Homework 1: representation and simulation of Inverted Pendulum

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Outline

Introduction

This presentation wants to cover the topics of Homework 1, of the discipline of advanced control, namely,

- Mathematical Modeling of the system,
- Linearization and system representation,
- Stability analysis.

The physical system which we will analyze is called "Inverted Pendulum".

- Codes and further explanation are available in `https://github.com/eddardd/Control-Theory/tree/master/Advanced-Control`

Intuition

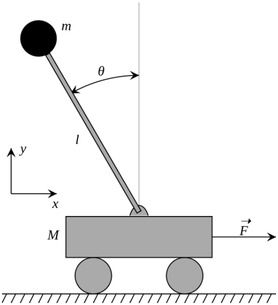

Figura: The physical schematic of our system

The system we have chosen is composed by a cart, with mass M, with a pole of length ℓ attached to it, with a ball of mass m at its extrema.

Intuition

The study of such system encounters applications, for example, in the development of devices called "Segways",

Figura: An example of a segway

Physical modeling

In order to describe the motion of our system, we shall adopt the Lagrangian formalism. To that effort, we need to define

$$\mathcal{L} = T - V$$

which accomplishes for the total energy in the system. Once we have it, we know it satisfies, for each degree of freedom, the **Euler-Lagrange** equation,

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0$$

Physical Modeling

Analyzing the system, we recognize two degrees of freedom:

- The cart's position, x,
- The pole's angle, θ

Being so,

- $T = \frac{1}{2}(M + m)\dot{x}^2 - m\dot{x}\dot{\theta}\ell cos(\theta) + \frac{1}{2}(m\ell^2 + I)\dot{\theta}^2$
- $V = mg\ell cos(\theta)$

Physical Modeling

With the last equations, and taking derivatives, we can write the Euler-Lagrange equations twice,

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = F - b_x \dot{x}$$
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = -b_\theta \dot{\theta}$$

These expressions leads to a system of (non-linear) equations,

$$(M+m)\ddot{x} - m\ell cos(\theta)\ddot{\theta} = F + m\ell sin(\theta)\dot{\theta} - b_x \dot{x}$$
$$(m\ell^2 + I)\ddot{\theta} - m\ell cos(\theta)\ddot{x} = mg\ell sin(\theta) - b_\theta \dot{\theta}$$

Physical Modeling

Those equations can be solved by using Cramer's Rule, yielding the following,

$$\ddot{x} = \frac{(m\ell^2 + I)(F + m\ell\dot{\theta} sin(\theta) - b_x\dot{x}) + m\ell cos(\theta)(mg\ell sin(\theta) - b_\theta\dot{\theta})}{(M + m)(m\ell^2 + I) - m^2\ell^2 cos^2(\theta)}$$

$$\ddot{\theta} = \frac{(M + m)(mg\ell sin(\theta) - b_\theta\dot{\theta}) + m\ell cos(\theta)(F + m\ell\dot{\theta} sin(\theta) - b_x\dot{x})}{(M + m)(m\ell^2 + I) - m^2\ell^2 cos^2(\theta)}$$

which are the non-linear equations that govern the system dynamics.

Mathematical Analysis

Since we have equations for $\ddot{x}$ and $\ddot{\theta}$, we notice that,

$$\begin{aligned}
\dot{x} &= f_1(t, x, \dot{x}, \theta, \dot{\theta}, u) \\
\ddot{x} &= f_2(t, x, \dot{x}, \theta, \dot{\theta}, u) \\
\dot{\theta} &= f_3(t, x, \dot{x}, \theta, \dot{\theta}, u) \\
\ddot{\theta} &= f_4(t, x, \dot{x}, \theta, \dot{\theta}, u)
\end{aligned}$$

That is, by defining $\mathbf{x} = (x, \dot{x}, \theta, \dot{\theta})$, $\mathbf{f} = (f_1, f_2, f_3, f_4)$, we have,

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, u)$$

Mathematical Analysis

Indeed, we have defined a (non-linear) vectorial field over $\mathbf{R}^4$. A quick simulation using numerical integration gives us the following results,

