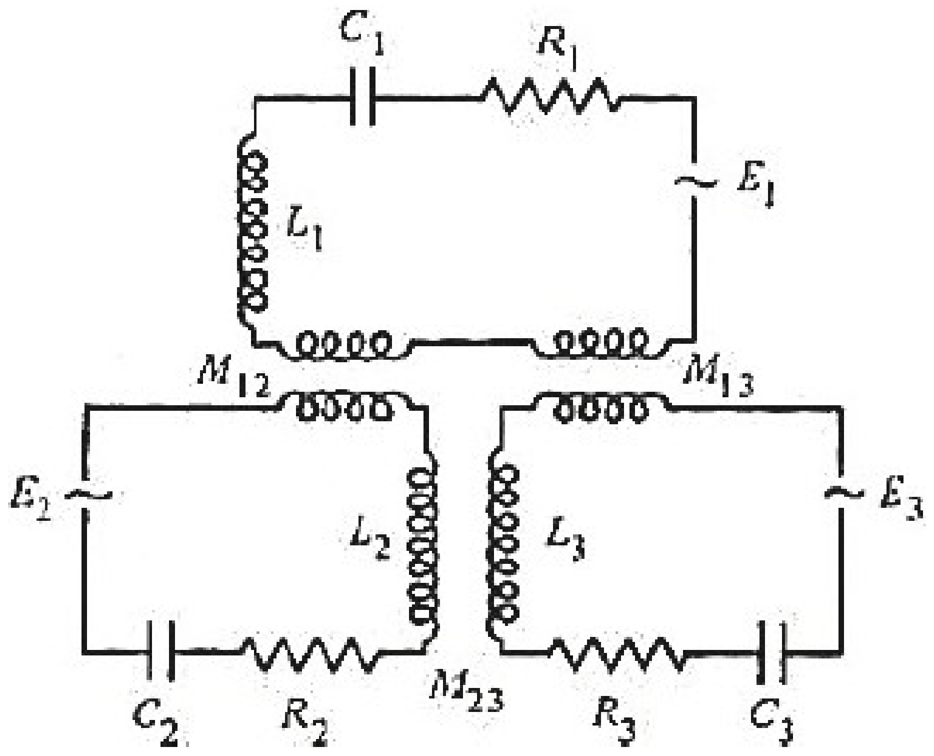
⇒ **Inductance L** plays an analogous role in electrical circuits that **mass m** plays in mechanical systems (***an inertial term***).

⇒ **Resistance R** plays an analogous role in electrical circuits that **viscosity η** plays in mechanical systems (***a frictional or drag term***).

⇒ **Capacitance C** (actually C^{-1}) plays an analogous role in electrical circuits that a Hooke's "Law" type **spring constant k** plays in mechanical systems (***a "stiffness" or tensile strength term***).

- With these analogies, consider the system of coupled electrical circuits (fig):

  $\mathbf{M_{jk}}$ = mutual inductances!



- Immediately, can write

  Lagrangian:

$$L = (½)\sum_j L_j(\dot{q}_j)^2 + (½)\sum_{j,k(\neq j)} M_{jk}\dot{q}_j\dot{q}_k - (½)\sum_j (1/C_j)(q_j)^2 + \sum_j E_j(t)q_j$$

Dissipation function: $\mathcal{F} = (½)\sum_j R_j(\dot{q}_j)^2$

- **Lagrangian:**

$$L = (½)\sum_j L_j(\dot{q}_j)^2 + (½)\sum_{j,k(\neq j)} M_{jk}\dot{q}_j\dot{q}_k - (½)\sum_j (1/C_j)(q_j)^2 + \sum_j E_j(t)q_j$$

Dissipation function: $\mathcal{F} = (½)\sum_j R_j(\dot{q}_j)^2$

Lagrange's Eqtns:

$$(d/dt)[(\partial L/\partial \dot{q}_j)] - (\partial L/\partial q_j) + (\partial \mathcal{F}/\partial \dot{q}_j) = 0$$

$\Rightarrow$ **Eqtns of motion** (the same as coupled, driven, damped harmonic oscillators!)

$$L_j(d^2q_j/dt^2) + \sum_{k(\neq j)} M_{jk}(d^2q_k/dt^2) + R_j(dq_j/dt) + (1/C_j)q_j = E_j(t)$$

- Describe 2 different physical systems by Lagrangians of the same mathematical form (circuits & harmonic oscillators):

⇒ **ALL results & techniques devised for studying & solving one system can be taken over directly & used to study & solve the other.**

⇒ Sophisticated studies of electrical circuits & techniques for solving them have been very well developed. **All such techniques can be taken over directly & used to study analogous mechanical (oscillator) systems.** These have wide applicability to acoustical systems. Also true in reverse.

- HP & resulting Lagrange formalism can be generalized to **apply to subfields of physics outside mechanics**.
- Similar variational principles exist in other subfields: Yielding

  Maxwell's Eqtns (E&M)

  the Schrödinger Eqtn

  Quantum Electrodynamics

  Quantum Chromodynamics, …..etc.

Conservation Theorems & Symmetry Properties

- **Lagrange Method:** A method to get the eqtns of motion. Solving them = math!

  n degrees of freedom.

  $\Rightarrow$ Need **n** eqtns of motion (2^{nd} order diff. eqtns)

  $\Rightarrow$ **2n** constants of integration (**2n** initial conditions).

