Study Notes on Optimal Control: Solving state space equations

Eduardo Fernandes Montesuma
email edumontesuma@gmail.com

22/03/2018

Contents

1 First Order Ordinary Differential Equations

1.1 Introduction

We develop a theory to treat a general, closed form solution for any first order (scalar) ordinary differential equation (ODE) because, as we shall see later on, an n-order ODE can be written as a first-order (vectorial) ODE. We start with the simplest kind of ODE, namely:

$$\frac{dx(t)}{dt} = Ax(t) \tag{1}$$

Here, the solution is straight-forward using the fundamental theorem of calculus, after noticing $\frac{d}{dt}[log(x(t))] = \frac{1}{x}\frac{dx}{dt}$:

$$\frac{1}{x}\frac{dx}{dt} = A \tag{2}$$

$$\frac{d}{dt}log[x(t)] = A \tag{3}$$

$$log[x(t)] = At + C_1 \tag{4}$$

$$x(t) = e^{At+C_1} \tag{5}$$

$$x(t) = C_2 e^{At} \tag{6}$$

Where we have used the notation $C_2 = e^{C_1}$. The solution, to be precisely is a family of solutions, indexed by two parameters, C_2 and A:

$$x(t|A, C) = Ce^{At} \tag{7}$$

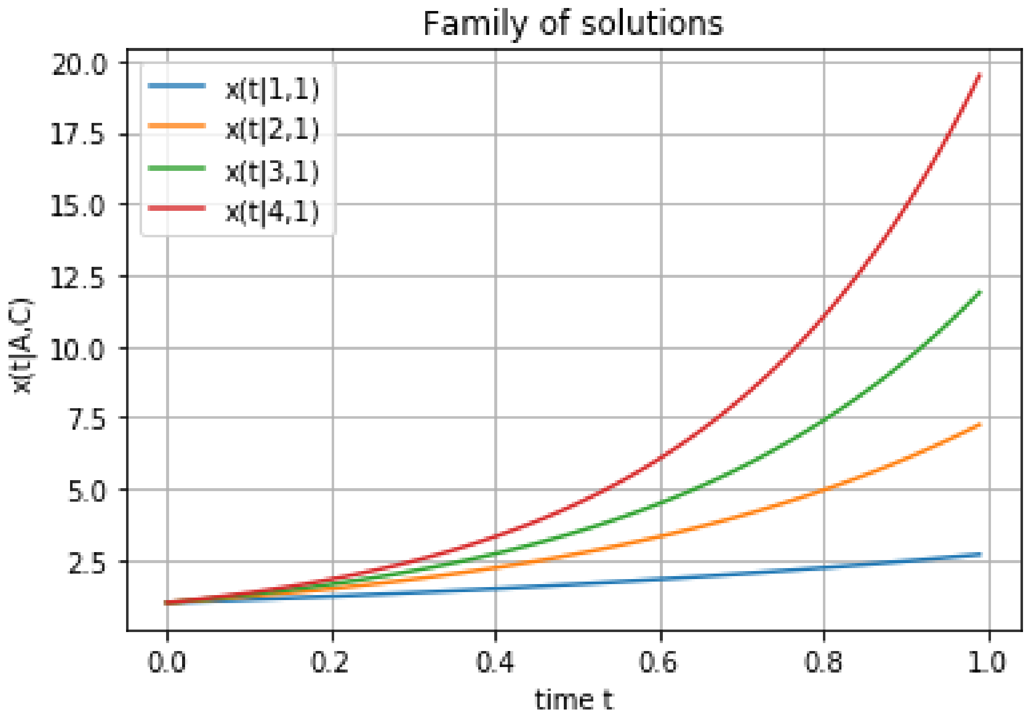

Figure 1: Form of solution's family

The shape of solutions, for each choice of parameters (A, C) is given in Figure 1. Although simple, the methodology to solve this particular kind of ODE can be carried out for more complex cases, that is, we try to manipulate the ODE in order to achieve some integrable function. This is exactly the nature of the integrating factor method, which is now discussed.

