Group Coursework Submission Form

Specialist Masters Programme

Please list all names of group members: (Surname, first name) 1. Alsari, Yusuf 2. Musci, Edoardo 3. Patel, Prachin 4. Shah, Siddh	GROUP NUMBER: **1**
MSc in: Financial Mathematics	
Module Code: SMM 281	
Module Title: Simulation Techniques and Financial Modelling	
Lecturer: Laura Ballota	**Submission Date: 22/03/2024**

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Q1: OPTION PRICE SIMULATION VIA THE HESTON MODEL

Premises

The Heston Model has been used to simulate the stock price. The simulated stock price is then used to estimate the European call and put implied volatilities via the Black Scholes Model. The prices can be found easily plugging the implied volatilities into the Black Scholes model, this analysis hasn't been carried out, but it is trivial to do so. The Heston model is a stochastic model used to assess the volatility of an underlying asset, the model assumes volatility follows a random process rather than a constant or deterministic process. Under the Heston model log-return process is given by:

$$ds_t = \left(\mu - \frac{v_t}{2} \right) d_t + \sqrt{v_t} dW_t \text{ , } \mu \in \mathbb{R}$$
$$dv_t = \kappa(\theta - v_t) d_t + \eta \sqrt{v_t} dZ_t$$

for two Brownian motions W, Z such that $[W, Z\}_t = \rho t, \rho \in (-1,1)$

where $s_t = price\ of\ the\ underlying\ asset$

$v_t = volatility\ of\ underlying\ asset$

$\mu = long\ run\ mean\ of\ underlying\ asset\ /\ drift\ parameter$

$\kappa = mean\ reversion\ speed\ parameter\ of\ volatility$

$\theta = long\ term\ average\ volatility$

$\eta = volatility\ of\ volatility\ parameter$

$\rho = correlation\ between\ two\ brownian\ motions\ W\ and\ Z$

The codes utilized for conducting the requisite analysis were primarily derived and customized from two main sources: "Tools for Stochastic Analysis" by Ballota & Fusai, and "Mathematical Modelling and Computation in Finance" by Oosterlee & Grzelak.

1.1. PDF of the Heston Model

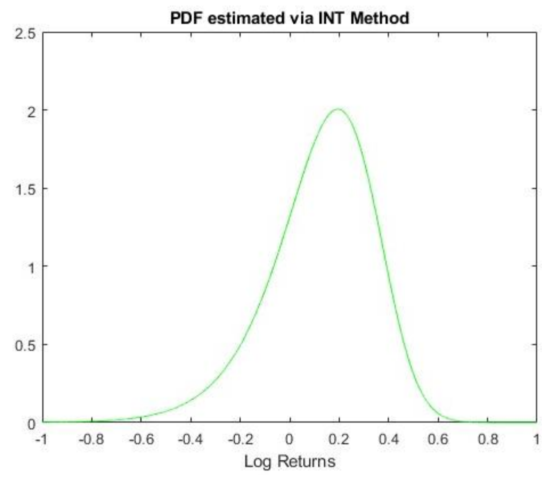
Probability Density Function (PDF) of the Heston model has no closed form solution however it can be solved numerically via inverting its characteristic function which is given as follows:

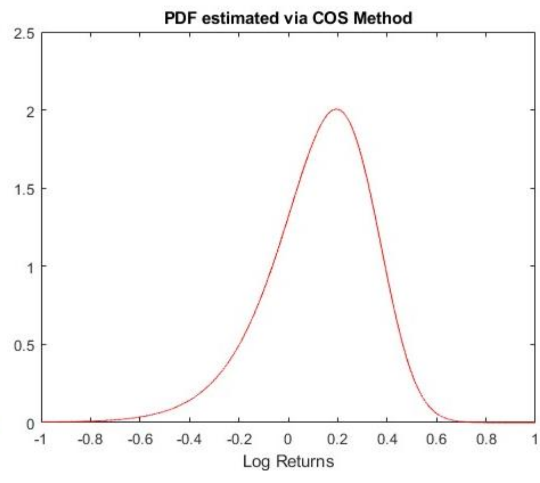
$$\phi(u; t) = e^{iu(\ln S_0 + rt) + A(t) + B(t) v_0},$$

with

$$\begin{aligned} A(t) &= \frac{\kappa\theta}{\eta^2}\left((\kappa^{\mathbb{M}} - d)t - 2\ln\frac{1 - ge^{-dt}}{1-g}\right) \\ B(t) &= \frac{\kappa^{\mathbb{M}} - d}{\eta^2}\frac{1 - e^{-dt}}{1 - ge^{-dt}} \\ d &= \sqrt{(\kappa^{\mathbb{M}})^2 + \eta^2(iu + u^2)} \\ g &= \frac{\kappa^{\mathbb{M}} - d}{\kappa^{\mathbb{M}} + d} \\ \kappa^{\mathbb{M}} &= \kappa - iu\eta\rho \end{aligned}$$

In the assessment of the density of Heston model, two distinct numerical methods were employed to estimate the probability density function (PDF) at t=1, initial stock price S0 was assumed to be 1. The methods used were the cosine Fourier transform (COS) and gaussian quadrature integration (INT) methods. The PDFs estimated by both methods are displayed in below figures:





Fourier-Cosine (COS) Series Method:

The Fourier-Cosine (COS) method is predicated on the Fourier series expansion. Specifically, it utilizes the cosine functions to perform an efficient and rapid calculation of the inverse Fourier transform, which recovers the probability density function (PDF) from the characteristic function. The characteristic function is essentially a complete representation of a probability distribution in the complex domain and is defined as the expected value of an exponential function of a stochastic variable.

Mathematically, the COS method can be succinctly expressed as follows:

$$f(x) = \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty} \times \cos\left(\frac{n\pi(x-L)}{U-L}\right) \mathrm{R}\left\{e^{-iunL}\phi\left(\frac{u}{n}\right)\right\},$$

In the context described, x represents the point at which the Probability Density Function (PDF) is evaluated. L and U are deterministic truncation bounds that define the domain of the PDF, ensuring the area of interest is precisely encapsulated. The characteristic function, denoted as ϕ(u), is evaluated at discrete points determined by the formula un =U−Lnπ, where n is the index of summation indicating the terms in the series. Additionally, the symbol $\Re$ signifies the process of taking the real part of a complex number, a crucial step in the evaluation of the series represented in this context. The efficiency of the COS method stems from its rapid convergence, which means that accurate approximations of the PDF are possible with a relatively small number of terms in the series. This approach is particularly advantageous when the characteristic function is known and can be calculated with ease.
When applying the COS method to the Heston model, one must first obtain the characteristic function of the model's log returns. The efficiency of the COS method allows for the rapid computation of the PDF and thus the pricing of options.