  - Sometimes, can easily integrate the eqtns of motion. Sometimes not (except numerically).
  - Even when we cannot integrate the eqtns of motion, can get a lot of info about *PHYSICS* of motion by considering **Conservation Theorems.**

- **1st Integrals of Motion** ≡ Relations between generalized coords, generalized velocities, & time which are **1st** order diff eqtns. Of the form:

$$\mathbf{f(q_1,q_2,\ldots,\dot{q}_1,\dot{q}_2,\ldots,\ t) = constant}$$

- **1st Integrals of Motion:** Very interesting because they tell us a lot about ***the system physics***. They come from **Conservation Theorems.**

- **Consider:** Point masses & conservative forces: Eqtns of motion in Cartesian coords:

$$L = T - V = (½)\sum_i m_i[(\dot{x}_i)^2+(\dot{y}_i)^2+(\dot{z}_i)^2] - V(r_1,r_2, \ldots r_N)$$

$$(d/dt)[(\partial L/\partial \dot{x}_i)] - (\partial L/\partial x_i) = 0$$

- Look at: $(\partial L/\partial \dot{x}_i) = (\partial T/\partial \dot{x}_i) - (\partial V/\partial \dot{x}_i) = (\partial T/\partial \dot{x}_i) =$ $(\partial/\partial \dot{x}_i)[\ (½)\sum_j m_j[(\dot{x}_j)^2+(\dot{y}_j)^2+(\dot{z}_j)^2] = m_i\dot{x}_i = p_{ix}$

(**x** component of momentum of **i**th particle)

⇒ ***DEFINE:*** **Generalized Momentum**

associated with Generalized Coord q_j:

$$p_j \equiv (\partial L/\partial \dot{q}_j)$$

Generalized Momentum

$\Rightarrow$ Generalized Momentum associated with (or *Momentum Conjugate* to) Generalized Coord $\mathbf{q_j}$:

$$\mathbf{p_j} \equiv (\partial L/\partial \dot{\mathbf{q}}_j)$$

Points worth noting:

- If $\mathbf{q_j}$ is not a Cartesian Coordinate, $\mathbf{p_j}$ is ***NOT* necessarily a linear momentum.**
- For a **velocity dependent potential** $\mathbf{U(q_j,\dot{q}_j,t)}$, then, even if $\mathbf{q_j}$ is a Cartesian Coordinate, the Generalized Momentum $\mathbf{p_j}$ is ***NOT* the usual *Mechanical* Momentum** $(\mathbf{p_j} \neq \mathbf{m_j\dot{q}_j})$

- **Example:** Velocity dependent potential with $\mathbf{q_j}$ a Cartesian Coord, & where $\mathbf{p_j}$ is ***not*** $\mathbf{m_j\dot{q}_j}$: Charged particles in E&M field $(\mathbf{A}, \phi)$. Ch. 1:

$$L \equiv \mathbf{T} - \mathbf{U} = (\tfrac{1}{2})\textstyle\sum_i[\mathbf{m_i(\dot{r}_i)^2} - \mathbf{q_i}\phi(\mathbf{r_i}) + \mathbf{q_i A(r_i)\bullet\dot{r}_i}]$$

$$\Rightarrow \qquad \mathbf{p_{ix}} \equiv (\partial L/\partial\dot{\mathbf{x}}_i) = \mathbf{m_i\dot{x}_i} + \mathbf{q_i A_x} \neq \mathbf{m_i\dot{x}_i}$$

Ignorable (Cyclic) Coordinates

- **Important special case!**

  Cyclic or Ignorable Coordinates ≡ Generalized Coordinates $\mathbf{q_j}$ not appearing in Lagrangian L (but the generalized velocity ***MAY*** still appear in L).

- Lagrange's Eqtn for a cyclic coordinate $\mathbf{q_j}$:

$$(d/dt)[(\partial L/\partial \dot{q}_j)] - (\partial L/\partial q_j) = 0$$

By definition of cyclic: $(\partial L/\partial q_j) = 0$

⇒ Lagrange Eqtn: $(d/dt)[(\partial L/\partial \dot{q}_j)] = 0$

Momentum Conjugate $\mathbf{p_j} \equiv (\partial L/\partial \dot{q}_j)$

⇒ Lagrange Eqtn for a cyclic coordinate:

$$(dp_j/dt) = 0$$

$\Rightarrow$ If a Generalized Coord $\mathbf{q_j}$ is cyclic or ignorable, the Lagrange Eqtn is $\mathbf{(dp_j/dt) = 0}$

where **Generalized Momentum** $\mathbf{p_j} \equiv (\partial L/\partial \dot{\mathbf{q}}_j)$

- $\mathbf{(dp_j/dt) = 0} \Rightarrow \mathbf{p_j}$ = ***constant*** (conserved)

$\Rightarrow$ A General

Conservation Theorem*: If the Generalized Coord q_j is cyclic or ignorable, the corresponding Generalized (or Conjugate) Momentum**, $\mathbf{p_j} \equiv (\partial L/\partial \dot{\mathbf{q}}_j)$ ***is conserved.