1.2 First-Order linear ODEs

The last case was a simple sub-case of a much more wider class of differential equations: the linear ones. A linear first-order ODE is given by,

$$\dot{x}(t) + p(t)x(t) = q(t) \tag{8}$$

where $p(t)$ and $q(t)$ are sufficiently smooth functions[1]. Again, recalling the last paragraph of last section, we want to express the left hand side as a derivative of something.

By force, we want to multiply the ODE by a function $\mu(t)$ such that $\mu(t)(\dot{x}(t) + p(t)x(t)) = \frac{d}{dt}\Big(\mu(t)x(t)\Big)$, that is,

$$\begin{aligned}
\dot{\mu}x + \mu\dot{x} &= \mu\dot{x} + \mu(t)p(t)x(t) &(9)\\
\dot{\mu}x(t) &= \mu(t)p(t)x(t) &(10)\\
\frac{\dot{\mu}}{\mu} &= p &(11)\\
\int_{t_0}^{t} \frac{\dot{\mu}(t)}{\mu} &= \int_{t_0}^{t} p(\tau)d\tau &(12)\\
log\ \mu(t) &= \int_{t_0}^{t} p(\tau)d\tau &(13)\\
\mu(t) &= e^{\int_{t_0}^{t} p(\tau)d\tau} &(14)
\end{aligned}$$

Putting it back into the ODE,

$$\begin{aligned}
\frac{d}{dt}\Big(x(t)e^{\int_{t_0}^{t} p(\tau)d\tau}\Big) &= q(t)e^{\int_{t_0}^{t} p(\tau)d\tau} &(15)\\
x(t)e^{\int_{t_0}^{t} p(\tau)d\tau} - x(t_0)e^{\int_{t_0}^{t_0} p(\tau)d\tau} &= \int_{t_0}^{t} q(r)e^{\int_{t_0}^{r} p(\tau)d\tau}dr &(16)
\end{aligned}$$

And therefore, the solution takes form:

$$x(t) = x(t_0)e^{-\int_{t_0}^{t} p(\tau)d\tau} + e^{-\int_{t_0}^{t} p(\tau)d\tau}\int_{t_0}^{t} q(r)e^{\int_{t_0}^{r} p(\tau)d\tau}dr \tag{17}$$

Such a solution is valid for any functions $p(t)$ and $q(t)$ satisfying our constraints. In practice, to describe systems with those concepts is to consider between the following classifications,

[1]We mean by "sufficiently smooth" functions that are both continuous and differentiable.

- Time-Invariant systems, in which $p(t) = a$, $q(t) = b$, for all $t \in \mathbb{R}$.
- Homogeneous systems, in which $q(t) = 0$.

For each of those classifications, there comes assumptions which simplifies our equations.

2 n-order differential equations and State-Space equations

2.1 Linear and Homogeneous case

Consider a differential equation of n^{th} order. For now, let us consider the linear homogeneous case, that is,

$$\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_1\frac{dx}{dt} + a_0 = 0 \tag{18}$$

We can transform this equation into a system of equations, using the following notation,

$$x_1 = x \qquad \dot{x}_1 = \frac{dx}{dt} \tag{19}$$

$$x_2 = \frac{dx}{dt} \qquad \dot{x}_2 = \frac{d^2x}{dt^2} \tag{20}$$

$$\vdots \tag{21}$$

$$x_n = \frac{d^{n-1}x}{dt^{n-1}} \qquad \dot{x}_n = \frac{d^n x}{dt^n} \tag{22}$$

Noticing that $\dot{x}_j = x_{j+1}$, for $0 \leq j < n$, and that $\dot{x}_n = \dfrac{d^n x}{d^n t} = \sum_j (-a_j)x_j$, we can express the system through matrices:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{23}$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{24}$$

Notice that this later differential equation resembles Equation 1. We argue that the solution of such a system is $\mathbf{x}(t) = \mathbf{x}(0)e^{\mathbf{A}t}$, just as before, but with the exponential of a matrix.

