Lecture Notes on Control Theory: Frequency Analysis of Electric Circuits

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1 Circuit Elements

1.1 Capacitor

Concerning capacitors, we know that the rate of electrical Charge and electrical voltage is given by the capacitance, C:

$$C = \frac{Q}{V_C} \tag{1}$$

We can rewrite this and differentiate, to achieve:

$$V_C(t) = \frac{1}{C}Q(t) \tag{2}$$
$$\dot{V}_C(t) = \frac{1}{C}i(t) \tag{3}$$

where $i(t)$ is the current function. We can, of course, use the Laplace Transform to achieve

$$sV_C(s) = \frac{1}{C}I(s) \tag{4}$$
$$\frac{V_C}{I}(s) = \frac{1}{Cs} \tag{5}$$

the rate $\mathcal{Z}(s) = \dfrac{V_C}{I}$ is called **impedance**, and it behaves as a variable resistence in the frequency domain.

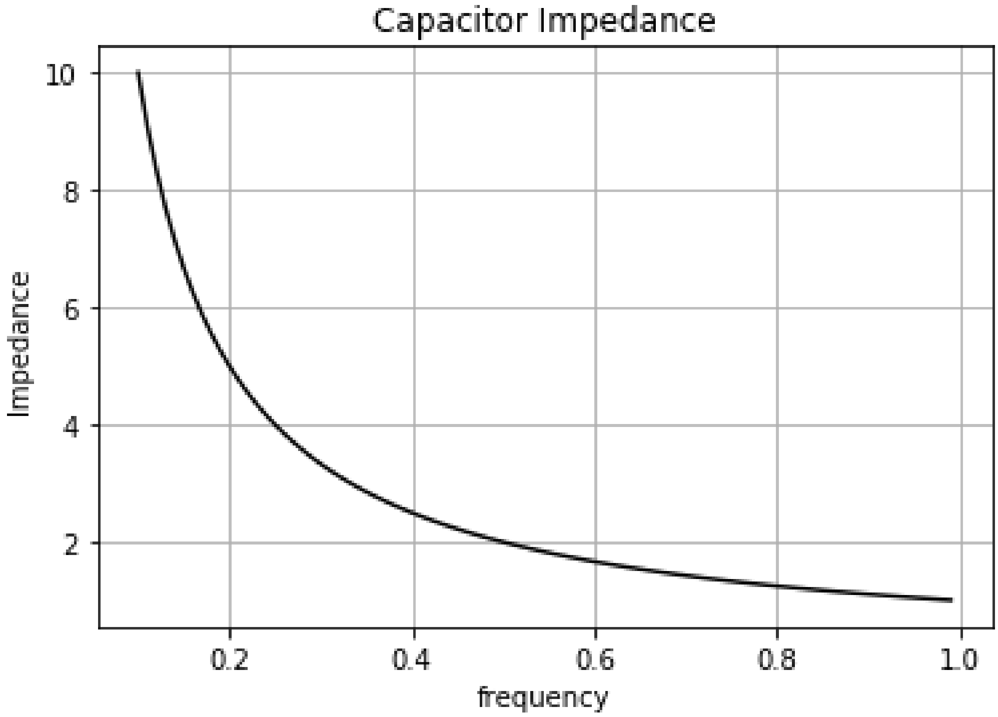

Figure 1: Impedance as a function of frequency for the capacitor.

As a remark, at low frequencies, the Capacitor impedance is high, and so, it opposes itself to the passage of current. In a ideal situation, $\mathcal{Z}(s) = \dfrac{V_C(s)}{I(s)} \to +\infty$, that is, it behaves like a open circuit, while at high frequency, $\mathcal{Z}(s) \to 0$, that is, it behaves like a short-circuit.

1.2 Inductor

Similar to the capacitor equations, we have for the inductor the following equation:

$$L = \frac{\Phi}{i} \tag{6}$$

Also, by Faraday's law of induction, we have that,

$$\begin{aligned} v_L &= \frac{d\Phi}{dt} && (7) \\ &= \frac{d}{dt}(Li) && (8) \\ &= L\frac{di}{dt} && (9) \end{aligned}$$

which is the differential equation for the inductor. Applying the Laplace transform,

