Constraints and Lagrangian Dynamics

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Constraints

- Discussion up to now ⇒ **All mechanics is reduced to solving a set of simultaneous, coupled, 2^{nd} order differential eqtns** which come from Newton's 2^{nd} Law applied to each mass individually:

$$(dp_i/dt) = m_i(d^2r_i/dt^2) = F_i^{(e)} + \sum_j F_{ji}$$

⇒ Given forces & initial conditions, problem is reduced to pure math!

- Oversimplification!! Many systems have ***CONSTRAINTS*** which limit their motion.
  - Example: Rigid Body. Constraints keep $\mathbf{r}_{ij}$ = constant.
  - Example: Particle motion on surface of sphere.

Types of Constraints

- In general, constraints are expressed as a mathematical relation or relations between particle coordinates & possibly the time.
  - Eqtns of constraint are relations like:

$$f(r_1,r_2,r_3,\ldots r_N,t) = 0$$

- Constraints which may be expressed as above:

$$\equiv$$ ***Holonomic Constraints***.

- Example of **Holonomic Constraint:** Rigid body. Constraints on coordinates are of the form:

$$(r_i - r_j)^2 - (c_{ij})^2 = 0$$

c_{ij} = some constant

- Constraints not expressible as $\mathbf{f(r_i,t) = 0}$

  ≡ ***Non-Holonomic Constraints***

- Example of **Non-Holonomic Constraint**: Particle confined to surface of rigid sphere, radius **a**: $\mathbf{r^2 - a^2 \geq 0}$

- Time dependent constraints:

  ≡ ***Rhenomic or Rhenomous Constraints***.

- If constraint eqtns don't explicitly contain time: ≡ ***Fixed or Scleronomic or Scleronomous Constraints***.

- Difficulties constraints introduce in problems:

1. Coordinates $\mathbf{r}_i$ are no longer all independent. Connected by constraint eqtns.

2. To apply Newton's 2nd Law, need ***TOTAL*** force acting on each particle. Forces of constraint aren't always known or easily calculated.

$\Rightarrow$ **With constraints, it's often difficult to *directly* apply Newton's 2nd Law.**

Put another way: **Forces of constraint are often among the unknowns of the problem!**

Generalized Coordinates

- To handle the 1st difficulty (with holonomic constraints), introduce ***Generalized Coordinates***.
  - – Alternatives to usual Cartesian coordinates.
- System (**3d**) **N** particles & no constraints.

  $\Rightarrow$ **3N degrees of freedom**

  (**3N** independent coordinates)
- With **k** holonomic constraints, each expressed by eqtn of form:

  $$f_m(r_1,r_2,r_3,\ldots r_N,t) = 0 \quad (m = 1, 2, \ldots k)$$

  $\Rightarrow$ **3N - k degrees of freedom**

  (**3N - k** independent coordinates)

- General mechanical system with **s = 3N - k** degrees of freedom (**3N - k** independent coordinates).
- Introduce **s = 3N - k** independent ***Generalized Coordinates*** to describe system:

Notation: $\mathbf{q_1,q_2, \ldots}$ Or: $\mathbf{q_\ell}$ (ℓ = **1,2,… s**)

- In principle, can always find relations between generalized coordinates & Cartesian (vector) coordinates of form: $\mathbf{r_i = r_i(q_1,q_2,q_3,..,t)}$ **(i = 1,2,3,…N)**
  - These are ***transformation eqtns*** from the set of coordinates $(\mathbf{r_i})$ to the set $(\mathbf{q_\ell})$. They are parametric representations of $(\mathbf{r_i})$
  - In principle, can combine with **k** constraint eqtns to obtain inverse relations $\mathbf{q_\ell = q_\ell(r_1,r_2,r_3,..t)}$ (ℓ = **1,2,… s**)

- ***Generalized Coordinates*** ≡ Any set of **s** quantities which ***completely specifies*** the state of the system (for a system with **s** degrees of freedom).
- These **s** generalized coords need not be Cartesian! Can choose ***any set of s coordinates*** which completely describes state of motion of system. **Depending on problem:**
  - Could have **s** curvilinear (spherical, cylindrical, ..) coords
  - Could choose ***mixture*** of rectangular coords (**m** **=** # rectangular coords) & curvilinear (**s - m =** # curvilinear coords)
  - The **s** generalized coords needn't have units of length! Could be dimensionless or have (almost) ***any units***.

- Generalized coords, $\mathbf{q}_\ell$ will (often) not divide into groups of 3 that can be associated with vectors.
  - **Example:** Particle on sphere surface: A convenient choice of

    $\mathbf{q}_\ell$ = latitude & longitude.
  - **Example:** Double pendulum:

    A convenient choice of

    $\mathbf{q}_\ell = \boldsymbol{\theta}_1$ & $\boldsymbol{\theta}_2$ **(Figure) →**

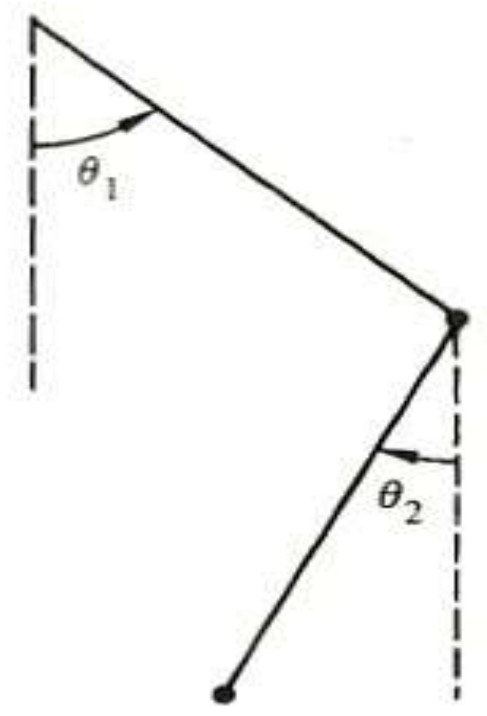


FIGURE 1.4 Double pendulum.

- Sometimes, it’s convenient & useful to use **Generalized Coords** (non-Cartesian) even in systems with no constraints.
  - **Example:** Central force field problems:
    V = V(r), it makes sense to use spherical coords!

- Generalized coords need not be orthogonal coordinates & need not be position coordinates.

- **Non-Holonomic constraint:**

⇒ Eqtns expressing constraint can’t be used to eliminate dependent coordinates.

- **Example:** Object rolling without slipping on a rough surface. Coordinates needed to describe motion**: Angular coords to specify body orientation + coords to describe location of point of contact of body & surface.** Constraint of rolling ⇒ Connects 2 coord sets: They aren’t independent. **BUT**, # coords cannot be reduced by the constraint, because cannot express rolling condition as eqtn between coords! Instead, (can show) **rolling constraint** is condition on the ***velocities***: a differential eqtn which can be integrated only after solution to problem is known!

Example: Rolling Constraint

- Disk, radius **a**, constrained to be vertical, rolling on the horizontal (**xy**) plane. Figure:

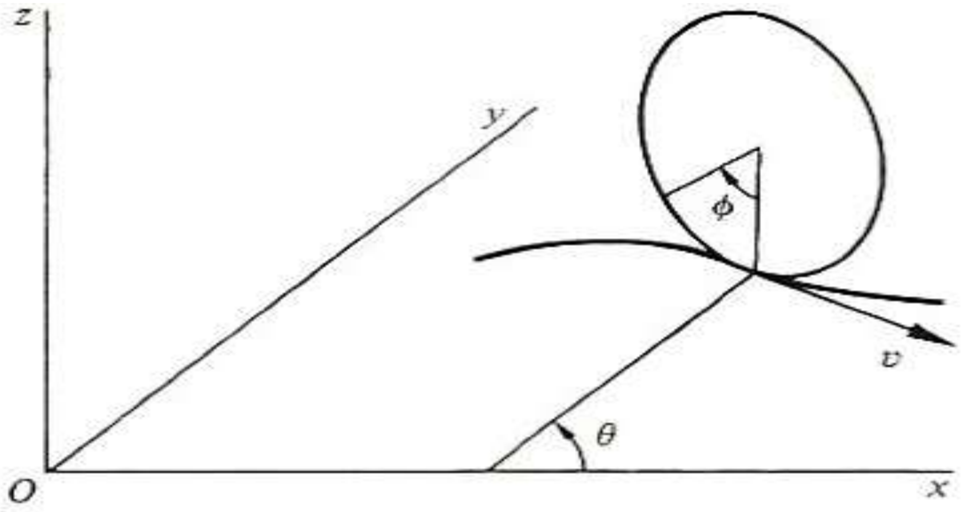


FIGURE 1.5 Vertical disk rolling on a horizontal plane.

- **Generalized coords:** **x, y** of point of contact of disk with plane **+ θ** = angle between disk axis & **x**-axis **+ φ** = angle of rotation about disk axis

- **Constraint:** Velocity **v** of disk center is related to angular velocity **(dφ/dt)** of disk rotation:

  $$v = a(d\varphi/dt) \quad (1)$$

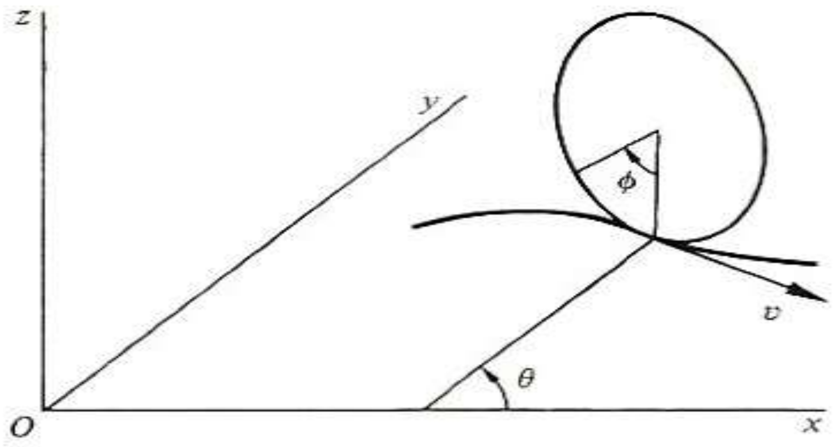


FIGURE 1.5 Vertical disk rolling on a horizontal plane.

Also Cartesian components of **v**:

$$v_x = (dx/dt) = v \sin\theta,\ v_y = (dy/dt) = -v \cos\theta \quad (2)$$

Combine **(1)** & **(2)** (multiplying through by dt):

$$\Rightarrow \quad dx - a \sin\theta\, d\varphi = 0 \qquad dy + a \cos\theta\, d\varphi = 0$$

Neither can be integrated without solving the problem! That is, a function **f(x,y,θ,φ) = 0** cannot be found. Physical argument that **φ** must be indep of **x,y,θ:** See pp. 15 & 16

- **Non-Holonomic constraints** can also involve higher order derivatives or inequalities.
- **Holonomic constraints** are preferred, since easiest to deal with. No general method to treat problems with Non-Holonomic constraints. Treat on case-by-case basis.
- **In special cases of Non-Holonomic constraints**, when constraint is expressed in differential form (as in example), can use method of Lagrange multipliers along with Lagrange's eqtns.
- Authors argue, except for some macroscopic physics textbook examples, most problems of practical interest to physicists are microscopic & the constraints are holonomic or do not actually enter the problem.

› **Difficulties constraints introduce:**

1. Coordinates $\mathbf{r}_i$ are no longer all independent. Connected by constraint eqtns.

– **Have now thoroughly discussed this problem**!

2. To apply Newton's 2nd Law, need the ***TOTAL*** force acting on each particle. Forces of constraint are not always known or easily calculated.

$\Rightarrow$ With constraints, it's often difficult to ***directly*** apply Newton's 2nd Law.

Put another way: Forces of constraint are often among the unknowns of the problem! To address this, long ago, people reformulated mechanics. **Lagrangian & Hamiltonian formulations.** No direct reference to forces of constraint.

D'Alembert's Principle & Lagrange's Equations

- ***Virtual* (infinitesimal) *displacement*** ≡ Change in the system configuration as result of an arbitrary infinitesimal change of coordinates $\delta \mathbf{r}_i$, ***consistent with the forces & constraints imposed on the system at a given time t.***

- ***"Virtual"*** distinguishes it from an **actual** displacement $d\mathbf{r}_i$, occurring in small time interval **dt** (during which forces & constraints may change)

- Consider the system at ***equilibrium*:** The total force on each particle is $\mathbf{F_i = 0}$. ***Virtual work*** done by $\mathbf{F_i}$ in displacement $\delta\mathbf{r_i}$:

$$\delta\mathbf{W_i} = \mathbf{F_i}\bullet\delta\mathbf{r_i} = \mathbf{0}.\ \text{Sum over } \boldsymbol{i}:$$

$$\Rightarrow \quad \delta\mathbf{W} = \textstyle\sum_i \mathbf{F_i}\bullet\delta\mathbf{r_i} = \mathbf{0}.$$

- Decompose $\mathbf{F_i}$ into **applied force** $\mathbf{F_i^{(a)}}$ & **constraint force** $\mathbf{f_i}$: $\mathbf{F_i = F_i^{(a)} + f_i}$

$$\Rightarrow \delta\mathbf{W} = \textstyle\sum_i (\mathbf{F_i^{(a)}} + \mathbf{f_i})\bullet\delta\mathbf{r_i} \equiv \delta\mathbf{W^{(a)}} + \delta\mathbf{W^{(c)}} = \mathbf{0}$$

- ***Special case*** (often true, see text discussion): Systems for which the net virtual work due to constraint forces is zero: $\sum_i \mathbf{f_i}\bullet\delta\mathbf{r_i} \equiv \delta\mathbf{W^{(c)}} = \mathbf{0}$

Principle of Virtual Work

⇒ ***Condition for system equilibrium:*** **Virtual work due to APPLIED forces vanishes:**

$$\delta W^{(a)} = \sum_i F_i^{(a)} \bullet \delta r_i = 0 \qquad (1)$$

≡ ***Principle of Virtual Work***

› **Note:** In general coefficients of δr_i , $F_i^{(a)} \neq 0$ even though $\sum_i F_i^{(a)} \bullet \delta r_i = 0$ because δr_i are not independent, but connected by constraints.

