Homework 1 solutions

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Contents

1 Exercise 1 **3**

2 Exercise 2 **6**

3 Exercise 3 **12**
3.1 Impulse Input 13
3.2 Step input 14

4 Exercise 4 **16**
4.1 Item A 16
4.2 Item B 16
4.3 Item C 17

1 Exercise 1

Remark: this solution is a little bit overdone, since i have worked with the complete solution of the system. For what matters, the homework asks you for the forced response, which is derived in these notes, although not plotted. For a matter of definition,

- Free-response: the response of the system, when $u(t) = 0, \forall t$.
- Forced-response: the response of the system, when $y_0 = 0$ and $\dot{y}_0$, subject to an arbitrary input.
- Complete-response: the response of the system in the presence of inputs and initial conditions.

For the first exercise, we consider the following differential equation:

$$\ddot{y} + 4\dot{y} + 5y = \dot{u} + 5u \tag{1}$$

with initial conditions $y_0 = 2$ and $\dot{y}_0 = 1$.

Item 1. *Find the free evolution of the system, assuming the already mentioned initial conditions.*

Solution. We mean by "free evolution" the response of the system in the absence of inputs, that is, $u(t) = 0, \forall t$. Being so, and using the Laplace transform,

$$(s^2Y(s) - sy_0 - \dot{y}_0) + 4(sY(s) - \dot{y}_0) + 5Y(s) = 0 \tag{2}$$

$$(s^2 + 4s + 5)Y(s) - 2s - 9 = 0 \tag{3}$$

$$Y(s) = \frac{2s+9}{s^2+4s+5} \tag{4}$$

this is the function, in the frequency domain, whose represent the evolution of the system. Moreover, it shows how the system behaves in the absence of causes, that is, by using its inner energy. You can think of it as a capacitor, which has accumulated energy and it has been let to discharge. To find the response, we use rules 19. and 20. of our Laplace Transform Table, which tells us that,

$$e^{\sigma t} sin(\omega t) \iff \frac{\omega}{(s-\sigma)^2 + \omega^2} \tag{5}$$

$$e^{\sigma t} cos(\omega t) \iff \frac{s-\sigma}{(s-\sigma)^2 + \omega^2} \tag{6}$$

being so, we rewrite 4 as,

$$Y(s) = \frac{1}{(s-(-2))^2+1^2} + 2\frac{s-(-2)}{(s-(-2))^2+1^2} + 4\frac{1}{(s-(-2))^2+1^2} \tag{7}$$

$$\tag{8}$$

And therefore, the response in time is $y(t) = 2e^{-2t}cos(t) + 5e^{-2t}sin(t)$, that is, is the free-force time evolution. □

Item 2. *Find the transfer function and, using the inverse Laplace transform the impulse response of the system.*

Solution. In that item, we need to proceed careful, because in such system we have two components: inputs, and initial conditions. By working in the frequency domain,

$$(s^2Y(s) - sy_0 - \dot{y}_0) + 4(sY(s) - \dot{y}_0) + 5Y(s) = sU(s) + 5U(s) \tag{9}$$

$$Y(s) = \frac{sy_0 + 4y_0 + \dot{y}_0}{s^2+4s+5} + \frac{s+5}{s^2+4s+5}U(s) \tag{10}$$

By definition, the transfer function, $T(s)$, can be found by setting $y_0 = 0$ and $\dot{y}_0 = 0$, hence,

$$Y(s) = \frac{s+5}{s^2+4s+5}U(s) \tag{11}$$

$$\frac{Y(s)}{U(s)} = \frac{s+5}{s^2+4s+5} \tag{12}$$

This is very curious, and indeed proves an important theorem in the theory of linear time-invariant systems: the general response of a system is composed by its free-force response, summed with its forced response with zero initial conditions[1]. Therefore, the impulse response of such system is given when $u(t) = \delta(t)$, which we known to have Laplace transform $U(s) = 1$. Therefore,

$$Y(s) = \frac{2s+9}{s^2+4s+5} + \frac{s+5}{s^2+4s+5} \tag{13}$$

which can be rewritten as,

$$Y(s) = 2\frac{s+2}{(s+2)^2+1^2} + 5\frac{1}{(s+2)^2+1^2} + \frac{s+2}{(s+2)^2+1^2} + 3\frac{1}{(s+2)^2+1^2} \tag{14}$$

