Data Science Math Skills
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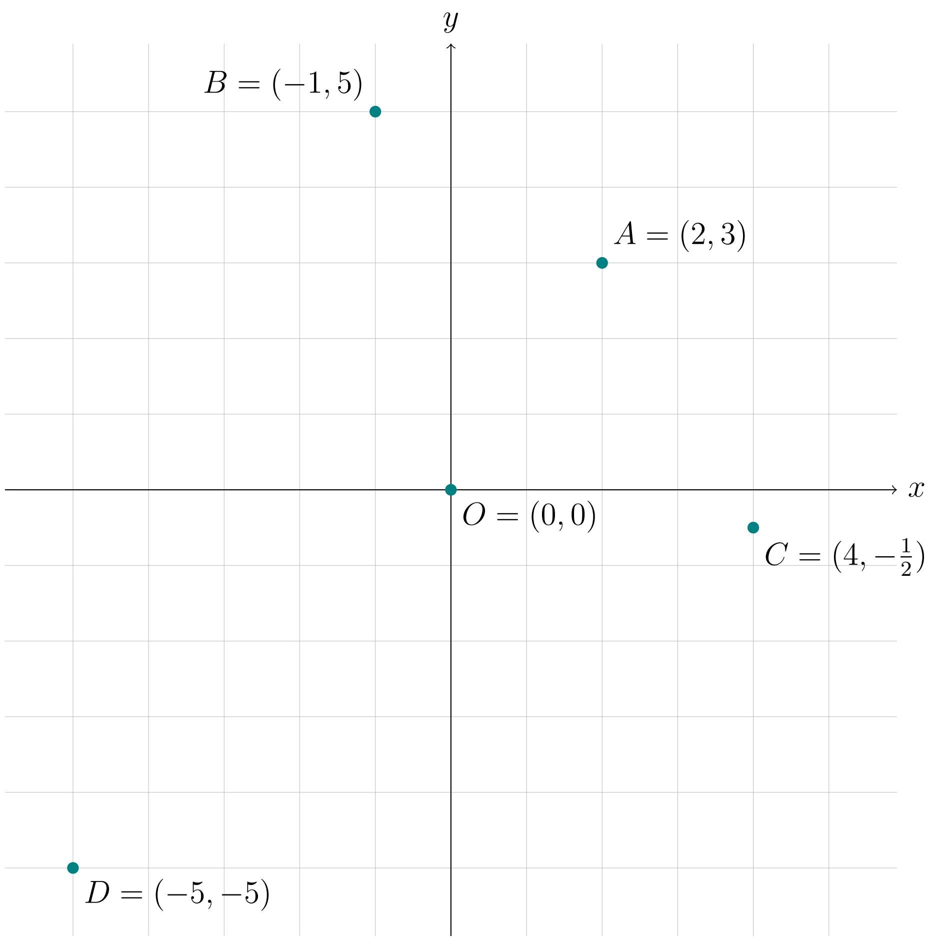
Cartesian Plane: Plotting Points

