Homework 1: representation and simulation of Inverted Pendulum

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Outline

Introduction

This presentation wants to cover the topics of Homework 1, of the discipline of advanced control, namely,

- Mathematical Modeling of the system,
- Linearization and system representation,
- Stability analysis.

The physical system which we will analyze is called "Inverted Pendulum".

- Codes and further explanation are available in `https://github.com/eddardd/Control-Theory/tree/master/Advanced-Control`

Intuition

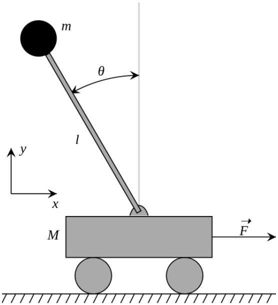

Figura: The physical schematic of our system

The system we have chosen is composed by a cart, with mass M, with a pole of length ℓ attached to it, with a ball of mass m at its extrema.

Intuition

The study of such system encounters applications, for example, in the development of devices called "Segways",

Figura: An example of a segway

Physical modeling

In order to describe the motion of our system, we shall adopt the Lagrangian formalism. To that effort, we need to define

$$\mathcal{L} = T - V$$

which accomplishes for the total energy in the system. Once we have it, we know it satisfies, for each degree of freedom, the **Euler-Lagrange** equation,

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0$$

Physical Modeling

Analyzing the system, we recognize two degrees of freedom:

- The cart's position, x,
- The pole's angle, θ

Being so,

- $T = \frac{1}{2}(M+m)\dot{x}^2 - m\dot{x}\dot{\theta}\ell cos(\theta) + \frac{1}{2}(m\ell^2 + I)\dot{\theta}^2$
- $V = mg\ell cos(\theta)$

Physical Modeling

With the last equations, and taking derivatives, we can write the Euler-Lagrange equations twice,

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = F - b_x \dot{x}$$
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = -b_\theta \dot{\theta}$$

These expressions leads to a system of (non-linear) equations,

$$(M+m)\ddot{x} - m\ell cos(\theta)\ddot{\theta} = F + m\ell sin(\theta)\dot{\theta} - b_x\dot{x}$$
$$(m\ell^2 + I)\ddot{\theta} - m\ell cos(\theta)\ddot{x} = mg\ell sin(\theta) - b_\theta\dot{\theta}$$

Physical Modeling

Those equations can be solved by using Cramer's Rule, yielding the following,

$$\ddot{x} = \frac{(m\ell^2 + I)(F + m\ell\dot{\theta} sin(\theta) - b_x\dot{x}) + m\ell cos(\theta)(mg\ell sin(\theta) - b_\theta\dot{\theta})}{(M+m)(m\ell^2 + I) - m^2\ell^2 cos^2(\theta)}$$

$$\ddot{\theta} = \frac{(M+m)(mg\ell sin(\theta) - b_\theta\dot{\theta}) + m\ell cos(\theta)(F + m\ell\dot{\theta} sin(\theta) - b_x\dot{x})}{(M+m)(m\ell^2 + I) - m^2\ell^2 cos^2(\theta)}$$

which are the non-linear equations that govern the system dynamics.

Mathematical Analysis

Since we have equations for $\ddot{x}$ and $\ddot{\theta}$, we notice that,

$$\begin{aligned}
\dot{x} &= f_1(t, x, \dot{x}, \theta, \dot{\theta}, u) \\
\ddot{x} &= f_2(t, x, \dot{x}, \theta, \dot{\theta}, u) \\
\dot{\theta} &= f_3(t, x, \dot{x}, \theta, \dot{\theta}, u) \\
\ddot{\theta} &= f_4(t, x, \dot{x}, \theta, \dot{\theta}, u)
\end{aligned}$$

That is, by defining $\mathbf{x} = (x, \dot{x}, \theta, \dot{\theta})$, $\mathbf{f} = (f_1, f_2, f_3, f_4)$, we have,

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, u)$$

Mathematical Analysis

Indeed, we have defined a (non-linear) vectorial field over $\mathbf{R}^4$. A quick simulation using numerical integration gives us the following results,

