Lecture Notes on Control Theory: simulation and representation of system through state space equations

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1 Problem Description

We want to model, and simulate a Spring-Mass system, which is given by the following figure:

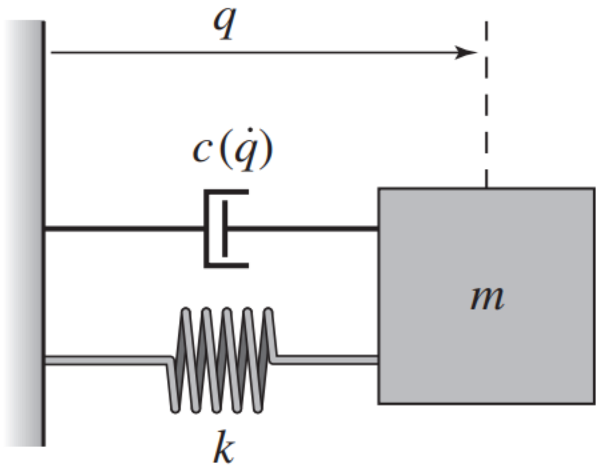

Figure 1: Mass attached to a spring.

In the above figure, we have three positive constants:

- k, the spring's constant.
- m, the block's mass.
- c, the friction coefficient.

Suppose, for a moment, that there is no driving force in the mass (we shall treat those cases later on). Thus, we can use Newton's second law to find a differential equation for the motion:

$$
\begin{aligned}
m\ddot{q} &= \sum F && (1)\\
&= -F_e - F_m && (2)\\
&= -c\dot{q} - kq && (3)
\end{aligned}
$$

Therefore, our differential equation is,

$$m\ddot{q} + c\dot{q} + kq = 0 \tag{4}$$

In a first attempt, we do not try to solve it. We want to see how can we model this through state-space equations. Henceforth, assume $\mathbf{x} = (q, \dot{q})$, namely, the position and velocity of the block. With this construction, we have:

$$\begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{k}{m} & \frac{c}{m} \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \tag{5}$$

$$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \tag{6}$$

This system has the following properties:

- It is linear,
- It is time invariant
- It is proper
- It is SIMO (single input, multiple output)

2 Code

In order to simulate[1] the state-space representation, we shall use the following codes:

```
# Libraries
import numpy as np # Basic python numeric library
import control as ctrl # Control systems library
import matplotlib.pyplot as plt # Python plots library
%matplotlib inline

# Parameters defining the system
m = 250 # system mass
k = 40  # spring constant
c = 60  # damping constant

# System matrices
A = np.array([[0, 1],[-k/m, -c/m]])
B = np.array([[0],[1/m]])
C = np.array([[1 , 0]])
sys = ctrl.ss(A, B, C, 0)
```

Listing 1: Representation

At its time, simulation uses the control library:

```
# Time domain simulation: natural response (zero input)
t = np.arange(0,100,0.1)
[t_anl, y_anl, _] = ctrl.forced_response(sys, T=t, X0 = [10,0])

# plotting the results
plt.figure(figsize=(7,7))
plt.plot(t_anl, y_anl, 'k-')
```

[1] Remark: try different values for (m,c,k)

```python
plt.title('Simulation of the system response')
plt.ylabel('Position x [m]')
plt.xlabel('Time t [sec]')
plt.grid()
plt.show()
```

Listing 2: Simulation

This last piece of code gives us the following output:

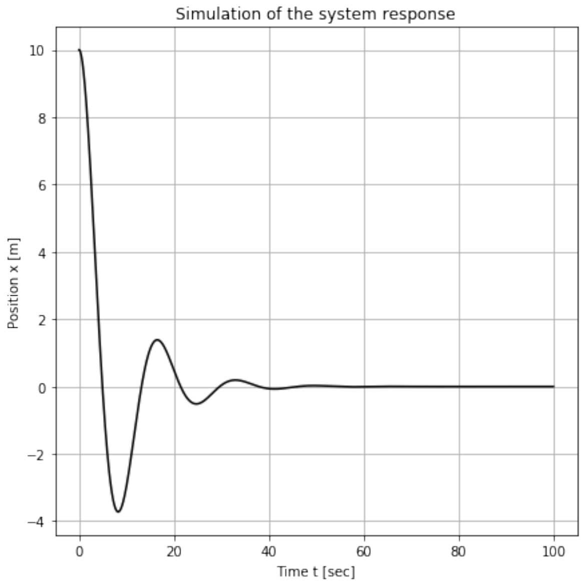

Figure 2: Output of our simulation code

3 Solving the Spring-Mass ODE

3.1 Equation Analysis

Up to now, we have not done much. Currently, we have only changed notation in our systems, and used a computer to simulate the answer. We may also care about the mathematical tools to solve the problem.

To begin with, it is noteworthy that the ODE is a linear combination of derivatives. Thus, if we are to have a function $x(t)$ that is a solution to the ODE, it must have derivatives that are linear combinations of itself. In other words, $\dot{x}(t) = ax(t)$. As you might guess, from calculus, we now that such functions have the form $x(t) = Ae^{rt}$. Let us try this kind of answer in the ODE:

$$
\begin{align}
ms^2e^{rt} + cse^{rt} + ke^{rt} = 0 \tag{7}\\
e^{rt}(mr^2 + cr + k) = 0 \tag{8}\\
mr^2 + cr + k = 0 \tag{9}
\end{align}
$$

Where to derive the last equation, we have used the fact that $e^{rt} \neq 0, \forall t$. The polynomial in r, $f(r) = mr^2 + cr + k$ is said the characteristic polynomial of the ODE, and its solution gives rise to the solution of the ODE itself. We can express its roots by:

$$
r = \frac{-c \overset{+}{-} \sqrt{c^2 - 4mk}}{2m} \tag{10}
$$

Particularly, our polynomial becomes,

$$
250r^2 + 60r + 40 = 0 \tag{11}
$$

which has roots:

$$
\begin{align}
r_1 = -\frac{3}{25} + i\frac{\sqrt{91}}{25} \tag{12}\\
r_2 = -\frac{3}{25} - i\frac{\sqrt{91}}{25} \tag{13}
\end{align}
$$

Expressing it as $r_1 = \sigma + i\omega$, $r_2 = \sigma - i\omega$, we have:

$$
\begin{align}
y(t) &= c_1e^{\sigma t}e^{i\omega t} + c_2e^{\sigma t}e^{-i\omega t} \tag{14}\\
&= e^{\sigma t}(c_1(cos(\omega t) + isin(\omega t)) + c_2(cos(\omega t) + isin(\omega t))) \tag{15}\\
&= e^{\sigma t}((c_1 + c_2)cos(\omega t) + i(c_1 - c_2)sin(\omega t)) \tag{16}\\
&= e^{\sigma t}(Acos(\omega t) + Bsin(\omega t)) \tag{17}
\end{align}
$$

The derivative, at its time,

$$
\dot{y}(t) = \sigma e^{\sigma t}(Acos(\omega t) + Bsin(\omega t)) + e^{\sigma t}(-A\omega sin(\omega t) + B\omega cos(\omega t)) \tag{18}
$$

Let us also consider the initial condition of $y(0) = A = 10$ and $\dot{y}(0) = \sigma + B\omega = 0$, that is, the block begins at 10m of distance, and with zero velocity. So, $A = 10$, $B = -\frac{\sigma}{\omega} = \frac{3\sqrt{91}}{91}$. We can, therefore, write our solution as:

$$y(t) = \frac{10}{91}e^{-\dfrac{3}{25}t}\left(91cos\left(\frac{\sqrt{91}}{25}t\right) + 3\sqrt{91}sin\left(\frac{\sqrt{91}}{25}t\right)\right) \tag{19}$$

$$y(t) = \frac{40}{\sqrt{91}}e^{-\dfrac{3}{25}t}sin\left(\frac{\sqrt{91}}{25}t\right) \tag{20}$$

