Advanced Control Homeworks

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Contents

Chapter 1

Homework 1: representation and simulation

Throughout the next sections, and also through the next chapters, we shall deal with the inverted control pendulum system, whose modeling, representation and simulation are discussed in this first chapter. However, before entering in the ideas behind the dynamic behavior of it, we shall make a few remarks.

First, as displayed in Figure 1.1, the inverted pendulum model finds applications in the control of segways and related devices. Second, it is a benchmark system, as stated in [1], for many control techniques. Being so, we shall use it to illustrate the concepts learned in class.

Figure 1.1: Example of a segway

Also, as a matter of clarity, we mean by,

- Modeling: the derivation and elucidation of dynamic equations of system's motion, through physics. After having the complete equations, we shall linearize them, to later on represent them.

- Representation: write the linearized equations through state-space models, and in frequency, with transfer functions.
- Simulation: to see how the system behaves in light of various inputs, in the time domain.

With those definitions, we intent to present a clear and objective guide, hoping to validate benchmark results. As tools to illustrate them, we shall employ programs in mainly two languages, Python and Matlab. You can also find out in the homework's website[1] the associated resources, as well as animations of the dynamic behavior of our system.

1.1 Modeling

1.1.1 Introduction

Systems like the already mentioned Segway device can be modeled as a cart (the basis, where one stands) with a pole attached to it (the person standing up). In such a representation, we have two bodies,

- The cart, with mass M, position x, velocity $\dot{x}$ and being under action of input force $u(t) = F(t)$ and friction, b_x.
- The pole, with a mass attached to its extrema, with length ℓ, angle deviation[2] θ, angular velocity ω and suffering from a friction between the cart and the junction b_θ.

Summing up those parts, we have the following diagram describing the system which we are dealing,

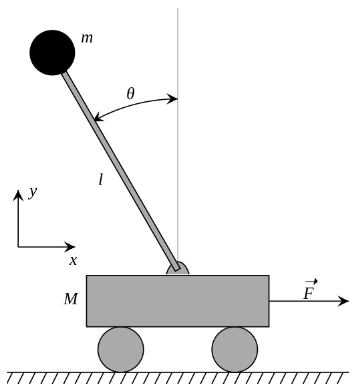

Figure 1.2: System's diagram

[1] `https://github.com/eddardd/Control-Theory/tree/master/Advanced-Control`
[2] from the upward position

Also, we have some constraints over the defined variables,

- We assume the joint between the cart and the pole to be free to spin around, in such a way that, if we assume θ to be in radians, $0 \leq \theta \leq 2\pi$, where 0 is the upward position, and π, the downward.
- The total length of the rail is 2 meters. Assuming the 0 position, the cart's position satisfies the following constraint, $-1 \leq x \leq 1$.
- The input force is limited to a value of $120N$, that is, $|u(t)| \leq 120$.

Except from the first constraint, all other can be found in [1]. Also based on such paper, 1.1, bellow, shows a complete list of constants used in our models, and their description,

Symbol	Description	Value	Unit
θ	Angular position of the Pendulum	variable	radians
x	Cart's linear position	variable	meters
F	external force applied to the cart	variable	Newton
M	Cart's mass	4.8	kilogram
m	Point mass of pendulum	0.356	kilogram
ℓ	pole length	0.56	meters
b_{th}	joint's friction	0.035	Nms/rad
b_x	track's friction	4.9	Newton
g	Gravitational constant	9.81	meter/second^2
I	Pole's moment of inertia	0.006	$kilogram \cdot meter^2$
L	total length of rail	2	meters
F_{max}	Maximum input value	120	Newton

Table 1.1: System constants

1.1.2 Dynamic Equations

We adopt as modeling strategy the Lagrangian formalism of classical mechanics, which is based on variational principles[3]. In order to take advantage of it, we recall the definition of Lagrangian, $\mathcal{L} = T - V$, where T is total kinetic energy in the system, and V, the total potential. Those later two quantities can be evaluated as,

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m||v_m||^2 + \frac{1}{2}I\omega^2 \tag{1.1}$$

$$V = mg\ell cos(\theta) \tag{1.2}$$

Here, we recognize that v_m, being the velocity vector of the mass attached to the pole's extrema, is given by:

[3] the necessary theory for the curious reader may be found in Appendix A

$$v_m = \left(\frac{d}{dt}\left(x - \ell sin(\theta)\right), \left(\frac{d}{dt}\left(\ell cos(\theta)\right)\right)\right) \tag{1.3}$$
$$= \left(x - \ell x\dot{\theta}cos(\theta), -\ell\dot{\theta}sin(\theta)\right) \tag{1.4}$$

By taking its squared euclidean norm, we conclude that,

$$||v_m||^2 = \dot{x}^2 - 2\dot{x}\dot{\theta}\ell cos(\theta) + \ell^2\dot{\theta}^2 \tag{1.5}$$

and so,

$$T = \frac{1}{2}(M+m)\dot{x}^2 - m\dot{x}\dot{\theta}\ell cos(\theta) + \frac{1}{2}(m\ell^2 + I)\dot{\theta}^2 \tag{1.6}$$

Therefore, we conclude that,

$$\mathcal{L} = \frac{1}{2}(M+m)\dot{x}^2 - m\dot{x}\dot{\theta}\ell cos(\theta) + \frac{1}{2}(m\ell^2 + I)\dot{\theta}^2 - mg\ell cos(\theta) \tag{1.7}$$

We shall now take our attention to the definition of action $\mathcal{S}$ within a physical system. The action can be formally defined as the integral of system's total energy through time[4],

$$\mathcal{S} = \int_{t_0}^{t} \mathcal{L}(\tau, x, \dot{x}, \theta, \dot{\theta})d\tau \tag{1.8}$$

With such definition, we state as an axiom the Principle of Least Action, which says: "The path taken by the system between times t_0 and t and configurations q_0 and q is the one for which the action is stationary to first order[2].

Under the Principle of Least Action, $\delta\mathcal{S} = 0$, therefore, by appendix A[5], the functional $\mathcal{L}$ needs to satisfy the Euler-Lagrange equations, that is,

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} - \frac{\partial\mathcal{L}}{\partial x} = F - b_x\dot{x} \tag{1.9}$$
$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta} = -b_\theta\dot{\theta} \tag{1.10}$$

Now, all we have to do is to take partial derivatives of $\mathcal{L}$. Indeed,

[4] Which, in such case, is the Lagrangian itself.

