Group Coursework Submission Form

Specialist Masters Programme

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MSc in: Financial Mathematics, Quantitative Finance	
Module Code: SMM269	
Module Title: Fixed Income	
Lecturer: Gianluca Fusai	**Submission Date: 22/03/2024**

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Q1. PRICING A STRUCTURED PRODUCT

Premises

The pricing of structured fixed income products has been based performed on particularly chosen bond. Given the provided bond description, the following caveats and assumptions are important to be considered:

1. The notional value is EUR 1,000 and coupon payments are made semi-annually where the coupon is as follows 6M EUR LIBOR + 100BP.
2. Day count convention is actual over 360 days and the reset date is 4 days before the start of the coupon period.
3. Follows the business day convention where the trade date is February 16th, 2024, issue date is October 5th, 2015, maturity is October 28th, 2028, and the spot lag is 2 days.

1.1. Historical Coupon Payments

This section outlines the methodology employed to calculate the historical coupons paid out. To initiate the computation process, it is necessary to access both current and past historical rates of EURIBOR 6M from the Issue Date minus Reset Days to the trade date via the Bloomberg terminal. Subsequently, the coupon payment start and end dates are determined based on the issue date and trade date, ensuring that all identified dates fall within working days and follow the end of month rule. Following this, the EURIBOR 6M rates at the reset dates for each coupon payment are retrieved from the data to compute the coupon rate. The coupon rate is then annualized by incorporating an additional 100 basis points. Further, the accrual factor is calculated by dividing the number of days between the start and end of the payments by 360 days. This accrual factor is then multiplied by the annualized coupon rate to derive the final coupon amounts. This results the following table for the bond at hand:

Reset Date	Payment starts	Payment ends	EUR6M	Spread	Annualized Coupon Rate	Coupon Days	Accrual Factor	Coupon Amount
22/10/15	28/10/15	28/04/16	0.0190%	1.0000%	2.048%	183	0.5083	10.4126
22/04/16	28/04/16	28/10/16	-0.1430%	1.0000%	1.721%	183	0.5083	10.4126
24/10/16	28/10/16	28/04/17	-0.2120%	1.0000%	1.582%	182	0.5056	10.3557
24/04/17	28/04/17	30/10/17	-0.2470%	1.0000%	1.512%	185	0.5139	10.5264
24/10/17	30/10/17	30/04/18	-0.2740%	1.0000%	1.457%	182	0.5056	10.3557
24/04/18	30/04/18	29/10/18	-0.0270%	1.0000%	1.955%	182	0.5056	10.3557
23/10/18	29/10/18	29/04/19	-0.2590%	1.0000%	1.487%	182	0.5056	10.3557
23/04/19	29/04/19	28/10/19	-0.2300%	1.0000%	1.546%	182	0.5056	10.3557
22/10/19	28/10/19	28/04/20	-0.3480%	1.0000%	1.308%	183	0.5083	10.4126
22/04/20	28/04/20	28/10/20	-0.1240%	1.0000%	1.760%	183	0.5083	10.4126
22/10/20	28/10/20	28/04/21	-0.4980%	1.0000%	1.007%	182	0.5056	10.3557
22/04/21	28/04/21	28/10/21	-0.5150%	1.0000%	0.972%	183	0.5083	10.4126
22/10/21	28/10/21	28/04/22	-0.5340%	1.0000%	0.934%	182	0.5056	10.3557
22/04/22	28/04/22	28/10/22	-0.2680%	1.0000%	1.469%	183	0.5083	10.4126
24/10/22	28/10/22	28/04/23	2.1320%	1.0000%	6.362%	182	0.5056	10.3557
24/04/23	28/04/23	30/10/23	3.6370%	1.0000%	9.489%	185	0.5139	10.5264
24/10/23	30/10/23	29/04/24	4.1020%	1.0000%	10.464%	182	0.5056	10.3557

Table 1

1.2. Bootstrapping Term Structure of Interest rates (Spot rates & Forward rates)

To establish the structure of interest rates, a process known as bootstrapping is employed. This entails obtaining EURIBOR 6M rates spanning tenors ranging from 6 months to 30 years. However, it is noted that data pertaining to spot rates of EURIBOR 6M is solely accessible for tenors ranging from 6 months to 18 months. Consequently, the subsequent analysis proceeds under the assumption that the term structure is bootstrapped utilizing EURIBOR 6M interest rate swaps, which offer tenors ranging from 2 years to 30 years. After that it enables one to compute the corresponding continuously compounded discount factors which are given by:

$$P(t,T) = e^{R(t,T) \times (T-t)}$$

$$\begin{aligned} \text{where } P(t,T) &= \text{Discount Factor} \\ R(t,T) &= \text{Spot Rate} \\ T-t &= \text{Time To Maturity} \end{aligned}$$

Linear interpolation is used to estimate the missing spot rates between observed maturities. Linear interpolation assumes a linear relationship between interest rates and time to maturity, allowing for the estimation of rates at intermediate maturities based on the known rates at adjacent maturities. Similarly, applying this method to get the interpolated daily forward rates which are formulated by:

$$F(t,T) = \left(\frac{R(t+1,T)}{R(t,T)} - 1\right) \times \frac{1}{365}$$

This results in the interpolated term structure of interest rates which is displayed by Figure 1 and interpolated Forward curve displayed by Figure 2: (only looks at the spot rates up until the days to the last coupon payment based on the maturity of the bond and the frequency of the payments):