– In order to have coefficients of $\delta r_i = 0$, must transform **Principle of Virtual Work** into a form involving virtual displacements of generalized coordinates $q_{\square}$, which are independent. **(1)** is good since it does not involve constraint forces f_i . But so far, only statics. Want to treat dynamics!

D'Alembert's Principle

- **Dynamics:** Start with **Newton's 2nd Law** for particle *i*: $\mathbf{F_i} = (\mathbf{dp_i/dt})$ Or: $\mathbf{F_i - (dp_i/dt) = 0}$

$\Rightarrow$ Can view system particles as in "equilibrium" under a force

= actual force + "reversed effective force" $\mathbf{= -(dp/dt)}$

- **Virtual work** done is

$$\delta W = \sum_i [F_i - (dp_i/dt)] \bullet \delta r_i = 0$$

- Again decompose $\mathbf{F_i}$: $\mathbf{F_i = F_i^{(a)}} + \mathbf{f_i}$

$$\Rightarrow \qquad \delta W = \sum_i [F_i^{(a)} - (dp_i/dt) + f_i] \bullet \delta r_i = 0$$

- Again restrict consideration to ***special case***: Systems for which the net virtual work due to constraint forces is zero: $\sum_i f_i \bullet \delta r_i \equiv \delta W^{(c)} = 0$

$\Rightarrow$ $\boldsymbol{\delta W = \sum_i [F_i - (dp_i/dt)] \bullet \delta r_i = 0}$ **(2)**

$\equiv$ ***D'Alembert's Principle***

– Dropped the superscript **(a)!**

› Transform **(2)** to an expression involving **virtual displacements** of $\mathbf{q}_\ell$ (which, for holonomic constraints, are indep of each other). Then, by linear independence, the coefficients of the $\boldsymbol{\delta q_\ell = 0}$

$$\delta W = \sum_i [F_i - (dp_i/dt)] \bullet \delta r_i = 0 \qquad (2)$$

- Much **manipulation** follows! Only highlights here!
- Transformation eqtns:

$$r_i = r_i(q_1, q_2, q_3, .., t) \qquad (i = 1,2,3, \ldots n)$$

- Chain rule of differentiation (velocities):

$$v_i \equiv (dr_i/dt) = \sum_k (\partial r_i/\partial q_k)(dq_k/dt) + (\partial r_i/\partial t) \qquad (a)$$

- Virtual displacements $\delta \mathbf{r}_i$ are connected to virtual displacements $\delta \mathbf{q}_\ell$: $\delta \mathbf{r}_i = \sum_j (\partial r_i/\partial q_j)\delta q_j \qquad (b)$

Generalized Forces

- 1^{st} term of **(2)** (Combined with **(b)**):

  $$\sum_i \mathbf{F_i} \bullet \boldsymbol{\delta}\mathbf{r_i} = \sum_{i,j} \mathbf{F_i} \bullet (\partial \mathbf{r_i}/\partial \mathbf{q_j})\boldsymbol{\delta}\mathbf{q_j} \equiv \sum_j \mathbf{Q_j}\boldsymbol{\delta}\mathbf{q_j} \qquad \textbf{(c)}$$

  Define ***Generalized Force*** (corresponding to Generalized Coordinate $\mathbf{q_j}$): $\mathbf{Q_j} \equiv \sum_i \mathbf{F_i} \bullet (\partial \mathbf{r_i}/\partial \mathbf{q_j})$

  – Generalized Coordinates $\mathbf{q_j}$ need not have units of length!

  $\Rightarrow$ Corresponding **Generalized Forces** $\mathbf{Q_j}$ need not have units of force!

  – For example: If $\mathbf{q_j}$ is an angle, corresponding $\mathbf{Q_j}$ will be a torque!

- 2^{nd} term of **(2)** (using **(b)** again):

$$\sum_i (dp_i/dt)\bullet\delta r_i = \sum_i [m_i\,(d^2r_i/dt^2)\bullet\delta r_i\,] =$$

$$\sum_{i,j} [m_i\,(d^2r_i/dt^2)\bullet(\partial r_i/\partial q_j)\delta q_j] \qquad \textbf{(d)}$$

- **Manipulate** with **(d):** $\sum_i [m_i\,(d^2r_i/dt^2)\bullet(\partial r_i/\partial q_j)] =$

$$\sum_i [d\{m_i(dr_i/dt)\bullet(\partial r_i/\partial q_j)\}/dt] - \sum_i [m_i(dr_i/dt)\bullet d\{(\partial r_i/\partial q_j)\}/dt]$$

Also: $d\{(\partial r_i/\partial q_j)\}/dt = \partial\{dr_i/dt\}/\partial q_j \equiv (\partial v_i/\partial q_j)$

Use **(a):** $(\partial v_i/\partial q_j) = \sum_k (\partial^2 r_i/\partial q_j \partial q_k)(dq_k/dt) + (\partial^2 r_i/\partial q_j \partial t)$

From **(a):** $(\partial v_i/\partial q_j) = (\partial r_i/\partial q_j)$

So: $\sum_i [m_i\,(d^2r_i/dt^2)\bullet(\partial r_i/\partial q_j)]$

$$= \sum_i [d\{m_i v_i \bullet(\partial v_i/\partial q_j)\}/dt] - \sum_i [m_i v_i \bullet(\partial v_i/\partial q_j)]$$

More manipulation $\Rightarrow$ **(2)** is: $\sum_i[F_i-(dp_i/dt)]\bullet\delta r_i = 0$

$$\sum_j\{d[\partial(\sum_i (½)m_i(v_i)^2)/\partial q_j]/dt - \partial(\sum_i(½)m_i(v_i)^2)/\partial q_j - Q_j\}\delta q_j = 0$$

- System kinetic energy is: $T \equiv (½)\sum_i m_i(v_i)^2$

$\Rightarrow$ ***D'Alembert's Principle*** becomes

$$\sum_j\{(d[\partial T/\partial q_j]/dt) - (\partial T/\partial q_j) - Q_j\}\delta q_j = 0 \qquad (3)$$

– Note: If q_j are Cartesian coords, $(\partial T/\partial q_j) = 0$

$\Rightarrow$ In generalized coords, $(\partial T/\partial q_j)$ comes from the curvature of the q_j. (**Example:** Polar coords, $(\partial T/\partial \theta)$ becomes the centripetal acceleration).

- So far, no restriction on constraints except that they do no work under virtual displacement. q_j are any set.

Special case: *Holonomic Constraints* $\Rightarrow$ It's possible to find sets of q_j for which each δq_j is independent.

$\Rightarrow$ **Each term in (3) is separately 0!**

- Holonomic constraints $\Rightarrow$ ***D'Alembert's Principle***:

$$(d[\partial T/\partial \dot{q}_j]/dt) - (\partial T/\partial q_j) = Q_j \qquad (4)$$

$(j = 1,2,3, \ldots n)$

- **Special case: *A Potential Exists*** $\Rightarrow \mathbf{F_i = - \nabla_i V}$
  - Needn't be conservative! **V** could be a function of **t**!

$\Rightarrow$ **Generalized forces have the form**

$$Q_j \equiv \sum_i F_i \bullet (\partial r_i/\partial q_j) = - \sum_i \nabla_i V \bullet (\partial r_i/\partial q_j) \equiv - (\partial V/\partial q_j)$$

- Put this in **(4)**: $(d[\partial T/\partial \dot{q}_j]/dt) - (\partial [T-V]/\partial q_j) = 0$
- So far, **V** doesn't depend on the velocities $\dot{q}_j$

$$\Rightarrow \qquad (d/dt)[\partial (T-V)/\partial \dot{q}_j] - \partial (T-V)/\partial q_j = 0 \qquad (4')$$

Lagrange's Equations

- **Define:** ***The Lagrangian*** ***L*** of the system:

$$L \equiv T - V$$

$\Rightarrow$ Can write ***D'Alembert's Principle*** as:

$$(d/dt)[(\partial L/\partial \dot{q}_j)] - (\partial L/\partial q_j) = 0 \qquad (5)$$

$(j = 1,2,3, \dots n)$

(5) ≡ *Lagrange's Equations*

Lagrange's Equations

- ***Lagrangian***: $L \equiv T - V$
- ***Lagrange's Eqtns:***

$$(d/dt)[(\partial L/\partial \dot{q}_j)] - (\partial L/\partial q_j) = 0 \qquad (j = 1,2,3, \ldots n)$$

- **Note:** L is **not unique**, but is arbitrary to within the addition of a derivative **(dF/dt)**. **F = F(q,t)** is ***any*** differentiable function of **q**'s & **t**.
- That is, if we define a new Lagrangian L'

$$L' = L + (dF/dt)$$

It is easy to show that L' satisfies ***the same*** Lagrange's Eqtns (above).

Velocity-Dependent Potentials & the Dissipation Function

- **Non-conservative** forces? It's still possible, in a ***Special Case***, to use Lagrange's Eqtns unchanged, provided a **Generalized or Velocity-Dependent Potential U = $U(q_j,\dot{q}_j)$** exists, where the generalized forces $\mathbf{Q_j}$ are obtained as:

$$Q_j \equiv -(\partial U/\partial q_j) + (d/dt)[(\partial U/\partial \dot{q}_j)]$$

- The **Lagrangian is now:** $L \equiv$ **T - U** & Lagrange's Eqtns are still:

$$(d/dt)[(\partial L/\partial \dot{q}_j)] - (\partial L/\partial q_j) = 0 \quad (j = 1,2,3, \dots n)$$

- **A very important application:** Electromagnetic forces on moving charges.

Electromagnetic Force Problem

- Particle, mass **m**, charge **q** moving with velocity **v** in combined electric (**E**) & magnetic (**B**) fields.
- **Lorentz Force** (SI units!):

$$\mathbf{F} = q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})] \qquad (1)$$

- ***E&M results that you should know***!

E = E(x,y,z,t) & **B = B(x,y,z,t)** are derivable from a scalar potential **ϕ = ϕ(x,y,z,t)** and a vector potential **A = A(x,y,z,t)** as:

$$\mathbf{E} \equiv -\nabla\phi - (\partial\mathbf{A}/\partial t) \qquad (2)$$

$$\mathbf{B} \equiv \nabla \times \mathbf{A} \qquad (3)$$

- **Can obtain the Lorentz Force (1) from the velocity dependent potential:** $\mathbf{U} \equiv \mathbf{q\phi - qA \bullet v}$

$$\mathbf{F = -\nabla U}$$

  - Proof: **Exercise for student!** Use **(1),(2),(3)** together.

- Lagrangian is: $L \equiv \mathbf{T - U = (½)mv^2 - q\phi + qA \bullet v}$
- Use Cartesian coords. Lagrange Eqtn for coord $\mathbf{x}$ (noting $\mathbf{v}^2 = (\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2$ & $\mathbf{v} = \dot{x}\,\hat{i} + \dot{y}\,\hat{j} + \dot{z}\,\hat{k}$)

$$\mathbf{(d/dt)[(\partial \mathit{L}/\partial \dot{x})] - (\partial \mathit{L}/\partial x) = 0}$$

$$\Rightarrow \mathbf{m\ddot{x} = q[\dot{x}(\partial A_x/\partial x) + \dot{y}(\partial A_y/\partial x) + \dot{z}(\partial A_z/\partial x)]}$$

$$\mathbf{- q[(\partial \phi/\partial x) + (dA_x/dt)]} \qquad \mathbf{(a)}$$

Note that: $\mathbf{(dA_x/dt) = v \bullet \nabla A_x + (\partial A_x/\partial t)}$

$$\Rightarrow \qquad m\ddot{x} = -\,q(\partial\phi/\partial x) - q(\partial A_x/\partial t)$$

$$+\, q[\dot{y}\{(\partial A_y/\partial x) - (\partial A_x/\partial y)\} + \dot{z}\{(\partial A_z/\partial x) - (\partial A_x/\partial z)\}]$$

- Using **(2)** & **(3)** this becomes:

$$m\ddot{x} = q[E_x + \dot{y}B_z - \dot{z}B_y]$$

Or: $\quad m\ddot{x} = q[E_x + (v \times B)_x] = F_x$ (Proven!)