Bellow we show the results for this approach:

```
# Solution
t = np.arange(0,100,0.1)

sigma = -(3/25)
omega = (np.sqrt(91)/25)

c1 = 10 / 91
c2 = 3 * np.sqrt(91)
c3 = 40 / np.sqrt(91)

y  = c1 * np.exp(sigma * t)*(91 * np.cos(omega * t) + 3 * np.sqrt(91) *
    np.sin(omega * t))
dy = c3 * np.exp(sigma * t) * np.sin(omega * t)

plt.figure(figsize=(7,7))
plt.plot(t, y, 'k-', label = 'y(t)')
plt.plot(t, dy, 'r-', label = r'$\dfrac{dy}{dt}(t)$')
plt.title('Simulation of the system response')
plt.ylabel('Position x [m]')
plt.xlabel('Time t [sec]')
plt.legend()
plt.grid()
plt.show()
```

Listing 3: Differential Equation representation

The output of these last lines of codes is displayed in figure 3

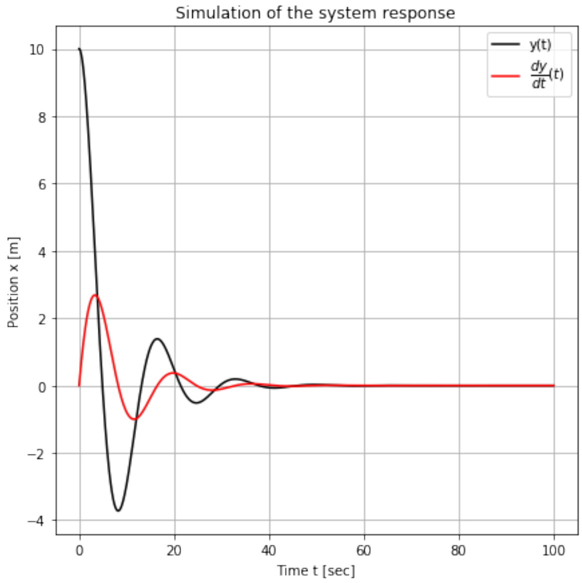

Figure 3: Result and derivative of the solution for the spring-mass system

4 Appendix: Actually solving ODEs

Remark: Before proceeding, let us make a little change of notation: we shall call (m, c, k) as (a, b, c), for a more natural representation (closely to bhaskara) of roots. The solutions of r can happen in three ways:

1. One or two real roots,
2. Purely imaginary roots,
3. Complex roots

We analyse each of these cases bellow:

4.1 Two real roots

If the polynomial $ar^2 + br + c = 0$ has to real roots, then we shall have $b > 2\sqrt{ac}$. In that case, denote $x_1 = c_1 e^{r_1 t}$, $x_2 = c_2 e^{r_2 t}$, we claim that $y(t) = x_1 + x_2$ is a solution for the our differential equation. Indeed:

$$\dot{y} = c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t} \tag{21}$$
$$\ddot{y} = c_1 r_1^2 e^{r_1 t} + c_2 r_2^2 e^{r_2 t} \tag{22}$$

And thus, we can rewrite the ODE as:

$$
\begin{aligned}
a\ddot{y} + b\dot{y} + cy &= a(c_1 r_1^2 e^{r_1 t} + c_2 r_2^2 e^{r_2 t}) + b(c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t}) + c(x_1 + x_2) && (23)\\
&= e^{r_1 t}(a r_1^2 + b r_1 + c) + e^{r_2 t}(a r_2^2 + b r_2 + c) && (24)\\
&= e^{r_1 t} \cdot 0 + e^{r_2 t} \cdot 0 && (25)\\
&= 0 && (26)
\end{aligned}
$$

So $y(t)$ is a solution as well. It is indeed, unique.

4.1.1 Example 1

Consider the following ODE:

$$\ddot{x}(t) + 5\dot{x}(t) + 6x(t) = 0 \tag{27}$$

Notice that its solution comes from solving the second order polynomial $s^2 + 5s + 6 = (s+2)(s+3)$, which has roots $s_1 = -2$ and $s_2 = -3$. Therefore:

$$x(t) = c_1 e^{-2t} + c_2 e^{-3t} \rightarrow \dot{x}(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t} \tag{28}$$