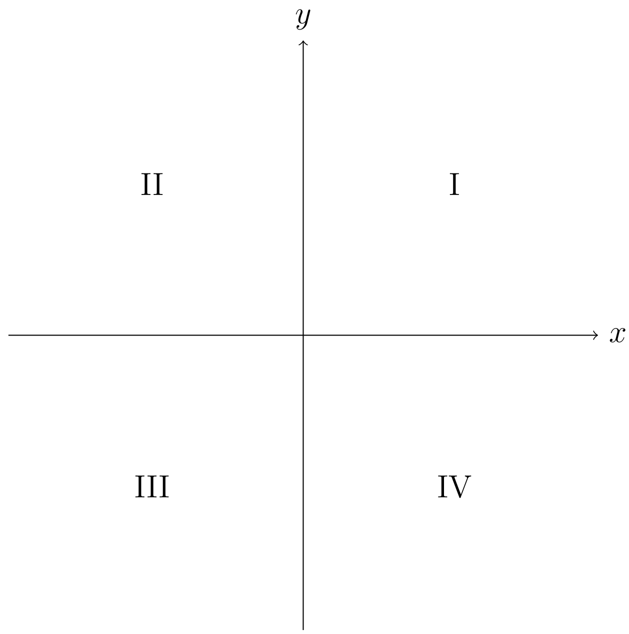
Video companion

1 Introduction

Cartesian plane denoted $\mathbb{R}^2$



2 Axes and quadrants



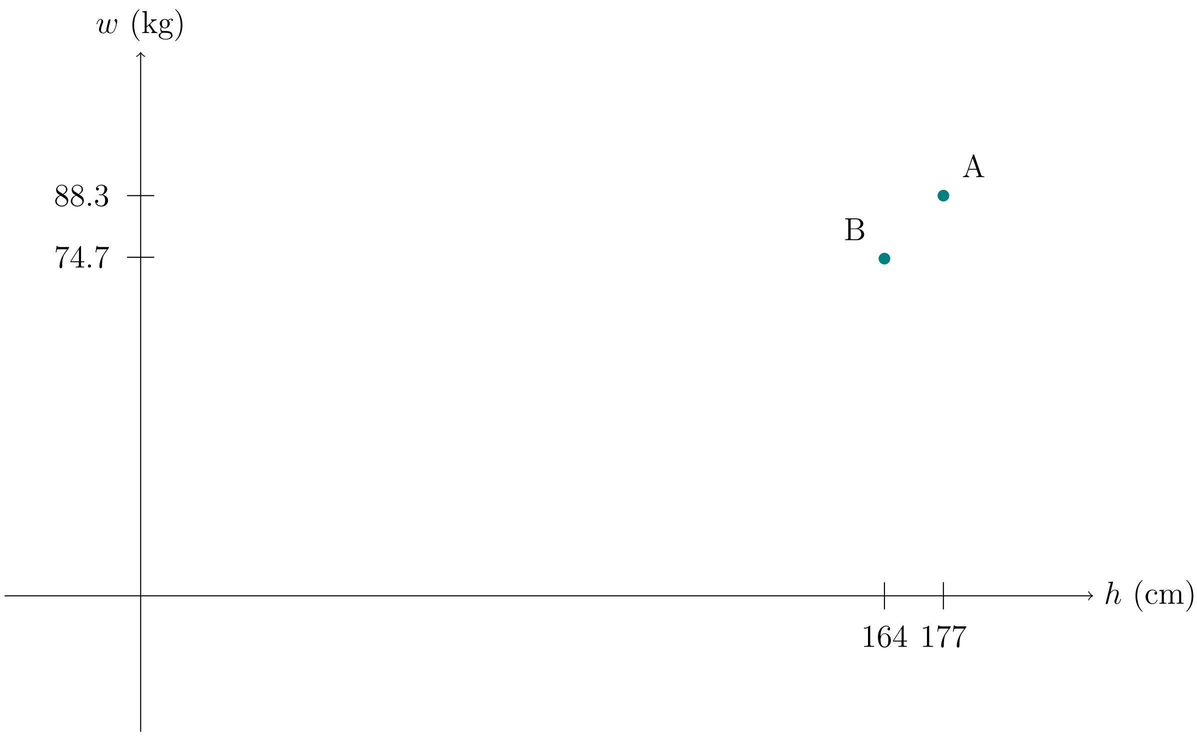
$$\begin{aligned} x\text{−axis} &= \left\{(x, y) \in \mathbb{R}^2 : y = 0\right\} \\ y\text{−axis} &= \left\{(x, y) \in \mathbb{R}^2 : x = 0\right\} \end{aligned}$$

$$\begin{aligned} \text{first quadrant} &= \left\{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\right\} \\ \text{second quadrant} &= \left\{(x, y) \in \mathbb{R}^2 : x < 0, y > 0\right\} \\ \text{third quadrant} &= \left\{(x, y) \in \mathbb{R}^2 : \qquad\qquad\qquad \right\} \\ \text{fourth quadrant} &= \left\{(x, y) \in \mathbb{R}^2 : \qquad\qquad\qquad \right\} \end{aligned}$$

3 Real-world example

Table of height and weight:

	h (cm)	w (kg)
A	177	88.3
B	164	74.7



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Cartesian Plane: Point-Slope Formula for Lines

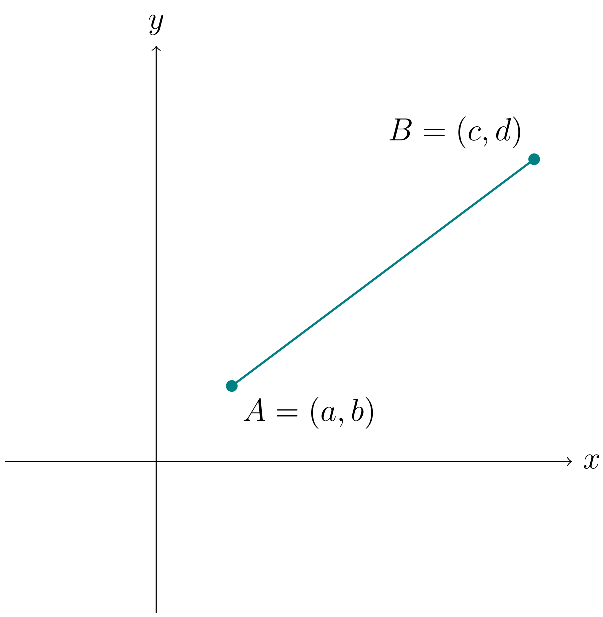
Video companion

1 Introduction

In this video: Demystify formulas for equations of lines

$y - y_0 = m(x - x_0)$	Point-slope form
$y = mx + b$	Slope-intercept form

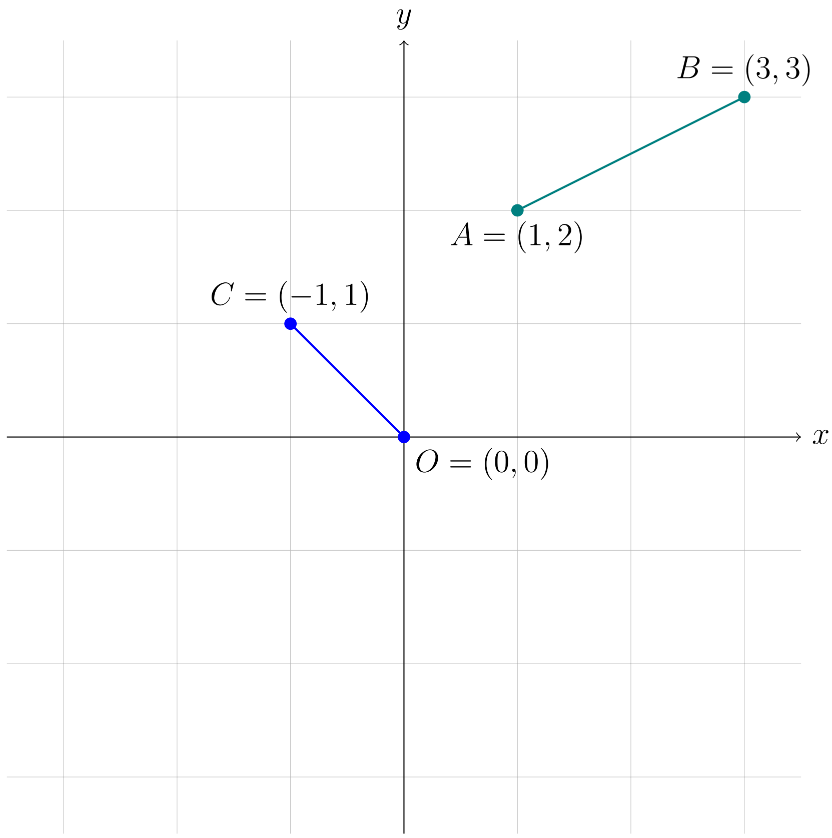
2 Slope of a line segment



Slope of $\overrightarrow{AB}$:

$$m = \frac{d-b}{c-a} = \frac{\text{“rise”}}{\text{“run”}}$$

3 Examples



Slope of $\overrightarrow{AB}$:

$$m = \frac{3-2}{3-1} = \frac{1}{2}$$

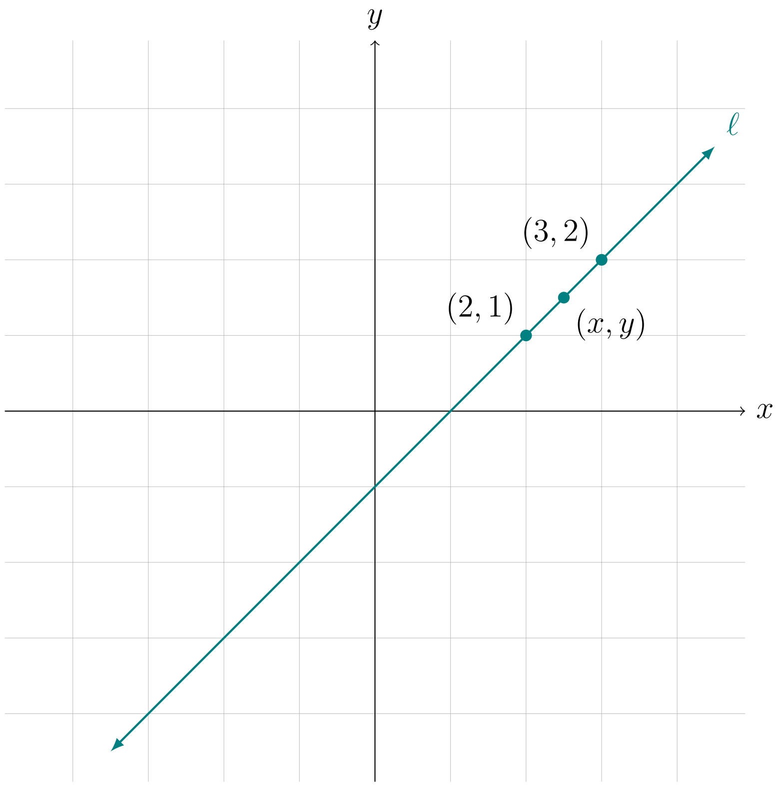
$m = \frac{1}{2}$ is a positive slope.

Slope of $\overrightarrow{CO}$:

$$m = \frac{0-1}{0-(-1)} = -1$$

$m = -1$ is a negative slope.

4 Equation of a line



For a point (x, y) to be on the line, the line segment from $(2, 1)$ to (x, y) need to have a slope of 1.

$$1 = \frac{y-1}{x-2}$$
$$y - 1 = 1(x - 2)$$

The line is defined by this formula:

$$\ell = \left\{(x, y) \in \mathbb{R}^2 : y - 1 = 1(x - 2)\right\}$$

Check that $(3, 2)$ is on the line:

$$(3, 2) \in \ell\,?$$
$$2 - 1 \stackrel{?}{=} 1(3 - 2)$$
$$1 \stackrel{?}{=} 1 \quad \checkmark$$

Check if $(5,1)$ is on the line:

$$\begin{aligned} &(5,1) \in \ell\,? \\ &1-1 \stackrel{?}{=} 1(5-2) \\ &\quad\;\, 0 \stackrel{?}{=} 3 \quad \times \end{aligned}$$

5 Point-slope formula

If a line ℓ has slope m, *and* if (x_0, y_0) is *any* point on ℓ, then ℓ has the equation

$$\boxed{y - y_0 = m(x - x_0).}$$

Cartesian Plane: Distance Formula

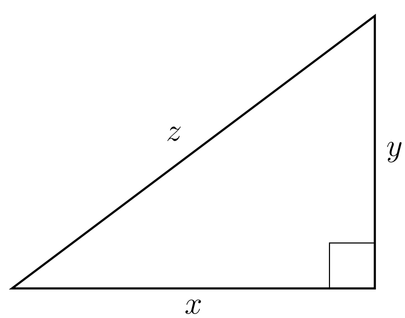
Video companion

1 Introduction

In this video:

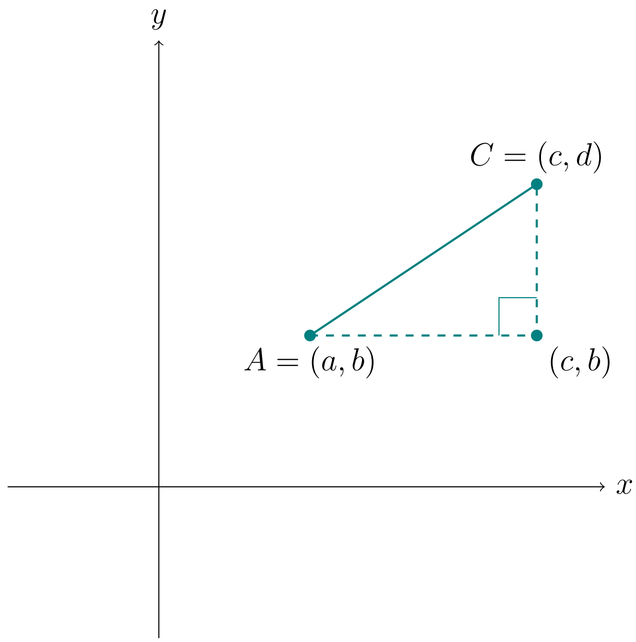
- The distance formula
- Nearest neighbors
- Clustering

