# CHAPTER 7

# *Hamilton's Principle—Lagrangian and Hamiltonian Dynamics*

## 7.1 Introduction

Experience has shown that a particle's motion in an inertial reference frame is correctly described by the Newtonian equation $\mathbf{F} = \dot{\mathbf{p}}$. If the particle is not required to move in some complicated manner and if rectangular coordinates are used to describe the motion, then usually the equations of motion are relatively simple. But if either of these restrictions is removed, the equations can become quite complex and difficult to manipulate. For example, if a particle is constrained to move on the surface of a sphere, the equations of motion result from the projection of the Newtonian vector equation onto that surface. The representation of the acceleration vector in spherical coordinates is a formidable expression, as the reader who has worked Problem 1-25 can readily testify.

Moreover, if a particle is constrained to move on a given surface, certain forces must exist (called **forces of constraint**) that maintain the particle in contact with the specified surface. For a particle moving on a smooth horizontal surface, the force of constraint is simply $\mathbf{F}_c = -m\mathbf{g}$. But, if the particle is, say, a bead sliding down a curved wire, the force of constraint can be quite complicated. Indeed, in particular situations it may be difficult or even impossible to obtain explicit expressions for the forces of constraint. But in solving a problem by using the Newtonian procedure, we must know *all* the forces, because the quantity $\mathbf{F}$ that appears in the fundamental equation is the *total* force acting on a body.

To circumvent some of the practical difficulties that arise in attempts to apply Newton's equations to particular problems, alternate procedures may be developed. All such approaches are in essence *a posteriori*, because we know beforehand that a result equivalent to the Newtonian equations must be obtained. Thus, to effect a simplification we need not formulate a *new* theory of mechanics—the Newtonian theory is quite correct—but only devise an alternate method of dealing with complicated problems in a general manner. Such a method is contained in **Hamilton's Principle,** and the equations of motion resulting from the application of this principle are called **Lagrange's equations.**

If Lagrange's equations are to constitute a proper description of the dynamics of particles, they must be equivalent to Newton's equations. On the other hand, Hamilton's Principle can be applied to a wide range of physical phenomena (particularly those involving *fields*) not usually associated with Newton's equations. To be sure, each of the results that can be obtained from Hamilton's Principle was *first* obtained, as were Newton's equations, by the correlation of experimental facts. Hamilton's Principle has not provided us with any new physical theories, but it has allowed a satisfying unification of many individual theories by a single basic postulate. This is not an idle exercise in hindsight, because it is the goal of physical theory not only to give precise mathematical formulation to observed phenomena but also to describe these effects with an economy of fundamental postulates and in the most unified manner possible. Indeed, Hamilton's Principle is one of the most elegant and far-reaching principles of physical theory.

In view of its wide range of applicability (even though this is an after-the-fact discovery), it is not unreasonable to assert that Hamilton's Principle is more "fundamental" than Newton's equations. Therefore, we proceed by first postulating Hamilton's Principle; we then obtain Lagrange's equations and show that these are equivalent to Newton's equations.

Because we have already discussed (in Chapters 2, 3, and 4) dissipative phenomena at some length, we henceforth confine our attention to *conservative* systems. Consequently, we do not discuss the more general set of Lagrange's equations, which take into account the effects of nonconservative forces. The reader is referred to the literature for these details.*

## 7.2 Hamilton's Principle

Minimal principles in physics have a long and interesting history. The search for such principles is predicated on the notion that nature always minimizes certain important quantities when a physical process takes place. The first such minimum principles were developed in the field of optics. Hero of Alexandria, in the second century B.C., found that the law governing the reflection of light could be obtained by asserting that a light ray, traveling from one point to another by a reflection from a plane mirror, always takes the shortest possible path. A simple geometric construction verifies that this minimum principle does indeed lead to

*See, for example, Goldstein (Go80, Chapter 2) or, for a comprehensive discussion, Whittaker (Wh37, Chapter 8).

the equality of the angles of incidence and reflection for a light ray reflected from a plane mirror. Hero's principle of the *shortest path* cannot, however, yield a correct law for *refraction.* In 1657, Fermat reformulated the principle by postulating that a light ray always travels from one point to another in a medium by a path that requires the least time.* Fermat's principle of *least time* leads immediately, not only to the correct law of reflection, but also to Snell's law of refraction (see Problem 6-7).†

Minimum principles continued to be sought, and in the latter part of the seventeenth century the beginnings of the calculus of variations were developed by Newton, Leibniz, and the Bernoullis when such problems as the brachistochrone (see Example 6.2) and the shape of a hanging chain (a catenary) were solved.

The first application of a general minimum principle in mechanics was made in 1747 by Maupertuis, who asserted that dynamical motion takes place with minimum action.‡ Maupertuis's **principle of least action** was based on theological grounds (action is minimized through the "wisdom of God"), and his concept of "action" was rather vague. (Recall that *action* is a quantity with the dimensions of *length* × *momentum* or *energy* × *time.*) Only later was a firm mathematic foundation of the principle given by Lagrange (1760). Although it is a useful form from which to make the transition from classical mechanics to optics and to quantum mechanics, the principle of least action is less general than Hamilton's Principle and, indeed, can be derived from it. We forego a detailed discussion here.§

In 1828, Gauss developed a method of treating mechanics by his principle of **least constraint;** a modification was later made by Hertz and embodied in his principle of **least curvature.** These principles‖ are closely related to Hamilton's Principle and add nothing to the content of Hamilton's more general formulation; their mention only emphasizes the continual concern with minimal principles in physics.

In two papers published in 1834 and 1835, Hamilton¶ announced the dynamical principle on which it is possible to base all of mechanics and, indeed, most of classical physics. Hamilton's Principle may be stated as follows**:

> *Of all the possible paths along which a dynamical system may move from one point to another within a specified time interval (consistent with any constraints), the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies.*

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*Pierre de Fermat (1601–1665), a French lawyer, linguist, and amateur mathematician.

†In 1661, Fermat correctly deduced the law of refraction, which had been discovered experimentally in about 1621 by Willebrord Snell (1591–1626), a Dutch mathematical prodigy.

‡Pierre-Louise-Moreau de Maupertuis (1698–1759), French mathematician and astronomer. The first use to which Maupertuis put the principle of least action was to restate Fermat's derivation of the law of refraction (1744).

§See, for example, Goldstein (Go80, pp. 365–371) or Sommerfeld (So50, pp. 204–209).

‖See, for example, Lindsay and Margenau (Li36, pp. 112–120) or Sommerfeld (So50, pp. 210–214).

¶Sir William Rowan Hamilton (1805–1865), Irish mathematician and astronomer, and later, Irish Astronomer Royal.

**The general meaning of "the path of a system" is made clear in Section 7.3.

In terms of the calculus of variations, Hamilton's Principle becomes

$$\delta \int_{t_1}^{t_2} (T - U)\, dt = 0 \tag{7.1}$$

where the symbol $\delta$ is a shorthand notation to describe the variation discussed in Sections 6.3 and 6.7. This variational statement of the principle requires only that the integral of $T - U$ be an *extremum*, not necessarily a *minimum*. But in almost all important applications in dynamics, the minimum condition occurs.

The kinetic energy of a particle expressed in fixed, rectangular coordinates is a function only of the $\dot{x}_i$, and if the particle moves in a conservative force field, the potential energy is a function only of the $x_i$:

$$T = T(\dot{x}_i), \quad U = U(x_i)$$

If we define the difference of these quantities to be

$$L \equiv T - U = L(x_i, \dot{x}_i) \tag{7.2}$$

then Equation 7.1 becomes

$$\boxed{\delta \int_{t_1}^{t_2} L(x_i, \dot{x}_i)\, dt = 0} \tag{7.3}$$

The function $L$ appearing in this expression may be identified with the function $f$ of the variational integral (see Section 6.5),

$$\delta \int_{x_1}^{x_2} f\{y_i(x), y_i'(x); x\}\, dx$$

if we make the transformations

$$\begin{aligned} x &\rightarrow t \\ y_i(x) &\rightarrow x_i(t) \\ y_i'(x) &\rightarrow \dot{x}_i(t) \\ f\{y_i(x), y_i'(x); x\} &\rightarrow L(x_i, \dot{x}_i) \end{aligned}$$

The Euler-Lagrange equations (Equation 6.57) corresponding to Equation 7.3 are therefore

$$\boxed{\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0, \quad i = 1, 2, 3} \quad \text{Lagrange equations of motion} \tag{7.4}$$

These are the **Lagrange equations of motion** for the particle, and the quantity $L$ is called the **Lagrange function** or **Lagrangian** for the particle.

By way of example, let us obtain the Lagrange equation of motion for the one-dimensional harmonic oscillator. With the usual expressions for the kinetic and potential energies, we have

$$
\begin{aligned}
L = T - U &= \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \\
\frac{\partial L}{\partial x} &= -kx \\
\frac{\partial L}{\partial \dot{x}} &= m\dot{x} \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) &= m\ddot{x}
\end{aligned}
$$

Substituting these results into Equation 7.4 leads to

$$m\ddot{x} + kx = 0$$

which is identical with the equation of motion obtained using Newtonian mechanics.

The Lagrangian procedure seems needlessly complicated if it can only duplicate the simple results of Newtonian theory. However, let us continue illustrating the method by considering the plane pendulum (see Section 4.4). Using Equation 4.23 for $T$ and $U$, we have, for the Lagrangian function

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta)$$

We now treat $\theta$ *as if it were a rectangular coordinate* and apply the operations specified in Equation 7.4; we obtain

$$
\begin{aligned}
\frac{\partial L}{\partial \theta} &= -mgl\sin\theta \\
\frac{\partial L}{\partial \dot{\theta}} &= ml^2\dot{\theta} \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= ml^2\ddot{\theta} \\
\ddot{\theta} + \frac{g}{l}\sin\theta &= 0
\end{aligned}
$$

which again is identical with the Newtonian result (Equation 4.21). This is a remarkable result; it has been obtained by calculating the kinetic and potential energies in terms of $\theta$ rather than $x$ and then applying a set of operations designed for use with rectangular rather than angular coordinates. We are therefore led to suspect that the Lagrange equations are more general and useful than the form of Equation 7.4 would indicate. We pursue this matter in Section 7.4.

Another important characteristic of the method used in the two preceding simple examples is that nowhere in the calculations did there enter any statement regarding *force*. The equations of motion were obtained only by specifying certain properties associated *with the particle* (the kinetic and potential energies), and without the necessity of explicitly taking into account the fact that there was an external agency acting *on the particle* (the force). Therefore, insofar as *energy* can be defined independently of Newtonian concepts, Hamilton's Principle allows us to calculate the equations of motion of a body completely without recourse to Newtonian theory. We shall return to this important point in Sections 7.5 and 7.7.

## 7.3 Generalized Coordinates

We now seek to take advantage of the flexibility in specifying coordinates that the two examples of the preceding section have suggested is inherent in Lagrange's equations.

We consider a general mechanical system consisting of a collection of $n$ discrete point particles, some of which may be connected to form rigid bodies. We discuss such systems of particles in Chapter 9 and rigid bodies in Chapter 11. To specify the state of such a system at a given time, it is necessary to use $n$ radius vectors. Because each radius vector consists of three numbers (e.g., the rectangular coordinates), $3n$ quantities must be specified to describe the positions of all the particles. If there exist equations of constraint that relate some of these coordinates to others (as would be the case, for example, if some of the particles were connected to form rigid bodies or if the motion were constrained to lie along some path or on some surface), then not all the $3n$ coordinates are independent. In fact, if there are $m$ equations of constraint, then $3n - m$ coordinates are independent, and the system is said to possess $3n - m$ *degrees of freedom.*

It is important to note that if $s = 3n - m$ coordinates are required in a given case, we need not choose $s$ rectangular coordinates or even $s$ curvilinear coordinates (e.g., spherical, cylindrical). We can choose *any* $s$ independent parameters, as long as they completely specify the state of the system. These $s$ quantities need not even have the dimensions of length. Depending on the problem at hand, it may prove more convenient to choose some of the parameters with dimensions of *energy*, some with dimensions of $(length)^2$, some that are *dimensionless*, and so forth. In Example 6.5, we described a disk rolling down an inclined plane in terms of one coordinate that was a length and one that was an angle. We give the name **generalized coordinates** to any set of quantities that completely specifies the state of a system. The generalized coordinates are customarily written as $q_1, q_2, \ldots$, or simply as the $q_j$. A set of independent generalized coordinates whose number equals the number $s$ of degrees of freedom of the system and not restricted by the constraints is called a *proper* set of generalized coordinates. In certain instances, it may be advantageous to use generalized coordinates whose number exceeds the number of degrees of freedom and to explicitly take into account the constraint relations through the use of the Lagrange undetermined multipliers. Such would be the case, for example, if we desired to calculate the forces of constraint (see Example 7.9).

The choice of a set of generalized coordinates to describe a system is not unique; there are in general many sets of quantities (in fact, an *infinite* number!) that completely specify the state of a given system. For example, in the problem of the disk rolling down the inclined plane, we might choose as coordinates the height of the center of mass of the disk above some reference level and the distance through which some point on the rim has traveled since the start of the motion. The ultimate test of the "suitability" of a particular set of generalized coordinates is whether the resulting equations of motion are sufficiently simple to allow a straightforward interpretation. Unfortunately, we can state no general rules for selecting the "most suitable" set of generalized coordinates for a given problem. A certain skill must be developed through experience, and we present many examples in this chapter.

In addition to the generalized coordinates, we may define a set of quantities consisting of the time derivatives of $q_j$: $\dot{q}_1, \dot{q}_2, \ldots$, or simply $\dot{q}_j$. In analogy with the nomenclature for rectangular coordinates, we call $\dot{q}_j$ the **generalized velocities.**

If we allow for the possibility that the equations connecting $x_{\alpha,i}$ and $q_j$ explicitly contain the time, then the set of transformation equations is given by*

$$\begin{aligned} x_{\alpha,i} &= x_{\alpha,i}(q_1, q_2, \ldots, q_s, t), \qquad &&\begin{cases} \alpha = 1, 2, \ldots, n \\ i = 1, 2, 3 \end{cases} \\ &= x_{\alpha,i}(q_j, t), && j = 1, 2, \ldots, s \end{aligned} \tag{7.5}$$

In general, the rectangular components of the velocities depend on the generalized coordinates, the generalized velocities, and the time:

$$\dot{x}_{\alpha,i} = \dot{x}_{\alpha,i}(q_j, \dot{q}_j, t) \tag{7.6}$$

We may also write the inverse transformations as

$$q_j = q_j(x_{\alpha,i}, t) \tag{7.7}$$

$$\dot{q}_j = \dot{q}_j(x_{\alpha,i}, \dot{x}_{\alpha,i}, t) \tag{7.8}$$

Also, there are $m = 3n - s$ equations of constraint of the form

$$f_k(x_{\alpha,i}, t) = 0, \quad k = 1, 2, \ldots, m \tag{7.9}$$

### EXAMPLE 7.1

Find a suitable set of generalized coordinates for a point particle moving on the surface of a hemisphere of radius $R$ whose center is at the origin.

***Solution.*** Because the motion always takes place on the surface, we have

$$x^2 + y^2 + z^2 - R^2 = 0, \quad z \geq 0 \tag{7.10}$$

Let us choose as our generalized coordinates the cosines of the angles between the $x$-, $y$-, and $z$-axes and the line connecting the particle with the origin.

*In this chapter, we attempt to simplify the notation by reserving the subscript $i$ to designate rectangular axes; therefore, we always have $i = 1, 2, 3$.

Therefore,

$$q_1 = \frac{x}{R}, \qquad q_2 = \frac{y}{R}, \qquad q_3 = \frac{z}{R} \tag{7.11}$$

But the sum of the squares of the direction cosines of a line equals unity. Hence,

$$q_1^2 + q_2^2 + q_3^2 = 1 \tag{7.12}$$

This set of $q_j$ does not constitute a proper set of generalized coordinates, because we can write $q_3$ as a function of $q_1$ and $q_2$:

$$q_3 = \sqrt{1 - q_1^2 - q_2^2} \tag{7.13}$$

We may, however, choose $q_1 = x/R$ and $q_2 = y/R$ as proper generalized coordinates, and these quantities, together with the equation of constraint (Equation 7.13)

$$z = \sqrt{R^2 - x^2 - y^2} \tag{7.14}$$

are sufficient to uniquely specify the position of the particle. This should be an obvious result, because only two coordinates (e.g., latitude and longitude) are necessary to specify a point on the surface of a sphere. But the example illustrates the fact that the equations of constraint can always be used to reduce a trial set of coordinates to a proper set of generalized coordinates.