That is represented in time as,

[1]Equation 10 should make you sure about that.

$$y(t) = (2e^{-2t}cos(t) + 5e^{-2t}sin(t)) + (e^{-2t}cos(t) + 3e^{-2t}sin(t)) \tag{15}$$
$$y(t) = 3e^{-2t}cos(t) + 8e^{-2t}sin(t) \tag{16}$$

Again, it is evident that our response is composed by the sum of the free-response with the forced-response. □

Item 3. *Find the forced response subject to an unit step input.*

Solution. Recalling Equation 10, we have the same scenario, but at this time, $U(s) = \frac{1}{s}$. Being so, we need to work out with partial fractions, in order to separate,

$$\frac{s+5}{s(s^2+4s+5)} = \frac{A}{s} + \frac{B}{s+2-j} + \frac{C}{s+2+j} \tag{17}$$

Using the relations,

$$A = \lim_{s \to 0} \frac{s+5}{s^2+4s+5} = 1 \tag{18}$$
$$B = \lim_{s \to -2+j} \frac{s+5}{s(s+2+i)} = \frac{-1+i}{2} \tag{19}$$
$$C = \lim_{s \to -2-i} \frac{s+5}{s(s+2-i)} = \frac{-1-i}{2} \tag{20}$$

So, we conclude that,

$$Y(s) = \frac{1}{s} + B\frac{1}{s+2-i} + C\frac{1}{s+2+i} \tag{21}$$
$$y(t) = 1 + e^{-2t}\Big((B+C)cos(t) + i(B-C)sin(t)\Big) \tag{22}$$

Where $B + C = -1$ and $i(B - C) = i \times i = -1$, therefore,

$$y_{forced}(t) = 1 - e^{-2t}\Big(cos(t) + sin(t)\Big) \tag{23}$$

But we already know that $y_{free}(t) = e^{-2t}(2cos(t) + 5sin(t))$. Therefore, the complete system's response is,

$$y_{complete}(t) = y_{free}(t) + y_{forced}(t) \tag{24}$$
$$= 1 + e^{-2t}\Big(cos(t) + 4sin(t)\Big) \tag{25}$$

□

As a final summary, we have the following plots produced by Python/Matlab, which you may find the code in the github repository[2]:

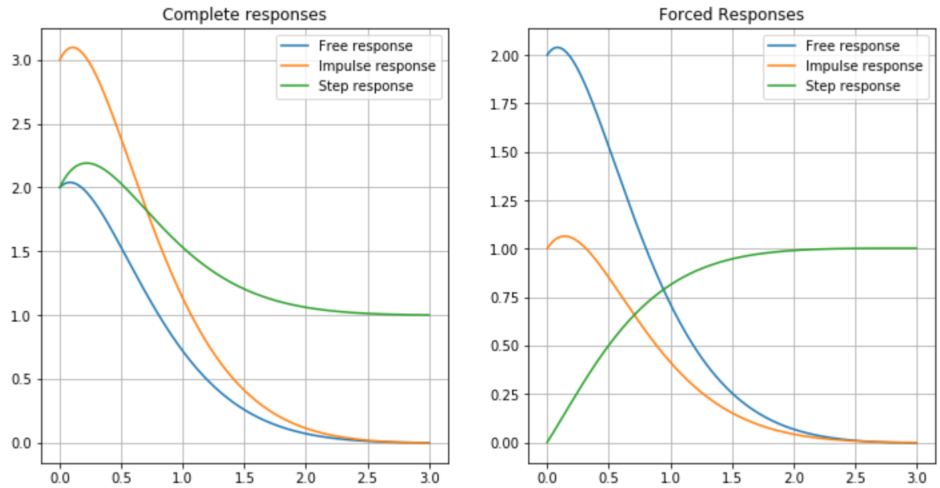

Figure 1: On the left, the complete responses for items 2. and 3., on the right, the forced responses for items 2. and 3.