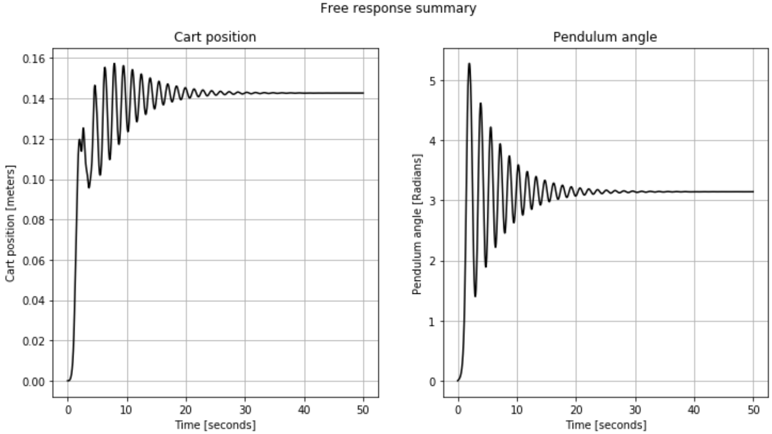

Figura: Step-response of the system

Mathematical Analysis

Indeed, two points are of our interest,

- The upward point, $\mathbf{x}_{sp} = (0, 0, 0, 0)$, which yields, by inspection, $\mathbf{f}(t, \mathbf{x}) = 0$.
- The downward point, $\mathbf{x}_{sp} = (0, 0, \pi, 0)$, which yields, by inspection, $\mathbf{f}(t, \mathbf{x}) = 0$.

These points gives rise of what we call **singular trajectories**, since for every t, with those initial conditions, $\mathbf{f}(\mathbf{x}) = 0$.

Stability

Assuming $\mathbf{x}_{sp}$ being a stationary point, we informally define stability as,

- Stable stationary points are those whose, after applied a disturbance, tends to get back to the original point.
- Unstable points are those whose, after applied a disturbance, tends to go away from the original point.

Under those definitions, we notice that $(0, 0, 0, 0)$ is a unstable point, while $(0, 0, \pi, 0)$ is a stable one, matching our common sense.

Linearization

- Linearization is a local technique analysis, which permits us to transform the nonlinear equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u)$ into a linear one,
- By Taylor's Expansion,

$$\mathbf{f}(\mathbf{x}_{sp}, u) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x} - \mathbf{x}_{sp}) + \frac{\partial f}{\partial u} u$$

in which we identify $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ as the Jacobian matrix of $\mathbf{f}$. This gives us the following state-space representation,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

State Space equations

For $\mathbf{x}_{sp} = (0, 0, 0, 0)$, we calculate the partial derivatives of f, to achieve:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\dfrac{(m\ell^2 + I)b_x}{\alpha} & \dfrac{(m\ell)^2 g}{\alpha} & \dfrac{-m\ell b_\theta}{\alpha} \\ 0 & 0 & 0 & 1 \\ 0 & -\dfrac{m\ell b_x}{\alpha} & \dfrac{(M+m)mg\ell}{\alpha} & \dfrac{-(M+m)b_\theta}{\alpha} \end{bmatrix}$$

with $\alpha = (M+m)(m\ell^2 + I) - (m\ell)^2$.

State Space equations

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with $\alpha = (M+m)(m\ell^2 + I) - (m\ell)^2$.

State Space equations

For **B**, we have:

$$\mathbf{B}_0 = \begin{bmatrix} 0 \\ \dfrac{m\ell^2 + I}{\alpha} \\ 0 \\ \dfrac{m\ell}{\alpha} \end{bmatrix} \qquad \mathbf{B}_\pi = \begin{bmatrix} 0 \\ \dfrac{m\ell^2 + I}{\alpha} \\ 0 \\ \dfrac{-m\ell}{\alpha} \end{bmatrix}$$

Transfer Function

The transfer functions of the system can be encountered by transforming the State-Space equation to frequency, and solving for $\frac{Y}{U}$, to do so, we need,

- To define an choice for input, which in our case is unique, that is, the force F.
- To define an choice for output, which in our case we shall define $y = x$ and $y = \theta$.

Therefore, we use linearize the equations from physical modeling to derive two transfer functions, T_x and T_θ

Transfer Function

Calculating and solving for each variable yields, for $\mathbf{x}_{sp} = (0, 0, 0, 0)$.

$$\frac{X(s)}{U(s)} = \frac{(m\ell^2 + I)s^2 + b_\theta s - mg\ell}{\alpha s^4 + \Big((M+m)b_\theta + (m\ell^2 + I)b_x\Big)s^3 + \Big(b_\theta b_x - (M+m)mg\ell\Big)s^2 - mg\ell b_x s}$$

$$\frac{\Theta(s)}{U(s)} = \frac{m\ell s^2}{\alpha s^4 + \Big((M+m)b_\theta + (m\ell^2 + I)b_x\Big)s^3 + \Big(b_\theta b_x - (M+m)mg\ell\Big)s^2 - mg\ell b_x s}$$

similar results can be done for $\mathbf{x}_{sp} = (0, 0, \pi, 0)$:

$$\frac{X(s)}{U(s)} = \frac{(m\ell^2 + I)s^2 + b_\theta s + mg\ell}{\alpha s^4 + \Big((M+m)b_\theta + (m\ell^2 + I)b_x\Big)s^3 + \Big(b_\theta b_x + (M+m)mg\ell\Big)s^2 + mg\ell b_x s}$$

$$\frac{\Theta(s)}{U(s)} = \frac{-m\ell s^2}{\alpha s^4 + \Big((M+m)b_\theta + (m\ell^2 + I)b_x\Big)s^3 + \Big(b_\theta b_x + (M+m)mg\ell\Big)s^2 + mg\ell b_x s}$$