## EXAMPLE 7.2

Use the $(x, y)$ coordinate system of Figure 7-1 to find the kinetic energy $T$, potential energy $U$, and the Lagrangian $L$ for a simple pendulum (length $\ell$, mass bob $m$) moving in the $x$, $y$ plane. Determine the transformation equations from the $(x, y)$ rectangular system to the coordinate $\theta$. Find the equation of motion.

***Solution.*** We have already examined this general problem in Sections 4.4 and 7.1. When using the Lagrangian method, it is often useful to begin with

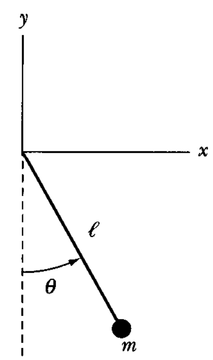


**FIGURE 7-1** Example 7.2. A simple pendulum of length $\ell$ and bob of mass $m$.

rectangular coordinates and transform to the most obvious system with the simplest generalized coordinates. In this case, the kinetic and potential energies and the Lagrangian become

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2$$

$$U = mgy$$

$$L = T - U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - mgy$$

Inspection of Figure 7-1 reveals that the motion can be better described by using $\theta$ and $\dot{\theta}$. Let's transform $x$ and $y$ into the coordinate $\theta$ and then find $L$ in terms of $\theta$.

$$x = \ell \sin\theta$$
$$y = -\ell \cos\theta$$

We now find for $\dot{x}$ and $\dot{y}$

$$\dot{x} = \ell\dot{\theta}\cos\theta$$

$$\dot{y} = \ell\dot{\theta}\sin\theta$$

$$L = \frac{m}{2}(\ell^2\dot{\theta}^2\cos^2\theta + \ell^2\dot{\theta}^2\sin^2\theta) + mg\ell\cos\theta = \frac{m}{2}\ell^2\dot{\theta}^2 + mg\ell\cos\theta$$

The only generalized coordinate in the case of the pendulum is the angle $\theta$, and we have expressed the Lagrangian in terms of $\theta$ by following a simple procedure of finding $L$ in terms of $x$ and $y$, finding the transformation equations, and then inserting them into the expression for $L$. If we do as we did in the previous section and treat $\theta$ *as if it were a rectangular coordinate*, we can find the equation of motion as follows:

$$\frac{\partial L}{\partial \theta} = -mg\ell\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = m\ell^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m\ell^2\ddot{\theta}$$

We insert these relations into Equation 7.4 to find the same equation of motion as found previously.

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$$

The state of a system consisting of $n$ particles and subject to $m$ constraints that connect some of the $3n$ rectangular coordinates is completely specified by $s = 3n - m$ generalized coordinates. We may therefore represent the state of such a system by a point in an $s$-dimensional space called **configuration space.** Each dimension of this space corresponds to one of the $q_j$ coordinates. We may represent the time history of a system by a curve in configuration space, each point specifying the *configuration* of the system at a particular instant. Through each such point passes an infinity of curves representing possible motions of the system; each curve corresponds to a particular set of initial conditions. We may therefore speak of the "path" of a system as it "moves" through configuration space. But we must be careful not to confuse this terminology with that applied to the motion of a particle along a path in ordinary three-dimensional space.

We should also note that a dynamical path in a configuration space consisting of proper generalized coordinates is automatically consistent with the constraints on the system, because the coordinates are chosen to correspond only to realizable motions of the system.

## 7.4 Lagrange's Equations of Motion in Generalized Coordinates

In view of the definitions in the preceding sections, we may now restate Hamilton's Principle as follows:

> *Of all the possible paths along which a dynamical system may move from one point to another in configuration space within a specified time interval, the actual path followed is that which minimizes the time integral of the Lagrangian function for the system.*

To set up the variational form of Hamilton's Principle in generalized coordinates, we may take advantage of an important property of the Lagrangian we have not so far emphasized. The Lagrangian for a system is defined to be the difference between the kinetic and potential energies. But *energy* is a scalar quantity and so the *Lagrangian is a scalar function.* Hence the Lagrangian must be *invariant with respect to coordinate transformations.* However, certain transformations that change the Lagrangian but *leave the equations of motion unchanged* are allowed. For example, equations of motion are unchanged if $L$ is replaced by $L + d/dt[f(q_i, t)]$ for a function $f(q_i, t)$ with continuous second partial derivatives. As long as we define the Lagrangian to be the difference between the kinetic and potential energies, we may use different generalized coordinates. (The Lagrangian is, however, indefinite to an additive constant in the potential energy $U$.) It is therefore immaterial whether we express the Lagrangian in terms of $x_{\alpha,i}$ and $\dot{x}_{\alpha,i}$ or $q_j$ and $\dot{q}_j$:

$$\begin{aligned} L &= T(\dot{x}_{\alpha,i}) - U(x_{\alpha,i}) \\ &= T(q_j, \dot{q}_j, t) - U(q_j, t) \end{aligned} \tag{7.15}$$

that is,

$$\begin{aligned} L &= L(q_1, q_2, \dots, q_s; \dot{q}_1, \dot{q}_2, \dots, \dot{q}_s; t) \\ &= L(q_j, \dot{q}_j, t) \end{aligned} \tag{7.16}$$

Thus, Hamilton's Principle becomes

$$\boxed{\delta \int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)\,dt = 0} \quad \text{Hamilton's Principle} \tag{7.17}$$

If we refer to the definitions of the quantities in Section 6.5 and make the identifications

$$\begin{aligned} x &\rightarrow t \\ y_i(x) &\rightarrow q_j(t) \\ y_i'(x) &\rightarrow \dot{q}_j(t) \\ f\{y_i, y_i'; x\} &\rightarrow L(q_j, \dot{q}_j, t) \end{aligned}$$

then the Euler equations (Equation 6.57) corresponding to the variational problem stated in Equation 7.17 become

$$\boxed{\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0, \qquad j = 1, 2, \dots, s} \tag{7.18}$$

These are the Euler-Lagrange equations of motion for the system (usually called simply **Lagrange's equations***). There are $s$ of these equations, and together with the $m$ equations of constraint and the initial conditions that are imposed, they completely describe the motion of the system.†

It is important to realize that the validity of Lagrange's equations requires the following two conditions:

1. The forces acting on the system (apart from any forces of constraint) must be derivable from a potential (or several potentials).
2. The equations of constraint must be relations that connect the *coordinates* of the particles and may be functions of the time—that is, we must have constraint relations of the form given by Equation 7.9.

If the constraints can be expressed as in condition 2, they are termed **holonomic** constraints. If the equations do not explicitly contain the time, the constraints are said to be **fixed** or **scleronomic;** moving constraints are **rheonomic.**

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*First derived for a mechanical system (although not, of course, by using Hamilton's Principle) by Lagrange and presented in his famous treatise *Mécanique analytique* in 1788. In this monumental work, which encompasses all phases of mechanics (statics, dynamics, hydrostatics, and hydrodynamics), Lagrange placed the subject on a firm and unified mathematical foundation. The treatise is mathematical rather than physical; Lagrange was quite proud of the fact that the entire work contains not a single diagram.

†Because there are $s$ second-order differential equations, $2s$ initial conditions must be supplied to determine the motion uniquely.

Here we consider only the motion of systems subject to conservative forces. Such forces can always be derived from potential functions, so that condition 1 is satisfied. This is not a necessary restriction on either Hamilton's Principle or Lagrange's equations; the theory can readily be extended to include nonconservative forces. Similarly, we can formulate Hamilton's Principle to include certain types of nonholonomic constraints, but the treatment here is confined to holonomic systems. We return to nonholonomic constraints in Section 7.5.

We now want to work several examples using Lagrange's equations. Experience is the best way to determine a set of generalized coordinates, realize the constraints, and set up the Lagrangian. Once this is done, the remainder of the problem is for the most part mathematical.

## EXAMPLE 7.3

Consider the case of projectile motion under gravity in two dimensions as was discussed in Example 2.6. Find the equations of motion in both Cartesian and polar coordinates.

***Solution.*** We use Figure 2-7 to describe the system. In Cartesian coordinates, we use $x$ (horizontal) and $y$ (vertical). In polar coordinates we use $r$ (in radial direction) and $\theta$ (elevation angle from horizontal). First, in Cartesian coordinates we have

$$\left.\begin{aligned} T &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 \\ U &= mgy \end{aligned}\right\} \tag{7.19}$$

where $U = 0$ at $y = 0$.

$$L = T - U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - mgy \tag{7.20}$$

We find the equations of motion by using Equation 7.18:

$x$:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$$

$$0 - \frac{d}{dt}m\dot{x} = 0$$

$$\ddot{x} = 0 \tag{7.21}$$

$y$:

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = 0$$

$$-mg - \frac{d}{dt}(m\dot{y}) = 0$$

$$\ddot{y} = -g \tag{7.22}$$

By using the initial conditions, Equations 7.21 and 7.22 can be integrated to determine the appropriate equations of motion.

In polar coordinates, we have

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(r\dot{\theta})^2$$

$$U = mgr\sin\theta$$

where $U = 0$ for $\theta = 0$.

$$L = T - U = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - mgr\sin\theta \tag{7.23}$$

$r$:

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$$

$$mr\dot{\theta}^2 - mg\sin\theta - \frac{d}{dt}(m\dot{r}) = 0$$

$$r\dot{\theta}^2 - g\sin\theta - \ddot{r} = 0 \tag{7.24}$$

$\theta$:

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$$

$$-mgr\cos\theta - \frac{d}{dt}(mr^2\dot{\theta}) = 0$$

$$-gr\cos\theta - 2r\dot{r}\dot{\theta} - r^2\ddot{\theta} = 0 \tag{7.25}$$

The equations of motion expressed by Equations 7.21 and 7.22 are clearly simpler than those of Equations 7.24 and 7.25. We should choose Cartesian coordinates as the generalized coordinates to solve this problem. The key in recognizing this was that the potential energy of the system only depended on the $y$ coordinate. In polar coordinates, the potential energy depended on both $r$ and $\theta$.

## EXAMPLE 7.4

A particle of mass $m$ is constrained to move on the inside surface of a smooth cone of half-angle $\alpha$ (see Figure 7-2). The particle is subject to a gravitational force. Determine a set of generalized coordinates and determine the constraints. Find Lagrange's equations of motion, Equation 7.18.

***Solution.*** Let the axis of the cone correspond to the $z$-axis and let the apex of the cone be located at the origin. Since the problem possesses cylindrical symmetry, we choose $r$, $\theta$, and $z$ as the generalized coordinates. We have, however, the equation of constraint

$$z = r\cot\alpha \tag{7.26}$$

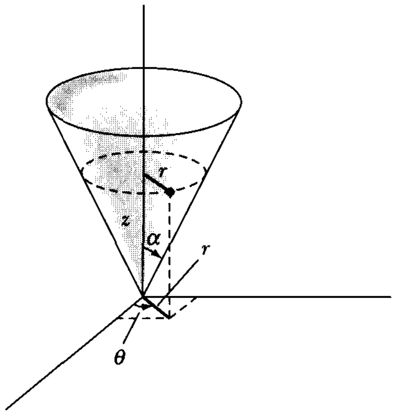


**FIGURE 7-2** Example 7.4. A smooth cone of half-angle $\alpha$. We choose $r$, $\theta$, and $z$ as the generalized coordinates.

so there are only two degrees of freedom for the system, and therefore only two proper generalized coordinates. We may use Equation 7.26 to eliminate either the coordinate $z$ or $r$; we choose to do the former. Then the square of the velocity is

$$\begin{aligned} v^2 &= \dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2 \\ &= \dot{r}^2 + r^2\dot{\theta}^2 + \dot{r}^2 \cot^2\alpha \\ &= \dot{r}^2 \csc^2\alpha + r^2\dot{\theta}^2 \end{aligned} \tag{7.27}$$

The potential energy (if we choose $U = 0$ at $z = 0$) is

$$U = mgz = mgr \cot \alpha$$

so the Langrangian is

$$L = \frac{1}{2}m\,(\dot{r}^2 \csc^2\alpha + r^2\dot{\theta}^2) - mgr \cot \alpha \tag{7.28}$$

We note first that $L$ does not explicitly contain $\theta$. Therefore $\partial L/\partial\theta = 0$, and the Lagrange equation for the coordinate $\theta$ is

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = 0$$

Hence

$$\frac{\partial L}{\partial\dot{\theta}} = mr^2\dot{\theta} = \text{constant} \tag{7.29}$$

But $mr^2\dot{\theta} = mr^2\omega$ is just the angular momentum about the $z$-axis. Therefore, Equation 7.29 expresses the conservation of angular momentum about the axis of symmetry of the system.

The Lagrange equation for $r$ is

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial\dot{r}} = 0 \tag{7.30}$$

Calculating the derivatives, we find

$$\ddot{r} - r\dot{\theta}^2 \sin^2\alpha + g \sin\alpha \cos\alpha = 0 \tag{7.31}$$

which is the equation of motion for the coordinate $r$.

We shall return to this example in Section 8.10 and examine the motion in more detail.

---

## EXAMPLE 7.5

The point of support of a simple pendulum of length $b$ moves on a massless rim of radius $a$ rotating with constant angular velocity $\omega$. Obtain the expression for the Cartesian components of the velocity and acceleration of the mass $m$. Obtain also the angular acceleration for the angle $\theta$ shown in Figure 7-3.

***Solution.*** We choose the origin of our coordinate system to be at the center of the rotating rim. The Cartesian components of mass $m$ become

$$\left.\begin{aligned} x &= a \cos \omega t + b \sin \theta \\ y &= a \sin \omega t - b \cos \theta \end{aligned}\right\} \tag{7.32}$$

The velocities are

$$\left.\begin{aligned} \dot{x} &= -a\omega \sin \omega t + b\dot{\theta} \cos \theta \\ \dot{y} &= a\omega \cos \omega t + b\dot{\theta} \sin \theta \end{aligned}\right\} \tag{7.33}$$

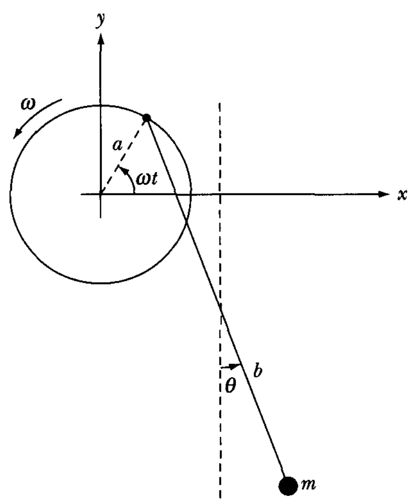


**FIGURE 7-3** Example 7.5. A simple pendulum is attached to a rotating rim.

Taking the time derivative once again gives the acceleration:

$$\ddot{x} = -a\omega^2 \cos \omega t + b(\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta)$$

$$\ddot{y} = -a\omega^2 \sin \omega t + b(\ddot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta)$$

It should now be clear that the single generalized coordinate is $\theta$. The kinetic and potential energies are

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$U = mgy$$

where $U = 0$ at $y = 0$. The Lagrangian is

$$L = T - U = \frac{m}{2}[a^2\omega^2 + b^2\dot{\theta}^2 + 2b\dot{\theta}a\omega \sin (\theta - \omega t)] - mg(a \sin \omega t - b \cos \theta) \tag{7.34}$$

The derivatives for the Lagrange equation of motion for $\theta$ are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = mb^2\ddot{\theta} + mba\omega(\dot{\theta} - \omega) \cos(\theta - \omega t)$$

$$\frac{\partial L}{\partial \theta} = mb\dot{\theta}a\omega \cos(\theta - \omega t) - mgb \sin \theta$$

which results in the equation of motion (after solving for $\ddot{\theta}$)

$$\ddot{\theta} = \frac{\omega^2 a}{b} \cos(\theta - \omega t) - \frac{g}{b} \sin \theta \tag{7.35}$$

Notice that this result reduces to the well-known equation of motion for a simple pendulum if $\omega = 0$.

## EXAMPLE 7.6

Find the frequency of small oscillations of a simple pendulum placed in a railroad car that has a constant acceleration $a$ in the $x$-direction.