$$V_L(s) = LsI(s) \tag{10}$$

$$\mathcal{Z}(s) = \frac{V_L(s)}{I(s)} = Ls \tag{11}$$

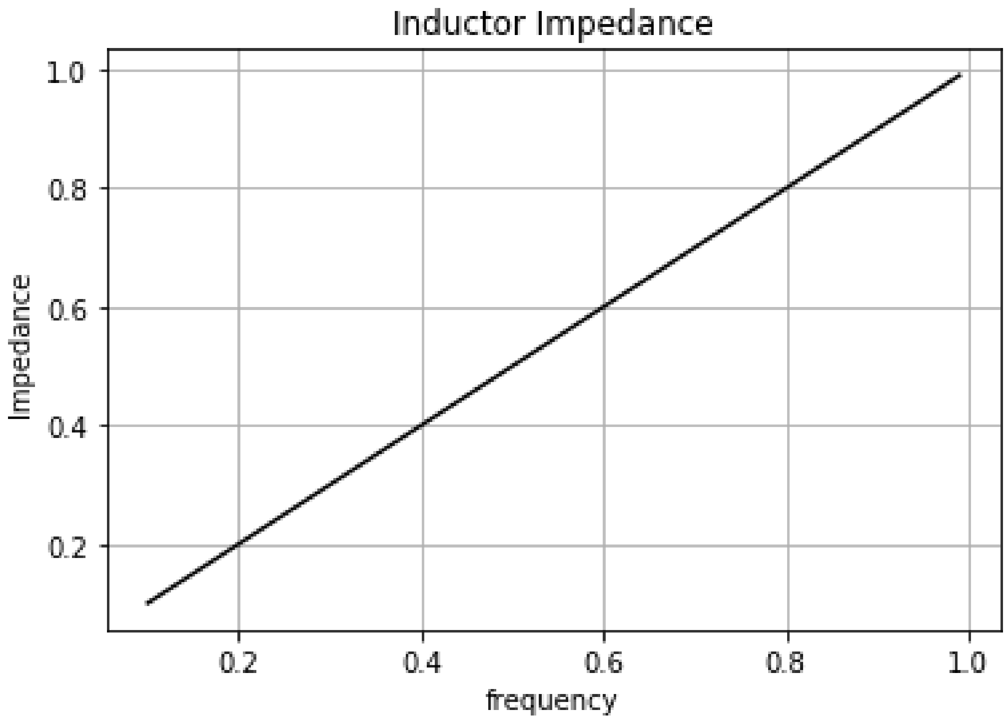

Figure 2: Impedance as a function of frequency for the inductor.

We can draw another kind of conclusion for the inductor, in comparison to the capacitor, that is, as $\lim_{s\to 0} \mathcal{Z}(s) = 0$ behaves like an short-circuit. Simillarly, $\lim_{s\to\infty} \mathcal{Z}(s) = \infty$, and thus, for high frequencies, the inductor behaves as a open circuit.

2 First-Order Circuit Analysis

2.1 RC Circuit

Consider the following circuit,

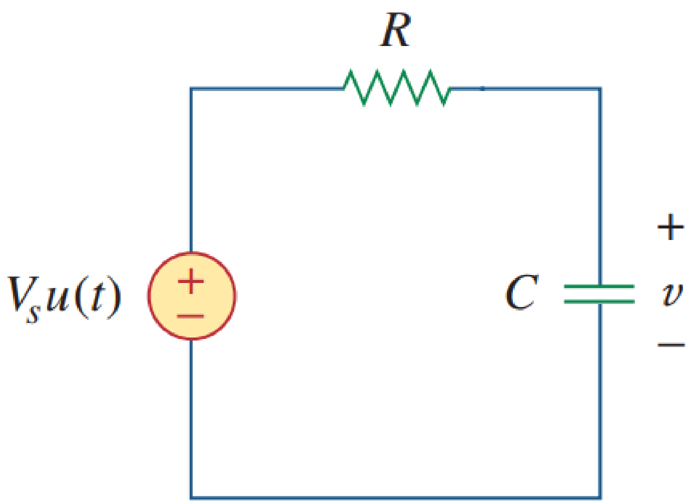

Figure 3: Circuit diagram for the problem

Using Kirchoff's Law of Voltages, we have:

$$V(t) - V_R(t) - V_C(t) = 0 \tag{12}$$

By ohm's law, we know $V_R(t) = Ri(t)$. Also, by the capacitor equation $V_c(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$. Therefore, we can rewrite this expression as:

$$\begin{aligned} V(t) &= V_R(t) + V_C(t) && (13)\\ &= Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau && (14)\\ & && (15) \end{aligned}$$

Being $V(t) = V_0 u(t)$, $V(s) = \frac{V_0}{s}$. Representing the above equation in the frequency domain:

$$\begin{aligned} V(s) &= RI(s) + \frac{I(s)}{sC} && (16)\\ \frac{V_0}{s} &= I(s)(\frac{RCs}{Cs} + \frac{1}{Cs}) && (17)\\ I(s) &= \frac{CV_0}{1+RCs} && (18)\\ I(s) &= \frac{V_0}{R}\frac{1}{s+(RC)^{-1}} && (19) \end{aligned}$$

From this, we conclude that $i(t) = \frac{V_0}{R}\mathcal{L}^{-1}\left\{\frac{1}{s+(RC)^{-1}}\right\}$, that is,

$$i(t) = \frac{V_0}{R}e^{-t/RC} \tag{20}$$

Notice that we can understand $i(t)$ as both the state of the system and the output. Concerning those two views:

- If we understand the current as a state variable, we are looking at the fact that, despite our input, the system acumulates energy in the capacitor. If the input cease its action, the capacitor will keep driving the system somewhere (through its stored energy, in form of current).
- If we understand the current as output, we can view it as a consequence of the input.