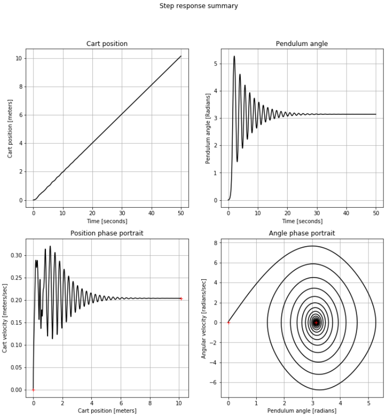

Figura: Step-response of the system

Mathematical Analysis

Indeed, two points are of our interest,

- The upward point, $\mathbf{x}_{sp} = (0, 0, 0, 0)$, which yields, by inspection, $\mathbf{f}(t, \mathbf{x}) = 0$.
- The downward point, $\mathbf{x}_{sp} = (0, 0, \pi, 0)$, which yields, by inspection, $\mathbf{f}(t, \mathbf{x}) = 0$.

These points gives rise of what we call **singular trajectories**, since for every t, with those initial conditions, $\mathbf{f}(\mathbf{x}) = 0$.

Stability

Assuming $\mathbf{x}_{sp}$ being a stationary point, we informally define stability as,

- Stable stationary points are those whose, after applied a disturbance, tends to get back to the original point.
- Unstable points are those whose, after applied a disturbance, tends to go away from the original point.

Under those definitions, we notice that $(0,0,0,0)$ is a unstable point, while $(0,0,\pi,0)$ is a stable one, matching our common sense.

Linearization

- Linearization is a local technique analysis, which permits us to transform the nonlinear equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u)$ into a linear one,
- By Taylor's Expansion,

$$\mathbf{f}(\mathbf{x}_{sp}, u) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x} - \mathbf{x}_{sp}) + \frac{\partial f}{\partial u} u$$

in which we identify $\dfrac{\partial \mathbf{f}}{\partial \mathbf{x}}$ as the Jacobian matrix of $\mathbf{f}$. This gives us the following state-space representation,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

State Space equations

For $\mathbf{x}_{sp} = (0,0,0,0)$, we calculate the partial derivatives of f, to achieve:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\dfrac{(m\ell^2 + I)b_x}{\alpha} & \dfrac{(m\ell)^2 g}{\alpha} & \dfrac{-m\ell b_\theta}{\alpha} \\ 0 & 0 & 0 & 1 \\ 0 & -\dfrac{m\ell b_x}{\alpha} & \dfrac{(M+m)mg\ell}{\alpha} & \dfrac{-(M+m)b_\theta}{\alpha} \end{bmatrix}$$

with $\alpha = (M+m)(m\ell^2 + I) - (m\ell)^2$.

State Space equations

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with $\alpha = (M + m)(m\ell^2 + I) - (m\ell)^2$.

State Space equations

For **B**, we have:

$$\mathbf{B}_0 = \begin{bmatrix} 0 \\ \dfrac{m\ell^2 + I}{\alpha} \\ 0 \\ \dfrac{m\ell}{\alpha} \end{bmatrix} \qquad \mathbf{B}_\pi = \begin{bmatrix} 0 \\ \dfrac{m\ell^2 + I}{\alpha} \\ 0 \\ \dfrac{-m\ell}{\alpha} \end{bmatrix}$$

Transfer Function

The transfer functions of the system can be encountered by transforming the State-Space equation to frequency, and solving for $\dfrac{Y}{U}$,

$$
\begin{aligned}
s\mathbf{X}(s) &= \mathbf{A}\mathbf{X}(s) + \mathbf{B}u(s) \\
(s\mathbf{I} - \mathbf{A})\mathbf{X} &= \mathbf{B}u \\
\mathbf{X}(s) &= (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}u
\end{aligned}
$$

And, being $\mathbf{Y} = \mathbf{I}\mathbf{X}$, we have:

$$
\frac{Y(s)}{U(s)} = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}
$$

Transfer Function

Calculating and solving for each variable yields, for $\mathbf{x}_{sp} = (0, 0, 0, 0)$.

