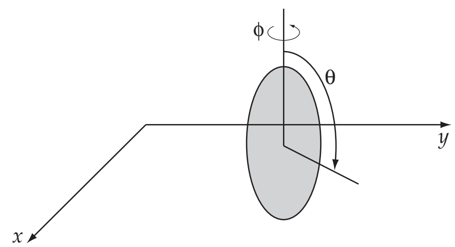
CHAPTER 7

Hamilton's Principle—Lagrangian and Hamiltonian Dynamics

7-1. Four coordinates are necessary to completely describe the disk. These are the x and y coordinates, the angle θ that measures the rolling, and the angle ϕ that describes the spinning (see figure).



Since the disk may only roll in one direction, we must have the following conditions:

$$dx \cos \phi + dy \sin \phi = R\, d\theta \tag{1}$$

$$\frac{dy}{dx} = \tan \phi \tag{2}$$

These equations are not integrable, and because we cannot obtain an equation relating the coordinates, the constraints are nonholonomic. This means that although the constraints relate the infinitesimal displacements, they do not dictate the relations between the coordinates themselves, e.g. the values of x and y (position) in no way determine θ or ϕ (pitch and yaw), and vice versa.

7-2. Start with the Lagrangian

$$L = \frac{m}{2}\left[\left(v_0 + at + \ell\dot{\theta}\cos\theta\right)^2 + \left(\ell\dot{\theta}\sin\theta\right)^2\right] + mg\ell\cos\theta \tag{1}$$

$$= \frac{m}{2}\left[\left(v_0 + at\right)^2 + 2\left(v_0 + at\right)\ell\dot{\theta}\cos\theta + \ell^2\dot{\theta}^2\right] + mg\ell\cos\theta \tag{2}$$

Now let us just compute

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left[m\left(v_0 + at\right)\ell \cos\theta + m\ell^2\dot{\theta}\right] \tag{3}$$

$$= ma\ell \cos\theta - m\left(v_0 - at\right)\ell\dot{\theta}\sin\theta + m\ell^2\ddot{\theta} \tag{4}$$

$$\frac{\partial L}{\partial \theta} = -m\left(v_0 + at\right)\ell\dot{\theta}\sin\theta - mg\ell\sin\theta \tag{5}$$

According to Lagrange's equations, (4) is equal to (5). This gives Equation (7.36)

$$\ddot{\theta} = \frac{g}{\ell}\sin\theta + \frac{a}{\ell}\cos\theta = 0 \tag{6}$$

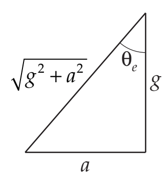
To get Equation (7.41), start with Equation (7.40)

$$\ddot{\eta} = -\frac{g\cos\theta_e - a\sin\theta_e}{\ell}\eta \tag{7}$$

and use Equation (7.38)

$$\tan\theta_e = -\frac{a}{g} \tag{8}$$

to obtain, either through a trigonometric identity or a figure such as the one shown here,



$$\cos\theta_e = \frac{g}{\sqrt{g^2 + a^2}} \qquad \sin\theta_e = \frac{a}{\sqrt{g^2 + a^2}} \tag{9}$$

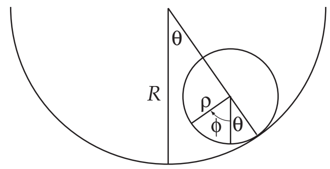
Inserting this into (7), we obtain

$$\ddot{\eta} = -\frac{\sqrt{a^2 + g^2}}{\ell}\eta \tag{10}$$

as desired.

We know intuitively that the period of the pendulum cannot depend on whether the train is accelerating to the left or to the right, which implies that the sign of a cannot affect the frequency. From a Newtonian point of view, the pendulum will be in equilibrium when it is in line with the effective acceleration. Since the acceleration is sideways and gravity is down, and the period can only depend on the magnitude of the effective acceleration, the correct form is clearly $\sqrt{a^2 + g^2}$.

7-3.



If we take angles θ and ϕ as our generalized coordinates, the kinetic energy and the potential energy of the system are

$$T = \frac{1}{2} m\left[(R-\rho)\dot{\theta}\right]^2 + \frac{1}{2} I\,\dot{\phi}^2 \tag{1}$$

$$U = \left[R - (R-\rho)\cos\theta\right] mg \tag{2}$$

where m is the mass of the sphere and where $U = 0$ at the lowest position of the sphere. I is the moment of inertia of sphere with respect to any diameter. Since $I = (2/5)\, m\rho^2$, the Lagrangian becomes

$$L = T - U = \frac{1}{2} m(R-\rho)^2\,\dot{\theta}^2 + \frac{1}{5} m\rho^2\dot{\phi}^2 - \left[R - (R-\rho)\cos\theta\right] mg \tag{3}$$

When the sphere is at its lowest position, the points A and B coincide. The condition $A0 = B0$ gives the equation of constraint:

$$f(\theta,\phi) = (R-\rho)\theta - \rho\phi = 0 \tag{4}$$

Therefore, we have two Lagrange's equations with one undetermined multiplier:

$$\left.\begin{aligned} \frac{\partial L}{\partial \theta} - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\theta}}\right] + \lambda\,\frac{\partial f}{\partial \theta} = 0 \\ \frac{\partial L}{\partial \phi} - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\phi}}\right] + \lambda\,\frac{\partial f}{\partial \phi} = 0 \end{aligned}\right] \tag{5}$$

After substituting (3) and $\partial f/\partial\theta = R-\rho$ and $\partial f/\partial\phi = -\rho$ into (5), we find

$$-(R-\rho)mg\sin\theta - m(R-\rho)^2\,\ddot{\theta} + \lambda(R-\rho) = 0 \tag{6}$$

$$-\frac{2}{5} m\rho^2\ddot{\phi} - \lambda\rho = 0 \tag{7}$$

From (7) we find λ:

$$\lambda = -\frac{2}{5} m\rho\ddot{\phi} \tag{8}$$

or, if we use (4), we have

$$\lambda = -\frac{2}{5} m(R-\rho)\ddot{\theta} \tag{9}$$

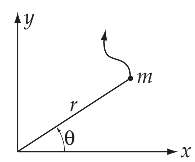
Substituting (9) into (6), we find the equation of motion with respect to θ:

$$\ddot{\theta} = -\omega^2 \sin\theta \tag{10}$$

where ω is the frequency of small oscillations, defined by

$$\omega = \sqrt{\frac{5g}{7(R-\rho)}} \tag{11}$$

7-4.



If we choose (r,θ) as the generalized coordinates, the kinetic energy of the particle is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \tag{1}$$

Since the force is related to the potential by

$$f = -\frac{\partial U}{\partial r} \tag{2}$$

we find

$$U = \frac{A}{\alpha}r^\alpha \tag{3}$$

where we let $U(r = 0) = 0$. Therefore, the Lagrangian becomes

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{A}{\alpha}r^\alpha \tag{4}$$

Lagrange's equation for the coordinate r leads to

$$\boxed{m\ddot{r} - mr\dot{\theta}^2 + Ar^{\alpha-1} = 0} \tag{5}$$

Lagrange's equation for the coordinate θ leads to

$$\boxed{\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0} \tag{6}$$

Since $mr^2\dot{\theta} = \ell$ is identified as the angular momentum, (6) implies that angular momentum is conserved. Now, if we use ℓ, we can write (5) as

$$m\ddot{r} - \frac{\ell^2}{mr^3} + Ar^{\alpha-1} = 0 \tag{7}$$

Multiplying (7) by $\dot{r}$, we have

$$m\dot{r}\ddot{r} - \frac{\dot{r}\ell^2}{mr^3} + A\dot{r}\,r^{\alpha-1} = 0 \tag{8}$$

which is equivalent to

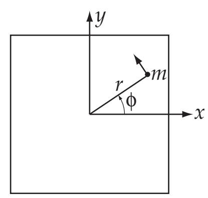
$$\frac{d}{dt}\left[\frac{1}{2}m\dot{r}^2\right]+\frac{d}{dt}\left[\frac{\ell^2}{2mr^2}\right]+\frac{d}{dt}\left[\frac{A}{\alpha}r^\alpha\right]=0 \tag{9}$$

Therefore,

$$\boxed{\frac{d}{dt}(T+U)=0} \tag{10}$$

and the total energy is conserved.

7-5.



Let us choose the coordinate system so that the x-y plane lies on the vertical plane in a gravitational field and let the gravitational potential be zero along the x axis. Then the kinetic energy and the potential energy are expressed in terms of the generalized coordinates (r,ϕ) as

$$T=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\phi}^2\right) \tag{1}$$

$$U=\frac{A}{\alpha}r^\alpha+mgr\sin\phi \tag{2}$$

from which the Lagrangian is

$$L=T-U=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\phi}^2\right)-\frac{A}{\alpha}r^\alpha-mgr\sin\phi \tag{3}$$

Therefore, Lagrange's equation for the coordinate r is

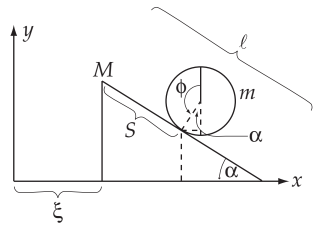
$$\boxed{m\ddot{r}-mr\dot{\phi}^2+Ar^{\alpha-1}+mg\sin\phi=0} \tag{4}$$

Lagrange's equation for the coordinate ϕ is

$$\boxed{\frac{d}{dt}\left(mr^2\dot{\phi}\right)+mgr\cos\phi=0} \tag{5}$$

Since $mr^2\dot{\phi}$ is the angular momentum along the z axis, (5) shows that the angular momentum is *not* conserved. The reason, of course, is that the particle is subject to a *torque* due to the gravitational force.

7-6.



Let us choose ξ, S as our generalized coordinates. The x,y coordinates of the center of the hoop are expressed by

$$\left.\begin{aligned} x &= \xi + S\cos\alpha + r\sin\alpha \\ y &= r\cos\alpha + (\ell - S)\sin\alpha \end{aligned}\right] \tag{1}$$

Therefore, the kinetic energy of the hoop is

$$\begin{aligned} T_{\text{hoop}} &= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}I\dot{\phi}^2 \\ &= \frac{1}{2}m\left[\left(\dot{\xi} + \dot{S}\cos\alpha\right)^2 + \left(-\dot{S}\sin\alpha\right)^2\right] + \frac{1}{2}I\dot{\phi}^2 \end{aligned} \tag{2}$$

Using $I = mr^2$ and $S = r\phi$, (2) becomes

$$T_{\text{hoop}} = \frac{1}{2}m\left[2\dot{S}^2 + \dot{\xi}^2 + 2\dot{\xi}\dot{S}\cos\alpha\right] \tag{3}$$

In order to find the total kinetic energy, we need to add the kinetic energy of the translational motion of the plane along the x-axis which is

$$T_{\text{plane}} = \frac{1}{2}M\,\dot{\xi}^2 \tag{4}$$

Therefore, the total kinetic energy becomes

$$T = m\dot{S}^2 + \frac{1}{2}(m + M)\dot{\xi}^2 + m\dot{\xi}\dot{S}\cos\alpha \tag{5}$$

The potential energy is

$$U = mgy = mg\left[r\cos\alpha + (\ell - S)\sin\alpha\right] \tag{6}$$

Hence, the Lagrangian is

$$l = m\dot{S}^2 + \frac{1}{2}(m + M)\dot{\xi}^2 + m\dot{\xi}\dot{S}\cos\alpha - mg\left[r\cos\alpha + (\ell - S)\sin\alpha\right] \tag{7}$$

from which the Lagrange equations for ξ and S are easily found to be

$$\boxed{2m\ddot{S} + m\ddot{\xi}\cos\alpha - mg\sin\alpha = 0} \tag{8}$$

$$\boxed{(m + M)\ddot{\xi} + m\ddot{S}\cos\alpha = 0} \tag{9}$$

or, if we rewrite these equations in the form of uncoupled equations by substituting for $\ddot{\xi}$ and $\ddot{S}$, we have

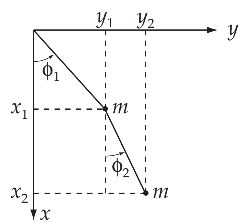
$$\left.\begin{aligned} \left[2-\frac{m\cos^2\alpha}{m+M}\right]\ddot{S}-g\sin\alpha=0 \\ \ddot{\xi}=-\frac{mg\sin\alpha\cos\alpha}{2(m+M)-m\cos^2\alpha} \end{aligned}\right] \tag{10}$$

Now, we can rewrite (9) as

$$\frac{d}{dt}\left[(m+M)\dot{\xi}+m\dot{S}\cos\alpha\right]=0 \tag{11}$$

where we can interpret $(m+M)\dot{\xi}$ as the x component of the linear momentum of the total system and $m\dot{S}\cos\alpha$ as the x component of the linear momentum of the hoop with respect to the plane. Therefore, (11) means that the x component of the total linear momentum is a constant of motion. This is the expected result because no external force is applied along the x-axis.