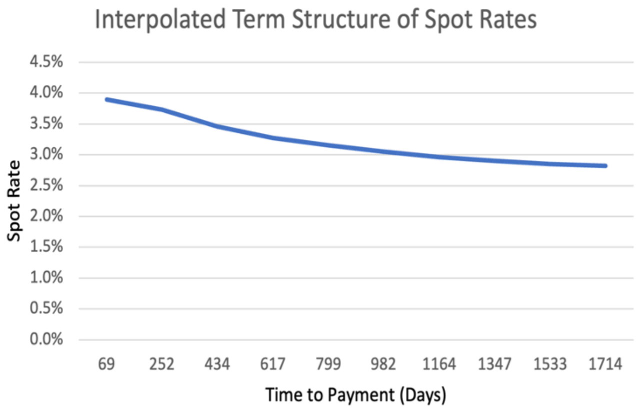

Figure 1

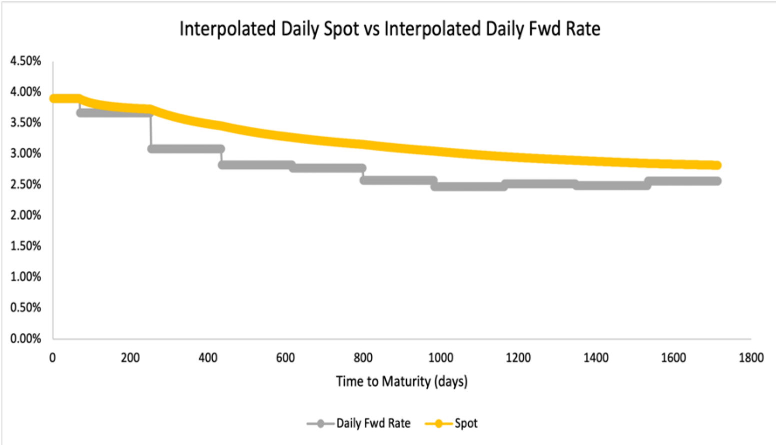

Figure 2

1.3. Pricing Floating Rate Note

A Floating Rate Note (FRN) is a type of bond or debt instrument whose interest payments fluctuate with changes in a specified benchmark interest rate. Unlike fixed-rate bonds, where the interest rate remains constant throughout the bond's life, FRNs have variable interest rates that adjust periodically according to a predetermined formula. The next step involves replicating the coupons detailed in Table 1 for pricing the FRN. To ascertain the expected coupon of the FRN, it is necessary to compute the forward rate as per the equation above which is slightly modified where it's multiplied the accrual factor which is given by:

$$Accrual\ Factor = \frac{Coupon\ Tenor\ (Days)}{360}$$

Then the expected coupons for the FRN is computed as follows:

$$C_i = (F(0,T) + F(t_i,T) + Spread) \times\ Accrual\ Factor\ \times\ Notional\ Value$$

Subsequently, discounting the anticipated coupons, summing them, and incorporating the present value of the notional amount results in the gross price and clean price as below:

$$Gross\ Price = \sum_{i=1}^{T}\left(\frac{Expected\ Coupon}{(1+R(t,T))^i}\right) + \left(\frac{Face\ Value}{(1+R(t,T)^T}\right)$$
$$Clean\ Price = Gross\ Price - Accrued\ Interest$$

$$where\ Accrued\ Interest = Coupon\ \times \frac{Days\ Since\ Last\ Coupon\ Payment}{Days\ Between\ Two\ Coupon\ Payment}$$

Risk Free Valuation			
	PV	PV(Coupons)	PV(Notional)
Gross Price	1060.2047	184.1663254	876.0383403
Current Coupon Rate	5.102%		
Days(LCD, VD)	113	Days since last coupon	
Days(VD, NCD)	69	Days to next coupon	
Days in the Coupon Period	182	Days in the coupon period	
Accrued Interest	31.7		
Clean Price	1028.5047		

Table 2

Looking closer at the expected cashflows without the present value of the notional yields the figure below:

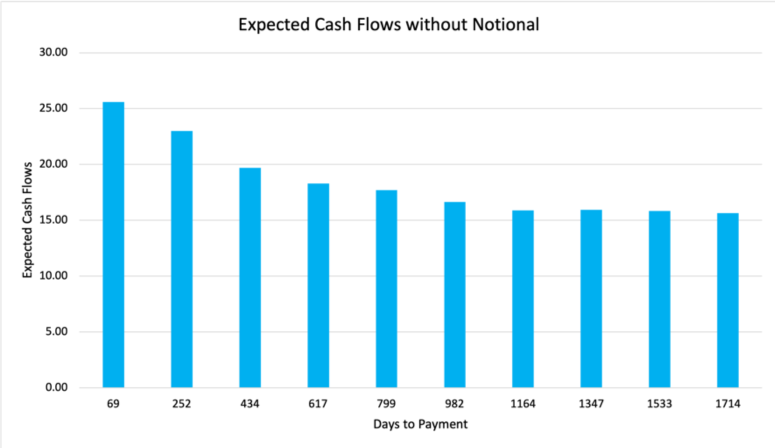

Figure 2

The trend observed in Figure 3, where expected cash flows decrease as days to payment increase, reflects a fundamental characteristic of FRNs: their cash flows are typically inversely correlated with the time to maturity. This inverse relationship between expected cash flows and days to payment can be attributed to the structure of FRNs, which often feature variable interest rates tied to a benchmark rate plus a spread. As the maturity date approaches, the uncertainty surrounding future interest rate movements increases. The declining trend in expected cash flows underscores an important aspect of FRN valuation: the impact of interest rate expectations on the present value of future cash flows. The decreasing bar heights reflect the diminishing present value of cash flows as the time horizon extends further into the future. This phenomenon is consistent with the concept of discounting, wherein cash flows further in the future are discounted more heavily due to the time value of money and increased uncertainty. Moreover, the variability in the size of the bars provides insight into the sensitivity of FRN cash flows to changes in the interest rate environment. Larger bars, corresponding to shorter time horizons, indicate relatively higher expected cash flows, reflecting the lower discounting effect and reduced uncertainty associated with nearer payment dates. Conversely, smaller bars representing longer time horizons signify lower expected cash flows, indicative of higher discounting and heightened uncertainty over a more extended period.

1.4. Pricing Caps & Floors

Caplets are interest rate derivatives that provide the buyer the right but not the obligation to receive payments at a predetermined interest rate if the EURIBOR 6M exceeds the strike i.e. it sets a minimum of 3% coupon rate in our case. Floorlets on the other hand, provide the buyer the right but not the obligation to receive payments at a predetermined interest rate if the EURIBOR 6M fall below the strike i.e. it sets a maximum of 6% coupon rate in our case. Thus they have the following payoff (intrinsic value):

$$Caplet\ Payoff = max(L(T_{i-1}, T_i) - K, 0)$$
$$Floorlet\ Payoff = max(K - L(T_{i-1}, T_i), 0)$$

$$where\ L(T_{i-1}, T_i) = EURIBOR\ 6M\ at\ time\ i$$

To initiate the valuation process for caplets and floorlets, one must first obtain the implied volatilities from the VCube of EURIBOR Caps/Floors. These volatilities correspond to various strikes and terms, offering a comprehensive view of the market's expectations regarding future interest rate movements.

