Lecture Notes on Control Theory: Response of LTI systems in time

Eduardo Fernandes Montesuma
email edumontesuma@gmail.com

19/04/2018

Contents

1 First Order systems

1.1 Response derivation

First order systems are represented, under I/O representation[1] is given by a first-order differential equation, in the form:

$$\dot{y}(t) + ay(t) = au(t)$$

In which $y(t)$ is the output function, and $u(t)$, the input function. To analyze such systems, we use the Laplace Transform,

$$\begin{aligned} sY(s) + aY(s) &= aU(s) \\ \frac{Y(s)}{U(s)} &= \frac{a}{s+a} \end{aligned} \tag{1}$$

Specially, we want to study the time response of systems defined by 1 when the input is a step function. In such cases, $U(s) = \dfrac{1}{s}$, then,

$$\begin{aligned} Y(s) &= \frac{a}{s(s+a)} \\ &= \frac{1}{s} - \frac{1}{s+a} \end{aligned}$$

taking the inverse Laplace transform, $y(t) = 1 - e^{-at}$. This response is composed by two terms, $y_1(t) = 1$ and $y_2(t) = e^{-at}$. We can see that, while y_2 comes from the original system (the inverse transform of Y/U is ae^{-at}), y_1 comes from the pole inserted at the origin, by $u(t)$. Moreover, every first-order system, subject to a step input will have such response, plotted bellow,

[1] Recall that the I/O representation of a system is given by an Ordinary Differential Equation

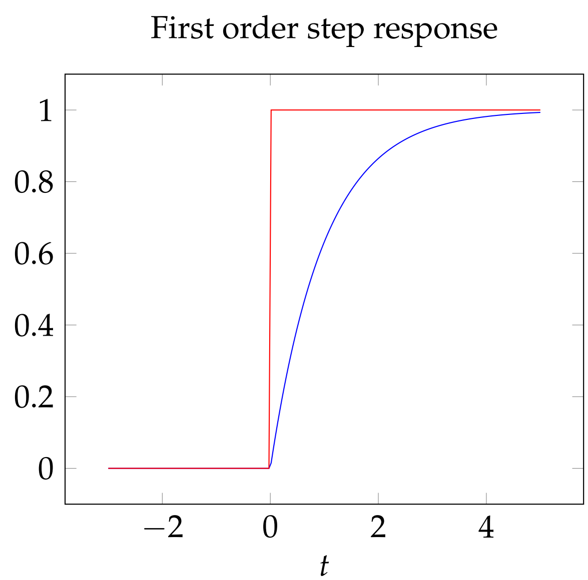

Figure 1: On blue, the output, on red, the input.

1.2 Response Analysis

To analyze the system's response, we define the following quantities,

Definition 1. *Given a first order system $C(s) = a/(s+a)$, we define: 1) The time constant, $\tau = \dfrac{1}{a}$, 2) The rise time, $T_r = \dfrac{2.2}{a}$, 3) The settling time, $T_s = \dfrac{4}{a}$.*

First, the time constant defines the rate in which our exponential decays. Being the response $y(t) = 1 - e^{-at}$, we can rewrite it in terms of the time-constant, $y(t) = 1 - e^{-t/\tau}$. Here, there comes a useful interpretation: since the exponent has no physical unit, τ has second units. Also, by letting $t = \tau$, we get:

$$\begin{aligned} y(\tau) &= 1 - e^{-a\tau} \\ &= 1 - e^{-1} \\ &= 1 - 0.367 \\ &= 0.632 \end{aligned}$$

So, we can rewrite the definition of time constant, as the time, in seconds, needed for the response to get to 63.2% of its final value.