- If **some forces in the problem are conservative & some are not:** $\Rightarrow$ Have potential **V** for conservative ones & thus have the Lagrangian $L \equiv T - V$ for these. For non-conservative ones, still have generalized forces:

$$Q_j \equiv \sum_i F_i \bullet (\partial r_i/\partial q_j)$$

Non-Conservative Forces

- $L \equiv$ **T - V** for conservative forces.
- **Generalized forces:** $\mathbf{Q_j} \equiv \sum_i \mathbf{F_i} \bullet (\partial \mathbf{r_i}/\partial \mathbf{q_j})$ for non-conservative forces.
- Follow derivation of Lagrange Eqtns & get:

$$(d/dt)[(\partial L/\partial \dot{q}_j)] - (\partial L/\partial q_j) = Q_j \quad (j = 1,2,3, ..n)$$

- **Friction:** A common non-conservative force.
- **Friction** (or air resistance): A common ***model:*** Components are proportional to some power of **v** (often the **1**st power): $\mathbf{F_{fx} = -k_x v_x}$ ($\mathbf{k_x}$ = const)

Frictional Forces

- ***Model* for Friction** (or air resistance): $\mathbf{F_{fx} = -k_x v_x}$
- Can Include such forces in Lagrangian formalism by introducing ***Rayleigh's Dissipation Function*** $\mathcal{F}$

  $\mathcal{F} \equiv (\frac{1}{2})\sum_i[k_x(v_{ix})^2 + k_y(v_{iy})^2 + k_z(v_{iz})^2]$
- Obtain components of the frictional force by:

  $\mathbf{F_{fxi}} \equiv -(\partial\mathcal{F}/\partial v_{ix})$, etc. Or, $\mathbf{F_f} = -\nabla_v\mathcal{F}$
- **Physical Interpretation** of $\mathcal{F}$: Work done by system *against* friction: $\mathbf{dW_f = -F_f \bullet dr = -F_f \bullet v\ dt}$

  $= -[k_x(v_{ix})^2 + k_y(v_{iy})^2 + k_z(v_{iz})^2]\ dt = -2\mathcal{F}\, dt$

$\Rightarrow$ **Rate of energy dissipation due to friction:**

$$(dW_f/dt) = -2\mathcal{F}$$

- ***Rayleigh's Dissipation Function $\mathcal{F}$***

$$\mathcal{F} \equiv (\tfrac{1}{2})\sum_i[k_x(v_{ix})^2 + k_y(v_{iy})^2 + k_z(v_{iz})^2]$$

- **Frictional force:** $\mathbf{F_{fi} = -\nabla_{vi}\mathcal{F}}$
- Corresponding **generalized force:**

$$Q_j \equiv \sum_i F_{fi}\bullet(\partial r_i/\partial q_j) = -\sum_i \nabla_{vi}\mathcal{F}\bullet(\partial r_i/\partial q_j)$$

Note that: $(\partial r_i/\partial q_j) = (\partial \dot{r}_i/\partial \dot{q}_j)$

$$Q_j = -\sum_i \nabla_{vi}\mathcal{F}\bullet(\partial \dot{r}_i/\partial \dot{q}_j) = -(\partial \mathcal{F}/\partial \dot{q}_j)$$

- Lagrange's Eqtns, with frictional (dissipative) forces:

$$(d/dt)[(\partial L/\partial \dot{q}_j)] - (\partial L/\partial q_j) = Q_j$$

Or

$$(d/dt)[(\partial L/\partial \dot{q}_j)] - (\partial L/\partial q_j) + (\partial \mathcal{F}/\partial \dot{q}_j) = 0$$

$$(j = 1,2,3, ..n)$$

Simple Applications of the Lagrangian Formulation

- **Lagrangian formulation: 2 *scalar functions*, T & V**
- **Newtonian formulation: *MANY* vector forces &** accelerations. (*Advantage of Lagrangian over Newtonian!*)
- **"Recipe"** for application of the Lagrangian method:
  - Choose appropriate generalized coordinates
  - Write **T** & **V** in terms of these coordinates
  - Form the Lagrangian $L = T - V$
  - Apply: ***Lagrange's Eqtns:***

    $$(d/dt)[(\partial L/\partial \dot{q}_j)] - (\partial L/\partial q_j) = 0 \qquad (j = 1,2,3, \dots n)$$
  - Equivalently ***D'Alembert's Principle***:

    $$(d/dt)[\partial T/\partial \dot{q}_j] - (\partial T/\partial q_j) = Q_j \qquad (j = 1,2,3, \dots n)$$

Examples

- Simple examples (for some, the Lagrangian method is "overkill"):

1. A single particle in space (subject to force **F**):

a. Cartesian coords

b. Plane polar coords.

2. The Atwood's machine

3. Time dependent constraint: A bead sliding on rotating wire

Particle in Space (Cartesian Coords)

- The Lagrangian method is "overkill" for this problem!
- Mass **m**, force **F**: Generalized coordinates $\mathbf{q_j}$ are Cartesian coordinates **x, y, z**! $\mathbf{q_1 = x}$, etc. Generalized forces $\mathbf{Q_j}$ are Cartesian components of force $\mathbf{Q_1 = F_x}$, etc.
- Kinetic energy: $\mathbf{T = (\frac{1}{2})m[(\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2]}$
- Lagrange eqtns which contain generalized forces (*D'Alembert's Principle*):

  $\mathbf{(d[\partial T/\partial \dot{q}_j]/dt) - (\partial T/\partial q_j) = Q_j}$ $\quad$ $\mathbf{(j = 1,2,3}$ or $\mathbf{x,y,z)}$

- $T = (\frac{1}{2})m[(\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2]$

  $(d[\partial T/\partial \dot{q}_j]/dt) - (\partial T/\partial q_j) = Q_j$

  $(j = 1,2,3 \text{ or } x,y,z)$

$$(\partial T/\partial x) = (\partial T/\partial y) = (\partial T/\partial z) = 0$$

$$(\partial T/\partial \dot{x}) = m\dot{x}, \ (\partial T/\partial \dot{y}) = m\dot{y}, \ (\partial T/\partial \dot{z}) = m\dot{z}$$

$$\Rightarrow \ d(m\dot{x})/dt = m\ddot{x} = F_x \ ; \ d(m\dot{y})/dt = m\ddot{y} = F_y$$

$$d(m\dot{z})/dt = m\ddot{z} = F_z$$

Identical results (of course!) to Newton's 2nd Law.

Particle in Plane (Plane Polar Coords)

- **Plane Polar Coordinates:**

$$q_1 = r,\ q_2 = \theta$$

- **Transformation eqtns:**

$$x = r\cos\theta,\quad y = r\sin\theta$$

$$\Rightarrow\quad x = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$

$$y = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

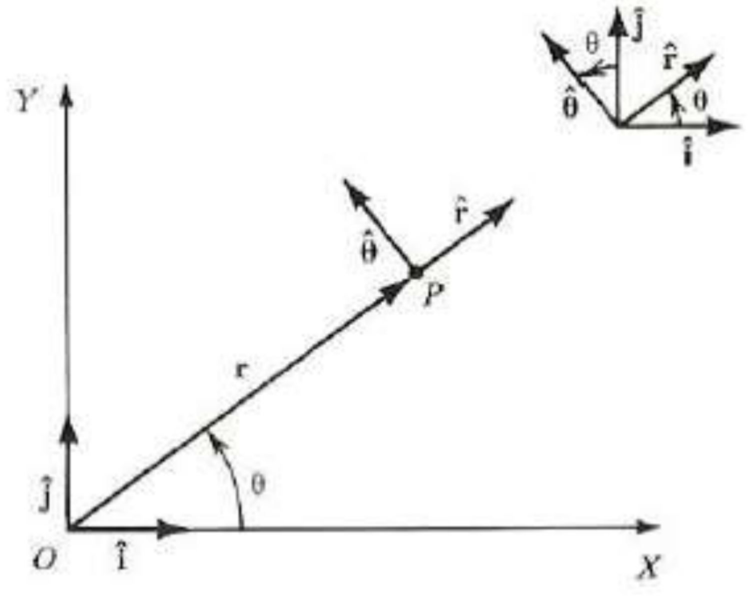


Figure 6.9 Unit vectors $\hat{r}$ and $\hat{\theta}$ in plane polar coordinates.

$\Rightarrow$ **Kinetic energy:**

$$T = (½)m[(\dot{x})^2 + (\dot{y})^2] = (½)m[(\dot{r})^2 + (r\dot{\theta})^2]$$

Lagrange: $(d[\partial T/\partial \dot{q}_j]/dt) - (\partial T/\partial q_j) = Q_j$ $(j = 1,2 \text{ or } r, \theta)$

Generalized forces: $Q_j \equiv \sum_i F_i \bullet (\partial \vec{r}_i/\partial q_j)$

$$\Rightarrow\quad Q_1 = Q_r = \vec{F}\bullet(\partial\vec{r}/\partial r) = \vec{F}\bullet\hat{r} = F_r$$

$$Q_2 = Q_\theta = \vec{F}\bullet(\partial\vec{r}/\partial\theta) = \vec{F}\bullet r\hat{\theta} = rF_\theta$$

$T = (½)m[(\dot{r})^2 + (r\dot{\theta})^2]$ **Forces:** $Q_r = F_r$, $Q_\theta = rF_\theta$

Lagrange: $(d[\partial T/\partial \dot{q}_j]/dt) - (\partial T/\partial q_j) = Q_j$ $(j = r, \theta)$

– *Physical interpretation:* $Q_r = F_r$ = radial force component.

$Q_r = F_r$ = radial component of force.

$Q_\theta = rF_\theta$ = torque about axis ⊥ plane through origin

- r: $(\partial T/\partial r) = mr(\dot{\theta})^2$; $(\partial T/\partial \dot{r}) = m\dot{r}$; $(d[\partial T/\partial \dot{r}]/dt) = m\ddot{r}$

$$\Rightarrow \qquad m\ddot{r} - mr(\dot{\theta})^2 = F_r \qquad (1)$$

– *Physical interpretation:* $- mr(\dot{\theta})^2$ = centripetal force

- θ: $(\partial T/\partial \theta) = 0$; $(\partial T/\partial \dot{\theta}) = mr^2\dot{\theta}$; (Note: $L = mr^2\dot{\theta}$)

$$(d[\partial T/\partial \dot{\theta}]/dt) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = (dL/dt) = N$$

$$\Rightarrow \qquad mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = rF_\theta \qquad (2)$$

– *Physical interpretation:* $mr^2\dot{\theta} = L$ = **angular momentum** about axis through origin ⇒ (2) ≡ $(dL/dt) = N = rF_\theta$

Atwood's Machine

- **M_1 & M_2** connected over a massless, frictionless pulley by a massless, extensionless string, length **ℓ**.

  Gravity acts, of course!

  $\Rightarrow$ ***Conservative system, holonomic, scleronomous constraints***

- 1 indep. coord. (1 deg. of freedom). Position **x** of **M_1**.

  Constraint keeps const. length **ℓ**.

- **PE:** $V = -M_1gx - M_2g(\ell - x)$
- **KE:** $T = (½)(M_1 + M_2)(\dot{x})^2$
- **Lagrangian:** $L = T - V = (½)(M_1+M_2)(\dot{x})^2 - M_1gx - M_2g(\ell - x)$

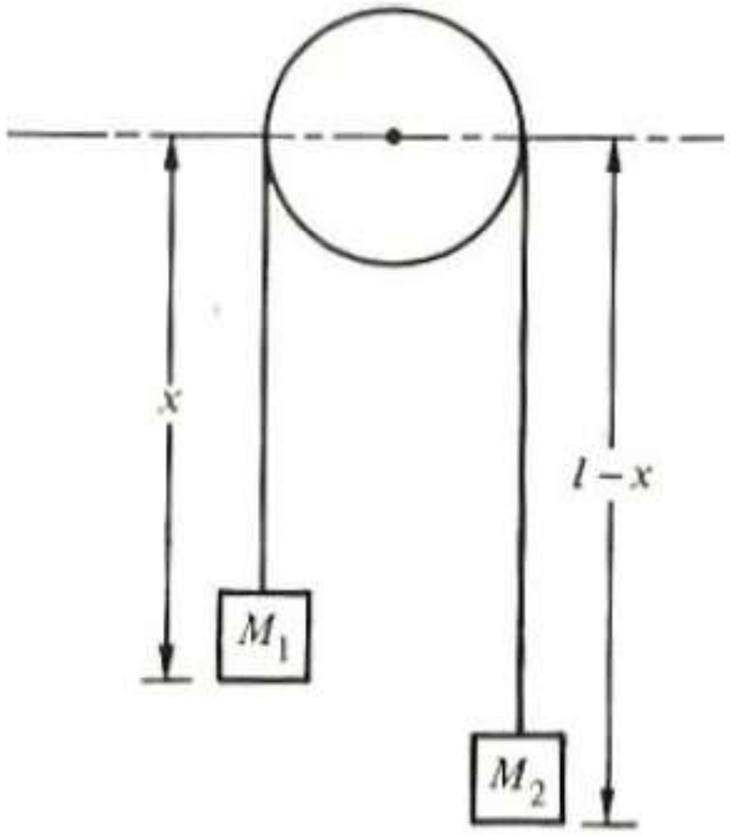


FIGURE 1.7 Atwood's machine.

$$L = (½)(M_1+M_2)(\dot{x})^2 - M_1gx - M_2g(\ell - x)$$

- **Lagrange:** $(d/dt)[(\partial L/\partial\dot{x})] - (\partial L/\partial x) = 0$

$$(\partial L/\partial x) = (M_2 - M_1)g \;;\; (\partial L/\partial\dot{x}) = (M_1+M_2)\dot{x}$$

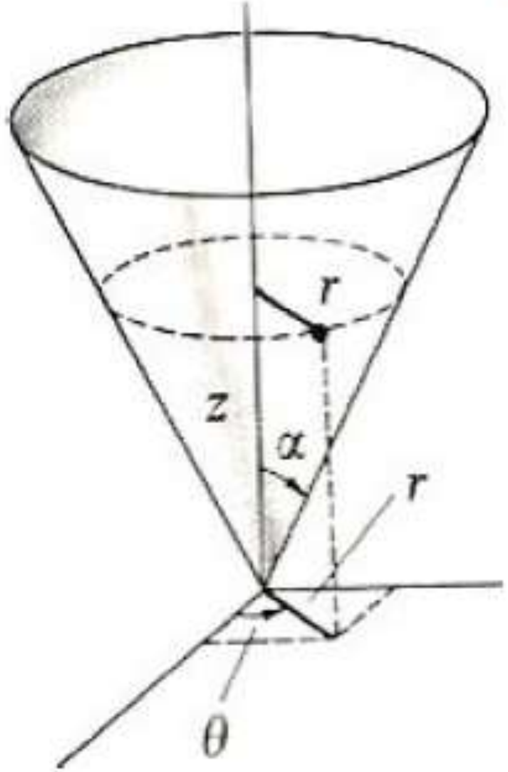
$$\Rightarrow \qquad (M_1+M_2)\ddot{x} = (M_2 - M_1)g$$

Or: $\ddot{x} = [(M_2 - M_1)/(M_1+M_2)]\, g$

Same as obtained in freshman physics!

- **Force of constraint = tension. Compute using Lagrange multiplier method (later!).**

- Particle, mass **m**, constrained to move on the inside surface of a smooth cone of half angle $\boldsymbol{\alpha}$ (Fig.). Subject to gravity. Determine a set of generalized coordinates & determine the constraints. Find the eqtns of motion.