***Solution.*** A schematic diagram is shown in Figure 7-4a for the pendulum of length $\ell$, mass $m$, and displacement angle $\theta$. We choose a fixed cartesian coordinate system with $x = 0$ and $\dot{x} = v_0$ at $t = 0$. The position and velocity of $m$ become

$$x = v_0 t + \frac{1}{2}at^2 + \ell \sin \theta$$

$$y = -\ell \cos \theta$$

$$\dot{x} = v_0 + at + \ell\dot{\theta} \cos \theta$$

$$\dot{y} = \ell\dot{\theta} \sin \theta$$

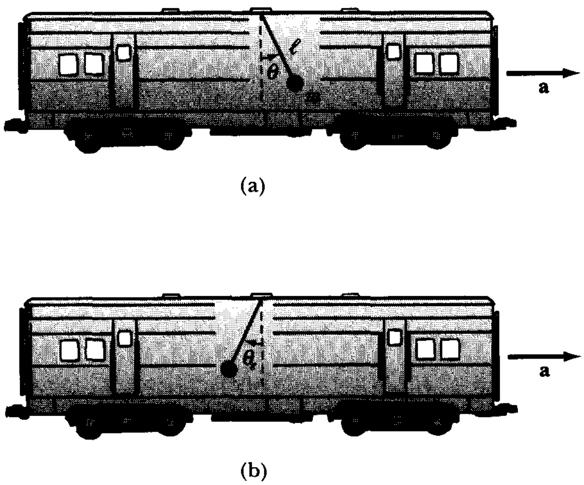


**FIGURE 7-4** Example 7.6. (a) A simple pendulum swings in an accelerating railroad car. (b) The angle $\theta_e$ is the equilibrium angle due to the car's acceleration $a$ and acceleration of gravity $g$.

The kinetic and potential energies are

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \qquad U = -mg\ell\cos\theta$$

and the Lagrangian is

$$L = T - U = \frac{1}{2}m(v_0 + at + \ell\dot{\theta}\cos\theta)^2 + \frac{1}{2}m(\ell\dot{\theta}\sin\theta)^2 + mg\ell\cos\theta$$

The angle $\theta$ is the only generalized coordinate, and after taking the derivatives for Lagrange's equations and suitable collection of terms, the equation of motion becomes (Problem 7-2)

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta - \frac{a}{\ell}\cos\theta \tag{7.36}$$

We determine the equilibrium angle $\theta = \theta_e$ by setting $\ddot{\theta} = 0$,

$$0 = g\sin\theta_e + \text{a}\cos\theta_e \tag{7.37}$$

The equilibrium angle $\theta_e$, shown in Figure 7-4b, is obtained by

$$\tan\theta_e = -\frac{a}{g} \tag{7.38}$$

Because the oscillations are small and are about the equilibrium angle, let $\theta = \theta_e + \eta$, where $\eta$ is a small angle.

$$\ddot{\theta} = \ddot{\eta} = -\frac{g}{\ell}\sin(\theta_e + \eta) - \frac{a}{\ell}\cos(\theta_e + \eta) \tag{7.39}$$

We expand the sine and cosine terms and use the small angle approximation for $\sin\eta$ and $\cos\eta$, keeping only the first terms in the Taylor series expansions.

$$\begin{aligned}\ddot{\eta} &= -\frac{g}{\ell}(\sin\theta_e\cos\eta + \cos\theta_e\sin\eta) - \frac{a}{\ell}(\cos\theta_e\cos\eta - \sin\theta_e\sin\eta)\\ &= -\frac{g}{\ell}(\sin\theta_e + \eta\cos\theta_e) - \frac{a}{\ell}(\cos\theta_e - \eta\sin\theta_e)\\ &= -\frac{1}{\ell}[(g\sin\theta_e + a\cos\theta_e) + \eta(g\cos\theta_e - a\sin\theta_e)]\end{aligned}$$

The first term in the brackets is zero because of Equation 7.37, which leaves

$$\ddot{\eta} = -\frac{1}{\ell}(g\cos\theta_e - a\sin\theta_e)\eta \tag{7.40}$$

We use Equation 7.38 to determine $\sin\theta_e$ and $\cos\theta_e$ and after a little manipulation (Problem 7-2), Equation 7.40 becomes

$$\ddot{\eta} = -\frac{\sqrt{a^2+g^2}}{\ell}\eta \tag{7.41}$$

Because this equation now represents simple harmonic motion, the frequency $\omega$ is determined to be

$$\omega^2 = \frac{\sqrt{a^2+g^2}}{\ell} \tag{7.42}$$

This result seems plausible, because $\omega \to \sqrt{g/\ell}$ for $a = 0$ when the railroad car is at rest.

---

## EXAMPLE 7.7

A bead slides along a smooth wire bent in the shape of a parabola $z = cr^2$ (Figure 7-5). The bead rotates in a circle of radius $R$ when the wire is rotating about its vertical symmetry axis with angular velocity $\omega$. Find the value of $c$.

***Solution.*** Because the problem has cylindrical symmetry, we choose $r$, $\theta$, and $z$ as the generalized coordinates. The kinetic energy of the bead is

$$T = \frac{m}{2}[\dot{r}^2 + \dot{z}^2 + (r\dot{\theta}^2)] \tag{7.43}$$

If we choose $U = 0$ at $z = 0$, the potential energy term is

$$U = mgz \tag{7.44}$$

But $r$, $z$, and $\theta$ are not independent. The equation of constraint for the parabola is

$$z = cr^2 \tag{7.45}$$

$$\dot{z} = 2c\dot{r}r \tag{7.46}$$

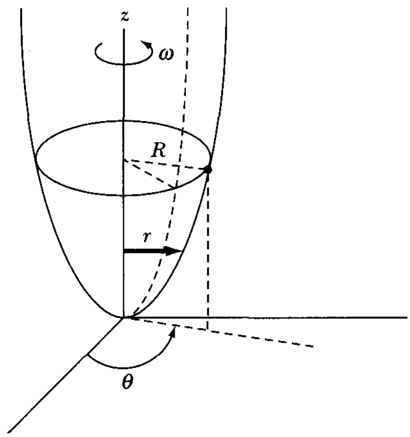


**FIGURE 7-5** Example 7.7. A bead slides along a smooth wire that rotates about the $z$-axis.

We also have an explicit time dependence of the angular rotation

$$\begin{aligned}\theta &= \omega t\\ \dot{\theta} &= \omega\end{aligned} \tag{7.47}$$

We can now construct the Lagrangian as being dependent only on $r$, because there is no direct $\theta$ dependence.

$$\begin{aligned}L &= T - U\\ &= \frac{m}{2}(\dot{r}^2 + 4c^2r^2\dot{r}^2 + r^2\omega^2) - mgcr^2\end{aligned} \tag{7.48}$$

The problem stated that the bead moved in a circle of radius $R$. The reader might be tempted at this point to let $r = R =$ const. and $\dot{r} = 0$. It would be a mistake to do this now in the Lagrangian. First, we should find the equation of motion for the variable $r$ and then let $r = R$ as a condition of the particular motion. This determines the particular value of $c$ needed for $r = R$.

$$\frac{\partial L}{\partial \dot{r}} = \frac{m}{2}(2\dot{r} + 8c^2r^2\dot{r})$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{m}{2}(2\ddot{r} + 16c^2r\dot{r}^2 + 8c^2r^2\ddot{r})$$

$$\frac{\partial L}{\partial r} = m(4c^2r\dot{r}^2 + r\omega^2 - 2gcr)$$

Lagrange's equation of motion becomes

$$\ddot{r}(1 + 4c^2r^2) + \dot{r}^2(4c^2r) + r(2gc - \omega^2) = 0 \tag{7.49}$$

which is a complicated result. If, however, the bead rotates with $r = R =$ constant, then $\dot{r} = \ddot{r} = 0$, and Equation 7.49 becomes

$$R(2gc - \omega^2) = 0$$

and

$$c = \frac{\omega^2}{2g} \tag{7.50}$$

is the result we wanted.

## EXAMPLE 7.8

Consider the double pulley system shown in Figure 7-6. Use the coordinates indicated, and determine the equations of motion.

***Solution.*** Consider the pulleys to be massless, and let $l_1$ and $l_2$ be the lengths of rope hanging freely from each of the two pulleys. The distances $x$ and $y$ are measured from the center of the two pulleys.

$m_1$:

$$v_1 = \dot{x} \tag{7.51}$$

$m_2$:

$$v_2 = \frac{d}{dt}(l_1 - x + y) = -\dot{x} + \dot{y} \tag{7.52}$$

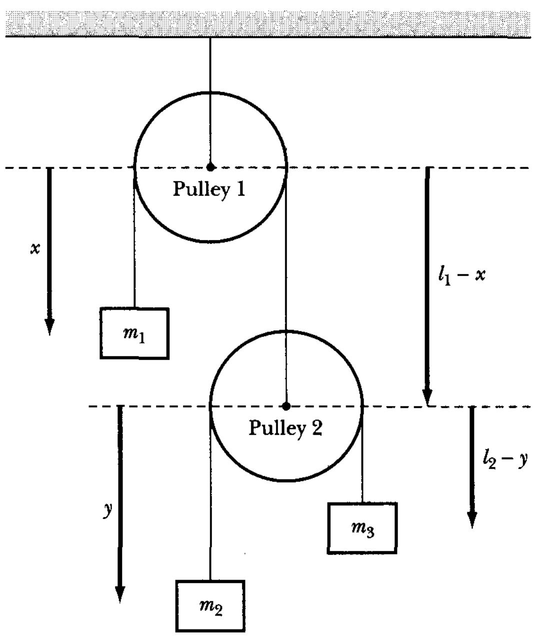


**FIGURE 7-6** Example 7.8. The double pulley system.

$m_3$:

$$v_3 = \frac{d}{dt}(l_1 - x + l_2 - y) = -\dot{x} - \dot{y} \tag{7.53}$$

$$\begin{aligned} T &= \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \frac{1}{2}m_3 v_3^2 \\ &= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(\dot{y} - \dot{x})^2 + \frac{1}{2}m_3(-\dot{x} - \dot{y})^2 \end{aligned} \tag{7.54}$$

Let the potential energy $U = 0$ at $x = 0$.

$$\begin{aligned} U &= U_1 + U_2 + U_3 \\ &= -m_1 gx - m_2 g(l_1 - x + y) - m_3 g(l_1 - x + l_2 - y) \end{aligned} \tag{7.55}$$

Because $T$ and $U$ have been determined, the equations of motion can be obtained using Equation 7.18. The results are

$$m_1\ddot{x} + m_2(\ddot{x} - \ddot{y}) + m_3(\ddot{x} + \ddot{y}) = (m_1 - m_2 - m_3)g \tag{7.56}$$

$$-m_2(\ddot{x} - \ddot{y}) + m_3(\ddot{x} + \ddot{y}) = (m_2 - m_3)g \tag{7.57}$$

Equations 7.56 and 7.57 can be solved for $\ddot{x}$ and $\ddot{y}$.

---

Examples 7.2–7.8 indicate the ease and usefulness of using Lagrange's equations. It has been said, probably unfairly, that Lagrangian techniques are simply recipes to follow. The argument is that we lose track of the "physics" by their use. Lagrangian methods, on the contrary, are extremely powerful and allow us to solve problems that otherwise would lead to severe complications using Newtonian methods. Simple problems can perhaps be solved just as easily using Newtonian methods, but the Lagrangian techniques can be used to attack a wide range of complex physical situations (including those occurring in quantum mechanics*).

## 7.5 Lagrange's Equations with Undetermined Multipliers

Constraints that can be expressed as algebraic relations among the coordinates are holonomic constraints. If a system is subject only to such constraints, we can always find a proper set of generalized coordinates in terms of which the equations of motion are free from explicit reference to the constraints.

Any constraints that must be expressed in terms of the *velocities* of the particles in the system are of the form

$$f(x_{\alpha,i}, \dot{x}_{\alpha,i}, t) = 0 \tag{7.58}$$

*See Feynman and Hibbs (Fe65).

and constitute nonholonomic constraints *unless* the equations can be integrated to yield relations among the coordinates.*

Consider a constraint relation of the form

$$\sum_i A_i \dot{x}_i + B = 0, \quad i = 1, 2, 3 \tag{7.59}$$

In general, this equation is nonintegrable, and therefore the constraint is nonholonomic. But if $A_i$ and $B$ have the forms

$$A_i = \frac{\partial f}{\partial x_i}, \quad B = \frac{\partial f}{\partial t}, \quad f = f(x_i, t) \tag{7.60}$$

then Equation 7.59 may be written as

$$\sum_i \frac{\partial f}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial f}{\partial t} = 0 \tag{7.61}$$

But this is just

$$\frac{df}{dt} = 0$$

which can be integrated to yield

$$f(x_i, t) - \text{constant} = 0 \tag{7.62}$$

so the constraint is actually holonomic.

From the preceding discussion, we conclude that constraints expressible in differential form as

$$\sum_j \frac{\partial f_k}{\partial q_j} dq_j + \frac{\partial f_k}{\partial t} dt = 0 \tag{7.63}$$

are equivalent to those having the form of Equation 7.9.

If the constraint relations for a problem are given in differential form rather than as algebraic expressions, we can incorporate them directly into Lagrange's equations by using the Lagrange undetermined multipliers (see Section 6.6) without first performing the integrations; that is, for constraints expressible as in Equation 6.71,

$$\sum_j \frac{\partial f_k}{\partial q_j} dq_j = 0 \quad \begin{cases} j = 1, 2, \ldots, s \\ k = 1, 2, \ldots, m \end{cases} \tag{7.64}$$

the Lagrange equations (Equation 6.69) are

$$\boxed{\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_k \lambda_k(t)\frac{\partial f_k}{\partial q_j} = 0} \tag{7.65}$$

In fact, because the variation process involved in Hamilton's Principle holds the time constant at the endpoints, we could add to Equation 7.64 a term $(\partial f_k/\partial t)dt$

*Such constraints are sometimes called "semiholonomic."

without affecting the equations of motion. Thus constraints expressed by Equation 7.63 also lead to the Lagrange equations given in Equation 7.65.

The great advantage of the Lagrangian formulation of mechanics is that the explicit inclusion of the forces of constraint is not necessary; that is, the emphasis is placed on the dynamics of the system rather than the calculation of the forces acting on each component of the system. In certain instances, however, it might be desirable to know the forces of constraint. For example, from an engineering standpoint, it would be useful to know the constraint forces for design purposes. It is therefore worth pointing out that in Lagrange's equations expressed as in Equation 7.65, **the undetermined multipliers $\lambda_k(t)$ are closely related to the forces of constraint.*** The generalized forces of constraint $Q_j$ are given by

$$Q_j = \sum_k \lambda_k \frac{\partial f_k}{\partial q_j} \tag{7.66}$$

## EXAMPLE 7.9

Let us consider again the case of the disk rolling down an inclined plane (see Example 6.5 and Figure 6-7). Find the equations of motion, the force of constraint, and the angular acceleration.

***Solution.*** The kinetic energy may be separated into translational and rotational terms†

$$\begin{aligned} T &= \frac{1}{2} M\dot{y}^2 + \frac{1}{2} I\dot{\theta}^2 \\ &= \frac{1}{2} M\dot{y}^2 + \frac{1}{4} MR^2 \dot{\theta}^2 \end{aligned}$$

where $M$ is the mass of the disk and $R$ is the radius; $I = \frac{1}{2} MR^2$ is the moment of inertia of the disk about a central axis. The potential energy is

$$U = Mg(l - y) \sin \alpha \tag{7.67}$$

where $l$ is the length of the inclined surface of the plane and where the disk is assumed to have zero potential energy at the bottom of the plane. The Lagrangian is therefore

$$\begin{aligned} L &= T - U \\ &= \frac{1}{2}M\dot{y}^2 + \frac{1}{4}MR^2\dot{\theta}^2 + Mg(y - l) \sin \alpha \end{aligned} \tag{7.68}$$

*See, for example, Goldstein (Go80, p. 47). Explicit calculations of the forces of constraint in some specific problems are carried out by Becker (Be54, Chapters 11 and 13) and by Symon (Sy71, p. 372ff).

†We anticipate here a well-known result from rigid-body dynamics discussed in Chapter 11.

