Classical Mechanics
(MP-301)

Nonlinear Oscillations & Chaos

Nonlinear Oscillations & Chaos

Confine general discussion to oscillations & oscillators. Most oscillators seen in undergrad mechanics are **Linear Oscillators:**

$\Rightarrow$ Obey Hooke's "Law", linear restoring force: **F(x)= -kx**

$\Rightarrow$ Potential energy **$V(x)= (½)kx^2$**

Real world: Most oscillators are, in fact, ***nonlinear!***

Techniques for solving linear problems might or might not be useful for nonlinear ones!

Often, rather than having a general technique for nonlinear problem solution, technique may be problem dependent.

Often need numerical techniques to solve diff. eqtns.

- Many techniques exist for solving (or at least approximately solving) nonlinear differential equations.
- Often, nonlinear systems reveal a ***rich & beautiful physics*** that is simply not there in linear systems. (e.g. Chaos).
- The related areas of nonlinear mechanics and Chaos are very modern & are (in some places) hot topics of ***current research!***

Non-Linearities - Examples

- **Chemistry And Physics**
  - **Chemical Reactions (Kinetics)**
  - **Thin Film Deposition (Laser Deposition)**
  - **Turbulent Fluid Flows**
  - **Crystalline Growth**
- **Medical & Biological**
  - Heart Disorders (Arrhythmia, Fibrillation)
  - Brain And Nervous System
  - Population Dynamics
  - Epidemiology Forecasting (Flu, Diseases)
  - Food Supply (Vs. Weather & Population)
- **Economy**
  - Stock Market
  - World Economy

what is nonlinear? mathematical definition

Linear Terms: one that is **first degree** in its **dependent** variables and derivatives

x is 1st degree and therefore a linear term

xt is 1st degree in x and therefore a linear term

x^2 is 2nd degree in x and there not a linear term

Nonlinear Terms: any term that contains higher powers, products and transcendentals of the dependent variable is nonlinear

x^2, e^x, $x(x+1)^{-1}$ all nonlinear terms

$\sin x$ nonlinear term

other examples where x & y are the dependent variables and time is the independent variable

- **Linear**

$$\frac{dy}{dt}$$

$$\frac{d^2y}{dt^2}$$

2nd order
not
2nd degree

$$\sin t\left(\frac{dy}{dt}\right)$$

- **Nonlinear**

$$\left(\frac{dy}{dt}\right)^2$$

$$\sin x\left(\frac{dy}{dt}\right)$$

$$xy$$

$$x^3$$

linear /nonlinear equations

linear equation: consists of a sum of linear terms

$y = x + 2$

$y(t) + x(t) = N$

$dy/dt = x + \sin t$

nonlinear equation: all other equations

$y + x^2 = 2$

$x(t) * y(t) = N$

$dy / dt = xy + \sin x$

most nonlinear differential equations are impossible to solve analytically!

So what do we do???

linear and nonlinear systems

system 1

$$\frac{dy_1}{dt} = 2y_1 + y_2$$

$$\frac{dy_2}{dt} = y_1 + 3y_2$$

linear system: system of linear equations

system 2

$$\frac{dy_1}{dt} = 2y_1y_2 + y_2$$

$$\frac{dy_2}{dt} = y_1y_2 + 3y_2$$

nonlinear system: system of equations containing at least 1 nonlinear term

we can use tools such as Laplace transformations to assist in solving linear systems of differential equations

can't use for nonlinear systems!!

what is nonlinear conceptually?

- nonlinear implies interactions!!

$$y = 2x_1 + x_2$$

$$y = 2x_1x_2 + x_2$$

the impact of x_1 is always the same

the impact of x_1 on y depends on the value of x_2

there is an *interaction* between x_1 and x_2

16.3. PHASE TRAJECTORIES-SINGULAR POINTS (*TOPOLOGICAL METHODS*)

If the nonlinear differential equation is of the second order and does not contain the independent variable t explicitly, a good amount of information, regarding the properties of the solution, may be obtained by a geometrical procedure without solving the equation itself. Dynamical systems, whose equations of motion do not contain the time t explicitly, are known as **autonomous systems**. Large amplitude oscillations of simple pendulum and the system represented by Van der Pol's equation are the examples of autonomous systems.

The equations of motion of second order automomous systems generally possess the following form:

$$\frac{d^2x}{dt^2}+F(\dot{x})+f(x)=0 \qquad \ldots(6)$$

where $F(\dot{x})$ is any function of velocity $\dot{x}(=dx/dt)$ and $f(x)$ that of displacement x.

If we introduce the velocity $y=\dot{x}$ as another dependent variable, the differential equation (6) may be written in the form of first order equations :

$$\frac{dx}{dt}=y \qquad \ldots(7)$$

and

$$\frac{dy}{dt}=-[F(y)+f(x)] \qquad \ldots(8)$$

For a more general system, we may write

$$\frac{dx}{dt} = P(x, y) \qquad ...(9)$$

$$\frac{dy}{dt} = Q(x, y) \qquad ...(10)$$

where P and Q are functions of x and y. Equations (7) and (8) are the special cases of eqs. (9) and (10) respectively.

In order to investigate the qualitative features of the solutions of (6), some definitions are given below.

Phase Plane : The quantities x and y can be considered as cartesian coordinates in x-y plane. This plane is called phase plane.