COS Method Results: Figure 1 presents the estimated PDF utilizing the COS method. The estimated density curve is smooth and bell-shaped, indicating that the log returns distribution is roughly symmetric about the mean. The peak of the distribution occurs near zero, which suggests that small log returns are more probable than large deviations from the mean. No significant skewness or heavy tails are observed, which would be indicative of a stable market with moderate kurtosis. The range of log returns is predominantly between -1 and 1, which is typical for daily returns in many financial markets.

Fourier Inversion Method (INT Method):

The INT method refers to the numerical integration approach used to compute the inverse Fourier transform of a characteristic function to obtain the PDF of a distribution. This method is more general and doesn't rely on specific series expansions.

The Fourier inversion theorem states that you can retrieve the PDF f(x) if you have the characteristic function ϕ(u):

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\phi(u)\ du,$$

where the integration is over the entire real line, which in practice, is truncated and discretized for numerical computation. Due to the oscillatory nature of the integrand, specialized numerical techniques such as the Fast Fourier Transform (FFT) or adaptive quadrature methods are often employed to handle the evaluation efficiently. Gaussian Quadrature is the method implemented to carry out the relevant analysis. In practice, the INT method may involve adjusting the path of integration or employing windowing techniques to manage the infinite range and ensure the integrand decays sufficiently at the bounds. The accuracy of the INT method hinges on these numerical strategies and the resolution of the discretization. Using the INT method with the Heston model entails setting up the Fourier inversion integral with the model's characteristic function. Although computationally more intensive than the COS method, the INT approach is versatile and does not necessitate a series expansion and generally proven to be more accurate than the COS approach.

INT Method Results: Figure 2 shows the estimated PDF derived from the INT method. Similar to the COS method, the estimated density via the INT method is also smooth and exhibits a bell-shaped curve, reinforcing the findings from the COS method. The peak and the symmetry are consistent with a standard log-normal distribution, a common assumption in financial models. The particular numerical integration technique used here is the adaptive Gauss-Kronrod quadrature. Gauss-Kronrod quadrature is a numerical integration method that expands the Gauss quadrature rule by adding extra points to the Gauss nodes. Thus, increasing the order of the quadrature rule and improving the accuracy without increasing the number of function evaluations significantly. This is particularly useful for integrating functions to a higher degree of accuracy.

Comparing the PDFs obtained from both methods, there is a notable consistency in the shape and range of estimated densities. This congruence suggests that both methods are appropriate for estimating the PDF of log returns under the Heston model with the given parameter set. It is crucial to note that while the PDFs are similar, the choice between the COS and INT method may depend on factors such as computational efficiency, ease of implementation, and numerical stability in different parameter regimes. The COS method, based on the Fourier cosine series expansion, is typically faster and can handle a wider range of option pricing scenarios with high accuracy. The INT method, while generally robust, might require more computational resources for integration, especially for complex models.

1.2. Comparing Moments of Heston Model

The statistical moments of the log returns estimated by both the INT and COS methods show close agreement, indicating the robustness of the numerical techniques used to approximate the underlying distribution defined by the Heston model:

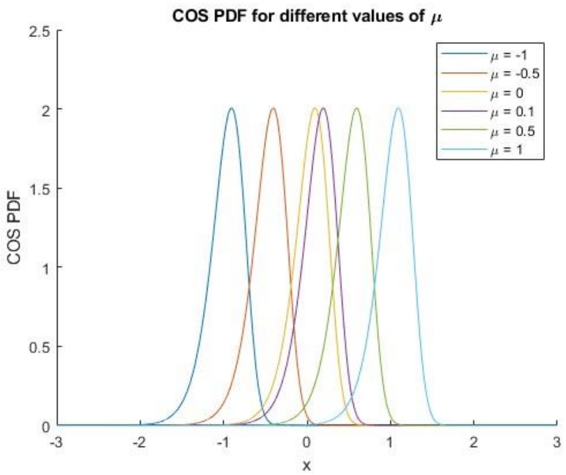
Method	Mean	Variance	Skewness	Excess Kurtosis
INT	0.12526	0.06261	0.012548	0.009677
COS	0.12506	0.06279	0.012385	0.0098258

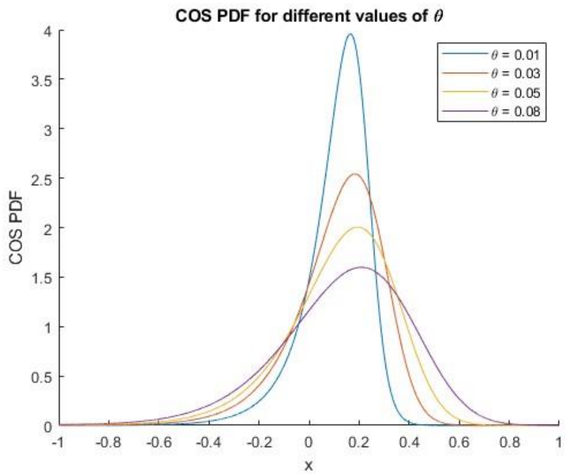
The INT and COS methods provide estimates for the moments of log returns under the Heston model with remarkable similarity, affirming their reliability. Each generates a mean of just over 0.125, with the INT method slightly higher, suggesting a consistent central tendency. Variance estimates are practically equivalent, reinforcing confidence in the model's assessment of volatility. Skewness is positively minimal, hinting at a nearly symmetrical distribution with a lean toward higher returns. The excess kurtosis is positive for both methods, this indicates a distribution with fatter tails than a normal distribution. This suggests the presence of a higher likelihood of extreme returns. However, since the values are close to zero, the tails are not significantly heavier than those of a normal distribution. The slight difference between the methods may be attributable to the numerical approximation and discretization inherent in each method, yet they agree closely enough to ensure robustness in practical financial analysis. These results demonstrate the nuanced capabilities of both methods in capturing the subtle behaviours of market returns within the stochastic volatility framework of the Heston model.