Upon gathering the implied volatilities, the next step involves interpolating these values to derive the appropriate volatility for the specific maturity and strike combinations relevant to our caplets and floorlets. In our case, we are interested in maturities of 4 and 5 years, with strikes of 3% and 4% for caplets, and strikes of 5% and 6% for floorlets. Utilizing linear interpolation allows us to estimate the volatility for these intermediate points accurately.

However, it's essential to acknowledge the limitations of the available data. Implied volatilities for strikes greater than 5% are not readily accessible through public sources. To circumvent this challenge, a pragmatic approach involves extrapolating these volatilities based on historical trends. By analysing past differences and deriving an average, one can reasonably estimate the implied volatilities for strikes beyond the available range. This extrapolation process, while pragmatic, carries inherent uncertainties.

For the analysis it is assumed that the caplets and floorlets are priced using the standard Black Scholes Merton (BSM) model. The primary reason of utilising it is that it provides an easy interpretable closed form solution which takes into account main financial theories like no arbitrage and risk neutral pricing. Accounting for the fact that the BSM model has a lot of drawback like constant implied volatility and assumes efficient market hypothesis which may not hold in reality. The model lead to the results below:

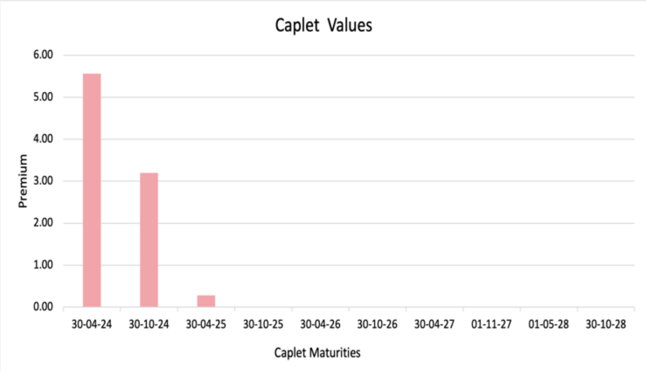

Figure 3

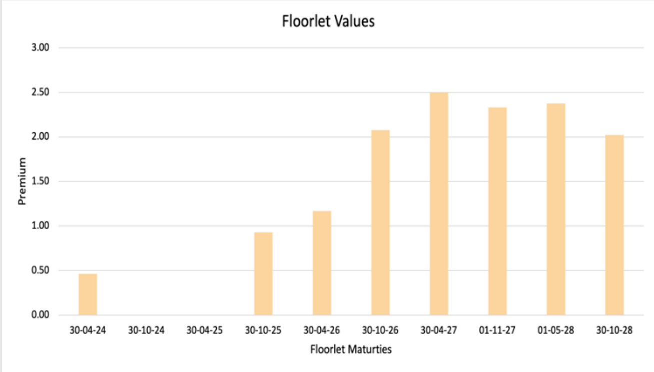

Figure 4

Upon comparing the plots of the values of floorlets and caplets at various points along the maturity spectrum, a clear pattern surfaces: as time progresses closer to maturity, the bootstrapped EURIBOR 6M forward rates exhibit a declining trend. This trend manifests itself through the observation that floorlet values are positive further into the future, while caplets exhibit positivity earlier on. When the values of caplets and floorlets are aggregated, they collectively represent the values of Caps and Floors. Specifically, the summation yields EUR 9.04366 for Caps and EUR 13.87489 for Floors.

1.5. Pricing Structured Bond

A structured bond is a type of financial instrument that combines traditional bonds with derivatives to create customized investment products tailored to specific investor needs. For the bond at hand there is a fixed component with an assumed rate of 2% and the derivatives portion is the embedded caplet with strike of 3%. The risk-free valuation technique involves a computation method similar to that employed in the previous section. In this approach, the fixed coupon component is discounted straightforwardly, reflecting its predetermined cash flows. Similarly as above, caplets are priced using the BSM model, which accounts for factors such as volatility and time to maturity. The valuation of caplets hinges on the determination of the forward premium. Once the fixed coupon component and the caplet component are individually priced, their respective values are summed to derive the expected cash flows. This summation encapsulates the total value of the interest rate derivative, incorporating both fixed coupon payments and the optionality embedded within caplets. Furthermore, one can conduct risk adjusted valuation by incorporating probabilities of survival and default. The survival probability can be computed as follows:

$$Q(t,T) = \frac{e^{-CDS(t,T)(T-t)} - R}{1-R}$$

$$\begin{aligned} where\ Q(t,T) &= Survival\ Probability \\ CDS(t,T) &= \text{Credit Default Swap (CDS) Level} \\ R &= Recovery\ Ratio \end{aligned}$$

For the premise it is assumed that CDS level is 50BP and the recovery ratio is 40%. This yields the following table of survival probabilities for each of the coupon payments:

The survival probabilities are consistent with the fact that coupon payments further into the future have higher probabilities of default. Hence, the decreasing survival probabilities as time to maturity increases. To calculate the expected cash flows from the caplet, adjusting for risk, a standard procedure is followed. Initially, the expected cash flow is multiplied by the survival probability, reflecting the likelihood of the counterparty fulfilling its obligations. Subsequently, this adjusted cash flow is discounted to present value terms. In parallel, to determine the cash flows in the event of default, the probability of default is multiplied by the expected cash flow, adjusted by the recovery ratio. The sum of these cash flows yields the risk-adjusted valuation of the structured bond which is given by:

TTM	I Survival Prob.
0.1890	99.843%
0.6904	99.426%
1.1890	99.014%
1.6904	98.601%
2.1890	98.192%
2.6904	97.783%
3.1890	97.377%
3.6904	96.971%
4.2000	96.561%
4.6959	96.162%

Table 3

$$V_D(t) = \underbrace{\sum_{i=1}^{n} Q(t,T_i) \times P(t,T_i) \times \alpha_{T_{i-1},T_i} \times \mathbb{E}_t(CF(T_i)) \times N}_{\text{expected coupon payment if survives up to } T_i}$$
$$+ \underbrace{Q(t,T_n) \times P(t,T_n) \times N}_{\text{notional payment if survives up to } T_n}$$
$$+ N \times R \times \sum_{i=1}^{n} \underbrace{(Q(t,T_{i-1}) - Q(t,T_i))}_{prob.\ of\ default\ in\ (T_{i-1},T_i]} \times P(t,T_i),$$
$$where\ CF(T_i) = Random\ Semiannual\ Coupon\ at\ time\ T_i$$
$$\alpha_{T_{i-1},T_i} \times E_t(CF(T_i)) = Forward\ Expected\ Cashflow$$