7-7.



If we take (ϕ_1, ϕ_2) as our generalized coordinates, the x,y coordinates of the two masses are

$$\left.\begin{aligned} x_1 &= \ell\cos\phi_1 \\ y_1 &= \ell\sin\phi_1 \end{aligned}\right] \tag{1}$$

$$\left.\begin{aligned} x_2 &= \ell\cos\phi_1+\ell\cos\phi_2 \\ y_2 &= \ell\sin\phi_1+\ell\sin\phi_2 \end{aligned}\right] \tag{2}$$

Using (1) and (2), we find the kinetic energy of the system to be

$$\begin{aligned} T &= \frac{m}{2}\left(\dot{x}_1^2+\dot{y}_1^2\right)+\frac{m}{2}\left(\dot{x}_2^2+\dot{y}_2^2\right) \\ &= \frac{m}{2}\ell^2\left[\dot{\phi}_1^2+\dot{\phi}_1^2+\dot{\phi}_2^2+2\dot{\phi}_1\dot{\phi}_2\left(\sin\phi_1\sin\phi_2+\cos\phi_1\cos\phi_2\right)\right] \\ &= \frac{m}{2}\ell^2\left[2\dot{\phi}_1^2+\dot{\phi}_2^2+2\dot{\phi}_1\dot{\phi}_2\cos\left(\phi_1-\phi_2\right)\right] \end{aligned} \tag{3}$$

The potential energy is

$$U = -mgx_1 - mgx_2 = -mg\ell\left[2\cos\phi_1 + \cos\phi_2\right] \tag{4}$$

Therefore, the Lagrangian is

$$L = m\ell^2\left[\dot{\phi}_1^2 + \frac{1}{2}\dot{\phi}_2^2 + \dot{\phi}_1\dot{\phi}_2\cos\left(\phi_1 - \phi_2\right)\right] + mg\ell\left[2\cos\phi_1 + \cos\phi_2\right] \tag{5}$$

from which

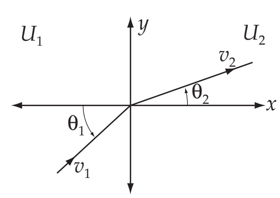
$$\left.\begin{aligned}
\frac{\partial L}{\partial \phi_1} &= m\ell^2\dot{\phi}_1\dot{\phi}_2\sin\left(\phi_1 - \phi_2\right) - 2mg\ell\sin\phi_1 \\
\frac{\partial L}{\partial \dot{\phi}_1} &= 2m\ell^2\dot{\phi}_1 + m\ell^2\dot{\phi}_2\cos\left(\phi_1 - \phi_2\right) \\
\frac{\partial L}{\partial \phi_2} &= -m\ell^2\dot{\phi}_1\dot{\phi}_2\sin\left(\phi_1 - \phi_2\right) - mg\ell\sin\phi_2 \\
\frac{\partial L}{\partial \dot{\phi}_2} &= m\ell^2\dot{\phi}_2 + m\ell^2\dot{\phi}_1\cos\left(\phi_1 - \phi_2\right)
\end{aligned}\right] \tag{6}$$

The Lagrange equations for ϕ_1 and ϕ_2 are

$$\boxed{2\phi_1 + \ddot{\phi}_2\cos\left(\phi_1 - \phi_2\right) + \dot{\phi}_2^2\sin\left(\phi_1 - \phi_2\right) + 2\frac{g}{\ell}\sin\phi_1 = 0} \tag{7}$$

$$\boxed{\ddot{\phi}_2 + \ddot{\phi}_1\cos\left(\phi_1 - \phi_2\right) - \dot{\phi}_1^2\sin\left(\phi_1 - \phi_2\right) + \frac{g}{\ell}\sin\phi_2 = 0} \tag{8}$$

7-8.



Let us choose the x,y coordinates so that the two regions are divided by the y axis:

$$U(x) = \begin{bmatrix} U_1 & x < 0 \\ U_2 & x > 0 \end{bmatrix}$$

If we consider the potential energy as a function of x as above, the Lagrangian of the particle is

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - U(x) \tag{1}$$

Therefore, Lagrange's equations for the coordinates x and y are

$$m\ddot{x} + \frac{dU(x)}{dx} = 0 \tag{2}$$

$$m\ddot{y} = 0 \tag{3}$$

Using the relation

$$m\ddot{x} = \frac{d}{dt} m\dot{x} = \frac{dP_x}{dt} = \frac{dP_x}{dx}\frac{dx}{dt} = \frac{P_x}{m}\frac{dp_x}{dx} \tag{4}$$

(2) becomes

$$\frac{P_x}{m}\frac{dP_x}{dx} + \frac{dU(x)}{dx} = 0 \tag{5}$$

Integrating (5) from any point in the region 1 to any point in the region 2, we find

$$\int_1^2 \frac{P_x}{m}\frac{dP_x}{dx}dx + \int_1^2 \frac{dU(x)}{dx}dx = 0 \tag{6}$$

$$\frac{P_{x_2}^2}{2m} - \frac{P_{x_1}^2}{2m} + U_2 - U_1 = 0 \tag{7}$$

or, equivalently,

$$\frac{1}{2}m\dot{x}_1^2 + U_1 = \frac{1}{2}m\dot{x}_2^2 + U_2 \tag{8}$$

Now, from (3) we have

$$\frac{d}{dt}m\dot{y} = 0$$

and $m\dot{y}$ is constant. Therefore,

$$m\dot{y}_1 = m\dot{y}_2 \tag{9}$$

From (9) we have

$$\frac{1}{2}m\dot{y}_1^2 = \frac{1}{2}m\dot{y}_2^2 \tag{10}$$

Adding (8) and (10), we have

$$\frac{1}{2}mv_1^2 + U_1 = \frac{1}{2}mv_2^2 + U_2 \tag{11}$$

From (9) we also have

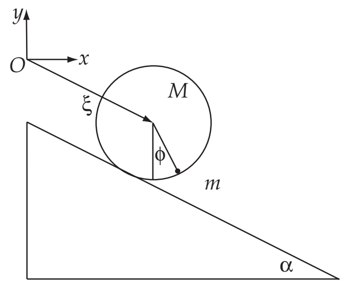
$$mv_1 \sin\theta_1 = mv_2 \sin\theta_2 \tag{12}$$

Substituting (11) into (12), we find

$$\boxed{\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_2}{v_1} = \left[1 + \frac{U_1 - U_2}{T_1}\right]^{1/2}} \tag{13}$$

This problem is the mechanical analog of the refraction of light upon passing from a medium of a certain optical density into a medium with a different optical density.

7-9.



Using the generalized coordinates given in the figure, the Cartesian coordinates for the disk are ($\xi\cos\alpha$, $-\xi\sin\alpha$), and for the bob they are ($\ell\sin\phi + \xi\cos\alpha$, $-\ell\cos\phi - \xi\sin\alpha$). The kinetic energy is given by

$$T = T_{\text{disk}} + T_{\text{bob}} = \left[\frac{1}{2}M\dot{\xi}^2 + \frac{1}{2}I\dot{\theta}^2\right] + \frac{1}{2}m\left(\dot{x}_{\text{bob}}^2 + \dot{y}_{\text{bob}}^2\right) \tag{1}$$

Substituting the coordinates for the bob, we obtain

$$T = \frac{1}{2}(M+m)\dot{\xi}^2 + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 + m\ell\dot{\phi}\dot{\xi}\cos(\phi + a) \tag{2}$$

The potential energy is given by

$$U = U_{\text{disk}} + U_{\text{bob}} = Mgy_{\text{disk}} + mgy_{\text{bob}} = -(M+m)g\xi\sin\alpha - mg\ell\cos\phi \tag{3}$$

Now let us use the relation $\xi = R\theta$ to reduce the degrees of freedom to two, and in addition substitute $I = MR^2/2$ for the disk. The Lagrangian becomes

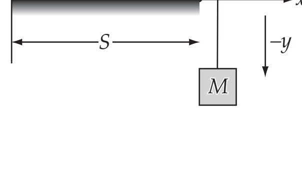
$$L = T - U = \left(\frac{3}{4}M + \frac{1}{2}m\right)\dot{\xi}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 + m\ell\dot{\phi}\dot{\xi}\cos(\phi + a) + (M+m)g\xi\sin\alpha + mg\ell\cos\phi \tag{4}$$

The resulting equations of motion for our two generalized coordinates are

$$\left(\frac{3}{2}M + m\right)\ddot{\xi} - (M+m)g\sin\alpha + m\ell\left[\ddot{\phi}\cos(\phi+\alpha) - \dot{\phi}^2\sin(\phi+\alpha)\right] = 0 \tag{5}$$

$$\ddot{\phi} + \frac{1}{\ell}\ddot{\xi}\cos(\phi+\alpha) + \frac{g}{\ell}\sin\phi = 0 \tag{6}$$

7-10.



Let the length of the string be ℓ so that

$$(S-x)-y=\ell \tag{1}$$

Then,

$$\dot{x}=-\dot{y} \tag{2}$$

a) The Lagrangian of the system is

$$L=\frac{1}{2}M\dot{x}^2+\frac{1}{2}M\dot{y}^2-Mgy=M\dot{y}^2-Mgy \tag{3}$$

Therefore, Lagrange's equation for y is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}}-\frac{\partial L}{\partial y}=2M\ddot{y}+Mg=0 \tag{4}$$

from which

$$\ddot{y}=-\frac{g}{2} \tag{5}$$

Then, the general solution for y becomes

$$y(t)=-\frac{g}{4}t^2+C_1t+C_2 \tag{6}$$

If we assign the initial conditions $y(t=0)=0$ and $\dot{y}(t=0)=0$, we find

$$\boxed{y(t)=-\frac{g}{4}t^2} \tag{7}$$

b) If the string has a mass m, we must consider its kinetic energy and potential energy. These are

$$T_{\text{string}}=\frac{1}{2}m\dot{y}^2 \tag{8}$$

$$U_{\text{string}}=-\frac{m}{\ell}yg\frac{y}{2}=-\frac{mg}{2\ell}y^2 \tag{9}$$

Adding (8) and (9) to (3), the total Lagrangian becomes

$$L=M\dot{y}^2-Mgy+\frac{1}{2}m\dot{y}^2+\frac{mg}{2\ell}y^2 \tag{10}$$

Therefore, Lagrange's equation for y now becomes

$$(2M+m)\ddot{y}-\frac{mg}{\ell}y+Mg=0 \tag{11}$$

In order to solve (11), we arrange this equation into the form

$$(2M+m)\ddot{y}=\frac{mg}{\ell}\left[y-\frac{M\ell}{m}\right] \tag{12}$$

Since $\dfrac{d^2}{dt^2}\left[y-\dfrac{M\ell}{m}\right]=\dfrac{d^2}{dt^2}y$, (12) is equivalent to

$$\frac{d^2}{dt^2}\left[y-\frac{M\ell}{m}\right]=\frac{mg}{\ell(2M+m)}\left[y-\frac{M\ell}{m}\right] \tag{13}$$

which is solved to give

$$y-\frac{M\ell}{m}=Ae^{\gamma t}+Be^{-\gamma t} \tag{14}$$

where

$$\gamma=\sqrt{\frac{mg}{\ell(2M+m)}} \tag{15}$$

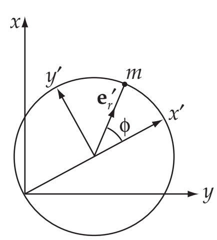
If we assign the initial condition $y(t=0)=0$; $\dot{y}(t=0)=0$, we have

$$A=+B=-\frac{M\ell}{2m}$$

Then,

$$\boxed{y(t)=\frac{M\ell}{m}\left(1-\cosh\gamma t\right)} \tag{16}$$

7-11.