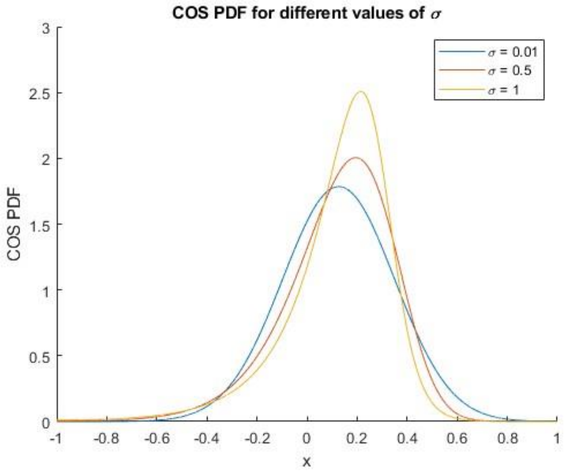
1.3. Sensitivity Analysis of Heston Model Parameters

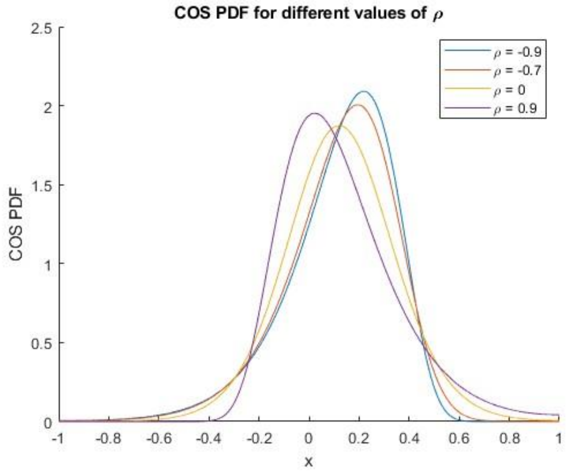
A sensitivity analysis was conducted by varying the input parameters (μ, θ, η, ρ, and κ) within the Heston model to observe their impact on the probability density function (PDF) of the model. Again, COS and INT methods were employed to estimate the PDF's. To conduct the sensitivity analysis one parameter was adjusted while keeping all other parameters constant.

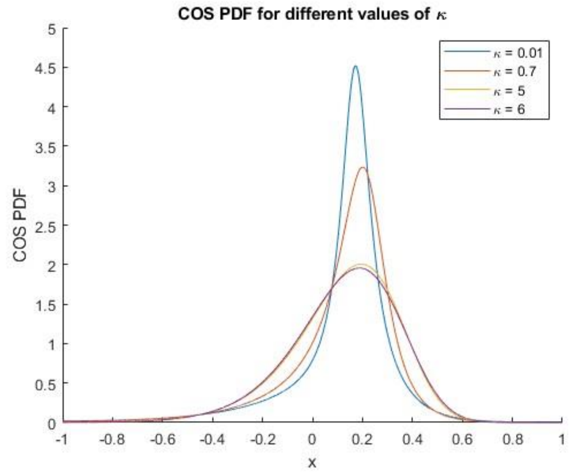
<u>*COS Method Sensitivity Analysis:*</u>











An increase in μ causes a rightward shift in the PDF, indicating higher expected returns by moving the distribution's apex towards the right, which aligns with its role in determining log returns' central tendency. θ alterations impact the PDF's height and spread; higher θ values lead to a flatter peak and a wider distribution, signalling an increase in long-term variance expectations and, consequently, in pricing uncertainty. Since θ signifies the target level for the asset return's variance across time, augmented θ values infer a wider dispersion of the asset's returns, thereby heightening long-term pricing uncertainty. Variations in η (depicted as σ in the image above and codes) flattens or broadens the PDF, reflecting greater volatility uncertainty by suggesting more varied asset return outcomes due to increased volatility variance. Given that η is the volatility of volatility parameter, its incrementation implies heightened instability in the variance, hence a broader spectrum of possible asset returns outcomes. The ρ parameter introduces subtle changes, mainly affecting the PDF's symmetry and tails, highlighting the leverage effect where asset prices and volatility are inversely related. ρ primarily influences the leverage effect—where asset prices and volatility share an inverse relationship—indicating that diverse correlation levels may not radically alter the forecasted log returns but do impact the distribution's symmetry and tail propensity. Variations in κ affect the PDF's peak sharpness, with higher κ indicating a quicker reversion to the long-term variance mean, leading to a more concentrated distribution around the expected variance. This denotes that with a greater κ, the variance in the asset's returns tends to swiftly revert to the long-term average θ, yielding a distribution more densely concentrated around the expected variance.