Phase Trajectory : If we divide eq. (7) by eq. (8), we obtain

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{F(y)+f(x)}{y} \qquad \text{...(11)}$$

From eq. (11) we may obtain an equation of a definite curve in the phase plane (x, y). This curve is called the **phase trajectory** or **phase path** corresponding to the differential equation (6). Further the phase trajectory of the system, defined by eqs. (9) and (10), is given by

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}, \qquad P(x,y) \neq 0 \qquad \text{...(12)}$$

By integrating eqs. (9) and (10), we can get

$$x = x(t) \text{ and } y = y(t) \qquad \text{...(13)}$$

These are called **parametric equations of the trajectory.**

Singular Point : A point (x_0, y_0) of the phase plane for which

$$P(x_0, y_0) = Q(x_0, y_0) = 0 \qquad \text{...(14)}$$

is called a ***singular point.***

Ordinary Points : All the points of the phase plane, for which the functions P and Q do not vanish simultaneously, are called ***ordinary points.***

Representative Point : We may consider the derivatives (9) and (10) as the x and y components of the velocity of a point in the phase plane. This point is called the ***representative point*** of the system. The x and y components of the velocity of the representative point are given by

$$v_x = \frac{dx}{dt} = P(x, y) \text{ and } v_y = \frac{dy}{dt} = Q(x, y) \qquad \text{...(15)}$$

Obviously at a singular point $v_x = v_y = 0$. Therefore, a singular point represents a position of equilibrium of the system with zero velocity.

16.4. PHASE TRAJECTORIES OF LINEAR SYSTEMS

(1) Linear Harmonic Oscillator. The differential equation of a linear harmonic oscillator having mass m and spring force constant k is given by

$$m\frac{d^2x}{dt^2}+kx=0 \quad \text{or} \quad \frac{d^2x}{dt^2}+\omega_0{}^2x=0 \qquad ...(16)$$

where $\omega_0=\sqrt{k/m}$.

Let $\dfrac{dx}{dt}=y$. Then eq. (16) is equivalent to

$$\frac{dy}{dt}=-\omega_0{}^2x \qquad ...(17)$$

and

$$\frac{dx}{dt}=y \qquad ...(18)$$

In fact, these are coupled first order differential equations.

From eqs. (17) and (18), we have

$$\frac{dy}{dx} = -\frac{kx}{my} \quad \text{or} \quad m\,ydy + k\,xdx = 0$$

Integrating, we get

$$m\frac{y^2}{2} + \frac{kx^2}{2} = C \qquad ...(19)$$

where C is a constant, representing the total energy E of the oscillator, *i.e.*,

$$\frac{1}{2}my^2 + \frac{1}{2}kx^2 = E \qquad ...(20)$$

This can be written in the form

$$\frac{x^2}{2E/k} + \frac{y^2}{2E/m} = 1 \qquad ...(21)$$

If we put $2E/k = a^2$ and $2E/m = b^2$, then eq. (21) represents an ellipse, given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad ...(22)$$

whose centre is at the origin (0,0) with semimajor axis $a = \sqrt{2E/k}$ and semiminor axis $b = \sqrt{2E/m}$ (Fig. 16.1).

The phase trajectory of the differential equations (17) and (18) [or eq. (16)] is a definite ellipse for a definite amount of total energy E. Different values of the total energy E fix correspondingly different ellipses. From the equation $y = dx/dt$, we see that the representative point P moves along the phase trajectory in a *clockwise direction.*

The **Parametric equations** of the ellipse (22) are obtained by integrating eqs. (17) and (18) with respect to time t as follows :

$$x = x_0 \cos\omega_0 t + \frac{y_0}{\omega}\sin\omega_0 t \qquad \text{...(23a)}$$

and $$y = -\omega_0 x_0 \sin\omega_0 t + y_0 \cos\omega_0 t \qquad \text{...(23b)}$$

where $\omega_0 = \sqrt{k/m}$ and at $t = 0$, $x = x_0$, $y = y_0$.

These equations can be obtained as follows :

$$\frac{dy}{dt} = -\omega_0{}^2 x,\ y = \frac{dx}{dt}$$

Let the solution be $x = a\sin(\omega_0 t + \phi)$

Therefore, $y = a\omega_0 \cos(\omega_0 t + \phi)$

At $t = 0$, $x = x_0$, $y = y_0$ and hence

$$x_0 = a\sin\phi,\ y = a\omega_0\cos\phi$$

Therefore, $x = a\sin\omega_0 t\cos\phi + a\cos\omega_0 t\sin\phi$

or $$x = x_0\cos\omega_0 t + \frac{y}{\omega_0}\sin\omega_0 t$$

Hence $$y = -\omega x_0 \sin\omega t + y\cos\omega t$$

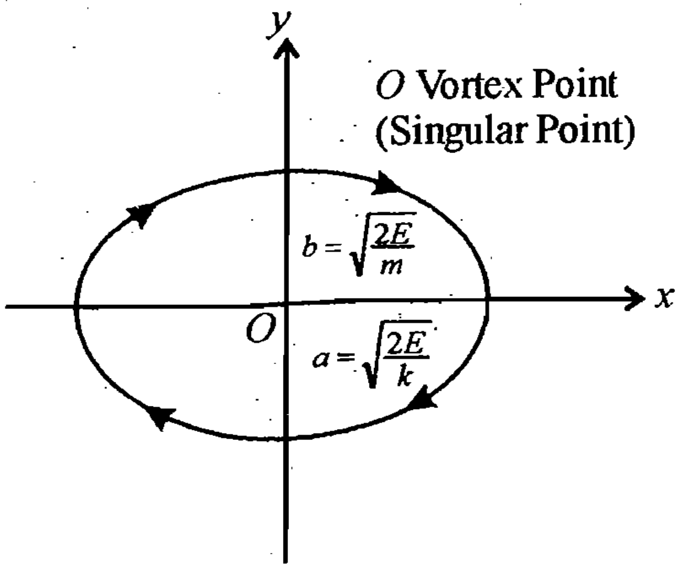


Fig. 16.1. Phase trajectory of linear harmonic oscillator

Obviously the periodicity of the solution (23) tells us that the representative point completes one revolution of the ellipse in time

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}} \qquad \text{...(24)*}$$

From eqs. (17) and (18) for the point (0,0)

$$\frac{dx}{dt} = P(0,0) = 0 \text{ and } \frac{dy}{dt} = Q(0,0) = 0 \qquad \text{...(25)}$$

Therefore the centre of the ellipse $x = 0, y = 0$ is a *singular point* of the differential eqs. (17) and (18). The possible phase trajectories are ellipses which enclose this point. A singular point of this type enclosed by the phase trajectories and approached by none is called a **vortex point.** This vortex point represents a position of stable equilibrium of the system.

