Lecture Notes on Control Theory: Exercise Solutions

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1 Exercise 1

1.1 Moving cart

Consider a moving cart of mass M, under a driving force F_m, attached to a spring whose force is given by F_e at displacement $x(t)$. Assume, also, that our spring follows Hooke's law with a variable damp factor, that is, $k = k(t) = k_{t_0}e^{-\alpha(t-t_0)}$. Thus, by Newton's second law:

$$F_m - F_e = m\ddot{x} \tag{1}$$
$$F_m = k_{t_0}e^{-\alpha(t-t_0)}x(t) + M\ddot{x}(t) \tag{2}$$

Notice also that the energy of our system is given by the sum of potential with kinetic energy. Thus:

$$E(t,x,\dot{x}) = \frac{1}{2}k_{t_0}e^{-\alpha(t-t_0)}x(t)^2 + \frac{1}{2}M\dot{x}(t) \tag{3}$$

In terms of state-space variables, we can define,

$$x_1 = x(t) \tag{4}$$
$$x_2 = \dot{x}(t) \tag{5}$$
$$u(t) = F_m(t) \tag{6}$$
$$y(t) = E(t) \tag{7}$$

In such a setup, we describe Equation 2 as:

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k(t)}{M} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{M} \end{bmatrix} u \tag{8}$$

Which is our state equation. Notice that solving it implies to discover how the system manages to change its inner state $(x, \dot{x})$. To retrieve our desired output, we have the transmission equation, which is given by Equation 3.

Also, despite the matrix representation, $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$, A is not a matrix of constants, that is, $A = A(t)$, and therefore our system is indeed linear, although not time independent.

We can prove linearity easily. Consider vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ for which Equation 8 holds, then,

$$\frac{d}{dt}(\mathbf{x}_1 + \mathbf{x}_2) = \frac{d}{dt}\mathbf{x}_1 + \frac{d}{dt}\mathbf{x}_2 \tag{9}$$
$$= \mathbf{A}\mathbf{x}_1 + \mathbf{B}u + \mathbf{A}\mathbf{x}_2 + \mathbf{B}u \tag{10}$$
$$= \mathbf{A}(\mathbf{x}_1 + \mathbf{x}_2) + \mathbf{B}(2u) \tag{11}$$

So, we conclude that the system is linear. From a theoretical point of view, we have one-dimensional inputs and outputs, which can be described by differential equations in a input-output fashion. This model, however, does not make clear how (or what is) the inner state of the system evolves.

Of course, solving the differential equation and calculating the output variable implies taking into account the displacement and velocity. The state-space only make it explicit in the equations. We shall explore this philosophy in the next examples.

1.2 RL circuit

Consider the following electric circuit,

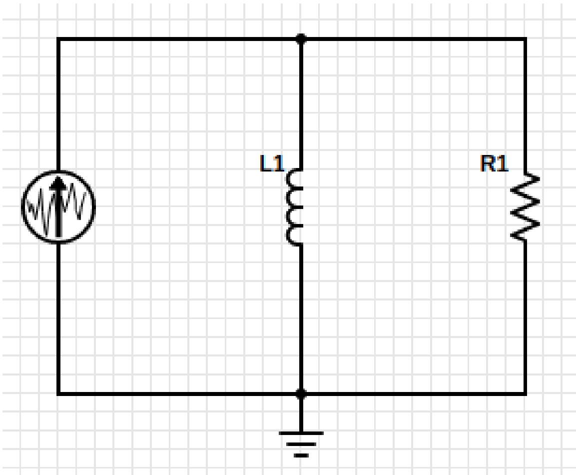

Figure 1: Electric Circuit for exercise 2

Additionally, we shall consider $x(t)$ and $y(t)$, the inductor's and resistor's current, respectively. Thus, from Ohm's law, and the inductor equation:

$$V_L(t) = L\dot{x}(t) \tag{12}$$
$$V_R(t) = Ry(t) \tag{13}$$

By applying Kirchoff's Law of nodes and tensions, we get that:

$$y(t) = x(t) - u(t) \tag{14}$$
$$L\dot{x}(t) = R(x(t) - u(t)) \tag{15}$$

Notice that Equation 14 is already our transmission equation. Also, Equation 15 is a ODE of first order.

2 Exercise 2

We shall explore the properties of the following systems:

2.1 Example 1

Suppose x_1, x_2, u, y_1, y_2 are functions of $t \in \mathbb{R}_+$,

$$\dot{x}_1 = x_2 + u \tag{16}$$
$$\dot{x}_2 = |x_1 + x_2| + u \tag{17}$$
$$y_1 = x_1 + u^2 \tag{18}$$
$$y_2 = x_1 + x_2 \tag{19}$$

- **Order:** The system is of second-order. Recall that $\dot{x}_1 = x_2 + u$, and thus, $\dot{x}_2 = \ddot{x}_1 - \dot{u} = |x_1 + x_2| + u$.
- **Quantity of variables:** We have one input ($u(t)$) and two outputs ($y_1(t), y_2(t)$). Also, our state is described by $(x_1(t), x_2(t))$.
- **Linearity:** It is nonlinear, by equations 17 and 18
- **Time invariance:** it is time-invariant since its equations are autonomous (they do not depend explicitly of t).