Solution: Let the axis of the cone correspond to the z-axis and let the apex of the cone be located at the origin. Since the problem possesses cylindrical symmetry, we choose r, θ, and z as the generalized coordinates. We have, however, the equation of constraint

$$z = r \cot \alpha \tag{7.26}$$

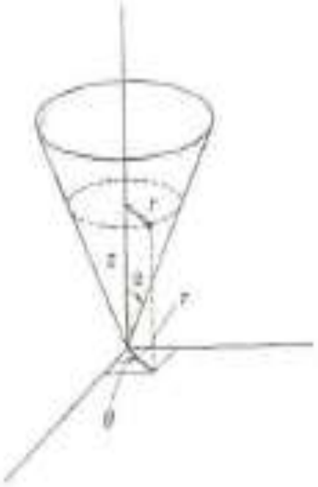


FIGURE 7-2

so there are only two degrees of freedom for the system, and therefore only two proper generalized coordinates. We may use Equation 7.26 to eliminate either the coordinate z or r; we choose to do the former. Then the square of the velocity is

$$\begin{aligned} v^2 &= \dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2 \\ &= \dot{r}^2 + r^2\dot{\theta}^2 + \dot{r}^2 \cot^2 \alpha \\ &= \dot{r}^2 \csc^2 \alpha + r^2\dot{\theta}^2 \end{aligned} \tag{7.27}$$

The potential energy (if we choose $V = 0$ at $z = 0$) is

$$V = mgz = mgr \cot \alpha$$

so the Langrangian is

$$L = \frac{1}{2}m(\dot{r}^2 \csc^2 \alpha + r^2\dot{\theta}^2) - mgr \cot \alpha \tag{7.28}$$

We note first that L does not explicitly contain θ. Therefore $\partial L/\partial\theta = 0$, and the Lagrange equation for the coordinate θ is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$$

Hence

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = \text{constant} \tag{7.29}$$

ut $mr^2\dot{\theta} = mr^2\omega$ is just the angular momentum about the z-axis. Therefore, Equa-
on 7.29 expresses the conservation of angular momentum about the axis of sym-
etry of the system.

The Lagrange equation for r is

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$$

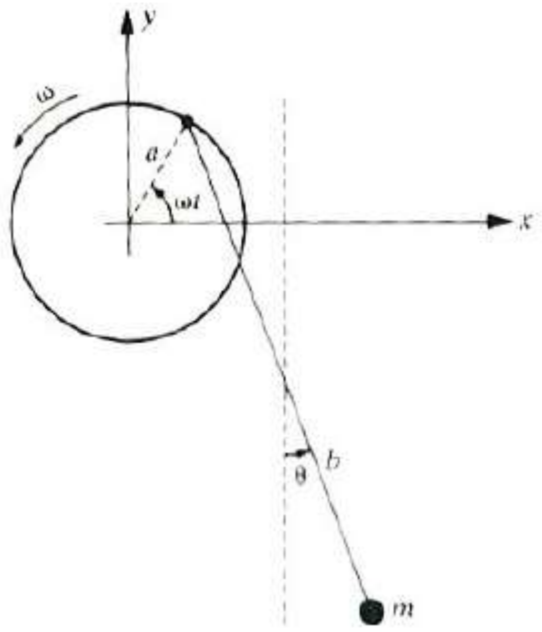
Calculating the derivatives, we find

$$\ddot{r} - r\dot{\theta}^2 \sin^2 \alpha + g \sin \alpha \cos \alpha$$

which is the equation of motion for the coordinate r.

We shall return to this example in Section 8.10
more detail.

- The point of support of a simple pendulum (length **b**) moves on massless rim (radius **a**) rotating with const angular velocity **ω**. Obtain expressions for the Cartesian components of velocity & acceleration of **m**. Obtain the angular acceleration for the angle **θ** shown in the figure.



Solution!

Solution: We choose the origin of our coordinate system to be at the center of the rotating rim. The Cartesian components of mass m become

$$\left.\begin{aligned} x &= a\cos\omega t + b\sin\theta \\ y &= a\sin\omega t - b\cos\theta \end{aligned}\right\} \tag{7.32}$$

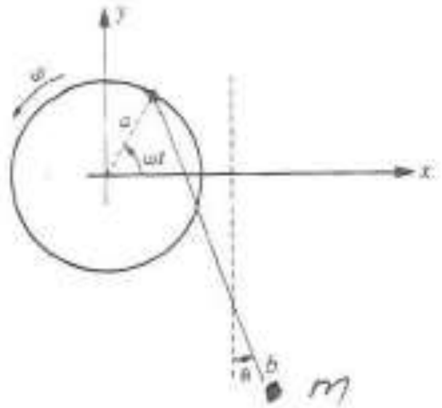
The velocities are

$$\left.\begin{aligned} \dot{x} &= -a\omega\sin\omega t + b\dot{\theta}\cos\theta \\ \dot{y} &= a\omega\cos\omega t + b\dot{\theta}\sin\theta \end{aligned}\right\} \tag{7.33}$$

Taking the time derivative once again gives the acceleration:

$$\ddot{x} = -a\omega^2\cos\omega t + b(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)$$

$$\ddot{y} = -a\omega^2\sin\omega t + b(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)$$



It should now be clear that the single generalized coordinate is θ. The kin potential energies are

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$V = mgy$$

where $V = 0$ at $y = 0$. The Lagrangian is

$$L = T - V = \frac{m}{2}[a^2\omega^2 + b^2\dot{\theta}^2 + 2b\dot{\theta}a\omega\sin(\theta - \omega t)] - mg(a\sin\omega t - b\cos\theta)$$

The derivatives for the Lagrange equation of motion for θ are

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = mb^2\ddot{\theta} + mba\omega(\dot{\theta} - \omega)\cos(\theta - \omega t)$$

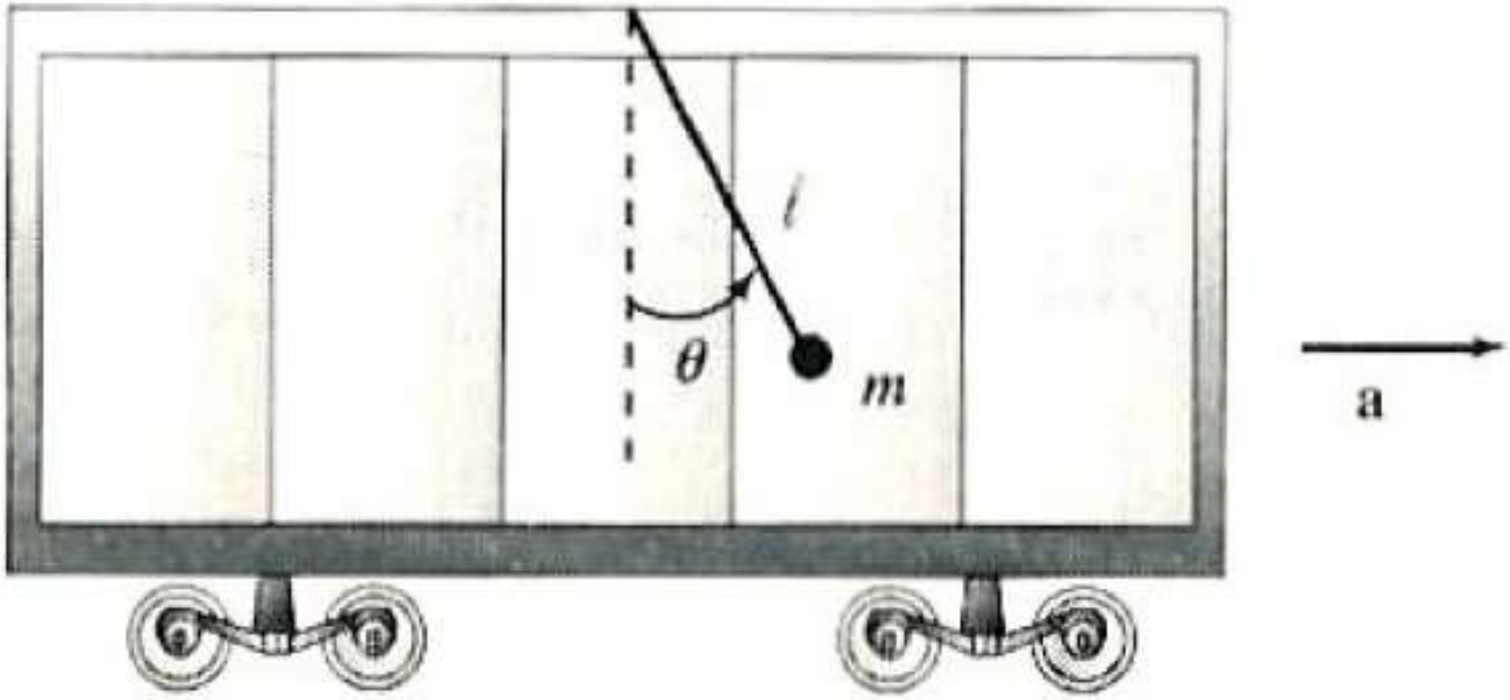
$$\frac{\partial L}{\partial\theta} = mb\dot{\theta}a\omega\cos(\theta - \omega t) - mgb\sin\theta$$

which results in the equation of motion (after solving for $\ddot{\theta}$)

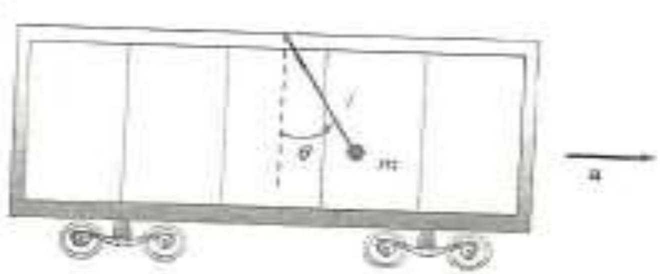
$$\ddot{\theta} = \frac{\omega^2 a}{b}\cos(\theta - \omega t) - \frac{g}{b}\sin\theta$$

Notice that this result reduces to the well-known equation of motion for pendulum if $\omega = 0$.

- Find the eqtn of motion for a simple pendulum placed in a railroad car that has a const **x**-directed acceleration **a**.



Solution!



Solution: A schematic diagram is shown in Figure 7-4a for the pendulum of length ℓ, mass m, and displacement angle θ. We choose a fixed cartesian coordinate system with $x = 0$ and $\dot{x} = v_0$ at $t = 0$. The position and velocity of m become

$$x = v_0 t + \frac{1}{2} a t^2 + \ell \sin \theta$$

$$y = -\ell \cos \theta$$

$$\dot{x} = v_0 + at + \ell \dot{\theta} \cos \theta$$

$$\dot{y} = \ell \dot{\theta} \sin \theta$$

The kinetic and potential energies are

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) \qquad V = -mg\ell \cos \theta$$

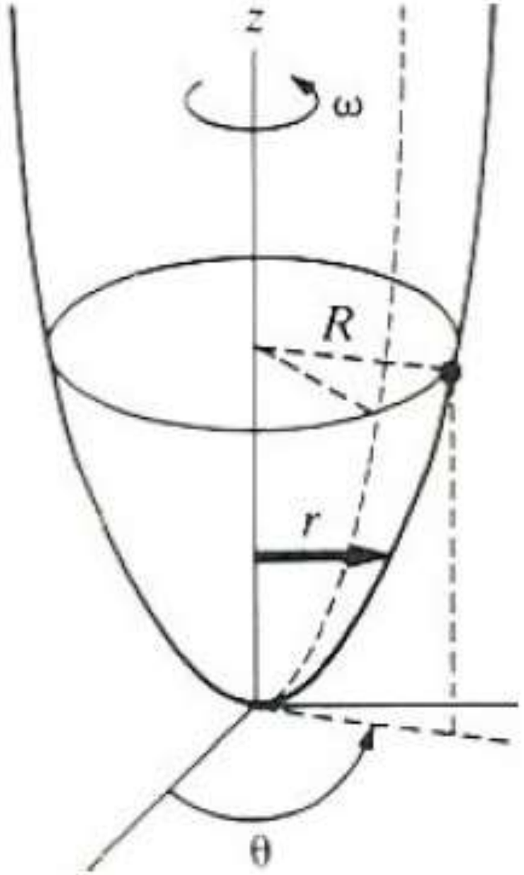
and the Lagrangian is

$$L = T - V = \frac{1}{2} m(v_0 + at + \ell \dot{\theta} \cos \theta)^2 + \frac{1}{2} m(\ell \dot{\theta} \sin \theta)^2 + mg\ell \cos \theta$$

The angle θ is the only generalized coordinate, and after taking the derivatives for Lagrange's equations and suitable collection of terms, the equation of motion becomes (Problem 7-2)

$$\ddot{\theta} = -\frac{g}{\ell} \sin \theta - \frac{a}{\ell} \cos \theta$$

- A bead slides along a smooth wire bent in the shape of a parabola, $\mathbf{z = cr^2}$ (Fig.) The bead rotates in a circle, radius **R,** when the wire is rotating about its vertical symmetry axis with angular velocity $\boldsymbol{\omega}$. Find the constant **c**.



Solution!