Now, let us consider the rise time, $T_r = 2.2/a = 2.31/a - 0.11/a$. Applying it to our response,

$$\begin{aligned}
y(2.31a) &= 1 - e^{-a \times 2.31/a} \\
&= 1 - e^{-2.31} \\
&= 1 - 0.1 \\
&= 0.9 \\
y(0.11a) &= 1 - e^{-a \times 0.11/a} \\
&= 1 - e^{-0.11} \\
&= 1 - 0.9 \\
&= 0.1
\end{aligned}$$

Hence, T_r is the time for the response to go from 10% of its final value, to 90%. Finally, for the settling time, we can replace T_s directly, yielding:

$$\begin{aligned}
y(T_s) &= 1 - e^{-a \times 4/a} \\
&= 1 - e^{-4} \\
&= 0.98
\end{aligned}$$

That is, it is the time necessary for the response to reach 98% of its final value, or, more precisely, to be within 2% of it steady state value. As a final remark, notice we can rewrite T_r and T_s as functions of the time constant, τ:

$$T_r = 2.2\tau; T_s = 4\tau$$

Hence, as the time constant grows, the time necessary for the system to rise, and to settle, increase. In such cases, we say that our system becomes less aggressive (it behaves more slowly). This can be viewed graphically,

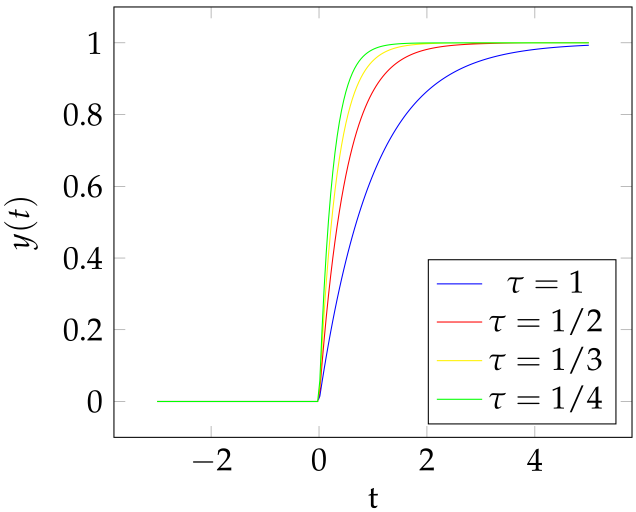

Figure 2: Responses for various values of time constants.

1.3 Automatically defining the step response

As we saw, for a generic transfer function

$$T(s) = \frac{a}{s+a}$$

The time response is given by $y(t) = 1 - e^{-at}$. More generally, we have:

$$\begin{aligned} T(s) &= \frac{K}{s+a} \\ &= \frac{K}{a}\frac{a}{s+a} \end{aligned}$$

Where $K_g = K/a$ is said to be the steady state gain of our system. To understand the origin of such nomenclature, notice that $\mathcal{L}(kT) = k\mathcal{L}$, by linearity. Hence, the step response of the last transfer function is $y(t) = (K/a)(1 - e^{-at}) = K_g(1 - e^{-at})$.

Another factor that is crucial in our system is the time constant $\tau = 1/a$, as we have saw. Being so, we claim that those two parameters fully characterizes any first-order response. Finally, in face of any transfer function of such kind, all we need to do is examine and find τ, and then determine the gain $K_g = \tau K = K/a$.

2 Second Order Systems

2.1 Transfer function characterization

As first order systems, the second order ones also relies on ordinary differential equations. That time, we have,

$$\ddot{y} + b\dot{y} + cy = bu(t)$$

Therefore,

$$\frac{Y(s)}{U(s)} = \frac{b}{s^2 + as + b}$$

Such system has poles as the roots of $s^2 + as + b$. We can easily compute them with Bhaskara's formula,

$$s = \frac{-a \overset{+}{-} \sqrt{a^2 - 4b}}{2}$$

The resulting analysis of this system implies the evaluation of whether $a^2 > 4b$ or not. Suppose, for the sake of argument, that $a = 0$. So, the roots are purely imaginary, with oscillating frequency $\omega_n = \sqrt{b}$.