The x,y coordinates of the particle are

$$\left.\begin{aligned} x&=R\cos\omega t+R\cos(\phi+\omega t)\\ y&=R\sin\omega t+R\sin(\phi+\omega t)\end{aligned}\right] \tag{1}$$

Then,

$$\left.\begin{aligned} \dot{x}&=-R\omega\sin\omega t-R\left(\dot{\phi}+\omega\right)\sin(\phi+\omega t)\\ \dot{y}&=R\omega\cos\omega t+R\left(\dot{\phi}+\omega\right)\cos(\phi+\omega t)\end{aligned}\right] \tag{2}$$

Since there is no external force, the potential energy is constant and can be set equal to zero. The Lagrangian becomes

$$L = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right)$$

$$= \frac{m}{2}\left[R^2\omega^2 + R^2\left(\phi + \omega\right)^2 + 2R^2\omega\left(\phi + \omega\right)\cos\phi\right] \tag{3}$$

from which

$$\frac{\partial L}{\partial \phi} = -mR^2\omega\left(\dot{\phi} + \omega\right)\sin\phi \tag{4}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{d}{dt}\left[mR^2\left(\dot{\phi} + \omega + \omega\cos\phi\right)\right] \tag{5}$$

Therefore, Lagrange's equation for ϕ becomes

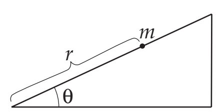
$$\boxed{\ddot{\phi} + \omega^2 \sin\phi = 0} \tag{6}$$

which is also the equation of motion for a simple pendulum. To make the result appear reasonable, note that we may write the acceleration felt by the particle in the rotating frame as

$$\mathbf{a} = \omega^2 R\left(\mathbf{i}' + \mathbf{e}_r'\right) \tag{7}$$

where the primed unit vectors are as indicated in the figure. The part proportional to $\mathbf{e}_r'$ does not affect the motion since it has no contribution to the torque, and the part proportional to $\mathbf{i}'$ is constant and does not contribute to the torque in the same way a constant gravitational field provides a torque to the simple pendulum.

7-12.



Put the origin at the bottom of the plane

$$L = T - U = \frac{1}{2} m\left(\dot{r}^2 + r^2\,\dot{\theta}^2\right) - mgr\sin\theta$$

$$\theta = \alpha t; \quad \dot{\theta} = \alpha$$

$$L = \frac{1}{2} m\left(\dot{r}^2 + \alpha^2 r^2\right) - mgr\sin\alpha t$$

Lagrange's equation for r gives

$$m\ddot{r} = m\alpha^2 r - mg\sin\alpha t$$

or

$$\ddot{r} - \alpha^2 r = -g\sin\alpha t \tag{1}$$

The general solution is of the form $r = r_p + r_h$ where r_h is the general solution of the homogeneous equation $\ddot{r} - \alpha^2 r = 0$ and r_p is a particular solution of Eq. (1).

So

$$r_h = Ae^{\alpha t} + Be^{-\alpha t}$$

For r_p, try a solution of the form $r_p = C \sin \alpha t$. Then $\ddot{r}_p = -C\,\alpha^2 \sin \alpha t$. Substituting into (1) gives

$$-C\,\alpha^2 \sin \alpha t - C\;\alpha^2 \sin \alpha t = -g \sin \alpha t$$

$$C = \frac{g}{2\alpha^2}$$

So

$$r(t) = Ae^{\alpha t} + Be^{-\alpha t} + \frac{g}{2\alpha^2} \sin \alpha t$$

We can determine A and B from the initial conditions:

$$r(0) = r_0 \tag{2}$$

$$\dot{r}(0) = 0 \tag{3}$$

(2) implies $r_0 = A + B$

(3) implies $0 = A - B + \dfrac{g}{2\alpha^2}$

Solving for A and B gives:

$$A = \frac{1}{2}\left[r_0 - \frac{g}{2\alpha^2}\right] \qquad B = \frac{1}{2}\left[r_0 + \frac{g}{2\alpha^2}\right]$$

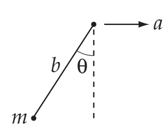
$$r(t) = \frac{1}{2}\left[r_0 - \frac{g}{2\alpha^2}\right]e^{\alpha t} + \frac{1}{2}\left[r_0 + \frac{g}{2\alpha^2}\right]e^{-\alpha t} + \frac{g}{2\alpha^2} \sin \alpha t$$

or

$$\boxed{r(t) = r_0 \cosh \alpha t + \frac{g}{2\alpha^2}\left(\sin \alpha t - \sinh \alpha t\right)}$$

7-13.

a)



$$x = \frac{1}{2}at^2 - b\sin\theta$$

$$y = -b\cos\theta$$

$$\dot{x} = at - b\dot{\theta}\cos\theta$$

$$\dot{y} = b\dot{\theta}\sin\theta$$

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$$

$$= \frac{1}{2}m\left(a^2t^2 - 2at\,b\dot{\theta}\cos\theta + b^2\dot{\theta}^2\right) + mgb\cos\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = \frac{\partial L}{\partial\theta} \quad \text{gives}$$

$$\frac{d}{dt}\left[-mat\,b\cos\theta + mb^2\dot{\theta}\right] = mat\,b\ddot{\theta}\sin\theta - mgb\sin\theta$$

This gives the equation of motion

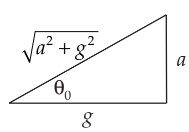
$$\boxed{\ddot{\theta} + \frac{g}{b}\sin\theta - \frac{a}{b}\cos\theta = 0}$$

b) To find the period for small oscillations, we must expand $\sin\theta$ and $\cos\theta$ about the equilibrium point θ_0. We find θ_0 by setting $\ddot{\theta} = 0$. For equilibrium,

$$g\sin\theta_0 = a\cos\theta_0$$

or

$$\tan\theta_0 = \frac{a}{g}$$



Using the first two terms in a Taylor series expansion for $\sin\theta$ and $\cos\theta$ gives

$$f(\theta) \simeq f(\theta_0) + f'(\theta)\Big]_{\theta=\theta_0}(\theta - \theta_0)$$

$$\sin\theta \simeq \sin\theta_0 + (\theta - \theta_0)\cos\theta_0$$

$$\cos\theta \simeq \cos\theta_0 - (\theta - \theta_0)\sin\theta_0$$

$$\tan\theta_0 = \frac{a}{g} \text{ implies } \sin\theta_0 = \frac{a}{\sqrt{a^2 + g^2}},$$

$$\cos\theta_0 = \frac{g}{\sqrt{a^2 + g^2}}$$

Thus

$$\sin\theta \simeq \frac{1}{\sqrt{a^2+g^2}}\left(a+g\theta-g\theta_0\right)$$

$$\cos\theta \simeq \frac{1}{\sqrt{a^2+g^2}}\left(g-a\theta+a\theta_0\right)$$

Substituting into the equation of motion gives

$$0=\ddot{\theta}+\frac{g}{b\sqrt{a^2+g^2}}\left(a+g\theta-g\theta_0\right)-\frac{a}{b\sqrt{a^2+g^2}}\left(g-a\theta+a\theta_0\right)$$

This reduces to

$$\ddot{\theta}+\frac{\sqrt{g^2+a^2}}{b}\theta=\frac{\sqrt{g^2+a^2}}{b}\theta_0$$

The solution to this inhomogeneous differential equation is

$$\theta=\theta_0+A\cos\omega\theta+B\sin\omega\theta$$

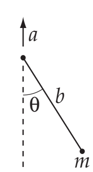
where

$$\omega=\frac{\left(g^2+a^2\right)^{1/4}}{b^{1/2}}$$

Thus

$$\boxed{T=\frac{2\pi}{\omega}=\frac{2\pi\, b^{1/2}}{\left(g^2+a^2\right)^{1/4}}}$$

7-14.



$$x=b\sin\theta$$

$$y=\frac{1}{2}at^2-b\cos\theta$$

$$\dot{x}=b\theta\cos\theta$$

$$\dot{y}=at+b\dot{\theta}\sin\theta$$

$$T=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)=\frac{1}{2}m\left(b^2\dot{\theta}^2+a^2t^2+2abt\dot{\theta}\sin\theta\right)$$

$$U = mgy = mg\left[\frac{1}{2}at^2 - b\cos\theta\right]$$

$$L = T - U = \frac{1}{2}m\left(b^2\dot{\theta}^2 + a^2t^2 + 2abt\dot{\theta}\sin\theta\right) + mg\left(b\cos\theta - \frac{1}{2}at^2\right)$$

Lagrange's equation for θ gives

$$\frac{d}{dt}\left[mb^2\dot{\theta} + mabt\sin\theta\right] = mabt\dot{\theta}\cos\theta - mgb\sin\theta$$

$$b^2\ddot{\theta} + ab\sin\theta + abt\dot{\theta}\cos\theta = abt\dot{\theta}\cos\theta - gb\sin\theta$$

$$\boxed{\ddot{\theta} + \frac{a+g}{b}\sin\theta = 0}$$

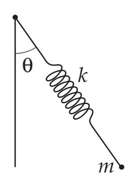
For small oscillations, $\sin\theta \simeq \theta$

$$\ddot{\theta} + \frac{a+g}{b}\theta = 0\ .$$

Comparing with $\ddot{\theta} + \omega^2\theta = 0$ gives

$$\boxed{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{b}{a+g}}}$$

7-15.



b = unextended length of spring

ℓ = variable length of spring

$$T = \frac{1}{2}m\left(\dot{\ell}^2 + \ell^2\,\dot{\theta}^2\right)$$

$$U = \frac{1}{2}k(\ell - b)^2 + mgy = \frac{1}{2}k(\ell - b)^2 - mg\,\ell\cos\theta$$

$$L = T - U = \frac{1}{2}m\left(\dot{\ell}^2 + \ell^2\,\dot{\theta}^2\right) - \frac{1}{2}(\ell - b)^2 + mg\,\ell\cos\theta$$

Taking Lagrange's equations for ℓ and θ gives

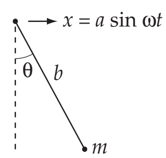
$$\ell : \frac{d}{dt}\left[m\dot{\ell}\right] = m\ell\dot{\theta}^2 - k(\ell - b) + mg\cos\theta$$

$$\theta: \frac{d}{dt}\left[m\ell^2\dot{\theta}\right] = -mg\ \ell \sin\theta$$

This reduces to

$$\boxed{\begin{aligned} &\ddot{\ell} - \ell\dot{\theta}^2 + \frac{k}{m}(\ell - b) - g\cos\theta = 0 \\ &\ddot{\theta} + \frac{2}{\ell}\dot{\ell}\dot{\theta} + \frac{g}{\ell}\sin\theta = 0 \end{aligned}}$$

7-16.



For mass m:

$$x = a\sin\omega t + b\sin\theta$$

$$y = -b\cos\theta$$

$$\dot{x} = a\omega\cos\omega t + b\dot{\theta}\cos\theta$$

$$\dot{y} = b\dot{\theta}\sin\theta$$

Substitute into

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$$

$$U = mgy$$

and the result is

$$L = T - U = \frac{1}{2}m\left(a^2\omega^2\cos^2\omega t + 2ab\omega\dot{\theta}\cos\omega t\cos\theta + b^2\dot{\theta}^2\right) + mgb\cos\theta$$

Lagrange's equation for θ gives

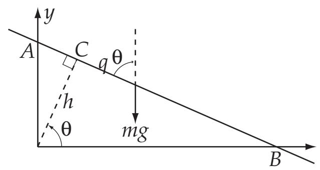
$$\frac{d}{dt}\left(mab\omega\cos\omega t\cos\theta + mb^2\dot{\theta}\right) = -mabw\dot{\theta}\cos\omega t\sin\theta - mgb\sin\theta$$

$$-ab\omega^2\sin\omega t\cos\theta - ab\omega\dot{\theta}\cos\omega t\sin\theta + b^2\ddot{\theta} = -\ ab\omega\dot{\theta}\cos\omega t\sin\theta - gb\sin\theta$$

or

$$\boxed{\ddot{\theta} + \frac{g}{b}\sin\theta - \frac{a}{b}\omega^2\sin\omega t\cos\theta = 0}$$

7-17.