Stability analysis in linear systems

The following theorem states the equivalence between stability of stationary points in non-linear and linear systems,

Lyapunov-Perron Theorem

Let $\mathbf{x}_{sp}$ be a stationary point of a field $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$. Thus, $\mathbf{x}_{sp}$ is a stable stationary point of $\mathbf{f}$ if, and only if $\mathbf{J}|_{\mathbf{x}=\mathbf{x}_{sp}}$ has only eigenvalues with negative real part.

- This indeed allows us to substitute the analysis of stability of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ by $\dot{\mathbf{x}} = \mathbf{J}\mathbf{x}$

Linear stable systems

We begin by noticing that the solution of $\dot{\mathbf{x}} = \mathbf{J}\mathbf{x}$ relies on the spectra of $\mathbf{J}$,

- If $\mathbf{J}$ is diagonalizable, then the solution is

$$\mathbf{x}(t) = \sum_{i=1}^{n} \boldsymbol{\xi}_i e^{\lambda_i t}$$

  for eigenvalues λ_i and eigenvectors ξ_i.
- If any λ_i has positive real part, then the solution diverges on at least one of its coordinates.
- The system is thus, unstable.

This is the criteria for Linear Stability.

Responses

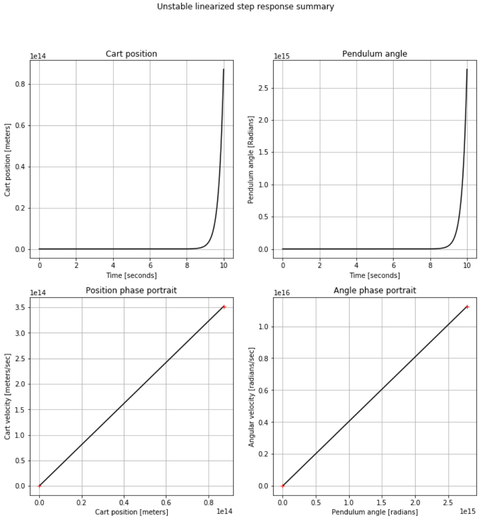

Figura: Linearized step response for $\theta_{sp} = 0$. Observe how it is unstable.

Responses

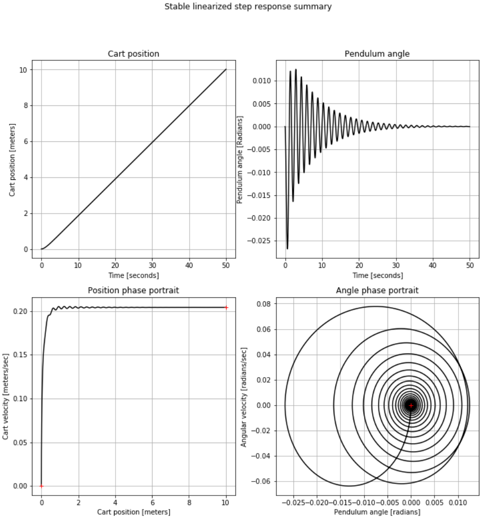

Figura: Linearized step response for $\theta_{sp} = \pi$. Confirming our claim that π is a stable stationary point.

Poles of linearizations

The later responses can be understood by looking at the eigenvalues of each linearized matrix, A:

Eigenvalue	A_0	A_π
λ_1	0	0
λ_2	-4.423	-1.0819
λ_3	-0.9469	-0.126 + 4.218i
λ_4	4.0349	-0.126 - 4.218i

Tabela: System's Eigenvalues

Since A_0 has a pole (or eigenvalue) with positive real part, we conclude that it is an unstable stationary point.

Methodology

To simulate our system, we have used Python language,

- The linearization were simulated using the Control package,
- We have used a hard-coded numerical ODE solver to obtain the non-linear results,
- The animations were done using Matplotlib's functions.