(2) Aperiodic Motion. Let a mass m be repulsed from the origin by a force proportional to its distance x from the origin O (Fig. 16·2), *i.e.*,

$$F \propto x \quad \text{or} \quad F = kx$$

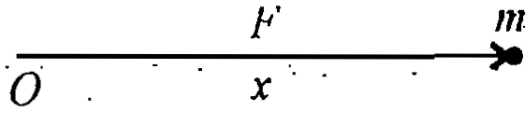


Fig. 16.2. Force proportional to x

The equation of motion is

$$m\frac{d^2x}{dt^2} = x \quad \text{or} \quad \frac{d^2x}{dt^2} = \alpha^2 x \qquad ...(26)$$

where $\alpha = \sqrt{k/m}$.

Eq. (26) can be represented as

$$\int \frac{dx}{y} = \int \frac{dx}{\sqrt{\frac{2}{ml^2}[E - V(x)]}}$$

$$\frac{dy}{dt} = \alpha^2 x \qquad ...(27)$$

and $$\frac{dx}{dt} = y \qquad ...(28)$$

Hence $$\frac{dy}{dx} = \frac{\alpha^2 x}{y} \quad \text{or} \quad y\,dy - \alpha^2 x\,dx = 0$$

Integrating it, we get

$$y^2 - \alpha^2 x^2 = C \qquad ...(29)$$

where C is a constant. Eq. (29) represents a hyperbola. For different values of C, eq. (29) gives a family of hyperbolas in the phase plane. If at time $t = 0$, $x = x_0$ and $y = y_0$, the solution of the differential equation gives

* From the equation $y = \frac{dx}{dt}$, we have $dt = \frac{dx}{y}$

Integrating for one revolution $T = \int \frac{dx}{y} = 4\int_0^1 \frac{dx}{\left(\frac{2E}{m} - \omega_0^2 x^2\right)} = \frac{2\pi}{\omega_0}$

because from eq. (20) $y = \sqrt{\frac{2E}{m} - \frac{k}{m}x^2}$.

$$x = x_0 \cosh \alpha t + \frac{y_0}{\alpha} \sinh \alpha t$$

$$y = \alpha x_0 \sinh \alpha t + y_0 \cosh \alpha t$$

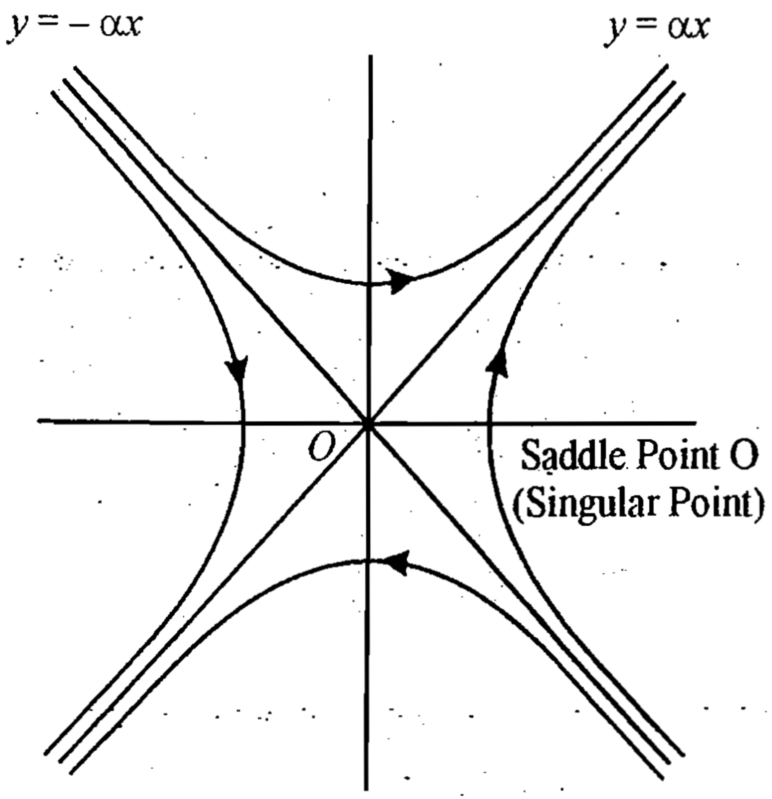


Fig. 16.3. Phase trajectory of aperiodic motion

Damped Harmonic Oscillator and Overdamped Motion

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = 0 \qquad ...(31)$$

or equivalently,

$$\frac{dy}{dt} = -2by - \omega_0^2 x \qquad ...(32)$$

and

$$\frac{dx}{dt} = y \qquad ...(33)$$

Remembering that by the substitution of the trial solution $x = Ae^{\alpha t}$ in eq. (31), we obtain a quadratic equation in α whose roots are

$$\alpha = -b \pm \sqrt{b^2 - \omega_0^2} \qquad ...(34)$$

The solution, therefore, depends on the nature of the quantity $\sqrt{b^2 - \omega_0^2}$. If $b^2 < \omega_0^2$, the roots are complex and the solution of equation (31) represents the damped harmonic oscillations.

From eqs. (32) and (33), we obtain

$$\frac{dy}{dx} = \frac{-2by - \omega_0^2 x}{y} \qquad ...(35)$$

If we put $\omega = \sqrt{\omega_0^2 - b^2}$, the general solution of eqn. (31), or equivalently eqs. (32) and (33), is

$$x = Ae^{-bt} \cos(\omega t + \phi) \qquad ...(36)$$

or

$$y = \dot{x} = -Ae^{-bt}[b\cos(\omega t + \phi) + \omega \sin(\omega t + \phi)] \qquad ...(37)$$

where A and ϕ are constants.