Using q and $\theta\,(=\omega t$ since $\theta(0)=0)$, the x,y coordinates of the particle are expressed as

$$\left.\begin{aligned} x &= h\cos\theta + q\sin\theta = h\cos\omega t + q(t)\sin\omega t \\ y &= h\sin\theta - q\cos\theta = h\sin\omega t - q(t)\cos\omega t \end{aligned}\right] \tag{1}$$

from which

$$\left.\begin{aligned} \dot{x} &= -h\omega\sin\omega t + q\omega\cos\omega t + \dot{q}\sin\omega t \\ \dot{y} &= h\omega\cos\omega t + q\omega\sin\omega t - \dot{q}\cos\omega t \end{aligned}\right] \tag{2}$$

Therefore, the kinetic energy of the particle is

$$\begin{aligned} T &= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) \\ &= \frac{1}{2}m\left(h^2\omega^2 + q^2\omega^2 + \dot{q}^2\right) - mh\omega\dot{q} \end{aligned} \tag{3}$$

The potential energy is

$$U = mgy = mg\left(h\sin\omega t - q\cos\omega t\right) \tag{4}$$

Then, the Lagrangian for the particle is

$$L = \frac{1}{2}mh^2\omega^2 + \frac{1}{2}mq^2\omega^2 + \frac{1}{2}m\dot{q}^2 - mgh\sin\omega t + mgq\cos\omega t - mh\omega\dot{q} \tag{5}$$

Lagrange's equation for the coordinate is

$$\ddot{q} - \omega^2 q = g\cos\omega t \tag{6}$$

The complementary solution and the particular solution for (6) are written as

$$\left.\begin{aligned} q_c(t) &= A\cos\left(i\omega t + \delta\right) \\ q_P(t) &= -\frac{g}{2\omega^2}\cos\omega t \end{aligned}\right] \tag{7}$$

so that the general solution is

$$q(t) = A\cos\left(i\omega t + \delta\right) - \frac{g}{2\omega^2}\cos\omega t \tag{8}$$

Using the initial conditions, we have

$$\left.\begin{aligned} q(0) &= A\cos\delta - \frac{g}{2\omega^2} = 0 \\ \dot{q}(0) &= -i\omega A\sin\delta = 0 \end{aligned}\right] \tag{9}$$

Therefore,

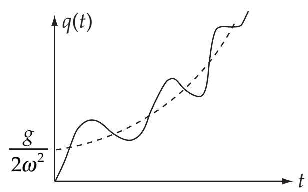
$$\delta = 0,\ A = \frac{g}{2\omega^2} \tag{10}$$

and

$$q(t) = \frac{g}{2\omega^2}\left(\cos i\omega t - \cos\omega t\right) \tag{11}$$

or,

$$\boxed{q(t) = \frac{g}{2\omega^2}\left(\cosh\omega t - \cos\omega t\right)} \tag{12}$$



In order to compute the Hamiltonian, we first find the canonical momentum of q. This is obtained by

$$p = \frac{\partial L}{\partial \dot{q}} = mq - m\omega h \tag{13}$$

Therefore, the Hamiltonian becomes

$$\begin{aligned} H &= p\dot{q} - L \\ &= m\dot{q}^2 - m\omega h\dot{q} - \frac{1}{2}m\omega^2h^2 - \frac{1}{2}m\omega^2q^2 - \frac{1}{2}m\dot{q}^2 + mgh\sin\omega t - mgq\cos\omega t + m\omega\dot{q}h \end{aligned}$$

so that

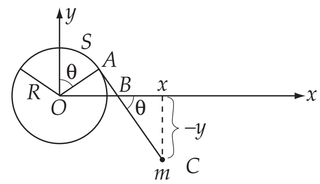
$$H = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2h^2 - \frac{1}{2}m\omega^2q^2 + mgh\sin\omega t - mgq\cos\omega t \tag{14}$$

Solving (13) for $\dot{q}$ and substituting gives

$$\boxed{H = \frac{p^2}{2m} + \omega hp - \frac{1}{2}m\omega^2q^2 + mgh\sin\omega t - mgq\cos\omega t} \tag{15}$$

The Hamiltonian is therefore different from the total energy, $T + U$. The energy is not conserved in this problem since the Hamiltonian contains time explicitly. (The particle gains energy from the gravitational field.)

7-18.



From the figure, we have the following relation:

$$\overline{AC} = \ell - s = \ell - R\theta \tag{1}$$

where θ is the generalized coordinate. In terms of θ, the x,y coordinates of the mass are

$$\left.\begin{aligned} x &= \overline{AC}\cos\theta + R\sin\theta = (\ell - R\theta)\cos\theta + R\sin\theta \\ y &= R\cos\theta - \overline{AC}\sin\theta = R\cos\theta - (\ell - R\theta)\sin\theta \end{aligned}\right] \tag{2}$$

from which

$$\left.\begin{aligned} \dot{x} &= R\theta\dot{\theta}\sin\theta - \ell\dot{\theta}\sin\theta \\ \dot{y} &= R\theta\dot{\theta}\cos\theta - \ell\dot{\theta}\cos\theta \end{aligned}\right] \tag{3}$$

Therefore, the kinetic energy becomes

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m\left[\ell^2\dot{\theta}^2 + R^2\theta^2\dot{\theta}^2 - 2R\ell\theta\dot{\theta}^2\right] \tag{4}$$

The potential energy is

$$U = mgy = mg\left[R\cos\theta - (\ell - R\theta)\sin\theta\right] \tag{5}$$

Then, the Lagrangian is

$$L = T - U = \frac{1}{2}m\left[\ell^2\dot{\theta}^2 + R^2\theta^2\dot{\theta}^2 - 2R\ell\theta\dot{\theta}^2\right] - mg\left[R\cos\theta - (\ell - R\theta)\sin\theta\right] \tag{6}$$

Lagrange's equation for θ is

$$(\ell - R\theta)\ddot{\theta} - R\dot{\theta}^2 - g\cos\theta = 0 \tag{7}$$

Now let us expand about some angle θ_0, and assume the deviations are small. Defining $\varepsilon \equiv \theta - \theta_0$, we obtain

$$\ddot{\varepsilon} + \frac{g\sin\theta_0}{\ell - R\theta_0}\varepsilon = \frac{g\cos\theta_0}{\ell - R\theta_0} \tag{8}$$

The solution to this differential equation is

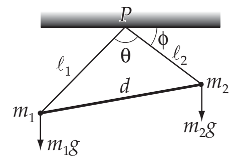
$$\varepsilon = A\sin(\omega t + \delta) + \frac{\cos\theta_0}{\sin\theta_0} \tag{9}$$

where A and δ are constants of integration and

$$\omega \equiv \sqrt{\frac{g \sin \theta_0}{\ell - R\theta_0}} \tag{10}$$

is the frequency of small oscillations. It is clear from (9) that θ extends equally about θ_0 when $\theta_0 = \pi/2$.

7-19.



Because of the various constraints, only one generalized coordinate is needed to describe the system. We will use ϕ, the angle between a plane through P perpendicular to the direction of the gravitational force vector, and one of the extensionless strings, e.g., ℓ_2, as our generalized coordinate.

The, the kinetic energy of the system is

$$T = \frac{1}{2} m_1 \left(\ell_1 \dot{\phi}\right)^2 + \frac{1}{2} m_2 \left(\ell_2 \dot{\phi}\right)^2 \tag{1}$$

The potential energy is given by

$$U = -m_1 g \ell_1 \sin\left(\pi - (\phi + \theta)\right) - m_2 g \ell_2 \sin \phi \tag{2}$$

from which the Lagrangian has the form

$$L = T - U = \frac{1}{2}\left(m_1 \ell_1^2 + m_2 \ell_2^2\right)\dot{\phi}^2 + m_1 g \ell_1 \sin(\phi + \theta) + m_2 g \ell_2 \sin \phi \tag{3}$$

The Lagrangian equation for ϕ is

$$m_2 g \ell_2 \cos \phi + m_1 g \ell_1 \cos(\phi + \theta) - \left(m_1 \ell_1^2 + m_2 \ell_2^2\right)\ddot{\phi} = 0 \tag{4}$$

This is the equation which describes the motion in the plane m_1, m_2, P.

To find the frequency of small oscillations around the equilibrium position (defined by $\phi = \phi_0$), we expand the potential energy U about ϕ_0:

$$\begin{aligned} U(\phi) &= U(\phi_0) + U'(\phi_0)\phi + \frac{1}{2} U''(\phi_0)\phi^2 + \ldots \\ &= \frac{1}{2} U''(\phi_0)\phi^2 \end{aligned} \tag{5}$$

where the last equality follows because we can take $U(\phi_0) = 0$ and because $U'(\phi_0) = 0$.

From (4) and (5), the frequency of small oscillations around the equilibrium position is

$$\omega^2 = \frac{U''(\phi_0)}{m_1\ell_1^2 + m_2\ell_2^2} \tag{6}$$

The condition $U'(\phi_0) = 0$ gives

$$\tan\phi_0 = \frac{m_2\ell_2 + m_1\ell_1\cos\theta}{m_1\ell_1\sin\theta} \tag{7}$$

or,

$$\sin\phi_0 = \frac{m_2\ell_2 + m_1\ell_1\cos\theta}{\left(m_1^2\ell_1^2 + m_2^2\ell_2^2 + 2m_1m_2\ell_1\ell_2\cos\theta\right)^{1/2}} \tag{8}$$

Then from (2), (7), and (8), $U''(\phi_0)$ is found to be

$$\begin{aligned} U''(\phi_0) &= g\sin\phi_0\left(m_2\ell_2 + m_1\ell_1\cos\theta + m_1\ell_1\sin\theta\cot\phi_0\right) \\ &= \frac{g\left(m_2\ell_2 + m_1\ell_1\cos\theta\right)}{\left(m_1^2\ell_1^2 + m_2^2\ell_2^2 + 2m_2m_1\ell_2\ell_1\cos\theta\right)^{1/2}}\left[m_2\ell_2 + m_1\ell_1\cos\theta + \frac{m_1^2\ell_1^2\sin^2\theta}{m_2\ell_2 + m_1\ell_1\cos\theta}\right] \\ &= g\left(m_1^2\ell_1^2 + m_2^2\ell_2^2 + 2m_2m_1\ell_2\ell_1\cos\theta\right)^{1/2} \end{aligned} \tag{9}$$

Finally, from (6) and (9), we have

$$\omega^2 = g\frac{\left(m_1^2\ell_1^2 + m_2^2\ell_2^2 + 2m_1m_2\ell_1\ell_2\cos\theta\right)^{1/2}}{\left(m_1\ell_1^2 + m_2\ell_2^2\right)} \tag{10}$$

which, using the relation,

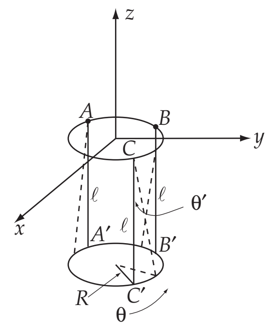
$$\cos\theta = \frac{\ell_1^2 + \ell_2^2 - d^2}{2\ell_1\ell_2} \tag{11}$$

can be written as

$$\boxed{\omega^2 = \frac{g\left[\left(m_1 + m_2\right)\left(m_1\ell_1^2 + m_2\ell_2^2\right) - d^2m_1m_2\right]^{1/2}}{\left(m_1\ell_1^2 + m_2\ell_2^2\right)}} \tag{12}$$

Notice that ω^2 degenerates to the value g/ℓ appropriate for a simple pendulum when d $\rightarrow 0$ (so that $\ell_1 = \ell_2$).

7-20. The x-y plane is horizontal, and A, B, C are the fixed points lying in a plane above the hoop. The hoop rotates about the vertical through its center.



The kinetic energy of the system is given by

$$T = \frac{1}{2} I\omega^2 + \frac{1}{2} M\dot{z}^2 = \frac{MR^2}{2}\dot{\theta}^2 + \frac{1}{2} M\left[\frac{\partial z}{\partial \theta}\right]^2 \dot{\theta}^2 \tag{1}$$

For small θ, the second term can be neglected since $\left(\partial z/\partial \theta\right)\Big|_{\theta=0} = 0$

The potential energy is given by

$$U = Mgz \tag{2}$$

where we take $U = 0$ at $z = -\ell$.

Since the system has only one degree of freedom we can write z in terms of θ. When $\theta = 0$, $z = -\ell$. When the hoop is rotated thorough an angle θ, then

$$z^2 = \ell^2 - \left(R - R\cos\theta\right)^2 - \left(R\sin\theta\right)^2 \tag{3}$$

so that

$$z = -\left[\ell^2 + 2R^2\left(\cos\theta - 1\right)\right]^{1/2} \tag{4}$$

and the potential energy is given by

$$U = -Mg\left[\ell^2 + 2R^2\left(\cos\theta - 1\right)\right]^{1/2} \tag{5}$$

for small θ, $\cos\theta - 1 \cong -\theta^2/2$; then,

$$\begin{aligned} U &\cong -Mg\ell\left[1 - \frac{R^2\theta^2}{\ell^2}\right]^{1/2} \\ &\cong -Mg\ell\left[1 - \frac{R^2\theta^2}{2\ell^2}\right] \end{aligned} \tag{6}$$

From (1) and (6), the Lagrangian is

$$L = T - U = \frac{1}{2} MR^2\dot{\theta}^2 + Mg\ell\left[1 - \frac{R^2\theta^2}{2\ell^2}\right], \tag{7}$$

for small θ. The Lagrange equation for θ gives

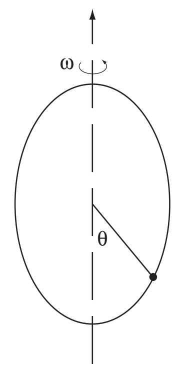
$$\ddot{\theta}+\frac{g}{\ell}\theta=0 \tag{8}$$

where

$$\boxed{\omega=\sqrt{\frac{g}{\ell}}} \tag{9}$$

which is the frequency of small rotational oscillations about the vertical through the center of the hoop and is the same as that for a simple pendulum of length ℓ.