- **Note:** Derivation assumes that $\mathbf{q_j}$ is a Generalized Coord & linearly independent of all other coords.
- As we've seen, when constraints exist, the $\mathbf{q_j}$ **are not** all linearly independent.
  - Recall hoop rolling without slipping in **xy** plane (Ch. 1): Generalized Coords **x, θ, ϕ** Constraint Eqtn: $\mathbf{r d\theta = dx}$ **(1)**

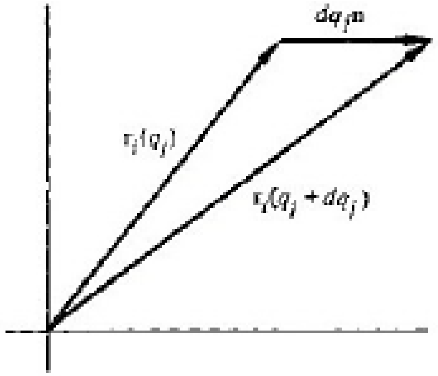


FIGURE 2.7 Change in a position vector under translation of the system.

$$L = (\tfrac{1}{2})M(\dot{x})^2 + (\tfrac{1}{2})I(\dot{\phi})^2$$

L : indep of **θ**. ***BUT*** because of the constraint eqtn **(1),** the corresponding conjugate momentum (angular momentum)

$$p_\theta = Mr^2\dot{\theta} \text{ is } \textit{NOT} \text{ conserved!}$$

Cyclic Coords & Conservation Theorems

- If $\mathbf{q_j}$ is cyclic, $\mathbf{p_j} = (\partial L/\partial \dot{q}_j)$ is conserved. In this case, $\mathbf{p_j} \equiv$ ***a 1st integral of the motion***

$\Rightarrow$ Can use to **formally eliminate the cyclic coordinate from the problem** (reducing # of diff eqtns of motion needed to solve from **n** to **n-1**).

$\equiv$ **Routh Method:** If $\mathbf{q_j}$ is cyclic, compute

$\mathbf{p_j} = (\partial L/\partial \dot{q}_j)$ (= constant). Solve for $\dot{\mathbf{q}}_j$ in terms of $\mathbf{p_j}$: $\dot{\mathbf{q}}_j = \mathbf{f(p_j)}$. Everywhere $\dot{\mathbf{q}}_j$ appears in the Lagrangian L, make the replacement:

$\Rightarrow$ $\mathbf{L}(\dot{\mathbf{q}}_j$, + rest of $\mathbf{q}$'s, $\dot{\mathbf{q}}$'s$) \rightarrow \mathbf{L}'(\mathbf{p_j}$,+rest of $\mathbf{q}$'s, $\dot{\mathbf{q}}$'s$)$

- Advantage: $\mathbf{p_j}$ = integration constant. Remaining eqtns involve only non-cyclic coords. Return to this later when discuss Hamiltonian formulation.

- Conditions for **conservation of generalized momenta** are actually ***MORE GENERAL*** than those used to derive conservation theorems for many particle systems.
  - Those in Ch. 1 needed assumptions of Newton's 3rd Law in weak & strong forms for internal forces.
  - **Present case:** Generalized Momentum conservation theorems **result even if these 3rd Law assumptions are violated!**

- **Example:** A single charged particle in an E&M field $(\mathbf{A},\phi)$. Assume $\mathbf{A}$ & ϕ are indep of $\mathbf{x}$ (but **DO** depend on $\mathbf{y}$ & $\mathbf{z}$, or else $\mathbf{E} = \mathbf{B} = \mathbf{0}$!)

$$L = (\tfrac{1}{2})m[(\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2] - q\phi(y,z) + q\mathbf{A}(y,z)\bullet(\dot{x}\,\hat{\mathbf{i}} + \dot{y}\,\hat{\mathbf{j}} + \dot{z}\,\hat{\mathbf{k}})$$

$$\Rightarrow \quad p_x \equiv (\partial L/\partial \dot{x}) = m\dot{x} + qA_x = \text{constant}$$

$$(\text{since } (\partial L/\partial x) = 0)$$

- $m\dot{x} + qA_x$ = Constant instead of $m\dot{x}$. *Physics:* From E&M can show: qA_x = $\mathbf{x}$ component of momentum of EM field associated with charge $\mathbf{q}$.

$\Rightarrow$ $m\dot{x} + qA_x$ = constant $\Rightarrow$ $\mathbf{x}$ component of mech. momentum + $\mathbf{x}$ component of field momentum is conserved (allowing for exchange of momentum between field & particle!)

Generalized Momenta & Forces

- Consider a **specific Generalized Coord** $\mathbf{q_j}$ for which infinitesimal change $\mathbf{dq_j}$ = translation of the whole system in some direction (change of origin). $\Rightarrow$ $\mathbf{q_j}$ cannot appear in the KE **T** because velocities aren't affected by the origin shift: $(\partial T/\partial q_j) = 0$
- Assume also, ***conservative forces***, so the PE **V** is not a function of the velocities: $(\partial V/\partial \dot{q}_j) = 0$

  $\Rightarrow (\partial L/\partial q_j) = -(\partial V/\partial q_j) \equiv Q_j$ (generalized force)

  Also: $\Rightarrow (\partial L/\partial \dot{q}_j) \equiv p_j = (\partial T/\partial \dot{q}_j)$
- **Lagrange's Eqtn** for this coordinate $\mathbf{q_j}$:

$$(d/dt)[(\partial L/\partial \dot{q}_j)] - (\partial L/\partial q_j) = 0$$

Combining $\Rightarrow$ This becomes: $(dp_j/dt) - Q_j = 0$

- **Summary:** For Generalized Coord $\mathbf{q_j}$ for which change $\mathbf{dq_j}$ = change of origin. $\Rightarrow$ $\mathbf{q_j}$ & for ***conservative forces***, Lagrange's Eqtn is:

$$(\mathbf{dp_j/dt}) = \dot{\mathbf{p}}_\mathbf{j} = \mathbf{Q_j}$$

(Newton's 2nd Law in generalized coord language)