Eqs. (36) and (37) are the parametric equations of phase trajectories representing a family of spirals and one of the spirals is shown in Fig 16·4. The origin O is a singular point of a type called a **focal point**.

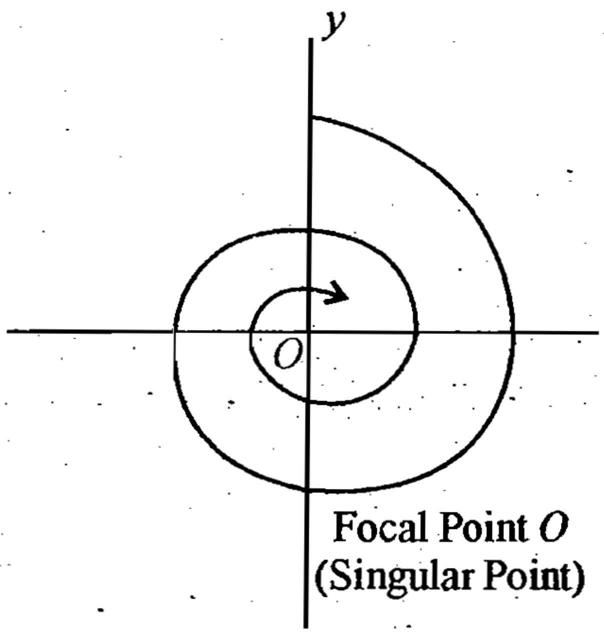


Fig. 16.4. Phase trajectory of an underdamped oscillator

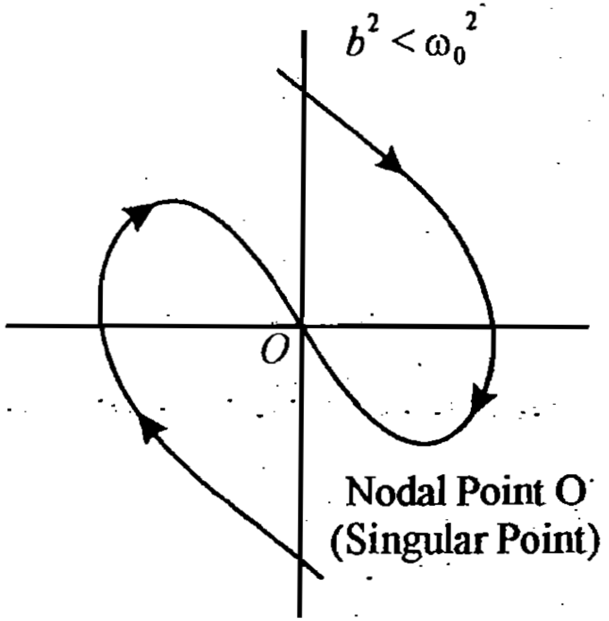


Fig. 16.5. Phase trajectory of an overdamped oscillator

The representative point moves along a spiral in the phase plane and approaches the focal point at the origin O in an asymplotic manner with no definite direction. The focal point O is a point of *stable equilibrium*.

In case, the damping is heavy, *i.e.*, $b^2 > \omega_0^2$ the motion is no longer oscillatory. If we put $\sqrt{b^2 - \omega_0^2} = \beta$, then the solutions are of the form

$$x = Ae^{-bt} \cos h(bt + \phi) \qquad \text{...(38}a\text{)}$$

and $$y = \dot{x} = Ae^{-bt}[\beta \sin h\,(\beta t + \phi) - b \cos h\,(\beta t + \phi)]$$

where A and ϕ are constants.

Eqs. (38) are the parametric equations, representing the phase trajectories of a overdamped case. One of the phase trajectories is shown in Fig 16·5. The origin is a singular point of the type, called a **nodal point**. We find that for each phase trajectory (curve), the representative point moves toward the nodal point with a *definite direction*.

We summarise below the four basic types of singular points :

Table : Singular Points

S.N.	Name	Type of Motion	Approach	Stability
1.	Vortex point	Oscillatory	None	Stable
2.	Saddle point	Aperiodic	Along asymptotes	Unstable
3.	Focal point	Damped oscillatory	With no definite direction	Stable
4.	Nodal point	Nonoscillatory	With a definite direction	Stable

16.5. PHASE TRAJECTORIES OF NONLINEAR SYSTEMS

We may effectively study the motion of nonlinear conservative and nonconservative systems with specific examples by topological methods.

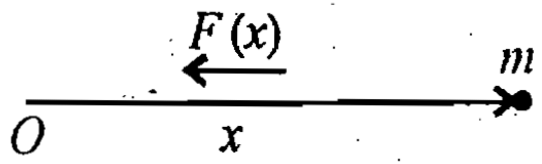


Fig. 16.6. Nonlinear force F(x)

(1) Nonlinear Conservative Systems

As an example of such a system, we consider the motion of a mass m.

attracted towards a fixed point O, the origin, by a nonlinear restoring force $F(x)$ at the displacement x (Fig 16·6). The equation of motion of the system is

$$m\frac{d^2x}{dt^2} = -F(x) \quad ...(39)$$

This is equivalent to the first order differential equations, given by

$$\frac{dy}{dt} = -\frac{F(x)}{m} = -f(x) \quad \text{(say)} \quad ...(40)$$

and $$\frac{dx}{dt} = y \quad ...(41)$$

Dividing (40) by (41), we get the following equation for the phase trajectory of the motion :

$$\frac{dy}{dx} = -\frac{F(x)}{my} \quad ...(42)$$

From eq. (42), we obtain

$$mydy = -F(x)dx$$

Integrating it, we get

$$\frac{1}{2}my^2 + \int_0^x F(x)dx = C \quad \text{...(43)}$$

where C is a constant of integration and the mass m is at $x = 0$, when $t = 0$,

This constant C can be evaluated, if we put the condition at $t = 0$, $x = 0$ and $y = y_0$ in eq. (43), *i.e.*,

$$\frac{1}{2}my_0^2 = C$$

Therefore, eq. (43) takes the form

$$\frac{1}{2}my^2 + \int_0^x F(x)dx = \frac{1}{2}my_0^2 \quad \text{...(44)}$$

In eq. (44), the quantity $\frac{1}{2}my^2$ is the kinetic energy of the system and

$$\int_0^x F(x)dx = V(x) \quad \text{...(45)}$$

is the potential energy of the system. Hence, eq. (44) can be written as

$$\frac{1}{2}my^2 + V(x) = \frac{1}{2}my_0^2 = E \quad \text{...(46)}$$

where E is the total energy of the system.