Solution: Because the problem has cylindrical symmetry, we choose r, θ, and z as the generalized coordinates. The kinetic energy of the bead is

$$T = \frac{m}{2}[\dot{r}^2 + \dot{z}^2 + (r\dot{\theta})^2] \quad (7.43)$$

If we choose $U = 0$ at $z = 0$, the potential energy term is

$$U = mgz$$

But r, z, and θ are not independent. The equation of constraint for the paraboloid

$$z = cr^2 \quad (7.45)$$

$$\dot{z} = 2c\dot{r}r \quad (7.46)$$

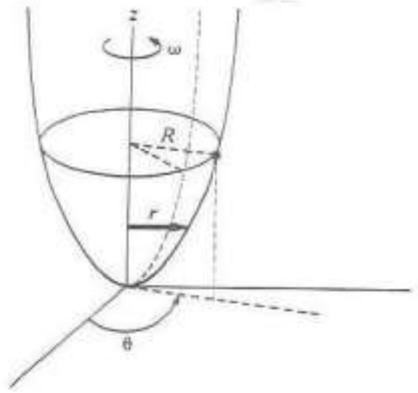


FIGURE 7-5

ve an explicit time dependence of the angular rotation

$$\theta = \omega t$$

$$\dot{\theta} = \omega \quad (7.47)$$

v construct the Lagrangian as being dependent only on r, because there θ dependence.

$$L = T - U$$

$$= \frac{m}{2}(\dot{r}^2 + 4c^2r^2\dot{r}^2 + r^2\omega^2) - mgcr^2 \quad (7.48)$$

v construct the Lagrangian as being dependent only on r, because there θ dependence.

$$L = T - U$$

$$= \frac{m}{2}(\dot{r}^2 + 4c^2r^2\dot{r}^2 + r^2\omega^2) - mgcr^2 \quad (7.48)$$

n stated that the bead moved in a circle of radius R. The reader might at this point to let $r = R =$ const. and $\dot{r} = 0$. It would be a mistake ow in the Lagrangian. First, we should find the equation of motion for r and then let $r = R$ as a condition of the particular motion. This the particular value of c needed for $r = R$.

$$\frac{\partial L}{\partial \dot{r}} = \frac{m}{2}(2\dot{r} + 8c^2r^2\dot{r})$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{m}{2}(2\ddot{r} + 16c^2r\dot{r}^2 + 8c^2r^2\ddot{r})$$

$$\frac{\partial L}{\partial r} = m(4c^2r\dot{r}^2 + r\omega^2 - 2gcr)$$

equation of motion becomes

$$\ddot{r}(1 + 4c^2r^2) + \dot{r}^2(4c^2r) + r(2gc - \omega^2) = 0$$

(7.49)

which is a complicated result. If, however, the bead rotates with then $\dot{r} = \ddot{r} = 0$, and Equation 7.49 becomes

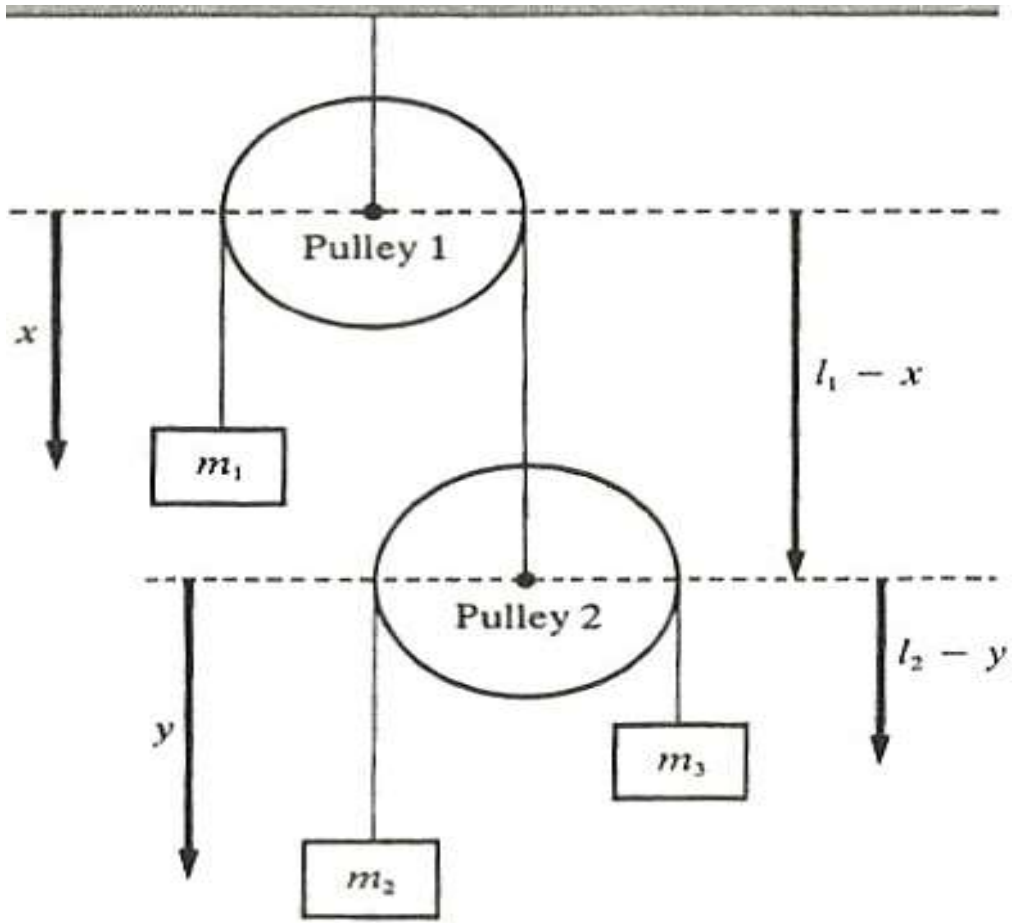
$$R(2gc - \omega^2) = 0$$

and

$$c = \frac{\omega^2}{2g}$$

is the result we wanted.

- Consider the double pulley system shown. Use the coordinates indicated & determine the eqtns of motion.

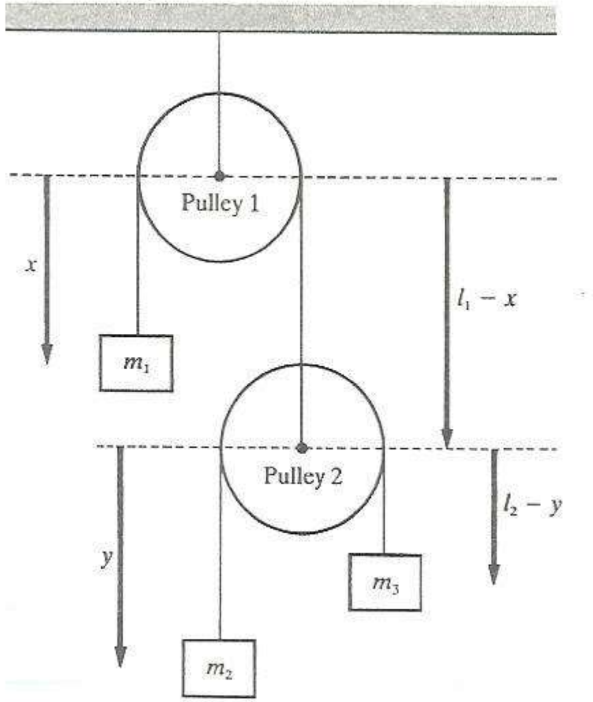


Solution!

Solution: Consider the pulleys to be massless, and let l_1 and l_2 be the lengths of rope hanging freely from each of the two pulleys. The distances x and y are measured from the center of the two pulleys.

m_1:

$$v_1 = \dot{x} \tag{7.51}$$



m_2:

$$v_2 = \frac{d}{dt}(l_1 - x + y) = -\dot{x} + \dot{y} \tag{7.52}$$

m_3:

$$v_3 = \frac{d}{dt}(l_1 - x + l_2 - y) = -\dot{x} - \dot{y} \tag{7.53}$$

$$T = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 + \tfrac{1}{2}m_3v_3^2$$

$$= \tfrac{1}{2}m_1\dot{x}^2 + \tfrac{1}{2}m_2(\dot{y} - \dot{x})^2 + \tfrac{1}{2}m_3(-\dot{x} - \dot{y})^2 \tag{7.54}$$

Let the potential energy $U = 0$ at $x = 0$.

$$U = U_1 + U_2 + U_3$$

$$= -m_1gx - m_2g(l_1 - x + y) - m_3g(l_1 - x + l_2 - y) \tag{7.55}$$

Because T and U have been determined, the equations of motion can be obtained using Equation 7.18. The results are

$$m_1\ddot{x} + m_2(\ddot{x} - \ddot{y}) + m_3(\ddot{x} + \ddot{y}) = (m_1 - m_2 - m_3)g \tag{7.56}$$

$$-m_2(\ddot{x} - \ddot{y}) + m_3(\ddot{x} + \ddot{y}) = (m_2 - m_3)g \tag{7.57}$$

Equations 7.56 and 7.57 can be solved for $\ddot{x}$ and $\ddot{y}$.

Variational Principles & Lagrange’s Eqtns

Hamilton’s Principle

- Our derivation of Lagrange’s Eqtns from D’Alembert’s Principle: Used Virtual Work - ***A Differential Principle.*** (A ***LOCAL*** principle).
- Here: An alternate derivation from **Hamilton’s Principle**: ***An Integral (or Variational) Principle*** (A ***GLOBAL*** principle). More general than D’Alembert’s Principle.
  - Based on techniques from the **Calculus of Variations**.
  - Brief discussion of derivation & of Calculus of Variations. More details: See the text!

- **System: n** generalized coordinates $q_1,q_2,q_3,..q_n$.
  - At time t_1: These all have some value.
  - At a later time t_2: They have changed according to the eqtns of motion & all have some other value.
- **System Configuration:** A point in **n**-dimensional space ("***Configuration Space***"), with q_i as **n** coordinate "axes".
  - At time t_1: Configuration of System is represented by a point in this space.
  - At a later time t_2: Configuration of System has changed & that point has moved (according to eqtns of motion) in this space.
  - Time dependence of System Configuration: The point representing this in Configuration Space traces out a path.

- **Monogenic Systems** ≡ All Generalized Forces (except constraint forces) are derivable from a **Generalized Scalar Potential** that ***may*** be a function of generalized coordinates, generalized velocities, & time:

$$\mathbf{U(q_i,\dot{q}_i,t)}: \;\; \mathbf{Q_i \equiv - (\partial U/\partial q_i) + (d/dt)[(\partial U/\partial \dot{q}_i)]}$$

  - If **U** depends only on $\mathbf{q_i}$ (& not on $\mathbf{\dot{q}_i}$ & **t**), **U** = **V** & the system is conservative.

- Monogenic systems, **Hamilton's Principle**:

***The motion of the system** (in configuration space)* ***from time t_1 to time t_2 is such that the line integral** (the action or action integral)*

$$\mathbf{I} = \int L\, \mathbf{dt} \qquad (\text{limits } \mathbf{t_1} < \mathbf{t} < \mathbf{t_2})$$

has a stationary value for the actual path of motion.

$$L \equiv \mathbf{T} - \mathbf{V} = \text{Lagrangian of the system}$$

$$L = \mathbf{T} - \mathbf{U}, \ (\text{if the potential depends on } \dot{\mathbf{q}}_{\mathbf{i}} \ \& \ \mathbf{t})$$

Hamilton's Principle (HP)

$$I = \int L\, dt \qquad (\text{limits } t_1 < t < t_2, L = T - V)$$

- **Stationary value** $\Rightarrow$ **I** is an extremum (maximum or minimum, ***almost always*** a minimum).
- In other words: Out of all possible paths by which the system point could travel in configuration space from t_1 to t_2, it will ***ACTUALLY*** travel along path for which **I** is an extremum (usually a minimum).

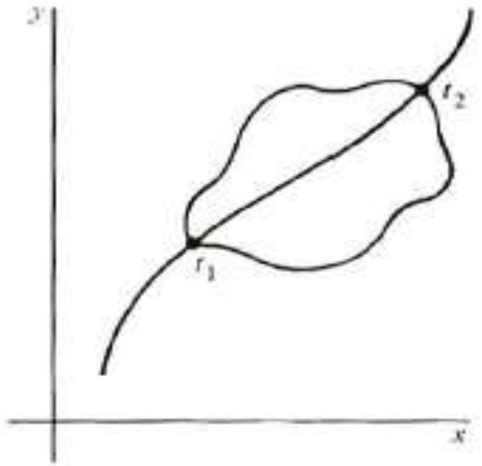


FIGURE 2.1 Path of the system point in configuration space.

$$\mathbf{I} = \int L\, \mathbf{dt} \qquad (\text{limits } \mathbf{t_1} < \mathbf{t} < \mathbf{t_2},\ \ L = \mathbf{T} - \mathbf{V}\ )$$

- In the terminology & notation from the **calculus of variations**:
  HP $\Rightarrow$ the motion is such that ***the variation of*** **I** (fixed $\mathbf{t_1}$ & $\mathbf{t_2}$) *is zero*:

$$\delta\int L\, \mathbf{dt} = \mathbf{0} \qquad (\text{limits } \mathbf{t_1} < \mathbf{t} < \mathbf{t_2}) \quad \mathbf{(1)}$$

$\boldsymbol{\delta} \equiv$ Arbitrary variation (calculus of variations).

$\boldsymbol{\delta}$ plays a role in the calculus of variations that the derivative plays in calculus.

- **Holonomic constraints** $\Rightarrow$ **(1)** is both a necessary & a sufficient condition for Lagrange's Eqtns.
  – That is, we can derive **(1)** from Lagrange's Eqtns.
  – However this text & (most texts) do it the other way around & derive Lagrange's Eqtns from **(1).**
  – Advantage: ***Valid in any system of generalized coords.!!***

- History, philosophy, & general discussion, which is worth briefly mentioning (not in Goldstein!).
- Historically, to overcome some practical difficulties of Newton's mechanics (e.g. needing all forces & not knowing the forces of constraint)

$\Rightarrow$ Alternate procedures were developed

Hamilton's Principle

$\Rightarrow$ ***Lagrangian Dynamics***

$\Rightarrow$ ***Hamiltonian Dynamics***

$\Rightarrow$ **Also Others!**

- **<u>All</u>** such procedures obtain results *100% equivalent* to *Newton's 2nd Law:* **F = dp/dt**

⇒ *Alternate procedures are NOT new theories!* But **reformulations of Newtonian Mechanics** in different math language.

- **Hamilton's Principle (HP):** Applicable outside particle mechanics! For example to fields in E&M.

- **HP:** Based on experiment!

• HP: Philosophical Discussion

HP: ⇒ No new physical theories, just new formulations of old theories

HP: Can be used to ***unify*** several theories: Mechanics, E&M, Optics, …

HP: ***Very elegant & far reaching!***

HP: "More fundamental" than Newton's Laws!

HP: Given as a (single, simple) postulate.

HP & Lagrange Eqtns apply (as we've seen) to non-conservative systems.