A few comments can be drawn here,

- All the responses have a decaying exponential term, multiplying sines and cosines. This means the system eventually converge as the time goes by (that is, its limit, as $t \to \infty$, is finite).
- The previous fact comes from the fact that the poles (roots of the denominator's polynomial) are all negative. Moreover, if they are complex, their real part is negative.
- Although, not all responses converge to zero. Indeed, the step response converge to 1.
- This fact comes from noticing that $U(s)$ inserts a pole at origin, which yields a constant term.

2 Exercise 2

For exercise two, we consider the following transfer function,

[2] `https://github.com/eddardd/Control-Theory/tree/master/Homework1Solutions`

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix} u \tag{26}$$

$$y = \begin{bmatrix} 13 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{27}$$

Item 4. *Considering the State-Space representation in Equations 23 and 24, find the corresponding transfer function.*

Solution. To convert the system into the corresponding transfer function, $\frac{Y}{U}$, we begin by expressing each of those equations in scalar form,

$$\dot{x}_1 = x_2 \tag{28}$$

$$\dot{x}_2 = -4x_1 - 5x_2 + \frac{1}{4}u \tag{29}$$

$$y = 13x_1 + 9x_2 \tag{30}$$

which in frequency, correspond to,

$$sX_1 = X_2 \tag{31}$$

$$sX_2 = -4X_1 - 5X_2 + \frac{1}{4}U \tag{32}$$

$$Y = 13X_1 + 9X_2 \tag{33}$$

therefore,

$$(s^2 + 5s)X_2 = -4sX_1 + \frac{s}{4}U \tag{34}$$

$$X_2 = \frac{s}{4(s^2 + 5s + 4)}U \tag{35}$$

Then, using the expression for Y,

$$sY = 13sX_1 + 9sX_2 \tag{36}$$

$$sY = (13 + 9s)X_2 \tag{37}$$

$$sY = \frac{s(13 + 9s)}{4(s^2 + 5s + 4)}U \tag{38}$$

$$\frac{Y}{U} = \frac{13 + 9s}{4(s^2 + 5s + 4)} \tag{39}$$

Which is the transfer function we have been looking for. □

Item 5. *Find the corresponding input-output representation.*

Solution. Basing ourselves in the transfer function obtained in the last item,

$$
\begin{aligned}
4(s^2+5s+4)Y(s) &= (13+9s)U(s) && (40)\\
4\ddot{y}+20\dot{y}+16y &= 13u+9\dot{u} && (41)
\end{aligned}
$$

where from the first to the second line we have converted the system back to time-domain according to the inverse Laplace transform. □

Item 6. *Find the state and output forced evolution subject to the input $u(t)=e^{-t}$*

Solution. Working with equation (36), we already know that $U(s)=\dfrac{1}{s+1}$. Then,

$$
\begin{aligned}
Y(s) &= \frac{13+9s}{4(s+1)(s+4)}U(s) && (42)\\
&= \frac{13+9s}{4(s+1)^2(s+4)} = \frac{A}{(s+1)^2}+\frac{B}{(s+1)}+\frac{C}{s+4} && (43)
\end{aligned}
$$

Therefore, we have,

$$
\begin{aligned}
13+9s &= 4A(s+4)+4B(s+1)(s+4)+4C(s+1)^2 && (44)\\
13+9(-1) &= 4A(s+(-1)) && (45)\\
13+9(-2) &= 4A(-2+4)+4B(-2+1)(-2+4)+4C(-2+1)^2 && (46)\\
13+9(-4) &= 4C(-4+1)^2 && (47)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
A &= \frac{13-9}{4(-1+4)} = \frac{1}{3} && (48)\\
C &= \frac{13-36}{36} = -\frac{23}{36} && (49)\\
B &= \frac{5+8\frac{1}{3}+4\frac{23}{36}}{8} = \frac{23}{36} && (50)
\end{aligned}
$$

$$
\begin{aligned}
Y(s) &= A\frac{1}{(s+1)^2}+B\frac{1}{s+1}+C\frac{1}{s+4} && (51)\\
y(t) &= \frac{1}{3}te^{-t}+\frac{23}{36}e^{-t}-\frac{23}{36}e^{-4t} && (52)
\end{aligned}
$$