From eq. (46), we obtain the expression for the velocity of the system as

$$y = \sqrt{\frac{2}{m}[E - V(x)]} \qquad \ldots(47)$$

We interpret this equation topologically by taking for a special case, where

$$F(x) = x(x+a) \qquad \ldots(48)$$

Hence the equation of motion of the system is

$$\frac{d^2x}{dt^2} = -\frac{x(x+a)}{m} \qquad \ldots(49)$$

which is equivalent to

$$\frac{dy}{dt} = -\frac{x(x+a)}{m} \text{ and } \frac{dx}{dt} = y \qquad \ldots(50)$$

So that $$\frac{dy}{dx} = -\frac{x(x+a)}{my} \qquad \ldots(51)$$

Also $$\frac{1}{2}my^2 + V(x) = E \qquad \ldots(52)$$

where $$V(x) = \int_0^x F(x)dx = \int_0^x x(x+a)dx \qquad \ldots(53)$$

The phase trajectories for the equation (49) is obtained from eq. (52) as

$$y = \sqrt{\frac{2}{m}[E - V(x)]} \qquad \ldots(54)$$

For the force of the form (48) , we have drawn the curves for $F(x)$ versus x, energy diagram $V(x)$ versus

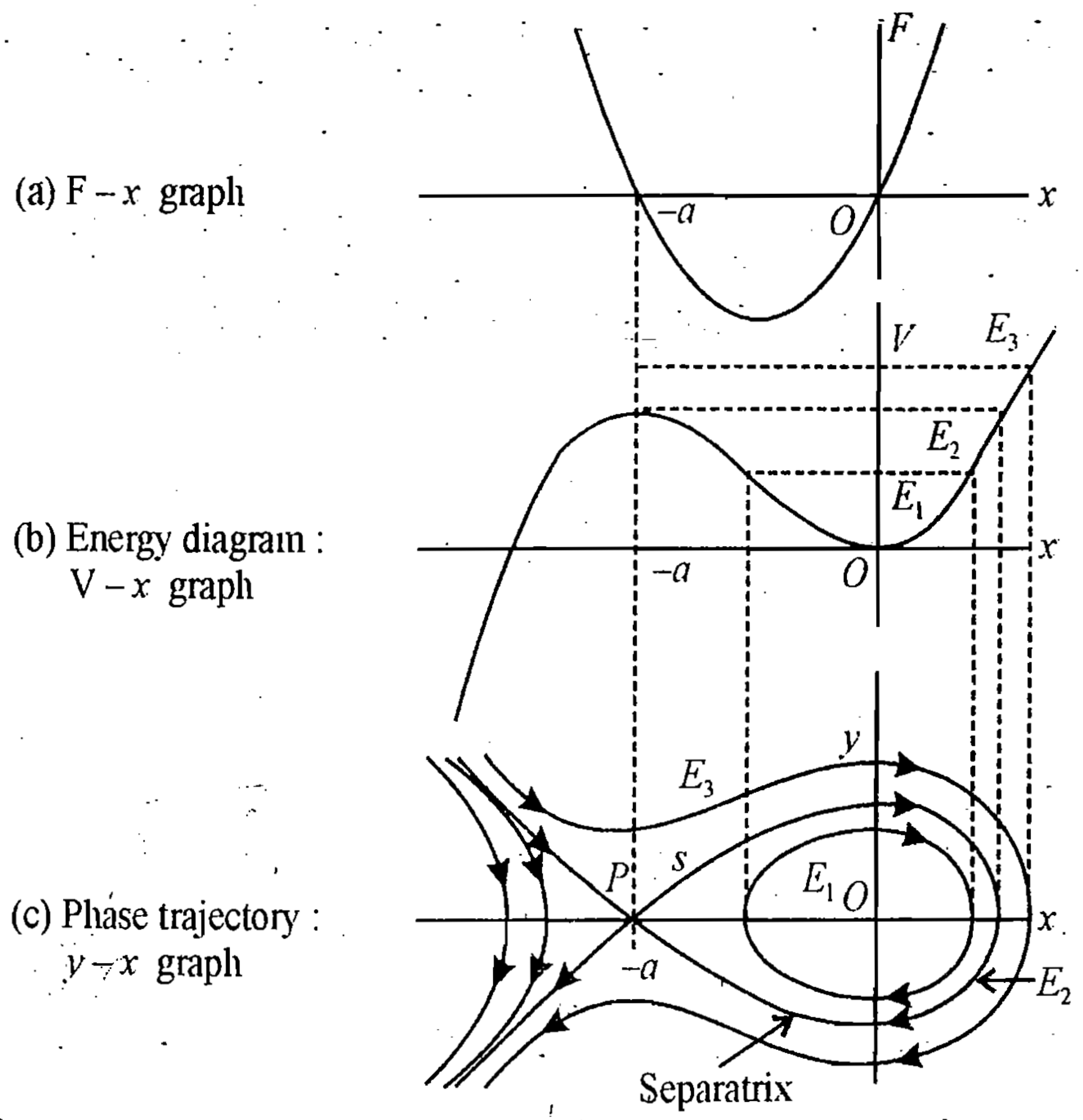


Fig. 16.7. Phase trajectory corresponding to equation $F(x) = x\,(x + a)$

Trajectory of the Equation of a Simple Pendulum. The restoring couple on a simple pendulum at angular displacement θ is given by

$$I\frac{d^2\theta}{dt^2} = -mgl\sin\theta \qquad \text{...(55)}$$

Since $I = ml^2$, we get from eq. (55)