Now, consider $b \neq 0$, but $4b > a^2$. The roots now are complex, with real part $-a/2$ and imaginary part $\overset{+}{-} \frac{\sqrt{a^2 - 4b}}{2}$. Hence, is meaningful to assign $\sigma = -a/2 = \sigma$, the rate of decay of our exponential. Finally, we define our damping ratio, as the ratio between the decay, and the oscillation frequency,

$$\zeta = \frac{|\sigma|}{\omega_n}$$
$$a = 2\zeta\omega_n$$

With variables a, b renamed to ζ, ω_n, which holds more physical sense, we may write,

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Now, we want to find the response for four cases of parameters ζ and ω_n: 1. $\zeta = 0$, which is the undamped case, 2. $0 < \zeta < 1$, which is the under damped case, $\zeta = 1$, which is the critically damped case and finally, $\zeta > 1$, the over damped case.

2.2 Undamped case

For the undamped second order system, $\zeta = 0$, hence,

$$\begin{aligned} Y(s) &= \frac{\omega_n^2}{s(s^2 + \omega_n^2)} \\ &= \frac{1}{s} + \frac{s}{s^2 + \omega_n^2} \end{aligned}$$

That is, $y(t) = 1 - cos(\omega_n t)$. For such response, we have only one parameters, ω_n, which tells the frequency of oscillation of our system, as can be saw in Figure 3.

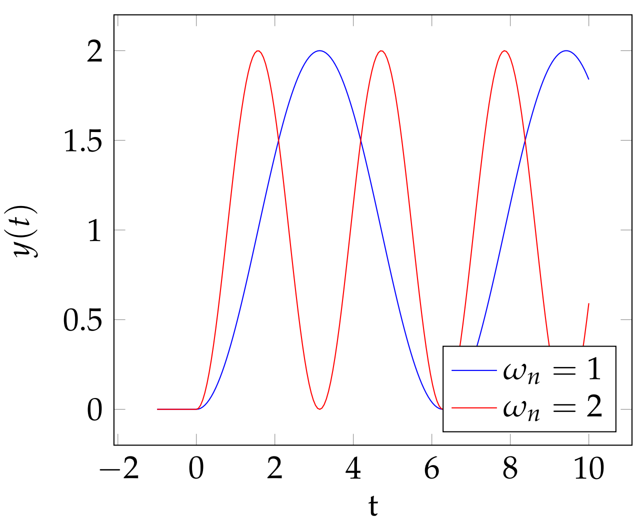

Figure 3: Step response for the undamped case.

2.3 Under-damped system

For the under-damped case, we have $0 < \zeta < 1$, hence, we have two complex roots for $s^2 + 2\zeta\omega_n + \omega_n^2$, $-\zeta w_n + j\omega_n\sqrt{1-\zeta^2}$ and $-\zeta w_n - j\omega_n\sqrt{1-\zeta^2}$. Working through the transfer function:

$$
\begin{aligned}
Y &= \frac{w_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} \\
&= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\
&= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + (\zeta\omega_n)^2 - \zeta^2\omega_n^2 + \omega_n^2} \\
&= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n\sqrt{1-\zeta^2}} \\
&= \frac{1}{s} - \left(\frac{s - (-\zeta\omega_n)}{(s - (-\zeta\omega_n))^2 + (\sqrt{1-\zeta^2}\omega_n)^2} + \frac{\zeta\omega_n}{(s - (-\zeta\omega_n))^2 + (\sqrt{1-\zeta^2}\omega_n)^2} \right) \\
&= \frac{1}{s} - \left(\frac{s - (-\zeta\omega_n)}{(s - (-\zeta\omega_n))^2 + (\sqrt{1-\zeta^2}\omega_n)^2} + \frac{\zeta}{\sqrt{1-\zeta^2}} \frac{\sqrt{1-\zeta^2}\omega_n}{(s - (-\zeta\omega_n))^2 + (\sqrt{1-\zeta^2}\omega_n)^2} \right)
\end{aligned}
$$

Hence, we can get back in the time-domain through the inverse Laplace transform, for $\omega_d = \omega_n\sqrt{1-\zeta^2}$,

$$
y(t) = 1 - e^{-\zeta\omega_n t}\left(cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}} sin(\omega_d t) \right)
$$

Here, we can understand more clearly the nomenclature of such response: the damping in our system comes from the exponentially decaying factor, $e^{-\zeta\omega_n t}$. For example, using $\zeta = 0.2$ and $\omega_n = 3$, we get,