This is the forced response of the output variable. We need to find the forced-response to each state variable, also, that is, x_1 and x_2. Working with transfer functions, Equation (32) indeed gives us,

$$X_2 = \frac{s}{4(s+1)(s+4)}U \tag{53}$$

$$X_2 = \frac{s}{4(s+1)^2(s+4)} = \frac{A}{(s+1)^2} + \frac{B}{(s+1)} + \frac{C}{s+4} \tag{54}$$

Working as before,

$$s = 4A(s+4) + 4B(s+1)(s+4) + 4C(s+1)^2 \tag{55}$$

$$-1 = 4A(-1+4) \tag{56}$$

$$-2 = 4A(-2+4) + 4B(-2+1)(-2+4) + 4C(-2+1)^2 \tag{57}$$

$$-4 = 4C(-4+1)^2 \tag{58}$$

And we conclude that,

$$A = -\frac{1}{12} \tag{59}$$

$$C = -\frac{1}{9} \tag{60}$$

$$B = \frac{2+8A+4C}{8} = \frac{1}{9} \tag{61}$$

So,

$$x_2 = \frac{-1}{12}te^{-t} + \frac{1}{9}e^{-t} + \frac{-1}{9}e^{-4t} \tag{62}$$

Finally, to find x_1, we notice that $X_1 = \frac{1}{s}X_2$ so,

$$X_1 = \frac{1}{4(s+1)^2(s+4)} = \frac{A}{(s+1)^2} + \frac{B}{(s+1)} + \frac{C}{(s+4)} \tag{63}$$

and, as usual,

$$1 = 4A(s+4) + 4B(s+1)(s+4) + 4C(s+1)^2 \tag{64}$$

$$1 = 4A(-1+4) \tag{65}$$

$$1 = 4A(-2+4) + 4B(-2+1)(-2+4) + 4C(-2+1)^2 \tag{66}$$

$$1 = 4C(-4+1)^2 \tag{67}$$

which yields,

$$A = \frac{1}{12} \tag{68}$$

$$B = -\frac{1}{36} \tag{69}$$

$$C = \frac{1}{36} \tag{70}$$

Hence,

$$x_1(t) = \frac{1}{12}te^{-t} - \frac{1}{36}e^{-t} + \frac{1}{36}e^{-4t} \tag{71}$$

As a summary,

$$x_1(t) = \frac{1}{12}te^{-t} - \frac{1}{36}e^{-t} + \frac{1}{36}e^{-4t} \tag{72}$$

$$x_2(t) = \frac{-1}{12}te^{-t} + \frac{1}{9}e^{-t} + \frac{-1}{9}e^{-4t} \tag{73}$$

$$y(t) = \frac{1}{3}te^{-t} + \frac{23}{36}e^{-t} - \frac{23}{36}e^{-4t} \tag{74}$$

Another way to find the same solutions in a smaller form, is to directly solve the state-space equations. For states,

$$\dot{\mathbf{x}} = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)d\tau \tag{75}$$

$$\overset{\mathbf{x}(0)=\mathbf{0}}{=} \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}e^{-\tau}d\tau \tag{76}$$

In order to do so, we need to calculate $e^{\mathbf{A}(t-\tau)}$. To calculate the matrix exponential, we need to first calculate its eigenvalues, which happens to be the same as the poles of the founded transfer function, namely, -1 and -4. That way,

$$e^{\mathbf{A}t} = \begin{bmatrix} K_1e^{-t} + K_2e^{-4t} & K_3e^{-t} + K_4e^{-4t} \\ K_5e^{-t} + K_6e^{-4t} & K_7e^{-t} + K_8e^{-4t} \end{bmatrix} \tag{77}$$