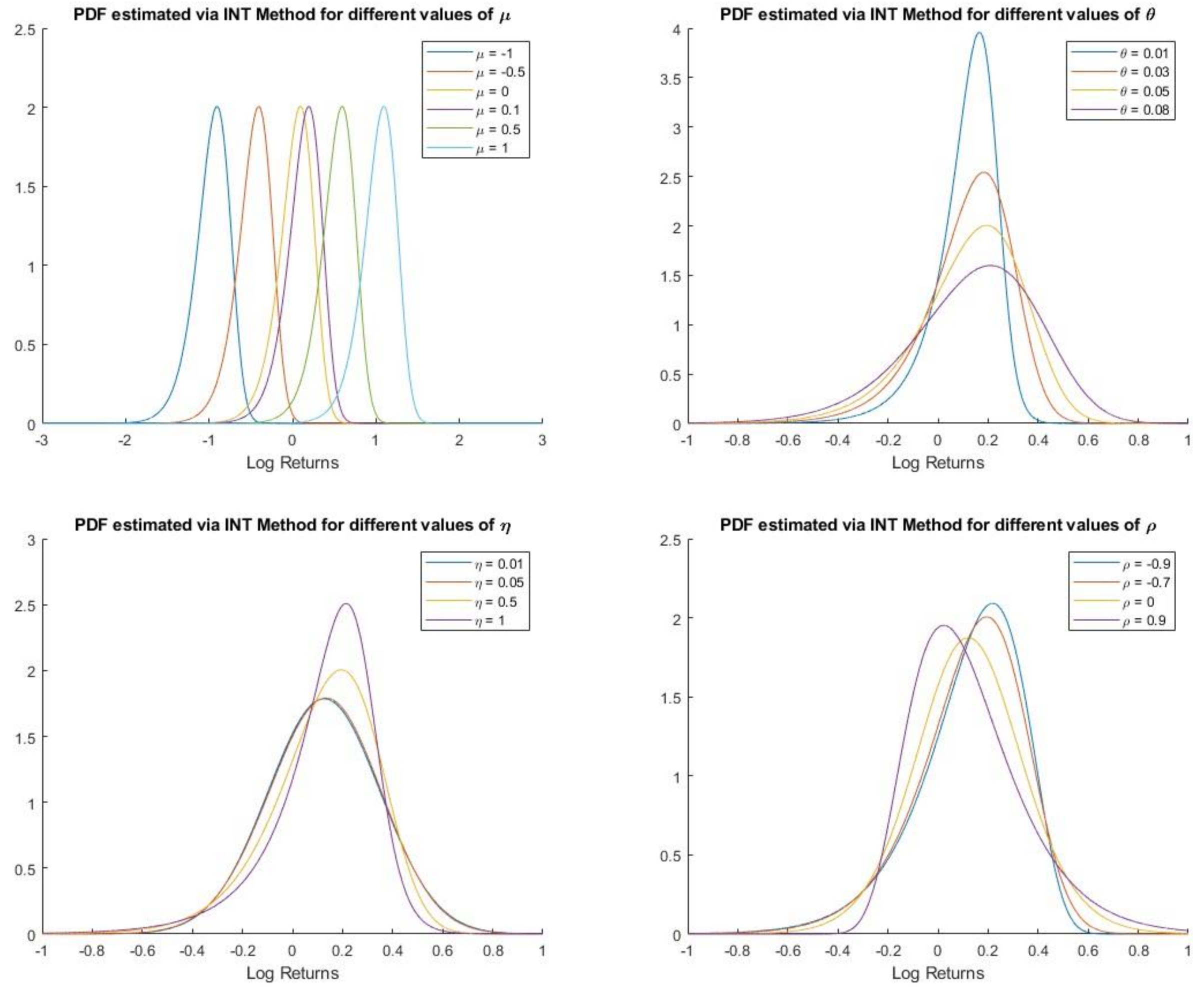
This sensitivity analysis underscores the distinct roles of each Heston model parameter in shaping the log returns' PDF, providing vital insights for financial practitioners. By understanding these effects, professionals can better manage risk and precisely price options, tailoring the Heston model to reflect the underlying asset's risk profile accurately and adjust to different market conditions or asset characteristics.

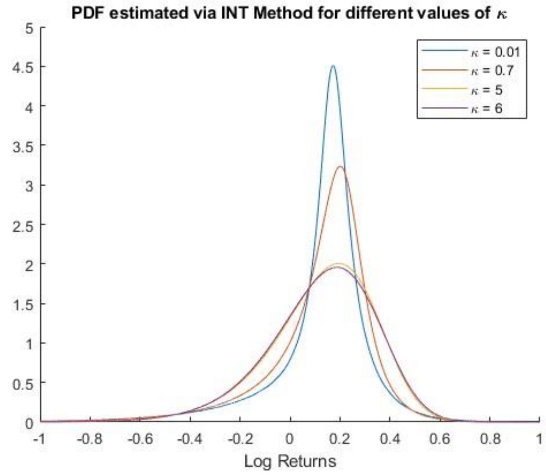
INT Method Sensitivity Analysis:

The INT method's PDF graphs, shown below, provide a nuanced view of the Heston model's parameters on log returns' probability density functions (PDFs), revealing distinct behavioural patterns for each parameter. Increases in the mean rate of return (μ) shift the distribution's apex rightwards, affirming the economic theory that higher mean returns boost expected log returns. This rightward shift, with minimal shape alteration, underscores μ's role in setting the returns distribution's central point without affecting its spread. Changes in the long-term mean variance (θ) visibly alter the distribution's peak and width. Lower θ values create a sharper, more concentrated peak, signalling a tighter return distribution around the mean and less uncertainty. Higher θ values result in a flatter, broader distribution, indicating greater return dispersion and increased long-term variance uncertainty, aligning with θ's purpose of setting the variance target over time. Furthermore, adjusting η expands the distribution, with higher η levels reflecting greater volatility uncertainty, highlighting η's impact on the stochastic volatility process. The correlation coefficient (ρ) significantly affects the PDF's skewness, with negative ρ values causing a leftward skew (emphasizing the leverage effect during price declines) and positive ρ values skewing it rightward. The rate of variance mean reversion (κ) impacts the PDF's peak sharpness, where higher κ values suggest quicker reversion to the long-term average, leading to a more focused distribution and less ambiguity in returns. Conversely, lower κ values indicate a slower reversion, resulting in a wider range of possible returns.

Having said that, this sensitivity analysis clearly delineates each Heston model parameter's influence on the log returns' PDF shape, highlighting the model's flexibility and precision in option pricing and risk management. Understanding these impacts allows for the model's effective adjustment based on empirical data, essential for informed financial decision-making.

Upon visual inspection, the sensitivity analysis conducted through two different methods (COS and INT) yields comparable results. This demonstrates the robustness and reliability of both methods in assessing the impact of various parameters on the Heston model. Furthermore, obtaining consistent results from different methodologies enhances confidence in the accuracy of the findings. It underscores the reliability of the conclusions drawn from the analysis, reinforcing the validity of the study's outcomes.