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta \text{ or } \frac{d^2\theta}{dt^2} = -\omega_0^2\sin\theta \qquad \text{...(56)}$$

where $\omega_0^2 = \sqrt{g/l}$. This is the *equation of motion of the simple pendulum.*

Now, writing $\theta = x$, eq. (56) can be written as

$$\frac{dy}{dt} = -\omega_0^2\sin x \qquad \text{...(57)}$$

and

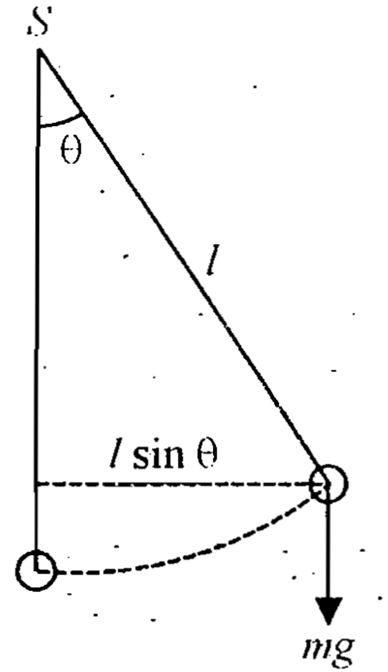
$$\frac{dx}{dt} = y \qquad \text{...(58)}$$

From eqs. (57) and (58), we get

$$\frac{dy}{dx} = -\frac{\omega_0^2\sin x}{y} \qquad \text{...(59)}$$

From eq. (59), we obtain

$$y\,dy = -\omega_0^2\sin x\,dx$$



$$\frac{y^2}{2}+\int_0^x \omega_0^2 \sin x \, dx = C \quad \text{or} \quad \frac{y^2}{2}+\omega_0^2(1-\cos x) = C$$

where C is a constant.

Multiply by ml^2, we obtain

$$\frac{ml^2 y^2}{2}+mgl(1-\cos x) = E \qquad \text{...(60)}$$

(Kinetic energy + Potential energy = Total energy)

where the constant E is the total energy for the conservative system.

The potential energy $V(x) = mgl(1-\cos x) = 2mgl\sin^2(x/2)$ ...(61)

From (60) and (61)

$$y = \sqrt{\frac{2}{ml^2}[E - V(x)]} \quad \text{or} \quad y = \sqrt{\frac{2}{ml^2}[E - 2mgl\sin^2\frac{x}{2}]} \qquad \text{...(62)}$$

In order to study, the topology of the equation, we plot the energy diagram $[V(x)$ against $x]$ by using eq. (61) and phase trajectory [y against x] by using eq. (62) in Fig 16.9.