The state-equation can be written, thus, in the following form:

$$i(t) = \frac{V(t)}{R} - \frac{1}{RC}\int_0^t i(\tau)d\tau \tag{21}$$

$$\frac{di}{dt} = -\frac{1}{RC}i(t) + \frac{1}{R}\frac{dV}{dt} \tag{22}$$

Also, analysing the step-response $i(t)$, we remark the following:

- $\lim_{t\to 0^+} i(t) = \frac{V_0}{R}$, which comes from the fact that the capacitor behaves like a short-circuit for low frequencies (or for little passage of time).
- $\lim_{t\to +\infty} i(t) = 0$, which comes from the fact that the capacitor behaves like a open-circuit for high frequencies (or for long passage of time).

We can also represent the voltage on the capacitor by,

$$V_C(t) = \frac{1}{C}\int_0^t i(r)dr \tag{23}$$

$$= \frac{1}{C}\int_0^t \frac{V_0}{R}e^{-r/RC}dr \tag{24}$$

$$= \frac{V_0}{RC}\left(- RCe^{-r/RC}\right)\Big|_0^t \tag{25}$$

$$= \frac{V_0}{RC}\left(RC - RCe^{-t/RC}\right) \tag{26}$$

$$= V_0(1 - e^{-t/RC}) \tag{27}$$

Both responses can be summarized bellow:

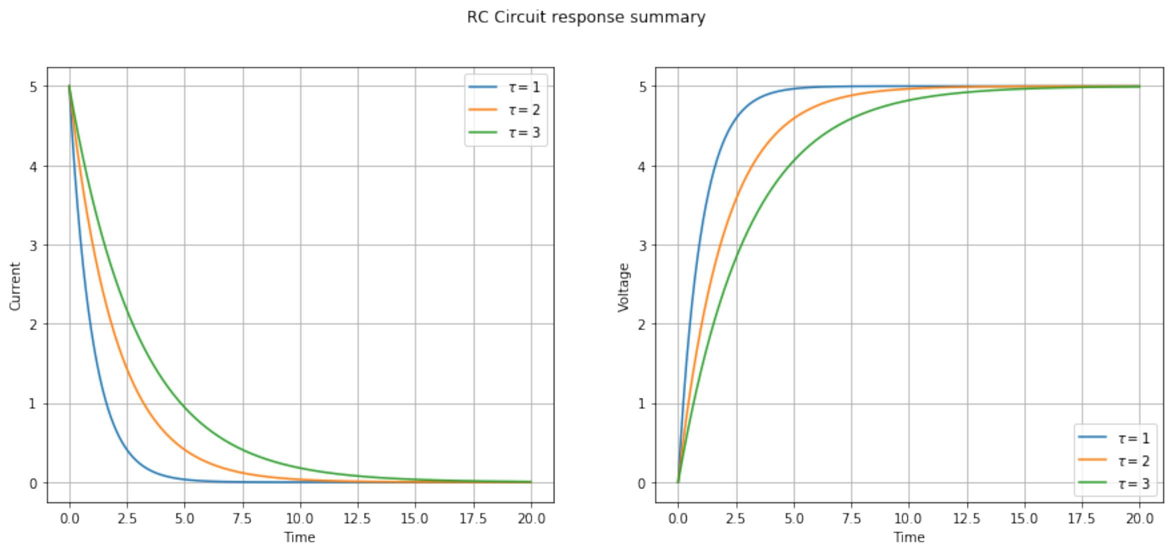

Figure 4: Summary of the system's step response. On the left, the current, and on the right, the voltage.

Here, we make a simple remark about the time constant τ:

- The larger τ is, slower is the decay in the current, and slower is the charge of the capacitor. It is called time constant for this, since it controls the rate in which the capacitor charges.

2.2 RL Circuit

Consider the following circuit,

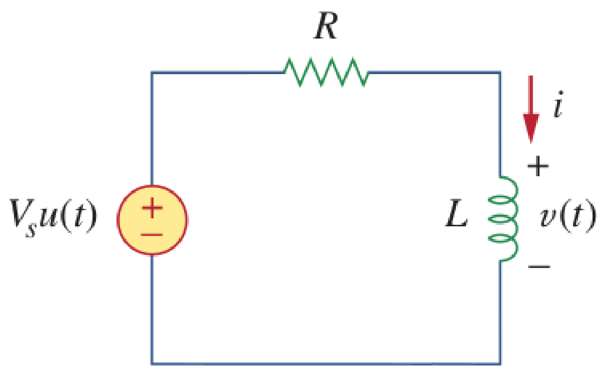

Figure 5: RL Circuit diagram

Writting the differential equation that arises from the use of Kirchoff's Voltage law, we acquire:

$$v(t) - v_R(t) - v_L(t) = 0 \tag{28}$$
$$v(t) = v_R(t) + v_L(t) \tag{29}$$

But we, again, know that $v_R(t) = Ri(t)$, and $v_L = L\frac{di}{dt}$, then:

$$v(t) = Ri(t) + L\frac{di}{dt} \tag{30}$$

Using the laplace transform,

$$\frac{V_0}{s} = RI(s) + L[sI(s) - i(0)] \tag{31}$$
$$\frac{V_0}{s} + Li(0) = I(s)(Ls + R) \tag{32}$$

then,

$$I(s) = \frac{V_0}{s(R + Ls)} + \frac{L}{R + Ls}i(0) \tag{33}$$
$$= \frac{V_0}{R}(\frac{1}{s} - \frac{L}{Ls + R}) + \frac{L}{R + Ls}i(0) \tag{34}$$
$$= \frac{V_0}{R}(\frac{1}{s} - \frac{1}{s + \frac{R}{L}}) + \frac{1}{s + \frac{R}{L}}i(0) \tag{35}$$

therefore,

$$i(t) = \frac{V_0}{R}(1 - e^{-(R/L)t}) + i(0)e^{-(R/L)t} \tag{36}$$

By current, we can draw other elements functions, like:
* Resistor voltage, $v_R = Ri = V_0(1 - e^{-(R/L)t}) + Ri(0)e^{-(R/L)t}$ * Inductor voltage, $v_L = v - v_R$
Also, we can represent the state-equation as:

$$\frac{di}{dt} = -\frac{R}{L}i + \frac{v}{L} \tag{37}$$

thus, $A = -\frac{R}{L}$ is a scalar, so is $B = \frac{1}{L}$. An example of curves $i(t)$ and $v_L(t)$ can be found bellow:

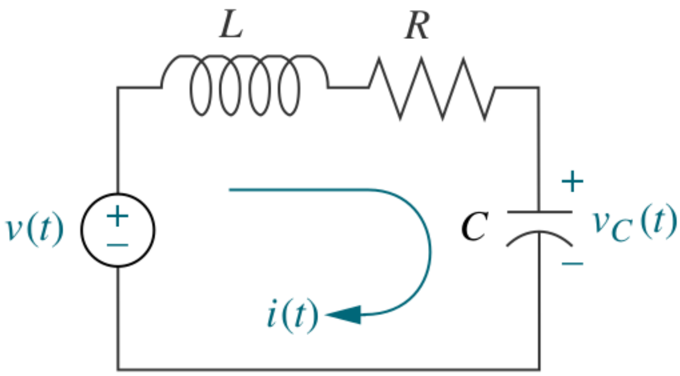

Figure 7: RLC Circuit Diagram

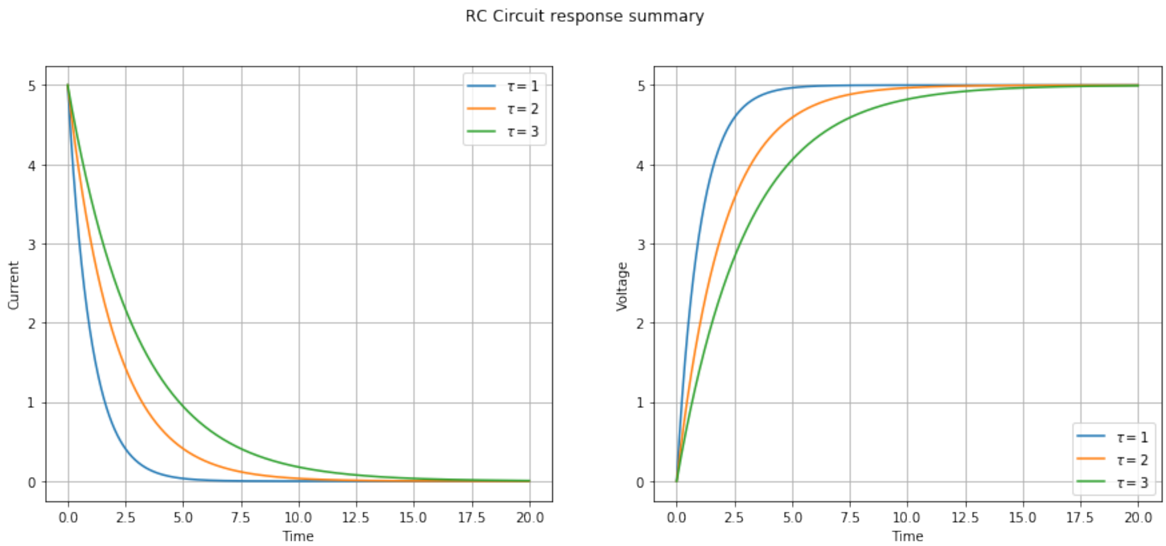

Figure 6: Summary of the system's step response. On the left, the current, and on the right, the voltage.