Before prove our claim, we shall provide the definition of matrix exponentials. Looking at the scalar case, through Taylor series, the exponential function can be expressed as:

$$e^{at} = 1 + \sum_{n=1}^{\infty} \frac{a^n t^n}{n!} \tag{25}$$

Under the assumption of convergence, this series gives us the given exponential. If we instead consider matrices, we can write:

$$e^{\mathbf{A}t} = \mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{A}^n t^n}{n!} \tag{26}$$

Where the convergence is done element-wise. Consider, now, the derivative:

$$\frac{d}{dt} e^{\mathbf{A}t} = \frac{d}{dt}\mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{A}^n}{n!} \frac{d}{dt} t^n \tag{27}$$

$$= \sum_{n=1}^{\infty} \frac{\mathbf{A}^n}{(n-1)!} t^{n-1} \tag{28}$$

$$= \mathbf{A}\left(\mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{A}^n t^n}{n!}\right) \tag{29}$$

$$= \mathbf{A} e^{\mathbf{A}t} \tag{30}$$

Therefore, if $\mathbf{x}(t) = \mathbf{x}(0) e^{\mathbf{A}t}$, we can write:

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{x}(0) \frac{d}{dt} e^{\mathbf{A}t} \tag{31}$$

$$= \mathbf{A}(\mathbf{x}(0) e^{\mathbf{A}t}) \tag{32}$$

$$= \mathbf{A}\mathbf{x}(t) \tag{33}$$

As we wanted. This case is indeed simple and without complications, since A does not depend on time, and the equation is homogeneous. We shall explore more complex cases henceforth.

2.2 Non-homogeneous time-invariant

In that case, we are considering the following ODE:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{34}$$

for a sufficiently smooth function $\mathbf{u} : \mathbb{R} \to \mathbb{R}^n$. To solve such a system, we perform just like we have done with the integrating factor[2]:

$$e^{-\mathbf{A}t}\dot{\mathbf{x}}(t) - e^{-\mathbf{A}t}\mathbf{A}\mathbf{x}(t) = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t) \tag{35}$$

Noticing that $\dfrac{d}{dt}e^{\mathbf{A}t}\mathbf{x}(t)$ yields the right-hand-side (RHS) of such equation,

$$\frac{d}{dt}e^{-\mathbf{A}t}\mathbf{x}(t) = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t) \tag{36}$$

$$e^{-\mathbf{A}t}\mathbf{x}(t) - e^{-\mathbf{A}t_0}\mathbf{x}_0(t_0) = \int_{t_0}^{t} e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau \tag{37}$$

Rearranging, gives us the full solution for the time-invariant system,

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}_0(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau \tag{38}$$

As we shall see in the next section, this is the solution for the state equations of time-invariant systems.

2.3 Non-homogeneous time-variant

This is the most general case for linear systems. they are governed by the ODE,

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \tag{39}$$

For such equation, however, the solution is not straightforward as before, since we need to solve the integrating factor

$$\mu(t) = e^{-\int_{t_0}^{t}\mathbf{A}(\tau)d\tau} \tag{40}$$

Indeed, we shall rather represent $\mu(t)$ as a transition matrix, between time t_0 and t,

$$\Phi(t, t_0) = e^{-\int_{t_0}^{t}\mathbf{A}(\tau)d\tau} \tag{41}$$

Which has the following properties,

Property 1: $\dfrac{\partial\Phi}{\partial t} = -\Phi(t, t_0)\mathbf{A}(t)$

Proof: By our definition,

[2]indeed, $\mu(t) = e^{-\int_0^t \mathbf{A}(\tau)d\tau}$, which leaves us with $e^{-\mathbf{A}t}$

$$\frac{\partial \Phi}{\partial t} = \frac{\partial}{\partial t} e^{-\int_{t_0}^{t} \mathbf{A}(\tau) d\tau} \tag{42}$$

$$= e^{-\int_{t_0}^{t} \mathbf{A}(\tau) d\tau} \frac{d}{dt}\left(- \int_{t_0}^{t} \mathbf{A}(\tau) d\tau \right) \tag{43}$$

$$= -\Phi(t, t_0)\mathbf{A}(t) \tag{44}$$

Where from the first equation, from the second, we have used the matrix exponential derivative with the chain rule.