- **HP:** One of many **"Minimal" Principles:**

  (Or variational principles)

  – Assume Nature always minimizes certain quantities when a physical process takes place

  – Common in the history of physics

- **History:** List of (some) other minimal principles:

  – **Hero, 200 BC:** Optics: ***Hero's Principle of Least Distance:*** A light ray traveling from one point to another by reflection from a plane mirror, always takes shortest path. By geometric construction:

$\Rightarrow$ **Law of Reflection.** $\boldsymbol{\theta_i = \theta_r}$

Says nothing about the Law of Refraction!

- **"Minimal" Principles:**
  - **Fermat, 1657:** Optics: ***Fermat's Principle of Least Time:*** A light ray travels in a medium from one point to another by a path that takes the least time.

    $\Rightarrow$ **Law of Reflection:** $\boldsymbol{\theta_i = \theta_r}$

    $\Rightarrow$ **Law of Refraction:** "Snell's Law"
  - **Maupertuis, 1747:** Mechanics: ***Maupertuis's Principle of Least Action:*** Dynamical motion takes place with minimum action:
    - **Action** $\equiv$ (Distance) $\times$ (Momentum) = (Energy) $\times$ (Time)
    - Based on ***Theological*** Grounds!!! (???)
    - Lagrange: Put on firm math foundation.
    - Principle of Least Action $\Rightarrow$ ***HP***

Hamilton's Principle

(As originally stated 1834-35)

– Of all possible paths along which a dynamical system may move from one point to another, in a given time interval (consistent with the constraints), the ***actual path*** followed is one which minimizes the time integral of the difference in the KE & the PE. That is, the one which makes the variation of the following integral vanish:

$$\boldsymbol{\delta\int[T - V]dt = \delta\int L dt = 0} \quad (\text{limits } \mathbf{t_1} < \mathbf{t} < \mathbf{t_2})$$

- Consider the following problem in the **xy** plane:

  The Basic Calculus of Variations Problem:

  Determine the function **y(x)** for which the integral

  $$\mathbf{J \equiv \int f[y(x),y'(x);x]dx} \quad (\text{fixed limits } \mathbf{x_1 < x < x_2})$$

  is an ***extremum*** (max or min)

  $\mathbf{y'(x) \equiv dy/dx}$ (Note: The text calls this $\dot{\mathbf{y}}\mathbf{(x)}$!)

  - Semicolon in **f** separates independent variable **x** from dependent variable **y(x)** & its derivative **y′(x)**
  - **f ≡ A GIVEN *functional*. Functional** ≡ Quantity **f[y(x),y′(x);x]** which depends on the *functional form* of the dependent variable y**(x).** "A function of a function".

- ***Basic problem restated:*** Given $\mathbf{f[y(x),y'(x);x]}$, find (for fixed $\mathbf{x_1}$, $\mathbf{x_2}$) the function(s) $\mathbf{y(x)}$ which minimize (or maximize) $\mathbf{J \equiv \int f[y(x),y'(x);x]dx}$ (limits $x_1 < x < x_2$)

$\Rightarrow$ Vary $\mathbf{y(x)}$ until an extremum (max or min; ***usually min!***) of $\mathbf{J}$ is found. Stated another way, vary $\mathbf{y(x)}$ so that the variation of $\mathbf{J}$ is zero or

$$\delta J = \delta\int f[y(x),y'(x);x]dx = 0$$

Suppose the function $\mathbf{y = y(x)}$ gives $\mathbf{J}$ a min value:

$\Rightarrow$ Every "***neighboring function***", no matter how close to $\mathbf{y(x)}$, must make $\mathbf{J}$ increase!

› ***Solution to basic problem :*** The text **proves** that to minimize (or maximize)

$$J \equiv \int f[y(x),y'(x);x]dx \qquad \text{(limits } x_1 < x < x_2\text{)}$$

or $$\delta J = \delta\int f[y(x),y'(x);x]dx = 0$$

$\Rightarrow$ The functional **f** must satisfy:

$$(\partial f/\partial y) - (d[\partial f/\partial y']/dx) = 0$$

$\equiv$ ***Euler's Equation***

- Euler, 1744. Applied to mechanics

  $\equiv$ ***Euler - Lagrange Equation***
- Various pure math applications,
- Read on your own!

- 1st, extension of calculus of variations results to **Functions with Several Dependent Variables**
- Derived **Euler Eqtn** = Solution to problem of finding path such that $\mathbf{J = \int f\, dx}$ is an extremum or $\mathbf{\delta J = 0}$. Assumed one dependent variable $\mathbf{y(x)}$.
- In mechanics, we often have problems with many dependent variables: $\mathbf{y_1(x), y_2(x), y_3(x), \ldots}$
- In general, have a functional like:

$$\mathbf{f = f[y_1(x), y_1'(x), y_2(x), y_2'(x), \ldots; x]}$$

$$\mathbf{y_i'(x) \equiv dy_i(x)/dx}$$

- *Abbreviate* as $\mathbf{f = f[y_i(x), y_i'(x); x]}$, $\mathbf{i = 1,2, \ldots, n}$

- Functional: $\mathbf{f = f[y_i(x),y_i'(x);x],\ \ i = 1,2, \ldots,n}$
- **Calculus of variations problem:** Simultaneously find the "**n** paths" $\mathbf{y_i(x),\ i = 1,2, \ldots,n}$, which minimize (or maximize) the integral:

$$\mathbf{J \equiv \int f[y_i(x),y_i'(x);x]dx}$$

$(\mathbf{i = 1,2, \ldots,n}$, fixed limits $\mathbf{x_1 < x < x_2})$

Or for which $\mathbf{\delta J = 0}$

- Follow the derivation for one independent variable & get:

$$\mathbf{(\partial f/\partial y_i) - (d[\partial f/\partial y_i']dx) = 0} \qquad \mathbf{(i = 1,2, \ldots,n)}$$

$\equiv$ ***Euler's Equations***

(Several dependent variables)

- **Summary:** Forcing $\mathbf{J} \equiv \int \mathbf{f}[\mathbf{y_i}(\mathbf{x}),\mathbf{y_i}'(\mathbf{x});\mathbf{x}]\mathbf{dx}$

  $(\mathbf{i} = 1,2, \ldots,\mathbf{n}$, fixed limits $\mathbf{x_1} < \mathbf{x} < \mathbf{x_2})$

  To have an extremum (or forcing

  $\delta\mathbf{J} = \delta\int \mathbf{f}[\mathbf{y_i}(\mathbf{x}),\mathbf{y_i}'(\mathbf{x});\mathbf{x}]\mathbf{dx} = 0$) requires **f** to satisfy:

  $$(\partial f/\partial y_i) - (d[\partial f/\partial y_i']dx) = 0 \qquad (i = 1,2, \ldots,n)$$

  $\equiv$ ***Euler's Equations***

- **HP** $\Rightarrow$ The system motion is such that $\mathbf{I} = \int L\ \mathbf{dt}$ is an extremum (fixed $\mathbf{t_1}$ & $\mathbf{t_2}$)

  $\Rightarrow$ The variation of this integral **I** is zero:

  $$\delta\int L\ dt = 0 \qquad (\text{limits } t_1 < t < t_2)$$

- HP $\Rightarrow$ Identical to abstract calculus of variations problem of with replacements:

$$J \rightarrow \int L\, dt; \quad \delta J \rightarrow \delta \int L\, dt$$

$$x \rightarrow t\ ;\ y_i(x) \rightarrow q_i(t)$$

$$y_i'(x) \rightarrow dq_i(t)/dt = q_i(t)$$

$$f[y_i(x), y_i'(x); x] \rightarrow L(q_i, q_i; t)$$

$\Rightarrow$ **The Lagrangian *L* satisfies Euler's eqtns with these replacements!**

$\Rightarrow$ Combining HP with Euler's eqtns gives:

$$(d/dt)[(\partial L/\partial \dot{q}_j)] - (\partial L/\partial q_j) = 0 \quad (j = 1,2,3, \ldots n)$$

- **Summary:** **HP gives *Lagrange's Eqtns:***

$$(d/dt)[(\partial L/\partial \dot{q}_j)] - (\partial L/\partial q_j) = 0$$

$$(j = 1,2,3, \dots n)$$

  – Stated another way, ***Lagrange's Eqtns ARE Euler's eqtns*** in the special case where the abstract functional **f** is the Lagrangian **L!**

$\Rightarrow$ They are sometimes called the

Euler-Lagrange Eqtns.

Advantages of a Variational Principle Formulation

- **HP** $\Rightarrow \delta\int L\,dt = 0$ (limits $t_1 < t < t_2$). An example of a ***variational principle*.**
- Most useful when a coordinate system-independent Lagrangian **L = T - V** can be set up.
- **HP**: "Elegant". **Contains all of mechanics of holonomic systems in which forces are derivable from potentials.**
- **HP**: Involves only physical quantities (**T, V**) which can be generally defined without reference to a specific set of generalized coords.

$\Rightarrow$ ***A formulation of mechanics which is independent of the choice of coordinate system!***

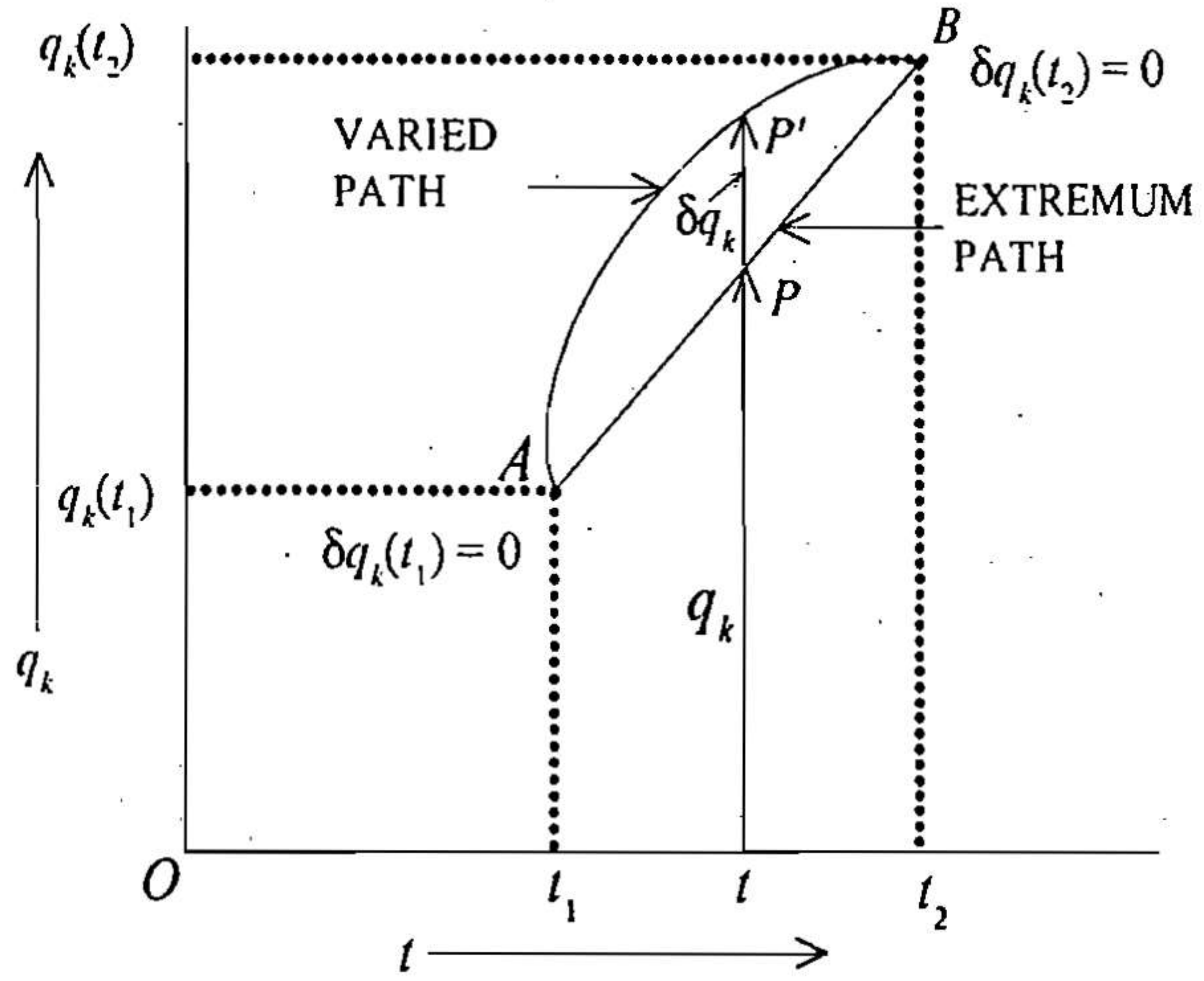


Fig. 2.9 : δ-variation - extremum path

Lagrange's equation from Hamilton's principle : The Lagrangian L is a function of generalized coordinates q_k's and generalized velocities $\dot{q}_k$'s and time t, *i.e.*,

$$L = L\,(q_1, q_2, \ldots, q_k, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_k, \ldots, \dot{q}_n, t\,)$$

If the Lagrangian does not depend on time t explicitly, then the variation δL can be written as

$$\delta L = \sum_{k=1}^{n} \frac{\partial L}{\partial q_k} \delta q_k + \sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k$$

Integrating both sides from $t = t_1$ to $t = t_2$, we get

$$\int_{t_1}^{t_2} \delta L\, dt = \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k\, dt + \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k\, dt$$

But in view of the Hamilton's principle

$$\delta \int_{t_1}^{t_2} L\, dt = 0$$

Therefore, $$\int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k\, dt + \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k\, dt = 0$$

where $\delta \dot{q}_k = \dfrac{d}{dt}(\delta q_k)$.

$$\int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \, dt = \sum_k \left[\frac{\partial L}{\partial \dot{q}_k} \delta q_k \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum_k \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k \, dt$$

At the end points of the path at the times t_1 and t_2, the coordinates must have definite values $q_k(t_1)$ and $q_k(t_2)$ respectively, *i.e.*, $\delta q_k(t_1) = \delta q_k(t_2) = 0$ (Fig. 2.9) and hence

$$\sum_k \left[\frac{\partial L}{\partial \dot{q}_k} \delta q_k \right]_{t_1}^{t_2} = 0$$

Therefore, eq. (72) takes the form

$$\int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k \, dt - \int_{t_1}^{t_2} \sum_k \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k \, dt = 0$$

$$\sum_k \int_{t_1}^{t_2} \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} \right] \delta q_k \, dt = 0$$

$$\sum_k \int_{t_1}^{t_2} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} \right] \delta q_k \, dt = 0$$

For holonomic system, the generalized coordinates δq_k are independent of each other. Therefore, the coefficient of each δq_k must vanish, *i.e.*,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0$$

where k =1, 2,...., n are the generalized coordinates.