7-21.



From the figure, we can easily write down the Lagrangian for this system.

$$T=\frac{mR^2}{2}\left(\dot{\theta}^2+\omega^2\sin^2\theta\right) \tag{1}$$

$$U=-mgR\cos\theta \tag{2}$$

The resulting equation of motion for θ is

$$\ddot{\theta}-\omega^2\sin\theta\cos\theta+\frac{g}{R}\sin\theta=0 \tag{3}$$

The equilibrium positions are found by finding the values of θ for which

$$0=\ddot{\theta}\Big|_{\theta=\theta_0}=\left(\omega^2\cos\theta_0-\frac{g}{R}\right)\sin\theta_0 \tag{4}$$

Note first that 0 and π are equilibrium, and a third is defined by the condition

$$\cos\theta_0=\frac{g}{\omega^2R} \tag{5}$$

To investigate the stability of each of these, expand using $\varepsilon=\theta-\theta_0$

$$\ddot{\varepsilon}=\omega^2\left(\cos\theta_0-\frac{g}{\omega^2R}-\varepsilon\sin\theta_0\right)\left(\sin\theta_0+\varepsilon\cos\theta_0\right) \tag{6}$$

For $\theta_0 = \pi$, we have

$$\ddot{\varepsilon} = \omega^2 \left(1 + \frac{g}{\omega^2 R}\right) \varepsilon \tag{7}$$

indicating that it is unstable. For $\theta_0 = 0$, we have

$$\ddot{\varepsilon} = \omega^2 \left(1 - \frac{g}{\omega^2 R}\right) \varepsilon \tag{8}$$

which is stable if $\omega^2 < g/R$ and unstable if $\omega^2 > g/R$. When stable, the frequency of small oscillations is $\sqrt{\omega^2 - g/R}$. For the final candidate,

$$\ddot{\varepsilon} = -\omega^2 \sin^2 \theta_0 \varepsilon \tag{9}$$

with a frequency of oscillations of $\sqrt{\omega^2 - (g/\omega R)^2}$, when it exists. Defining a critical frequency $\omega_c^2 \equiv g/R$, we have a stable equilibrium at $\theta_0 = 0$ when $\omega < \omega_c$, and a stable equilibrium at $\theta_0 = \cos^{-1}\left(\omega_c^2/\omega^2\right)$ when $\omega \geq \omega_c$. The frequencies of small oscillations are then $\omega\sqrt{1 - (\omega_c/\omega)^2}$ and $\omega\sqrt{1 - (\omega_c/\omega)^4}$, respectively.

To construct the phase diagram, we need the Hamiltonian

$$H \equiv \dot{\theta} \frac{\partial L}{\partial \dot{\theta}} - L \tag{10}$$

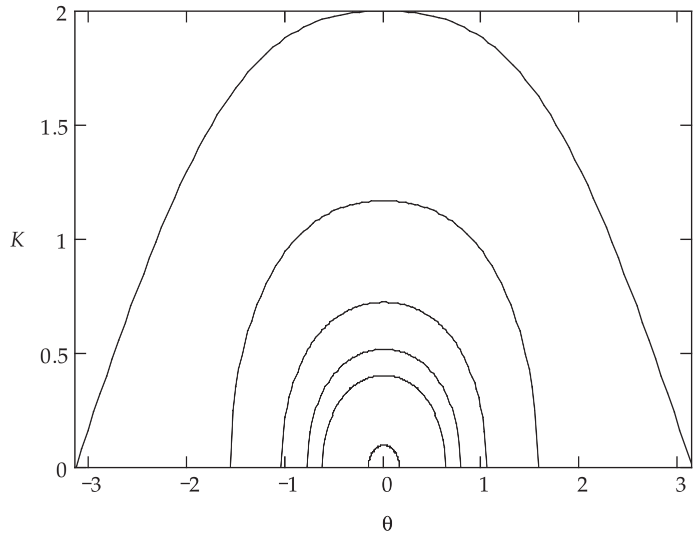
which is not the total energy in this case. A convenient parameter that describes the trajectory for a particular value of *H* is

$$K \equiv \frac{H}{m\omega_c^2 R^2} = \frac{1}{2}\left[\left(\frac{\dot{\theta}}{\omega_c}\right)^2 - \left(\frac{\omega}{\omega_c}\right)^2 \sin^2 \theta\right] - \cos\theta \tag{11}$$

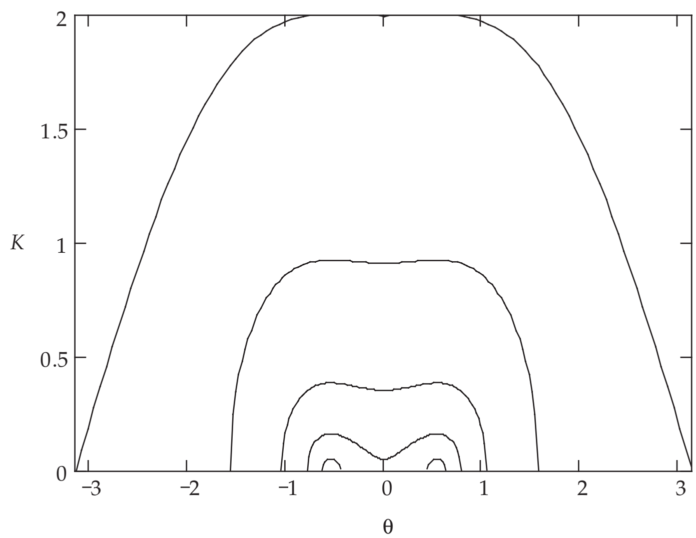
so that we'll end up plotting

$$\left(\frac{\dot{\theta}}{\omega_c}\right)^2 = 2\left(K + \cos\theta\right) + \left(\frac{\omega}{\omega_c}\right)^2 \sin^2 \theta \tag{12}$$

for a particular value of ω and for various values of *K*. The results for $\omega < \omega_c$ are shown in figure (b), and those for $\omega > \omega_c$ are shown in figure (c). Note how the origin turns from an attractor into a separatrix as ω increases through ω_c. As such, the system could exhibit chaotic behavior in the presence of damping.



(b)



(c)

7-22. The potential energy U which gives the force

$$F(x,t) = \frac{k}{x^2}\, e^{-(t/\tau)} \tag{1}$$

must satisfy the relation

$$F = -\frac{\partial U}{\partial x} \tag{2}$$

we find

$$U = \frac{k}{x}\, e^{-t/\tau} \tag{3}$$

Therefore, the Lagrangian is

$$\boxed{L = T - U = \frac{1}{2} m\dot{x}^2 - \frac{k}{x} e^{-t/\tau}} \tag{4}$$

The Hamiltonian is given by

$$H = p_x \dot{x} - L = \dot{x}\frac{\partial L}{\partial \dot{x}} - L \tag{5}$$

so that

$$\boxed{H = \frac{p_x^2}{2m} + \frac{k}{x} e^{-t/\tau}} \tag{6}$$

The Hamiltonian is equal to the total energy, $T + U$, because the potential does not depend on velocity, but the total energy of the system is not conserved because H contains the time explicitly.

7-23. The Hamiltonian function can be written as [see Eq. (7.153)]

$$H = \sum_j p_j \dot{q}_j - L \tag{1}$$

For a particle which moves freely in a conservative field with potential U, the Lagrangian in rectangular coordinates is

$$L = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - U$$

and the linear momentum components in rectangular coordinates are

$$\left.\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} = m\dot{x} \\ p_y &= m\dot{y} \\ p_z &= m\dot{z} \end{aligned}\right] \tag{2}$$

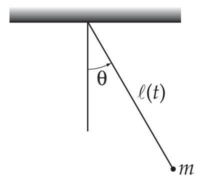
$$\begin{aligned} H &= \left[m\dot{x}^2 + m\dot{y}^2 + m\dot{z}^2\right] - \left[\frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - U\right] \\ &= \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + U = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) \end{aligned} \tag{3}$$

which is just the total energy of the particle. The canonical equations are [from Eqs. (7.160) and (7.161)]

$$\boxed{\begin{aligned} \dot{p}_x &= m\ddot{x} = -\frac{\partial U}{\partial x} = F_x \\ \dot{p}_y &= m\ddot{y} = -\frac{\partial U}{\partial y} = F_y \\ \dot{p}_z &= m\ddot{z} = -\frac{\partial U}{\partial z} = F_z \end{aligned}} \tag{4}$$

These are simply Newton's equations.

7-24.



The kinetic energy and the potential energy of the system are expressed as

$$\left.\begin{aligned} T &= \frac{1}{2} m\left(\dot{\ell}^2 + \ell^2\dot{\theta}^2\right) = \frac{1}{2} m\left(\alpha^2 + \ell^2\dot{\theta}^2\right) \\ U &= -mg\ell\cos\theta \end{aligned}\right] \tag{1}$$

so that the Lagrangian is

$$L = T - U = \frac{1}{2} m\left(\alpha^2 + \ell^2\dot{\theta}^2\right) + mg\ell\cos\theta \tag{2}$$

The Hamiltonian is

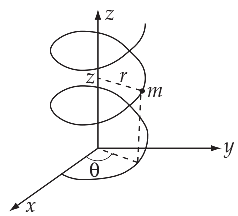
$$H = p_\theta\dot{\theta} - L = \frac{\partial L}{\partial\dot{\theta}}\dot{\theta} - L$$

$$= \boxed{\frac{p_\theta^2}{2m\ell^2} - \frac{1}{2} m\alpha^2 - mg\ell\cos\theta} \tag{3}$$

which is different from the total energy, $T + U$. The total energy is not conserved in this system because work is done on the system and we have

$$\frac{d}{dt}(T + U) \neq 0 \tag{4}$$

7-25.



In cylindrical coordinates the kinetic energy and the potential energy of the spiraling particle are expressed by

$$\left.\begin{aligned} T &= \frac{1}{2} m\left[\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right] \\ U &= mgz \end{aligned}\right] \tag{1}$$

Therefore, if we use the relations,

$$\left.\begin{aligned} &z = k\theta \qquad \text{i.e., } \dot{z} = k\dot{\theta} \\ &r = \text{const.} \end{aligned}\right] \tag{2}$$

the Lagrangian becomes

$$L = \frac{1}{2} m\left[\frac{r^2}{k^2}\dot{z}^2 + \dot{z}^2\right] - mgz \tag{3}$$

Then the canonical momentum is

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\left[\frac{r^2}{k^2} + 1\right]\dot{z} \tag{4}$$

or,

$$\dot{z} = \frac{p_z}{m\left[\dfrac{r^2}{k^2} + 1\right]} \tag{5}$$

The Hamiltonian is

$$H = p_z\,\dot{z} - L = p_z\,\frac{p_z}{m\left[\dfrac{r^2}{k^2} + 1\right]} - \frac{p_z^2}{2m\left[\dfrac{r^2}{k^2} + 1\right]} + mgz \tag{6}$$

or,

$$H = \frac{1}{2}\,\frac{p_z^2}{m\left[\dfrac{r^2}{k^2} + 1\right]} + mgz \tag{7}$$

Now, Hamilton's equations of motion are

$$-\frac{\partial H}{\partial z} = \dot{p}_z\,; \qquad \frac{\partial H}{\partial p_z} = \dot{z} \tag{8}$$

so that

$$-\frac{\partial H}{\partial z} = -mg = \dot{p}_z \tag{9}$$

$$\frac{\partial H}{\partial p_z} = \frac{p_z}{m\left[\dfrac{r^2}{k^2}+1\right]} = \dot{z} \tag{10}$$

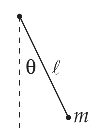
Taking the time derivative of (10) and substituting (9) into that equation, we find the equation of motion of the particle:

$$\boxed{\ddot{z} = \frac{g}{\left[\dfrac{r^2}{k^2}+1\right]}} \tag{11}$$

It can be easily shown that Lagrange's equation, computed from (3), gives the same result as (11).