Hence, the price of the structured bond is as follows from which one can compute Credit Value Adjustment (CVA i.e. the difference between the risk free value and risky value):

Risk Free Valuation			
	PV		
Face Value	1,000.0000	Fixed Comp.	Option Component
PV Coupons	119.2501	93.8732	25.3770
PV Notional	876.0383		
Bond Gross Price	995.2885		
Current Coupon	15.6823		
Accrued Interest	9.7368		
Clean Price	985.5516		
Risky Adjusted Valuation		CVA Calculation	
PV Coupons on Survival	117.0810		
PV Notional on Survival	842.4189	Risk Free Value	995.2885
PV Cash Flow on Default	14.2168	Risky Value	973.7167
Bond Gross Price	973.7167	CVA	21.5718

Table 4

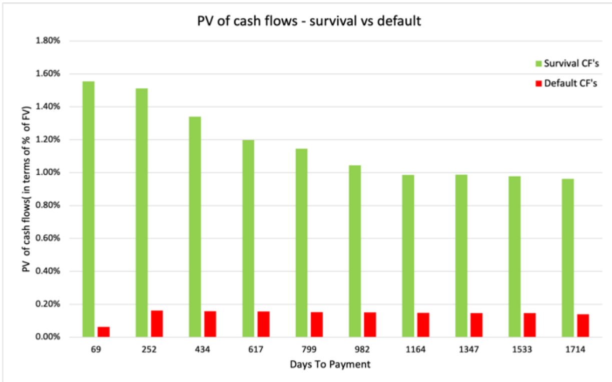

Figure 4

Upon close inspection of Figure 4 one observes that the cashflows of survival decreases as days to payment increase this is because of the decreasing probability of survival. However, the cashflows in the event of a default is almost constant as days to payments increase. This is because the recovery ratio is assumed to be constant which may not hold true in reality.

One can hedge by utilizing CDS contracts to mitigate the credit risk exposure associated with the bond. A CDS functions as a financial agreement between two parties, where the protection buyer makes periodic payments to the protection seller in exchange for protection against credit events, such as default. The CDS would provide insurance against potential losses arising from credit events associated with the bond issuer.

To further the analysis and explain the dynamics of how the two types of cashflows behave when the yield curve faces a parallel upward shift of 0.2% can be seen from the graph below:

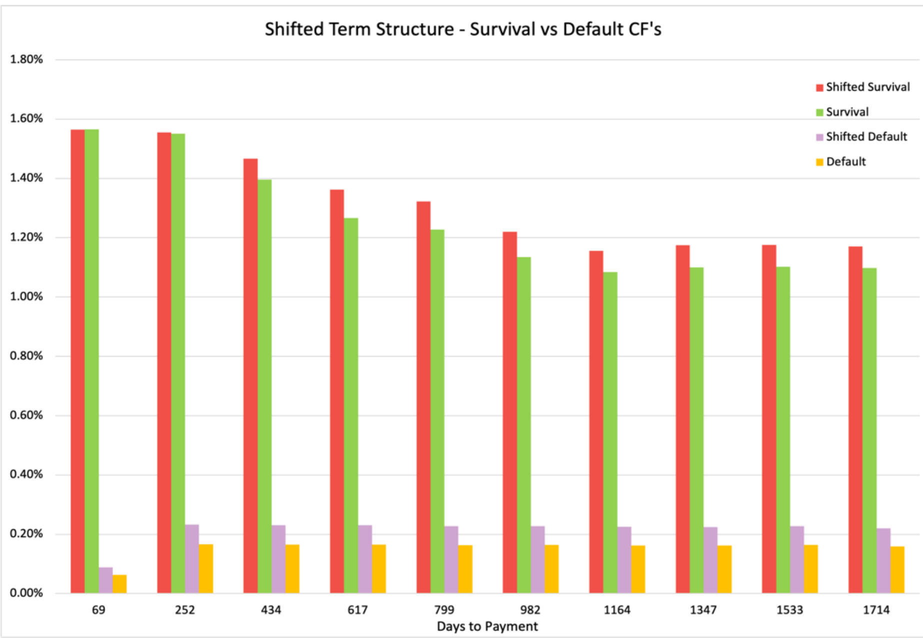

Figure 5

Figure 5 illustrates a notable observation regarding cash flows in the event of default when considering the shifted term structure of interest rates compared to no shift. It discerns a substantial rise in cash flows with the upward shift of the yield curve. This indicates that when the yield curve shifts upwards, the payoff of the caplet component of the structured bond experiences a notable increase. Additionally, there is an increase in the probability of default, albeit not as drastic as to diminish the default cash flows compared to those without a shift. This finding underscores the impact of changes in the term structure of interest rates on the risk profile and potential payoffs of structured bonds. While an upward shift in the yield curve enhances the profitability of the caplet component, it also corresponds to a heightened likelihood of default.

1.6. Pricing A Swap Contract

A swap contract is a financial agreement between two parties to exchange a series of cash flows or other financial instruments over a specified period. The swap contract in question is the plain vanilla swap that is the there are two different legs of the swap where one is floating and the other is fixed where the analysis assumes that the initial guess for the fair fixed rate is 2.784%. We adopt the previous procedure to compute the values of the fixed and floating leg where we multiply the fixed rate and the bootstrapped forward rates with the accrual factor and the discount rate to get the present value of the swap. The methodology employed utilizes a standard procedure for computing the values of the fixed and floating legs. This involves multiplying the fixed rate and the bootstrapped forward rates by the accrual factor and the discount rate, respectively, to ascertain the present value of the swap which is presented in the equation and table below:

$$FV_t(IRS_{payer}) = P(t,T_0) - P(t,T_n) - S(t) \times \sum_{i=1}^{n} P(t,T_i) \times \alpha_{(T_{i-1},T_i)}$$

$$where\ S(t) = Fixed\ Swap\ Rate$$

Valuation	
	PV
Notional	1,000
Floating Leg	179.89
Fixed Leg	120.39
Net Value	59.50
Annuity	4,324
Fair Swap Rate	4.1600%

Table 5

For the net value of the swap to be 0 i.e. the swap to have the same expected cash flows for both the legs the fair swap rate is 4.16%. To gain deeper insight into why the fair swap rate exceeds the initial estimate, it can be highlighted using the graph below:

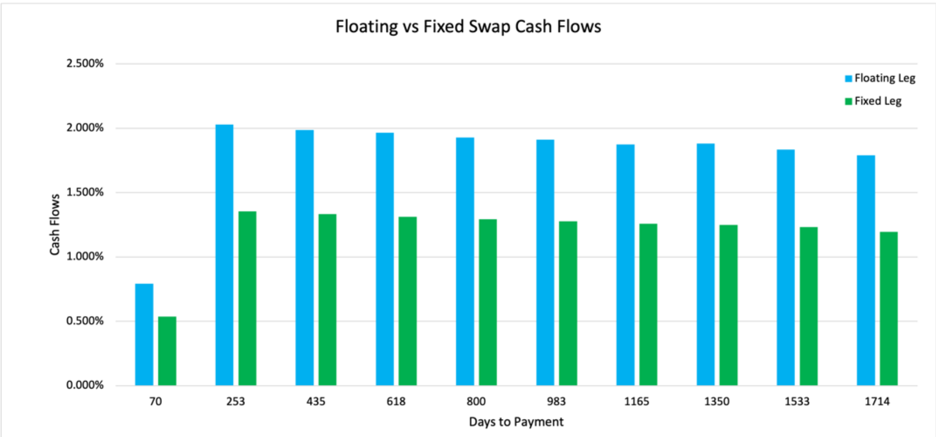

Figure 6

Given that the cash flows from the floating leg exceed those from the fixed leg for each coupon payment, the fixed rate must be higher for the net value to equate to zero. Continuing the discussion on hedging the interest rate risk associated with the long position in a structured bond using a plain vanilla swap, it's crucial to address the concerns regarding the caplet component of the structured bond, which relies on the prevailing EURIBOR 6M rate in the market on the reset date.

Given the dependency of the caplet component on the EURIBOR 6M rate, fluctuations in the term structure can potentially lead to adverse effects on the cash flows of the structured bond. Specifically, if the spot rate decreases due to shifts in the term structure, this would likely result in a reduction in cash flows from the structured bond. To mitigate this risk, the investor may choose to enter into a receiver swap concurrently. In a receiver swap arrangement, the investor receives fixed cash flows while paying floating cash flows. This strategy serves as a hedge against potential decreases in cash flows from the structured bond due to declining spot rates.

By entering into a receiver swap, the investor effectively locks in a predetermined fixed rate of return, irrespective of fluctuations in the market interest rates. This provides a degree of certainty and stability to the investor's cash flows, thereby mitigating the impact of adverse movements in the term structure on the overall investment portfolio. Furthermore, the receiver swap allows the investor to capitalize on any decreases in the prevailing market interest rates. As the investor is receiving fixed cash flows at a predetermined rate, they stand to benefit if the floating rates decline below the fixed rate specified in the swap agreement.

1.7. Sensitivity Analysis

To perform a sensitivity analysis on the structured bond and swap prices concerning shifts in the term structure of interest rates, it's imperative to model three distinct types of shifts: parallel, slope, and convex shifts in the yield curves. Parallel shift represents a general increase or decrease in interest rates across the board, affecting both short-term and long-term rates proportionally. On the other hand, slope shift indicates a change in the yield curve's slope, reflecting alterations in the market's expectations regarding future interest rate movements whereas convex shifts represent changes in the curvature of the yield curve. They have been modelled as follows:

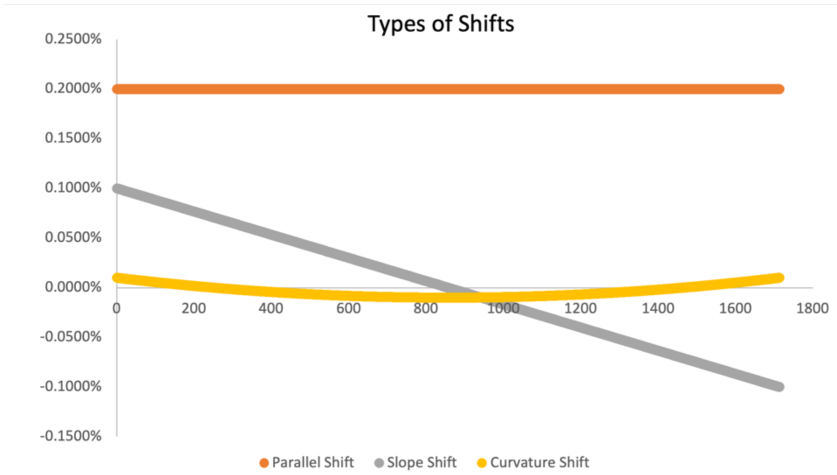

Figure 7

In Figure 7 the parallel shift is of 0.2%, slope shift is of 0.1% and the curvature shift is 0.01%. Applying the respective up and down shifts to calculate the modified structured bond prices and swap prices allows for the determination of the ideal hedge ratio for the structured bonds to swaps, as given by:

$$\mathrm{HR} = \frac{DV01_{Bond}}{DV01_{Swap}}$$

$$\text{where HR} = \text{Hedge Ratio}$$
$$\mathrm{DV01_{Bond}} = Difference\ in\ the\ Bond\ Prices\ between\ Up\ shift\ \&\ Down\ shift$$
$$\mathrm{DV01_{Swap}} = Difference\ in\ the\ Swap\ Prices\ between\ Up\ shift\ \&\ Down\ shift$$

For the bond in question for the research found the following results for the respective shifts:

	Up Shifts		Down Shifts		DV01		
	Bond	Swap	Bond	Swap	Bond	Swap	HR
Parallel	993.5209	59.5007	998.0446	60.1099	4.5237	0.6092	-7.4256
Slope	994.3376	59.6522	996.4966	59.9569	2.1590	0.3047	-7.0857
Convexity	995.1836	59.7891	995.3960	59.8195	0.2124	0.0304	-6.9868

Table 6

The derived hedge ratio from Table 6 indicates that for each long position in the structured bond, the investor should hedge by taking a long position in 8 receiver swaps for a parallel shift, 7 for a slope shift, and 7 for a convexity shift. Conversely, the investor can opt to take a short position in the payer swaps to achieve an equivalent hedging outcome. However, this position should only be made when entering into the long structured bond. As the investor would need to dynamically hedge in order to avoid losses from the shifts in the interest rates. Hence, the investor will have to regularly update the position as per the market dynamics. If the yield curve shifts upwards the value of the structured bond increases thus the swap require to hedge the risk would now be lower and vice versa in the case for a decrease in the interest rates.

