Group Coursework Submission Form

Specialist Masters Programme

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Lecturer: Ales Cérny	**Submission Date: 25/03/2024**

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FINANCIAL MODELLING WITH LÉVY PROCESSES

Premises

For the analysis an α stable Levy measure was truncated at a finite level $\bar{x}$, i.e. the TRUST model. Jumps were truncated at the 1% threshold for the purposes of this analysis. The parameterization is as follows:

$$\Pi(dx) = Ax^{-1-\alpha}\mathbf{1}_{0<\mathrm{x}<\bar{\mathrm{x}}}dx$$

For a given value of α the parameters A and $\bar{x}$ allows one to match the desired variance and kurtosis. The model is calibrated based on an arbitrarily chosen values of variance and kurtosis, 0.04 and 0.02 respectively. For convenience and ease of computation an alternative parametrization was employed as follows:

$$\Pi(dx) = \bar{A}\left(\frac{x}{\bar{x}}\right)^{-1-\alpha}\mathbf{1}_{0<\mathrm{x}<\bar{\mathrm{x}}}dx$$

$$where\ \bar{A} = \frac{A}{\bar{x}^{1+\alpha}} = \frac{(2-\alpha)^{2.5}}{(4-\alpha)^{1.5}0.02^{1.5}0.2}$$

$$\bar{x} = 0.2\sqrt{\frac{4-\alpha}{2-\alpha}\times 0.02}$$

The analysis has been conducted considering the parameter values $\alpha \in \{1.999, 1.95, 1, -1, -100\}$. For model to be effective $\alpha < 2$ is a necessary condition.

1. Mean of $L(S)_1$ under the 2-sided TRUST Model

$L(S)_T$ denotes the cumulative arithmetic rate of return also known as the stochastic logarithm.
Under the Black-Scholes economy one would obtain:

$$E[L(S)_1]_{BS} = 0.1 + \frac{1}{2}0.25^2 = 0.13125$$

Under the 2-sided TRUST model the following is obtained:

$$E[L(S)_1]_T = 0.1 + \frac{1}{2}0.15^2 + \int_{-\bar{x}}^{\bar{x}}(e^x - 1 - x)\,\bar{A}\left(\frac{x}{\bar{x}}\right)^{-1-\alpha}dx$$

The difference between the two models is computed as follows:

$$E[L(S)_1]_T - E[L(S)_1]_{BS}$$

The below differences were numerically obtained for the given α values.

α	1.999	1.95	1	-1	-100
$E[L(S)_1]_T - E[L(S)_1]_{BS}$	1.37e-06	1.33e-06	1.32e-06	1.33e-06	1.33e-06

The jump portion of the TRUST model mainly attributes to the differences, since under the Black-Scholes environment it is assumed stock prices do not jump and is just a continuous process. The differences are so small because the jump sizes have been truncated at 1%.

2.1. Calibrated TRUST Model

For the given α values the following table have been reproduced as required:

α	$\bar{x}$	A	$\bar{A}$	$\Pi(\mathbb{R})$	$\Pi([0.5,1]\bar{x})$	Var split	Exkurt split
1.999	126.5%	4.00e-05	1.97e-05	Inf	3.75e-05	99.9%	25%
1.95	18.1%	2.18e-03	3.37e-01	Inf	0.09	96.6%	24.1%
1	4.9%	8.16e-01	3.40e+02	Inf	16.7	50%	12.5%
-1	3.7%	2.46e+03	2.46e+03	90	45	12.5%	3.1%
-100	2.9%	1.33e+158	1.75e+05	50	50	0%	0%

The $\bar{x}$, A, and $\bar{A}$ computations are done as per the formulations mentioned in premises. Π(R) denotes the total intensity of jumps for various alpha values It was computed as follows :

$$\Pi(\mathrm{R}) = A\frac{\bar{x}^{-\alpha}}{-\alpha}$$

As α decreases the total intensity of the jumps decreases. $\Pi([0.5,1]\bar{x})$ denotes the total intensity of jumps between the interval $0.5\bar{x}$ and $\bar{x}$ ie the number of jumps occurring between jump size of 0.5 and 1. It was computed as follows:

$$\Pi([0.5,1]\bar{x}) = \int_{0.5\bar{x}}^{\bar{x}} \bar{A}\left(\frac{x}{\bar{x}}\right)^{-1-\alpha} dx$$