- Ch. 1: Generalized Force $\mathbf{Q_j}$, in terms of Cartesian vector forces $\vec{\mathbf{F}}_\mathbf{i}$: $\mathbf{Q_j} \equiv \sum_i \vec{\mathbf{F}}_\mathbf{i} \bullet (\partial \vec{\mathbf{r}}_\mathbf{i} / \partial \mathbf{q_j})$
- For the special case considered, $\mathbf{dq_j}$ = change of origin, so $\vec{\mathbf{r}}_\mathbf{i}(\mathbf{q_j})$ & $\vec{\mathbf{r}}_\mathbf{i}(\mathbf{q_j}+\mathbf{dq_j})$ are related as in the figure:

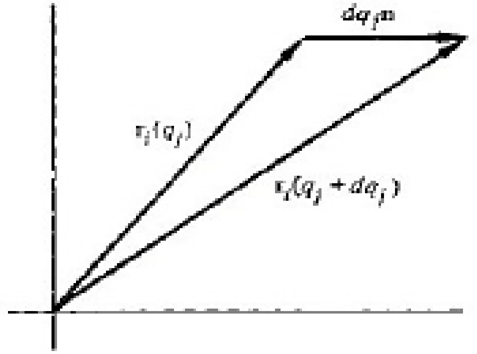


FIGURE 2.7 Change in a position vector under translation of the system.

- So: $(d\mathbf{p}_j/dt) = \dot{\mathbf{p}}_j = \mathbf{Q}_j$ and $\mathbf{Q}_j \equiv \sum_i \vec{\mathbf{F}}_i \bullet (\partial \vec{\mathbf{r}}_i/\partial \mathbf{q}_j)$
- In this case, $\mathbf{r}_i(\mathbf{q}_j)$ & $\mathbf{r}_i(\mathbf{q}_j+d\mathbf{q}_j)$ are as in figure:

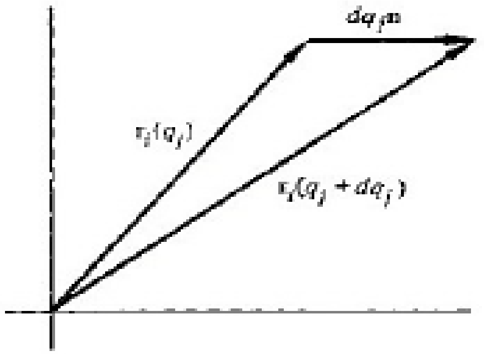


FIGURE 2.7 Change in a position vector under translation of the system.

By definition: $(\partial \vec{\mathbf{r}}_i/\partial \mathbf{q}_j)$

$= \lim_{(d\mathbf{q}_j \to 0)} [\vec{\mathbf{r}}_i(\mathbf{q}_j+d\mathbf{q}_j) - \vec{\mathbf{r}}_i(\mathbf{q}_j)]/(d\mathbf{q}_j) = (d\mathbf{q}_j/d\mathbf{q}_j)\hat{\mathbf{n}} = \hat{\mathbf{n}}$

$\hat{\mathbf{n}}$ = unit vector in direction of system translation

$$\Rightarrow \qquad \mathbf{Q}_j \equiv \sum_i \vec{\mathbf{F}}_i \bullet (\partial \vec{\mathbf{r}}_i/\partial \mathbf{q}_j) = \sum_i \vec{\mathbf{F}}_i \bullet \hat{\mathbf{n}} = \hat{\mathbf{n}} \bullet \vec{\mathbf{F}}$$

- In special the case considered, **Lagrange** gives **Newton's 2nd Law:**

$$(dp_j/dt) = \dot{p}_j = Q_j = \hat{n}\bullet\vec{F}$$

Generalized Force = Component of Cartesian vector force $\vec{F}$ in direction of origin shift.

- Also, in this case, KE: $T = (½)\sum_i m_i(\dot{\vec{r}}_i)^2$.

Conjugate momentum:

$$p_j = (\partial T/\partial \dot{q}_j) = \sum_i m_i \dot{\vec{r}}_i \bullet (\partial \vec{r}_i/\partial q_j)$$

(note that $(\partial \dot{r}_i/\partial \dot{q}_j) = (\partial r_i/\partial q_j)$)

$$\Rightarrow p_j = \sum_i m_i \vec{v}_i \bullet (\partial \vec{r}_i/\partial q_j) \equiv n\bullet \sum_i m_i \vec{v}_i = \hat{n}\bullet\vec{p}$$

Generalized Momentum = Component of Cartesian vector momentum **p** in direction of origin shift.

- In special case considered, **Newton's 2nd Law** from **Lagrange**:

$$\hat{\mathbf{n}}\bullet\dot{\vec{\mathbf{p}}} = \dot{\mathbf{p}}_j = \mathbf{Q}_j = \hat{\mathbf{n}}\bullet\vec{\mathbf{F}}$$

- Suppose, in this same case, Generalized Coord $\mathbf{q}_j$ is cyclic: $\mathbf{Q}_j = -(\partial V/\partial q_j) = 0 \Rightarrow \dot{\mathbf{p}}_j = \hat{\mathbf{n}}\bullet\dot{\vec{\mathbf{p}}} = 0$

$\Rightarrow$ **Conservation of linear momentum:** If a component of total force vanishes, $\hat{\mathbf{n}}\bullet\vec{\mathbf{F}} = 0$, then corresponding component of total linear momentum $\hat{\mathbf{n}}\bullet\vec{\mathbf{p}}$ = **constant** (is conserved)

Angular Momentum & Torque

- Do a similar analysis to show that, if a cyclic coordinate $\mathbf{q_j}$ is such that $\mathbf{dq_j}$ corresponds to a **rotation of the entire particle system**, then conservation of the conjugate momentum corresponds to **conservation of angular momentum**.