The equation of constraint is

$$f(y, \theta) = y - R\theta = 0 \tag{7.69}$$

The system has only one degree of freedom if we insist that the rolling takes place without slipping. We may therefore choose either $y$ or $\theta$ as the proper coordinate and use Equation 7.69 to eliminate the other. Alternatively, we may continue to consider *both* $y$ and $\theta$ as generalized coordinates and use the method of undetermined multipliers. The Lagrange equations in this case are

$$\left.\begin{aligned} \frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} + \lambda\frac{\partial f}{\partial y} = 0 \\ \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \lambda\frac{\partial f}{\partial \theta} = 0 \end{aligned}\right\} \tag{7.70}$$

Performing the differentiations, we obtain, for the equations of motion,

$$Mg\sin\alpha - M\ddot{y} + \lambda = 0 \tag{7.71a}$$

$$-\frac{1}{2}MR^2\ddot{\theta} - \lambda R = 0 \tag{7.71b}$$

Also, from the constraint equation, we have

$$y = R\theta \tag{7.72}$$

These equations (Equations 7.71 and 7.72) constitute a soluble system for the three unknowns $y$, $\theta$, $\lambda$. Differentiating the equation of constraint (Equation 7.72), we obtain

$$\ddot{\theta} = \frac{\ddot{y}}{R} \tag{7.73}$$

Combining Equations 7.71b and 7.73, we find

$$\lambda = -\frac{1}{2}M\ddot{y} \tag{7.74}$$

and then using this expression in Equation 7.71a there results

$$\ddot{y} = \frac{2g\sin\alpha}{3} \tag{7.75}$$

with

$$\lambda = -\frac{Mg\sin\alpha}{3} \tag{7.76}$$

so that Equation 7.71b yields

$$\ddot{\theta} = \frac{2g\sin\alpha}{3R} \tag{7.77}$$

Thus, we have three equations for the quantities $\ddot{y}$, $\ddot{\theta}$, and $\lambda$ that can be immediately integrated.

We note that if the disk were to slide without friction down the plane, we would have $\ddot{y} = g\sin\alpha$. Therefore, the rolling constraint reduces the acceleration to $\frac{2}{3}$ of the value of frictionless sliding. The magnitude of the force of friction producing the constraint is just $\lambda$—that is, $(Mg/3)\sin\alpha$.

The generalized forces of constraint, Equation 7.66, are

$$Q_y = \lambda\frac{\partial f}{\partial y} = \lambda = -\frac{Mg\sin\alpha}{3}$$

$$Q_\theta = \lambda\frac{\partial f}{\partial \theta} = -\lambda R = \frac{MgR\sin\alpha}{3}$$

Note that $Q_y$ and $Q_\theta$ are a force and a torque, respectively, and they are the generalized forces of constraint required to keep the disk rolling down the plane without slipping.

Note that we may eliminate $\dot{\theta}$ from the Lagrangian by substituting $\dot{\theta} = \dot{y}/R$ from the equation of constraint:

$$L = \frac{3}{4}M\dot{y}^2 + Mg(y - l)\sin\alpha \tag{7.78}$$

The Lagrangian is then expressed in terms of only one proper coordinate, and the single equation of motion is immediately obtained from Equation 7.18:

$$Mg\sin\alpha - \frac{3}{2}M\ddot{y} = 0 \tag{7.79}$$

which is the same as Equation 7.75. Although this procedure is simpler, it cannot be used to obtain the force of constraint.

## EXAMPLE 7.10

A particle of mass $m$ starts at rest on top of a smooth fixed hemisphere of radius $a$. Find the force of constraint, and determine the angle at which the particle leaves the hemisphere.

***Solution.*** See Figure 7-7. Because we are considering the possibility of the particle leaving the hemisphere, we choose the generalized coordinates to be $r$ and $\theta$. The constraint equation is

$$f(r,\theta) = r - a = 0 \tag{7.80}$$

The Lagrangian is determined from the kinetic and potential energies:

$$T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$U = mgr\cos\theta$$

$$L = T - U$$

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta \tag{7.81}$$

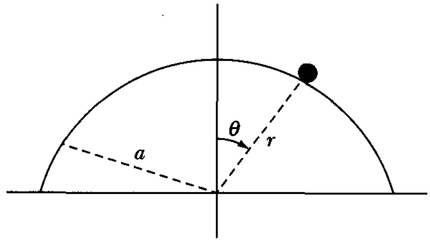


**FIGURE 7-7** Example 7.10. A particle of mass $m$ moves on the surface of a fixed smooth hemisphere.

where the potential energy is zero at the bottom of the hemisphere. The Lagrange equations, Equation 7.65, are

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} + \lambda\frac{\partial f}{\partial r} = 0 \tag{7.82}$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \lambda\frac{\partial f}{\partial \theta} = 0 \tag{7.83}$$

Performing the differentiations on Equation 7.80 gives

$$\frac{\partial f}{\partial r} = 1, \quad \frac{\partial f}{\partial \theta} = 0 \tag{7.84}$$

Equations 7.82 and 7.83 become

$$mr\dot{\theta}^2 - mg\cos\theta - m\ddot{r} + \lambda = 0 \tag{7.85}$$

$$mgr\sin\theta - mr^2\ddot{\theta} - 2mr\dot{r}\dot{\theta} = 0 \tag{7.86}$$

Next, we apply the constraint $r = a$ to these equations of motion:

$$r = a, \quad \dot{r} = 0 = \ddot{r}$$

Equations 7.85 and 7.86 then become

$$ma\dot{\theta}^2 - mg\cos\theta + \lambda = 0 \tag{7.87}$$

$$mga\sin\theta - ma^2\ddot{\theta} = 0 \tag{7.88}$$

From Equation 7.88, we have

$$\ddot{\theta} = \frac{g}{a}\sin\theta \tag{7.89}$$

We can integrate Equation 7.89 to determine $\dot{\theta}^2$.

$$\ddot{\theta} = \frac{d}{dt}\frac{d\theta}{dt} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{d\dot{\theta}}{d\theta} \tag{7.90}$$

We integrate Equation 7.89,

$$\int \dot{\theta}\, d\dot{\theta} = \frac{g}{a}\int \sin\theta\, d\theta \tag{7.91}$$

which results in

$$\frac{\dot{\theta}^2}{2} = \frac{-g}{a}\cos\theta + \frac{g}{a} \tag{7.92}$$

where the integration constant is $g/a$, because $\dot{\theta} = 0$ at $t = 0$ when $\theta = 0$. Substituting $\dot{\theta}^2$ from Equation 7.92 into Equation 7.87 gives, after solving for $\lambda$,

$$\lambda = mg(3\cos\theta - 2) \tag{7.93}$$

which is the force of constraint. The particle falls off the hemisphere at angle $\theta_0$ when $\lambda = 0$.

$$\lambda = 0 = mg(3\cos\theta_0 - 2) \tag{7.94}$$

$$\theta_0 = \cos^{-1}\left(\frac{2}{3}\right) \tag{7.95}$$

As a quick check, notice that the constraint force is $\lambda = mg$ at $\theta = 0$ when the particle is perched on top of the hemisphere.

---

The usefulness of the method of undetermined multipliers is twofold:

1. The Lagrange multipliers are closely related to the forces of constraint that are often needed.
2. When a proper set of generalized coordinates is not desired or too difficult to obtain, the method may be used to increase the number of generalized coordinates by including constraint relations between the coordinates.

## 7.6 Equivalence of Lagrange's and Newton's Equations

As we have emphasized from the outset, the Lagrangian and Newtonian formulations of mechanics are equivalent: The viewpoint is different, but the content is the same. We now explicitly demonstrate this equivalence by showing that the two sets of equations of motion are in fact the same.

In Equation 7.18, let us choose the generalized coordinates to be the rectangular coordinates. Lagrange's equations (for a single particle) then become

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0, \qquad i = 1, 2, 3 \tag{7.96}$$

or

$$\frac{\partial(T - U)}{\partial x_i} - \frac{d}{dt}\frac{\partial(T - U)}{\partial \dot{x}_i} = 0$$

But in rectangular coordinates and for a conservative system, we have $T = T(\dot{x}_i)$ and $U = U(x_i)$, so

$$\frac{\partial T}{\partial x_i} = 0 \quad \text{and} \quad \frac{\partial U}{\partial \dot{x}_i} = 0$$

Lagrange's equations therefore become

$$-\frac{\partial U}{\partial x_i} = \frac{d}{dt}\frac{\partial T}{\partial \dot{x}_i} \tag{7.97}$$

We also have (for a conservative system)

$$-\frac{\partial U}{\partial x_i} = F_i$$

and

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_i} = \frac{d}{dt}\frac{\partial}{\partial \dot{x}_i}\left(\sum_{j=1}^{3} \frac{1}{2} m\dot{x}_j^2\right) = \frac{d}{dt}(m\dot{x}_i) = \dot{p}_i$$

so Equation 7.97 yields the Newtonian equations, as required:

$$F_i = \dot{p}_i \tag{7.98}$$

Thus, the Lagrangian and Newtonian equations are identical if the generalized coordinates are the rectangular coordinates.

Now let us derive Lagrange's equations of motion using Newtonian concepts. Consider only a single particle for simplicity. We need to transform from the $x_i$-coordinates to the generalized coordinates $q_j$. From Equation 7.5, we have

$$x_i = x_i(q_j, t) \tag{7.99}$$

$$\dot{x}_i = \sum_j \frac{\partial x_i}{\partial q_j}\dot{q}_j + \frac{\partial x_i}{\partial t} \tag{7.100}$$

and

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_i} = \frac{\partial x_i}{\partial q_i} \tag{7.101}$$

A generalized momentum $p_j$ associated with $q_j$ is easily determined by

$$p_j = \frac{\partial T}{\partial \dot{q}_j} \tag{7.102}$$

For example, for a particle moving in plane polar coordinates, $T = (\dot{r}^2 + r^2\dot{\theta}^2)\, m/2$, we have $p_r = m\dot{r}$ for coordinate $r$ and $p_\theta = mr^2\dot{\theta}$ for coordinate $\theta$. Obviously $p_r$ is a linear momentum and $p_\theta$ is an angular momentum, so our generalized momentum definition seems consistent with Newtonian concepts.

We can determine a generalized force by considering the *virtual* work $\delta W$ done by a varied path $\delta x_i$ as described in Section 6.7.

$$\delta W = \sum_i F_i \delta x_i = \sum_{i,j} F_i \frac{\partial x_i}{\partial q_j} \delta q_j \tag{7.103}$$

$$\equiv \sum_j Q_j \delta q_j \tag{7.104}$$

so that the generalized force $Q_j$ associated with $q_j$ is

$$Q_j = \sum_i F_i \frac{\partial x_i}{\partial q_j} \tag{7.105}$$

Just as work is always energy, so is the product of $Qq$. If $q$ is length, $Q$ is force; if $q$ is an angle, $Q$ is torque. For a conservative system, $Q_j$ is derivable from the potential energy:

$$Q_j = -\frac{\partial U}{\partial q_j} \tag{7.106}$$

Now we are ready to obtain Lagrange's equations:

$$p_j = \frac{\partial T}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}\left(\sum_i \frac{1}{2} m\dot{x}_i^2\right)$$

$$= \sum_i m\dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_j}$$

$$p_j = \sum_i m\dot{x}_i \frac{\partial x_i}{\partial q_j} \tag{7.107}$$

where we use Equation 7.101 for the last step. Taking the time derivative of Equation 7.107 gives

$$\dot{p}_j = \sum_i \left( m\ddot{x}_i \frac{\partial x_i}{\partial q_j} + m\dot{x}_i \frac{d}{dt}\frac{\partial x_i}{\partial q_j} \right) \tag{7.108}$$

Expanding the last term gives

$$\frac{d}{dt}\frac{\partial x_i}{\partial q_j} = \sum_k \frac{\partial^2 x_i}{\partial q_k \partial q_j}\dot{q}_k + \frac{\partial^2 x_i}{\partial q_j \partial t}$$

and Equation 7.108 becomes

$$\dot{p}_j = \sum_i m\ddot{x}_i \frac{\partial x_i}{\partial q_j} + \sum_{i,k} m\dot{x}_i \frac{\partial^2 x_i}{\partial q_k \partial q_j}\dot{q}_k + \sum_i m\dot{x}_i \frac{\partial^2 x_i}{\partial q_j \partial t} \tag{7.109}$$

The first term on the right side of Equation 7.109 is just $Q_j (F_i = m\ddot{x}_i$ and Equation 7.105). The sum of the other two terms is $\partial T/\partial q_j$:

$$\frac{\partial T}{\partial q_j} = \sum_i m\dot{x}_i \frac{\partial \dot{x}_i}{\partial q_j}$$

$$= \sum_i m\dot{x}_i \frac{\partial}{\partial q_j}\left(\sum_k \frac{\partial x_i}{\partial q_k}\dot{q}_k + \frac{\partial x_i}{\partial t}\right) \tag{7.110}$$

where we have used $T = \sum_i 1/2\, m\dot{x}_i^2$ and Equation 7.100.

Equation 7.109 can now be written as

$$\dot{p}_j = Q_j + \frac{\partial T}{\partial q_j} \tag{7.111}$$

or, using Equations 7.102 and 7.106,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j = -\frac{\partial U}{\partial q_j} \tag{7.112}$$

Because $U$ does not depend on the generalized velocities $\dot{q}_j$, Equation 7.112 can be written

$$\frac{d}{dt}\left[\frac{\partial(T - U)}{\partial \dot{q}_j}\right] - \frac{\partial(T - U)}{\partial q_j} = 0 \tag{7.113}$$

and using $L = T - U$,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \tag{7.114}$$

which are Lagrange's equations of motion.

## 7.7 Essence of Lagrangian Dynamics

In the preceding sections, we made several general and important statements concerning the Lagrange formulation of mechanics. Before proceeding further, we should summarize these points to emphasize the differences between the Lagrange and Newtonian viewpoints.

Historically, the Lagrange equations of motion expressed in generalized coordinates were derived before the statement of Hamilton's Principle.* We elected to deduce Lagrange's equations by postulating Hamilton's Principle because this is the most straightforward approach and is also the formal method for unifying classical dynamics.

First, we must reiterate that Lagrangian dynamics does not constitute a *new* theory in any sense of the word. The results of a Lagrangian analysis or a Newtonian analysis must be the same for any given mechanical system. The only difference is the method used to obtain these results.

Whereas the Newtonian approach emphasizes an outside agency acting *on* a body (the *force*), the Lagrangian method deals only with quantities associated *with* the body (the kinetic and potential *energies*). In fact, nowhere in the Lagrangian formulation does the concept of *force* enter. This is a particularly important property—and for a variety of reasons. First, because energy is a scalar quantity, the Lagrangian function for a system is invariant to coordinate transformations. Indeed, such transformations are not restricted to be between various

*Lagrange's equations, 1788; Hamilton's Principle, 1834.

orthogonal coordinate systems in ordinary space; they may also be transformations between *ordinary* coordinates and *generalized* coordinates. Thus, it is possible to pass from ordinary space (in which the equations of motion may be quite complicated) to a configuration space that can be chosen to yield maximum simplification for a particular problem. We are accustomed to thinking of mechanical systems in terms of *vector* quantities such as force, velocity, angular momentum, and torque. But in the Lagrangian formulation, the equations of motion are obtained entirely in terms of *scalar* operations in configuration space.

Another important aspect of the force-versus-energy viewpoint is that in certain situations it may not even be possible to state explicitly all the forces acting on a body (as is sometimes the case for forces of constraint), whereas it is still possible to give expressions for the kinetic and potential energies. It is just this fact that makes Hamilton's Principle useful for quantum-mechanical systems where we normally know the energies but not the forces.

The differential statement of mechanics contained in Newton's equations or the integral statement embodied in Hamilton's Principle (and the resulting Lagrangian equations) have been shown to be entirely equivalent. Hence, no distinction exists between these viewpoints, which are based on the description of *physical effects.* But from a philosophical standpoint, we can make a distinction. In the Newtonian formulation, a certain force on a body produces a definite motion—that is, we always associate a definite *effect* with a certain *cause.* According to Hamilton's Principle, however, the motion of a body results from the attempt of nature to achieve a certain *purpose,* namely, to minimize the time integral of the difference between the kinetic and potential energies. The operational solving of problems in mechanics does not depend on adopting one or the other of these views. But historically such considerations have had a profound influence on the development of dynamics (as, for example, in Maupertuis's principle, mentioned in Section 7.2). The interested reader is referred to Margenau's excellent book for a discussion of these matters.*

## 7.8 A Theorem Concerning the Kinetic Energy

If the kinetic energy is expressed in fixed, rectangular coordinates, the result is a homogeneous quadratic function of $\dot{x}_{\alpha,i}$:

$$T = \frac{1}{2} \sum_{\alpha=1}^{n} \sum_{i=1}^{3} m_\alpha \dot{x}_{\alpha,i}^2 \tag{7.115}$$

We now wish to consider in more detail the dependence of $T$ on the generalized coordinates and velocities. For many particles, Equations 7.99 and 7.100 become

$$x_{\alpha,i} = x_{\alpha,i}(q_j, t), \quad j = 1, 2, \ldots, s \tag{7.116}$$

$$\dot{x}_{\alpha,i} = \sum_{j=1}^{s} \frac{\partial x_{\alpha,i}}{\partial q_j} \dot{q}_j + \frac{\partial x_{\alpha,i}}{\partial t} \tag{7.117}$$

*Margenau (Ma77, Chapter 19).