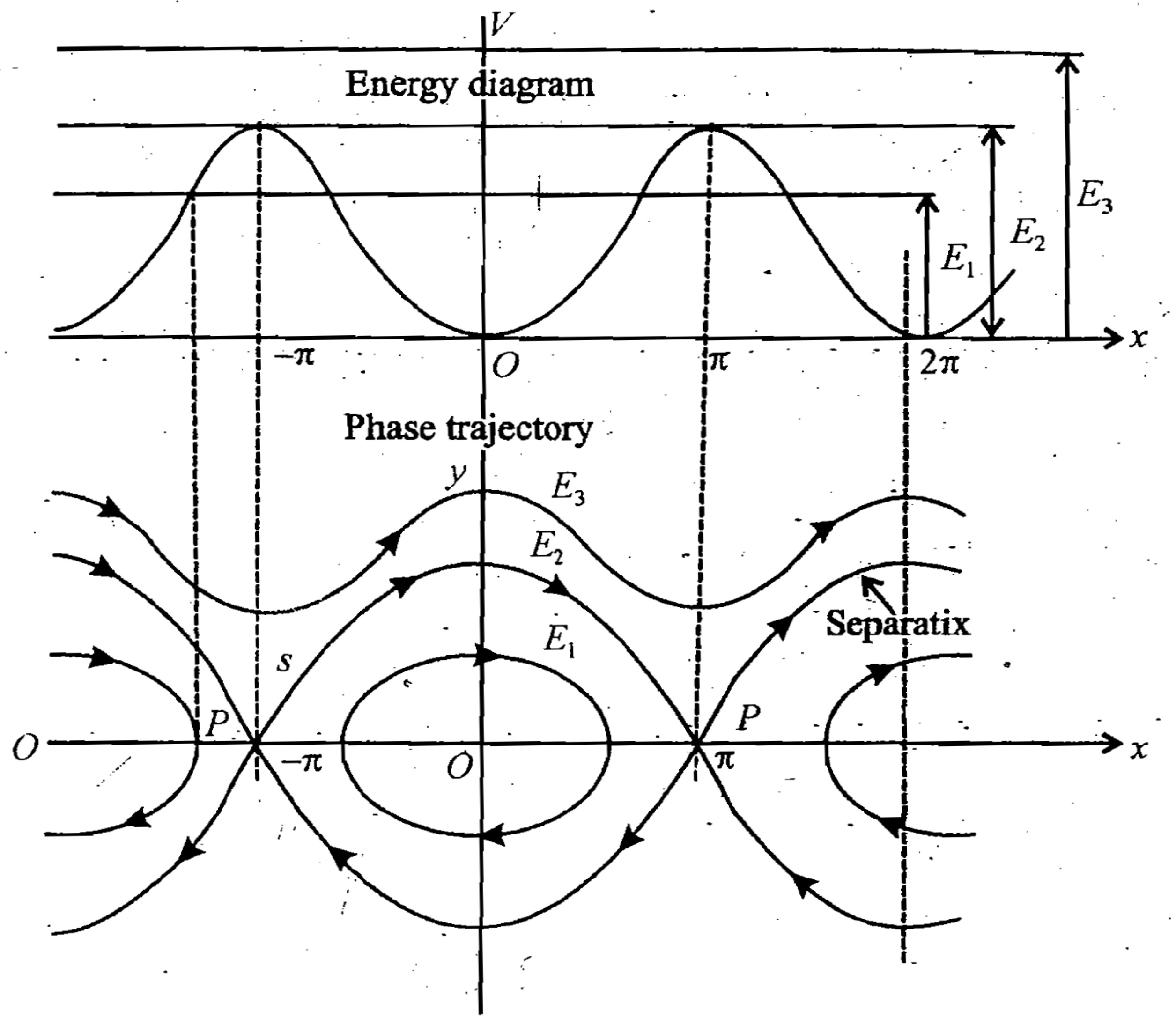


Fig. 16.9. Phase trajectories of simple pendulum

In Fig 16·9 (*a*), the quantity $V(x) = 2mgl\sin^2(x/2)$ has been plotted against x. We have drawn three horizontal lines on the energy diagram corresponding to the three distinct values of total energy E_1, E_2 and E_3, where $E_1 < E_2 < E_3$.

Case (i) $E_1 < 2mgl$: This represents the phase trajectory for E_1 of a oscillating pendulum, whose energy (E_1) is less than the energy necessary ($2mgl$) to go over the top.

Case (ii), $E_2 = 2mgl$: The phase trajectory in this case is the one in which the system has just enough total energy to take it to the top at $x = \pi$ and it is called **separatrix** because it separates the motion of one type (oscillatory) from those of entirely different type (nonoscillatory). The point P of the phase trajectory is a **saddle point,** where $V(x)$ is maximum ($2mgl$). This point represents **a point of unstable equilibrium,** corresponding to the inverted position of the pendulum. The point P has completely different properties, when compared to the point O. The point O is a **vortex point,** corresponding to a position of stable equilibrium, the bottom position of the pendulum.

Case (iii), $E_3 > 2mgl$: When the pendulum has total energy E_3, which is greater than the critical energy E_2 to take it over the top then the phase trajectory is a curve that continues with x and the motion is no longer oscillatory. In such a case, the pendulum revolves about its point of support continuously with **maximum angular velocity** at $x = 0, 2\pi, 4\pi$ ....*etc.* (at bottom) and **minimum angular velocity** at $x = \pi, 3\pi, 5\pi$ .... *etc.* (at top).

Period of nonlinear oscillations, when the phase trajectory is a closed path, can be obtained by using eq. (58) and (62), *i.e.*,

$$T = \int \frac{dx}{y} = \int \frac{dx}{\sqrt{\frac{2}{ml^2}[E - V(x)]}} \quad \text{...(63)}$$

Remember that for simple pendulum $x = \theta$, the angular displacement. Hence from (60) at maximum angular displacement θ_0, $y = \dot{\theta} = 0$,

$$mgl(1-\cos\theta_0) = E$$

Also $$V(x) = mgl(1-\cos x) = mgl(1-\cos\theta)$$

Therefore, $$T = 2\int_{-\theta_0}^{\theta_0} \frac{d\theta}{\sqrt{\frac{2}{ml^2}[mgl(1-\cos\theta_0) - mgl(1-\cos\theta)]}}$$

because the period is twice the time taken by the pendulum to oscillate from $\theta = -\theta_0$ to $\theta = +\theta_0$.

Now, $$T = 2\int_{-\theta_0}^{\theta_0} \frac{d\theta}{\sqrt{2\frac{g}{l}(\cos\theta - \cos\theta_0)}} = \frac{2}{\sqrt{g/l}}\int_{-\theta_0}^{\theta} \frac{d\theta}{\sqrt{2(\cos\theta - \cos\theta_0)}} \quad \text{...(64)}$$

$$= \frac{1}{\omega_0}\int_{-\theta_0}^{\theta} \frac{d\theta}{\sqrt{[\sin^2(\theta_0/2) - \sin^2(\theta/2)]}}$$

Let $k = \sin\theta_0/2$ and $\sin\theta/2 = k\sin\alpha$

So that $$d\theta = \frac{2k\cos\alpha\, d\alpha}{\cos(\theta/2)} = \frac{2k\cos\alpha\, d\alpha}{\sqrt{1-k^2\sin^2\alpha}}$$

Hence $$T \frac{2}{\omega_0}\int_0^{\pi/2} \frac{2k\cos\alpha\, d\alpha}{\sqrt{k^2 - k^2\sin^2\alpha}\sqrt{1-k^2\sin^2\alpha}}$$

or $$T = \frac{4}{\omega_0}\int_0^{\pi/2} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}} \qquad ...(65)$$

The integral $$\int_0^{\pi/2} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}} = K(k) \qquad ...(66)$$

is called the **elliptical integral of first kind**. This $K(k)$ is given by

$$K(k) = \frac{\pi}{2}\left[1+(1/2)^2k^2+(1\cdot 3/2\cdot 4)^2k^4+......\right]$$

Thus $$T = \frac{4}{\omega_0} K(k) = \frac{2\pi}{\omega_0}\left[1+\left(\frac{1}{2}\right)^2 k^2 + \left(\frac{3}{8}\right)^2 k^4 + \ldots\ldots\right]$$

But $k = \sin(\theta_0/2)$ and $\theta_0 =$ angular amplitude, hence

$$T = \frac{2\pi}{\omega_0}\left[1+\frac{1}{4}\sin^2\frac{\theta_0}{2}+\frac{9}{64}\sin^4\frac{\theta_0}{2}+\ldots\ldots\right] \qquad \ldots(67)$$

For $\theta_0 = 2°$ and $\theta_0 = 60°$,

$$T_{(2°)} = 1\cdot 000076\, T_0 \text{ and } T_{(60°)} = 1\cdot 07 T_0 \qquad \ldots(68)$$

where $T_0 = 2\pi/\omega_0$. Thus **the period of a pendulum depends on amplitude.**

Further for relatively small amplitude, $k = \sin\theta_0/2 = \theta_0/2$ and neglecting higher order terms more than k^2, we have

$$T = \frac{2\pi}{\omega_0}\left[1+\frac{\theta_0^{\,2}}{16}\right] \text{ or } T = T_0\left[1+\frac{\theta_0^{\,2}}{16}\right] \qquad \ldots(69)$$

From the above discussion we see the power of the topological aspects of the phase trajectories to provide general qualitative information regarding the behaviour of autonomous systems.