[5] Appendix A contains a more formal discussion of those concepts, based on references [5] and [6]

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = (M+m)\ddot{x} - m\ell\ddot{\theta}cos(\theta) - m\ell\dot{\theta}^2 sin(\theta) \tag{1.11}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = -m\ell cos(\theta)\ddot{x} + m\ell\dot{\theta}^2\dot{x}sin(\theta) + (m\ell^2 + I)\ddot{\theta} - m\ell\dot{\theta}^2\dot{x}sin(\theta) - mg\ell sin(\theta) \tag{1.12}$$

Being so, the dynamic behavior of our system is described by two second-order differential equations,

$$(M+m)\ddot{x} - m\ell cos(\theta)\ddot{\theta} = F + m\ell\dot{\theta}sin(\theta) - b_x\dot{x} \tag{1.13}$$

$$-m\ell cos(\theta)\ddot{x} + (m\ell^2 + I)\ddot{\theta} = mg\ell sin(\theta) - b_\theta\dot{\theta} \tag{1.14}$$

These are two non-linear differential equations in terms of $\ddot{x}$ and $\ddot{\theta}$. We can effectively separate them using Cramer's Rule. Doing so implies into calculate the following determinants,

$$\begin{aligned}
\alpha &= \begin{vmatrix} M+m & -m\ell cos(\theta) \\ -m\ell cos(\theta) & (m\ell^2+I) \end{vmatrix} \\
&= (M+m)(m\ell^2+I) - m^2\ell^2 cos^2(\theta) && (1.15)\\
N_x &= \begin{vmatrix} F + m\ell\dot{\theta}sin(\theta) - b_x\dot{x} & -m\ell cos(\theta) \\ mg\ell sin(\theta) - b_\theta\dot{\theta} & (m\ell^2+I) \end{vmatrix} \\
&= (m\ell^2+I)(F + m\ell\dot{\theta}sin(\theta) - b_x\dot{x}) + m\ell cos(\theta)(mg\ell sin(\theta) - b_\theta\dot{\theta}) && (1.16)\\
N_\theta &= \begin{vmatrix} M+m & F + m\ell\dot{\theta}sin(\theta) - b_x\dot{x} \\ -m\ell cos(\theta) & mg\ell sin(\theta) - b_\theta\dot{\theta} \end{vmatrix} \\
&= (M+m)(mg\ell sin(\theta) - b_\theta\dot{\theta}) + m\ell cos(\theta)(F + m\ell\dot{\theta}sin(\theta) - b_x\dot{x}) && (1.17)
\end{aligned}$$

With those relations, we can effectively express our ODEs as,

$$\ddot{x} = \frac{(m\ell^2+I)(F + m\ell\dot{\theta}sin(\theta) - b_x\dot{x}) + m\ell cos(\theta)(mg\ell sin(\theta) - b_\theta\dot{\theta})}{(M+m)(m\ell^2+I) - m^2\ell^2cos^2(\theta)} \tag{1.18}$$

$$\ddot{\theta} = \frac{(M+m)(mg\ell sin(\theta) - b_\theta\dot{\theta}) + m\ell cos(\theta)(F + m\ell\dot{\theta}sin(\theta) - b_x\dot{x})}{(M+m)(m\ell^2+I) - m^2\ell^2cos^2(\theta)} \tag{1.19}$$

those are, indeed, the nonlinear equations for the systems dynamics. Now, we can simulate how the system behaves in the presence of various force inputs. We display a diagram representing our system in Figure 1.3,

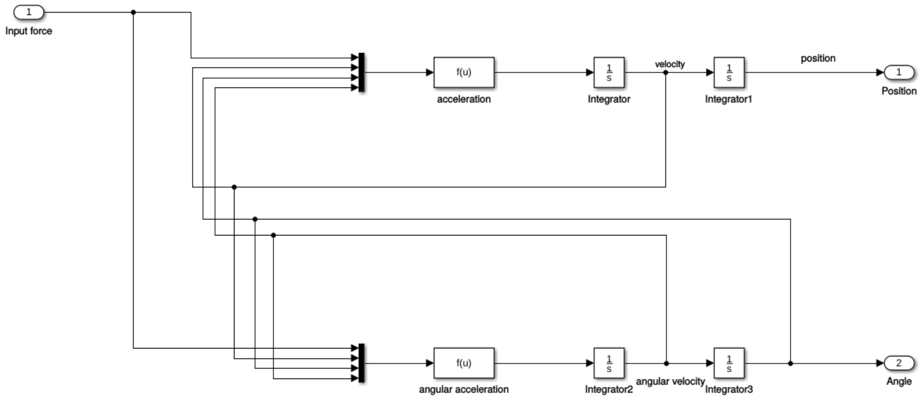

Figure 1.3: System's Diagram for inverted pendulum

With such representation, we can see the cart's position and pendulum's angle responses in Figure 1.4,

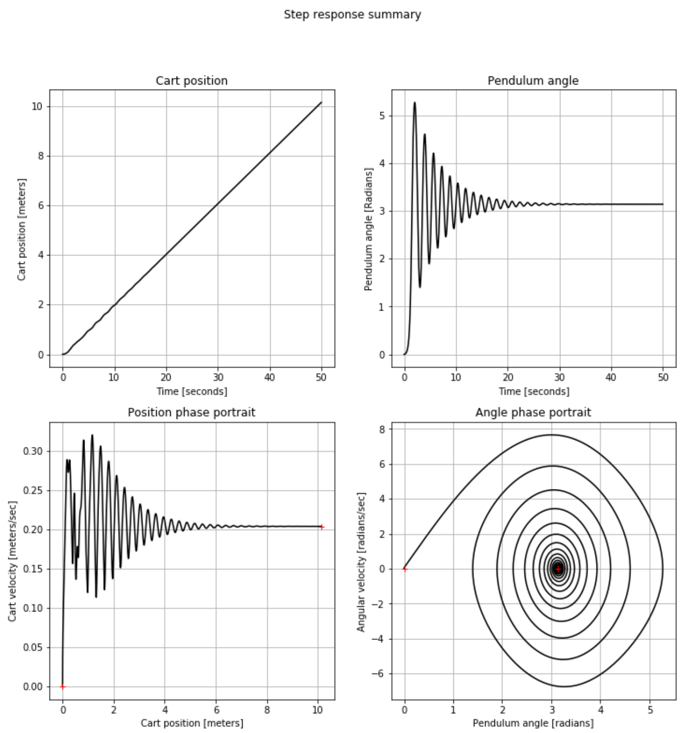

Figure 1.4: Nonlinear system's response to an step

We can point out that while the position variable, x_1 is unstable to an step input, the pendulum angle is stable, since it converges to $\theta = \pi$. This is indeed what our common sense tells us: if we suppose, for simplicity, the track as infinite, pushing the cart in any direction shall cause the pendulum, by inertia, to fall down to the downward position.

Another important comment to be made is that the cart goes off the track, since its position grows to values superior to 1 meter.