Q2. VAR OF A BOND

2.1. Introduction

The objective of this investigation is to employ the endogenous term-structure model of Vasicek (1977), to simulate stochastic interest rates. It is a simple one-factor Gaussian model that helps with describing the dynamics of the instantaneous short rate r(t) over time according to a mean-reverting Gaussian process. Where those estimated rates are further used for the estimation of bond pricing.
The Vasicek model is represented as the following SDE:

$$dr = \alpha(\mu - r)dt + \sigma dW(t) \qquad (1)$$

or equivalently as

$$r(t + \Delta) = A + Br(t) + C\varepsilon(t) \qquad (2)$$

$$where\ A = e^{-\alpha\Delta},\ B = \left(1 - e^{-\alpha\Delta}\right)\mu,\ C = \sqrt{V(t, t+\Delta)} \text{ and } \Delta = time\ step$$

Equation (1) is the common Vasicek representation, offering a framework which allows for financial interpretation of its parameters. Within this equation, α assumes a pivotal role, denoting the speed at which mean reversion occurs. Meanwhile μ signifies the long term mean of the interest rate and the variable r assumes the interest rate, subject to the stochastic fluctuations over time. Finally, the term $\sigma dW(t)$ encapsulates the stochastic element of the diffusion process, where $W(t)$ is based on the Brownian Motion, also known as the Wiener Process.

2.2. Real World Estimation

In the initial stages of formulating the model, the estimation of interest rates begins with the formulation of an Auto-regressive (AR1) framework. Using historical data of 10 years EURIBOR 6M market rate at daily frequency, is considered the initial step in formulating the model for subsequent analysis. The selection of the EURIBOR 6M as a reference rate, to proxy the market, is aligning with the overall objective of simulating the pricing dynamics of a semi-annual Euro-denominated bond.

The AR (1) estimation process is implemented through an Ordinary Least Square (OLS) regression, with the objective of deriving a slope coefficients' estimated counterpart $\hat{\beta}$.
The AR (1) OLS estimation was implemented as the following regression:

$$R(t + \Delta) = a + \beta r(t) + c\,\varepsilon(t) \qquad (3)$$

$$where\ R(t+\Delta)\ is\ Euribor\ after\ \Delta, \qquad \varepsilon(t) = residual\ error\ terms$$

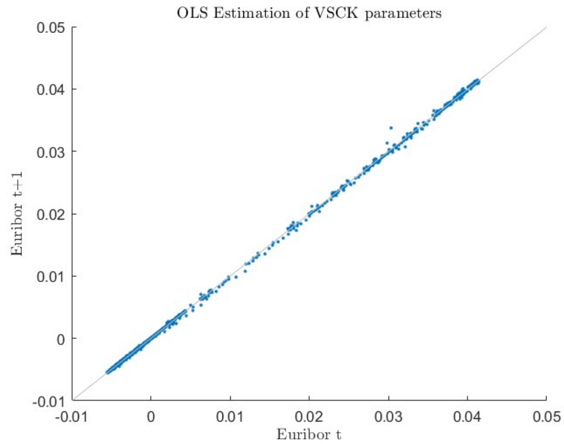

Figure 8

The coefficient estimation of β captures the Euribor rate's proportional reliance on earlier observations. This provides insight on the relative influence of the rate on subsequent values.
The extraction of the value described in the above regression initiates the procedure of modelling the real-world estimation of the Vasicek parameters A and B specified in equation (2).
The computation of $\hat{\alpha}$ is encapsulated by the formula: $\hat{\alpha} = -\frac{\log(\hat{\beta})}{dt}$ allowed the estimation of A and B in equation (2), which are crucial components in modelling the dynamics of the interest rate. Furthermore, the determination of the parameter C is calculated through the variance of the residuals obtained from the initial OLS AR(1) estimation, which encapsulates the volatility of the interest rate in the real market. Moreover, estimating $\hat{\alpha}$ allowed

Beta Hat	Mu Hat	Alpha Hat	Sigma Hat
1.0013	-0.0070075	-0.10853	0.0014222

Table 7

for the estimation of $\hat{\mu}$ for estimating B as part of the parameter such that; $\hat{\mu} = \frac{\hat{\alpha}}{(1-e^{-\hat{\alpha}dt})}$. In addition, the Vasicek models' volatility sigma hat is estimated a $\hat{\sigma} = \frac{\hat{c}}{\sqrt{\frac{(1-e^{-2\hat{\alpha}dt})}{2\hat{\alpha}}}}$. Within this study the parameters above obtained the following values:

2.3. Real world simulation

Following these steps for the estimation of the parameters, 1000 simulations are conducted with the above estimated parameters for A, B, and C, where;

$$r(i)(T) = A + Br(t) + C\varepsilon(t) \qquad (4)$$

with ε(t) being standard normal drawn values

The instantaneous rate has therefore been computed by summing A+B, adding a standard normal variable scaled by $\sqrt{Vts}$,which is the variance of the jumps in the Brownian Motion.
As there is no observable short rate in the market, that can be utilised for calibration purposes, following the previous simulation using the Vasicek Model, the approximation of a Spot Rate has been estimated to allow for being able to compare to an observable rate, such as and to best fit, hence calibrate the modelling procedure.

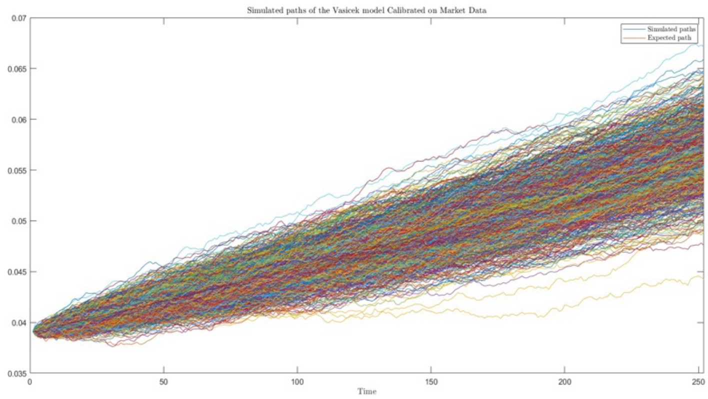

Figure 9: Simulated Paths of Calibrated Vasicek Model

However, since the Vasicek model directly deals with instantaneous interest rates, the approximated short rates are converted into spot rates, which are named Rproxy. Then, a differenced data AR(1) model is applied to estimate the new parameters of the Vasicek model. This involves modelling the first differences of said spot interest rates as a function of its own values. By fitting this autoregressive model to the approximated data, one can estimate the calibrated Vasicek parameters.
The difference between those two estimations can be observed in the figure below.