2 Pythagorean theorem



$$z^2 = x^2 + y^2$$
$$z = \sqrt{x^2 + y^2}$$

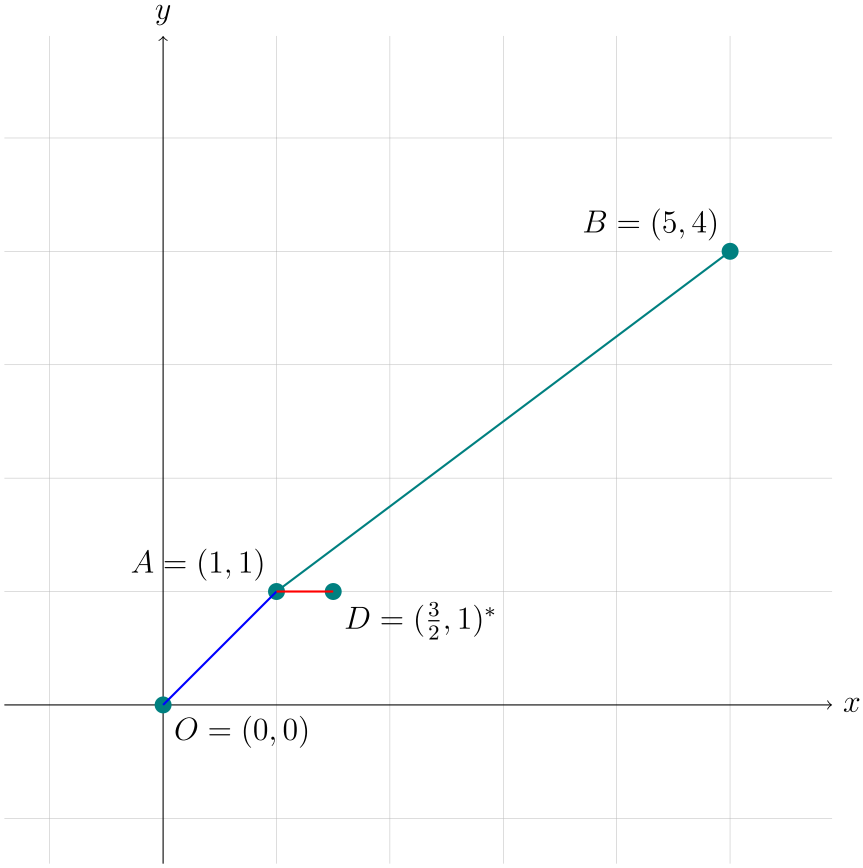
3 Graph of distance formula



Distance formula:

$$\text{dist}(A, C) = \sqrt{(c-a)^2 + (d-b)^2}$$

4 Example and nearest neighbors



$$\begin{aligned} \text{dist}(A, B) &= \sqrt{(5-1)^2 + (4-1)^2} \\ &= 5 \end{aligned}$$

$$\begin{aligned} \text{dist}(A, O) &= \sqrt{(1-0)^2 + (1-0)^2} \\ &= \sqrt{2} \approx 1.4 \end{aligned}$$

$$\begin{aligned} \text{dist}(A, D) &= \sqrt{\left(\frac{3}{2} - 1\right)^2 + (1-1)^2} \\ &= \frac{1}{2} \end{aligned}$$

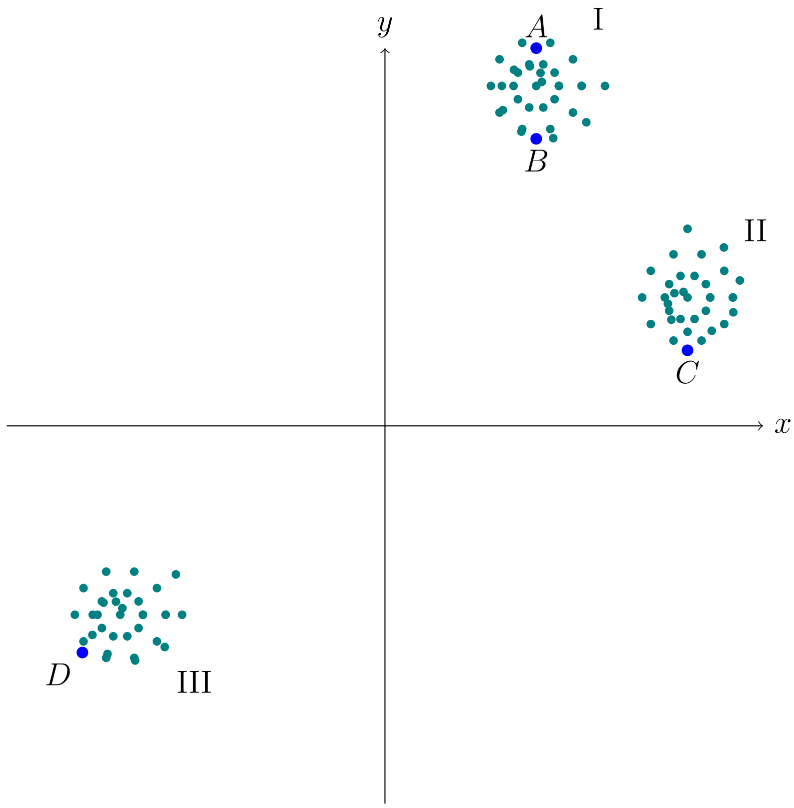
*Note that the x and y values of point D are reversed in the video, but it does not matter in calculating the distance from A.

Consider set S:

$$S = \{O, B, D\}$$

The *nearest neighbor* of A in S is D.
The second nearest neighbor of A in S is O.
The third nearest neighbor of A in S is B.