1.2 Representation

1.2.1 Linearization and State-Space representation

In order to linearize Equations 1.18 and 1.19, we shall adopt the following notation: let $\mathbf{x}$ be a vector constituted by $(x, \dot{x}, \theta, \dot{\theta})$[6]. By doing so, we can express the ODE's variables derivatives as functions of themselves,

$$\dot{x}_1 = \dot{x} = x_2 \tag{1.20}$$
$$\dot{x}_2 = \ddot{x} = \text{Equation 1.18} \tag{1.21}$$
$$\dot{x}_3 = \dot{\theta} = x_4 \tag{1.22}$$
$$\dot{x}_4 = \ddot{\theta} = \text{Equation 1.19} \tag{1.23}$$

This induces a vectorial field, $\mathbf{f} : \mathbb{R}^6 \to \mathbb{R}^4$ that satisfies, for a time t, a state vector $\mathbf{x}$ and an input force u, the vectorial equation,

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, u) \tag{1.24}$$

Linearization is done around the points such that $\mathbf{f}(t, \mathbf{x}, 0) = 0$. By inspection, the interesting cases where either $\theta = 0$, the upward position, and $\theta = \pi$, the downward position, among with $\dot{x} = 0$ and $\dot{\theta} = 0$, gives us stationary points for such field.

The idea behind stationary points is that, if $\mathbf{f}(\mathbf{x}) = 0$, then $\dot{\mathbf{x}}$ is zero, and therefore, the point does not move. For that reason, they are called stationary - since they do not move, if they are exactly in that point.

Moreover, the concept of stability comes naturally from the idea of stationarity: suppose we are in a stationary point, and we give the system a little disturbance δ. If the system returns, naturally, to the stationary point, we shall call it a stable singular point - or a sink -. However, if the system goes away from such position, we shall call it a unstable singular point.

By Figure 1.4, we can draw some conclusions. The solution of our differential equation starts in the point $(x, 0, 0, 0)$ - the source -, and goes out, until it reaches another stationary point, that time, a stable stationary point - the sink -. A more precise definition of what those terms means is given in the section about stability, and in Appendix B.

Hence, by taking derivatives,

[6]Indeed, as we shall see before, this shall be our **state vector**

$$\frac{\partial f_1}{\partial x_1} = 0 \qquad \frac{\partial f_1}{\partial x_2} = 1 \qquad \frac{\partial f_1}{\partial x_3} = 0 \qquad \frac{\partial f_1}{\partial x_4} = 0 \tag{1.25}$$

$$\frac{\partial f_3}{\partial x_1} = 0 \qquad \frac{\partial f_3}{\partial x_2} = 0 \qquad \frac{\partial f_3}{\partial x_3} = 0 \qquad \frac{\partial f_3}{\partial x_4} = 1 \tag{1.26}$$

and now, let us consider $\ddot{x} = f_2(t, \mathbf{x}, u) = \dfrac{\beta(x_2, x_3, x_4, u)}{\alpha(x_3)}$ and $\ddot{\theta} = f_4(t, \mathbf{x}, u) = \dfrac{\gamma(x_2, x_3, x_4, u)}{\alpha(x_3)}$. Being so, all derivatives with respect to x_1 are zero, and,

$$\frac{\partial f_2}{\partial x_2} = \frac{-(m\ell^2 + I)b_x}{(m\ell^2 + I)(M + m) - m^2\ell^2 cos^2(\theta)} \tag{1.27}$$

$$\frac{\partial f_2}{\partial x_3} = \frac{\frac{\partial \beta}{\partial x_3}\alpha - \frac{d\alpha}{dx_3}\beta}{\alpha^2} \tag{1.28}$$

$$\frac{\partial f_2}{\partial x_4} = \frac{(m\ell^2 + I)m\ell sin(\theta) - m\ell cos(\theta) b_\theta}{(m\ell^2 + I)(M + m) - m^2\ell^2 cos^2(\theta)} \tag{1.29}$$

Since we want those derivatives on the stationary points, we have,

$$\begin{aligned} \alpha(0) &= \alpha(\pi) = (m\ell^2 + I)(M + m) - m^2\ell^2 \\ \dot{\alpha}(0) &= \dot{\alpha}(\pi) = 0 \end{aligned}$$

and for β:

$$\begin{aligned} \frac{\partial \beta}{\partial x_2}(x, 0, 0, 0) &= -(m\ell^2 + I)b_x & \frac{\partial \beta}{\partial x_2}(x, 0, \pi, 0) &= -(m\ell^2 + I)b_x \\ \frac{\partial \beta}{\partial x_3}(x, 0, 0, 0) &= (m\ell)^2 g & \frac{\partial \beta}{\partial x_3}(x, 0, \pi, 0) &= -(m\ell)^2 g \\ \frac{\partial \beta}{\partial x_4}(x, 0, 0, 0) &= -m\ell b_\theta & \frac{\partial \beta}{\partial x_4}(x, 0, \pi, 0) &= m\ell b_\theta \end{aligned}$$

From where we conclude that,

$$\begin{aligned} \frac{\partial f_2}{\partial x_1}(x, 0, 0, 0) &= 0 & \frac{\partial f_2}{\partial x_1}(x, 0, \pi, 0) &= 0 \\ \frac{\partial f_2}{\partial x_2}(x, 0, 0, 0) &= \frac{-(m\ell^2 + I)b_x}{(M + m)(m\ell^2 + I) - (m\ell)^2} & \frac{\partial f_2}{\partial x_2}(x, 0, \pi, 0) &= \frac{-(m\ell^2 + I)b_x}{(M + m)(m\ell^2 + I) - (m\ell)^2} \\ \frac{\partial f_2}{\partial x_3}(x, 0, 0, 0) &= \frac{(m\ell)^2 g}{(M + m)(m\ell^2 + I) - (m\ell)^2} & \frac{\partial f_2}{\partial x_3}(x, 0, \pi, 0) &= \frac{-(m\ell)^2 g}{(M + m)(m\ell^2 + I) - (m\ell)^2} \\ \frac{\partial f_2}{\partial x_4}(x, 0, 0, 0) &= \frac{-m\ell b_\theta}{(M + m)(m\ell^2 + I) - (m\ell)^2} & \frac{\partial f_2}{\partial x_4}(x, 0, \pi, 0) &= \frac{m\ell b_\theta}{(M + m)(m\ell^2 + I) - (m\ell)^2} \end{aligned}$$

with addition of gamma derivatives,

$$\frac{\partial \gamma}{\partial x_2} = -m\ell cos(\theta) b_x \tag{1.30}$$
$$\frac{\partial \gamma}{\partial x_4} = (m\ell)^2 sin(\theta)cos(\theta) - (M+m)b_\theta \tag{1.31}$$

$$\frac{\partial \gamma}{\partial x_3} = (M+m)mg\ell cos(\theta) - m\ell sin(\theta)(F + m\ell\dot{\theta} sin(\theta) - b_x\dot{x}) + m^2\ell^2 cos^2(\theta)\dot{\theta} \tag{1.32}$$