The coefficients $K_1, \cdots, K_8$ can be determined by equations $e^{\mathbf{A0}} = \mathbf{I}$ and $\left.\frac{d}{dt}e^{\mathbf{At}}\right|_{t=0} = A$, therefore,

$$
\begin{aligned}
K_1 + K_2 &= 1\\
K_3 + K_4 &= 0\\
K_5 + K_6 &= 0\\
K_7 + K_8 &= 1
\end{aligned}
$$

and,

$$
\begin{aligned}
-K_1 - 4K_2 &= 0\\
-K_3 - 4K_4 &= 1\\
-K_5 - 4K_6 &= -4\\
-K_7 - 4K_8 &= -5
\end{aligned}
$$

whose solution yields,

$$
\begin{aligned}
K_1 &= \frac{4}{3} & K_2 &= -\frac{1}{3}\\
K_3 &= \frac{1}{3} & K_4 &= -\frac{1}{3}\\
K_5 &= -\frac{4}{3} & K_6 &= \frac{4}{3}\\
K_7 &= -\frac{1}{3} & K_8 &= \frac{4}{3}
\end{aligned}
$$

Being $B = \begin{bmatrix} 0 & \frac{1}{4} \end{bmatrix}^T$, then,

$$
e^{\mathbf{A}(t-\tau)}\mathbf{B}e^{-\tau} = \begin{bmatrix} \frac{4}{3}e^{-(t-\tau)} - \frac{1}{3}e^{-4(t-\tau)} & \frac{1}{3}e^{-(t-\tau)} - \frac{1}{3}e^{-4(t-\tau)} \\ -\frac{4}{3}e^{-(t-\tau)} + \frac{4}{3}e^{-4(t-\tau)} & -\frac{1}{3}e^{-(t-\tau)} + \frac{4}{3}e^{-4(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix} e^{-\tau} \tag{78}
$$

$$
= \begin{bmatrix} \frac{1}{12}e^{-t} - \frac{1}{12}e^{-4t+3\tau} \\ \frac{1}{12}e^{-t} - \frac{1}{3}e^{-4t+3\tau} \end{bmatrix} \tag{79}
$$

And so, by taking integrals,

$$
\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}e^{-t}d\tau = \begin{bmatrix} \frac{1}{12}\int_0^t e^{-t}d\tau - \frac{1}{12}\int_0^t e^{-4t+3\tau}d\tau \\ \frac{1}{12}\int_0^t e^{-t}d\tau - \frac{1}{3}\int_0^t e^{-4t+3\tau}d\tau \end{bmatrix} \tag{80}
$$

and the problem indeed reduces to evaluate two integrals, namely:

$$\int_0^t e^{-t} d\tau = e^{-t} \int_0^t d\tau = te^{-t} \tag{81}$$

$$\int_0^t e^{-4t+3\tau} d\tau = e^{-4t} \int_0^t e^{3\tau} d\tau = \frac{1}{3}e^{-t} - \frac{1}{3}e^{-4t} \tag{82}$$

Plugging back those into equation (77), and using equation (72),

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{12}te^{-t} - \frac{1}{36}e^{-t} + \frac{1}{36}e^{-4t} \\ -\frac{1}{12}te^{-t} + \frac{23}{36}e^{-t} - \frac{23}{36}e^{-4t} \end{bmatrix} \tag{83}$$

And now, equation (24) yields $y(t) = 13x_1 + 9x_2$. Applying those to the previous results shall yield $y(t) = \frac{1}{3}te^{-t} + \frac{23}{36}e^{-t} - \frac{23}{36}e^{-4t}$. □

Those two derivations indeed show how the system's representation are equivalent. As a last remark, we point that that, although having a dense notation, the results with State-Space equations are certainly more straightforwardly general (for MIMO systems, for instance). In Figure 2, we can see the responses to the applied input,