5 Clustering



Three clusters: I, II, and III

If A and B are in cluster I,
and C is in cluster II,
and D is in cluster III,

Then $\text{dist}(A, B) << \text{dist}(A, C)$,
$<< \text{dist}(A, D)$

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Functions: Composition and Inverse

Video companion

1 Introduction

- Composing two functions
  - Basic identity
  - A warning
- Inverse functions
  - Basic identity
  - A neat picture
  - A warning

2 Composing functions

Definition: Given functions f and g, $(g \circ f)(x) = g(f(x))$, and $(f \circ g)(x) = f(g(x))$

Example:

$$f(x) = x^2$$
$$g(x) = x + 5$$

$$\begin{aligned}
(g \circ f)(x) &= g(f(x)) \\
&= g(x^2) = x^2 + 5 \\
g(f(2)) &= g(2^2) = 2^2 + 5 = 9 \\
(f \circ g)(x) &= f(g(x)) \\
&= f(x+5) = (x+5)^2 \neq x^2 + 5
\end{aligned}$$

3 Inverse functions

Example:

$$f(x) = 2x$$
$$g(x) = \frac{1}{2}x$$

$$(g \circ f)(x) = g(f(x)) = g(2x) = \frac{1}{2}(2x) = x$$

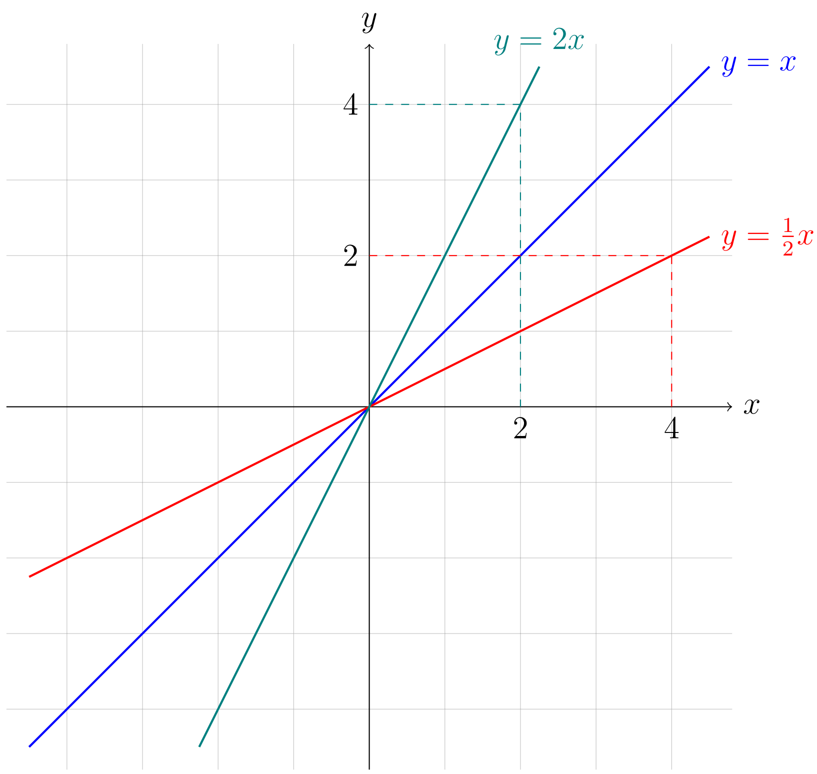
Notice: true for all x

$$(g \circ f)(3) = g(f(3)) = g(2 * 3) = \frac{1}{2}(2 * 3) = 3$$
$$(g \circ f)(\pi) = g(f(\pi)) = g(2 * \pi) = \frac{1}{2}(2 * \pi) = \pi$$

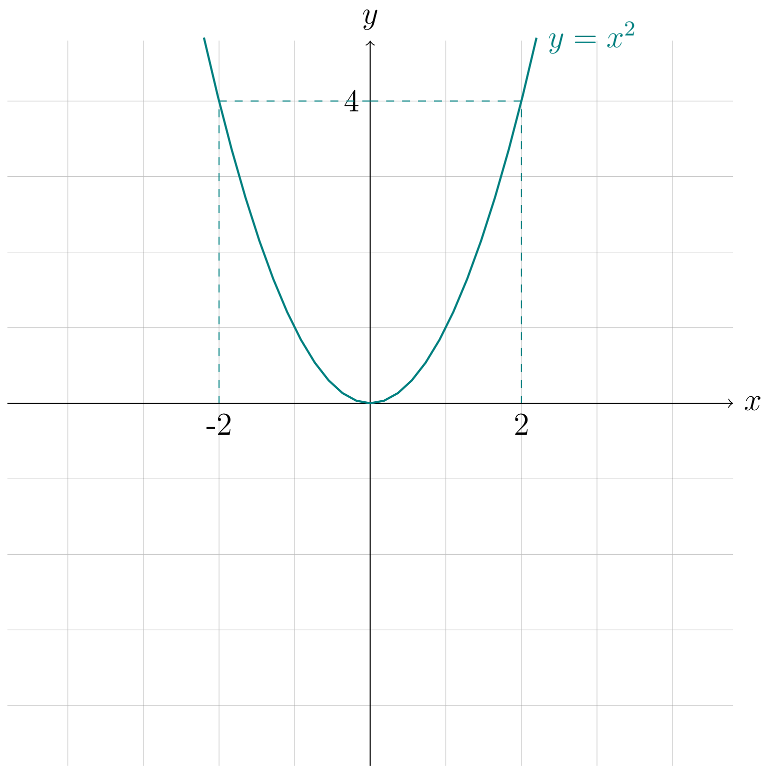
f and g are *inverses* of each other, i.e. f undoes what g does.

$$g = f^{-1}$$

4 Graphical depiction



Warning: not every function $f : \mathbb{R} \to \mathbb{R}$ has an inverse.



Warning: if the graph of f fails the horizontal line test, then f has no inverse. The only invertible functions are those that are either strictly increasing or strictly decreasing.