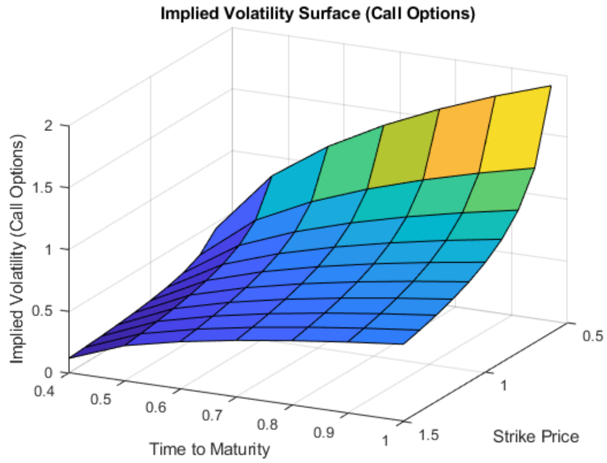


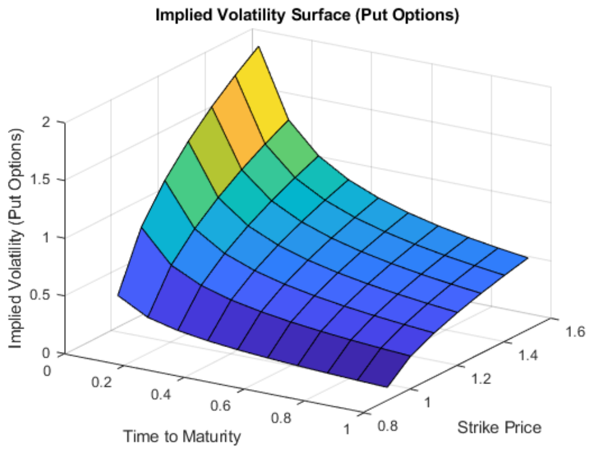


1.4. Obtaining Implied Volatility Surface

The stock prices simulated via the Heston model were subsequently used to compute implied volatilities using the Black-Scholes-Merton (BSM) formula. Initially, diverse stock prices were simulated, and then implied volatilities were calculated by varying maturities and strike prices, with the risk-free interest rate held constant at 1% p.a. throughout the process. The COS method was employed for computing the stock prices within the Heston model

The results are presented graphically below, illustrating the implied volatility surface for European call and put options.





For call options, the implied volatility surface depicts a smooth curve, exhibiting a notable increase in implied volatility for lower strike prices and longer time to maturity. This behaviour aligns with market dynamics, where implied volatility tends to be higher for options with lower strike prices and longer maturities. Moreover, the surface reveals a steep rise in implied volatility for short maturities, followed by a gradual decline as maturity extends—a pattern consistent with market observations.

Similarly, the implied volatility surface for put options also manifests as a smooth curve, demonstrating a sharp rise in implied volatility for lower strike prices and longer maturities. Again, this aligns with market behaviour, where implied volatility typically increases for options with lower strike prices and longer maturities. Notably, the surface exhibits a gradual increase in implied volatility for short maturities, followed by a steep ascent as maturity extends—a pattern echoing the behaviour observed in call options.

These implied volatility surfaces offer valuable insights into market dynamics, particularly the phenomenon of the volatility smile, wherein implied volatility tends to be higher for options with lower strike prices and longer maturities. The smooth curves and consistent patterns observed in the surfaces underscore the robustness of the implied volatility estimation via the COS method. This consistency, coupled with alignment with market observations, validates the reliability and accuracy of the COS method in capturing the intricate dynamics of financial markets.

1.5. Sensitivity Analysis on Implied Volatilities

A sensitivity analysis was conducted by varying the input parameters (θ, η, ρ, and κ) within the Heston model to assess their influence on the Black Scholes implied volatilities of European call options. The COS method was used to estimate the stock prices. To conduct the sensitivity analysis one parameter was adjusted while keeping all other parameters constant. The graphical results are presented on the following page.

The sensitivity analysis conducted on the Heston model parameters concerning their influence on the implied volatility of European call options and the consequent shaping of the log-return distribution demonstrates a clear correlation between η and the implied volatility level. This relationship is expected, given that η represents the volatility of volatility parameter, implying that any alterations in η will directly affect implied volatility. Higher values of η indicate increased uncertainty regarding significant movements in asset prices, particularly for out-of-the-money options. This finding reaffirms the significant impact of η on the fat tails of the distribution. Nevertheless, it's crucial to exercise caution and refrain from overestimating the frequency and impact of extreme market events, as this could lead to the overpricing of tail risk.

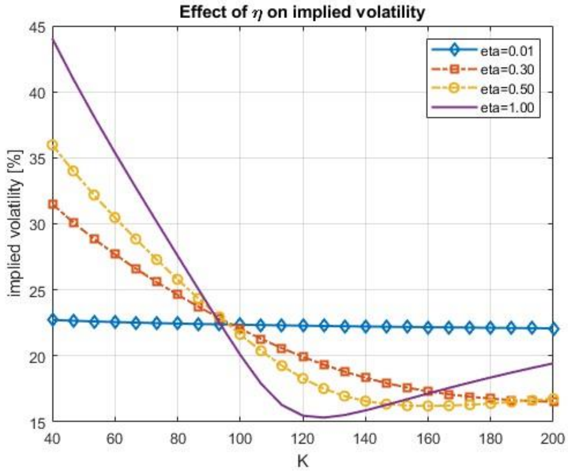
Furthermore, the parameter κ exhibits a significant impact on implied volatility, particularly noticeable at extreme strike prices. A high κ indicates a steeper and more pronounced peak in the distribution, suggesting a swift reversion to mean variance. However, the variability of κ's influence across strike prices suggests that it may not uniformly capture the mean reversion behaviour in different market conditions, thus questioning its reliability in a dynamic financial environment.

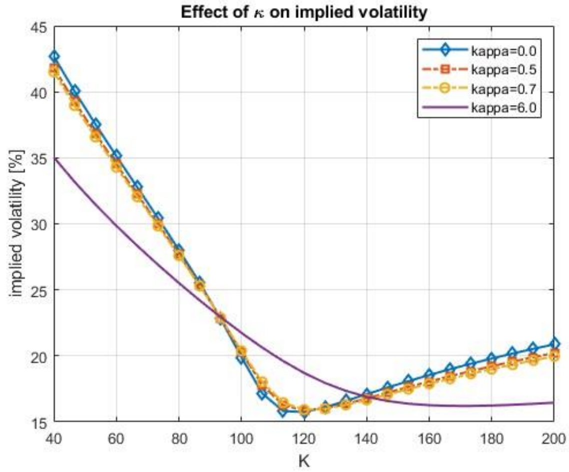
The role of ρ is crucial in explaining the asymmetry in the implied volatility curve. Negative ρ values increase volatility for in-the-money options, confirming the presence of the leverage effect. The relationship between ρ and the market's pricing of risk during downturns (left skew) is evident, yet it is essential to recognize that ρ is often estimated with considerable error, and the assumption of constant correlation may not hold in turbulent markets, limiting the model's effectiveness.