Evaluating the square of $\dot{x}_{\alpha,i}$, we obtain

$$\dot{x}_{\alpha,i}^2 = \sum_{j,k} \frac{\partial x_{\alpha,i}}{\partial q_j} \frac{\partial x_{\alpha,i}}{\partial q_k} \dot{q}_j \dot{q}_k + 2\sum_j \frac{\partial x_{\alpha,i}}{\partial q_j} \frac{\partial x_{\alpha,i}}{\partial t} \dot{q}_j + \left(\frac{\partial x_{\alpha,i}}{\partial t}\right)^2 \tag{7.118}$$

and the kinetic energy becomes

$$T = \sum_\alpha \sum_{i,j,k} \frac{1}{2} m_\alpha \frac{\partial x_{\alpha,i}}{\partial q_j} \frac{\partial x_{\alpha,i}}{\partial q_k} \dot{q}_j \dot{q}_k + \sum_\alpha \sum_{i,j} m_\alpha \frac{\partial x_{\alpha,i}}{\partial q_j} \frac{\partial x_{\alpha,i}}{\partial t} \dot{q}_j + \sum_\alpha \sum_i \frac{1}{2} m_\alpha \left(\frac{\partial x_{\alpha,i}}{\partial t}\right)^2 \tag{7.119}$$

Thus, we have the general result

$$T = \sum_{j,k} a_{jk} \dot{q}_j \dot{q}_k + \sum_j b_j \dot{q}_j + c \tag{7.120}$$

A particularly important case occurs when the system is *scleronomic*, so that the time does not appear explicitly in the equations of transformation (Equation 7.116); then the partial time derivatives vanish:

$$\frac{\partial x_{\alpha,i}}{\partial t} = 0, \qquad b_j = 0, \qquad c = 0$$

Therefore, under these conditions, the kinetic energy is a *homogeneous quadratic function* of the generalized velocities:

$$\boxed{T = \sum_{j,k} a_{jk} \dot{q}_j \dot{q}_k} \tag{7.121}$$

Next, we differentiate Equation 7.121 with respect to $\dot{q}_l$:

$$\frac{\partial T}{\partial \dot{q}_l} = \sum_k a_{lk} \dot{q}_k + \sum_j a_{jl} \dot{q}_j$$

Multiplying this equation by $\dot{q}_l$ and summing over $l$, we have

$$\sum_l \dot{q}_l \frac{\partial T}{\partial \dot{q}_l} = \sum_{k,l} a_{lk} \dot{q}_k \dot{q}_l + \sum_{j,l} a_{jl} \dot{q}_j \dot{q}_l$$

In this case, *all* the indices are dummies, so both terms on the right-hand side are identical:

$$\boxed{\sum_l \dot{q}_l \frac{\partial T}{\partial \dot{q}_l} = 2\sum_{j,k} a_{jk} \dot{q}_j \dot{q}_k = 2T} \tag{7.122}$$

This important result is a special case of *Euler's theorem*, which states that if $f(y_k)$ is a homogeneous function of the $y_k$ that is of degree $n$, then

$$\sum_k y_k \frac{\partial f}{\partial y_k} = nf \tag{7.123}$$

## 7.9 Conservation Theorems Revisited

### Conservation of Energy

We saw in our previous arguments* that *time* is homogeneous within an inertial reference frame. Therefore, the Lagrangian that describes a *closed system* (i.e., a system not interacting with anything outside the system) cannot depend explicitly on time,† that is,

$$\frac{\partial L}{\partial t} = 0 \tag{7.124}$$

so that the total derivative of the Lagrangian becomes

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j \tag{7.125}$$

where the usual term, $\partial L/\partial t$, does not now appear. But Lagrange's equations are

$$\frac{\partial L}{\partial q_j} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} \tag{7.126}$$

Using Equation 7.126 to substitute for $\partial L/\partial q_j$ in Equation 7.125, we have

$$\frac{dL}{dt} = \sum_j \dot{q}_j \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j$$

or

$$\frac{dL}{dt} - \sum_j \frac{d}{dt}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) = 0$$

so that

$$\frac{d}{dt}\left(L - \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) = 0 \tag{7.127}$$

The quantity in the parentheses is therefore constant in time; denote this constant by $-H$:

$$L - \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} = -H = \text{constant} \tag{7.128}$$

If the potential energy $U$ does not depend explicitly on the velocities $\dot{x}_{\alpha,i}$ or the time $t$, then $U = U(x_{\alpha,i})$. The relations connecting the rectangular coordinates and the generalized coordinates are of the form $x_{\alpha,i} = x_{\alpha,i}(q_j)$ or $q_j = q_j(x_{\alpha,i})$,

*See Section 2.3.

†The Lagrangian is likewise independent of the time if the system exists in a uniform force field.

where we exclude the possibility of an explicit time dependence in the transformation equations. Therefore, $U = U(q_j)$, and $\partial U/\partial \dot{q}_j = 0$. Thus

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial (T - U)}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}$$

Equation 7.128 can then be written as

$$(T - U) - \sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = -H \tag{7.129}$$

and, using Equation 7.122, we have

$$(T - U) - 2T = -H$$

or

$$T + U = E = H = \text{constant} \tag{7.130}$$

The total energy $E$ is a constant of the motion for this case.

The function $H$, called the **Hamiltonian** of the system, may be defined as in Equation 7.128 (but see Section 7.10). It is important to note that the Hamiltonian $H$ is equal to the total energy $E$ only if the following conditions are met:

1. The equations of the transformation connecting the rectangular and generalized coordinates (Equation 7.116) must be independent of the time, thus ensuring that the kinetic energy is a homogeneous quadratic function of the $\dot{q}_j$.
2. The potential energy must be velocity independent, thus allowing the elimination of the terms $\partial U/\partial \dot{q}_j$ from the equation for $H$ (Equation 7.129).

The questions "Does $H = E$ for the system?" and "Is energy conserved for the system?," then, pertain to two *different* aspects of the problem, and each question must be examined separately. We may, for example, have cases in which the Hamiltonian does not equal the total energy, but nevertheless, the energy is conserved. Thus, consider a conservative system, and let the description be made in terms of generalized coordinates in motion with respect to fixed, rectangular axes. The transformation equations then contain the time, and the kinetic energy is *not* a homogeneous quadratic function of the generalized velocities. The choice of a mathematically convenient set of generalized coordinates cannot alter the physical fact that energy is conserved. But in the moving coordinate system, the Hamiltonian is no longer equal to the total energy.

## Conservation of Linear Momentum

Because space is homogeneous in an inertial reference frame, the Lagrangian of a closed system is unaffected by a translation of the entire system in space. Consider an infinitesimal translation of every radius vector $\mathbf{r}_\alpha$ such that $\mathbf{r}_\alpha \to \mathbf{r}_\alpha + \delta\mathbf{r}$; this amounts to translating the entire system by $\delta\mathbf{r}$. For simplicity, let us examine a system consisting of only a single particle (by including a summation over $\alpha$ we could consider an $n$-particle system in an entirely equivalent manner), and let us write the Lagrangian in terms of rectangular coordinates $L = L(x_i, \dot{x}_i)$. The change in $L$ caused by the infinitesimal displacement $\delta\mathbf{r} = \sum_i \delta x_i \mathbf{e}_i$ is

$$\delta L = \sum_i \frac{\partial L}{\partial x_i}\delta x_i + \sum_i \frac{\partial L}{\partial \dot{x}_i}\delta \dot{x}_i = 0 \tag{7.131}$$

We consider only a varied *displacement*, so that the $\delta x_i$ are not explicit or implicit functions of the time. Thus,

$$\delta \dot{x}_i = \delta\frac{dx_i}{dt} = \frac{d}{dt}\delta x_i \equiv 0 \tag{7.132}$$

Therefore, $\delta L$ becomes

$$\delta L = \sum_i \frac{\partial L}{\partial x_i}\delta x_i = 0 \tag{7.133}$$

Because each of the $\delta x_i$ is an independent displacement, $\delta L$ vanishes identically only if each of the partial derivatives of $L$ vanishes:

$$\frac{\partial L}{\partial x_i} = 0 \tag{7.134}$$

Then, according to Lagrange's equations,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0 \tag{7.135}$$

and

$$\frac{\partial L}{\partial \dot{x}_i} = \text{constant} \tag{7.136}$$

or

$$\frac{\partial(T-U)}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} = \frac{\partial}{\partial \dot{x}_i}\left(\frac{1}{2}m\sum_j \dot{x}_j^2\right)$$
$$= m\dot{x}_i = p_i = \text{constant} \tag{7.137}$$

Thus, the homogeneity of space implies that the linear momentum $\mathbf{p}$ of a closed system is constant in time.

This result may also be interpreted according to the following statement: If the Lagrangian of a system (not necessarily *closed*) is invariant with respect to translation in a certain direction, then the linear momentum of the system in that direction is constant in time.

## Conservation of Angular Momentum

We stated in Section 2.3 that one characteristic of an inertial reference frame is that space is *isotropic*—that is, that the mechanical properties of a closed system are unaffected by the orientation of the system. In particular, the Lagrangian of a closed system does not change if the system is rotated through an infinitesimal angle.*

*We limit the rotation to an infinitesimal angle because we wish to be able to represent the rotation by a vector; see Section 1.15.

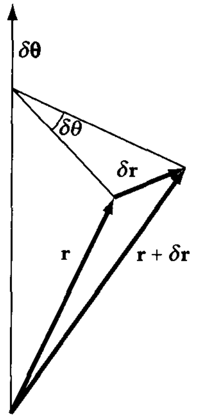


**FIGURE 7-8** A system is rotated by an infinitesimal angle $\delta\theta$.

If a system is rotated about a certain axis by an infinitesimal angle $\delta\theta$ (see Figure 7-8), the radius vector $\mathbf{r}$ to a given point changes to $\mathbf{r} + \delta\mathbf{r}$, where (see Equation 1.106)

$$\delta\mathbf{r} = \delta\boldsymbol{\theta} \times \mathbf{r} \tag{7.138}$$

The velocity vectors also change on rotation of the system, and because the transformation equation for all vectors is the same, we have

$$\delta\dot{\mathbf{r}} = \delta\boldsymbol{\theta} \times \dot{\mathbf{r}} \tag{7.139}$$

We consider only a single particle and express the Lagrangian in rectangular coordinates. The change in $L$ caused by the infinitesimal rotation is

$$\delta L = \sum_i \frac{\partial L}{\partial x_i}\delta x_i + \sum_i \frac{\partial L}{\partial \dot{x}_i}\delta\dot{x}_i = 0 \tag{7.140}$$

Equations 7.136 and 7.137 show that the rectangular components of the momentum vector are given by

$$p_i = \frac{\partial L}{\partial \dot{x}_i} \tag{7.141}$$

Lagrange's equations may then be expressed by

$$\dot{p}_i = \frac{\partial L}{\partial x_i} \tag{7.142}$$

Hence, Equation 7.140 becomes

$$\delta L = \sum_i \dot{p}_i\delta x_i + \sum_i p_i\delta\dot{x}_i = 0 \tag{7.143}$$

or

$$\dot{\mathbf{p}}\cdot\delta\mathbf{r} + \mathbf{p}\cdot\delta\dot{\mathbf{r}} = 0 \tag{7.144}$$

Using Equations 7.138 and 7.139, this equation may be written as

$$\dot{\mathbf{p}} \cdot (\delta\boldsymbol{\theta} \times \mathbf{r}) + \mathbf{p} \cdot (\delta\boldsymbol{\theta} \times \dot{\mathbf{r}}) = 0 \tag{7.145}$$

We may permute in cyclic order the factors of a triple scalar product without altering the value. Thus,

$$\delta\boldsymbol{\theta} \cdot (\mathbf{r} \times \dot{\mathbf{p}}) + \delta\boldsymbol{\theta} \cdot (\dot{\mathbf{r}} \times \mathbf{p}) = 0$$

or

$$\delta\boldsymbol{\theta} \cdot [(\mathbf{r} \times \dot{\mathbf{p}}) + (\dot{\mathbf{r}} \times \mathbf{p})] = 0 \tag{7.146}$$

The terms in the brackets are just the factors that result from the differentiation with respect to time of $\mathbf{r} \times \mathbf{p}$:

$$\delta\boldsymbol{\theta} \cdot \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = 0 \tag{7.147}$$

Because $\delta\boldsymbol{\theta}$ is arbitrary, we must have

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = 0 \tag{7.148}$$

so

$$\mathbf{r} \times \mathbf{p} = \text{constant} \tag{7.149}$$

But $\mathbf{r} \times \mathbf{p} = \mathbf{L}$; the angular momentum of the particle in a closed system is therefore constant in time.

An important corollary of this theorem is the following. Consider a system in an external force field. If the field possesses an axis of symmetry, then the Lagrangian of the system is invariant with respect to rotations about the symmetry axis. Hence, the angular momentum of the system about the axis of symmetry is constant in time. This is exactly the case discussed in Example 7.4; the vertical direction was an axis of symmetry of the system, and the angular momentum about that axis was conserved.

The importance of the connection between *symmetry* properties and the *invariance* of physical quantities can hardly be overemphasized. The association goes beyond momentum conservation—indeed beyond classical systems—and finds wide application in modern theories of field phenomena and elementary particles.

We have derived the conservation theorems for a closed system simply by considering the properties of an inertial reference frame. The results, summarized in Table 7-1, are generally credited to Emmy Noether.*

There are then seven constants (or integrals) of the motion for a closed system: total energy, linear momentum (three components), and angular momentum (three components). These and only these seven integrals have the property that they are *additive* for the particles composing the system; they possess this property whether or not there is an interaction among the particles.

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*Emmy Noether (1882–1935), one of the first female German mathematical physicists, endured poor treatment by German mathematicians early in her career. She is the originator of Noether's Theorem, which proves a relationship between symmetries and conservation principles.

**TABLE 7-1**

| Characteristic of inertial frame | Property of Lagrangian | Conserved quantity |
|---|---|---|
| Time homogeneous | Not explicit function of time | Total energy |
| Space homogeneous | Invariant to translation | Linear momentum |
| Space isotropic | Invariant to rotation | Angular momentum |

## 7.10 Canonical Equations of Motion—Hamiltonian Dynamics

In the previous section, we found that if the potential energy of a system is velocity independent, then the linear momentum components in rectangular coordinates are given by

$$p_i = \frac{\partial L}{\partial \dot{x}_i} \tag{7.150}$$

By analogy, we extend this result to the case in which the Lagrangian is expressed in generalized coordinates and define the **generalized momenta*** according to

$$\boxed{p_j \equiv \frac{\partial L}{\partial \dot{q}_j}} \tag{7.151}$$

(Unfortunately, the customary notations for ordinary momentum and generalized momentum are the same, even though the two quantities may be quite different.) The Lagrange equations of motion are then expressed by

$$\boxed{\dot{p}_j = \frac{\partial L}{\partial q_j}} \tag{7.152}$$

Using the definition of the generalized momenta, Equation 7.128 for the Hamiltonian may be written as

$$H = \sum_j p_j \dot{q}_j - L \tag{7.153}$$

The Lagrangian is considered to be a function of the generalized coordinates, the generalized velocities, and possibly the time. The dependence of $L$ on the time may arise either if the constraints are time dependent or if the transformation equations connecting the rectangular and generalized coordinates explicitly contain the time. (Recall that we do not consider time-dependent potentials.) We may solve Equation 7.151 for the generalized velocities and express them as

$$\dot{q}_j = \dot{q}_j(q_k, p_k, t) \tag{7.154}$$

*The terms *generalized coordinates, generalized velocities,* and *generalized momenta* were introduced in 1867 by Sir William Thomson (later, Lord Kelvin) and P. G. Tait in their famous treatise *Natural Philosophy.*

Thus, in Equation 7.153, we may make a change of variables from the $(q_j, \dot{q}_j, t)$ set to the $(q_j, p_j, t)$ set* and express the Hamiltonian as

$$\boxed{H(q_k, p_k, t) = \sum_j p_j\dot{q}_j - L(q_k, \dot{q}_k, t)} \tag{7.155}$$

This equation is written in a manner that stresses the fact that *the Hamiltonian is always considered as a function of the* $(q_k, p_k, t)$ *set, whereas the Lagrangian is a function of the* $(q_k, \dot{q}_k, t)$ *set*:

$$\boxed{H = H(q_k, p_k, t), \qquad L = L(q_k, \dot{q}_k, t)} \tag{7.156}$$

The total differential of $H$ is therefore

$$dH = \sum_k \left(\frac{\partial H}{\partial q_k}\, dq_k + \frac{\partial H}{\partial p_k}\, dp_k\right) + \frac{\partial H}{\partial t}\, dt \tag{7.157}$$

According to Equation 7.155, we can also write

$$dH = \sum_k \left(\dot{q}_k\, dp_k + p_k\, d\dot{q}_k - \frac{\partial L}{\partial q_k}\, dq_k - \frac{\partial L}{\partial \dot{q}_k}\, d\dot{q}_k\right) - \frac{\partial L}{\partial t}\, dt \tag{7.158}$$

Using Equations 7.151 and 7.152 to substitute for $\partial L/\partial q_k$ and $\partial L/\partial \dot{q}_k$, the second and fourth terms in the parentheses in Equation 7.158 cancel, and there remains

$$dH = \sum_k (\dot{q}_k dp_k - \dot{p}_k dq_k) - \frac{\partial L}{\partial t}\, dt \tag{7.159}$$

If we identify the coefficients† of $dq_k$, $dp_k$, and $dt$ between Equations 7.157 and 7.159, we find

$$\boxed{\begin{aligned} \dot{q}_k &= \frac{\partial H}{\partial p_k} \\ -\dot{p}_k &= \frac{\partial H}{\partial q_k} \end{aligned}} \qquad \text{Hamilton's equations of motion} \qquad \begin{aligned} (7.160) \\ (7.161) \end{aligned}$$

and

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \tag{7.162}$$

Furthermore, using Equations 7.160 and 7.161 in Equation 7.157, the term in the parentheses vanishes, and it follows that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \tag{7.163}$$

---

*This change of variables is similar to that frequently encountered in thermodynamics and falls in the general class of the so-called Legendre transformations (used first by Euler and perhaps even by Leibniz). A general discussion of Legendre transformations with emphasis on their importance in mechanics is given by Lanczos (La49, Chapter 6).