3 Second-Order Circuits

3.1 ODE Modeling

Consider a circuit composed by a voltage source, a inductor, a resistor and a capacitor. Such a circuit is displayed bellow,

Using Kirchoff Voltage Law (KVL), we achieve:

$$
\begin{align}
v(t) &= v_R + v_L + v_C \tag{38}\\
&= Ri(t) + L\frac{di}{dt}(t) + \frac{1}{C}\int_0^t i(r)dr \tag{39}\\
& \tag{40}
\end{align}
$$

The above equation has the drawback of having an integral in its initial form. By changing variables, and considering the charge instead of a current, $q(t) = \int_0^t i(r)dr, \dot{q} = i(r)$, we have:

$$
\begin{align}
v &= L\ddot{q} + R\dot{q} + \frac{1}{C}q \tag{41}\\
\frac{1}{L}v &= \ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}q \tag{42}
\end{align}
$$

Under the assumption that $q(0) = 0, \dot{q}(0) = i(0) = 0$, we can use the laplace transform to find the transfer function for such equation:

$$
\begin{align}
\frac{1}{L}V(s) &= Q(s)\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) \tag{43}\\
\frac{Q(s)}{V(s)} &= \frac{\frac{1}{L}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \tag{44}
\end{align}
$$

Although not common, we can use $q(t)$ as our output (the response to v(t)). A more common choice for output would be the Capacitor's current, which can be retrieved by $v_C(t) = Cq(t) \xrightarrow{\mathcal{L}} V_C(s) = CQ(s)$, therefore, we can rewrite:

$$
\frac{V_C(s)}{V(s)} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \tag{45}
$$

As a block diagram, this system is represent as,

3.2 Mesh Analysis

We could model the same system, and achieve the same results by another kind of method. Notice that in the ODE modelling, we start working in the time domain, where we achieve a ODE for the system's dynamics, transform it to frequency domain and find the Transfer Function.

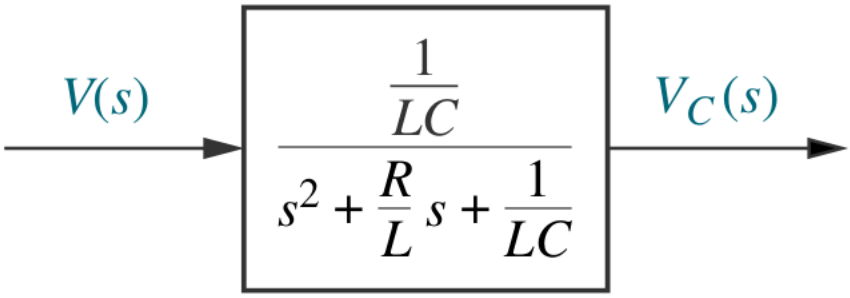

Figure 8: Block Diagram for the RLC Circuit

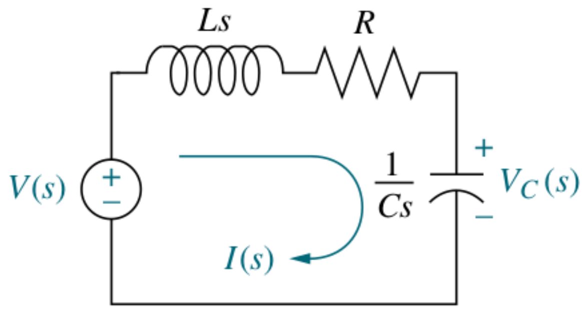

Figure 9: RLC circuit Frequency representation

A much more direct approach could be to directly transform each component in the system to their frequency counterpart, and then work with impedances, rather than with resistencs, capacitances and indutancies. Such approach relies in the linear structure of our systems. Consider, then, the following diagram:

Using, again, the KVL:

$$V(s) = (Ls + R + \frac{1}{Cs})I(s) \tag{46}$$

$$\frac{I(s)}{V(s)} = \frac{1}{Ls + R + \frac{1}{Cs}} \tag{47}$$

$$\frac{I(s)}{V(s)} = \frac{\frac{s}{L}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \tag{48}$$

$$\tag{49}$$

Here, instead of a ratio between $V_C(s)$ and $V(s)$, we have achieved a ratio between

$I(s)$ and $V(s)$. To retrieve $V_c(s)$ we recall that $\frac{1}{C}i(t) = \dot{v}_C$, then $I(s) = CsV_c(s)$. We conclude that:

$$\frac{V_c(s)}{V(s)} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \tag{50}$$

4 Operational Amplifiers

An operational amplifier is a digital device which is designed to amplify signals.