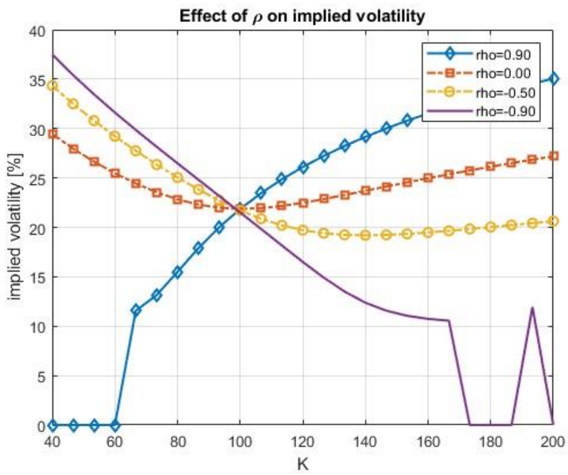
Finally, Higher θ values raise the general level of implied volatilities, pointing to increased long-term uncertainty. The parameter θ is thus integral to defining the central plateau of the implied volatility surface. However, this parameter's influence is somewhat broad and unspecific, making it a less precise tool for fine-tuning the volatility smile or smirk that options markets often exhibit.

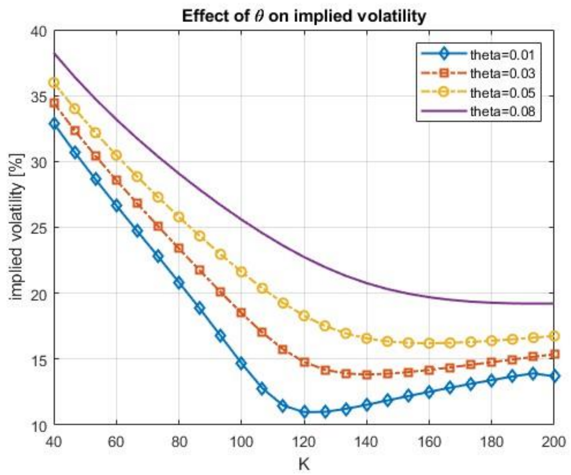
The sensitivity analysis undoubtedly emphasizes the individual and combined effects of the Heston model parameters on the implied volatility landscape. Each parameter distinctly contributes to the characteristics of the implied volatility surface: η governs the tails, κ affects the peak, ρ determines the skew, and θ sets the baseline level. Nonetheless, the model's static nature regarding these parameters may not encapsulate the evolving conditions of financial markets, potentially leading to mismatches between modelled and observed market behaviour. A dynamic calibration of the model, accounting for time-varying parameters, could offer a solution, albeit at the cost of computational complexity.

Moreover, the analysis reveals that while the parameters critically define the shape and behaviour of the log-return distribution, their real-world applicability is contingent on the accuracy of their estimation. Empirical fitting of these parameters is fraught with challenges, such as market noise and estimation biases, which must be carefully managed. In conclusion, the Heston model's parameters are indispensable in crafting a comprehensive view of the risks inherent in option pricing. Still, practitioners must exercise critical judgment in their application, remaining mindful of the model's limitations and the nuanced behaviour of financial markets.









1.6. Conclusion

In conclusion, the Heston model's parameters exert significant influence over both the log-return distribution and the implied volatility surface, pivotal aspects in option pricing and risk assessment. Through meticulous sensitivity analysis, the distinct contributions of each parameter (μ, θ, η, ρ, and κ) on shaping these critical metrics have been revealed.

η, as the volatility of volatility parameter, directly impacts the implied volatility level, particularly affecting the fat tails of the distribution. Meanwhile, κ, representing the mean reversion speed parameter, plays a significant role in shaping the implied volatility curve, especially noticeable at extreme strike prices. Additionally, ρ, the correlation coefficient, dictates the asymmetry observed in the implied volatility curve, emphasizing the leverage effect and its implications during market downturns. Moreover, θ, the long-term mean variance parameter, defines the baseline level of implied volatilities, indicating long-term uncertainty. Its broad influence extends to the log-return distribution, influencing the distribution's peak and spread.

The analysis highlights the nuanced interplay of these parameters in shaping market dynamics, providing valuable insights for option pricing and risk management. Nevertheless, it is crucial to acknowledge the model's limitations, particularly in capturing evolving market conditions and the challenges associated with parameter estimation.

Overall, the Heston model parameters serve as indispensable tools in comprehending and managing the risks inherent in option pricing. Their judicious application, informed by empirical data and critical judgment, is essential for navigating the complexities of financial markets and ensuring robust risk management practices.

Q2. VAR VIA MONTE CARLO SIMULATION

Premises

The analysis explores the computation of Value at Risk (VaR) for a short position in a call option on an equity index S expiring in 3 months, utilizing Monte Carlo Simulation (MCS) techniques. This computation is pivotal for determining appropriate capital requirements. It assumes the following:

1. The initial stock price is $100 and a risk-free interest rate of 1% per annum.
2. The pricing of the European call option is under the real-world probability measure instead of the risk neutral probability measure. However, the analysis still utilizes the Black Scholes Merton (BSM) model which assumes the latter.
3. All the VaR computations are done under the Gaussian Parametric approach using 1 million simulations and using the trading days day count convention.