$$\frac{X(s)}{U(s)} = \frac{-((m\ell^2 + I)s^2 + b_\theta s - \dfrac{((m\ell^2 + I) - (m\ell)^2)mg\ell}{\alpha})}{\alpha s^4 + ((M+m)b_\theta + (m\ell^2 + I)b_x)s^3 + (b_\theta b_x - (M+m)mg\ell)s^2 - (mg\ell b_x)s}$$

$$\frac{\Theta(s)}{U(s)} = \frac{\dfrac{(m\ell^2 + I)m\ell}{\alpha}s^2 + \dfrac{(m\ell^2 + I)m\ell(b_x - 1)}{\alpha}s}{\alpha s^4 + ((M+m)b_\theta + (m\ell^2 + I)b_x)s^3 + (b_\theta b_x - (M+m)mg\ell)s^2 - (mg\ell b_x)s}$$

similar results can be done for $\mathbf{x}_{sp} = (0, 0, \pi, 0)$

Stability analysis in linear systems

The following theorem states the equivalence between stability of stationary points in non-linear and linear systems,

Lyapunov-Perron Theorem

Let $\mathbf{x}_{sp}$ be a stationary point of a field $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$. Thus, $\mathbf{x}_{sp}$ is a stable stationary point of $\mathbf{f}$ if, and only if $\mathbf{J}|_{\mathbf{x}=\mathbf{x}_{sp}}$ has only eigenvalues with negative real part.

- This indeed allows us to substitute the analysis of stability of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ by $\dot{\mathbf{x}} = \mathbf{J}\mathbf{x}$

Linear stable systems

We begin by noticing that the solution of $\dot{\mathbf{x}} = \mathbf{J}\mathbf{x}$ relies on the spectra of $\mathbf{J}$,

- If $\mathbf{J}$ is diagonalizable, then the solution is

$$\mathbf{x}(t) = \sum_{i=1}^{n} \boldsymbol{\xi}_i e^{\lambda_i t}$$

  for eigenvalues λ_i and eigenvectors ξ_i.
- If any λ_i has positive real part, then the solution diverges on at least one of its coordinates.
- The system is thus, unstable.

This is the criteria for Linear Stability.

Responses

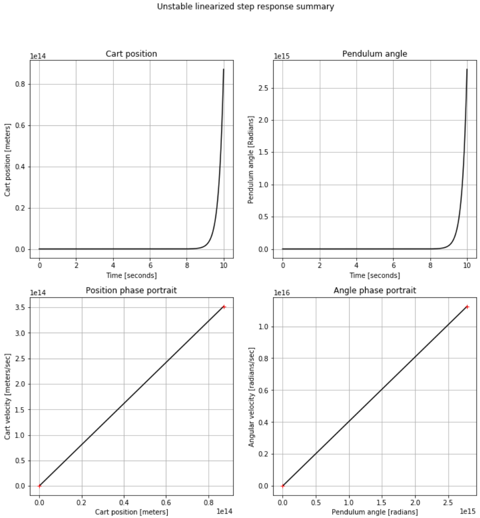

Figura: Linearized step response for $\theta_{sp} = 0$. Observe how it is unstable.

Responses

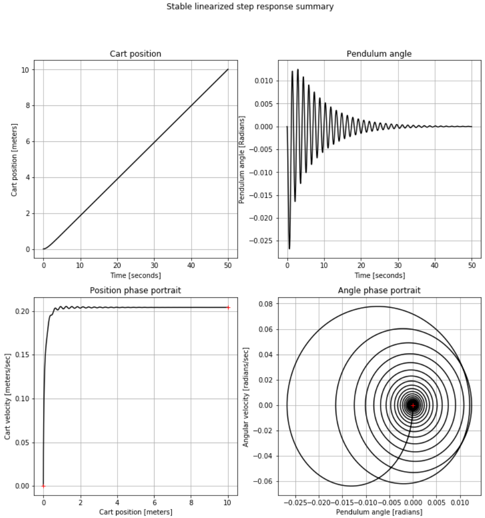

Figura: Linearized step response for $\theta_{sp} = \pi$. Confirming our claim that π is a stable stationary point.

Poles of linearizations

The later responses can be understood by looking at the eigenvalues of each linearized matrix, A:

Eigenvalue	A_0	A_π
λ_1	0	0
λ_2	-4.423	-1.0819
λ_3	-0.9469	-0.126 + 4.218i
λ_4	4.0349	-0.126 - 4.218i

Tabela: System's Eigenvalues

Since A_0 has a pole (or eigenvalue) with positive real part, we conclude that it is an unstable stationary point.