Free Response

We have simulated the free response of our system to an initial condition slightly moving in the anti-clockwise direction,

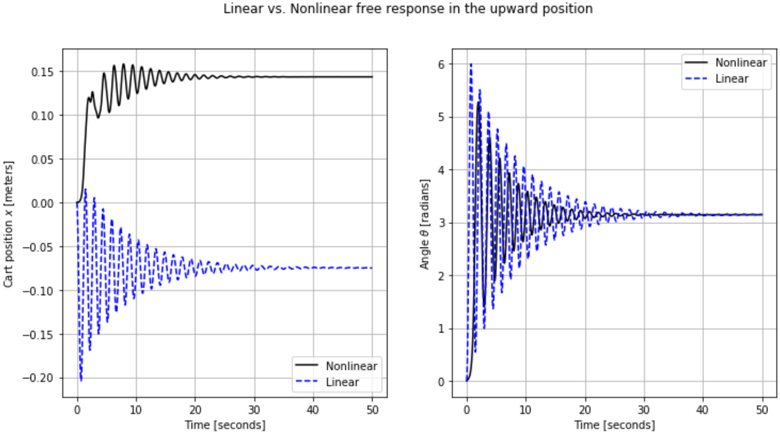

Figura: Linear and non-linear free-responses to initial conditions

Free Response

Those plots shows us that,

- The linearization response resembles the non-linear response for the angle, although for the cart's position, the behavior does not capt the original response,
- The angle tends to go to the stable point, that is, π.
- The cart's position response is stable. After a few oscillations, the pole movement amplitude is small, and the cart settles in a fixed position.

Impulse Response

We have simulated the impulse response of our system to initial conditions $(0, 0, \pi, 0)$, that is, the upward position.

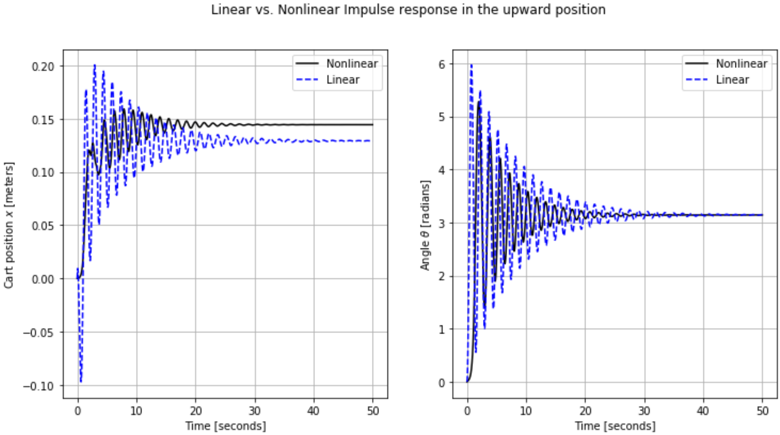

Figura: Linear and non-linear free-responses to an impulse input

Impulse Response

Those plots shows us that,

- Again, we have the same phenomena that the cart's position has a strange linear behavior.
- The angle still settles in π,
- Although given an initial force to the right, the pole's oscillations compensate the driving force to the right, making the cart to settle as the time goes by.

Step Response

We have simulated the step response of our system to initial conditions $(0, 0, \pi, 0)$, that is, the upward position.

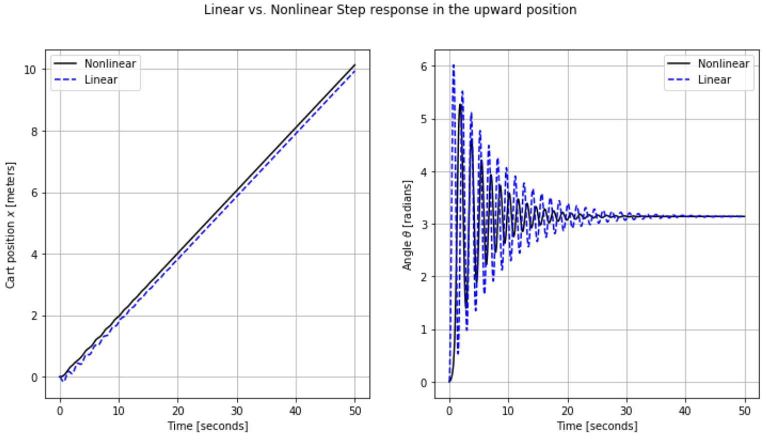

Figura: Linear and non-linear free-responses to a step input

Step Response

From the last figure, we can draw the following comments,

- That time, the linear response approximates quite well the cart's position behavior.
- If we have a continuous driving force to the right, on the cart, no compensation can stop the position to grow without bounds.

Still, we can estimate empirically some time-domain information,

Measure	Linear	Nonlinear
t_r (seconds)	0.25	0.60
t_s (seconds)	22.88	12.51
M_p	91.60%	67.83%
t_p (seconds)	0.74	2.06

Tabela: Time domain response information summary