Appendix A: Calculus of Variations

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1 Variational Calculus and Euler-Lagrange Equations

1.1 Introduction

We begin our discussion with the intent to define the stationary points of a functional. This, as we shall seen, gives the basis for the so-called Euler-Lagrange equations, which are used to model our physical problems.

Definition 1. *Consider a vector space $\mathcal{V}$, of arbitrary dimension. An function is any function $J : \mathcal{V} \rightarrow \mathbb{R}$.*

Henceforth, we shall pay attention to a special kind of functional, defined as:

$$J(y) = \int_{x_0}^{x_1} f(x, y, \dot{y}) dx \tag{1}$$

Specially, the domain in which J takes its values is $\mathcal{C}^2[x_0, x_1]$[1]. Although we have the definition of our functional, we still do not have the concept of maxima and minima for it:

Defitinion 2. *Let $(\mathcal{V}, ||\cdot||)$ be a metric space, and let $\mathcal{S} \subset \mathcal{V}$. We say that J attains a local maximum on $\mathcal{S}$, at $\hat{y} \in S$ if there exists $\epsilon > 0$ such that $J(\hat{y}) - J(y) \leq 0$ for all $\hat{y} \in S$, such that $||\hat{y} - y|| < \epsilon$. The definition for local minimum is analogous.*

Notice that, for functions $\hat{y} \in \mathcal{S}$, if y is in the neighborhood of $\hat{y}$, we represent $\hat{y}$ as a perturbation[2] of y:

[1] The space of twice differentiable functions
[2] Indeed, we could also talk about a intrinsic and uncontrollabe noise

$$\hat{y} = y + \epsilon\eta$$

This notion of perturbation indeed defines a topology in $\mathcal{C}^2$. We shall deal with special cases of perturbations, that is, we restrict η to suffice $\eta(x_0) = \eta(x_1) = 0$. Under this kind of assumption, our problem is called **fixed endpoint variation problem**. Graphically, we display this relation in Figure 1

Hence, we shall work within two sets of functions:

$$\mathcal{S} = \{y \in \mathcal{C}^2[x_0, x_1] : y(x_0) = y_0, y(x_1) = y_1\} \tag{2}$$
$$\mathcal{H} = \{\eta \in \mathcal{C}^2[x_0, x_1] : \eta(x_0) = \eta(x_1) = 0\} \tag{3}$$

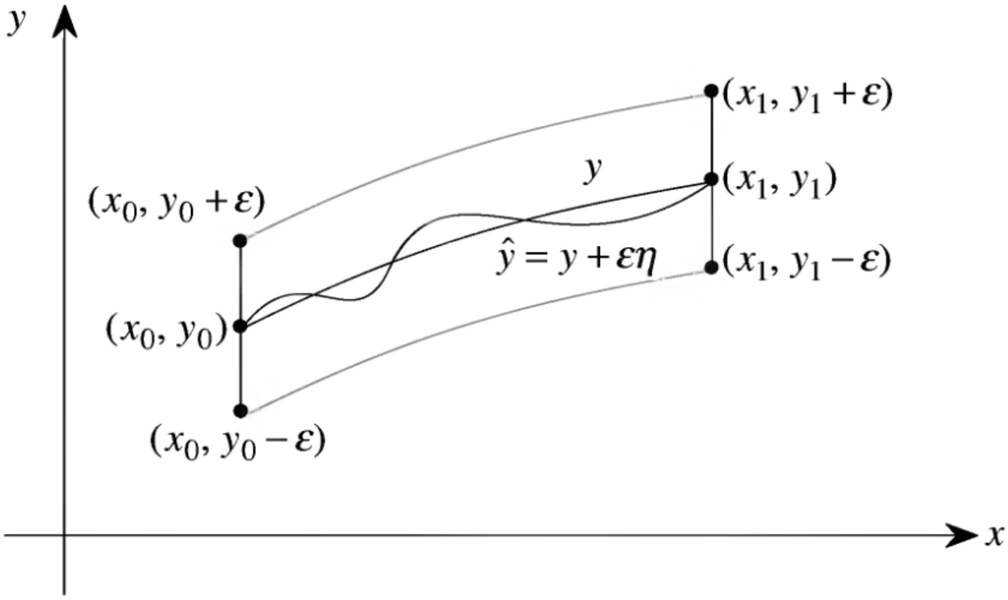

Figure 1: Region within $\mathbb{R}^2$ delimited by the perturbation in y

1.2 First Variation

Let us consider $f(x, y, \hat{y}')$ for small perturbations in $\hat{y}$:

$$f(x, \hat{y}, \hat{y}') = f(x, y + \epsilon\eta, y' + \epsilon\eta') \tag{4}$$
$$= f(x, y, y') + \epsilon(\eta\frac{\partial f}{\partial y} + \frac{\partial f}{\partial y'}) + \mathcal{O}(\epsilon^2) \tag{5}$$

