Appendix B: Stability of Dynamic Systems

Eduardo Fernandes Montesuma
email edumontesuma@gmail.com

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Contents

The concepts presented here are a synthesis from references [3] and [4]. For a further explanation of those concepts, the readers are invited to look more careful inside those books.

1 Introductory Notions

In order to discuss the stability of nonlinear systems, we need to establish some notions. We start with the definitions about nonlinear state-space representation.

1.1 Nonlinear State Space representation

We begin with the definition of a vector field,

Definition 1. *Let $E \subset \mathbb{R}^n$ and $F \subset \mathbb{R}^m$ be euclidean vectorial spaces of dimensions n and m, respectively. We call **vectorial field** any function $f : E \to F$.*

In practice, vectorial fields assigns, to each point in $E \subset \mathbb{R}^n$, a vector $\mathbf{f} \subset \mathbb{R}^m$. Being so, they can be represented by m functions of n variables, that is,

$$\mathbf{f}(\mathbf{x}) = \left[f_1(\mathbf{x}), \cdots, f_m(\mathbf{x})\right]^T$$

Such definition provide us a mathematical tool to analyze non-linear differential equations. For instance, let $x_1(t), \cdots, x_n(t)$ be n functions of time, each of them satisfying its non-linear differential equation, $f_j(t, x_1, \cdots, x_n)$. We can create a system, in the following way,

$$\begin{cases} \dot{x}_1 & = f_1(t, x_1, \cdots, x_n) \\ \dot{x}_2 & = f_2(t, x_1, \cdots, x_n) \\ & \vdots \\ \dot{x}_n & = f_n(t, x_1, \cdots, x_n) \end{cases}$$

This can be represented, in a more compact form, by denoting,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

indeed, this was the approach we have done through section 1.2 in Chapter 1. Equations 1.20-1.23 are, also, the system's representation as a vector field, from $\mathbb{R}^4$ to $\mathbb{R}^4$. Indeed, as time goes by, the vectors $\dot{\mathbf{x}} \in \mathbb{R}^n$ attached to the point $\mathbf{x}$ through the time index t changes. Therefore, it makes sense to give the following definition,

Definition 2. *Consider the non-linear differential equations,*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

*Any solution is a curve $\mathbf{x}$ in $\mathbb{R}^n$ with $\mathbf{x}(0) = \mathbf{x}_0$, and for each time t, $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$ is called the trajectory of f, by $\mathbf{x}_0$. Also, the sets defined as $\{\mathbf{x}(t) : t \in \mathbb{R}^n\} \subset \mathbb{R}^n$ are called **the orbits** of $\mathbf{f}$ by $\mathbf{x}_0$.*

Also important to mention, we shall adopt the notation $\phi_t(\mathbf{x}) = \mathbf{x}(t)$, which is known as **the flux** of f. Our last definition gives us precisely what are the singularities of the vector field,

Definition 3. *Given a non-linear differential equation, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, a singularity of f, or stationary point $\mathbf{x}_0$, is a point such that $\mathbf{f}(\mathbf{x}_0) = 0$.*

The notion of stationarity is given by the fact that, if $\mathbf{f}$ is zero in $\mathbf{x}_0$, then the trajectory $\mathbf{x}(t)$, passing through $\mathbf{x}_0$ is constant after reaching $\mathbf{x}_0$. Now, we turn to the stability in dynamic systems,

Definition 4. *Given a non-linear differential equation, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and a stationary point $\mathbf{x}_0$. We say $\mathbf{x}_0$ is stable if, for every $\epsilon > 0$ there exists a δ such that*

$$|\mathbf{x} - \mathbf{x}_0| < \delta \to |\phi_t(\mathbf{x}) - \mathbf{x}_0| < \epsilon, \forall t.$$

Moreover, $\mathbf{x}_0$ is asymptotically stable if, for every $\epsilon > 0$,

$$|\phi_t(\mathbf{x}) - \mathbf{x}_0| < \epsilon \to \lim_{t \to \infty} \phi_t(\mathbf{x}) = \mathbf{x}_0$$

The main difference from stability to asymptotically stability is that in the first, we are guaranteed to never leave an ϵ-neighborhood of $\mathbf{x}_0$, provided we are δ-close to it. In the second, provided that we have started ϵ close, we may leave this neighborhood, but we are guaranteed that, somewhere in the future, we will return.