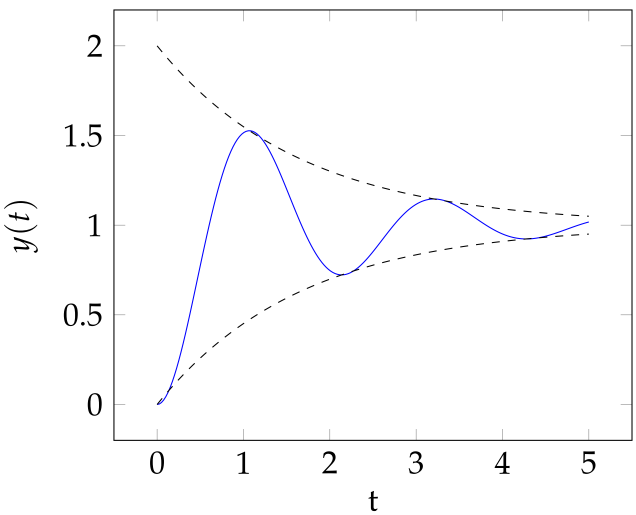

Figure 4: Step response for the under-damped case.

Still, parameters ζ and ω_n have a huge effect in how the response will be, as we can see in Figure 5,

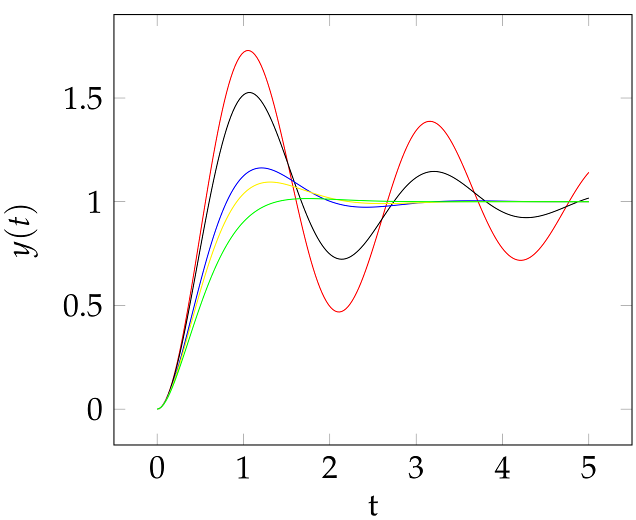

Figure 5: Step response for the under-damped case.

2.4 Critically Damped system

The critically damped case happens when $\zeta = 1$. In that case, our transfer function becomes $T(s) = \dfrac{\omega_n^2}{(s+\omega_n)^2}$, with identical roots. Again, we want to characterize the step response in such case:

$$\begin{aligned} Y(s) &= \frac{\omega_n^2}{s(s+\omega_n)^2} \\ &= \frac{A}{s} + \frac{Bs + C}{(s+w_n)^2} = \frac{(A+B)s^2 + (2w_n + C)s + Aw_n^2}{s(s+w_n)^2} \end{aligned}$$

From where $A = 1$, $B = -1$ and $C = -2\omega_n$. That is, we can write:

$$Y(s) = \frac{1}{s} - \left(\frac{1}{s+\omega_n} + \frac{\omega_n}{(s+\omega_n)^2} \right)$$

Using the inverse Laplace transform yields,

$$y(t) = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}$$

And the responses looks like,

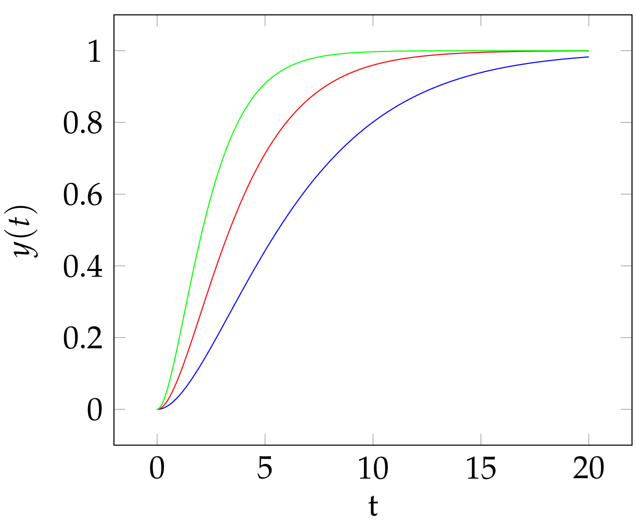

Figure 6: Step response for the critically damped case.