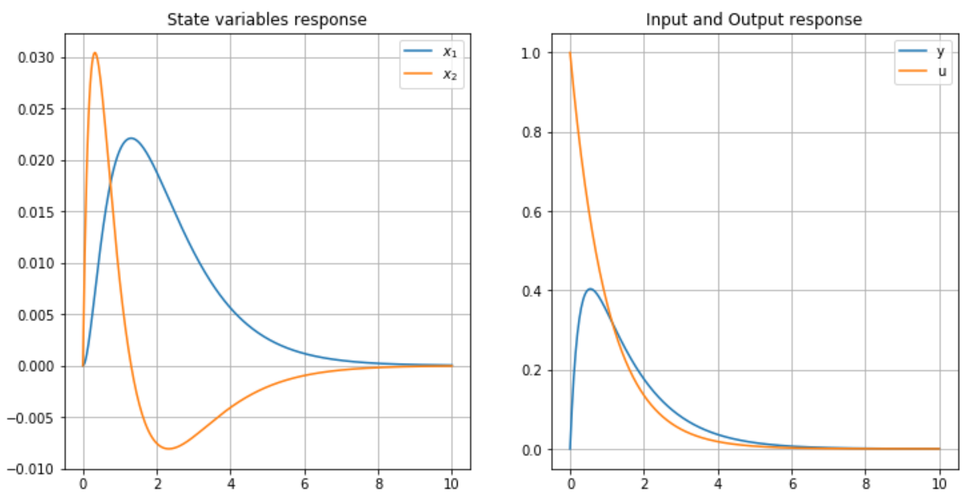

Figure 2: On the left, the state variables response to the input. On the right, the input and the system's output.

3 Exercise 3

For this exercise, we shall work with the following transfer function,

$$G(s) = \frac{\alpha s + 3}{s^2 + 4s + 12} = \frac{\alpha s + 3}{(s - p)(s - \bar{p})} \tag{84}$$

Where $p = -2 - 2\sqrt{2}i$ and $\bar{p} = -2 + 2\sqrt{2}i$ are the roots of $s^2 + 4s + 12$. By using partial fraction expansion,

$$\frac{\alpha s + 3}{(s - p)(s - \bar{p})} = \frac{A}{s - p} + \frac{B}{s - \bar{p}} \tag{85}$$

with,

$$A = \lim_{s \to p} \frac{\alpha s + 3}{s - \bar{p}} = -\frac{\alpha p + 3}{4\sqrt{2}i} \tag{86}$$

$$B = \lim_{s \to \bar{p}} \frac{\alpha s + 3}{s - p} = \frac{\alpha \bar{p} + 3}{4\sqrt{2}i} \tag{87}$$

Therefore,

$$G(s) = \frac{A}{s - p} + \frac{B}{s - \bar{p}} \tag{88}$$

We divide our analysis in two cases, for an impulse input, and for an unit step input.

3.1 Impulse Input

Considering an Impulse, $Y(s) = G(s)U(s) = G(s)$. Therefore,

$$Y(s) = \frac{A}{s - p} + \frac{B}{s - \bar{p}} \tag{89}$$

By denoting $p = \sigma + i\omega$ and $\bar{p} = \sigma - i\omega$, we achieve:

$$\begin{aligned}
y(t) &= Ae^{(\sigma + i\omega)t} + Be^{(\sigma - i\omega)t} && (90)\\
&= e^{\sigma t}(Ae^{i\omega t} + Be^{-i\omega t}) && (91)\\
&= e^{\sigma t}(A(cos(\omega t) + isin(\omega t)) + B(cos(\omega t) - isin(\omega t))) && (92)\\
&= e^{\sigma t}((A + B)cos(\omega t) + i(A - B)sin(\omega t)) && (93)
\end{aligned}$$

But $A + B = \frac{\alpha(\bar{p} - p)}{4\sqrt{2}i} = \frac{\alpha 4\sqrt{2}i}{4\sqrt{2}i} = \alpha$ and $i(A - B) = i\frac{-\alpha(p + \bar{p}) + 6}{4\sqrt{2}i} = \frac{6 - 4\alpha}{4\sqrt{2}}$ that is, the response is characterized as,

$$y(t|\alpha) = e^{-2t}(\alpha cos(2\sqrt{2}t) + \frac{6-4\alpha}{4\sqrt{2}} sin(2\sqrt{2}t)) \tag{94}$$

And being so, α does not have influence into the system's poles, and therefore, does not affect the decaying exponential term, nor the frequency of oscillation of each sine. For the provided range of α's values, we have the following plot,