At $(x, 0, 0, 0)$ and $(x, 0, \pi, 0)$, those becomes:

$$\begin{aligned}
\frac{\partial \gamma}{\partial x_2}(x,0,0,0) &= -m\ell b_x & \frac{\partial \gamma}{\partial x_2}(x,0,\pi,0) &= m\ell b_x \\
\frac{\partial \gamma}{\partial x_3}(x,0,0,0) &= (M+m)mg\ell & \frac{\partial \gamma}{\partial x_3}(x,0,\pi,0) &= -(M+m)mg\ell \\
\frac{\partial \gamma}{\partial x_4}(x,0,0,0) &= -(M+m)b_\theta & \frac{\partial \gamma}{\partial x_4}(x,0,\pi,0) &= -(M+m)b_\theta
\end{aligned}$$

also, we may write for $(x, 0, 0, 0)$,

$$\frac{\partial f_4}{\partial x_1} = 0 \qquad \frac{\partial f_4}{\partial x_2} = \frac{-m\ell b_x}{(M+m)(m\ell^2+I) - m^2\ell^2} \tag{1.33}$$
$$\frac{\partial f_4}{\partial x_3} = \frac{(M+m)mg\ell}{(M+m)(m\ell^2+I) - m^2\ell^2} \qquad \frac{\partial f_4}{\partial x_4} = \frac{-(M+m)b_\theta}{(M+m)(m\ell^2+I) - m^2\ell^2} \tag{1.34}$$

or still, for $(x, 0, \pi, 0)$,

$$\frac{\partial f_4}{\partial x_1} = 0 \qquad \frac{\partial f_4}{\partial x_2} = \frac{m\ell b_x}{(M+m)(m\ell^2+I) - m^2\ell^2} \tag{1.35}$$
$$\frac{\partial f_4}{\partial x_3} = \frac{-(M+m)mg\ell}{(M+m)(m\ell^2+I) - m^2\ell^2} \qquad \frac{\partial f_4}{\partial x_4} = \frac{-(M+m)b_\theta}{(M+m)(m\ell^2+I) - m^2\ell^2} \tag{1.36}$$

Finally, we need to compute the partial derivatives of each f_j with respect to the input. Those are easily found to be,

$$\begin{aligned}
\frac{\partial f_1}{\partial u} &= 0 & \frac{\partial f_2}{\partial u} &= \frac{m\ell^2 + I}{(m\ell^2+I)(M+m) - m^2\ell^2 cos^2(\theta)} \\
\frac{\partial f_3}{\partial u} &= 0 & \frac{\partial f_4}{\partial u} &= \frac{m\ell cos(\theta)}{(m\ell^2+I)(M+m) - m^2\ell^2 cos^2(\theta)}
\end{aligned}$$

In fact, those calculations gave us what we call **the Jacobian matrices** of the field, $\mathbf{f}$, with respect to the state-vector $\mathbf{x}$, and the input u. We remark that those are particular important because, through Taylor series expansion, one can write,

$$\mathbf{f}(t, \mathbf{x}_{sp} + \mathbf{x}_d, u_{sp} + u_d) = \mathbf{f}(t, \mathbf{x}_{sp}, u_{sp}) + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{(\mathbf{x}_{sp}, u_{sp})} \mathbf{x}_d + \left.\frac{\partial \mathbf{f}}{\partial u}\right|_{(\mathbf{x}_{sp}, u_{sp})} u_d \tag{1.37}$$

but, as hypothesis, $\mathbf{f}(t, \mathbf{x}_{sp}, u_{sp}) = 0$, since both $\mathbf{x}_{sp}$ and u_{sp} are stationary points. Therefore, what is left is,

$$\frac{d}{dt}\left(\mathbf{x}_{sp} + \mathbf{x}_d\right) = \mathbf{f}(t, \mathbf{x}_{sp} + \mathbf{x}_d, u_{sp} + u_d) \tag{1.38}$$

$$\dot{\mathbf{x}}_d = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{(\mathbf{x}_{sp}, u_{sp})} \mathbf{x}_d + \left.\frac{\partial \mathbf{f}}{\partial u}\right|_{(\mathbf{x}_{sp}, u_{sp})} u_d \tag{1.39}$$

$$\dot{\mathbf{x}}_d = \mathbf{A}\mathbf{x}_d + \mathbf{B}u_d \tag{1.40}$$

which we identify as the state-space representation of our system, with matrices:

- $\mathbf{A}$, the Jacobian matrix of $\mathbf{f}$, given by:

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} & \frac{\partial f_3}{\partial x_4} \\ \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \frac{\partial f_4}{\partial x_3} & \frac{\partial f_4}{\partial x_4} \end{bmatrix}$$

- $\mathbf{B}$, the derivative vector of $\mathbf{f}$ with respect to u, given by:

$$\begin{bmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_2}{\partial u} & \frac{\partial f_3}{\partial u} & \frac{\partial f_4}{\partial u} \end{bmatrix}^T$$

To see how $\theta_{sp} = 0$ behaves like a unstable stationary point, and $\theta_{sp} = \pi$ like an stable stationary one, we display in Figures 1.5 and 1.6 their responses,

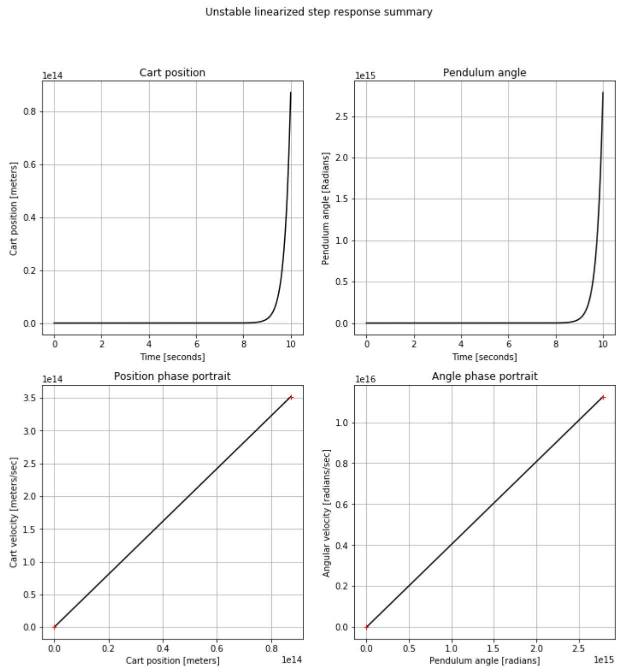

Figure 1.5: Linearized step response for $\theta_{sp} = 0$

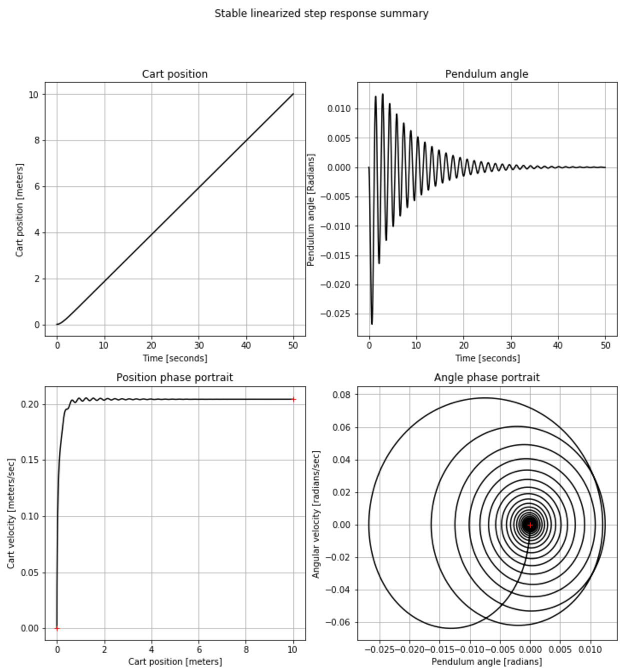

Figure 1.6: Linearized step response for $\theta_{sp} = \pi$

1.2.2 Transfer Function derivation

To express the system in terms of transfer functions, we perform a similar procedure we have done in the linearization section, but, at this time, in Equations 1.13 and 1.14. The procedure come from noticing that the only coordinate that change between the stationary points is $\theta = 0$ and $\theta = \pi$. Effectively,