- Consider a specific Generalized Coord $\mathbf{q_j}$ for which the infinitesimal change $\mathbf{dq_j}$ = rotation of whole system about some axis. $\Rightarrow$ $\mathbf{q_j}$ cannot appear in KE **T** because velocities aren't affected by rotation: $\mathbf{(\partial T/\partial q_j) = 0}$

- Also, *conservative forces*. PE **V** is not a function of the velocities: $(\partial \mathbf{V}/\partial \dot{\mathbf{q}}_j) = 0$

$\Rightarrow (\partial L/\partial \mathbf{q}_j) = -(\partial \mathbf{V}/\partial \mathbf{q}_j) \equiv \mathbf{Q}_j$ (generalized force)

Also: $\Rightarrow (\partial L/\partial \dot{\mathbf{q}}_j) \equiv \mathbf{p}_j = (\partial \mathbf{T}/\partial \dot{\mathbf{q}}_j)$

- **Lagrange's Eqtn** for this coordinate $\mathbf{q}_j$:

$$(\mathbf{d}/\mathbf{dt})[(\partial L/\partial \dot{\mathbf{q}}_j)] - (\partial L/\partial \mathbf{q}_j) = 0$$

$\Rightarrow$ This again becomes: $(\mathbf{dp}_j/\mathbf{dt}) - \mathbf{Q}_j = 0$

- Now show: If $\mathbf{q_j}$ = **rotation coord,** $\mathbf{Q_j}$ = **torque component about axis of rotation** & $\mathbf{p_j}$ = **angular momentum component** about same axis.
- **Generalized force** is again: $\mathbf{Q_j} \equiv \sum_i \vec{\mathbf{F}}_i \bullet (\partial \vec{\mathbf{r}}_i / \partial \mathbf{q_j})$. Now, $\mathbf{q_j}$ is a rotation coord. $\Rightarrow$ $\mathbf{dq_j}$ = infinitesimal rotation of vector $\mathbf{r_i}$, keeping magnitude of vector constant (fig).

  Rotation axis

  direction is $\mathbf{n}$.

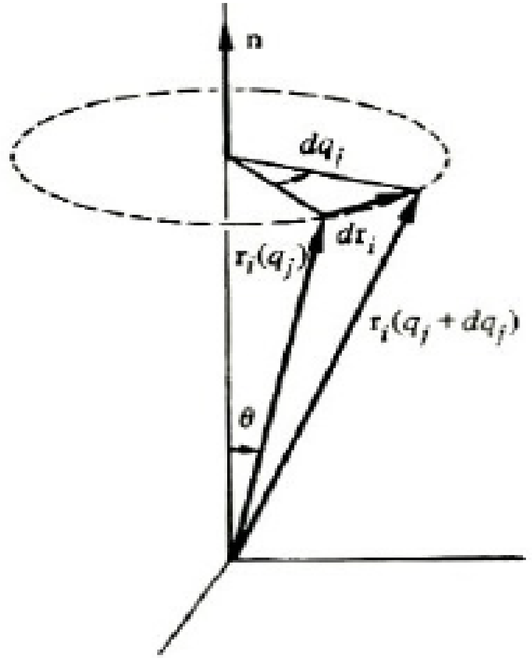


FIGURE 2.8 Change of a position vector under rotation of the system

- $\mathbf{Q_j} \equiv \sum_i \vec{\mathbf{F}}_i \bullet (\partial\vec{\mathbf{r}}_i/\partial \mathbf{q_j})$. $\mathbf{dq_j}$ = infinitesimal rotation of $\vec{\mathbf{r}}_i$, keeping magnitude constant.

$$|\mathbf{d}\vec{\mathbf{r}}_i| = \mathbf{r_i}\,\mathbf{sin\theta dq_j}$$

$$\Rightarrow \quad |\partial\vec{\mathbf{r}}_i/\partial \mathbf{q_j}| = \mathbf{r_i}\,\mathbf{sin\theta}$$

Also $(\partial\vec{\mathbf{r}}_i/\partial \mathbf{q_j})$ is $\perp$ to *both* $\vec{\mathbf{r}}_i$ & rotation axis direction **n**

$$\Rightarrow (\partial\vec{\mathbf{r}}_i/\partial \mathbf{q_j}) = \widehat{\mathbf{n}} \times \vec{\mathbf{r}}_i$$