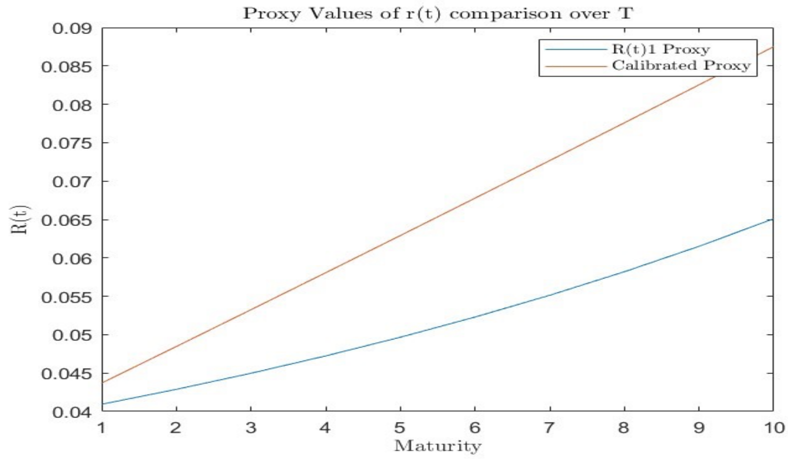

Figure 10

2.4. Risk Neutral Calibration

In order to move from a real world-based simulation to a risk neutral setting, randomly assigned parameters have been introduced into the Vasicek Model to estimate a new set of spot rates. These rates have been further processed, such that they have been compared to bootstrapped market spot rates from the previous exercise. The aim of this comparison is to minimise the difference between those spot rates, to allow for risk neutral parameter calibration. To solve a non-linear equation system, where the residuals are reduced to a minimum.
Hence, considering these new parameters employed within the model we can observe the following evolution across different maturities.

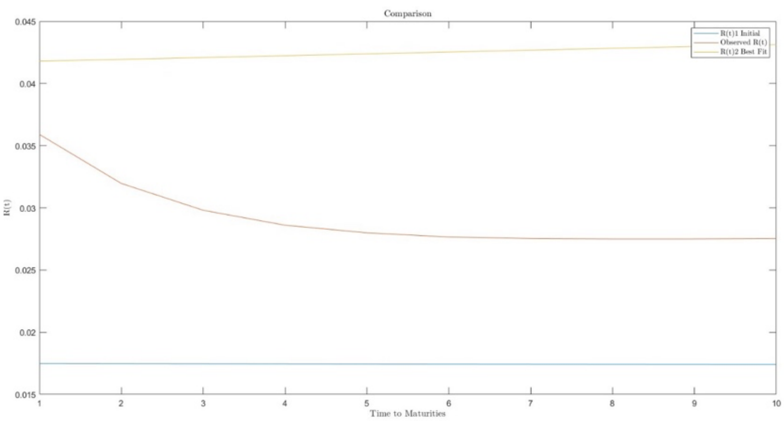

Figure 11

Regarding this optimisation, different processes have been examined. The unconstrained optimisation was unsuccessful. Indeed, iterating on the residuals, in a linear manner attempting to find exact parameters for estimating the spot rate have shown to be inadequate, which can be observed in the table below, as the results indicate imaginary values.

2.5. Risk Neutral Pricing

Within this study a 4% semi-annual coupon paying bond was priced with a Face Value of 1000EUR and Maturity of 3 years. To price the above bond, an estimation of the Money Market Account (MMA) has been employed, using the trapezium rule with the estimated spot rates of the risk neutrally calibrated estimation. The price of the bond has been determined as the sum of the coupons divided by the MMAs, with the respective maturity, in addition to the face value over the MMA for the time to maturity. This calculation resulted in a bond price of 1083.50EUR.

2.6. Determining Profit and Loss Distribution in Risk Neutral Setting

To continue, the analytical pricing for zero coupon bonds, using the model parameters estimated in the above calibrated procedure implemented such that the price of the zero-coupon bond *P(t,T)* follows

$$P(t,T) = e^{-B(t,T)r(t)+A(t,T)}, \qquad (5)$$

Where $B(t,T) = \frac{(1 - e^{-\alpha(T-t)})}{\alpha}$ and $A(t,T) = \frac{(B(t,T) - (T-t))}{\alpha^2} - \frac{\sigma^2}{4\alpha} B(t,T)^2$

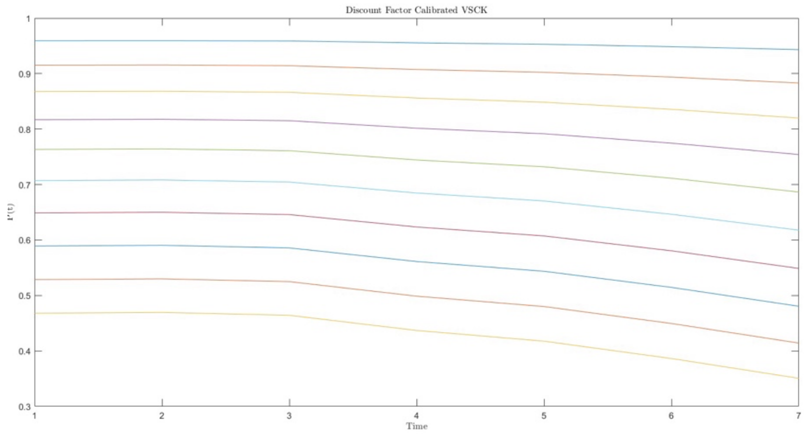

Figure 12

Utilising the newly calibrated parameters to estimate these zero-coupon bond prices is crucial as they are used as discount factors when calculating present values of structured/ coupon paying bonds.

$$BP(t,T) = \sum_{i=1}^{n} c_i * P(t,T_i) * \alpha_{T_{i-1},T_1} \quad (6)$$

To advance the analysis on the bonds, the price of the structured bond in section 2.5 has been computed under the risk-neutral measure considering a given real-world interest rate scenario and further for all the real world simulated interest rate scenarios at the Value at Risk (VaR) time horizon, which is 1 month. The bond prices have been computed once following the Vasicek bond pricing formula (6) and once utilising the MMA. The reason for the two computations is due to the Vasicek bond pricing not capturing the fluctuations of the rates within P(t,T), as the aim is to estimate the Profit and Loss distribution such that the difference between a reference bond and all other priced bonds, that make up the P&L. Unfortunately, using Vasicek's bond pricing revealed minimal to no difference among the prices. On the other hand, utilising the MMA resulted in better estimates for P&L. The P&L follows a Gaussian distribution, which is highlighted by the acceptance of the null hypothesis of the Jarque-Bera Test.