7-26.

a)



$$L = T - U = \frac{1}{2}m\ell^2\dot{\theta}^2 - mgy$$

$$L = \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell\cos\theta$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m\ell^2\dot{\theta}$$

so

$$\dot{\theta} = \frac{p_\theta}{m\ell^2}$$

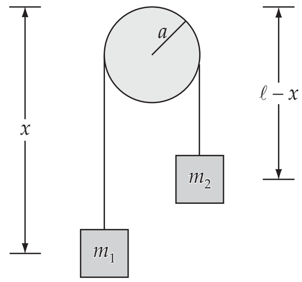
Since U is velocity-independent and the coordinate transformations are time-independent, the Hamiltonian is the total energy

$$\boxed{H = T + U = \frac{p_\theta^2}{2m\ell^2} - mg\ell\cos\theta}$$

The equations of motion are

$$\boxed{\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m\ell^2} \text{ and } \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mg\ell \sin\theta}$$

b)



$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} I \frac{\dot{x}^2}{a^2}$$

where I = moment of inertia of the pulley

$$U = -m_1 g x - m_2 g(\ell - x)$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = \left[m_1 + m_2 + \frac{I}{a^2} \right] \dot{x}$$

So

$$\dot{x} = \frac{p_x}{\left[m_1 + m_2 + \frac{I}{a^2} \right]}$$

$$H = T + U$$

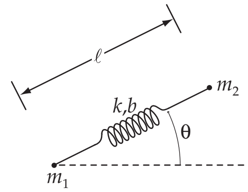
$$\boxed{H = \frac{p_x^2}{2\left[m_1 + m_2 + \frac{I}{a^2} \right]} - m_1 g x - m_2 g(\ell - x)}$$

The equations of motion are

$$\boxed{\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p_x} = \frac{p_x}{2\left[m_1 + m_2 + \frac{I}{a^2} \right]} \\ \dot{p}_x &= -\frac{\partial H}{\partial p_x} = m_1 g - m_2 g = g(m_1 - m_2) \end{aligned}}$$

7-27.

a)



$$x_i,\, y_i = \text{coordinates of } m_i$$

Using ℓ, θ as polar coordinates

$$x_2 = x_1 + \ell \cos\theta$$

$$y_2 = y_1 + \ell \sin\theta$$

$$\dot{x}_2 = \dot{x}_1 + \dot{\ell}\cos\theta - \ell\dot{\theta}\sin\theta \tag{1}$$

$$\dot{y}_2 = \dot{y}_1 + \dot{\ell}\sin\theta + \ell\dot{\theta}\cos\theta \tag{2}$$

If we substitute (1) and (2) into

$$L = T - U = \frac{1}{2}m_1\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2}m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right) - \frac{1}{2}k\left(\ell - b\right)^2$$

the result is

$$L = \frac{1}{2}\left(m_1 + m_2\right)\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2}m_2\left(\dot{\ell}^2 + \ell^2\dot{\theta}^2\right)$$
$$+m_2\dot{\ell}\left(\dot{x}_1\cos\theta + \dot{y}_1\sin\theta\right) + m_2\ell\dot{\theta}\left(\dot{y}_1\cos\theta - \dot{x}_1\sin\theta\right) - \frac{1}{2}k\left(\ell - b\right)^2$$

The equations of motion are

$$x_1: \frac{d}{dt}\left[\left(m_1 + m_2\right)\dot{x}_1 + m_2\dot{\ell}\cos\theta - m_2\ell\dot{\theta}\sin\theta\right] = 0$$

$$= m_1\dot{x}_1 + m_2\dot{x}_2 = p_x$$

So p_x = constant

$$y_1: \frac{d}{dt}\left[\left(m_1 + m_2\right)\dot{y}_1 + m_2\dot{\ell}\sin\theta + m_2\ell\dot{\theta}\cos\theta\right] = 0$$

$$= m_1\dot{y}_1 + m_2\dot{y}_2 = p_y$$

So p_y = constant

$$\ell: \frac{d}{dt}\left[m_2\dot{\ell} + m_2\left(\dot{x}_1\cos\theta + \dot{y}_1\sin\theta\right)\right] = m_2\ell\dot{\theta}^2 - k\left(\ell - b\right) + m_2\dot{\theta}\left(\dot{y}_1\cos\theta - \dot{x}_1\sin\theta\right)$$

which reduces to

$$\boxed{\ddot{\ell} - \ell\dot{\theta}^2 + \ddot{x}_1 \cos\theta + \ddot{y}_1 \sin\theta + \frac{k}{m_2}(\ell - b) = 0}$$

$$\theta: \frac{d}{dt}\left[m_2\ell^2\dot{\theta} + m_2\ell\left(\dot{y}_1 \cos\theta - \dot{x}_1 \sin\theta\right)\right]$$
$$= -m_2\dot{\ell}\left(\dot{x}_1 \sin\theta - \dot{y}_1 \cos\theta\right) + m_2\ell\dot{\theta}\left(-\dot{x}_1 \cos\theta - \dot{y}_1 \sin\theta\right)$$

which reduces to

$$\boxed{\ddot{\theta} + \frac{2}{\ell}\dot{\ell}\dot{\theta} + \frac{\cos\theta}{\ell}\ddot{y}_1 - \frac{\sin\theta}{\ell}\ddot{x}_1 = 0}$$

b) As was shown in (a)

$$\frac{\partial L}{\partial \dot{x}_1} = p_x = \text{constant}$$

$$\frac{\partial L}{\partial \dot{y}_1} = p_y = \text{constant} \qquad \text{(total linear momentum)}$$

c) Using L from part (a)

$$p_{x_1} = \frac{\partial L}{\partial \dot{x}_1} = \left(m_1 + m_2\right)\dot{x}_1 + m_2\dot{\ell}\cos\theta - m_2\ell\dot{\theta}\sin\theta$$

$$p_{y_1} = \frac{\partial L}{\partial \dot{y}_1} = \left(m_1 + m_2\right)\dot{y}_1 + m_2\dot{\ell}\sin\theta - m_2\ell\dot{\theta}\cos\theta$$

$$p_\ell = \frac{\partial L}{\partial \dot{\ell}} = m_2\dot{x}_1 \cos\theta + m_2\dot{y}_1 \sin\theta + m_2\dot{\ell}$$

$$p_\theta = -m_2\ell\dot{x}_1 \sin\theta + m_2\ell\dot{y}_1 \cos\theta + m_2\ell^2\dot{\theta}$$

Inverting these equations gives (after much algebra)

$$\dot{x}_1 = \frac{1}{m_1}\left[p_{x_1} - p_\ell \cos\theta + \frac{\sin\theta}{\ell}p_\theta\right]$$

$$\dot{y}_1 = \frac{1}{m_1}\left[p_{y_1} - p_\ell \sin\theta - \frac{\cos\theta}{\ell}p_\theta\right]$$

$$\dot{\ell} = \frac{1}{m_1}\left[-p_{x_1}\cos\theta - p_{y_1}\sin\theta + \frac{m_1 + m_2}{m_2}p_\ell\right]$$

$$\dot{\theta} = \frac{1}{m_1\ell}\left[p_{x_1}\sin\theta - p_{y_1}\cos\theta + \frac{m_1 + m_2}{m_2\ell}p_\theta\right]$$

Since the coordinate transformations are time independent, and U is velocity independent,

$$H = T + U$$

Substituting using the above equations for $\dot{q}_i$ in terms of p_i gives

$$H = \frac{1}{2m_1}\left[p_{x_1}^2 + p_{y_1}^2 + \frac{m_1+m_2}{m_2}\left[p_\ell^2 + \frac{p_\theta^2}{\ell^2}\right] - 2p_\ell\left(p_{x_1}\cos\theta + p_{y_1}\sin\theta\right) + 2\frac{p_\theta}{\ell}\left(p_{x_1}\sin\theta - p_{y_1}\cos\theta\right)\right] + \frac{1}{2}k(\ell-b)^2$$

The equations of motion are

$$\dot{x}_1 = \frac{\partial H}{\partial p_{x_1}} = \frac{1}{m_1}\left[p_{x_1} - p_\ell\cos\theta + \frac{\sin\theta}{\ell}p_\theta\right]$$

$$\dot{y}_1 = \frac{\partial H}{\partial p_{y_1}} = \frac{1}{m_1}\left[p_{y_1} - p_\ell\sin\theta - \frac{\cos\theta}{\ell}p_\theta\right]$$

$$\dot{\ell} = \frac{\partial H}{\partial p_\ell} = \frac{1}{m_1}\left[\frac{m_1+m_2}{m_2}p_\ell - p_{x_1}\cos\theta - p_{y_1}\sin\theta\right]$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{1}{m_1\ell}\left[\frac{m_1+m_2}{m_2\ell}p_\theta + p_{x_1}\sin\theta - p_{y_1}\cos\theta\right]$$

$$\dot{p}_{x_1} = -\frac{\partial H}{\partial x_1} = 0 \qquad \dot{p}_{y_1} = -\frac{\partial H}{\partial y_1} = 0$$

$$\dot{p}_\ell = -\frac{\partial H}{\partial \ell} = \frac{(m_1+m_2)p_\theta^2}{m_1m_2\ell^3} + \frac{p_\theta}{m_1\ell^2}\left(p_{x_1}\sin\theta - p_{y_1}\cos\theta\right) - k(\ell-b)$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\ell}{m_1}\left(-p_{x_1}\sin\theta + p_{y_1}\cos\theta\right) - \frac{p_\theta}{m_1\ell}\left[p_{x_1}\cos\theta + p_{y_1}\sin\theta\right]$$

Note: This solution chooses as its generalized coordinates what the student would most likely choose at this point in the text. If one looks ahead to Section 8.2 and 8.3, however, it would show another choice of generalized coordinates that lead to three cyclic coordinates (x_{CM}, y_{CM}, and θ), as shown in those sections.

7-28. $F = -kr^{-2}$ so $U = -kr^{-1}$

$$L = T - U = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{k}{r}$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad \text{so} \quad \dot{r} = \frac{p_r}{m}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \quad \text{so} \quad \dot{\theta} = \frac{p_\theta}{mr^2}$$

Since the coordinate transformations are independent of t, and the potential energy is velocity-independent, the Hamiltonian is the total energy.

$$H = T + U = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{k}{r}$$

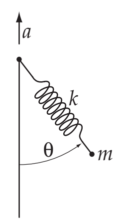
$$= \frac{1}{2} m\left[\frac{p_r^2}{m^2} + r^2 \frac{p_\theta^2}{m^2 r^4}\right] - \frac{k}{r}$$

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{k}{r}$$

Hamilton's equations of motion are

$$\boxed{\begin{aligned} &\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \qquad\qquad \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \\ &\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - \frac{k}{r^2} \\ &\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \end{aligned}}$$

7-29.



b = unextended length of spring

ℓ = variable length of spring

a) $x = \ell \sin\theta \qquad \dot{x} = \dot{\ell}\sin\theta + \ell\dot{\theta}\cos\theta$

$$y = \frac{1}{2}at^2 - \ell\cos\theta \qquad \dot{y} = at - \dot{\ell}\cos\theta + \ell\dot{\theta}\sin\theta$$

Substituting into $T = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right)$

$$U = mgy + \frac{1}{2}k\left(\ell - b\right)^2$$

gives

$$L = T - U = \frac{1}{2}m\left[\dot{\ell}^2 + \ell^2\dot{\theta}^2 + a^2t^2 + 2at\left(\ell\dot{\theta}\sin\theta - \dot{\ell}\cos\theta\right)\right] + mg\left(\ell\cos\theta - \frac{at^2}{2}\right) - \frac{k}{2}\left(\ell - b\right)^2$$

Lagrange's equations give:

$$\ell : \frac{d}{dt}\left[m\dot{\ell} - amt\cos\theta\right] = m\ell\dot{\theta}^2 + mat\dot{\theta}\sin\theta + mg\cos\theta - k(\ell - b)$$

$$\theta : \frac{d}{dt}\left[m\ell^2\dot{\theta} + mat\ell\sin\theta\right] = mat\dot{\ell}\sin\theta - mg\ell\sin\theta + mat\ell\dot{\theta}\cos\theta$$

Upon simplifying, the equations of motion reduce to:

$$\boxed{\begin{aligned} \ddot{\ell} - \ell\dot{\theta}^2 - (a+g)\cos\theta + \frac{k}{m}(\ell - b) = 0 \\ \ddot{\theta} + \frac{2}{\ell}\dot{\ell}\dot{\theta} + \frac{a+g}{\ell}\sin\theta = 0 \end{aligned}}$$

b)

$$p_\ell = \frac{\partial L}{\partial \dot{\ell}} = m\dot{\ell} - mat\cos\theta \quad \text{or} \quad \dot{\ell} = \frac{p_\ell}{m} + at\cos\theta \tag{1}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m\ell^2\dot{\theta} + mat\ell\sin\theta$$

$$\text{or} \quad \dot{\theta} = \frac{p_\theta}{m\ell^2} - \frac{at\sin\theta}{\ell} \tag{2}$$