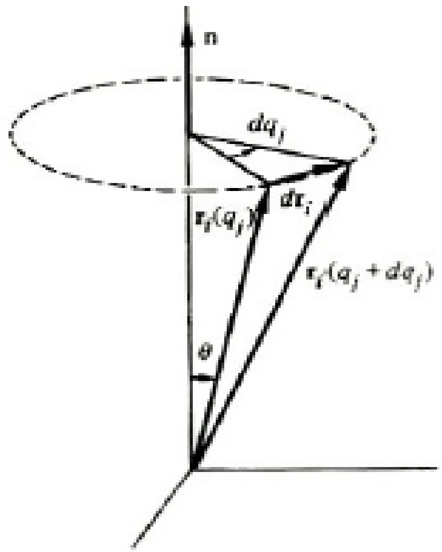


FIGURE 2.8 Change of a position vector under rotation of the system

- **Generalized Force:** $\mathbf{Q_j} \equiv \sum_i \vec{\mathbf{F}}_i \bullet (\partial\vec{\mathbf{r}}_i/\partial \mathbf{q_j})$. =

$$\sum_i \vec{\mathbf{F}}_i \bullet (\widehat{\mathbf{n}} \times \vec{\mathbf{r}}_i) = \sum_i \widehat{\mathbf{n}} \bullet (\vec{\mathbf{r}}_i \times \vec{\mathbf{F}}_i) = \widehat{\mathbf{n}} \bullet \sum_i (\vec{\mathbf{r}}_i \times \vec{\mathbf{F}}_i) = \widehat{\mathbf{n}} \bullet \sum_i \vec{\mathbf{N}}_i$$

$$\Rightarrow \quad \mathbf{Q_j} = \widehat{\mathbf{n}} \bullet \vec{\mathbf{N}} \qquad (\vec{\mathbf{N}} \equiv \text{total torque about rotation axis})$$

Generalized Force = Component of Cartesian vector torque N in rotation direction.

- In this case, **Conjugate momentum:**

$$\mathbf{p_j = (\partial T/\partial \dot{q}_j) = \sum_i m_i \vec{v}_i \bullet (\partial \vec{r}_i/\partial q_j)} \text{ with } \mathbf{(\partial \vec{r}_i/\partial q_j) = n \times \vec{r}_i}$$

$$\mathbf{p_j = \sum_i m_i \vec{v}_i \bullet (\hat{n} \times \vec{r}_i) = \hat{n} \bullet \sum_i (\vec{r}_i \times m_i \vec{v}_i) = \hat{n} \bullet \sum_i \vec{L}_i = \hat{n} \bullet \vec{L}}$$

Generalized Momentum = Component of Cartesian vector **angular momentum L** in direction of rotation.

- In special case considered, **Newton's 2nd Law** (rotational motion) from **Lagrange**:

$$\mathbf{\hat{n} \bullet \dot{\vec{L}} = \dot{p}_j = Q_j} = \mathbf{\hat{n} \bullet \vec{N}}$$

- Suppose, in this case, Generalized Coord. $\mathbf{q_j}$ is **cyclic.**

$$Q_j = -(\partial V/\partial q_j) = 0 \Rightarrow \dot{p}_j = \hat{n} \bullet \vec{N} = 0$$

⇒ **Conservation of angular momentum:**

If a component of the total torque vanishes, $\hat{n} \bullet \vec{N} = 0$, then the corresponding component of total angular momentum

$$\hat{n} \bullet \vec{L} = \textbf{constant } (\textit{is conserved})$$

Physics/Philosophy

- *Physical Significance* of cyclic coords (translation or rotation):
- If the $\mathbf{q_j}$ corresponding to a displacement is cyclic (& thus conjugate momentum is conserved):

$\Rightarrow$ System translation is *as if* it were rigid & has no effect on the problem. That is (we've shown), if the system is *invariant under translation* along a given direction $\mathbf{\hat{n}}$, the corresponding linear momentum along that direction is conserved.

- Also, if $\mathbf{q_j}$ corresponding to a rotation is cyclic (& thus conjugate angular momentum is conserved):

$\Rightarrow$ System rotation is *as if* it were rigid & has no effect on the problem. That is (we've shown), if the system is *invariant under rotation* about a given direction $\mathbf{\hat{n}}$, the corresponding angular momentum along that direction is conserved.

Conservation Laws & Symmetry

- <u>Bottom line:</u> *The Generalized Momentum Conservation Theorems are closely connected with the Symmetry Properties of the system.*
  - For example, spherically symmetric systems *ALWAYS* have all components of angular momentum conserved.
  - If the system has symmetry only about one axis (say, **z**) then angular momentum about that axis (L_z) is conserved.
  - If, by inspection, a system has some symmetry, can often use it to *<u>IMMEDIATELY</u>* conclude that some Generalized Momentum component is conserved.

  $\Rightarrow$ There is an *intimate connection between symmetry properties and conservation theorems.* (More later on this!)

Energy Function & Energy Conservation

- One more conservation theorem which we would expect to get from the Lagrange formalism is: ***CONSERVATION OF ENERGY.***
- Consider a general Lagrangian L, a function of the coords $\mathbf{q_j}$, velocities $\mathbf{\dot{q}_j}$, & time $\mathbf{t}$:

$$L = L(q_j,\dot{q}_j,t) \qquad (j = 1,\ldots n)$$

- The total time derivative of L (chain rule):

$$(dL/dt) = \sum_j(\partial L/\partial q_j)(dq_j/dt) + \sum_j(\partial L/\partial \dot{q}_j)(d\dot{q}_j/dt) + (\partial L/\partial t)$$

Or:

$$(dL/dt) = \sum_j(\partial L/\partial q_j)\dot{q}_j + \sum_j(\partial L/\partial \dot{q}_j)\ddot{q}_j + (\partial L/\partial t)$$

- **Total time derivative** of L:

$$(dL/dt) = \sum_j(\partial L/\partial q_j)\dot{q}_j + \sum_j(\partial L/\partial \dot{q}_j)\ddot{q}_j + (\partial L/\partial t) \quad (1)$$