Now, suppose the initial conditions impose that $c_1, c_2 > 0$, then we have the following limits:

$$
\begin{aligned}
\lim_{t\to+\infty} x(t) &= 0 && (29)\\
\lim_{t\to-\infty} x(t) &= +\infty && (30)\\
\lim_{t\to+\infty} \dot{x}(t) &= 0 && (31)\\
\lim_{t\to-\infty} \dot{x}(t) &= -\infty && (32)
\end{aligned}
$$

4.1.2 Example 2

Consider the following ODE:

$$\ddot{x}(t) - x(t) = 0 \tag{33}$$

Its polynomial is given by $s^2 - 1 = (s-1)(s+1)$, which yields the solution:

$$x(t) = c_1 e^{t} + c_2 e^{-t} \rightarrow \dot{x}(t) = c_1 e^{t} - c_2 e^{-t} \tag{34}$$

Now, suppose the initial conditions impose that $c_1, c_2 > 0$, then we have the following limits:

$$\lim_{t \to +\infty} x(t) = +\infty \tag{35}$$
$$\lim_{t \to -\infty} x(t) = +\infty \tag{36}$$
$$\lim_{t \to +\infty} \dot{x}(t) = +\infty \tag{37}$$
$$\lim_{t \to -\infty} \dot{x}(t) = -\infty \tag{38}$$

4.1.3 Example 3

Consider the following ODE:

$$\ddot{x}(t) - 5\dot{x}(t) + 6x(t) = 0 \tag{39}$$

Its polynomial is given by $s^2 - 5s + 6 = (s-3)(s-2)$, which yields the solution:

$$x(t) = c_1 e^{2t} + c_2 e^{3t} \rightarrow \dot{x}(t) = 2c_1 e^{2t} + 3c_2 e^{3t} \tag{40}$$

Now, suppose the initial conditions impose that $c_1, c_2 > 0$, then we have the following limits:

$$\lim_{t \to +\infty} x(t) = +\infty \tag{41}$$
$$\lim_{t \to -\infty} x(t) = 0 \tag{42}$$
$$\lim_{t \to +\infty} \dot{x}(t) = +\infty \tag{43}$$
$$\lim_{t \to -\infty} \dot{x}(t) = 0 \tag{44}$$

4.2 Repeated Roots

Remark: this section offers some algebraic difficulties, so it can be skipped.

It happens when the characteristic equation

$$ar^2 + br + c = 0 \tag{45}$$

has only one solution, that is, $b = 2\sqrt{ac}$. In that case, we have only one solution $s = -\frac{b}{2a}$. The solution generated by this approach is then:

$$x(t) = ce^{-\frac{b}{2a}} \tag{46}$$

However, since this is a second-order equation, the space of solutions should have dimension two (that is, we should have two linearly independent functions generating the space of solutions). Therefore, we look forward another solution $x_1(t)$ such that it differs from $x(t)$ more than a constant. That is, we are interested in $x_1(t) = \eta(t)x(t)$. Taking derivatives:

$$\dot{x}_1(t) = \dot{\eta}(t)x(t) + \eta(t)\dot{x}(t) \tag{47}$$
$$\ddot{x}_1(t) = \ddot{\eta}(t)x(t) + 2\dot{\eta}(t)\dot{x}(t) + \eta(t)\ddot{x}(t) \tag{48}$$

Therefore:

$$\dot{x}_1(t) = e^{-\frac{b}{2a}}(\dot{\eta} - \frac{b}{2a}\eta) \tag{49}$$
$$\ddot{x}_1(t) = e^{-\frac{b}{2a}}(\ddot{\eta}(t) - \frac{b}{a}\dot{\eta}(t) + \frac{b^2}{4a^2}\eta(t)) \tag{50}$$

So:

$$a\ddot{x}_1 + b\dot{x}_1 + cx_1 = 0 \tag{51}$$
$$ae^{-\frac{b}{2a}}(\ddot{\eta}(t) - \frac{b}{a}\dot{\eta}(t) + \frac{b^2}{4a^2}\eta(t)) + be^{-\frac{b}{2a}}(\dot{\eta} - \frac{b}{2a}\eta) + c\eta e^{-\frac{b}{2a}} = 0 \tag{52}$$
$$a\ddot{\eta}(t) - b\dot{\eta}(t) + \frac{b^2}{4a}\eta(t) + b\dot{\eta}(t) - \frac{2b^2}{4a}\eta(t) + \frac{4ac}{4a}\eta(t) = 0 \tag{53}$$
$$\ddot{\eta}(t) = 0 \tag{54}$$