Here the intensity of small jumps increases as α decreases, which aligns with expectations since for a lower alpha the levy density is an increasing function of x in the interval $(0, \bar{x}]$ and $0.5\bar{x}$ falls within this interval. The Variance split and the Excess Kurtosis split shows the proportion of variance & excess kurtosis of small jumps v/s total jumps. The computationis done as follows:

$$Var\ split = \frac{\int_{0.5\bar{x}}^{\bar{x}} x^2 \bar{A}\left(\frac{x}{\bar{x}}\right)^{-1-\alpha} dx}{A\frac{\bar{x}^{2-\alpha}}{2-\alpha}}$$

$$ExKurt\ split = \frac{\int_{0.5\bar{x}}^{\bar{x}} x^4 \bar{A}\left(\frac{x}{\bar{x}}\right)^{-1-\alpha} dx}{A\frac{\bar{x}^{4-\alpha}}{4-\alpha}}$$

As α decreases the Variance split and Excess Kurtosis split also decreases which once again aligns with expectations i.e. the proportion of variance & excess kurtosis due to small jumps decreases as jump sizes increase.

Similarly conducting the analysis $L(S)_t$ under the 2-sided TRUST Model

α	$\bar{x}$	$\bar{A}$	$E[L(S)_1]$	$\sqrt{Var[L(S)_1]}$	$ExKurt[L(S)_1]$
1.999	126.5%	1.97e-05	0.13125137	0.2500421	0.031738
1.95	18.1%	3.37e-01	0.13125133	0.2500373	0.008463
1	4.9%	3.40e+02	0.13125132	0.2500370	0.008143
-1	3.7%	2.46e+03	0.13125133	0.2500373	0.008191
-100	2.9%	1.75e+05	0.13125133	0.2500373	0.008201

The computations were done as follows:

$$k(v) = 0.1v + \frac{1}{2}(0.15v)^2 + \int_{-\bar{x}}^{\bar{x}} (e^{vx} - 1 - vx)\,\bar{A}\left(\frac{x}{\bar{x}}\right)^{-1-\alpha} dx$$

$$E[L(S)_1] = k(1),\ Var[L(S)_1] = k(2) - 2k(1),\ ExKurt[L(S)_1] = \frac{k(4)-4k(3)+6k(2)-4k(1)}{Var[L(S)_1]^2}$$

2.2. T-forward risk neutral measures

Now since the analysis is not conducted in the Black-Scholes setting, there now exists multiple risk neutral measures for the same numeraire under which one can formulate the cumulant generating function in multiple ways. This results in the following parameters under the given T-forward risk neutral measures:

α	*Correlation*	*Esscher Parameter*	*Minimal Entropy Parameter*	*Variance Optimal Parameter*
1.999	0.99998	2.09956	2.10024	2.09931
1.95	0.99998	2.09962	2.10030	2.09939
1	0.99998	2.09962	2.10029	2.09940
-1	0.99998	2.09962	2.10030	2.09940
-100	0.99998	2.09962	2.10030	2.09939

The computations for the above table were done as follows:

$$Corr\left(E\left(\tfrac{1}{2}L(S)_1\right), S_1^{\frac{1}{2}}\right) = \frac{e^{\frac{k(1.5)-k(1)-k(0.5)}{2}}-1}{\sqrt{e^{\frac{k(2)-2k(1)}{4}}-1}\sqrt{e^{k(1)-2k(0.5)}-1}},$$

$$0 = k(1-\gamma) - k(-\gamma), where\ \gamma = Esscher\ Parameter,$$

$$0 = 0.1 + \frac{-2\lambda+1}{2}\left(0.15^2 + \frac{2\bar{A} \times 0.01^{2-\alpha}}{(2-\alpha)\bar{x}^{1+\alpha}}\right) + \int_{-\bar{x}}^{\bar{x}} \left((e^x - 1)\left(e^{-\lambda(e^x-1)}\right) - x\right) \bar{A}\left(\frac{x}{\bar{x}}\right)^{-1-\alpha} dx\ ,$$

$$where\ \lambda = Minimal\ Entropy\ Parameter$$

$$Variance\ Optimal\ Parameter\ (\phi) = \frac{E[L(S)_1])}{Var[L(S)_1]}$$

In order to calculate the Esscher and Minimal Entropy parameter one needs to numerically solve the above equations conditioned to 0.

2.3. Short Log Contract

Let's delve into the quadratic hedging strategy applied to a short log contract. This approach indicates that the investor in this position is effectively long volatility. In other words, as volatility rises, so does the value of the short log contract. Upon deeper investigation into quadratic hedging, we uncover the following insights:

α	*Short Log Contract Parameter*	*Hedging Coefficient*	*Std Dev of Hedging Error to Maturity*
1.999	0.031219	99.975%	31.5 BP
1.95	0.031223	99.979%	28.3 BP
1	0.031224	99.979%	28.2 BP
-1	0.031223	99.979%	28.3 BP
-100	0.031224	99.979%	28.3 BP