Function	$\theta = 0$	$\theta = \pi$
$sin(\theta)$	θ	$-\theta$
$cos(\theta)$	1	-1
$\dot{\theta}\theta$	0	0

Table 1.2: Approximation table

Working first for $\theta = 0$,

$$(M+m)\ddot{x} - m\ell\ddot{\theta} = F - b_x\dot{x} \tag{1.41}$$

$$-m\ell\ddot{x} + (m\ell^2 + I)\ddot{\theta} = mg\ell\theta - b_\theta\dot{\theta} \tag{1.42}$$

now, in the frequency domain,

$$\Big((M+m)s^2 + b_x s\Big)X - m\ell s^2\Theta = F \tag{1.43}$$

$$-m\ell s^2 X + \Big((m\ell^2 + I)s^2 + b_\theta s - mg\ell\Big)\Theta = 0 \tag{1.44}$$

Solving for X and Θ results in calculating the following determinants,

$$\begin{aligned}
D(s) &= \begin{vmatrix} (M+m)s^2 + b_x s & -m\ell s^2 \\ -m\ell s^2 & (m\ell^2+I)s^2 + b_\theta s - mg\ell \end{vmatrix} \\
&= \alpha s^4 + \Big((M+m)b_\theta + (m\ell^2+I)b_x\Big)s^3 + \Big(b_\theta b_x - (M+m)mg\ell\Big)s^2 - mg\ell b_x s \\
N_x(s) &= \begin{vmatrix} F & -m\ell s^2 \\ 0 & (m\ell^2+I)s^2 + b_\theta s - mg\ell \end{vmatrix} \\
&= F\Big((m\ell^2+I)s^2 + b_\theta s - mg\ell\Big) \\
N_\theta(s) &= \begin{vmatrix} (M+m)s^2 + b_x s & F \\ -m\ell s^2 & 0 \end{vmatrix} \\
&= F\Big(m\ell s^2\Big)
\end{aligned}$$

Then, we can write, for $\mathbf{x}_{sp} = (0,0,0,0)$

$$T_x = \frac{X(s)}{U(s)} = \frac{(m\ell^2+I)s^2 + b_\theta s - mg\ell}{\alpha s^4 + \Big((M+m)b_\theta + (m\ell^2+I)b_x\Big)s^3 + \Big(b_\theta b_x - (M+m)mg\ell\Big)s^2 - mg\ell b_x s} \tag{1.45}$$

$$T_\theta = \frac{\Theta(s)}{U(s)} = \frac{m\ell s^2}{\alpha s^4 + \Big((M+m)b_\theta + (m\ell^2+I)b_x\Big)s^3 + \Big(b_\theta b_x - (M+m)mg\ell\Big)s^2 - mg\ell b_x s} \tag{1.46}$$

Now, for $\mathbf{x}_{sp} = (0,0,\pi,0)$, we can repeat the above procedure to find,

$$T_x = \frac{X(s)}{U(s)} = \frac{(m\ell^2+I)s^2 + b_\theta s + mg\ell}{\alpha s^4 + \Big((M+m)b_\theta + (m\ell^2+I)b_x\Big)s^3 + \Big(b_\theta b_x + (M+m)mg\ell\Big)s^2 + mg\ell b_x s} \tag{1.47}$$

$$T_\theta = \frac{\Theta(s)}{U(s)} = \frac{-m\ell s^2}{\alpha s^4 + \Big((M+m)b_\theta + (m\ell^2+I)b_x\Big)s^3 + \Big(b_\theta b_x + (M+m)mg\ell\Big)s^2 + mg\ell b_x s} \tag{1.48}$$

These equations are the counterpart in frequency of the linear state-space representation. It is relatively easy to see that $D(s) = det(s\mathbf{I} - \mathbf{A})$, implying that the poles of both transfer functions are the eigenvalues of $\mathbf{A}$, allowing us to make the same inference we did using the matrix spectra, for the polynomial of T_x and T_θ.

A last comment concerns the step response we have presented during this report. For T_x, the pole at origin introduced by the step input does not cancels with the numerator - thing that happens, when we talk about T_θ. Indeed, it is exactly this pole that introduces the instability in the plant for the cart's position.

1.3 Stability Analysis

In this section, we discuss further results about the stability of our system. Up to now, we have made a few comments based on Figures 1.5 and 1.6, that is, we have claimed that $\mathbf{x}_{sp} = (x,0,0,0)$ is a unstable stationary point, and $\mathbf{x}_{sp} = (x,0,\pi,0)$. First, we need to formalize the concepts of stability,

- A stationary point $\mathbf{x}^* = (x_1^*, x_2^*, x_3^*, x_4^*) \in \mathbb{R}^4$ is said to be stable if, and only if, for each $\epsilon > 0$ there exists a $\delta > 0$ such that for every solution $\mathbf{x}(t) = \boldsymbol{\phi}(t,\mathbf{x})$,

$$|\boldsymbol{\phi}(0,\mathbf{x}) - \mathbf{x}^*| < \delta \overset{\forall t \geq 0}{\rightarrow} |\boldsymbol{\phi}(t,\mathbf{x}) - \mathbf{x}^*| < \epsilon$$

Following the common-sense, stability says that trajectories which initiates δ-close to stationary points $\mathbf{x}^*$ stay ϵ-close, as time goes by. We say a point is unstable if it is not stable. Another kind of stability is asymptotically stability, which may be defined as,

- A stationary point $\mathbf{x}^* = (x_1^*, x_2^*, x_3^*, x_4^*) \in \mathbb{R}^4$ is said to be asymptotically stable if, and only if, there exists a δ such that each trajectory $\boldsymbol{\phi}(t, \mathbf{x})$,

$$|\boldsymbol{\phi}(0,\mathbf{x}) - \mathbf{x}^*| < \delta \to \lim_{t\to\infty} \boldsymbol{\phi}(t,\mathbf{x}) = \mathbf{x}^*$$

This has the same sense as before, but with a little bit more of flexibility, in the sense that solutions are only guaranteed to approach $\mathbf{x}^*$ in the infinity.