1. *Shortest distance between two points in a plane.* An element of length in a plane is

$$ds = \sqrt{dx^2 + dy^2}$$

and the total length of any curve going between points 1 and 2 is

$$I = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

The condition that the curve be the shortest path is that I be a minimum. This is an example of the extremum problem as expressed by Eq. (2.3), with

$$f = \sqrt{1 + \dot{y}^2}.$$

$$\frac{\partial f}{\partial y} = 0, \qquad \frac{\partial f}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{1+\dot{y}^2}},$$

we have

$$\frac{d}{dx}\left(\frac{\dot{y}}{\sqrt{1+\dot{y}^2}}\right) = 0$$

or

$$\frac{\dot{y}}{\sqrt{1+\dot{y}^2}} = c,$$

where c is constant. This solution can be valid only if

$$\dot{y} = a,$$

where a is a constant related to c by

$$a = \frac{c}{\sqrt{1-c^2}}.$$

But this is clearly the equation of a straight line,

$$y = ax + b,$$

Lagrange Applied to Circuit Theory

- **System: LR Circuit** (Fig.) Battery, voltage V, in series with inductor **L** & resistor **R** (which will give dissipation). Dynamical variable = charge **q**.

PE = **V** = **q**V

KE = **T** = $(\frac{1}{2})L(\dot{q})^2$

Lagrangian:

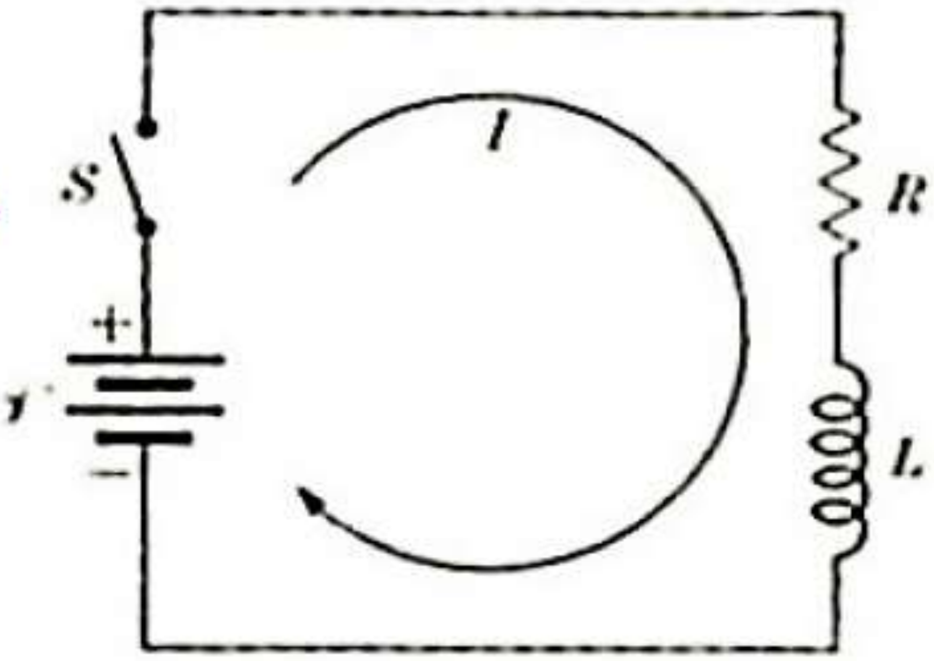
$$L = T - V$$

Dissipation Function:

(last chapter!)

$$\mathcal{F} = (\tfrac{1}{2})R(\dot{q})^2 = (\tfrac{1}{2})R(I)^2$$

switch →



Lagrange's Eqtn (with dissipation):

$$(d/dt)[(\partial L/\partial \dot{q})] - (\partial L/\partial q) + (\partial \mathcal{F}/\partial \dot{q}) = 0$$

Lagrange Applied to RL circuit

- **Lagrange's Eqtn (with dissipation):**

$$(d/dt)[(\partial L/\partial \dot{q})] - (\partial L/\partial q) + (\partial F/\partial \dot{q}) = 0$$

$$\Rightarrow V = L\ddot{q} + R\dot{q}$$

$$I = \dot{q} = (dq/dt)$$

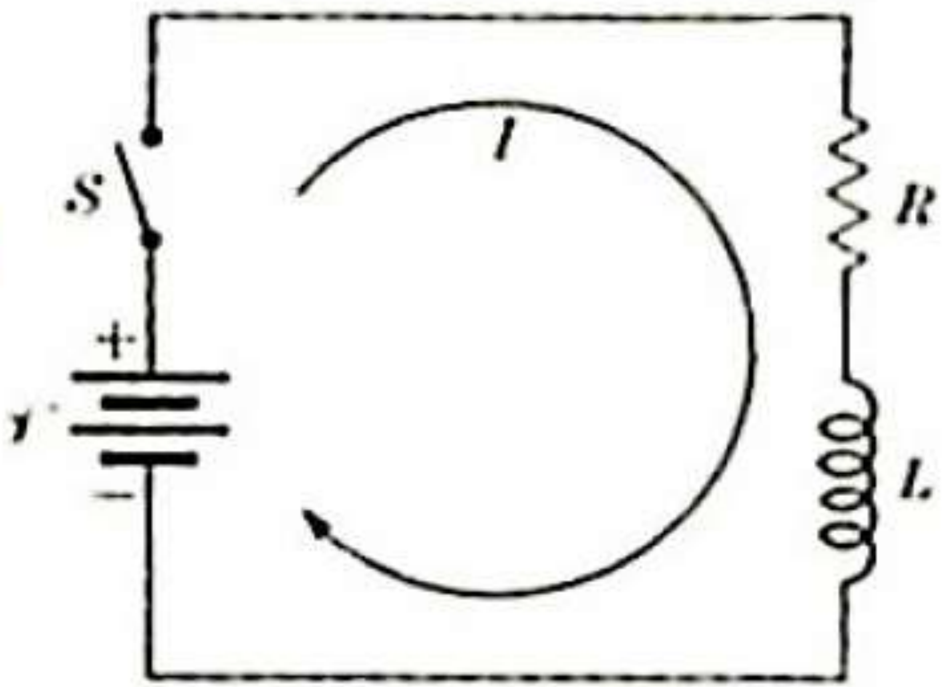
$$\Rightarrow V = L\dot{I} + RI$$

Solution, for switch closed at $\mathbf{t = 0}$ is:

$$I = (V/R)[1 - e^{(-Rt/L)}]$$

Steady state ($t \to \infty$):

$$I = I_0 = (V/R)$$



Mechanical Analogue to RL circuit

- **Mechanical analogue:**

  Sphere, radius **a**, (effective) mass **m´**, falling in a const density viscous fluid, viscosity η under gravity.

  $\mathbf{m'} \equiv \mathbf{m} - \mathbf{m_f}$, $\mathbf{m} \equiv$ actual mass, $\mathbf{m_f} \equiv$ mass of displaced fluid (buoyant force acting upward: Archimedes' principle)

- $V = m'gy$, $T = (\frac{1}{2})m'v^2$, $L = T - V$ $\quad (v = \dot{y})$

  Dissipation Function: $\quad \mathcal{F} = 3\pi\eta a v^2$

  Comes from Stokes' Law of frictional drag force:

  $\mathbf{F_f} = 6\pi\eta a v$ and (Ch. 1 result that) $\mathbf{F_f} = -\nabla_v \mathcal{F}$

  Lagrange's Eqtn (with dissipation):

  $$(d/dt)[(\partial L/\partial \dot{y})] - (\partial L/\partial y) + (\partial \mathcal{F}/\partial \dot{y}) = 0$$

- $V = m'gy$, $T = (\frac{1}{2})m'v^2$, $L = T - V$ $\quad (v = \dot{y})$

Dissipation Function: $\mathcal{F} = 3\pi\eta a v^2$

Comes from Stokes' Law frictional drag force:

$F_f = 6\pi\eta a v$ and (Ch. 1 result that) $F_f = -\nabla_v \mathcal{F}$

Lagrange's Eqtn (with dissipation):

$$(d/dt)[(\partial L/\partial y)] - (\partial L/\partial y) + (\partial \mathcal{F}/\partial y) = 0$$

$$\Rightarrow \qquad m'g = m'y + 6\pi\eta a\dot{y}$$

Solution, for $v = \dot{y}$ starting from rest at $t = 0$:

$v = v_0\,[1 - e^{(-t/\tau)}]$. $\tau \equiv m'\,(6\pi\eta a)^{-1} \equiv$ Time it takes sphere to reach e^{-1} of its terminal speed v_0. Steady state $(t \rightarrow \infty)$: $v = v_0 = (m'g)(6\pi\eta a)^{-1} = g\tau =$ terminal speed.

Lagrange Applied to Circuit Theory

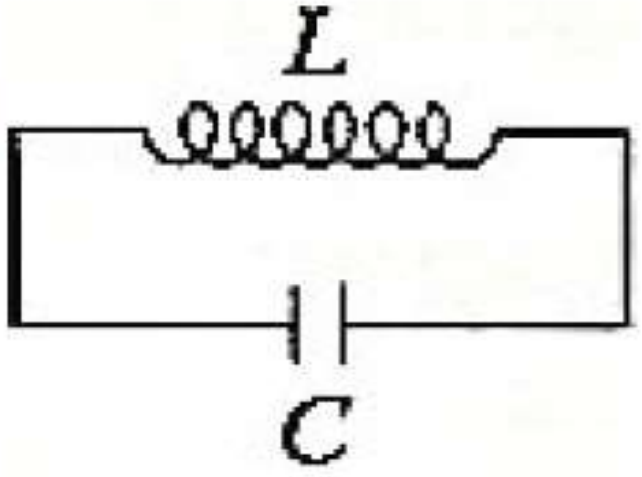
- **System: LC Circuit** (Fig.) Inductor **L** & capacitor **C** in series. Dynamical variable = charge **q**.

  Capacitor acts a PE source:

  PE = $(\frac{1}{2})q^2C^{-1}$, KE = $\mathbf{T} = (\frac{1}{2})L(\dot{q})^2$

  Lagrangian: $\mathcal{L} = \mathbf{T} - \mathbf{V}$

  (No dissipation!)



  Lagrange's Eqtn:

  $$(d/dt)[(\partial\mathcal{L}/\partial\dot{q})] - (\partial\mathcal{L}/\partial q) = 0 \Rightarrow L\ddot{q} + qC^{-1} = 0$$

  Solution (for $\mathbf{q = q_0}$ at $\mathbf{t = 0}$):

  $$q = q_0 \cos(\omega_0 t),\ \omega_0 = (LC)^{-(\frac{1}{2})}$$

  $\omega_0 \equiv$ natural or resonant frequency of circuit

Mechanical Analogue to LC Circuit

- **Mechanical analogue:**

  Simple harmonic oscillator (no damping) mass **m**, spring constant **k**.

- $V = (½)kx^2$, $T = (½)mv^2$, $L = T - V$ $\quad (v = \dot{x})$

  Lagrange's Eqtn:

  $$(d/dt)[(\partial L/\partial \dot{x})] - (\partial L/\partial x) = 0$$

  $$\Rightarrow m\ddot{x} + kx = 0$$

  Solution (for $\mathbf{x} = \mathbf{x_0}$ at $\mathbf{t} = \mathbf{0}$):

  $$x = x_0 \cos(\omega_0 t),\ \omega_0 = (k/m)^{½}$$

  $\omega_0 \equiv$ natural or resonant frequency of circuit

- Circuit theory examples give analogies:

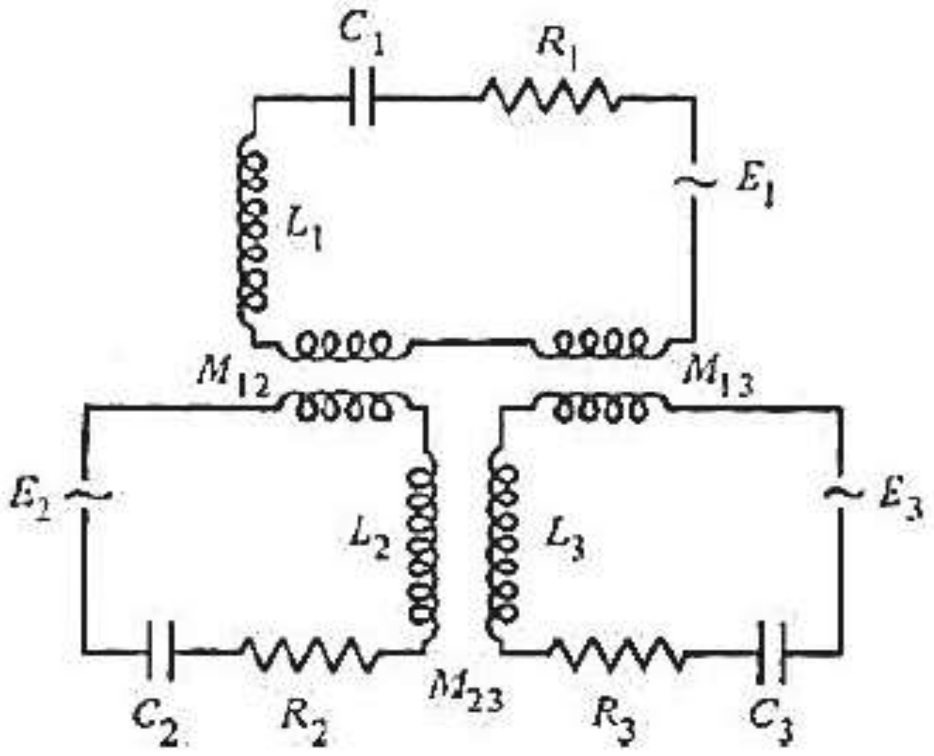
$\Rightarrow$ **Inductance L** plays an analogous role in electrical circuits that **mass m** plays in mechanical systems (***an inertial term***).