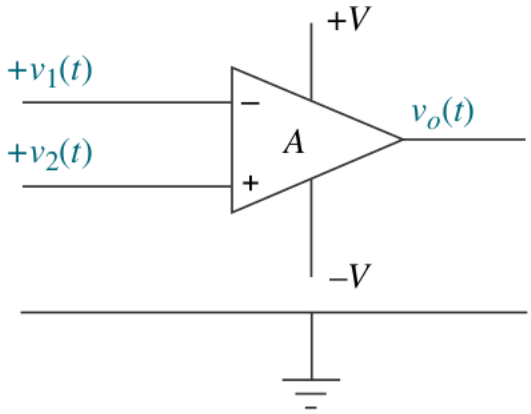

Figure 10: Diagram for a conventional operational amplifier

It has the following characteristics,

- High input impedance, that is, $Z_1 = \infty$,
- Negligible output impedance, $Z_0 = 0$,
- High gain, $A \to \infty$.

In those conditions, the output voltage is given by,

$$v_0(t) = A(v_2(t) - v_1(t)) \tag{51}$$

Under the hypothesis of a grounded positive input, that is, $v_2 = 0$, then this equation becomes:

$$v_0(t) = -Av_1(t) \tag{52}$$

4.1 Inverting Operational Amplifier

The circuit for an inverting operational amplifier is,

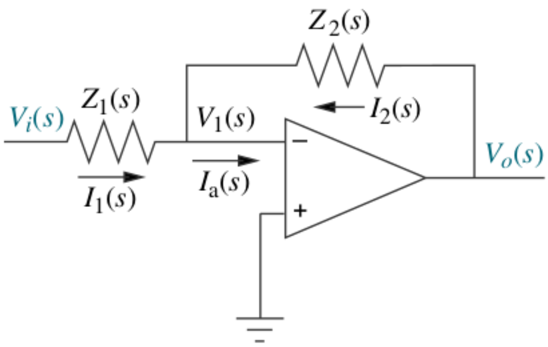

Figure 11: Inverting Operational Amplifier

Under the assumption that $Z_{in} \to \infty$, we can assume that $I_a = 0$. Therefore,

$$I_2 = \frac{V_0 - V_1}{Z_2} \tag{53}$$

$$I_1 = \frac{V_i - V_1}{Z_1} \tag{54}$$

Being $I_a = I_1 + I_2$, then $I_1 = -I_2$, through this equation:

$$\frac{V_1 - V_0}{Z_2} = \frac{V_i - V_1}{Z_1} \tag{55}$$

For large gains, $V_1 \to V_+ = 0$, then,

$$\frac{-V_0}{Z_2} = \frac{V_i}{Z_1} \tag{56}$$

$$\frac{V_0}{V_i} = -\frac{Z_2}{Z_1} \tag{57}$$

Therefore, given an arbitrary Operational Amplifier, its transfer function can be achieved by the ratio of impedances.

4.2 Non-inverting Operational Amplifier

The circuit for an non-inverting operational amplifier is,

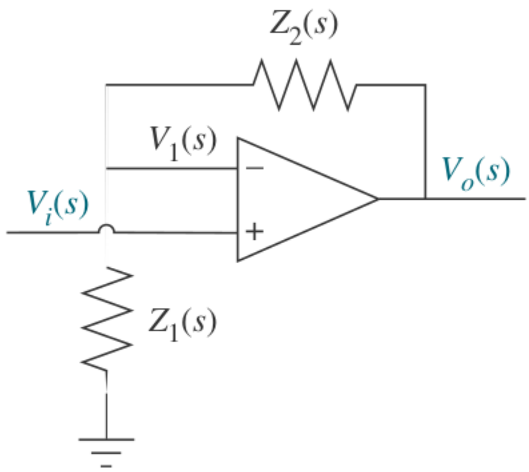

Figure 12: Circuit diagram for the non-inverting operational amplifier

Writing the Operational Amplifier's equation,

$$V_0 = A(V_i - V_1) \tag{58}$$

Using voltage division,

$$V_1 = \frac{Z_1}{Z_1 + Z_2} V_0 \tag{59}$$

Therefore, reapplying it in the last equation:

$$V_0 = AV_i - A\frac{Z_1}{Z_1 + Z_2} V_0 \tag{60}$$

$$V_i = \frac{\frac{1}{A} + Z_1}{Z_1 + Z_2} V_0 \tag{61}$$

$$V_i \overset{A \to \infty}{=} \frac{Z_1}{Z_1 + Z_2} V_0 \tag{62}$$

$$\frac{V_0}{V_i} = \frac{Z_1 + Z_2}{Z_1} \tag{63}$$

5 Exercises

5.1 Exercise 1

Explained how the frequency analysis of electrical circuits is done, we depart to circuits with more complexity. Then consider the following circuit,