With that, we conclude that $\eta(t) = c_1 t + c_2$, and the general solution looks like:

$$x(t) = c_1te^{rt} + c_2e^{rt} \tag{55}$$
$$\dot{x}(t) = c_1e^{rt} + sc_1te^{rt} + sc_2e^{rt} \tag{56}$$
$$= sc_1te^{rt} + (c_1 + rc_2)e^{rt} \tag{57}$$

4.3 Complex roots

By the fundamental theorem of algebra, any second order polynomial with real roots has two (possible complex) roots. Thus, we have explored the case where both roots of $as^2 + bs + c = 0$ are real (different or equal). Now, we suppose that $b < 2\sqrt{ac}$. If, additionally $b = 0$, the roots of this polynomial shall have pure imaginary roots, as we will view.

4.3.1 Pure imaginary roots

If $b = 0$, and $c \neq 0$, then $\sqrt{\Delta} = \sqrt{-4ac} = 2i\sqrt{ac}$. Then the roots of our polynomial are:

$$s_1 = \frac{i\sqrt{ac}}{2a} \tag{58}$$
$$s_2 = -\frac{i\sqrt{ac}}{2a} \tag{59}$$

We shall take, in order to shorten our notation, $\omega = \frac{\sqrt{ac}}{2a}$. Therefore, our solution looks like:

$$x(t) = c_1 e^{i\omega t} + c_2 e^{-i\omega t} \tag{60}$$

Rearranging terms:

$$\begin{aligned}
x(t) &= c_1 e^{i\omega t} + c_2 e^{-i\omega t} && (61)\\
&= c_1(cos(\omega t) + isen(\omega t)) + c_2(cos(\omega t) - isin(\omega t)) && (62)\\
&= (c_1 + c_2)cos(\omega t) + i(c_1 - c_2)sin(\omega t) && (63)\\
&= Acos(\omega t) + Bsin(\omega t) && (64)
\end{aligned}$$

Where D was a carefully choosen constant such that $cos(\varphi) = \frac{A}{D} < 1$ and $sin(\varphi) = \frac{B}{D} < 1$. Differentiating $x(t)$ gives us $\dot{x}(t)$:

$$\begin{aligned}
\dot{x}(t) &= -A\omega sin(\omega t) + B\omega cos(\omega t) && (65)\\
&= Ccos(\omega t) + Dsin(\omega t) && (66)
\end{aligned}$$

Thus we have oscilatory responses.

4.3.2 Complex roots

We shall consider the case where we have complex roots $s_1 = \sigma + i\omega, s_2 = \sigma - i\omega$, being so , we have:

$$\begin{aligned}
x(t) &= c_1 e^{\sigma t} e^{i\omega t} + c_2 e^{\sigma t} e^{-i\omega t} && (67)\\
&= e^{\sigma t}(c_1(cos(\omega t) + isin(\omega t)) + c_2(cos(\omega t) - isin(\omega t))) && (68)\\
&= e^{\sigma t}((c_1 + c_2)cos(\omega t) + i(c_1 - c_2)sin(\omega t)) && (69)\\
&= e^{\sigma t}(Acos(\omega t) + Bsin(\omega t)), A = c_1 + c_2, B = i(c_1 - c_2) && (70)
\end{aligned}$$

Having chosen D as in the last section. For the derivative:

$$\dot{x}(t) = \sigma e^{\sigma t}(Acos(\omega t) + Bsin(\omega t)) + e^{\sigma t}(-A\omega sin(\omega t) + B\omega cos(\omega t)) \tag{71}$$
$$= e^{\sigma t}((A\sigma + B\omega)cos(\omega t) + (B\sigma - A\omega)sin(\omega t)) \tag{72}$$
$$= e^{\sigma t}(Ccos(\omega t) + Dsin(\omega t)) \tag{73}$$

Which gives us another kind of oscilatory response: one that decreases or increases the oscilations magnitude.