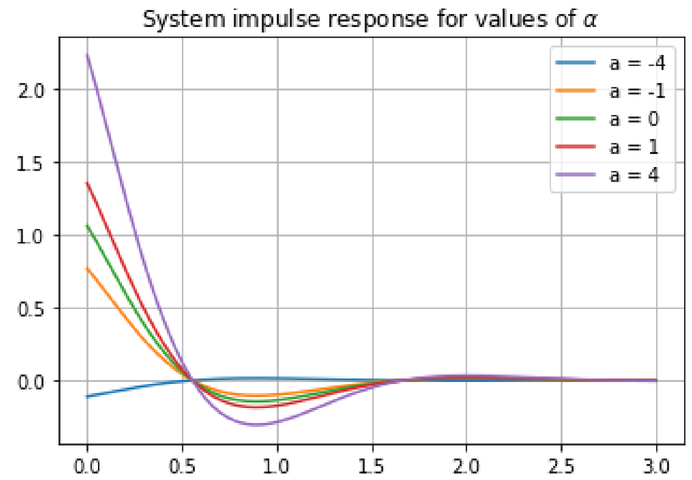

Figure 3: The behavior of $y(t|\alpha)$ for various values of α

3.2 Step input

For a step input, we have the following,

$$Y(s) = \frac{\alpha s + 3}{s(s-p)(s-\bar{p})} = \frac{A}{s} + \frac{B}{s-\bar{p}} + \frac{C}{s-p} \tag{95}$$

in such a way that,

$$A = \lim_{s\to 0} \frac{\alpha s + 3}{(s-p)(s-\bar{p})} = \frac{3}{p\bar{p}} = \frac{1}{4} \tag{96}$$

$$B = \lim_{s\to \bar{p}} \frac{\alpha s + 3}{s(s-p)} = \frac{\alpha\bar{p} + 3}{\bar{p}(p-\bar{p})} = -\frac{\alpha\bar{p} + 3}{4\bar{p}i} \tag{97}$$

$$C = \lim_{s\to p} \frac{\alpha s + 3}{s(s-\bar{p})} = \frac{\alpha p + 3}{p(\bar{p}-p)} = \frac{\alpha p + 3}{4pi} \tag{98}$$

Once again,

$$
\begin{aligned}
y(t) &= A + Be^{\sigma-\omega t} + Ce^{\sigma+\omega t} && (99)\\
&= A + e^{\sigma t}\bigg((C+B)cos(\omega t) + i(C-B)sin(\omega t)\bigg) && (100)
\end{aligned}
$$

And calculating,

$$
\begin{aligned}
C + B &= \frac{\alpha p\bar{p} + 3\bar{p}}{4p\bar{p}i} - \frac{\alpha p\bar{p} + 3p}{4p\bar{p}i} && (101)\\
&= \frac{3(\bar{p} - p)}{4p\bar{p}i} && (102)\\
&= \frac{3 \times 4\sqrt{2} \times i}{4 \times 12 \times i} && (103)\\
&= \frac{3\sqrt{2}}{12} && (104)
\end{aligned}
$$

Likewise,

$$
\begin{aligned}
i(C - B) &= -\frac{\alpha p\bar{p} + 3\bar{p}}{4p\bar{p}} - \frac{\alpha p\bar{p} + 3p}{4p\bar{p}} && (105)\\
&= \frac{2\alpha p\bar{p} + 3(p + \bar{p})}{4p\bar{p}} && (106)\\
&= \frac{12(2\alpha - 1)}{4 \times 12} && (107)\\
&= \frac{2\alpha - 1}{4} && (108)
\end{aligned}
$$

and the response is written as,

$$
y(t|\alpha) = \frac{1}{4} + \frac{e^{-2t}}{4}\bigg(\sqrt{2}cos(2\sqrt{2}t) + (2\alpha - 1)sin(2\sqrt{2}t)\bigg) \qquad (109)
$$

and again, α does not affect ω nor σ, but it influences only one amplitude: the sine amplitude, instead of the two. Also, the step response have introduced an off-set term in the response. We can view the solutions behavior in the bellow figure,