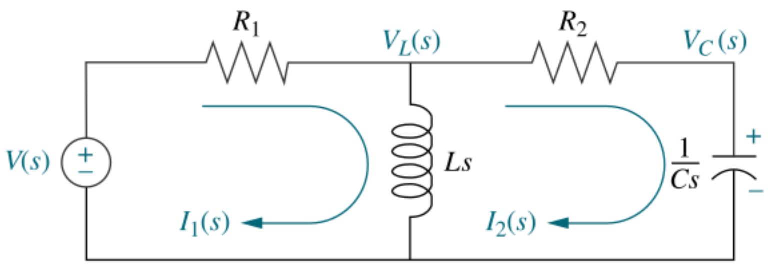

Figure 13: Circuit diagram for Exercise 1

Noticing that the current that passes through the inductor, $I = I_1 - I_2$, we can write two equations for the two meshes:

$$V = R_1 I_1 + LsI_1 - LsI_2 \tag{64}$$

$$0 = LsI_2 - LsI_1 + R_2 I_2 + \frac{1}{Cs} I_2 \tag{65}$$

From equation (2), we can conclude,

$$I_1 = (1 + \frac{R_2}{Ls} + \frac{1}{LCs^2}) I_2 \tag{66}$$

Reapplying in equation (1),

$$V = R_1(1 + \frac{R_2}{Ls} + \frac{1}{LCs^2}) I_2 + Ls(1 + \frac{R_2}{Ls} + \frac{1}{LCs^2}) I_2 - LsI_2 \tag{67}$$

$$= (\frac{R_1 LCs^2}{LCs^2} + \frac{R_1 R_2 Cs}{LCs^2} + \frac{R_1}{LCs^2}) I_2 + (\frac{R_2 LCs^2}{LCs^2} + \frac{Ls}{LCs^2}) I_2 \tag{68}$$

$$= \frac{(R_1 + R_2)LCs^2 + (R_1 R_2 C + L)s + R_1}{LCs^2} I_2 \tag{69}$$

From where we conclude that,

$$\frac{I_2}{V} = \frac{LCs^2}{(R_1 + R_2)LCs^2 + (R_1 R_2 C + L)s + R_1} \tag{70}$$

5.2 Exercise 2

For example 2, consider the following circuit,

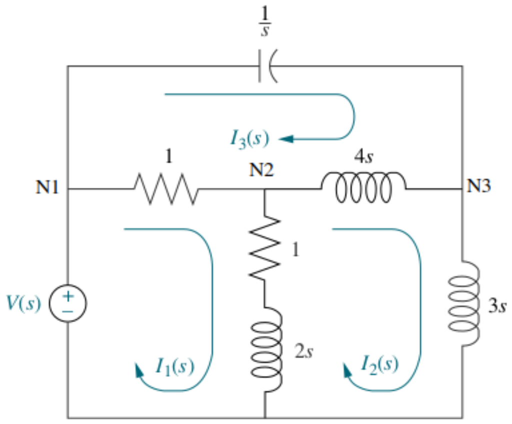

Figure 14: Circuit diagram for Exercise 2

Consider, as a matter of effect, consider the nodes N1, N2 and N3,

* Using the KCL in N1, implies that the current through the alone resistor is $I_4 = I_1 - I_3$ * Using the KCL in N2, implies that the current through the ensemble resistor-inductor is $I_5 = I_1 - I_2$. * Using the KCL in N3, implies that the current through the alone inductor is $I_6 = I_2 - I_3$

Using KVL in each mesh gives,

$$V = I_4 + (2s+1)I_5 \tag{71}$$
$$0 = 4sI_6 + 3sI_2 - (2s+1)I_5 \tag{72}$$
$$0 = \frac{1}{s}I_3 - 4sI_2 - I_4 \tag{73}$$

Substituting the expressions for I_4 and I_5,

$$V = (2s+2)I_1 - (2s+1)I_2 - I_3 \tag{74}$$
$$0 = -(2s+1)I_1 + (9s+1)I_2 - 3sI_3 \tag{75}$$
$$0 = -I_1 - 4sI_2 + (1 + 4s + \frac{1}{s})I_3 \tag{76}$$

We can express such system in matricial form as,

$$\begin{bmatrix} 2s+2 & -(2s+1) & -1 \\ -(2s+1) & 9s+1 & -3s \\ -1 & -4s & 1+4s+\frac{1}{s} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V(s) \\ 0 \\ 0 \end{bmatrix} \tag{77}$$

Solving such system, we achieve:

$$\frac{I_1}{V} = \frac{24s^3 + 13s^2 + 10s + 1}{s(32s^3 + 16s^2 + 5s + 6)} \tag{78}$$

$$\frac{I_2}{V} = \frac{8s^3 + 9s^2 + 3s + 1}{s(32s^3 + 16s^2 + 5s + 6))} \tag{79}$$