2.5 Over-damped systems

Finally, we have the over damped system, where $\zeta > 1$. In such case, our system has two real roots, and therefore it can be factored into the sum of two first order systems. To see that, consider the roots of our polynomial,

$$
\begin{aligned}
s_1 &= -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} \\
s_2 &= -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}
\end{aligned}
$$

Now, by working on the transfer function,

$$
\begin{aligned}
T(s) &= \frac{\omega_n^2}{(s - s_1)(s - s_2)} \\
&= \frac{A}{s - s_1} + \frac{B}{s - s_2}
\end{aligned}
$$

With,

$$
\begin{aligned}
A &= \lim_{s \to s_1} \frac{\omega_n^2}{s - s_2} = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \\
B &= \lim_{s \to s_2} \frac{\omega_n^2}{s - s_1} = -\frac{\omega_n}{2\sqrt{\zeta^2 - 1}}
\end{aligned}
$$

Which yields,

$$
\begin{aligned}
T(s) &= \frac{A}{s_1}\frac{s_1}{s - s_1} + \frac{B}{s_2}\frac{s_2}{s - s_2} \\
&= T_1(s) + T_2(s)
\end{aligned}
$$

Where we know that both T_1 and T_2 are from first order. As done through Section 1, we know by inspection that the resulting response will be the sum of responses from T_1 and T_2:

$$
y(t) =
$$

2.6 Time domain specifications

Given an arbitrary transfer function, we can classify it between one of our four cases simply by looking at the parameters ζ and ω_n. Moreover, it is instructive to consider four time-domain specifications for our systems: 1. Rise time, T_r, which is the time required for the response to go from 0.1 to 0.9 of its final value, T_p, the peak time, which is the time at which the system achieve its maximum and T_s, the settling time, the time needed for the system to rest at 2% from its steady state value.

We begin by calculating the peak time, T_p. From calculus, we know that the maximum point of a function is given when $\dot{y} = 0$. To find such point, we calculate the inverse Laplace transform of $sY(s)$,

$$\begin{aligned}
\mathcal{L}[\dot{y}] &= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\
&= \frac{\omega_n}{\sqrt{1-\zeta^2}} \frac{\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} \\
\dot{y}(t) &= \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} sin(\omega_n\sqrt{1-\zeta^2}t)
\end{aligned}$$

now, by setting $\dot{y}(t) = 0$, we have,

$$\begin{aligned}
\omega_n\sqrt{1-\zeta^2}t &= n\pi \\
T_p &= \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}
\end{aligned}$$

With the peak time at hand, we can reapply it to $y(t)$ to find the overshoot,

$$\begin{aligned}
y(T_p) &= 1 - e^{-\frac{\pi}{\sqrt{1-\zeta^2}}}\left(cos(\pi) + \frac{\zeta}{\sqrt{1-\zeta^2}} sin(\pi)\right) \\
&= 1 + e^{-\frac{\pi}{\sqrt{1-\zeta^2}}}
\end{aligned}$$

Now, to calculate the percentage of overshooting, $\%OS$, we use the fact that $y(\infty) = 1$,

$$
\begin{aligned}
\%OS &= \frac{y(T_p) - y(\infty)}{y(\infty)} \times 100 \\
&= e^{-\frac{\pi}{\sqrt{1-\zeta^2}}} \times 100
\end{aligned}
$$

To find the Settling time, we assume that having passed a large time, $cos(\omega_d t) = 1$ and $\zeta sin(\omega_d t) = 1$. Therefore, we want

$$
\begin{aligned}
\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} &= 0.02 \\
T_s &= \frac{-ln(0.02\sqrt{1-\zeta^2})}{\zeta\omega_n} \\
&= \frac{4}{\zeta\omega_n}
\end{aligned}
$$