Property 2: $\dfrac{\partial \Phi}{\partial t_0} = \mathbf{A}(t)\Phi(t, t_0)$

Proof: Again, using the definition,

$$\frac{\partial \Phi}{\partial t} = \frac{\partial}{\partial t_0} e^{-\int_{t_0}^{t} \mathbf{A}(\tau) d\tau} \tag{45}$$

$$= \frac{d}{dt_0}\left(- \int_{t_0}^{t} \mathbf{A}(\tau) d\tau \right) e^{-\int_{t_0}^{t} \mathbf{A}(\tau) d\tau} \tag{46}$$

$$= \mathbf{A}(t)\Phi(t, t_0) \tag{47}$$

Now, proceeding as before,

$$e^{-\int_0^t \mathbf{A}(\tau) d\tau} \dot{\mathbf{x}}(t) - e^{-\int_0^t \mathbf{A}(\tau) d\tau} \mathbf{A}(t)\mathbf{x}(t) = e^{-\int_0^t \mathbf{A}(\tau) d\tau} \mathbf{B}(t)\mathbf{u}(t) \tag{48}$$

$$\frac{d}{dt}(e^{-\int_0^t \mathbf{A}(\tau) d\tau} \mathbf{x}(t)) = e^{-\int_0^t \mathbf{A}(\tau) d\tau} \mathbf{B}(t)\mathbf{u}(t) \tag{49}$$

$$e^{-\int_0^t \mathbf{A}(\tau) d\tau} \mathbf{x}(t) - e^{-\int_0^{t_0} \mathbf{A}(\tau) d\tau} \dot{\mathbf{x}}(t_0) = \int_{t_0}^{t} e^{-\int_0^{\tau} \mathbf{A}(r) dr} \mathbf{B}(\tau)\mathbf{u}(\tau) d\tau \tag{50}$$

From which we conclude that,

$$\mathbf{x}(t) = e^{\int_{t_0}^{t} \mathbf{A}(\tau) d\tau} \mathbf{x}(t_0) + \int_{t_0}^{t} e^{\int_{\tau}^{t} \mathbf{A}(r) dr} \mathbf{B}(\tau)\mathbf{u}(\tau) d\tau \tag{51}$$

$$\mathbf{x}(t) = \Phi(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \Phi(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau) d\tau \tag{52}$$

3 Systems representation

Representation of dynamic systems can come in two flavors:

- Input-Output models, which are described by a direct relationship between inputs and outputs,

- State-Space models, which involves, beyond inputs and outputs, the concept of a inner-state in the system.

We shall see that those kinds of representations are rather equivalent, but state-space models leave the inner dynamics of the system in a more explicit form.

3.1 Input-Output representation

This kind of representation is based on causality, that is,

- Causes are the inputs
- Effects are the outputs

so, it leaves explicit how inputs relate themselves to create the outputs. In a black-box representation, we can see the action of the system as,