The stationary points are indeed the more valuable points in any differential equation, since they provide information about the behavior of our system (namely, from where we start, to where we go). As we shall see,

- Unstable stationary points are equivalent to sources, that is, the dynamics of the system has origin there,
- Since the last ones are unstable, they are repelled from there, in the direction of stable points - the sinks -. The stable stationary points are, thus, equivalent to sinks.

1.2 Linearization

From elementary calculus, we know the concept of Taylor series expansion, that is, given an analytic function[1], it can be expanded as a power series of x, that is,

$$f(x) = \sum_{n=0}^{\infty} \frac{(x - x_0)^n}{n!} \frac{d^n}{dt^n} f(x_0)$$

This series can be truncated, in order to generate the best n-th order polynomial approximation of f. For example, the best first-order approximation of f is,

$$f(x) = f(x_0) + \left.\frac{df}{dt}\right|_{x=x_0} (x - x_0) + \mathcal{O}((x - x_0)^2)$$

If x_0 is a stationary point of f, then $f(x_0) = 0$, and the approximation is linear, that is, provided that $\left.\frac{df}{dt}\right|_{x=x_0} = m$, $f(x) = m(x - x_0)$. This, although, only guarantees linearizations for the univariate case, which needs to be generalized to vectorial fields, $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$. Before doing so, let us suppose proved the result for surfaces[2], that is, given a function $f : \mathbb{R}^n \to \mathbb{R}$,

[1] Infinitely many differentiable functions

[2] Most calculus textbooks will employ what we are doing to prove this result for surfaces: they suppose the result is valid for the univariate case, and then they construct the Taylor series for most complex functions. We repeat, here, the idea of the argument.

$$f(\mathbf{x}) = f(\mathbf{x}_0) + \nabla f|^T_{\mathbf{x}=\mathbf{x}_0}(\mathbf{x} - \mathbf{x}_0)$$

where $\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}^T$ is the gradient vector of f. To begin with, we notice $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is a vector composed by n surfaces, that is, $\mathbf{f} = (f_j)_{j=1}^n$, each with its gradient, ∇f_j. We notice that,

$$\begin{aligned} \dot{x}_1 &= f_1(\mathbf{x}) \\ &\vdots \\ \dot{x}_n &= f_n(\mathbf{x}) \end{aligned}$$

Each f_j can be linearized as $f_j(\mathbf{x}) = f_j(\mathbf{x}_0) + \nabla f_j|_{\mathbf{x}=\mathbf{x}_0}(\mathbf{x} - \mathbf{x}_0)$, therefore,

$$\begin{aligned} \dot{x}_1 &= f_1(\mathbf{x}_0) + \nabla f_1^T|_{\mathbf{x}=\mathbf{x}_0}(\mathbf{x} - \mathbf{x}_0) \\ &\vdots \\ \dot{x}_n &= f_n(\mathbf{x}_0) + \nabla f_n^T|_{\mathbf{x}=\mathbf{x}_0}(\mathbf{x} - \mathbf{x}_0) \end{aligned}$$

gathering those together,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}_0) + \mathbf{J}|_{\mathbf{x}=\mathbf{x}_0}(x - \mathbf{x}_0)$$

in which $\mathbf{J}$ is the Jacobian matrix, given by,

$$\mathbf{J} = \begin{bmatrix} \nabla f_1^T \\ \nabla f_2^T \\ \vdots \\ \nabla f_n^T \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