†The assumptions implicitly contained in this procedure are examined in the following section.

Equations 7.160 and 7.161 are **Hamilton's equations of motion.*** Because of their symmetric appearance, they are also known as the **canonical equations of motion.** The description of motion by these equations is termed **Hamiltonian dynamics.**

Equation 7.163 expresses the fact that if $H$ does not explicitly contain the time, then the Hamiltonian is a conserved quantity. We have seen previously (Section 7.9) that the Hamiltonian equals the total energy $T + U$ if the potential energy is velocity independent and the transformation equations between $x_{\alpha,i}$ and $q_j$ do not explicitly contain the time. Under these conditions, and if $\partial H/\partial t = 0$, then $H = E =$ constant.

There are $2s$ canonical equations and they replace the $s$ Lagrange equations. (Recall that $s = 3n - m$ is the number of degrees of freedom of the system.) But the canonical equations are *first-order* differential equations, whereas the Lagrange equations are of *second order.*† To use the canonical equations in solving a problem, we must first construct the Hamiltonian as a function of the generalized coordinates and momenta. It may be possible in some instances to do this directly. In more complicated cases, it may be necessary first to set up the Lagrangian and then to calculate the generalized momenta according to Equation 7.151. The equations of motion are then given by the canonical equations.

## EXAMPLE 7.11

Use the Hamiltonian method to find the equations of motion of a particle of mass $m$ constrained to move on the surface of a cylinder defined by $x^2 + y^2 = R^2$. The particle is subject to a force directed toward the origin and proportional to the distance of the particle from the origin: $\mathbf{F} = -k\mathbf{r}$.

***Solution.*** The situation is illustrated in Figure 7-9. The potential corresponding to the force $\mathbf{F}$ is

$$\begin{aligned} U &= \frac{1}{2}kr^2 = \frac{1}{2}k(x^2 + y^2 + z^2) \\ &= \frac{1}{2}k(R^2 + z^2) \end{aligned} \tag{7.164}$$

We can write the square of the velocity in cylindrical coordinates (see Equation 1.101) as

$$v^2 = \dot{R}^2 + R^2\dot{\theta}^2 + \dot{z}^2 \tag{7.165}$$

But in this case, $R$ is a constant, so the kinetic energy is

$$T = \frac{1}{2}m(R^2\dot{\theta}^2 + \dot{z}^2) \tag{7.166}$$

*This set of equations was first obtained by Lagrange in 1809, and Poisson also derived similar equations in the same year. But neither recognized the equations as a basic set of equations of motion; this point was first realized by Cauchy in 1831. Hamilton first derived the equations in 1834 from a fundamental variational principle and made them the basis for a far-reaching theory of dynamics. Thus the designation "Hamilton's" equations is fully deserved.

†This is not a special result; any set of $s$ second-order equations can always be replaced by a set of $2s$ first-order equations.

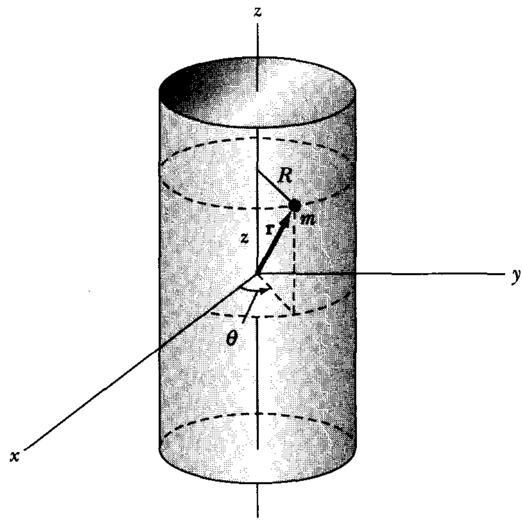


**FIGURE 7-9** Example 7.11. A particle is constrained to move on the surface of a cylinder.

We may now write the Lagrangian as

$$L = T - U = \frac{1}{2} m(R^2\dot{\theta}^2 + \dot{z}^2) - \frac{1}{2} k(R^2 + z^2) \tag{7.167}$$

The generalized coordinates are $\theta$ and $z$, and the generalized momenta are

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta} \tag{7.168}$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \tag{7.169}$$

Because the system is conservative and because the equations of transformation between rectangular and cylindrical coordinates do not explicitly involve the time, the Hamiltonian $H$ is just the total energy expressed in terms of the variables $\theta$, $p_\theta$, $z$, and $p_z$. But $\theta$ does not occur explicitly, so

$$\begin{aligned} H(z, p_\theta, p_z) &= T + U \\ &= \frac{p_\theta^2}{2mR^2} + \frac{p_z^2}{2m} + \frac{1}{2} kz^2 \end{aligned} \tag{7.170}$$

where the constant term $\frac{1}{2} kR^2$ has been suppressed. The equations of motion are therefore found from the canonical equations:

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \tag{7.171}$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = -kz \tag{7.172}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mR^2} \tag{7.173}$$

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \tag{7.174}$$

Equations 7.173 and 1.174 just duplicate Equations 7.168 and 7.169. Equations 7.168 and 7.171 give

$$p_\theta = mR^2\dot{\theta} = \text{constant} \tag{7.175}$$

The angular momentum about the $z$-axis is thus a constant of the motion. This result is ensured, because the $z$-axis is the symmetry axis of the problem. Combining Equations 7.169 and 7.172, we find

$$\ddot{z} + \omega_0^2 z = 0 \tag{7.176}$$

where

$$\omega_0^2 \equiv k/m \tag{7.177}$$

The motion in the $z$ direction is therefore simple harmonic.

---

The equations of motion for the preceding problem can also be found by the Lagrangian method using the function $L$ defined by Equation 7.167. In this case, the Lagrange equations of motion are easier to obtain than are the canonical equations. In fact, it is quite often true that the Lagrangian method leads more readily to the equations of motion that does the Hamiltonian method. But because we have greater freedom in choosing the variable in the Hamiltonian formulation of a problem (the $q_k$ and the $p_k$ are independent, whereas the $q_k$ and the $\dot{q}_k$ are not), we often gain a certain practical advantage by using the Hamiltonian method. For example, in celestial mechanics—particularly in the event that the motions are subject to perturbations caused by the influence of other bodies—it proves convenient to formulate the problem in terms of Hamiltonian dynamics. Generally speaking, however, the great power of the Hamiltonian approach to dynamics does not manifest itself in simplifying the solutions to mechanics problems; rather, it provides a base we can extend to other fields.

The generalized coordinate $q_k$ and the generalized momentum $p_k$ are **canonically conjugate** quantities. According to Equations 7.160 and 7.161, if $q_k$ does not appear in the Hamiltonian, then $\dot{p}_k = 0$, and the conjugate momentum $p_k$ is a constant of the motion. Coordinates not appearing explicitly in the expressions for $T$ and $U$ are said to be *cyclic*. A coordinate cyclic in $H$ is also cyclic in $L$. But, even if $q_k$ does not appear in $L$, the generalized velocity $\dot{q}_k$ related to this coordinate is in general still present. Thus

$$L = L(q_1, \ldots, q_{k-1}, q_{k+1}, \ldots, q_s, \dot{q}_1 \ldots, \dot{q}_s, t)$$

and we accomplish no reduction in the number of degrees of freedom of the system, even though one coordinate is cyclic; there are still $s$ second-order equations to be solved. However, in the canonical formulation, if $q_k$ is cyclic, $p_k$ is constant, $p_k = \alpha_k$, and

$$H = H(q_1, \ldots, q_{k-1}, q_{k+1}, \ldots, q_s, p_1, \ldots, p_{k-1}, \alpha_k, p_{k+1}, \ldots, p_s, t)$$

Thus, there are $2s - 2$ first-order equations to be solved, and the problem has, in fact, been reduced in complexity; there are in effect only $s - 1$ degrees of freedom remaining. The coordinate $q_k$ is completely separated, and it is *ignorable* as far as the remainder of the problem is concerned. We calculate the constant $\alpha_k$ by applying the initial conditions, and the equation of motion for the cyclic coordinate is

$$\dot{q}_k = \frac{\partial H}{\partial \alpha_k} \equiv \omega_k \tag{7.178}$$

which can be immediately integrated to yield

$$q_k(t) = \int \omega_k \, dt \tag{7.179}$$

The solution for a cyclic coordinate is therefore trivial to reduce to quadrature. Consequently, the canonical formulation of Hamilton is particularly well suited for dealing with problems in which one or more of the coordinates are cyclic. The simplest possible solution to a problem would result if the problem could be formulated in such a way that *all* the coordinates were cyclic. Then, each coordinate would be described in a trivial manner as in Equation 7.179. It is, in fact, possible to find transformations that render all the coordinates cyclic,* and these procedures lead naturally to a formulation of dynamics particularly useful in constructing modern theories of matter. The general discussion of these topics, however, is beyond the scope of this book.†

## EXAMPLE 7.12

Use the Hamiltonian method to find the equations of motion for a spherical pendulum of mass $m$ and length $b$ (see Figure 7-10).

***Solution.*** The generalized coordinates are $\theta$ and $\phi$. The kinetic energy is

$$T = \frac{1}{2} mb^2\dot{\theta}^2 + \frac{1}{2} mb^2 \sin^2 \theta \dot{\phi}^2$$

The only force acting on the pendulum (other than at the point of support) is gravity, and we define the potential zero to be at the pendulum's point of attachment.

$$U = -mgb \cos \theta$$

---

*Transformations of this type were derived by Carl Gustav Jacob Jacobi (1804–1851). Jacobi's investigations greatly extended the usefulness of Hamilton's methods, and these developments are known as *Hamilton-Jacobi theory*.

†See, for example, Goldstein (Go80, Chapter 10).

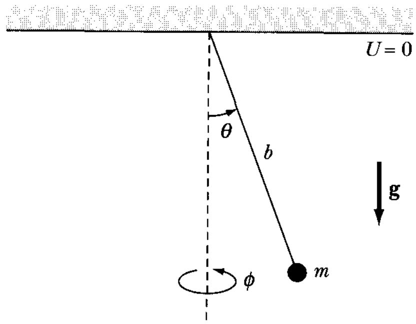


**FIGURE 7-10** Example 7.12. A spherical pendulum with generalized coordinates $\theta$ and $\phi$.

The generalized momenta are then

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mb^2\dot{\theta} \tag{7.180}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mb^2 \sin^2\theta\dot{\phi} \tag{7.181}$$

We can solve Equations 7.180 and 7.181 for $\dot{\theta}$ and $\dot{\phi}$ in terms of $p_\theta$ and $p_\phi$.

We determine the Hamiltonian from Equation 7.155 or from $H = T + U$ (because the conditions for Equation 7.130 apply).

$$\begin{aligned} H &= T + U \\ &= \frac{1}{2}mb^2\frac{p_\theta^2}{(mb^2)^2} + \frac{1}{2}\frac{mb^2\sin^2\theta p_\phi^2}{(mb^2\sin^2\theta)^2} - mgb\cos\theta \\ &= \frac{p_\theta^2}{2mb^2} + \frac{p_\phi^2}{2mb^2\sin^2\theta} - mgb\cos\theta \end{aligned}$$

The equations of motion are

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mb^2}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mb^2\sin^2\theta}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\phi^2\cos\theta}{mb^2\sin^3\theta} - mgb\sin\theta$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0$$

Because $\phi$ is cyclic, the momentum $p_\phi$ about the symmetry axis is constant.

## 7.11 Some Comments Regarding Dynamical Variables and Variational Calculations in Physics

We originally obtained Lagrange's equations of motion by stating Hamilton's Principle as a variational integral and then using the results of the preceding chapter on the calculus of variations. Because the method and the application were thereby separated, it is perhaps worthwhile to restate the argument in an orderly but abbreviated way.

Hamilton's Principle is expressed by

$$\delta \int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)\, dt = 0 \tag{7.182}$$

Applying the variational procedure specified in Section 6.7, we have

$$\int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q_j} \delta q_j + \frac{\partial L}{\partial \dot{q}_j} \delta \dot{q}_j \right) dt = 0$$

Next, we assert that the $\delta q_j$ and the $\delta \dot{q}_j$ are *not* independent, so the variation operation and the time differentiation can be interchanged:

$$\delta \dot{q}_j = \delta \left( \frac{dq_j}{dt} \right) = \frac{d}{dt} \delta q_j \tag{7.183}$$

The varied integral becomes (after the integration by parts in which the $\delta q_j$ are set equal to zero at the endpoints)

$$\int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q_j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} \right) \delta q_j\, dt = 0 \tag{7.184}$$

The requirement that the $\delta q_j$ be independent variations leads immediately to Lagrange's equations.

In Hamilton's Principle, expressed by the variational integral in Equation 7.182, the Lagrangian is a function of the generalized coordinates and the generalized velocities. But only the $q_j$ are considered as independent variables; the generalized velocities are simply the time derivatives of the $q_j$. When the integral is reduced to the form given by Equation 7.184, we state that the $\delta q_j$ are independent variations; thus the integrand must vanish identically, and Lagrange's equations result. We may therefore pose this question: Because the dynamical motion of the system is completely determined by the initial conditions, what is the meaning of the variations $\delta q_j$? Perhaps a sufficient answer is that the variables are to be considered geometrically feasible within the limits of the given constraints—although they are not dynamically possible; that is, when using a variational procedure to obtain Lagrange's equations, it is convenient to ignore temporarily the fact that we are dealing with a physical system whose motion is completely determined and subject to no variation and to consider instead only a certain abstract mathematical problem. Indeed, this is the spirit in which any variational calculation relating to a physical process must be carried out. In adopting such a viewpoint, we must not be overly concerned with the fact that the variational procedure may be contrary to certain known physical properties of the system. (For example, energy is generally not conserved in passing from the true path to the varied path.) A variational calculation simply tests various *possible* solutions to a problem and prescribes a method for selecting the *correct* solution.

The canonical equations of motion can also be obtained directly from a variational calculation based on the so-called **modified Hamilton's Principle.** The Lagrangian function can be expressed as (see Equation 7.153):

$$L = \sum_j p_j \dot{q}_j - H(q_j, p_j, t) \tag{7.185}$$

and the statement of Hamilton's Principle contained in Equation 7.182 can be modified to read

$$\boxed{\delta \int_{t_1}^{t_2} \left( \sum_j p_j \dot{q}_j - H \right) dt = 0} \tag{7.186}$$

Carrying out the variation in the standard manner, we obtain

$$\int_{t_1}^{t_2} \sum_j \left( p_j \delta \dot{q}_j + \dot{q}_j \delta p_j - \frac{\partial H}{\partial q_j} \delta q_j - \frac{\partial H}{\partial p_j} \delta p_j \right) dt = 0 \tag{7.187}$$

In the Hamiltonian formulation, the $q_j$ and the $p_j$ are considered to be independent. The $\dot{q}_j$ are again not independent of the $q_j$, so Equation 7.183 can be used to express the first term in Equation 7.187 as

$$\int_{t_1}^{t_2} \sum_j p_j \delta \dot{q}_j \, dt = \int_{t_1}^{t_2} \sum_j p_j \frac{d}{dt} \delta q_j \, dt$$

Integrating by parts, the integrated term vanishes, and we have

$$\int_{t_1}^{t_2} \sum_j p_j \delta \dot{q}_j \, dt = -\int_{t_1}^{t_2} \sum_j \dot{p}_j \delta q_j \, dt \tag{7.188}$$

Equation 7.187 then becomes

$$\int_{t_1}^{t_2} \sum_j \left\{ \left( \dot{q}_j - \frac{\partial H}{\partial p_j} \right) \delta p_j - \left( \dot{p}_j + \frac{\partial H}{\partial q_j} \right) \delta q_j \right\} dt = 0 \tag{7.189}$$

If $\delta q_j$ and $\delta p_j$ represent *independent variations,* the terms in the parentheses must separately vanish and Hamilton's canonical equations result.