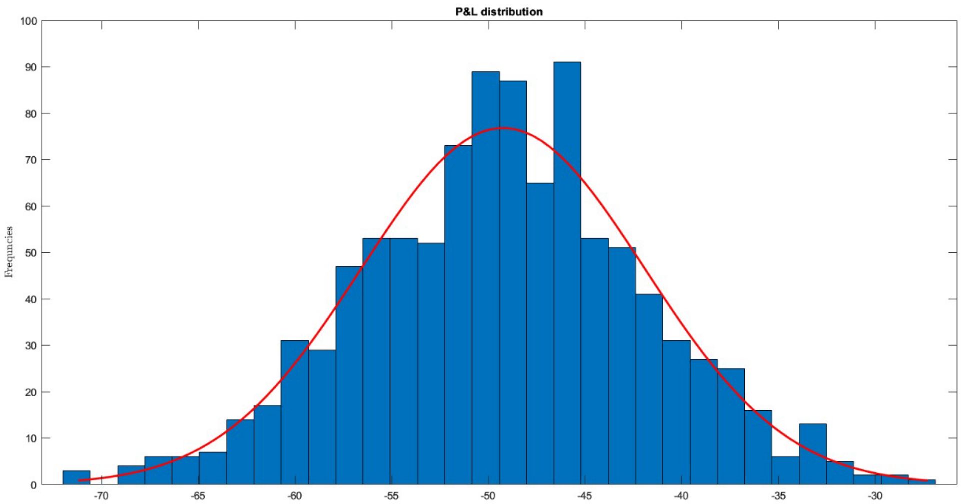

Figure 13

2.7. Value at Risk and Expected Shortfall

Further the computation of the Value at Risk (VaR) and Expected Shortfall (ES) aid in the evaluation of the riskiness of bonds. As the VaR quantifies the extent of financial loss, regarding the value of the bond, based on a confidence interval. The VaR estimation can be conducted using 2 approaches, either non-parametric or parametric, where the parametric approach is conducted by considering a specific distribution, here the Gaussian.

$$VaR_\alpha = -(\mu_{P\&L} + \sigma_{P\&L}(z_{1-\alpha})) \quad (7)$$

The parametric VaR computed on the values of the bond price was -18EUR, whereas the non-parametric accounts for -68.30EUR. The non-parametric approach considers the value in position of the quantile based on the confidence interval for sorted Profit and Loss values in ascending order. Violations of VaR, are occurrences, where the value of the profit and loss exceeds the VaR value. Here, the VaR violations under the non-parametric approach are very low, which means that the computation of the non-parametric VaR estimator has not been conservative. Contrarily, utilising the parametric VaR indicates that all P&L values violate that estimate.

Expected Shortfall (ES), also referred to as Conditional Value at Risk (CVaR), emerges as a superior and more coherent risk metric compared to VaR. Unlike VaR, which primarily focuses on a specific quantile of the loss distribution, ES considers the entire left tail of the profit and loss (P&L) distribution. By computing the average of all values beyond the VaR threshold, ES inherently incorporates extreme events, offering a more comprehensive assessment of tail risk. This distinction is crucial because extreme events, although less probable, can have significant impacts on portfolio performance during times of market stress or heightened volatility. The Gaussian parametric ES is formulated as below:

$$ES\alpha = -\left(\mu_{P\&L} + \sigma_{P\&L} \frac{\phi(Z_{1-\alpha})}{1-\alpha} \right) \ (8)$$

The values obtained for the Expected Shortfall in the parametric and non-parametric estimation are -28EUR and 105.78EUR respectively.

2.8. Advantages, Limitations and Possible Extensions of Vasicek Model

The advantages of the short rate modelling approach using the Vasicek (1977) model under the real-world measure allowing to address the dynamic nature of interest rates. When pricing, the common pricing approach for bonds is obtained by using a constant discount rate for the life of the bond. However, in reality interest rates are stochastic and fluctuate which would be overlooked in the traditional approach. This makes using the rates simulated with the Vasicek more precise and allows for the capturing of a more exact term structure and therefore positively impacts the analytical bond pricing. Additionally, the model's mathematical compliance allows equation (1) to be solved linearly and explicitly, which makes the models accessibility attractive and therefore practical to implement in real world applications. Finally, the models' interoperability enhances its versatility by allowing its adaptation to other financial models and frameworks, such that it can be adapted to be used for pricing other financial instruments. This is useful to incorporate in several investment strategy and risk management applications.

On the other hand, the major drawback of the Vasicek model is that the interest rates simulated can assume negative values with positive probability. The probability of the occurrence of negative interest rates is very low and almost impossible. The only time negative rates have been observed in the market was during financial crises and failure of attempts to stimulate economic activity by
The limitations of this approach include the endogenous nature, which means that if the market considers the initial zero-coupon bond curve, the model should incorporate such curve such that it is possible to model the market curve as close as possible utilising the estimated parameters. Within this estimation this disadvantage is attempted to mitigated by the calibration/ optimisation of the estimated spot rates with the bootstrapped market spot rates. Furthermore, the limited model parameters prevent the exact calibration of model to market, as regardless of how precise the parameters are chosen, the number of parameters is insufficient to model the exact dynamics of the term structure. Additionally, as the short rates modelled are mean reverting the expected rates tend to its long-term mean (mu) as the time approaches infinity, which overall impacts dynamic qualities of the model estimation.

A possible extension of this approach is to implement the exponential Vasicek model which will account for the limitation of having negative interest rates with positive probabilities, as it is a natural way of obtaining lognormal short rates. Although this would model would not imply explicit formulas for zero-coupon bond pricing as well as for option pricing. Furthermore, to tackle the issue of the poor fitting of the initial term structure of the interest rates Hull and White (1990) introduced first a single time varying parameter to the Vasicek model and then further they extended such model to allocate term structures of volatilities to said parameter, which they named an Extension of the Vasicek model. This model fits term structures of interest rates and term structures of spot or forward-rate volatilities.

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