Since the transformation equations relating the generalized coordinates to rectangular coordinates are not time-independent, the Hamiltonian is not the total energy.

$$H = \sum p_i\dot{q}_i - L = p_\ell\dot{\ell} + p_\theta\dot{\theta} - L$$

Substituting (1) and (2) for $\dot{\ell}$ and $\dot{\theta}$ and simplifying gives

$$H = \frac{p_\ell^2}{2m} + \frac{p_\theta^2}{2m\ell^2} - \frac{at}{\ell}p_\theta\sin\theta + atp_\ell\cos\theta + \frac{1}{2}k(\ell - b)^2 + \frac{1}{2}mgat^2 - mg\ell\cos\theta$$

The equations for $\dot{\theta}$ and $\dot{\ell}$ are

$$\boxed{\begin{aligned} &\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m\ell^2} - \frac{at}{\ell}\sin\theta \\ &\dot{\ell} = \frac{\partial H}{\partial p_\ell} = \frac{p_\ell}{m} + at\cos\theta \qquad \text{agreeing with (1) and (2)} \end{aligned}}$$

The equations for $\dot{p}_\ell$ and $\dot{p}_\theta$ are

$$\dot{p}_\ell = -\frac{\partial H}{\partial \ell} = -\frac{at}{\ell^2}p_\theta\sin\theta - k(\ell - b) + mg\cos\theta + \frac{p_\theta^2}{m\ell^3}$$

or

$$\boxed{\dot{p}_\ell + \frac{at}{\ell^2}p_\theta\sin\theta + k(\ell - b) - mg\cos\theta + \frac{p_\theta^2}{m\ell^3} = 0}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -\frac{at}{\ell}p_\theta\cos\theta + at\,p_\ell\sin\theta - mg\ell\sin\theta$$

or

$$\boxed{\dot{p}_\theta - \frac{at}{\ell}\, p_\theta \cos\theta - at\, p_\ell \sin\theta + mg\ell \sin\theta = 0}$$

c) $\sin\theta \simeq \theta, \cos\theta \simeq 1 - \dfrac{\theta^2}{2}$

Substitute into Lagrange's equations of motion

$$\ddot{\ell} - \ell\dot{\theta}^2 - (a+g)\left[1 - \frac{\theta^2}{2}\right] + \frac{k}{m}(\ell - b) = 0$$

$$\ddot{\theta} + \frac{2}{\ell}\,\dot{\ell}\dot{\theta} + \frac{a+g}{\ell}\,\theta - \frac{at\dot{\ell}\theta}{\ell} = 0$$

For small oscillations, $\theta \ll 1, \dot{\theta} \ll 1, \dot{\ell} \ll 1$. Dropping all second-order terms gives

$$\ddot{\ell} + \frac{k}{m}\,\ell = a + g + \frac{k}{m}\,b$$

$$\ddot{\theta} + \frac{a+g}{\ell}\,\theta = 0$$

For θ,

$$\boxed{T_\theta = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\ell}{a+g}}}$$

The solution to the equation for ℓ is

$$\ell = \ell_{\text{homogeneous}} + \ell_{\text{particular}}$$

$$= A\cos\sqrt{\frac{k}{m}}\,t + B\sin\sqrt{\frac{k}{m}}\,t + \frac{m}{k}(a+g) + b$$

So for ℓ,

$$\boxed{T_\ell = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}}$$

7-30.

a) From the definition of a total derivative, we can write

$$\frac{dg}{dt} = \frac{\partial g}{\partial t} + \sum_k \left[\frac{\partial g}{\partial q_k}\frac{\partial q_k}{\partial t} + \frac{\partial g}{\partial p_k}\frac{\partial p_k}{\partial t}\right] \tag{1}$$

Using the canonical equations

$$\left.\begin{aligned}\frac{\partial q_k}{\partial t} &= \dot{q}_k = \frac{\partial H}{\partial p_k} \\ \frac{\partial p_k}{\partial t} &= \dot{p}_k = -\frac{\partial H}{\partial q_k}\end{aligned}\right] \tag{2}$$

we can write (1) as

$$\frac{dg}{dt} = \frac{\partial g}{\partial t} + \sum_k \left[\frac{\partial g}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial g}{\partial p_k}\frac{\partial H}{\partial q_k} \right] \tag{3}$$

or

$$\boxed{\frac{dg}{dt} = \frac{\partial g}{\partial t} + \left[g, H \right]} \tag{4}$$

b)

$$\dot{q}_j = \frac{\partial q_j}{\partial t} = \frac{\partial H}{\partial p_j} \tag{5}$$

According to the definition of the Poisson brackets,

$$\left[q_j, H \right] = \sum_k \left[\frac{\partial q_j}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial g_j}{\partial p_k}\frac{\partial H}{\partial q_k} \right] \tag{6}$$

but

$$\frac{\partial q_j}{\partial q_k} = \delta_{jk} \text{ and } \frac{\partial q_j}{\partial p_k} = 0 \text{ for any } j,k \tag{7}$$

then (6) can be expressed as

$$\boxed{\left[q_j, H \right] = \frac{\partial H}{\partial p_j} = \dot{q}_j} \tag{8}$$

In the same way, from the canonical equations,

$$\dot{p}_j = -\frac{\partial H}{\partial q_j} \tag{9}$$

so that

$$\left[p_j, H \right] = \sum_k \left[\frac{\partial p_j}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial p_j}{\partial p_k}\frac{\partial H}{\partial q_k} \right] \tag{10}$$

but

$$\frac{\partial p_j}{\partial p_k} = \delta_{jk} \text{ and } \frac{\partial p_j}{\partial q_k} = 0 \text{ for any } j,k \tag{11}$$

then,

$$\boxed{\dot{p}_j = -\frac{\partial H}{\partial q_j} = \left[p_j, H\right]} \tag{12}$$

c)

$$\left[p_k, p_j\right] = \sum_\ell \left[\frac{\partial p_k}{\partial q_\ell}\frac{\partial p_j}{\partial p_\ell} - \frac{\partial p_k}{\partial p_\ell}\frac{\partial p_j}{\partial q_\ell}\right] \tag{13}$$

since,

$$\frac{\partial p_k}{\partial q_\ell} = 0 \text{ for any } k,\ell \tag{14}$$

the right-hand side of (13) vanishes, and

$$\boxed{\left[p_k, p_j\right] = 0} \tag{15}$$

In the same way,

$$\left[q_k, q_j\right] = \sum_\ell \left[\frac{\partial q_k}{\partial q_\ell}\frac{\partial q_j}{\partial p_\ell} - \frac{\partial q_k}{\partial p_\ell}\frac{\partial q_j}{\partial q_\ell}\right] \tag{16}$$

since

$$\frac{\partial q_j}{\partial p_\ell} = 0 \text{ for any } j,\ell \tag{17}$$

the right-hand side of (16) vanishes and

$$\boxed{\left[q_k, q_j\right] = 0} \tag{18}$$

d)

$$\left[q_k, p_j\right] = \sum_\ell \left[\frac{\partial q_k}{\partial q_\ell}\frac{\partial p_j}{\partial p_\ell} - \frac{\partial q_k}{\partial p_\ell}\frac{\partial p_j}{\partial q_\ell}\right]$$

$$= \sum_\ell \delta_{k\ell}\,\delta_{j\ell} \tag{19}$$

or,

$$\boxed{\left[q_k, p_j\right] = \delta_{kj}} \tag{20}$$

e) Let $g(p_k, q_k)$ be a quantity that does not depend explicitly on the time. If $g(p_k, q_k)$ commutes with the Hamiltonian, i.e., if

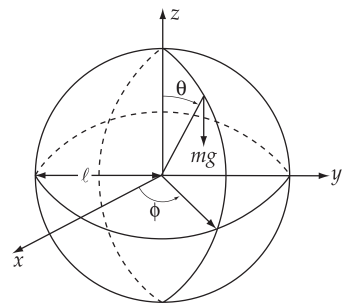
$$\left[g, H\right] = 0 \tag{21}$$

then, according to the result in a) above,

$$\boxed{\frac{dg}{dt} = 0} \tag{22}$$

and g is a constant of motion.

7-31. A spherical pendulum can be described in terms of the motion of a point mass m on the surface of a sphere of radius ℓ, where ℓ corresponds to the length of the pendulum support rod. The coordinates are as indicated below.



The kinetic energy of the pendulum is

$$T = \frac{1}{2} I_1 \dot{\phi}^2 + \frac{1}{2} I_2 \dot{\theta}^2 = \frac{1}{2} m\ell^2 \left(\dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2 \right) \tag{1}$$

and the potential energy is

$$U = mg\ell \cos\theta \tag{2}$$

The Lagrangian is

$$L = \frac{1}{2} m\ell^2 \left(\dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2 \right) - mg\ell \cos\theta \tag{3}$$

so that the momenta are

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m\ell^2 \dot{\theta} \tag{4}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m\ell^2 \dot{\phi} \sin^2 \theta \tag{5}$$

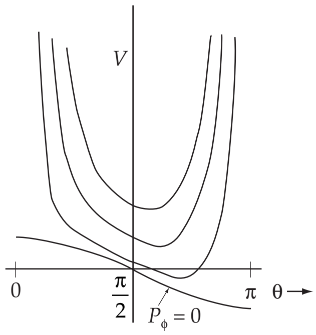
The Hamiltonian then becomes

$$\begin{aligned} H &= p_\theta \dot{\theta} + p_\phi \dot{\phi} - \frac{1}{2} m\ell^2 \left(\dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2 \right) + mg\ell \cos\theta \\ &= \frac{p_\theta^2}{2m\ell^2} + V\left(\theta, p_\phi\right) \end{aligned} \tag{6}$$

which is just the total energy of the system and where the effective potential is

$$V\left(\theta, p_\phi\right) = \frac{p_\theta^2}{2m\ell^2 \sin^2 \theta} + mg\ell \cos\theta \tag{7}$$

When $p_\phi = 0$, $V(\theta,0)$ is finite for all θ, with a maximum at $\theta = 0$ (top of the sphere) and a minimum at $\theta = \pi$ (bottom of the sphere); this is just the case of the ordinary pendulum. For different values of p_ϕ, the V–θ diagram has the appearance below:



When $p_\phi > 0$, the pendulum never reaches $\theta = 0$ or $\theta = \pi$ because V is infinite at these points. The V–θ curve has a single minimum and the motion is oscillatory about this point. If the total energy (and therefore V) is a minimum for a given p_ϕ, θ is a constant, and we have the case of a conical pendulum.

For further details, see J. C. Slater and N. H. Frank, *Mechanics*, McGraw-Hill, New York, 1947, pp. 79–86.

7-32. The Lagrangian for this case is

$$L = T - U = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2 \dot{\theta}\dot{\phi}^2\right) + \frac{k}{r} \tag{1}$$

where spherical coordinates have been used due to the symmetry of U.

The generalized coordinates are r, θ, and ϕ, and the generalized momenta are

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \tag{2}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \tag{3}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}\sin^2 \theta \tag{4}$$

The Hamiltonian can be constructed as in Eq. (7.155):

$$\begin{aligned} H &= p_r\dot{r} + p_\theta\dot{\theta} + p_\phi\ddot{\phi} - L \\ &= \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2 \theta\right) - \frac{k}{r} \\ &= \frac{1}{2}\left[\frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} + \frac{p_\phi^2}{mr^2 \sin^2 \theta}\right] - \frac{k}{r} \end{aligned} \tag{5}$$

Eqs. (7.160) applied to H as given in (5) reproduce equations (2), (3), and (4). The canonical equations of motion are obtained applying Eq. (7.161) to H:

$$\dot{p}_r = -\frac{\partial H}{\partial r} = -\frac{k}{r^2} + \frac{p_\theta^2}{mr^3} + \frac{p_\phi^2}{mr^3 \sin^2 \theta} \tag{6}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\phi^2 \cot\theta}{mr^2 \sin^2\theta} \tag{7}$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0 \tag{8}$$

The last equation implies that $p_\phi = \text{const}$, which reduces the number of variables on which H depends to four: r, θ, p_r, p_θ:

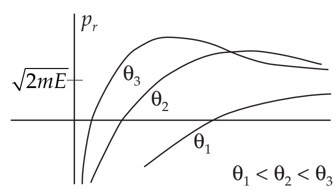
$$H = \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{r^2} + \frac{\text{const}}{r^2 \sin^2\theta}\right] - \frac{k}{r} \tag{9}$$

For motion with constant energy, (9) fixes the value of any of the four variables when the other three are given.