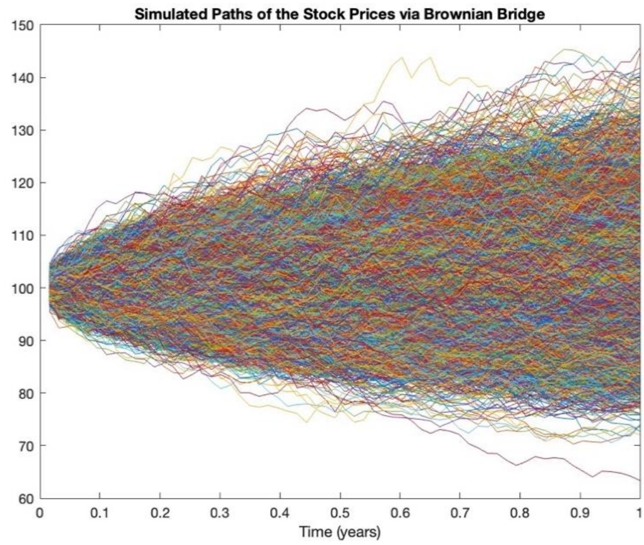
2.1. VaR via MCS using the Brownian Bridge

The underlying stock of the call option is assumed to follow the process under the real probability measure as below:

$$S_t = S_0 e^{\mu t + \sigma W_t}$$

$$\begin{aligned} where\ S_t &= Stock\ Price\ at\ time\ t \\ S_0 &= Initial\ Stock\ Price \\ \mu &= Longrun\ Average\ Stock\ Return \\ \sigma &= Volatility\ of\ Stock\ Return \\ W_t &= Brownian\ Motion \end{aligned}$$

The procedure begins with simulating the Brownian Bridge. A Brownian bridge is a type of stochastic process that is derived from a standard Brownian motion. While a Brownian motion is a random process that exhibits continuous, unpredictable movement, a Brownian bridge is constructed to maintain certain constraints, typically starting, and ending at specific points. It is essentially simulated by calculating the weights by taking the linear interpolation between the Brownian motion at time t and T. The simulated stock prices for 10000 simulations can be visualised as follows:



Upon simulating the stock prices, the team calculated the simulated log returns for both 1-day and 10-day periods. This enabled the determination of the 5% quantile of the returns and, consequently, the holding period returns for both 1-day and 10-day stock prices, which are:

```
5% quantiles for 1 & 10 day holding period log returns generated via brownian bridge respectively:
   -0.0204

   -0.0627
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These quantiles provide insights into the potential downside risk associated with the respective holding periods. A negative quantile indicates that there is a 5% probability that the actual log returns will fall below these values over the specified holding periods, suggesting potential losses in the simulated scenarios. Now evaluating the precision of the method by comparing it to the closed-form expression for quantiles derived from the Gaussian distribution where the stock price is modelled as below:

$$S_t = S_0 e^{\mu t + \sigma Z_\alpha}$$
$$where\ Z_\alpha = \Phi^{-1}(\alpha)$$
$$\alpha = \ Quantile$$

Then again computing the log returns for the above simulated stock prices to obtain the quantiles. The statistical measure used to assess the accuracy of the estimates is the difference between the two which is:

Error between Brownian Bridge & Gaussian

1 Day	10 Day
0.015482	0.013265

Utilizing the 5% quantile values derived from the simulated stock prices for both 1-day and 10-day periods, one can proceed to compute the corresponding call option prices under the BSM model. It is important to note that the day count convention employed in this computation is based on actual days, ensuring accurate valuation within the BSM framework. The (1-α)% VaR for the call option is formulated as below:

$$VaR_\alpha^h = \ C\ (Sh, K, T-h, r) \ - \ C\ (S0, K, T, r)$$

Now computing the 1-day and 10-day VaR for a range of strike prices from 80 to 120 yields the following output:

Strike Price	1 day 95% VaR	10 day 95% VaR
80	-1.9958	-5.9869
85	-1.9131	-5.6772
90	-1.7179	-4.9786
95	-1.3941	-3.8882
100	-0.99674	-2.6423
105	-0.62068	-1.5509
110	-0.33646	-0.78891
115	-0.15976	-0.35109
120	-0.067069	-0.13831

Examining the table reveals a nuanced relationship between strike prices and VaR values. As strike prices increase, the corresponding VaR tends to decrease. This phenomenon aligns with the intrinsic nature of short call options, where higher strike prices imply a reduced likelihood of the option being exercised, thus limiting potential losses for the option seller. Moreover, the table highlights the impact of time horizons on VaR estimates. Over a 10-day period, the potential losses associated with short call options tend to be higher compared to a 1-day period. This temporal effect is consistent with the concept of time decay in options trading, where longer timeframes allow for increased volatility and uncertainty, consequently amplifying risk exposure. Thus there would be higher chances of the call option to end in-the-money for the counterparty.

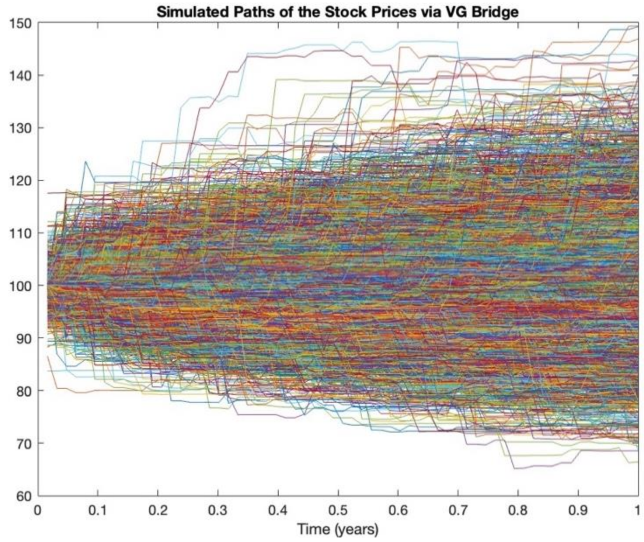
2.2. VaR via MCS using the Variance-Gamma Bridge

The underlying stock of the call option is assumed to follow the process as below:

$$S_t = S_0 e^{\mu t + X_t}$$

$$where\ X_t\ is\ Variance\ Gamma\ Process\ given\ by$$
$$X_t = \ \theta\, G(t) + \ \sigma W\left(G(t)\right), \theta\ \in\ R, \sigma\ >\ 0$$
$$G(t) is\ a\ Gamma\ process\ with\ parameters\ \alpha\ =\ \lambda\ =\ k^{-1}, k\ >\ 0$$

The procedure follows similarly as in the previous section i.e. it begins by simulating the Variance-Gamma Bridge (VG Bridge). The VG bridge technique involves generating correlated Gaussian random variables and then transforming them into Variance Gamma distributed random variables. This approach leverages the properties of the VG distribution to simulate asset price paths efficiently while preserving the key features of the VG process, including jumps and volatility clustering. For the VG process the analysis assumes the following parametric values: (μ, θ, σ, κ) = (0.11, −0.01, 0.2, 0.045). θ is the mean of the VG process and it dictates the sign of the skewness. σ is the variance of the VG process and κ is the variance rate of the Gamma process which controls the level of excess kurtosis . The simulated paths for the VG Bridge can be seen as follows:



By uniformly applying the previously employed methodology, which involves utilizing simulated paths to generate stock prices and subsequently computing log returns, one can derive the 5% quantile log returns for both 1-day and 10-day stock price movements. This methodology involves first simulating stock price paths using the previously established simulated paths. Then, by computing the logarithmic returns from these simulated price paths, one obtains a distribution of returns. From this distribution, the 5% quantile represents the log return value below which only 5% of the simulated returns fall. Hence, the holding period log return at 5% quantile is as below:

```
5% quantiles for 1 & 10 day holding period log returns generated via variance gamma bridge respectively:
   -0.0139

   -0.0624
```

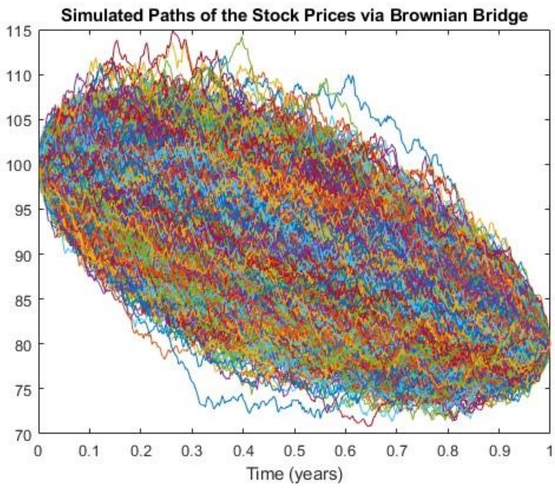
Now calculating the 1-day and 10-day Value at Risk (VaR) for European call options across strike prices ranging from 80 to 120 using the VG simulations produces the subsequent results:

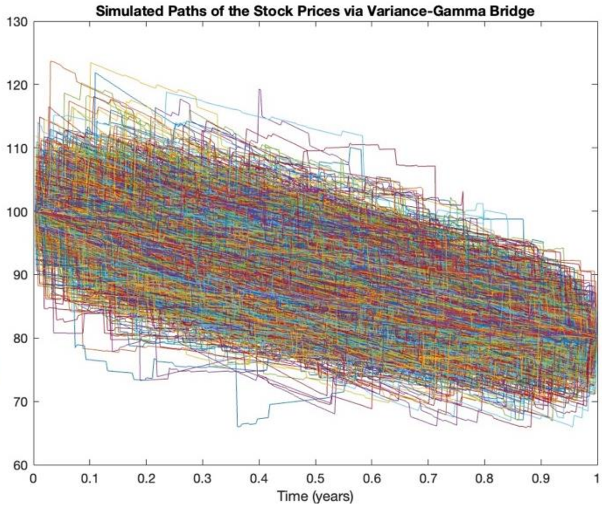
Strike Price	1 day 95% VaR	10 day 95% VaR
80	-1.3639	-5.9522
85	-1.3122	-5.6458
90	-1.1869	-4.9535
95	-0.9733	-3.8711
100	-0.70491	-2.6325
105	-0.44526	-1.5462
110	-0.24496	-0.78702
115	-0.11804	-0.35044
120	-0.050274	-0.13812

The observations and conclusions made from the previous analysis all still hold. Upon comparing the two tables, it is evident that the VaR values derived using the VG Bridge for each strike price and time horizon are lower than those obtained from the Brownian Bridge. This is due to the fact that under the VG Bridge it simulates small jumps whereas the Brownian Bridge is Gaussian in nature, and it will have 95% of its values within the interval of 2 standard deviations.

2.3. Stress Testing of VaR via MCS using Brownian Bridge & Variance-Gamma Bridge

Conducting stress testing on Value at Risk (VaR) involves utilizing the various Bridge's methodologies, wherein simulated stock prices begin at an initial value of $100 and eventually decline to $80 within one year. Logically, this scenario implies that VaR values should lean towards the lower end for a short European Call position, as there is a higher likelihood of the option being in-the-money at maturity. However, it's crucial to critically evaluate this expectation. Initially, the reduction in VaR values might not be significantly pronounced, primarily due to the modelling of extreme tail events. Despite the anticipated downward trend in stock prices over the year, the option expiration period is only three months. During this initial phase, the stock price could potentially increase before experiencing a subsequent decline. This temporal mismatch between the option's expiration and the gradual decline in stock price could mitigate the immediate impact on VaR values. So, the simulated stock prices under the Brownian Bridge & VG Bridge can be observed graphically below:





The above shows how the stock price begins at the initial price and the final price is at the stressed scenario of $80. Similarly adopting the procedure to compute the VaR produces the results below:

a) Using Brownian Bridge:

Strike Price	1 day 95% VaR	10 day 95% VaR
80	-0.90686	-3.2286
85	-0.87545	-3.1307
90	-0.79704	-2.862
95	-0.6601	-2.3679
100	-0.48395	-1.7188
105	-0.30986	-1.0786
110	-0.17289	-0.58424
115	-0.084495	-0.27504
120	-0.036488	-0.11375

b) Using VG Bridge:

Strike Price	1 day 95% VaR	10 day 95% VaR
80	-0.13777	-2.7867
85	-0.13665	-2.7136
90	-0.13106	-2.5015
95	-0.1169	-2.0947
100	-0.09345	-1.5422
105	-0.065438	-0.982
110	-0.039821	-0.53946
115	-0.021107	-0.25725
120	-0.0098231	-0.10761

As expected, the VaR values are lower compared to the previous results however, there is not that much of a discrepancy. Despite the drastic decline in simulated stock prices to $80, the VaR values do not exhibit a proportional decrease across all strike prices. This discrepancy can be attributed to various factors, including the option's intrinsic value, implied volatility, and time decay dynamics.

3.1. References

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