The computations were done as follows:

$$SLC\ Param = -0.1 + \frac{2\phi}{2}\left(0.15^2 + \frac{2\bar{A} \times 0.01^{2-\alpha}}{(2-\alpha)\bar{x}^{1+\alpha}}\right) + \int_{-\bar{x}}^{\bar{x}} \left(-x\left(1-\phi(e^x-1)\right) + x\right) \bar{A}\left(\frac{x}{\bar{x}}\right)^{-1-\alpha} dx\ ,$$

$$Hedging\ Coefficient\ (\beta_1) = \frac{\left(0.15^2 + \frac{2\bar{A} \times 0.01^{2-\alpha}}{(2-\alpha)\bar{x}^{1+\alpha}}\right) + \int_{-\bar{x}}^{\bar{x}} x(e^x-1)\, \bar{A}\left(\frac{x}{\bar{x}}\right)^{-1-\alpha} dx}{\left(0.15^2 + \frac{2\bar{A} \times 0.01^{2-\alpha}}{(2-\alpha)\bar{x}^{1+\alpha}}\right) + \int_{-\bar{x}}^{\bar{x}} (e^x-1)^2\, \bar{A}\left(\frac{x}{\bar{x}}\right)^{-1-\alpha} dx}$$

$$b_1^{[V,V]} = \left(0.15^2 + \frac{2\bar{A} \times 0.01^{2-\alpha}}{(2-\alpha)\bar{x}^{1+\alpha}}\right) + \int_{-\bar{x}}^{\bar{x}} x^2\, \bar{A}\left(\frac{x}{\bar{x}}\right)^{-1-\alpha} dx$$

$$Std\ Dev\ of\ Hedging\ Error\ (\varepsilon_0) = \sqrt{b_1^{[V,V]} - \beta_1^2 Var[L(S)_1]}$$

3. Variance Future Contract

Now if the payoff of the variance future contract delivered at maturity T is based on the quadratic variation of the cumulative arithmetic return would be given by $H = [L(S), L(S)]_T$. Then the mean value of the process by utilizing the Girsanov theorem via the variance-optimal measure, Q is given by:

$$V_t = E_t^Q[[L(S), L(S)]_T] = [L(S), L(S)]_t + b^{(e^{id}-1)\left(1-\gamma\left(e^{id}-1\right)\right)\circ \log(S)}(T-t)$$

The modified tracking parameter is the result of:

$$E[L(S)_1] = b^{(e^{id}-1)\circ \log(S)}$$

This yields the results below:

α	*Variance Future Contract Parameter*	*Hedging Coefficient*	*Std Dev of Hedging Error to Maturity*
1.999	0.062371	11.468 BP	111.35 BP
1.95	0.062417	7.721 BP	57.48 BP
1	0.062419	7.618 BP	56.38 BP
-1	0.062418	7.670 BP	56.55 BP
-100	0.062418	7.681 BP	56.59 BP

The computations were done as follows:

$$VF\ Param = \left(0.15^2 + \frac{2\bar{A}\ \times\ 0.01^{2-\alpha}}{(2-\alpha)\bar{x}^{1+\alpha}}\right) + \int_{-\bar{x}}^{\bar{x}} (e^x - 1)^2\,(1 - \phi(e^x - 1)\ \bar{A}\left(\frac{x}{\bar{x}}\right)^{-1-\alpha} dx\ ,$$

$$Hedging\ Coefficient\ (\beta_2) = \frac{\int_{-\bar{x}}^{\bar{x}}(e^x-1)^3\ \bar{A}\left(\frac{x}{\bar{x}}\right)^{-1-\alpha} dx}{\left(0.15^2 + \frac{2\bar{A}\ \times\ 0.01^{2-\alpha}}{(2-\alpha)\bar{x}^{1+\alpha}}\right) + \int_{-\bar{x}}^{\bar{x}}(e^x-1)^2\ \bar{A}\left(\frac{x}{\bar{x}}\right)^{-1-\alpha} dx}$$

$$b_2^{[V,V]} = \int_{-\bar{x}}^{\bar{x}} (e^x-1)^4\ \bar{A}\left(\frac{x}{\bar{x}}\right)^{-1-\alpha} dx$$

$$Std\ Dev\ of\ Hedging\ Error\ (\varepsilon_1) = \sqrt{b_2^{[V,V]} - \beta_2^2 Var[L(S)_1]}$$

4. Hedging Error Comparison

Now in light of the cash gamma we repeated the analysis of section 3 again for the variance future contract. This led to the following results:

α	*Cash Gamma Variance Parameter*	*Hedging Coefficient*	*Std Dev of Hedging Error to Maturity*
1.999	0.062464	0.02736%	111.4 BP
1.95	0.062466	0.02556%	57.5 BP
1	0.062467	0.02538%	56.4 BP
-1	0.062466	0.02555%	56.6 BP
-100	0.062466	0.02559%	56.6 BP