2.2 Example 2

Suppose x_1, x_2, u_1, u_2, y are functions of $t \in \mathbb{R}_+$,

$$\dot{x}_1(t) = sin(x_2(t)) + u_1(t) \tag{20}$$
$$\dot{x}_2(t) = cos(t)x_1(t) + u_2(t) \tag{21}$$
$$y(t) = x_1(t) \tag{22}$$

- **Order:** Notice that:

  $$\ddot{x}_1 = cos(x_2(t))\dot{x}_2(t) + \dot{u}_1(t) = cos(x_2(t))cos(t)x_1(t) + cos(x_2(t))u_2(t) + \dot{u}_1(t)$$

  which is a second order differential equation.
- **Quantity of variables:** We have two inputs u_1, u_2, two state variables, x_1, x_2 and one output, y.
- **Linearity:** It is nonlinear, since Equation 20 has a sine function of x_2.
- **Time invariance:** It is time variant, since

  $$\dot{x}_1(t - t_0) = sin(x_2(t - t_0)) \neq sin(x_2(t) - t_0)$$

2.3 Example 3

Consider the following system:

$$\dot{x}_1(t) = 9x_1(t) + 6x_2(t) - u(t) + \alpha \tag{23}$$
$$\dot{x}_2(t) = 4x_1(t) - 2x_2(t) + 5u(t) \tag{24}$$
$$y(t) = 8x_1(t) + 3x_2(t) \tag{25}$$

- **Order:** Again,

$$\ddot{x}_1(t) = 9\dot{x}_1(t) + 6\dot{x}_2(t) - \dot{u}(t) \tag{26}$$
$$= 9\dot{x}_1(t) + 24x_1(t) - 12x_2(t) + 30u(t) - \dot{u}(t) \tag{27}$$

  Which is a second order differential equation.

- **Quantity of variables:** We have one input, $u(t)$, one output, $y(t)$ and two state variables.

- **Linearity:** It is linear if, and only if $\alpha = 0$.

- **Time invariance:** It is time invariant, since the system is autonomous.

3 Exercise 3

Given the following system

$$\dot{x}_1(t) = x_2^2(t) + \alpha u(t) \tag{28}$$
$$\dot{x}_2(t) = x_1 x_2(t) + u(t) \tag{29}$$
$$y(t) = \beta x_1(t) \tag{30}$$

Suppose we have $\lim_{t\to\infty} u(t) = \bar{u} = 1$, We want to determine α, β such that $\lim_{t\to\infty} x_2(t) = \bar{x}_2 = 2$ and $\lim_{t\to\infty} y(t) = \bar{y} = 8$. To do so, we rewrite the system in terms of steady state values:

$$\bar{x}_1' = \bar{x}_2^2 + \alpha\bar{u} \tag{31}$$
$$\bar{x}_2' = \bar{x}_1\bar{x}_2 + \bar{u} \tag{32}$$
$$\bar{y} = \beta\bar{x}_1 \tag{33}$$

The system of differential equations has now turned into a (non-linear) system of algebraic equations. Solving for α and β yields: $\bar{x}_1 = -\frac{1}{2}$, $\alpha = -4$ and $\beta = -16$.

4 Appendix: Wronskian

The intents of this appendix is to serve as a argument to determine whether or not a set of functions is linear independent. This concept finds its motivation when it comes to analyze the order of a given system: it is the number of linearly independent state variables (functions).

Consider, for simplicity, two functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, a linear combination of f and g is the function:

$$h(t) = c_1 f(t) + c_2 g(t) \tag{34}$$

In allusion to the concept of linear dependency of two vectors, the pair (f, g) is independent if, and only if the only solution for $h(t) = 0$ is $c_1 = c_2 = 0$, otherwise, they are dependent. Now consider that it is the case, that is,

$$c_1 f + c_2 g = 0, \forall t \tag{35}$$

By differentiating, we arrive at,

$$c_1 \dot{f} + c_2 \dot{g} = 0, \forall t \tag{36}$$

Indeed, this is a system of two equation with two unknowns (c_1, c_2). It has a solution if, and only if the determinant of the matrix,

$$\begin{bmatrix} f(t) & g(t) \\ \dot{f}(t) & \dot{g}(t) \end{bmatrix} \tag{37}$$

Is equal to zero, for all $t \in \mathbb{R}$. This determinant is called **the Wronskian**. It has the form $W(f, g) = f\dot{g} - \dot{f}g$. It is straightforward to generalize this fact for a set of $n \in \mathbb{N}$ equations, $(f_1, \cdots, f_n)$.

With this concept at hand, all we have to do is to test, for state variables $(x_1, \cdots, x_n)$, whether or not $W(x_1, \cdots, x_n)$ is non-zero within $\mathbb{R}_+$, to determine the order of the system.

If it is the case where the set of state variables is non-zero, we can exclude some variables, to arrive at a subset $(x_1, \cdots, x_m) \subset (x_1, \cdots, x_n)$. This is the exactly analogous for functions, of the process with vectors.