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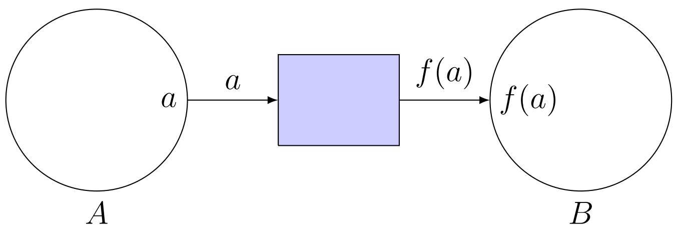
Functions: Graphing in the Cartesian Plane

Video companion

1 Introduction

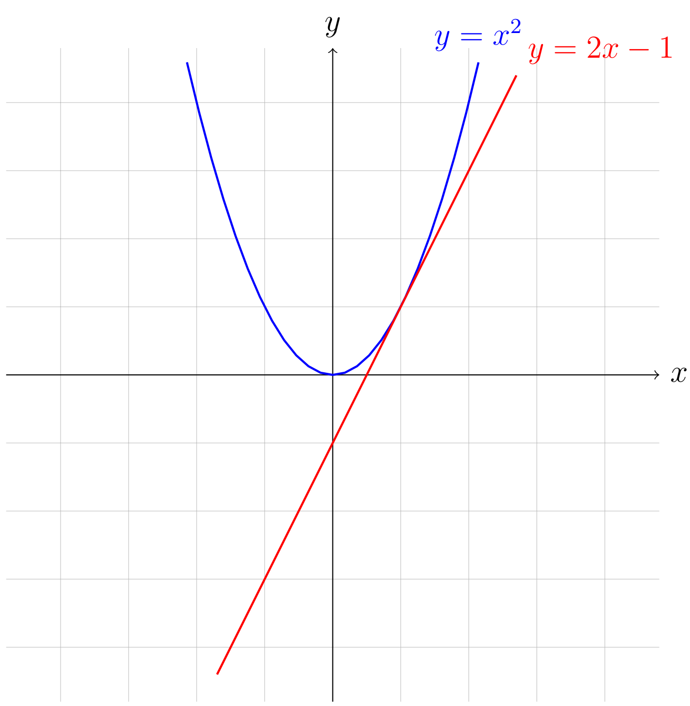
Last time: abstract depiction of a function as a machine

$f : A \to B$



This video: graphs of functions

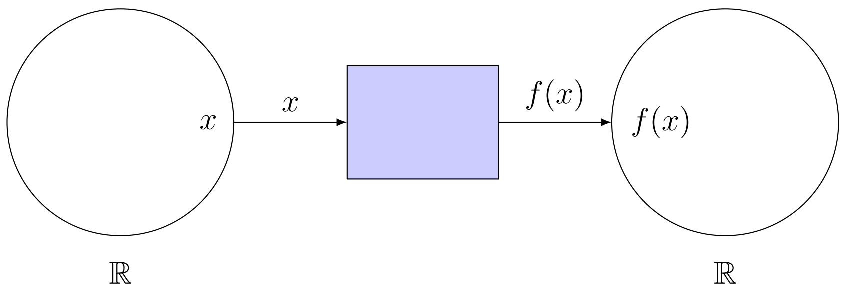
$f : \mathbb{R} \to \mathbb{R}$



You will learn the difference between a *graph* of a function and the function itself.

2 Map real line to real line

$f : \mathbb{R} \to \mathbb{R}$



A function is a formula, a rule for how to operate the machine.

$$\begin{aligned} f(x) &= 2x - 1 \\ f(1) &= 2(1) - 1 = 1 \\ f(0) &= 2(0) - 1 = -1 \\ f(5.1) &= 2(5.1) - 1 = 9.2 \end{aligned}$$

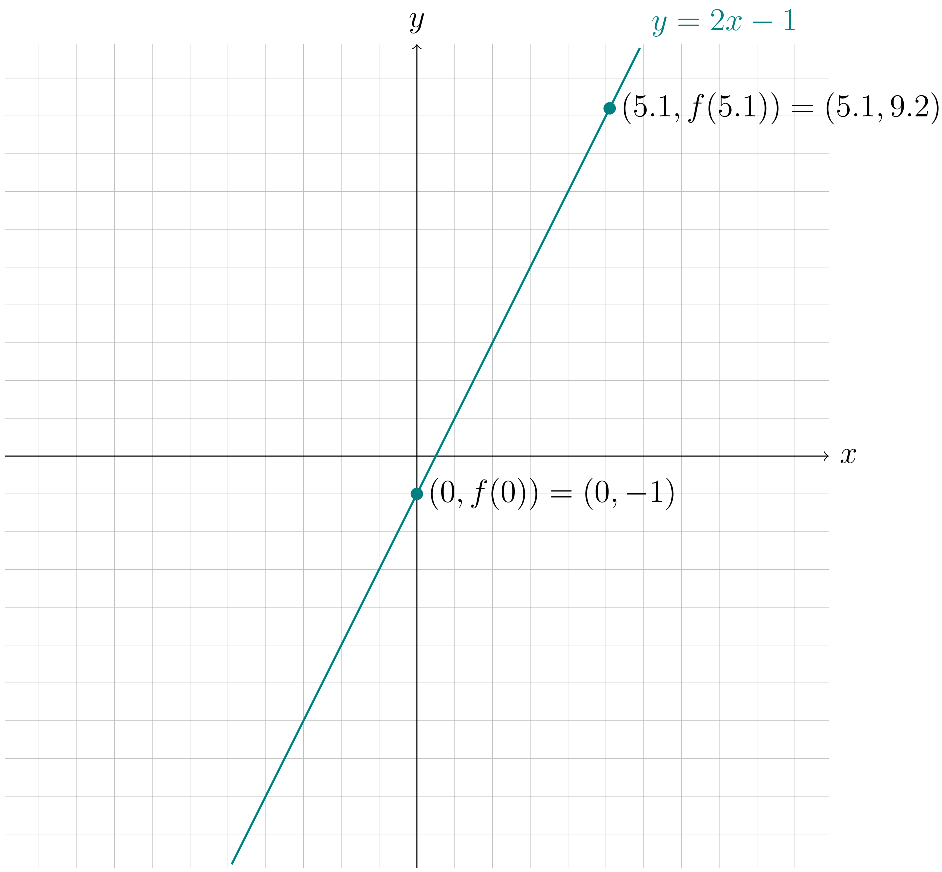
More complicated formulas, like absolute value:

$$\begin{aligned} g(x) &= |x| \\ &= \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases} \end{aligned}$$

Both f and g are functions, with a formula for how to compute the result.