The values in the above table are calculated via the following formulations:

$$VF\ Param = \left(0.15^2 + \frac{2\bar{A}\ \times\ 0.01^{2-\alpha}}{(2-\alpha)\bar{x}^{1+\alpha}}\right) + \int_{-\bar{x}}^{\bar{x}} x^2\,(1 - \phi(e^x - 1)\ \bar{A}\left(\frac{x}{\bar{x}}\right)^{-1-\alpha} dx\ ,$$

$$Hedging\ Coefficient\ (\beta_3) = \frac{\int_{-\bar{x}}^{\bar{x}} x^2\,(e^x-1)\ \bar{A}\left(\frac{x}{\bar{x}}\right)^{-1-\alpha} dx}{\left(0.15^2 + \frac{2\bar{A}\ \times\ 0.01^{2-\alpha}}{(2-\alpha)\bar{x}^{1+\alpha}}\right) + \int_{-\bar{x}}^{\bar{x}}(e^x-1)^2\ \bar{A}\left(\frac{x}{\bar{x}}\right)^{-1-\alpha} dx}$$

$$Std\ Dev\ of\ Hedging\ Error\ (\varepsilon_2) = \sqrt{b_3^{[V,V]} - \beta_2^2 Var[L(S)_1]}$$

An alternative way of analysing short log contract and quadratic variation future involves breaking down the hedging error into the average cash gamma squared over the lifespan. The hedging coefficient and standard deviation of hedging errors for short log contract remain relatively stable across different phi values showing an upward trend. This suggests hedging strategy isn't overly influenced by the tail thickness parameterized by alpha. The same holds for variance future as well. Comparing the hedging error for the variance future under the 2 method one observes that they are very similar that they exhibit comparable level of uncertainty or risk in terms of hedging effectiveness overtime. In practical terms investors may expect comparable levels of risk when employing the different hedging methods. However, it is important to note that the hedging coefficients for the both the methods are vastly differing. In cash gamma it is significantly higher suggesting it places more weight on the tail events occurring. Hence, hedging coefficient is higher for cash gamma.

5. CGMY

The selected model has been the CGMY process, that has been introduced in the paper by Carr, P. *et al.* (2002) implementing the model for equity returns. The report will consider this model as the original model for the conduction of the truncation.

The three important characteristics of the Levy triplet

The Drift rate of part with small jumps, b^x which has been computed directly by integrating the intensity:

$$\pi(\mathrm{dx})_{gcmy} \coloneqq \int_{-\infty}^{\infty} c \cdot e^{-\mathrm{M}x} \frac{1}{x^{1+y}} \cdot 1_{\{x>0\}} dx + c \cdot e^{\alpha x} \frac{1}{|x^{1+y}|} \cdot 1_{\{x>0\}} dx$$

The intensity is then multiplied by x (where $x \in R$). Thus, we obtain:

$$b^X = b^{X[1]} + \int_{\{|x|>1\}} x \, \Pi(dx)$$

The continuous quadratic rate assigned to the quadratic Brownian motion component is set to be zero. Furthermore, the Levy measure is set to be π(dx), which has been valued above. The drift assigned to all jumps smaller than 0.01 (Truncation point) has been 0.363.

Additionally, the whole estimation has been run by setting decay by large negative jumps equal to 4 (G=4). And its counterpart M has been set to three (M=3). Lastly, the measure of the overall activity 'C' was set equal to one (C=1), and the tail index that is controlling the stability of the model 'Y' is 0.5 (Y=5).

Comparing the moments between the Original and Truncated model

Moment	Original	Truncated
Mean	0.0137	0.082836
Variance	0.28133	0.034340
Skewness	0.043735	0.000069
Kurtosis	0.097028	0.000138

Looking at the table above, the truncation alters the higher moments, which characterize the tail behaviour and the peak of the return distribution. The second moment, or variance, decreases from 0.28133 in the original model to 0.03434 in the truncated version.

This significant reduction indicates that truncating the jumps to 0.1 constrains the distribution, leading to lower variability and a tighter concentration of returns around the mean. Furthermore, the third moment, skewness, shows a substantial decrease from 0.043735 in the original to a mere 6.8972e-05 in the truncated model. Such a decline points to the moderation of the asymmetric tail behaviour, resulting in a more symmetric distribution of returns.

As the kurtosis, the original model's value of 0.097028 is significantly higher than the truncated model's value of 0.00013802. The kurtosis reduction reflects the diminished probability of extreme outliers in the returns, as the truncation reduces the likelihood of observing returns with large magnitudes, thereby lowering the tail heaviness of the distribution. Overall, we see reductions in variance, skewness, and kurtosis, indicating a decrease in tail thickness and asymmetry, which aligns with the expectations of the truncation process.

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