In addition to those concepts, we shall define the sinks of a given field $\mathbf{f}$ as the points $\mathbf{x}^*$ whose Jacobian matrix of $\mathbf{f}$ is negative definite, that is,

- A stationary point $\mathbf{x}^*$ of the field $\mathbf{f}$ is a sink, if and only if all eigenvalues of $\dfrac{\partial \mathbf{f}}{\partial \mathbf{x}}$ are negative.

An important theorem gives us the connection between sinks and asymptotically stable points, which is the Lyapunov-Perron theorem, whose enunciate may be found bellow. With such theorem, we substitute the stability analysis of the field $\mathbf{f}$, by the stability analysis of the Jacobian, which is easier.

Theorem 1. *Let $\mathbf{x}^* \in E \subset \mathbb{R}^n$ be a stationary point of the field $\mathbf{f} : E \to \mathbb{R}^n$. If $\mathbf{x}^*$ is a sink of $\mathbf{f}$, then it is an asymptotically stable point of $\mathbf{f}$.*

A more detailed discussion about the proof, and stability analysis of linearized models can be found in the appendix. For now, it is sufficient to say that the time response of the system relies on the spectra of matrix $A = \dfrac{\partial \mathbf{f}}{\partial \mathbf{x}}$. Indeed, let λ_i be the eigenvalue associated with eigenvector $\boldsymbol{\xi}_i$, then, being $\mathbf{x} = \sum_i \boldsymbol{\xi}_i e^{\lambda_i t}$,

$$\begin{aligned} \mathbf{A}\mathbf{x} &= \sum_i e^{\lambda_i t} \boldsymbol{A}\boldsymbol{\xi}_i && (1.49)\\ &= \sum_i \lambda_i e^{\lambda_i t} \xi_i && (1.50)\\ &= \dot{\mathbf{x}} && (1.51) \end{aligned}$$

As we wanted. Thus, the stability will be determined by the real part of each eigenvalue, λ_i. The proof of Theorem 1, indeed, only establishes the intuitive fact that, in a neighborhood of each domain's point, the non-linear solution resembles the linear one.

Therefore, we need to look into the spectra of A. A quick evaluation using the benchmark values, gives us the eigenvalues displayed in Table 1.3 and Figure 1.7

Eigenvalue	A_0	A_π
λ_1	0	0
λ_2	-4.423	-1.0819
λ_3	-0.9469	-0.126 + 4.218i
λ_4	4.0349	-0.126 - 4.218i

Table 1.3: System's Eigenvalues

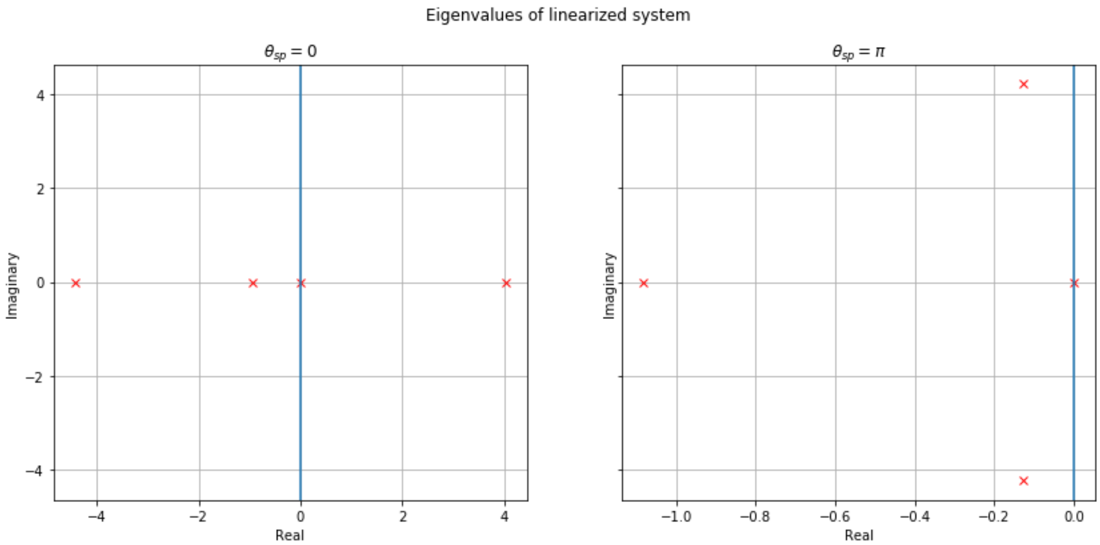

Figure 1.7: Eigenvalues Positions

Those results indeed confirm our conclusion that $\mathbf{x}_{sp} = (x, 0, 0, 0)$ is a source, or an unstable stability point, while $\mathbf{x}_{sp} = (x, 0, \pi, 0)$ is a stable one, and therefore a sink. The effort in the next chapters will be to analyze and build controllers to stabilize the system near $\mathbf{x}_{sp} = (x, 0, 0, 0)$, so we can have a pendulum with stable upward position.

To point out, we make a more formal discussion of those concepts in Appendix B. These formalizations are based on [3] and [4].

1.4 Simulation

In this section, we discuss and compare the behavior of the non-linear, and the linearized system due to several inputs. In particular, we simulate our system in three cases,

1. The free-response, with initial conditions $\mathbf{x}_0 = (0, 0, 0, 0.1)$, that is, a slightly velocity in the anti-clockwise direction.

2. The impulse response, with initial conditions $\mathbf{x}_0 = (0, 0, 0, 0)$, to illustrate how a push in the cart drive the pendulum to the downward position,

3. The step response, with initial conditions $\mathbf{x}_0 = (0,0,0,0)$, in the same spite of the last example,

4. A sinusoidal signal, $Asin(\omega t)$, to analyze how the system responds to a oscillating force.

First, the intents with the free-response, with an initial condition slightly moving in the anti-clockwise direction is to show how the upward position is unstable. Specially, from previous discussion, we already know that the upward position is unstable. We can see that this is indeed the case in Figure 1.8