In the preceding section, we obtained the canonical equations by writing two different expressions for the total differential of the Hamiltonian (Equations 7.157 and 7.159) and then equating the coefficients of $dq_j$ and $dp_j$. Such a procedure is valid if the $q_j$ and the $p_j$ are independent variables. Therefore, both in the previous derivation and in the preceding variational calculation, we obtained the canonical equations by exploring the independent nature of the generalized coordinates and the generalized momenta.

The coordinates and momenta are not actually "independent" in the ultimate sense of the word. For if the time dependence of each of the coordinates is known, $q_j = q_j(t)$, the problem is completely solved. The generalized velocities can be calculated from

$$\dot{q}_j(t) = \frac{d}{dt} q_j(t)$$

and the generalized momenta are

$$p_j = \frac{\partial}{\partial \dot{q}_j} L(q_j, \dot{q}_j, t)$$

The essential point is that, whereas the $q_j$ and the $\dot{q}_j$ are related by a simple time derivative *independent of the manner in which the system behaves,* the connection between the $q_j$ and the $p_j$ are the *equations of motion themselves.* Finding the relations that connect the $q_j$ and the $p_j$ (and thereby eliminating the assumed independence of these quantities) is therefore tantamount to solving the problem.

## 7.12 Phase Space and Liouville's Theorem (Optional)

We pointed out previously that the generalized coordinates $q_j$ can be used to define an $s$-dimensional *configuration space* with every point representing a certain state of the system. Similarly, the *generalized momenta* $p_j$ define an $s$-dimensional *momentum space* with every point representing a certain condition of motion of the system. A given point in configuration space specifies only the position of each of the particles in the system; nothing can be inferred regarding the motion of the particles. The reverse is true for momentum space. In Chapter 3, we found it profitable to represent geometrically the dynamics of simple oscillatory systems by phase diagrams. If we use this concept with more complicated dynamical systems, then a $2s$-dimensional space consisting of the $q_j$ and the $p_j$ allows us to represent both the positions *and* the momenta of all particles. This generalization is called **Hamiltonian phase space** or, simply, **phase space.***

### EXAMPLE 7.13

Construct the phase diagram for the particle in Example 7.11.

***Solution.*** The particle has two degrees of freedom $(\theta, z)$, so the phase space for this example is actually four dimensional: $\theta$, $p_\theta$, $z$, $p_z$. But $p_\theta$ is constant and therefore may be suppressed. In the $z$ direction, the motion is simple harmonic, and so the projection onto the $z$-$p_z$ plane of the phase path for any total energy $H$ is just an ellipse. Because $\dot{\theta}$ = constant, the phase path must represent motion increasing uniformly with $\theta$. Thus, the phase path on any surface $H$ = constant is a **uniform elliptic spiral** (Figure 7-11).

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*We previously plotted in the phase diagrams the position versus a quantity proportional to the velocity. In Hamiltonian phase space, this latter quantity becomes the generalized momentum.

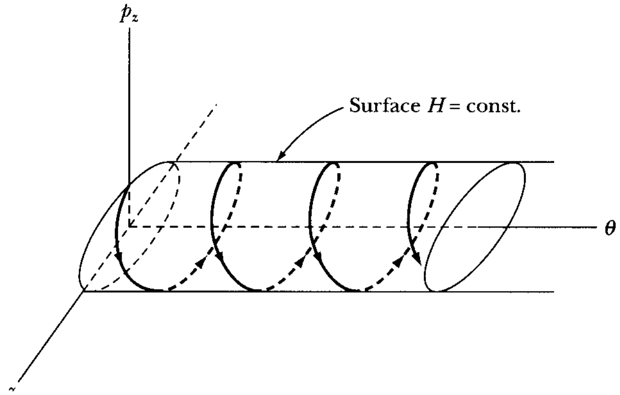


**FIGURE 7-11** Example 7.13. The phase path for the particle in Example 7.11.

If, at a given time, the position and momenta of all the particles in a system are known, then with these quantities as initial conditions, the subsequent motion of the system is completely determined; that is, starting from a point $q_j(0), p_j(0)$ in phase space, the representative point describing the system moves along a unique phase path. In principle, this procedure can always be followed and a solution obtained. But if the number of degrees of freedom of the system is large, the set of equations of motion may be too complicated to solve in a reasonable time. Moreover, for complex systems, such as a quantity of gas, it is a practical impossibility to determine the initial conditions for each constituent molecule. Because we cannot identify any particular point in phase space as representing the actual conditions at any given time, we must devise some alternative approach to study the dynamics of such systems. We therefore arrive at the point of departure of statistical mechanics. The Hamiltonian formulation of dynamics is ideal for the statistical study of complex systems. We demonstrate this in part by now proving a theorem that is fundamental for such investigations.

For a large collection of particles—say, gas molecules—we are unable to identify the particular point in phase space correctly representing the system. But we may fill the phase space with a collection of points, each representing a *possible* condition of the system; that is, we imagine a large number of systems (each consistent with the known constraints), any of which could conceivably be the actual system. Because we are unable to discuss the details of the particles' motion in the actual system, we substitute a discussion of an *ensemble* of equivalent systems. Each representative point in phase space corresponds to a single system of the ensemble, and the motion of a particular point represents the independent motion of that system. Thus, no two of the phase paths may ever intersect.

We may consider the representative points to be sufficiently numerous that we can define a *density in phase space* $\rho$. The volume elements of the phase space defining the density must be sufficiently large to contain a large number of representative points, but they must also be sufficiently small so that the density

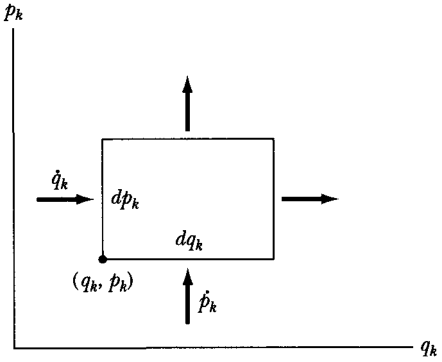


**FIGURE 7-12** An element of area $dA = dq_k dp_k$ in the $q_k$–$p_k$ plane in phase space.

varies continuously. The number $N$ of systems whose representative points lie within a volume $dv$ of phase space is

$$N = \rho\, dv \tag{7.190}$$

where

$$dv = dq_1\, dq_2 \cdots dq_s\, dp_1\, dp_2 \cdots dp_s \tag{7.191}$$

As before, $s$ is the number of degrees of freedom of each system in the ensemble.

Consider an element of area in the $q_k$–$p_k$ plane in phase space (Figure 7-12). The number of representative points moving across the left-hand edge into the area per unit time is

$$\rho \frac{dq_k}{dt} dp_k = \rho \dot{q}_k\, dp_k$$

and the number moving across the lower edge into the area per unit time is

$$\rho \frac{dp_k}{dt} dq_k = \rho \dot{p}_k\, dq_k$$

so that the total number of representative points moving *into* the area $dq_k\, dp_k$ per unit time is

$$\rho(\dot{q}_k dp_k + \dot{p}_k dq_k) \tag{7.192}$$

By a Taylor series expansion, the number of representative points moving *out* of the area per unit time is (approximately)

$$\left[\rho \dot{q}_k + \frac{\partial}{\partial q_k}(\rho \dot{q}_k)\, dq_k\right] dp_k + \left[\rho \dot{p}_k + \frac{\partial}{\partial p_k}(\rho \dot{p}_k)\, dp_k\right] dq_k \tag{7.193}$$

Thus, the total increase in density in $dq_k\, dp_k$ per unit time is the difference between Equations 7.192 and 7.193:

$$\frac{\partial \rho}{\partial t} dq_k\, dp_k = -\left[\frac{\partial}{\partial q_k}(\rho \dot{q}_k) + \frac{\partial}{\partial p_k}(\rho \dot{p}_k)\right] dq_k\, dp_k \tag{7.194}$$

After dividing by $dq_k\,dp_k$ and summing this expression over all possible values of $k$, we find

$$\frac{\partial\rho}{\partial t} + \sum_{k=1}^{s}\left(\frac{\partial\rho}{\partial q_k}\dot{q}_k + \rho\frac{\partial\dot{q}_k}{\partial q_k} + \frac{\partial\rho}{\partial p_k}\dot{p}_k + \rho\frac{\partial\dot{p}_k}{\partial p_k}\right) = 0 \tag{7.195}$$

From Hamilton's equations (Equations 7.160 and 7.161), we have (if the second partial derivatives of $H$ are continuous)

$$\frac{\partial\dot{q}_k}{\partial q_k} + \frac{\partial\dot{p}_k}{\partial p_k} = 0 \tag{7.196}$$

so Equation 7.195 becomes

$$\frac{\partial\rho}{\partial t} + \sum_{k}\left(\frac{\partial\rho}{\partial q_k}\frac{dq_k}{dt} + \frac{\partial\rho}{\partial p_k}\frac{dp_k}{dt}\right) = 0 \tag{7.197}$$

But this is just the total time derivative of $\rho$, so we conclude that

$$\boxed{\frac{d\rho}{dt} = 0} \tag{7.198}$$

This important result, known as **Liouville's theorem,*** states that the density of representative points in phase space corresponding to the motion of a system of particles remains constant during the motion. It must be emphasized that we have been able to establish the invariance of the density $\rho$ only because the problem was formulated in *phase space*; an equivalent theorem for configuration space does not exist. Thus, we must use Hamiltonian dynamics (rather than Lagrangian dynamics) to discuss ensembles in statistical mechanics.

Liouville's theorem is important not only for aggregates of microscopic particles, as in the statistical mechanics of gaseous systems and the focusing properties of charged-particle accelerators, but also in certain macroscopic systems. For example, in stellar dynamics, the problem is inverted and by studying the distribution function $\rho$ of stars in the galaxy, the potential $U$ of the galactic gravitational field may be inferred.

## 7.13 Virial Theorem (Optional)

Another important result of a statistical nature is worthy of mention. Consider a collection of particles whose position vectors $\mathbf{r}_\alpha$ and momenta $\mathbf{p}_\alpha$ are both bounded (i.e., remain finite for all values of the time). Define a quantity

$$S \equiv \sum_{\alpha}\mathbf{p}_\alpha \cdot \mathbf{r}_\alpha \tag{7.199}$$

*Published in 1838 by Joseph Liouville (1809–1882).

The time derivative of $S$ is

$$\frac{dS}{dt} = \sum_{\alpha}(\mathbf{p}_{\alpha} \cdot \dot{\mathbf{r}}_{\alpha} + \dot{\mathbf{p}}_{\alpha} \cdot \mathbf{r}_{\alpha}) \tag{7.200}$$

If we calculate the average value of $dS/dt$ over a time interval $\tau$, we find

$$\left\langle \frac{dS}{dt} \right\rangle = \frac{1}{\tau}\int_0^{\tau} \frac{dS}{dt}\,dt = \frac{S(\tau) - S(0)}{\tau} \tag{7.201}$$

If the system's motion is periodic—and if $\tau$ is some integer multiple of the period—then $S(t) = S(0)$, and $\langle \dot{S} \rangle$ vanishes. But even if the system does not exhibit any periodicity, then—because $S$ is by hypothesis a bounded function—we can make $\langle \dot{S} \rangle$ as small as desired by allowing the time $\tau$ to become sufficiently long. Therefore, the time average of the right-hand side of Equation 7.201 can always be made to vanish (or at least to approach zero). Thus, in this limit, we have

$$\left\langle \sum_{\alpha} \mathbf{p}_{\alpha} \cdot \dot{\mathbf{r}}_{\alpha} \right\rangle = -\left\langle \sum_{\alpha} \dot{\mathbf{p}}_{\alpha} \cdot \mathbf{r}_{\alpha} \right\rangle \tag{7.202}$$

On the left-hand side of this equation, $\mathbf{p}_{\alpha} \cdot \dot{\mathbf{r}}_{\alpha}$ is twice the kinetic energy. On the right-hand side, $\dot{\mathbf{p}}_{\alpha}$ is just the force $\mathbf{F}_{\alpha}$ on the $\alpha$th particle. Hence,

$$\left\langle 2\sum_{\alpha} T_{\alpha} \right\rangle = -\left\langle \sum_{\alpha} \mathbf{F}_{\alpha} \cdot \dot{\mathbf{r}}_{\alpha} \right\rangle \tag{7.203}$$

The sum over $T_{\alpha}$ is the total kinetic energy $T$ of the system, so we have the general result

$$\langle T \rangle = -\frac{1}{2}\left\langle \sum_{\alpha} \mathbf{F}_{\alpha} \cdot \mathbf{r}_{\alpha} \right\rangle \tag{7.204}$$

The right-hand side of this equation was called by Clausius* the **virial** of the system, and the **virial theorem** states that *the average kinetic energy of a system of particles is equal to its virial.*

## EXAMPLE 7.14

Consider an ideal gas containing $N$ atoms in a container of volume $V$, pressure $P$, and absolute temperature $T_1$ (not to be confused with the kinetic energy $T$). Use the virial theorem to derive the equation of state for a perfect gas.

***Solution.*** According to the equipartition theorem, the average kinetic energy of each atom in the ideal gas is $3/2\ kT_1$, where $k$ is the Boltzmann constant. The total average kinetic energy becomes

$$\langle T \rangle = \frac{3}{2}NkT_1 \tag{7.205}$$

* Rudolph Julius Emmanuel Clausius (1822–1888), a German physicist and one of the founders of thermodynamics.

The right-hand side of the virial theorem (Equation 7.204) contains the forces $\mathbf{F}_\alpha$. For an ideal perfect gas, no force of interaction occurs between atoms. The only force is represented by the force of constraint of the walls. The atoms bounce elastically off the walls, which are exerting a pressure on the atoms.

Because the pressure is force per unit area, we find the instantaneous differential force over a differential area to be

$$d\mathbf{F}_\alpha = -\mathbf{n}PdA \tag{7.206}$$

where $\mathbf{n}$ is a unit vector normal to the surface $dA$ and pointing outward. The right-hand side of the virial theorem becomes

$$-\frac{1}{2}\left\langle\sum_\alpha \mathbf{F}_\alpha \cdot \mathbf{r}_\alpha\right\rangle = \frac{P}{2}\int \mathbf{n}\cdot\mathbf{r}\,dA \tag{7.207}$$

We use the divergence theorem to relate the surface integral to a volume integral.

$$\int \mathbf{n}\cdot\mathbf{r}\,dA = \int \nabla\cdot\mathbf{r}\,dV = 3\int dV = 3V \tag{7.208}$$

The virial theorem result is

$$\begin{aligned}\frac{3}{2}NkT &= \frac{3PV}{2}\\ NkT &= PV\end{aligned} \tag{7.209}$$

which is the ideal gas law.

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If the forces $\mathbf{F}_\alpha$ can be derived from potentials $U_\alpha$, Equation 7.204 may be rewritten as

$$\langle T\rangle = \frac{1}{2}\left\langle\sum_\alpha \mathbf{r}_\alpha\cdot\nabla U_\alpha\right\rangle \tag{7.210}$$

Of particular interest is the case of two particles that interact according to a central power-law force: $F \propto r^n$. Then, the potential is of the form

$$U = kr^{n+1} \tag{7.211}$$

Therefore

$$\mathbf{r}\cdot\nabla U = \frac{dU}{dr} = k(n+1)r^{n+1} = (n+1)U \tag{7.212}$$

and the virial theorem becomes

$$\langle T\rangle = \frac{n+1}{2}\langle U\rangle \tag{7.213}$$

If the particles have a gravitational interaction, then $n = -2$, and

$$\langle T \rangle = -\frac{1}{2} \langle U \rangle, \quad n = -2$$

This relation is useful in calculating, for example, the energetics in planetary motion.