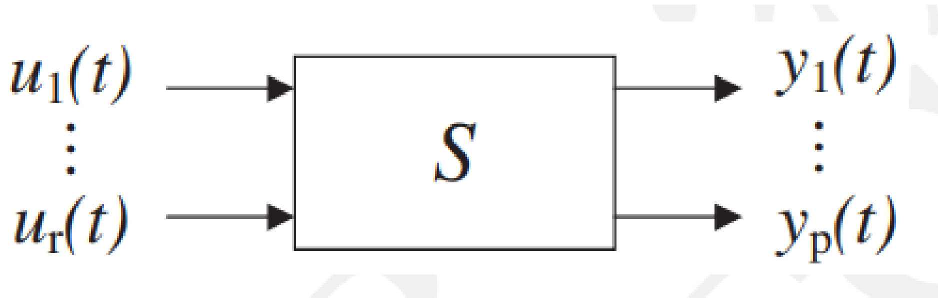

Figure 2: Transformation of causes into consequences

The idea, is that the system acts as an operator (or transformation), that is, being $\mathcal{U}$ the input-space, and $\mathcal{O}$, the output space, a system S is a mapping $S : \mathcal{U} \to \mathcal{O}$. In the above example, specifically, $\mathcal{U} = \mathbb{R}^r$, $\mathcal{O} = \mathbb{R}^p$.

The mathematical model involved in such systems is given by a differential equation, relating inputs and outputs. In the most general case, of a non-linear multiple input multiple output (MIMO) system, we have:

$$h_1\left[y_1, \cdots, \frac{d^{n_1}y_1}{dt^{n_1}}, u_1, \cdots, \frac{d^{m_{1,1}}u_1}{dt^{m_{1,1}}}, \cdots, u_r, \cdots, \frac{d^{m_{1,r}}u_r}{dt^{m_{1,r}}}\right] = 0 \tag{53}$$

$$\vdots \tag{54}$$

$$h_p\left[y_p, \cdots, \frac{d^{n_p}y_p}{dt^{n_p}}, u_1, \cdots, \frac{d^{m_{p,1}}u_1}{dt^{m_{p,1}}}, \cdots, u_r, \cdots, \frac{d^{m_{p,r}}u_r}{dt^{m_{p,r}}}\right] = 0 \tag{55}$$

For instance, consider the case where we have one input, and one output related linearly, with orders n and m, respectively, then our model becomes,

$$a_1 y + a_2 \dot{y} + \cdots + a_n y^{(n-1)} + y^{(n)} = b_1 u + b_2 \dot{u} + \cdots + b_m u^{(m-1)} + u^{(m)} \tag{56}$$

3.2 State-Space equations

Although useful, a major drawback in the last kind of representation is the lack of explicit meaning of system's inner dynamics. For example, in absence of inputs, that is, $u = 0$, how would the system behaves?

Indeed, if $u = 0$, then all that's left is the RHS, which, if we assume $y(t_0) = 0, \cdots, y^{(n-1)} = 0$ has as solution $y \equiv 0$. But at the moment one of these initial conditions is non-zero, the system has a non-null response, in the sense that it has accumulated energy[3] and spends energy to behave.

We could also proceed as in section 2.2, and write:

$$x_1 = y \tag{57}$$
$$x_2 = \dot{y} \tag{58}$$
$$\vdots \tag{59}$$
$$x_n = \frac{d^{n-1}y}{dt^{n-1}} \tag{60}$$

in such a way that we have the following equations:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{61}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \tag{62}$$

which is the linear time-invariant case. As you may wonder, the time-variant case is when both $\mathbf{A} = \mathbf{A}(t)$, $\mathbf{B} = \mathbf{B}(t)$. For the non-linear general case, we simply substitute the previous equations by mappings,

$$\dot{x} = \mathbf{f}(t, \mathbf{x}, \mathbf{u}) \tag{63}$$
$$\mathbf{y} = \mathbf{g}(t, \mathbf{x}, \mathbf{u}) \tag{64}$$

The previous sections have discussed the case in which those equations were linear. Moreover, they concern the solution of the state equation, namely, Equation 61. Given that we know the solution, $\mathbf{x}(t)$, then we can plug it, along with $\mathbf{u}$ (which we are supposed to know a priori) to get observations, $\mathbf{y}(t)$.

[3]Imagine the case of an Capacitor, in a RC circuit