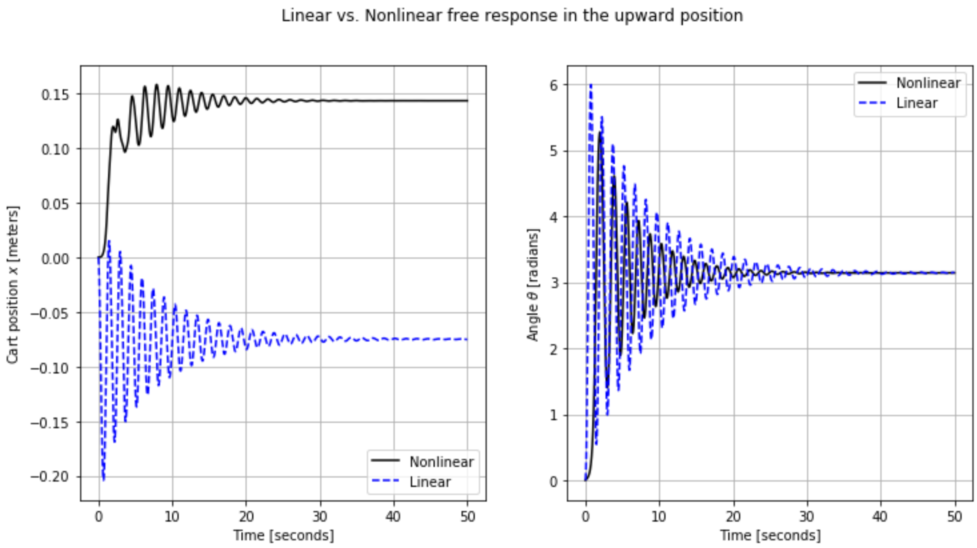

Figure 1.8: Free response of inverted pendulum

Although the linear response approximated quite well the angle position of the pendulum, we can not say the same about the cart's position, since it converges to another point, with a different kind of behavior. In the pendulum angle's response, we start in the upward position, then, since we have a counter-clockwise velocity, the pendulum falls to the stable position, π. Both linear and non-linear responses exhibits this behavior, although the nonlinear appears to have a delay.

In a similar way, the impulse response is closely related to the dynamics of the response to initial conditions. This is motivated by the fact that the impulse acts imposing initial conditions to the system at $t = 0$. For $t > 0$, the input force ceases, and thus, the system behaves like in the free response. This can be saw in Figure 1.9,

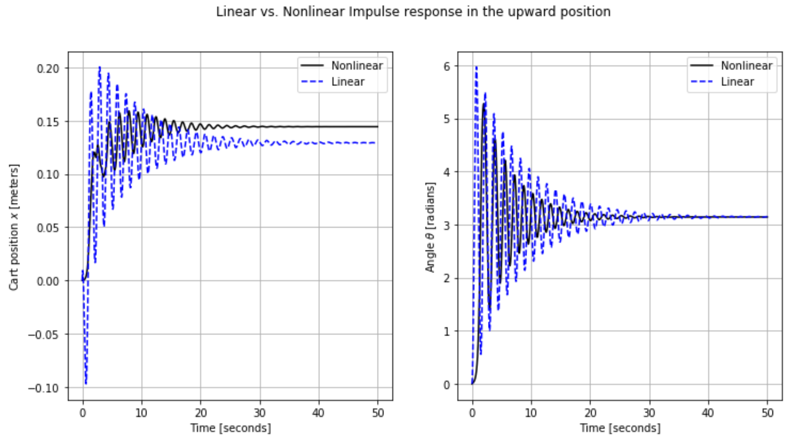

Figure 1.9: Impulse Response

In that case, the behavior only is not exactly the same because the impulse have imposed different initial conditions. Also, we see that the linear response begins to resemble more the nonlinear one. In both cases, the force or initial condition has made the cart to move a little to the right, and the pendulum to fall down to the downward position.

For the step response, we have Figure 1.10,

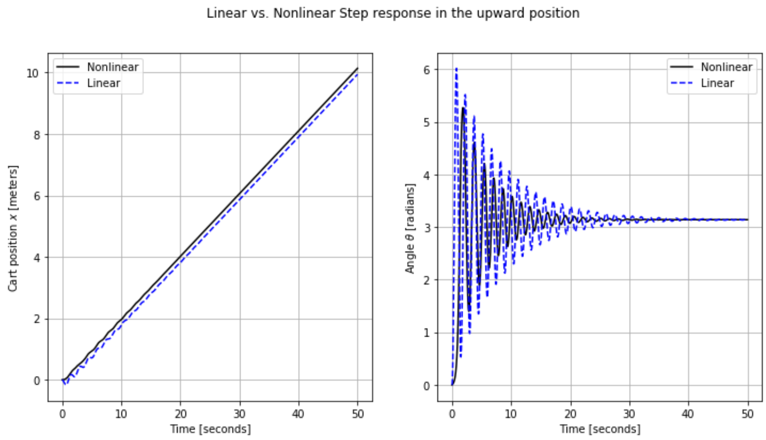

Figure 1.10: Step Response

The similarity in the last three figures highlights the fact that, being the unstable equilibrium broken, the system oscillates until it reaches $\theta = \pi$. Also important to mention, being the driving force a step function, the cart is constantly being pushed to the right. As consequence, the cart's position is unstable. From the angle's curve, we can also obtain a few information about the curves shape, as:

1. The rise time, t_r, which is the time the system takes to go from 0.1 of its final value,

to 0.9.

2. The settling time, t_s, which is the time it takes the system transients to decay,

3. The overshoot, M_p, which is the value that the curve overshoots its final value, in percentage.

4. The peak time, t_p, which is the time at which the response achieves its maximum value.

We display those results in Table 1.4, to compare the linear and nonlinear responses,

Measure	Linear	Nonlinear
t_r (seconds)	0.25	0.60
t_s (seconds)	22.88	12.51
M_p	91.60%	67.83%
t_p (seconds)	0.74	2.06

Table 1.4: Time domain response information summary

To calculate those values, we have approximated the response as second order, estimating ω_n and ξ from the formulas found in [7]. Specially, M_p, t_p and t_r were found manually using python. From those, we estimate,

$$\omega_n = \frac{1.8}{t_r} \tag{1.52}$$

$$\xi = \frac{-ln(M_p)}{\sqrt{\pi^2 + \Big(ln(M_p)\Big)^2}} \tag{1.53}$$

We also point out that, despite the resemblance of general shape and behavior of our linearization, the values displayed in Table 1.4 are quite different.

As a final remark, we have simulated the systems response has consequence of various sinusoidal waves, $u(t) = Asin(\omega t)$. The results for the cart's position and pendulum's angle are displayed separately in Figures 1.11 and 1.12,