(2) Non-linear Non-conservative Systems

As an example of non-linear non-conservative systems, we write the **Van Der Pol Equation**, represented by

$$\frac{d^2x}{dt^2} - \in(1-x^2)\frac{dx}{dt} + x = 0 \qquad ...(70)$$

The parametric equation of the phase trajectories of eqn. (70) are

$$\frac{dy}{dt} = \in(1-x^2)y - x \qquad ...(71a)$$

and

$$\frac{dx}{dt} = y \qquad ...(72b)$$

The equation of the phase trajectory is

$$\frac{dy}{dx} = \in(1-x^2) - \frac{x}{y} \qquad ...(72)$$

This equation can be plotted for different values of the parameter $\in$. When $\in = 0$, the phase path is a circle, because

$$\frac{dy}{dx} = -\frac{x}{y} \text{ or } xdx + ydy = 0 \text{ or } x^2 + y^2 = a^2 \qquad ...(73)$$

In such a case the motion will be simple harmonic.

When $\in = 0\cdot1$, the phase trajectory is sightly different from a circle. If we increase the value of $\in$, the phase trajectories differ much from circles. The phase trajectory of Van der Pol's equation for small values

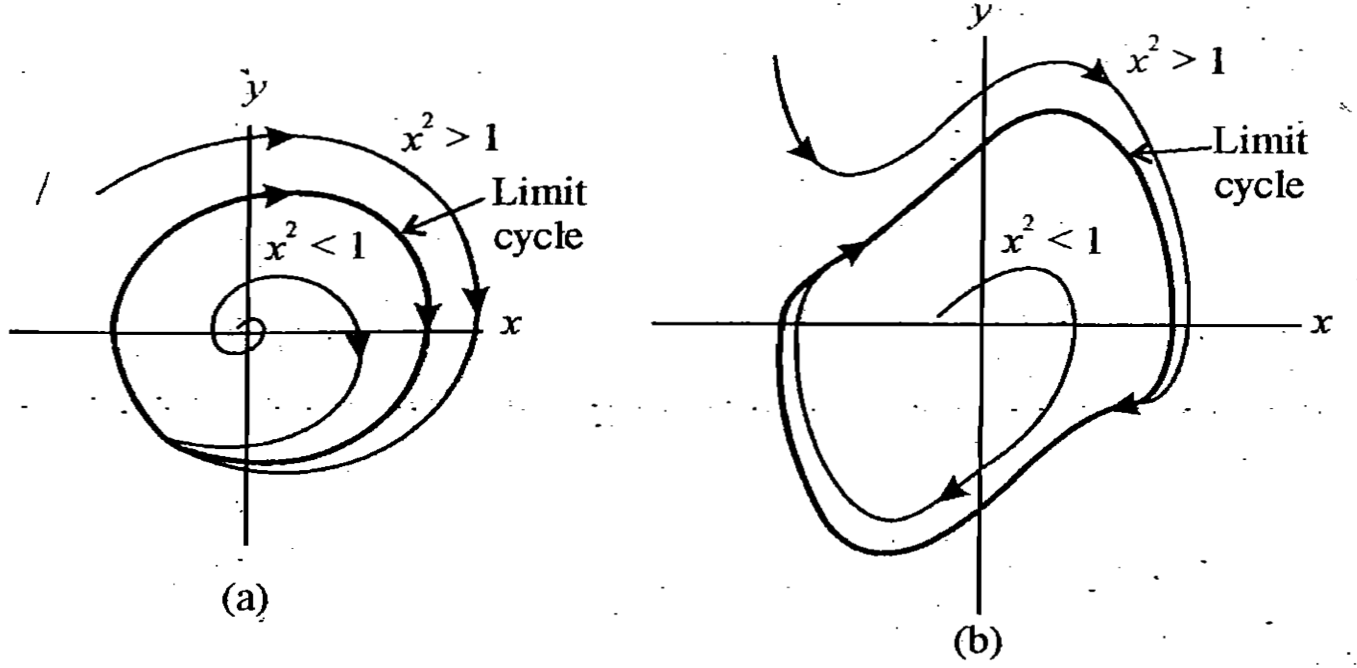


Fig. 16.10. Limit cycles of the Van der Pol equation : (a) approximately circle for small damping coefficient $\in$ and (b) distorted curve for high damping coefficient.

of $\in$ (say $\in = 0{\cdot}2$) is shown in Fig 16·10(a) . If $x^2 > 1$, the damping is positive and the motion spirals inward toward a stable path in phase space. This stable path is called the **limit cycle.** For $x^2 < 1$, the damping is negative and the motion spirals inward toward the limit cycle [Fig 16.10(a)]. The final state of motion is stabilized as the damping vanishes for $x = 1$ and the system moves on a closed path approximately circular.

When $\in$ is large enough, the damping term becomes comparable in magnitude to the other terms in eq. (70). The trajectories are still drawn toward the limit cycle but the cycle becomes sufficiently distorted from circular shape [Fig 16.10(b)]. Also, due to strong damping, the frequency relative to that of said simple harmonic motion, decreases and the oscillatons become distorted. For large damping, the shape of the trajectory is very much distorted from circular shape.

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Thank You