2 Stability of dynamic systems

2.1 Stability in linear fields

A linear field is a linear transform from $\mathbf{R}^n$ to itself. Thus, it makes sense to use the terminology to denote a matrix, which we will call here, in reference to state-space equations, as A. Moreover, let us assume that our differential equation does not have inputs, thus,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

We want to show that the solutions of such equation relies on the spectra of A. Indeed, assume A is diagonalizable, that is, A has eigenvalues $\lambda_1, \cdots, \lambda_n$, associated with eigenvectors $\boldsymbol{\xi}_1, \cdots, \boldsymbol{\xi}_n$. By building functions $\mathbf{g}_i = \boldsymbol{\xi}_i e^{\lambda_i t}$, each of them is a solution of our ODE, since,

$$\begin{aligned}
\dot{\mathbf{g}}_i &= \boldsymbol{\xi}_i \frac{d}{dt} e^{\lambda_i t} \\
&= \boldsymbol{\xi}_i \lambda_i e^{\lambda_i t} \\
&= \mathbf{A}\boldsymbol{\xi}_i e^{\lambda_i t} \\
&= \mathbf{A}\mathbf{g}_i
\end{aligned}$$

Moreover, since the vectors $\boldsymbol{\xi}_i$ are linear independent, the solutions $\mathbf{g}_i$ are linearly independent. We conclude that the solution of our ODE is given by $\mathbf{x}(t) = \sum_{i=1}^{n} \boldsymbol{\xi}_i e^{\lambda_i t}$. We have just proven the following theorem,

Theorem 1. *Let $A \in \mathbb{R}^{n \times n}$ be the linear diagonalizable field of the ODE $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. Then,*

$$\mathbf{x}(t) = \sum_{i=1}^{n} \boldsymbol{\xi}_i e^{\lambda_i t}$$

Although not in its general form (we have assumed that A is diagonalizable), this theorem indeeds proves that the solutions of a linear ODE relies on the spectra of $\mathbf{A}$. It can be proven even if the field is not diagonalizable, using Jordan's canonical form. We then, state the following corollary,

Corollary 1. *If $\mathbf{A}$ have an eigenvalue with positive real part, then the response is asymptotically unstable.*

A proof of such corollary comes from noticing that, if any $\lambda_i > 0$, then the limit $\lim_{t \to \infty} \mathbf{x}(t) = \infty$, in the sense that at least one of its coordinates grows without bounds. We conclude that a full characterization of stable linear differential equations is to identify those as having negative definite fields.

2.2 Stability of nonlinear dynamic systems

The intents of this section is to characterize the equivalence between stationary points which are asymptotically stable (according to definition 4) and sink points, discussed in the corollary 1. To that end, we need a theorem to establish this equivalence.

Before presenting the theorem without its proof, it is noteworthy that, at the bottom, what the theorem really says is that in a sufficiently small neighborhood of each singularity, the nonlinear field $\mathbf{f}$ resembles the linear one, $\mathbf{A}$. This is exactly the procedure of linearization we have just described.

Being so, we have the famous,

Theorem 2. *Let $\mathbf{x}_0 \in \mathbb{R}^n$ be a singularity of the field $f : \mathbb{R}^n \to \mathbb{R}^n$. If x_0 is a sink of f, then x_0 is a asymptotically stable point of f.*

We do not present a proof here, because we believe it is out of the scope of the homework, although [3] has a detailed proof and discussion about such result. As a final remark, this theorem allows us to study the behavior of system's stability in the singularities of it.

References

[1] Adreas Kroll, Horst Schulte *Benchmark problems for nonlinear systems identification and control using Soft Computing methods: Need and overview*. Applied Soft Computing, 2014.

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[5] Bruce van Brunt *The Calculus of Variations*. Springer, Universitext, 2004.

[6] John L. Troutman *Variational Calculus and Optimal Control*. Springer, 1995.