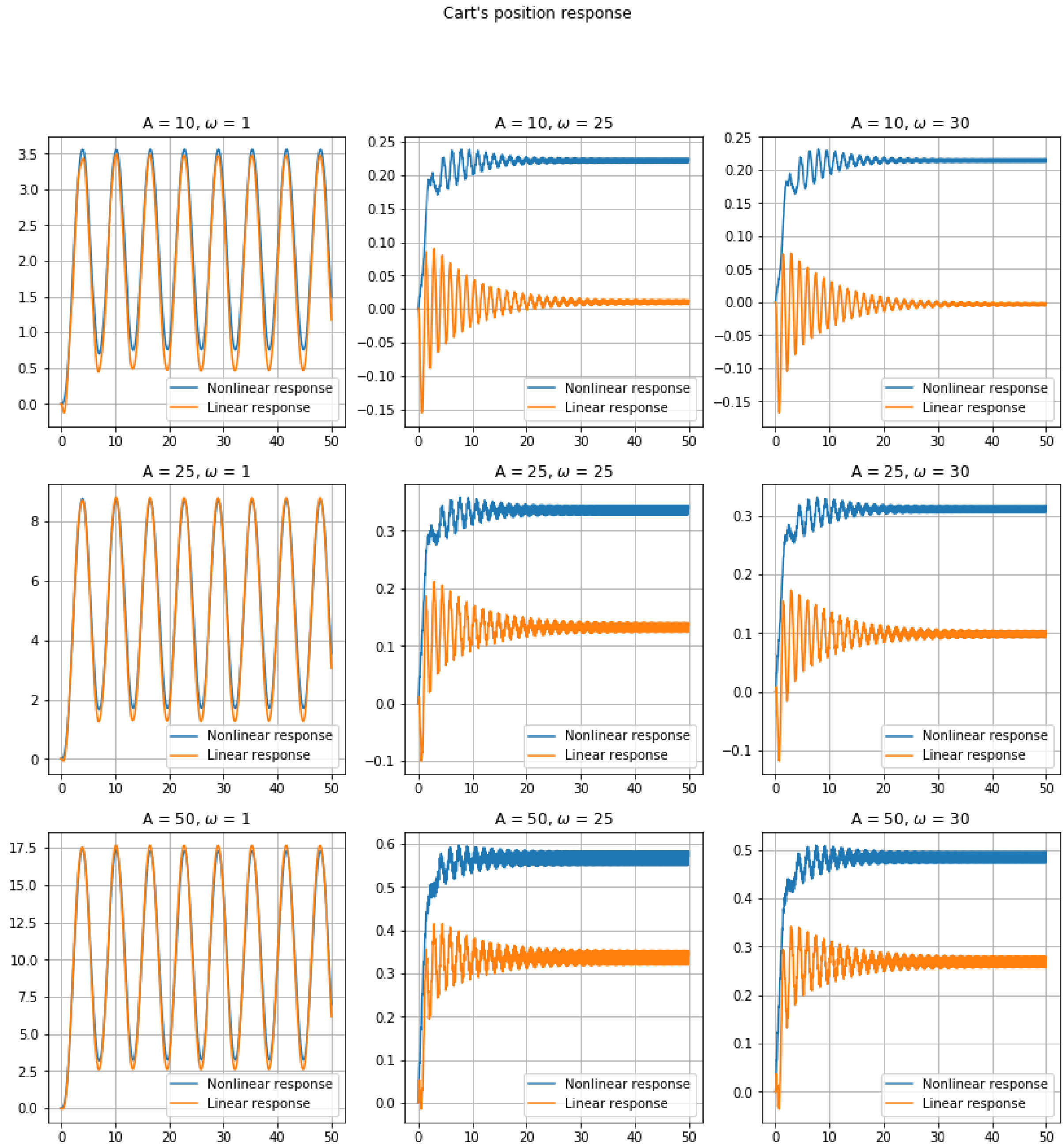

Figure 1.11: Cart's position response due to sinusoidal input

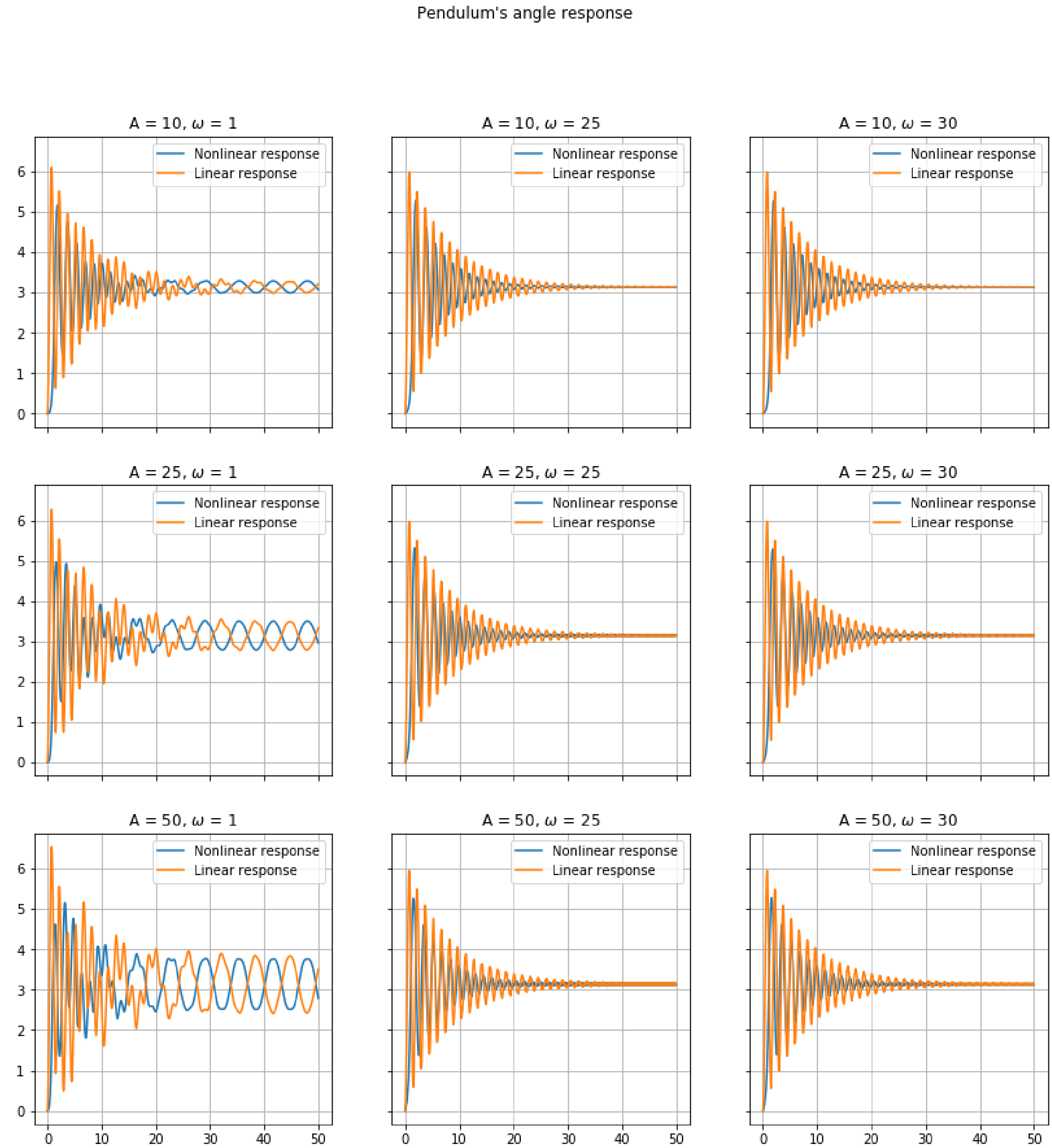

Figure 1.12: Pendulum's angle response due to sinusoidal input

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