- **Lagrange's Eqtns:** $(d/dt)[(\partial L/\partial \dot{q}_j)] - (\partial L/\partial q_j) = 0$

Put into **(1)**

$$(dL/dt) = \sum_j(d/dt)[(\partial L/\partial \dot{q}_j)]\dot{q}_j + \sum_j(\partial L/\partial \dot{q}_j)\ddot{q}_j + (\partial L/\partial t)$$

Identity: 1st 2 terms combine

$$(dL/dt) = \sum_j(d/dt)[\dot{q}_j(\partial L/\partial \dot{q}_j)] + (\partial L/\partial t)$$

Or: $$(d/dt)[\sum_j\dot{q}_j(\partial L/\partial \dot{q}_j) - L] + (\partial L/\partial t) = 0 \qquad (2)$$

$$(d/dt)[\sum_j \dot{q}_j(\partial L/\partial \dot{q}_j) - L] + (\partial L/\partial t) = 0 \quad (2)$$

- Define the ***Energy Function*** **h**:

$$h \equiv \sum_j \dot{q}_j(\partial L/\partial \dot{q}_j) - L = h(q_1,..q_n;\dot{q}_1,..\dot{q}_n,t)$$

- **(2)** $\Rightarrow$ $(dh/dt) = -(\partial L/\partial t)$

$\Rightarrow$ For a Lagrangian L which is **not an explicit function of time** (so that $(\partial L/\partial t) = 0$)

$$(dh/dt) = 0 \ \& \ h = \text{constant (conserved)}$$

- **Energy Function** $h = h(q_1,..q_n;\dot{q}_1,..\dot{q}_n,t)$
  - Identical ***Physically*** to what we later will call **the Hamiltonian H**. ***However,*** here, **h** is a function of **n** indep coords q_j & velocities $\dot{q}_j$. The Hamiltonian **H** is ***ALWAYS*** considered a function of **2n** indep coords q_j & momenta p_j

- ***Energy Function*** $\mathbf{h} \equiv \sum_j \dot{q}_j (\partial L/\partial \dot{q}_j) - L$

- We had $\mathbf{(dh/dt) = - (\partial \mathit{L}/\partial t)}$

$\Rightarrow$ For a Lagrangian for which $\mathbf{(\partial \mathit{L}/\partial t) = 0}$

$\mathbf{(dh/dt) = 0}$ **&** $\mathbf{h =}$ **constant** (conserved)

- For this to be useful, we need a

Physical Interpretation of **h.**

  – Will now show that, under *certain circumstances,* **h = total mechanical energy of the system.**

Physical Interpretation of h

- ***Energy Function*** $\mathbf{h} \equiv \sum_j \dot{q}_j(\partial L/\partial \dot{q}_j) - L$

- Recall (Sect. 1.6) that we can always write KE as:

$$\mathbf{T} = \mathbf{M_0} + \sum_j \mathbf{M_j}\dot{\mathbf{q}}_j + \sum_j \mathbf{M_{jk}}\dot{\mathbf{q}}_j\dot{\mathbf{q}}_k$$

$$\mathbf{M_0} \equiv (\tfrac{1}{2})\sum_i \mathbf{m_i}(\partial\vec{\mathbf{r}}_i/\partial \mathbf{t})^2, \quad \mathbf{M_j} \equiv \sum_i \mathbf{m_i}(\partial\vec{\mathbf{r}}_i/\partial \mathbf{t})\bullet(\partial\vec{\mathbf{r}}_i/\partial \mathbf{q_j})$$

$$\mathbf{M_{jk}} \equiv \sum_i \mathbf{m_i}(\partial\vec{\mathbf{r}}_i/\partial \mathbf{q_j})\bullet(\partial\vec{\mathbf{r}}_i/\partial \mathbf{q_k})$$

Or (schematically) $\mathbf{T} = \mathbf{T_0(q)} + \mathbf{T_1(q,\dot{q})} + \mathbf{T_2(q,\dot{q})}$

  - $\mathbf{T_0} \equiv \mathbf{M_0}$ independent of generalized velocities
  - $\mathbf{T_1} \equiv \sum_j \mathbf{M_j}\dot{\mathbf{q}}_j$ linear in generalized velocities
  - $\mathbf{T_2} \equiv \sum_j \mathbf{M_{jk}}\dot{\mathbf{q}}_j\dot{\mathbf{q}}_k$ quadratic in generalized velocities

- With almost ***complete generality***, we can write (schematically) the Lagrangian for most problems of interest in mechanics as:

$$L = L_0(q,t) + L_1(q,\dot{q},t) + L_2(q,\dot{q},t)$$

$L_0 \equiv$ independent of the generalized velocities

$L_1 \equiv$ linear in generalized the velocities

$L_2 \equiv$ quadratic in generalized the velocities

– For ***conservative forces***, L has this form. Also does for some velocity dependent potentials, such as for EM fields.

$$L = L_0(q,t) + L_1(q,\dot{q},t) + L_2(q,\dot{q},t) \qquad (1)$$

- **Euler's Theorem** from mathematics:

  If $\mathbf{f} = \mathbf{f}(\mathbf{x}_1,\mathbf{x}_2,\ldots \mathbf{x}_N)$ = a homogeneous function of degree **n** of the variables $\mathbf{x}_i$, then

$$\sum_i x_i(\partial f/\partial x_i) = n\, f \qquad (2)$$

- **Energy Function** $\mathbf{h} \equiv \sum_j \dot{q}_j(\partial L/\partial \dot{q}_j) - L \qquad (3)$
- For L of form **(1)**:

$$(2) \Rightarrow h = 0L_0 + 1L_1 + 2L_2 - [L_0 + L_1 + L_2]$$

or $\qquad h = \quad L_2 - L_0$

$$L = L_0(q,t) + L_1(q,\dot{q},t) + L_2(q,\dot{q},t)$$

$\Rightarrow$ **Energy function** $h \equiv \sum_j \dot{q}_j(\partial L/\partial \dot{q}_j) - L = L_2 - L_0$

- ***Special case*** *(both conditions!):*

a.) The **transformation eqtns** from Cartesian to Generalized Coords **are time indep.**

$\Rightarrow$ In the KE, $T_0 = T_1 = 0 \quad \Rightarrow \quad T = T_2$

b.) V is velocity indep. $\Rightarrow L_2 = T = T_2$ & $L_0 = -V$

$\Rightarrow$ $h = T + V = E \equiv$ ***Total Mechanical Energy***

- Under these conditions, *if* V does not depend on t, neither does L & thus $(\partial L/\partial t) = 0 = (dh/dt)$

so $h = E =$ **constant** (conserved)

Energy Conservation

- **Summary: Different Conditions:**

Energy Function $\mathbf{h} \equiv \sum_j \dot{q}_j(\partial L/\partial \dot{q}_j) - L$

ALWAYS: $(\mathrm{dh/dt}) = -(\partial L/\partial t)$

SOMETIMES: L does not depend on t

$\Rightarrow$ $(\partial L/\partial t) = 0$, $(\mathrm{dh/dt}) = 0$ & $\mathbf{h} =$ **const.** (conserved)

USUALLY: $L = L_0(q,t) + L_1(q,\dot{q},t) + L_2(q,\dot{q},t)$

$\Rightarrow$ $\mathbf{h} \equiv \sum_j \dot{q}_j(\partial L/\partial \dot{q}_j) - L = L_2 - L_0$

SOMETIMES: $\mathrm{T} = \mathrm{T}_2 = L_2$ AND $L_0 = -\mathrm{V}$

$\Rightarrow$ $\mathbf{h} = \mathrm{T} + \mathrm{V} = \mathrm{E} \equiv$ ***Total Mechanical Energy***

$\Rightarrow$ ***Conservation Theorem for Mechanical Energy:***

If h = E *AND* L does not depend on t, E is conserved!

Symmetry Properties & Conservation Laws (From Marion's Book!)

- In general, in physical systems:

A Symmetry Property of the System

⇒ *Conservation of Some Physical Quantity*

Also: *Conservation of Some Physical Quantity*

⇒ *A Symmetry Property of the System*

- Not just valid in classical mechanics! Valid in quantum mechanics also! Forms the foundation of modern field theories (Quantum Field Theory, Elementary Particles,…)

We've seen in general that:

Conservation Theorem: *If the Generalized Coord q_j is cyclic or ignorable, the corresponding Generalized (or Conjugate) Momentum*, $\mathbf{p_j} \equiv (\partial L/\partial \dot{\mathbf{q}}_j)$ *is conserved.*

- **An underlying symmetry property of the system:**

If $\mathbf{q_j}$ is cyclic, the system is unchanged (invariant) under a translation (or rotation) in the "$\mathbf{q_j}$ direction".

$\Rightarrow$ $\mathbf{p_j}$ **is conserved**

Linear Momentum Conservation

- **Conservation of linear momentum:**

  If a component of the total force vanishes, $\hat{\mathbf{n}}\bullet\vec{\mathbf{F}} = \mathbf{0}$, the corresponding component of total linear momentum $\hat{\mathbf{n}}\bullet\vec{\mathbf{p}} = \mathbf{const}$ (is conserved)

- **Underlying symmetry property of the system:**

  The system is **unchanged** (invariant) **under a translation** in the "**n** direction".

  $\Rightarrow$ $\hat{\mathbf{n}}\bullet\vec{\mathbf{p}}$ is **conserved**

Angular Momentum Conservation

- **Conservation of angular momentum:**

  If a component of total the torque vanishes, $\hat{\mathbf{n}}\bullet\vec{\mathbf{N}} = \mathbf{0}$, the corresponding component of total angular momentum $\hat{\mathbf{n}}\bullet\vec{\mathbf{L}}$ = **const** (conserved)

- **Underlying symmetry property of the system:**

  The system is **unchanged** (invariant) **under a rotation** about the "**n** direction".

  $\Rightarrow \qquad \hat{\mathbf{n}}\bullet\vec{\mathbf{L}}$ is conserved

Energy Conservation

- **Conservation of mechanical energy:**

  If all forces in the system are conservative, the total mechanical energy **E** = **const** (conserved)

- **Underlying symmetry property of the system:**

  (More subtle than the others!) The system is **unchanged** (invariant) **under a time reversal.**

  (Changing **t** to **-t** in all eqtns of motion)

$\Rightarrow$ **E** is **conserved**

Summary: Conservation Laws

- Under the proper conditions, there can be ***up to* 7 "Constants of the Motion" ≡ "1st Integrals of the Motion" ≡ Quantities which are Conserved** (const in time):

Total Mechanical Energy ($\mathbf{E}$)

3 vector components of Linear Momentum ($\vec{\mathbf{p}}$)

3 vector components of Angular Momentum ($\vec{\mathbf{L}}$)

Thank You