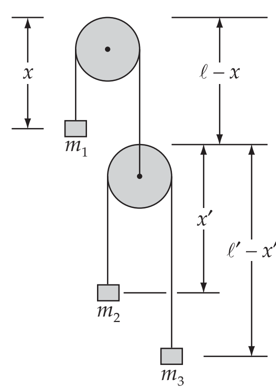
From (9), for a given constant value of $H = E$, we obtain

$$\boxed{p_r = \left[2mE - \frac{p_\theta^2 \sin^2\theta + const}{r^2 \sin^2\theta} + \frac{2mk}{r}\right]^{1/2}} \tag{10}$$

and so the projection of the phase path on the $r - p_r$ plane are as shown below.



7-33.



Neglect the masses of the pulleys

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(\dot{x}' - \dot{x})^2 + \frac{1}{2}m_3(-\dot{x} - \dot{x}')^2$$

$$U = -m_1 g x - m_2 g(\ell - x + x') - m_3 g(\ell - x + \ell' - x')$$

$$L = \frac{1}{2}(m_1 + m_2 + m_3)\dot{x}^2 + \frac{1}{2}(m_2 + m_3)\dot{x}'^2 + \dot{x}\dot{x}'(m_3 - m_2)$$
$$+ g(m_1 - m_2 - m_3)x + g(m_2 - m_3)x' + \text{constant}$$

We redefine the zero in U such that the constant in $L = 0$.

$$p_x = \frac{\partial L}{\partial \dot{x}} = (m_1 + m_2 + m_3)\dot{x} + (m_3 - m_2)\dot{x}' \tag{1}$$

$$p_{x'} = \frac{\partial L}{\partial \dot{x}'} = (m_3 - m_2)\dot{x} + (m_2 + m_3)\dot{x}' \tag{2}$$

Solving (1) and (2) for p_x and $p_{x'}$ gives

$$\dot{x} = D^{-1}\left[(m_2 + m_3)p_x + (m_2 - m_3)p_{x'}\right]$$

$$\dot{x}' = D^{-1}\left[(m_2 + m_3)p_x + (m_1 + m_2 + m_3)p_{x'}\right]$$

where $D = m_1 m_3 + m_1 m_2 + 4m_2 m_3$

$$H = T + U$$

$$= \frac{1}{2}(m_1 + m_2 + m_3)\dot{x}^2 + \frac{1}{2}(m_2 + m_3)\dot{x}'^2 + (m_3 - m_2)x\dot{x}$$
$$- g(m_1 - m_2 - m_3)x - g(m_2 - m_3)x'$$

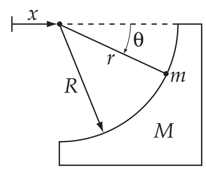
Substituting for $\dot{x}$ and $\dot{x}'$ and simplifying gives

$$\boxed{\begin{aligned} H = {} & \frac{1}{2}(m_2 + m_3)D^{-1}\, p_x^2 + \frac{1}{2}(m_1 + m_2 + m_3)D^{-1}\, p_{x'}^2 \\ & + (m_2 - m_3)D^{-1}\, p_x p_{x'} - g(m_1 - m_2 + m_3)x - g(m_2 - m_3)x' \\ & \text{where } D = m_1 m_3 + m_1 m_2 + 4m_2 m_3 \end{aligned}}$$

The equations of motion are

$$\boxed{\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p_x} = (m_2 + m_3)D^{-1}\, p_x + (m_2 - m_3)D^{-1}\, p_{x'} \\ \dot{x}' &= \frac{\partial H}{\partial p_{x'}} = (m_2 - m_3)D^{-1}\, p_{x'} + (m_1 + m_2 + m_3)D^{-1}\, p_{x'} \\ \dot{p}_x &= -\frac{\partial H}{\partial x} = g(m_1 - m_2 - m_3) \\ \dot{p}_{x'} &= -\frac{\partial H}{\partial x'} = g(m_2 - m_3) \end{aligned}}$$

7-34.



The coordinates of the wedge and the particle are

$$x_M = x \qquad x_m = r\cos\theta + x$$
$$y_M = 0 \qquad y_m = -r\sin\theta \tag{1}$$

The Lagrangian is then

$$L = \frac{M+m}{2}\dot{x}^2 + \frac{m}{r}\left(\dot{r}^2 + r^2\dot{\theta}^2 + 2\dot{x}\dot{r}\cos\theta - 2\dot{x}r\dot{\theta}\sin\theta\right) + mgr\sin\theta \tag{2}$$

Note that we do not take r to be constant since we want the reaction of the wedge on the particle. The constraint equation is $f(x,\theta,r) = r - R = 0$.

a) Right now, however, we may take $r = R$ and $\dot{r} = \ddot{r} = 0$ to get the equations of motion for x and θ. Using Lagrange's equations,

$$\ddot{x} = aR\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right) \tag{3}$$

$$\ddot{\theta} = \frac{\ddot{x}\sin\theta + g\cos\theta}{R} \tag{4}$$

where $a \equiv m/(M+m)$.

b) We can get the reaction of the wedge from the Lagrange equation for r

$$\lambda = m\ddot{x}\cos\theta - mR\dot{\theta}^2 - mg\sin\theta \tag{5}$$

We can use equations (3) and (4) to express $\ddot{x}$ in terms of θ and $\dot{\theta}$, and substitute the resulting expression into (5) to obtain

$$\lambda = \left[\frac{a-1}{1-a\sin^2\theta}\right]\left(R\dot{\theta}^2 + g\sin\theta\right) \tag{6}$$

To get an expression for $\dot{\theta}$, let us use the conservation of energy

$$H = \frac{M+m}{2}\dot{x}^2 + \frac{m}{2}\left(R^2\dot{\theta}^2 - 2\dot{x}R\dot{\theta}\sin\theta\right) - mgR\sin\theta = -mgR\sin\theta_0 \tag{7}$$

where θ_0 is defined by the initial position of the particle, and $-mgR\sin\theta_0$ is the total energy of the system (assuming we start at rest). We may integrate the expression (3) to obtain $\dot{x} = aR\dot{\theta}\sin\theta$, and substitute this into the energy equation to obtain an expression for $\dot{\theta}$

$$\dot{\theta}^2 = \frac{2g\left(\sin\theta - \sin\theta_0\right)}{R\left(1 - a\sin^2\theta\right)} \tag{8}$$

Finally, we can solve for the reaction in terms of only θ and θ_0

$$\lambda = -\frac{mMg\left(3\sin\theta - a\sin^3\theta - 2\sin\theta_0\right)}{(M+m)\left(1-a\sin^2\theta\right)^2} \tag{9}$$

7-35. We use z_i and p_i as our generalized coordinates, the subscript i corresponding to the ith particle. For a uniform field in the z direction the trajectories $z = z(t)$ and momenta $p = p(t)$ are given by

$$\left.\begin{aligned} z_i &= z_{i0} + v_{i0}t - \frac{1}{2}gt^2 \\ p_i &= p_{i0} - mgt \end{aligned}\right] \tag{1}$$

where z_{i0}, p_{i0}, and $v_{i0} = p_{i0}/m$ are the initial displacement, momentum, and velocity of the ith particle.

Using the initial conditions given, we have

$$z_1 = z_0 + \frac{p_0 t}{m} - \frac{1}{2}gt^2 \tag{2a}$$

$$p_1 = p_0 - mgt \tag{2b}$$

$$z_2 = z_0 + \Delta z_0 + \frac{p_0 t}{m} - \frac{1}{2}gt^2 \tag{2c}$$

$$p_2 = p_0 - mgt \tag{2d}$$

$$z_3 = z_0 + \frac{(p_0 + \Delta p_0)t}{m} - \frac{1}{2}gt^2 \tag{2e}$$

$$p_3 = p_0 + \Delta p_0 - mgt \tag{2f}$$

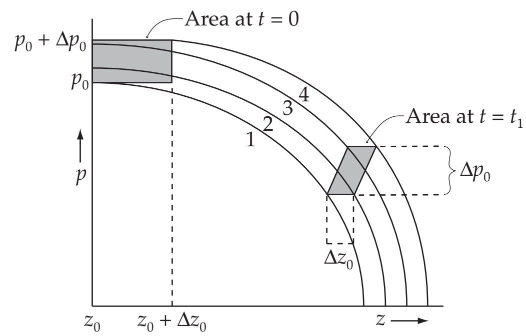
$$z_4 = z_0 + \Delta Z_0 + \frac{(p_0 + \Delta p_0)t}{m} - \frac{1}{2}gt^2 \tag{2g}$$

$$p_4 = p_0 + \Delta p_0 - mgt \tag{2h}$$

The Hamiltonian function corresponding to the ith particle is

$$H_i = T_i + V_i = \frac{m\dot{z}_i^2}{2} + mgz_i = \frac{p_i^2}{2m} + mgz_i = \text{const.} \tag{3}$$

From (3) the phase space diagram for any of the four particles is a parabola as shown below.

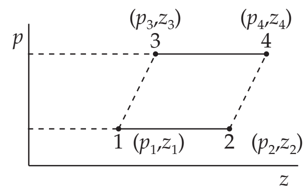


From this diagram (as well as from 2b, 2d, 2f, and 2h) it can be seen that *for any time t*,

$$p_1 = p_2 \tag{4}$$

$$p_3 = p_4 \tag{5}$$

Then for a certain time t the shape of the area described by the representative points will be of the general form



where the base $\overline{1\ 2}$ must parallel to the top $\overline{3\ 4}$. At time $t = 0$ the area is given by $\Delta z_0 \Delta p_0$, since it corresponds to a rectangle of base Δz_0 and height Δp_0. At any other time the area will be given by

$$A = \left\{ \text{base of parallelogram}\Big|_{t=t_1} = (z_2 - z_1)\Big|_{t=t_1} = (z_4 - z_3)\Big|_{t=t_1} = \Delta z_0 \right\}$$

$$x\ \left\{ \text{height of parallelogram}\Big|_{t=t_1} = (p_3 - p_1)\Big|_{t=t_1} = (p_4 - p_2)\Big|_{t=t_1} = \Delta p_0 \right\}$$

$$= \Delta p_0\ \Delta z_0 \tag{6}$$

Thus, the area occupied in the phase plane is constant in time.

7-36. The initial volume of phase space accessible to the beam is

$$V_0 = \pi R_0^2 \pi p_0^2 \tag{1}$$

After focusing, the volume in phase space is

$$V_1 = \pi R_1^2 \pi p_1^2 \tag{2}$$

where now p_1 is the resulting radius of the distribution of transverse momentum components of the beam with a circular cross section of radius R_1. From Liouville's theorem the phase space accessible to the ensemble is invariant; hence,

$$V_0 = \pi R_0^2 \pi p_0^2 = V_1 = \pi R_1^2 \pi p_1^2 \tag{3}$$

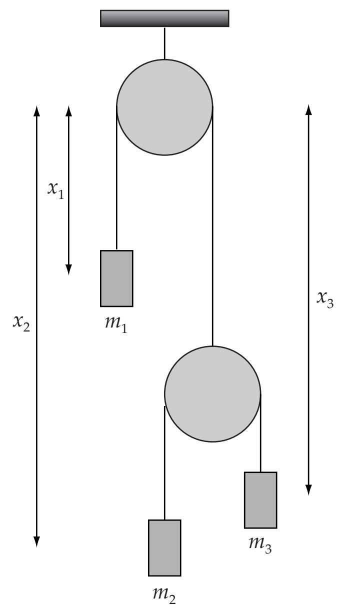
from which

$$p_1 = \frac{R_0 p_0}{R_1} \tag{4}$$

If $R_1 < R_0$, then $p_1 > p_0$, which means that the resulting spread in the momentum distribution has *increased*.

This result means that when the beam is better focused, the transverse momentum components are increased and there is a subsequent divergence of the beam past the point of focus.

7-37. Let's choose the coordinate system as shown:



The Lagrangian of the system is

$$L = T - U = \frac{1}{2}\left(m_1 \left(\frac{dx_1}{dt}\right)^2 + m_2 \left(\frac{dx_2}{dt}\right)^2 + m_3 \left(\frac{dx_3}{dt}\right)^2 \right) + g\left(m_1 x_1 + m_2 x_2 + m_3 x_3\right)$$

with the constraints

$$x_1 + y = l_1 \quad \text{and} \quad x_2 - y + x_3 - y = l_2$$

which imply $2x_1 + x_2 + x_3 - (2l_1 + l_2) = 0 \Rightarrow 2\dfrac{d^2 x_1}{dt^2} + \dfrac{d^2 x_2}{dt^2} + \dfrac{d^2 x_3}{dt^2} = 0$ (1)

The motion equations (with Lagrange multiplier λ) are