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## PROBLEMS

**7-1.** A disk rolls without slipping across a horizontal plane. The plane of the disk remains vertical, but it is free to rotate about a vertical axis. What generalized coordinates may be used to describe the motion? Write a differential equation describing the rolling constraint. Is this equation integrable? Justify your answer by a physical argument. Is the constraint holonomic?

**7-2.** Work out Example 7.6 showing all the steps, in particular those leading to Equations 7.36 and 7.41. Explain why the sign of the acceleration $a$ cannot affect the frequency $\omega$. Give an argument why the signs of $a^2$ and $g^2$ in the solution of $\omega^2$ in Equation 7.42 are the same.

**7-3.** A sphere of radius $\rho$ is constrained to roll without slipping on the lower half of the inner surface of a hollow cylinder of inside radius $R$. Determine the Lagrangian function, the equation of constraint, and Lagrange's equations of motion. Find the frequency of small oscillations.

**7-4.** A particle moves in a plane under the influence of a force $f = -Ar^{\alpha-1}$ directed toward the origin; $A$ and $\alpha$ ($> 0$) are constants. Choose appropriate generalized coordinates, and let the potential energy be zero at the origin. Find the Lagrangian equations of motion. Is the angular momentum about the origin conserved? Is the total energy conserved?

**7-5.** Consider a vertical plane in a constant gravitational field. Let the origin of a coordinate system be located at some point in this plane. A particle of mass $m$ moves in the vertical plane under the influence of gravity and under the influence of an additional force $f = -Ar^{\alpha-1}$ directed toward the origin ($r$ is the distance from the origin; $A$ and $\alpha$ [$\neq 0$ or 1] are constants). Choose appropriate generalized coordinates, and find the Lagrangian equations of motion. Is the angular momentum about the origin conserved? Explain.

**7-6.** A hoop of mass $m$ and radius $R$ rolls without slipping down an inclined plane of mass $M$, which makes an angle $\alpha$ with the horizontal. Find the Lagrange equations and the integrals of the motion if the plane can slide without friction along a horizontal surface.

**7-7.** A double pendulum consists of two simple pendula, with one pendulum suspended from the bob of the other. If the two pendula have equal lengths and have bobs of equal mass and if both pendula are confined to move in the same plane, find Lagrange's equations of motion for the system. Do not assume small angles.

**7-8.** Consider a region of space divided by a plane. The potential energy of a particle in region 1 is $U_1$ and in region 2 it is $U_2$. If a particle of mass $m$ and with speed $v_1$ in region 1 passes from region 1 to region 2 such that its path in region 1 makes an angle $\theta_1$ with the normal to the plane of separation and an angle $\theta_2$ with the normal when in region 2, show that

$$\frac{\sin\theta_1}{\sin\theta_2} = \left(1 + \frac{U_1 - U_2}{T_1}\right)^{1/2}$$

where $T_1 = \frac{1}{2}mv_1^2$. What is the optical analog of this problem?

**7-9.** A disk of mass $M$ and radius $R$ rolls without slipping down a plane inclined from the horizontal by an angle $\alpha$. The disk has a short weightless axle of negligible radius. From this axis is suspended a simple pendulum of length $l < R$ and whose bob has a mass $m$. Consider that the motion of the pendulum takes place in the plane of the disk, and find Lagrange's equations for the system.

**7-10.** Two blocks, each of mass $M$, are connected by an extensionless, uniform string of length $l$. One block is placed on a smooth horizontal surface, and the other block hangs over the side, the string passing over a frictionless pulley. Describe the motion of the system **(a)** when the mass of the string is negligible and **(b)** when the string has a mass $m$.

**7-11.** A particle of mass $m$ is constrained to move on a circle of radius $R$. The circle rotates in space about one point on the circle, which is fixed. The rotation takes place in the plane of the circle and with constant angular speed $\omega$. In the absence of a gravitational force, show that the particle's motion about one end of a diameter passing through the pivot point and the center of the circle is the same as that of a plane pendulum in a uniform gravitational field. Explain why this is a reasonable result.

**7-12.** A particle of mass $m$ rests on a smooth plane. The plane is raised to an inclination angle $\theta$ at a constant rate $\alpha$ ($\theta = 0$ at $t = 0$), causing the particle to move down the plane. Determine the motion of the particle.

**7-13.** A simple pendulum of length $b$ and bob with mass $m$ is attached to a massless support moving horizontally with constant acceleration $a$. Determine **(a)** the equations of motion and **(b)** the period for small oscillations.

**7-14.** A simple pendulum of length $b$ and bob with mass $m$ is attached to a massless support moving vertically upward with constant acceleration $a$. Determine **(a)** the equations of motion and **(b)** the period for small oscillations.

**7-15.** A pendulum consists of a mass $m$ suspended by a massless spring with unextended length $b$ and spring constant $k$. Find Lagrange's equations of motion.

**7-16.** The point of support of a simple pendulum of mass $m$ and length $b$ is driven horizontally by $x = a\sin\omega t$. Find the pendulum's equation of motion.

**7-17.** A particle of mass $m$ can slide freely along a wire $AB$ whose perpendicular distance to the origin $O$ is $h$ (see Figure 7-A, page 282). The line $OC$ rotates about the origin

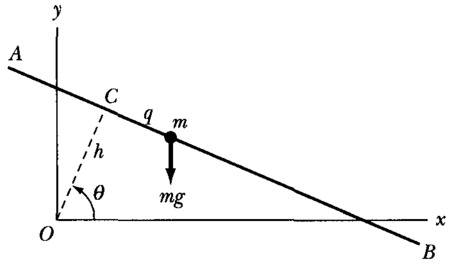


**FIGURE 7-A** Problem 7-17.

at a constant angular velocity $\dot{\theta} = \omega$. The position of the particle can be described in terms of the angle $\theta$ and the distance $q$ to the point $C$. If the particle is subject to a gravitational force, and if the initial conditions are

$$\theta(0) = 0, \quad q(0) = 0, \quad \dot{q}(0) = 0$$

show that the time dependence of the coordinate $q$ is

$$q(t) = \frac{g}{2\omega^2}(\cosh \omega t - \cos \omega t)$$

Sketch this result. Compute the Hamiltonian for the system, and compare with the total energy. Is the total energy conserved?

**7-18.** A pendulum is constructed by attaching a mass $m$ to an extensionless string of length $l$. The upper end of the string is connected to the uppermost point on a vertical disk of radius $R$ ($R < l/\pi$) as in Figure 7-B. Obtain the pendulum's equation of motion, and find the frequency of small oscillations. Find the line about which the angular motion extends equally in either direction (i.e., $\theta_1 = \theta_2$).

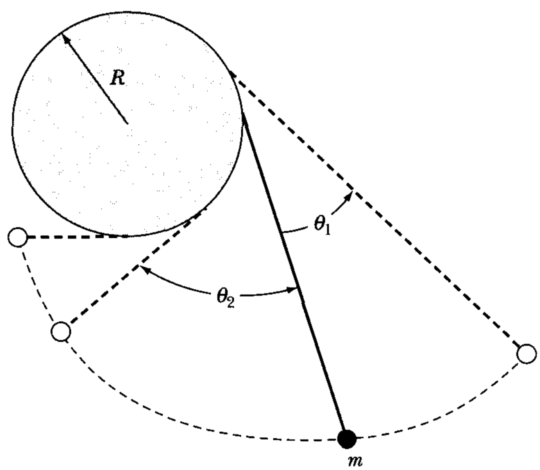


**FIGURE 7-B** Problem 7-18.

**7-19.** Two masses $m_1$ and $m_2$ ($m_1 \neq m_2$) are connected by a rigid rod of length $d$ and of negligible mass. An extensionless string of length $l_1$ is attached to $m_1$ and connected to a fixed point of support $P$. Similarly, a string of length $l_2$ ($l_1 \neq l_2$) connects $m_2$ and $P$. Obtain the equation describing the motion in the plane of $m_1$, $m_2$, and $P$, and find the frequency of small oscillations around the equilibrium position.

**7-20.** A circular hoop is suspended in a horizontal plane by three strings, each of length $l$, which are attached symmetrically to the hoop and are connected to fixed points lying in a plane above the hoop. At equilibrium, each string is vertical. Show that the frequency of small rotational oscillations about the vertical through the center of the hoop is the same as that for a simple pendulum of length $l$.

**7-21.** A particle is constrained to move (without friction) on a circular wire rotating with constant angular speed $\omega$ about a vertical diameter. Find the equilibrium position of the particle, and calculate the frequency of small oscillations around this position. Find and interpret physically a critical angular velocity $\omega = \omega_c$ that divides the particle's motion into two distinct types. Construct phase diagrams for the two cases $\omega < \omega_c$ and $\omega > \omega_c$.

**7-22.** A particle of mass $m$ moves in one dimension under the influence of a force

$$F(x, t) = \frac{k}{x^2} e^{-(t/\tau)}$$

where $k$ and $\tau$ are positive constants. Compute the Lagrangian and Hamiltonian functions. Compare the Hamiltonian and the total energy, and discuss the conservation of energy for the system.

**7-23.** Consider a particle of mass $m$ moving freely in a conservative force field whose potential function is $U$. Find the Hamiltonian function, and show that the canonical equations of motion reduce to Newton's equations. (Use rectangular coordinates.)

**7-24.** Consider a simple plane pendulum consisting of a mass $m$ attached to a string of length $l$. After the pendulum is set into motion, the length of the string is shortened at a constant rate

$$\frac{dl}{dt} = -\alpha = \text{constant}$$

The suspension point remains fixed. Compute the Lagrangian and Hamiltonian functions. Compare the Hamiltonian and the total energy, and discuss the conservation of energy for the system.

**7-25.** A particle of mass $m$ moves under the influence of gravity along the helix $z = k\theta$, $r =$ constant, where $k$ is a constant and $z$ is vertical. Obtain the Hamiltonian equations of motion.

**7-26.** Determine the Hamiltonian and Hamilton's equations of motion for **(a)** a simple pendulum and **(b)** a simple Atwood machine (single pulley).

**7-27.** A massless spring of length $b$ and spring constant $k$ connects two particles of masses $m_1$ and $m_2$. The system rests on a smooth table and may oscillate and rotate.

**(a)** Determine Lagrange's equations of motion.
**(b)** What are the generalized momenta associated with any cyclic coordinates?
**(c)** Determine Hamilton's equations of motion.

**7-28.** A particle of mass $m$ is attracted to a force center with the force of magnitude $k/r^2$. Use plane polar coordinates and find Hamilton's equations of motion.

**7-29.** Consider the pendulum described in Problem 7-15. The pendulum's point of support rises vertically with constant acceleration $a$.
**(a)** Use the Lagrangian method to find the equations of motion.
**(b)** Determine the Hamiltonian and Hamilton's equations of motion.
**(c)** What is the period of small oscillations?

**7-30.** Consider any two continuous functions of the generalized coordinates and momenta $g(q_k, p_k)$ and $h(q_k, p_k)$. The **Poisson brackets** are defined by

$$[g, h] \equiv \sum_k \left( \frac{\partial g}{\partial q_k}\frac{\partial h}{\partial p_k} - \frac{\partial g}{\partial p_k}\frac{\partial h}{\partial q_k} \right)$$

Verify the following properties of the Poisson brackets:

**(a)** $\dfrac{dg}{dt} = [g, H] + \dfrac{\partial g}{dt}$ **(b)** $\dot{q}_j = [q_i, H], \ \dot{p}_j = [p_j, H]$

**(c)** $[p_l, p_j] = 0, \ [q_l, q_j] = 0$ **(d)** $[q_l, p_j] = \delta_{lj}$

where $H$ is the Hamiltonian. If the Poisson bracket of two quantities vanishes, the quantities are said to *commute*. If the Poisson bracket of two quantities equals unity, the quantities are said to be *canonically conjugate*. **(e)** Show that any quantity that does not depend explicitly on the time and that commutes with the Hamiltonian is a constant of the motion of the system. Poisson-bracket formalism is of considerable importance in quantum mechanics.

**7-31.** A spherical pendulum consists of a bob of mass $m$ attached to a weightless, extensionless rod of length $l$. The end of the rod opposite the bob pivots freely (in all directions) about some fixed point. Set up the Hamiltonian function in spherical coordinates. (If $p_\phi = 0$, the result is the same as that for the plane pendulum.) Combine the term that depends on $p_\phi$ with the ordinary potential energy term to define as *effective* potential $V(\theta, p_\phi)$. Sketch $V$ as a function of $\theta$ for several values of $p_\phi$, including $p_\phi = 0$. Discuss the features of the motion, pointing out the differences between $p_\phi = 0$ and $p_\phi \neq 0$. Discuss the limiting case of the conical pendulum ($\theta$ = constant) with reference to the $V$-$\theta$ diagram.

**7-32.** A particle moves in a spherically symmetric force field with potential energy given by $U(r) = -k/r$. Calculate the Hamiltonian function in spherical coordinates, and obtain the canonical equations of motion. Sketch the path that a representative point for the system would follow on a surface $H$ = constant in phase space. Begin by showing that the motion must lie in a plane so that the phase space is four dimensional ($r, \theta, p_r, p_\theta$, but only the first three are nontrivial). Calculate the projection of the phase path on the $r$-$p_r$ plane, then take into account the variation with $\theta$.

**7-33.** Determine the Hamiltonian and Hamilton's equations of motion for the double Atwood machine of Example 7.8.

**7-34.** A particle of mass $m$ slides down a smooth circular wedge of mass $M$ as shown in Figure 7-C. The wedge rests on a smooth horizontal table. Find (**a**) the equation of motion of $m$ and $M$ and (**b**) the reaction of the wedge on $m$.

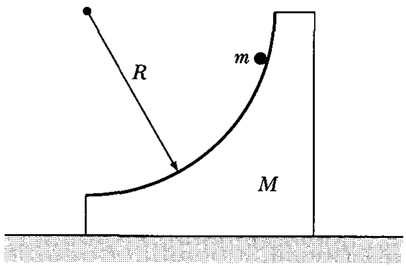


**FIGURE 7-C** Problem 7-34.

**7-35.** Four particles are directed upward in a uniform gravitational field with the following initial conditions:

$$\begin{array}{lll}
(1) & z(0) = z_0; & p_z(0) = p_0 \\
(2) & z(0) = z_0 + \Delta z_0; & p_z(0) = p_0 \\
(3) & z(0) = z_0; & p_z(0) = p_0 + \Delta p_0 \\
(4) & z(0) = z_0 + \Delta z_0; & p_z(0) = p_0 + \Delta p_0
\end{array}$$

Show by direct calculation that the representative points corresponding to these particles always define an area in phase space equal to $\Delta z_0 \, \Delta p_0$. Sketch the phase paths, and show for several times $t > 0$ the shape of the region whose area remains constant.

**7-36.** Discuss the implications of Liouville's theorem on the focusing of beams of charged particles by considering the following simple case. An electron beam of circular cross section (radius $R_0$) is directed along the $z$-axis. The density of electrons across the beam is constant, but the momentum components transverse to the beam ($p_x$ and $p_y$) are distributed uniformly over a circle of radius $p_0$ in momentum space. If some focusing system reduces the beam radius from $R_0$ to $R_1$, find the resulting distribution of the transverse momentum components. What is the physical meaning of this result? (Consider the angular divergence of the beam.)

**7-37.** Use the method of Lagrange undetermined multipliers to find the tensions in both strings of the double Atwood machine of Example 7.8.

**7-38.** The potential for an anharmonic oscillator is $U = kx^2/2 + bx^4/4$ where $k$ and $b$ are constants. Find Hamilton's equations of motion.

**7-39.** An extremely limber rope of uniform mass density, mass $m$ and total length $b$ lies on a table with a length $z$ hanging over the edge of the table. Only gravity acts on the rope. Find Lagrange's equation of motion.

**7-40.** A double pendulum is attached to a cart of mass $2m$ that moves without friction on a horizontal surface. See Figure 7-D. Each pendulum has length $b$ and mass bob $m$. Find the equations of motion.

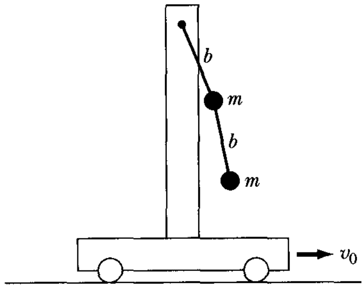


**FIGURE 7-D** Problem 7-40.

**7-41.** A pendulum of length $b$ and mass bob $m$ is oscillating at small angles when the length of the pendulum string is shortened at a velocity of $\alpha$ ($db/dt = -\alpha$). Find Lagrange's equations of motion.