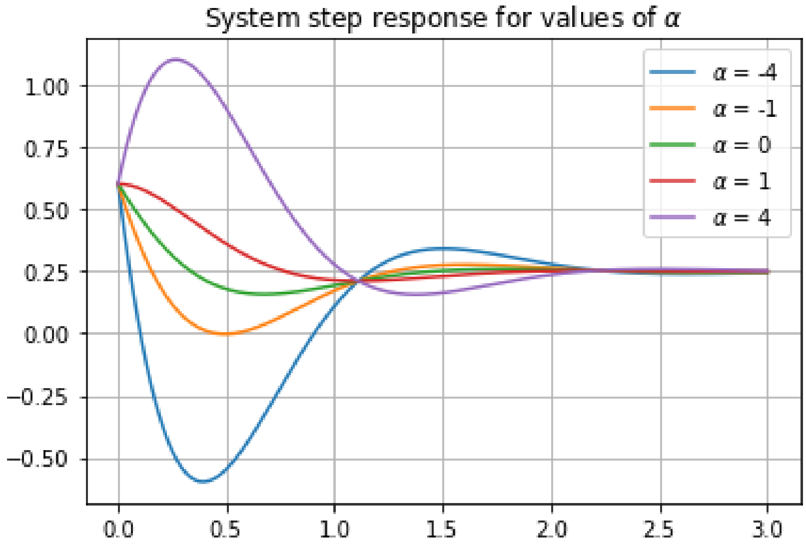

Figure 4: $y(t|\alpha)$ for varying values of α, subject to an step input.

4 Exercise 4

4.1 Item A

Considering,

$$G(s) = \frac{s-10}{(s+2)(s+5)} = \frac{A}{s+2} + \frac{B}{s+5} \tag{110}$$

and therefore,

$$A = \lim_{s \to -2} \frac{s-10}{s+5} = -4 \tag{111}$$

$$B = \lim_{s \to -5} \frac{s-10}{s+2} = 5 \tag{112}$$

we conclude, then, that in time,

$$g(t) = 5e^{-5t} - 4e^{-2t} \tag{113}$$

4.2 Item B

Being,

$$G(s) = \frac{100}{(s+1)(s^2+4s+13)} = \frac{A}{s+1} + \frac{B}{s+2+3i} + \frac{C}{s+2-3i} \tag{114}$$

Therefore,

$$A = \lim_{s \to -1} \frac{100}{s^2+4s+13} = 10 \tag{115}$$
$$B = \lim_{s \to -2-3i} \frac{100}{(s+1)(s+2-3i)} = \frac{10(6i-18)}{36} \tag{116}$$
$$C = \lim_{s \to -2+3i} \frac{100}{(s+1)(s+2+3i)} = -\frac{10(18+6i)}{36} \tag{117}$$

That way, we have,

$$G(s) = 10\frac{1}{s+1} + \frac{10(6i-18)}{36}\frac{1}{s+2+3i} - \frac{10(18+6i)}{36}\frac{1}{s+2-3i}$$
$$g(t) = 10e^{-t} + e^{-2t}\left((\frac{10(6i-18)}{36} - \frac{10(18+6i)}{36})cos(3t) + i(\frac{10(6i-18)}{36} + \frac{10(18+6i)}{36})sin(3t) \right)$$
$$g(t) = 10e^{-t} - e^{-2t}\left(10cos(3t) + \frac{10}{3}sin(3t) \right)$$

4.3 Item C

Finally,

$$G(s) = \frac{s+18}{s(s+3)^2} = \frac{A}{s} + \frac{B}{(s+3)^2} + \frac{C}{s+3} \tag{118}$$

therefore,

$$A = \lim_{s \to 0} \frac{s+18}{(s+3)^2} = 2 \tag{119}$$
$$B = \lim_{s \to -3} \frac{s+18}{s} = -5 \tag{120}$$

finally, to determine C,

$$\frac{s+18}{s(s+3)} = \frac{(s+3)A}{s} + \frac{B}{s+3} + C \tag{121}$$

by using $s = 2$,

$$
\begin{aligned}
C &= \frac{2+18}{2(2+3)} - \frac{(2+3)2}{2} + \frac{5}{2+3} && (122)\\
&= \frac{20}{10} - \frac{10}{2} + \frac{5}{5} && (123)\\
&= -2 && (124)
\end{aligned}
$$

therefore

$$g(t) = 2 - 5te^{-3t} - 2e^{-3t} \tag{125}$$