$$\frac{I_3}{V} = \frac{8s^2 + 13s + 1}{s(32s^3 + 16s^2 + 5s + 6)} \tag{80}$$

5.3 Exercise 3

For the third exercise, consider the following circuit,

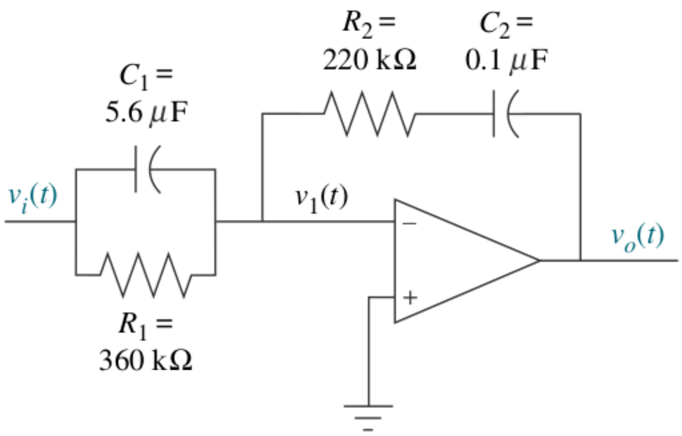

Figure 15: Circuit Diagram for Exercise 3

Following the previous discussed results about Operational Amplifiers, all we have to do is to derive the impedances Z_1 and Z_2:

$$Z_1 = \frac{1}{\frac{1}{R_1} + C_1 s} = \frac{360 \times 10^3}{2.016s + 1} \tag{81}$$

$$Z_2 = R_2 + \frac{1}{C_2 s} = 220 \times 10^3 + \frac{10^7}{s} \tag{82}$$

then,

$$\frac{V_0}{V_i} = -1.232 \frac{s^2 + 45.95s + 22.55}{s} \tag{83}$$

5.4 Exercise 4

For the fourth exercise, consider the following circuit,

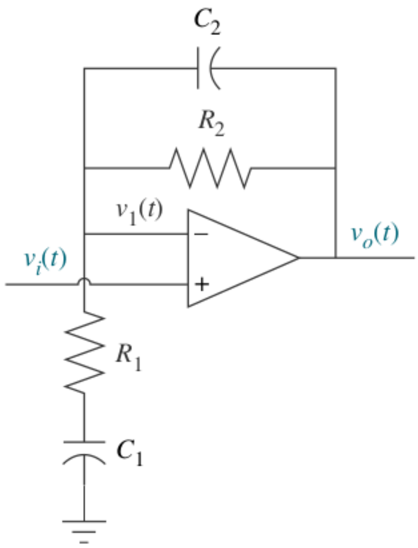

Figure 16: Circuit Diagram for Exercise 4

At this time, we recognize the impedances as,

$$Z_1(s) = R_1 + \frac{1}{C_1 s} \tag{84}$$

$$Z_2(s) = \frac{R_2(1/C_2 s)}{R_2 + (1/C_2 s)} \tag{85}$$

Applying this to the non-inverting amplifier equation:

$$\frac{V_0}{V_i} = \frac{C_1 C_2 R_1 R_2 s^2 + (C_2 R_2 + C_1 R_2 + C_1 R_1)s + 1}{C_1 C_2 R_1 R_2 s^2 + (C_2 R_2 + C_1 R_1)s + 1} \tag{86}$$

5.5 Exercise 5

For the fifth exercise, consider the following circuit,

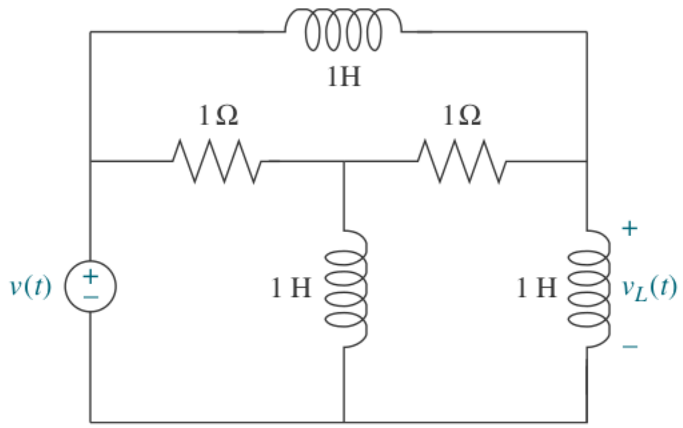

Figure 17: Diagram for exercise 5

Using a same kind analysis as in exercise 2 (and numerating the meshes as before), we have:

$$V = (s+1)I_1 - sI_2 - I_3 \tag{87}$$
$$0 = -sI_1 + (2s+1)I_2 - I_3 \tag{88}$$
$$0 = -I_1 - I_2 + (s+2)I_3 \tag{89}$$

Solving for I_2, we get,

$$I_2 = V\frac{s^2+2s+1}{s^2+5s+2} \tag{90}$$

By the inductor equation, $\dfrac{di}{dt} = v_L \rightarrow I(s) = \dfrac{1}{s}V_L$, from which we conclude that

$$\frac{V_L}{V} = \frac{s^2+2s+1}{s^3+5s^2+2s} \tag{91}$$