In which, from (4) to (5), we have used Taylor's approximation. We want to investigate $\Delta J(y) = J(\hat{y}) - J(y)$. This quantity can be expressed as:

$$\Delta J(y) = \int_{x_0}^{x_1} f(x,\hat{y},\hat{y}')dx - \int_{x_0}^{x_1} f(x,y,y')dx \tag{6}$$

$$= \int_{x_0}^{x_1} \{f(x,y,y') + \epsilon(\eta\frac{\partial f}{\partial y} + \frac{\partial f}{\partial y'}) + \mathcal{O}(\epsilon^2) - f(x,y,y')\}dx \tag{7}$$

$$= \epsilon\int_{x_0}^{x_1} (\eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'})dx + \mathcal{O}(\epsilon^2) \tag{8}$$

$$= \epsilon\delta J(\eta,y) + \mathcal{O}(\epsilon^2) \tag{9}$$

Where $\delta J(\eta,y) = \int_{x_0}^{x_1}(\eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'})dx$ is called the first variation of J. Now, since the boundary values of η are zero, $\eta \in \mathcal{H} \to -\eta \in \mathcal{H}$, and $\delta J(\eta,y) = -\delta J(-\eta,y)$. For small values of ϵ, the sign of $\Delta J(y)$ is determined by $\delta J(\eta,y)$, thus, if it is supposed to have a local maximum in $\mathcal{S}$, $J(\hat{y}) - J(y)$ does not change sign for any $\hat{y} \in S$, $||\hat{y} - y|| < \epsilon$. Thus:

$$\delta J(\eta,y) = \int_{x_0}^{x_1} (\eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'})dx = 0 \tag{10}$$

For all $\eta \in \mathcal{H}$. We could use similar arguments for the case in which J attains a local minima in $\mathcal{S}$. Equation 10, indeed, stablish the infinite-dimensional case for stationary points of J. In order to make this expression more tractable, we use integration by parts:

$$\int_{x_0}^{x_1} (\eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'})dx = \eta\frac{\partial f}{\partial y}\bigg|_{x_0}^{x_1} - \int_{x_0}^{x_1} \eta\frac{d}{dx}\frac{\partial f}{\partial y'}dx \tag{11}$$

$$= -\int_{x_0}^{x_1} \eta\frac{d}{dx}\frac{\partial f}{\partial y'}dx \tag{12}$$

Where, from Equation 11 to Equation 12 we have used the fact that η is zero at boundary values. With this result, the first variation takes the form:

$$\int_{x_0}^{x_1} \eta\left\{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right\}dx \tag{13}$$

Now, defining $E(x) = \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}$, we can understand Equation 13 as an inner product within a Hilbert Space[3], that is:

$$\langle\eta, E\rangle = \int_{x_0}^{x_1} \eta(x)E(x)dx \tag{14}$$

Indeed, $\langle\eta, E\rangle = 0$ stablish the orthogonality between η and E. We are now interested in the proposition that, is those two functions are indeed orthogonal, and η non-zero inside an open interval of $\mathbb{R}$, then $E(x) = 0$ for all x.

[3]Indeed, $\mathcal{C}^2[x_0, x_1]$ is a complete space with a inner product, so it is, by definition, a Hilber Space

1.3 The Euler-Lagrange equations

Proposition 1. *Suppose that* $\langle \eta, g \rangle = 0$*, for all* $\eta \in \mathcal{H}$*. If* $g : [x_0, x_1] \to \mathbb{R}$ *is continuous, then* $g = 0$ *on the interval* $[x_0, x_1]$*.*

Proof. Suppose that $g \neq 0$ for some $c \in [x_0, x_1]$. Without loss of generality, assume $g(c) > 0$, and by continuity $c \in (x_0, x_1)$. Then, there exists a subinterval $(\alpha, \beta) \subset (x_0, x_1)$, such that $c \in (\alpha, \beta)$, this implies that $g(x) > 0$, in (α, β). Notice that there exists a function[4] $\nu \in \mathcal{C}^2[x_0, x_1]$ such that $\nu > 0, \forall x \in (\alpha, \beta)$, and $\nu = 0, \forall x \in [x_0, x_1] - (\alpha, \beta)$. Therefore, since $\nu \in \mathcal{H}$:

$$\langle \nu, g \rangle = \int_{x_0}^{x_1} \nu(x) g(x) dx \tag{15}$$

$$= \int_{\alpha}^{\beta} \nu(x) g(x) dx \tag{16}$$

$$= 0 \tag{17}$$

Which contradicts the fact that $\langle \eta, g \rangle = 0, \forall \eta \in \mathcal{H}$. Thus, $g(x) = 0, \forall x \in [x_0, x_1]$. □

This proposition stablish the conditions in which we can conclude that $E(x) = 0, \forall x \in [x_0, x_1]$. This implies in the following corollary:

Corollary 1. *Let* $J : \mathcal{C}^2 \to \mathbb{R}$ *be a functional of the form,*

$$J(y) = \int_{x_0, x_1} f(x, y, y') dx \tag{18}$$

where f *has continuous partial derivatives of second order with respect to* x, y *and* y'*. Let,*

$$\mathcal{S} = \{y \in \mathcal{C}^2[x_0, x_1] : y(x_0) = y_0, y(x_1) = y_1\}$$

if $y \in \mathcal{S}$ *is a extremum point of* J*, then:*

$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0 \tag{19}$$

This latter equation is called **The Euler-Lagrange Equation**. It is the infinite-dimensional analogue for the conditions $\nabla f = 0$ and $\frac{d}{dx} f = 0$.

For now on, we shall be interested in describing physical systems through this theory[5]. Specifically, we shall define a quantity,

$$L = T - V \tag{20}$$

Where T is the total kinetic energy of the system, and V, the total potential energy of the system. Depending on the model we use for whichever system we want to control, we shall employ different technique in order to adequate L to be f, in Euler-Lagrange equations.

[4] We can choose $\nu(x) = (x - \alpha)^3 (\beta - x)^3$, for instance

[5] This is called Lagrangian Mechanics

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