$\Rightarrow$ **Resistance R** plays an analogous role in electrical circuits that **viscosity η** plays in mechanical systems (***a frictional or drag term***).

$\Rightarrow$ **Capacitance C** (actually C^{-1}) plays an analogous role in electrical circuits that a Hooke's "Law" type **spring constant k** plays in mechanical systems (***a "stiffness" or tensile strength term***).

- With these analogies, consider the system of coupled electrical circuits (fig):

  $\mathbf{M}_{jk}$ = mutual inductances!



- Immediately, can write

  Lagrangian:

$$\boldsymbol{L} = (½)\textstyle\sum_j \mathbf{L}_j(\dot{\mathbf{q}}_j)^2 + (½)\sum_{j,k(\neq j)} \mathbf{M}_{jk}\dot{\mathbf{q}}_j\dot{\mathbf{q}}_k - (½)\sum_j (1/\mathbf{C}_j)(\mathbf{q}_j)^2 + \sum_j \mathbf{E}_j(t)\mathbf{q}_j$$

  Dissipation function: $\boldsymbol{F} = (½)\sum_j \mathbf{R}_j(\dot{\mathbf{q}}_j)^2$

- **Lagrangian:**

$$L = (½)\sum_j L_j(\dot{q}_j)^2 + (½)\sum_{j,k(\neq j)} M_{jk}\dot{q}_j\dot{q}_k - (½)\sum_j (1/C_j)(q_j)^2 + \sum_j E_j(t)q_j$$

Dissipation function: $\mathcal{F} = (½)\sum_j R_j(\dot{q}_j)^2$

Lagrange's Eqtns:

$$(d/dt)[(\partial L/\partial \dot{q}_j)] - (\partial L/\partial q_j) + (\partial \mathcal{F}/\partial \dot{q}_j) = 0$$

$\Rightarrow$ **Eqtns of motion** (the same as coupled, driven, damped harmonic oscillators!)

$$L_j(d^2q_j/dt^2) + \sum_{k(\neq j)} M_{jk}(d^2q_k/dt^2) + R_j(dq_j/dt) + (1/C_j)q_j = E_j(t)$$

- **1^{st} Integrals of Motion** ≡ Relations between generalized coords, generalized velocities, & time which are **1^{st}** order diff eqtns. Of the form:

$$\mathbf{f(q_1,q_2,\ldots,\dot{q}_1,\dot{q}_2,\ldots, t) = constant}$$

- **1^{st} Integrals of Motion:** Very interesting because they tell us a lot about ***the system physics***. They come from **Conservation Theorems.**

- **Consider:** Point masses & conservative forces: Eqtns of motion in Cartesian coords:

$$L = \mathbf{T} - \mathbf{V} = (\tfrac{1}{2})\sum_i m_i[(\dot{x}_i)^2+(\dot{y}_i)^2+(\dot{z}_i)^2] - \mathbf{V}(\mathbf{r}_1,\mathbf{r}_2, \ldots \mathbf{r}_N)$$

$$(d/dt)[(\partial L/\partial \dot{x}_i)] - (\partial L/\partial x_i) = 0$$

- Look at: $(\partial L/\partial \dot{x}_i) = (\partial T/\partial \dot{x}_i) - (\partial V/\partial \dot{x}_i) = (\partial T/\partial \dot{x}_i) =$ $(\partial/\partial \dot{x}_i)[\,(\tfrac{1}{2})\sum_j m_j[(\dot{x}_j)^2+(\dot{y}_j)^2+(\dot{z}_j)^2] = m_i\dot{x}_i = p_{ix}$

(**x** component of momentum of **i**th particle)

⇒ ***DEFINE:*** **Generalized Momentum**

associated with Generalized Coord $\mathbf{q}_j$:

$$\mathbf{p}_j \equiv (\partial L/\partial \dot{\mathbf{q}}_j)$$

Generalized Momentum

$\Rightarrow$ **Generalized Momentum** associated with (or *Momentum Conjugate* to) Generalized Coord $\mathbf{q_j}$:

$$\mathbf{p_j} \equiv (\partial L/\partial \dot{\mathbf{q}}_\mathbf{j})$$

Points worth noting:

- If $\mathbf{q_j}$ is not a Cartesian Coordinate, $\mathbf{p_j}$ is ***NOT* necessarily a linear momentum.**
- For a **velocity dependent potential** $\mathbf{U(q_j,\dot{q}_j,t)}$, then, even if $\mathbf{q_j}$ is a Cartesian Coordinate, the Generalized Momentum $\mathbf{p_j}$ is ***NOT* the usual *Mechanical* Momentum** ($\mathbf{p_j} \neq \mathbf{m_j\dot{q}_j}$)

Ignorable (Cyclic) Coordinates

- **Important special case!**

  Cyclic or Ignorable Coordinates $\equiv$ Generalized Coordinates $\mathbf{q_j}$ not appearing in Lagrangian L (but the generalized velocity ***MAY*** still appear in L).

- Lagrange's Eqtn for a cyclic coordinate $\mathbf{q_j}$:

$$(d/dt)[(\partial L/\partial \dot{q}_j)] - (\partial L/\partial q_j) = 0$$

By definition of cyclic: $(\partial L/\partial q_j) = 0$

$\Rightarrow$ Lagrange Eqtn: $\quad (d/dt)[(\partial L/\partial \dot{q}_j)] = 0$

Momentum Conjugate $\mathbf{p_j} \equiv (\partial L/\partial \dot{q}_j)$

$\Rightarrow$ Lagrange Eqtn for a cyclic coordinate:

$$(dp_j/dt) = 0$$

$\Rightarrow$ If a Generalized Coord $\mathbf{q_j}$ is cyclic or ignorable, the Lagrange Eqtn is $\mathbf{(dp_j/dt) = 0}$

where **Generalized Momentum** $\mathbf{p_j} \equiv (\partial L/\partial \dot{q}_j)$

- $\mathbf{(dp_j/dt) = 0} \Rightarrow \mathbf{p_j}$ = ***constant*** (conserved)

$\Rightarrow$ A General

***Conservation Theorem**: If the Generalized Coord q_j is cyclic or ignorable, the corresponding Generalized (or Conjugate) Momentum*, $\mathbf{p_j} \equiv (\partial L/\partial \dot{q}_j)$ *is conserved.*

Energy Function & Energy Conservation

- One more conservation theorem which we would expect to get from the Lagrange formalism is: ***CONSERVATION OF ENERGY.***
- Consider a general Lagrangian L, a function of the coords $\mathbf{q_j}$, velocities $\dot{\mathbf{q}}_j$, & time $\mathbf{t}$:

$$L = L(q_j,\dot{q}_j,t) \qquad\qquad (j = 1,\ldots n)$$

- The total time derivative of L (chain rule):

$$(dL/dt) = \sum_j(\partial L/\partial q_j)(dq_j/dt) + \sum_j(\partial L/\partial \dot{q}_j)(d\dot{q}_j/dt) + (\partial L/\partial t)$$

Or:

$$(dL/dt) = \sum_j(\partial L/\partial q_j)\dot{q}_j + \sum_j(\partial L/\partial \dot{q}_j)\ddot{q}_j + (\partial L/\partial t)$$

- **Total time derivative** of L:

$$(dL/dt) = \sum_j(\partial L/\partial q_j)\dot{q}_j + \sum_j(\partial L/\partial \dot{q}_j)\ddot{q}_j + (\partial L/\partial t) \qquad (1)$$

- **Lagrange's Eqtns:** $(d/dt)[(\partial L/\partial \dot{q}_j)] - (\partial L/\partial q_j) = 0$

Put into **(1)**

$$(dL/dt) = \sum_j(d/dt)[(\partial L/\partial \dot{q}_j)]\dot{q}_j + \sum_j(\partial L/\partial \dot{q}_j)\ddot{q}_j + (\partial L/\partial t)$$

Identity: 1st 2 terms combine

$$(dL/dt) = \sum_j(d/dt)[\dot{q}_j(\partial L/\partial \dot{q}_j)] + (\partial L/\partial t)$$

Or: $$(d/dt)[\sum_j\dot{q}_j(\partial L/\partial \dot{q}_j) - L] + (\partial L/\partial t) = 0 \qquad (2)$$

$$(d/dt)[\sum_j \dot{q}_j(\partial L/\partial \dot{q}_j) - L] + (\partial L/\partial t) = 0 \quad (2)$$

- Define the ***Energy Function*** **h**:

$$h \equiv \sum_j \dot{q}_j(\partial L/\partial \dot{q}_j) - L = h(q_1,..q_n;\dot{q}_1,..\dot{q}_n,t)$$

- **(2)** $\Rightarrow$ $(dh/dt) = - (\partial L/\partial t)$

$\Rightarrow$ For a Lagrangian L which is **not an explicit function of time** (so that $(\partial L/\partial t) = 0$)

$$(dh/dt) = 0 \;\&\; h = \text{constant (conserved)}$$

- **Energy Function** $h = h(q_1,..q_n;\dot{q}_1,..\dot{q}_n,t)$
  - Identical ***Physically*** to what we later will call **the Hamiltonian H**. ***However,*** here, **h** is a function of **n** indep coords q_j & velocities $\dot{q}_j$. The Hamiltonian **H** is ***ALWAYS*** considered a function of **2n** indep coords q_j & momenta p_j

- ***Energy Function*** $\mathbf{h} \equiv \sum_j \dot{q}_j(\partial L/\partial \dot{q}_j) - L$

- We had $(dh/dt) = -(\partial L/\partial t)$

$\Rightarrow$ For a Lagrangian for which $(\partial L/\partial t) = 0$

$(dh/dt) = 0$ & $h =$ **constant** (conserved)

- For this to be useful, we need a

Physical Interpretation of **h.**

  – Will now show that, under ***certain circumstances,*** **h = total mechanical energy of the system.**

$$\sum_k \frac{\partial T}{\partial \dot{q}_k} \dot{q}_k = 2T$$

$$H = 2T - L = 2T - (T - V)$$

$$H = T + V = E, \text{ constant}$$

Thus *the Hamiltonian H represents the total energy of the system E and is conserved, provided the system is conservative and T is a homogeneous quadratic function.*

* For a system of N particles, when $\mathbf{r}_i$ does not depend on time explicitly,

then $$\mathbf{v}_i = \sum_{k=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k$$

Therefore, $$T = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2 = \sum_{i=1}^{N} \frac{1}{2} m_i \left(\sum_k \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k \right)^2 = \sum_{i=1}^{N} \frac{1}{2} m_i \left(\sum_k \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k \right) \cdot \left(\sum_l \frac{\partial \mathbf{r}_i}{\partial q_l} \dot{q}_l \right)$$

$$= \sum_{i=1}^{N} \frac{1}{2} m_i \sum_k \sum_l \left[\frac{\partial \mathbf{r}_i}{\partial q_k} \cdot \frac{\partial \mathbf{r}_i}{\partial q_l} \right] \dot{q}_k \dot{q}_l$$

$$\therefore \quad \frac{\partial T}{\partial \dot{q}_k} = \sum_{i=1}^{N} \sum_l m_i \left[\frac{\partial \mathbf{r}_i}{\partial q_k} \cdot \frac{\partial \mathbf{r}_i}{\partial q_l} \right] \dot{q}_l$$

Multiplying by $\dot{q}_k$ and summing over k, we get

$$\sum_k \dot{q}_k \frac{\partial T}{\partial \dot{q}_k} = \sum_{i=1}^{N} \sum_k \sum_l m_i \left[\frac{\partial \mathbf{r}_i}{\partial q_k} \cdot \frac{\partial \mathbf{r}_i}{\partial q_l} \right] \dot{q}_k \dot{q}_l = 2T$$

where each k and l run from 1 to n.

Hamiltonian's Equations

The Hamiltonian, in general, is a function of generalized coordinates q_k, generalized momenta p_k and time t, *i.e.*,

$$H = H(q_1, q_2, \dots, q_k, \dots, q_n, p_1, p_2, \dots, p_k, \dots, p_n, t)$$

We may write the differential dH as

$$dH = \sum_k \frac{\partial H}{\partial q_k} dq_k + \sum_k \frac{\partial H}{\partial p_k} dp_k + \frac{\partial H}{\partial t} dt$$

But as defined in eq. (27), $H = \sum_k p_k \dot{q}_k - L$ and hence

$$dH = \sum_k \dot{q}_k \, dp_k + \sum_k p_k \, d\dot{q}_k - dL$$

Also, $L = L(q_1, q_2, \dots, q_k, \dots, q_n, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_k, \dots, \dot{q}_n, t)$

Therefore, $$dL = \sum_k \frac{\partial L}{\partial q_k} dq_k + \sum_k \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k + \frac{\partial L}{\partial t} dt$$

But $\dot{p}_k = \frac{\partial L}{\partial q_k}$ [eq. (5)] and $p_k = \frac{\partial L}{\partial \dot{q}_k}$ [eq. (3)].

Therefore,
$$dL = \sum_k \dot{p}_k \, dq_k + \sum_k p_k \, d\dot{q}_k + \frac{\partial L}{\partial t} dt$$

Substituting for dL from eq. (35) in eq. (34), we get

$$dH = \sum_k \dot{q}_k \, dp_k - \sum_k \dot{p}_k \, dq_k - \frac{\partial L}{\partial t} dt$$

Comparing the coefficients of dp_k, dq_k and dt in eqs. (33) and (36), we obtain

$$\dot{q}_k = \frac{\partial H}{\partial p_k}$$

$$-\dot{p}_k = \frac{\partial H}{\partial q_k}$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}$$

Thank You