3 What is a graph?

Graph of the function $f : \mathbb{R} \to \mathbb{R}$

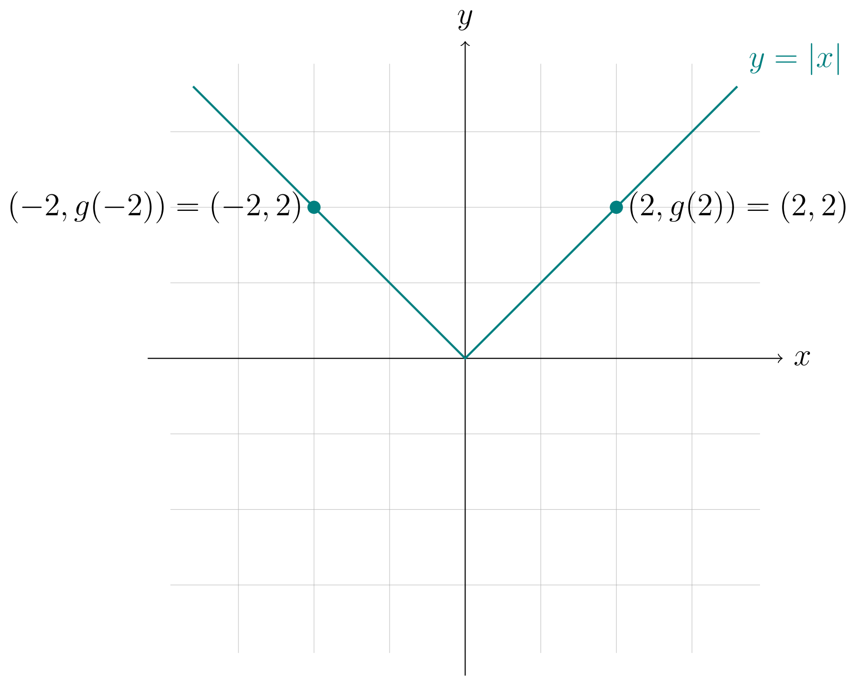


If g is a function $: \mathbb{R} \to \mathbb{R}$, the graph of $g = \{(x, y) \in \mathbb{R}^2 : y = g(x)\}$

4 Examples

Absolute value function

$$\begin{aligned} g(x) &= |x| \\ &= \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases} \end{aligned}$$



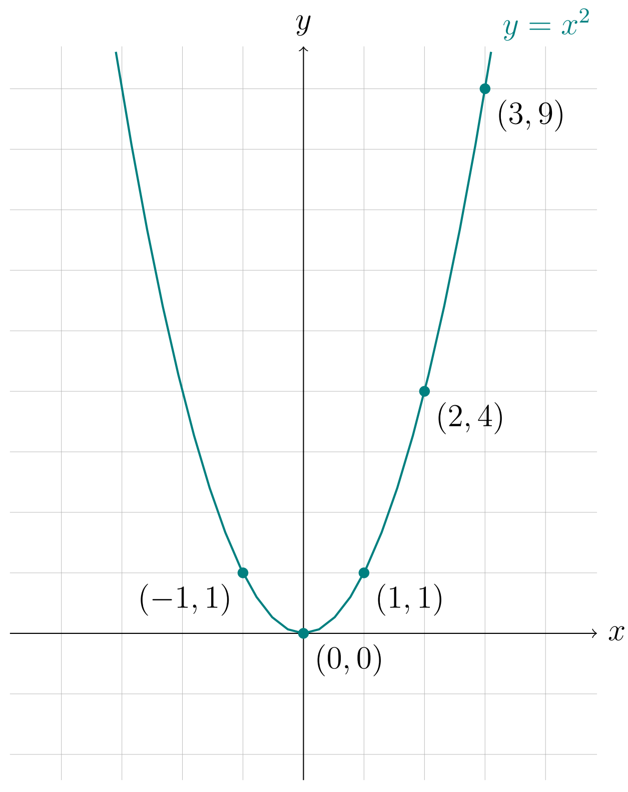
Quadratic function

$$h(x) = x^2$$

Graph a function by testing input and output pairs, see a pattern, and try to draw a curve through it. This is similar to *querying* in supervised learning.

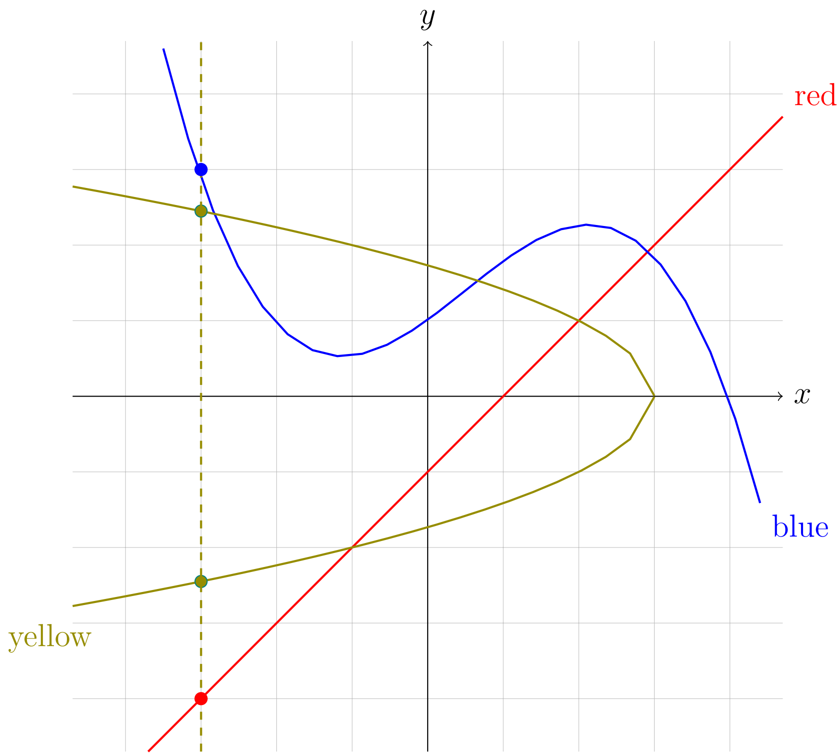
Table of values:

x	$h(x)$
0	$0^2 = 0$
1	$1^2 = 1$
2	$2^2 = 4$
3	$3^2 = 9$
-1	$(-1)^2 = 1$



$h(x) = x^2$ is a *quadratic* function.

5 Vertical line test



Red and blue could be graphs of functions. Yellow could not be the graph of a function because it violates the *vertical line test*, which states that *any vertical line intersects the graph of a function once.*

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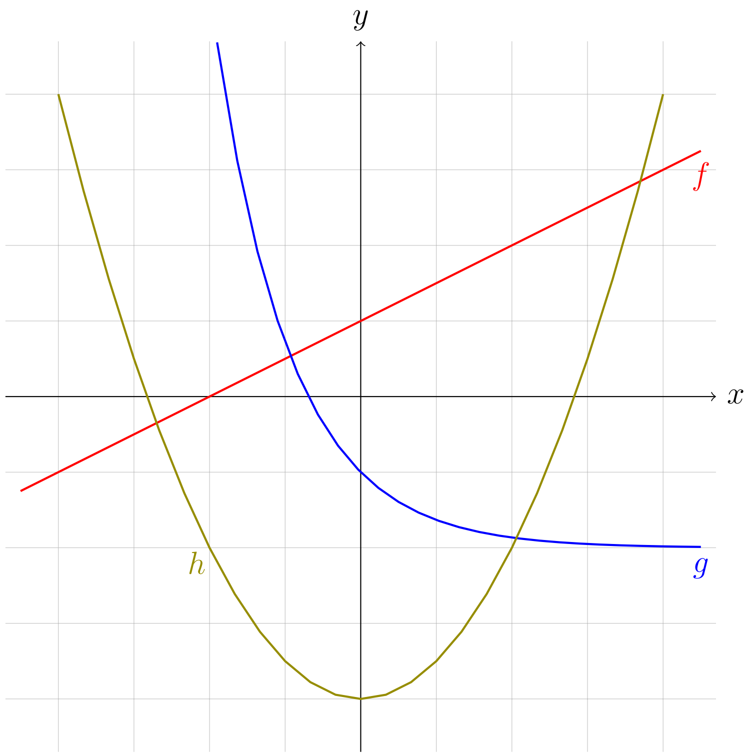
Functions: Increasing and Decreasing Functions

Video companion

1 Introduction



- f is *strictly increasing*
- g is *strictly decreasing*
- h is neither

Let $f : \mathbb{R} \to \mathbb{R}$,
f is strictly increasing if whenever $a < b$, we have $f(a) < f(b)$.
f is strictly decreasing if whenever $a < b$, we have $f(a) > f(b)$.

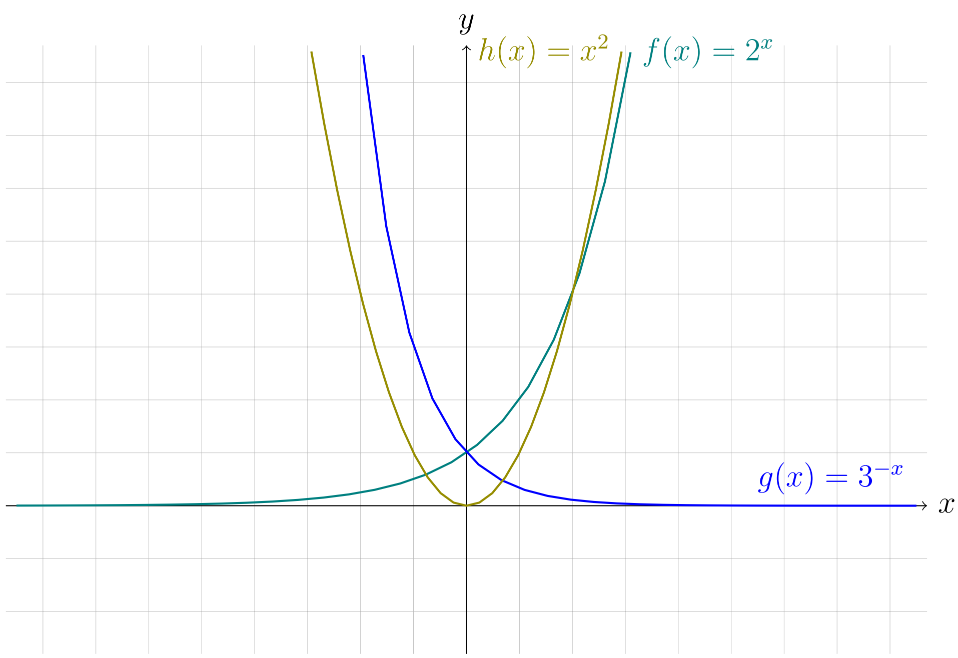
2 Examples

$f(x) = 2^x \quad$ (exponential function)
$g(x) = 3^{-x}$
$h(x) = x^2$

x	$f(x)$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$
-1	$2^{-1} = \frac{1}{2}$

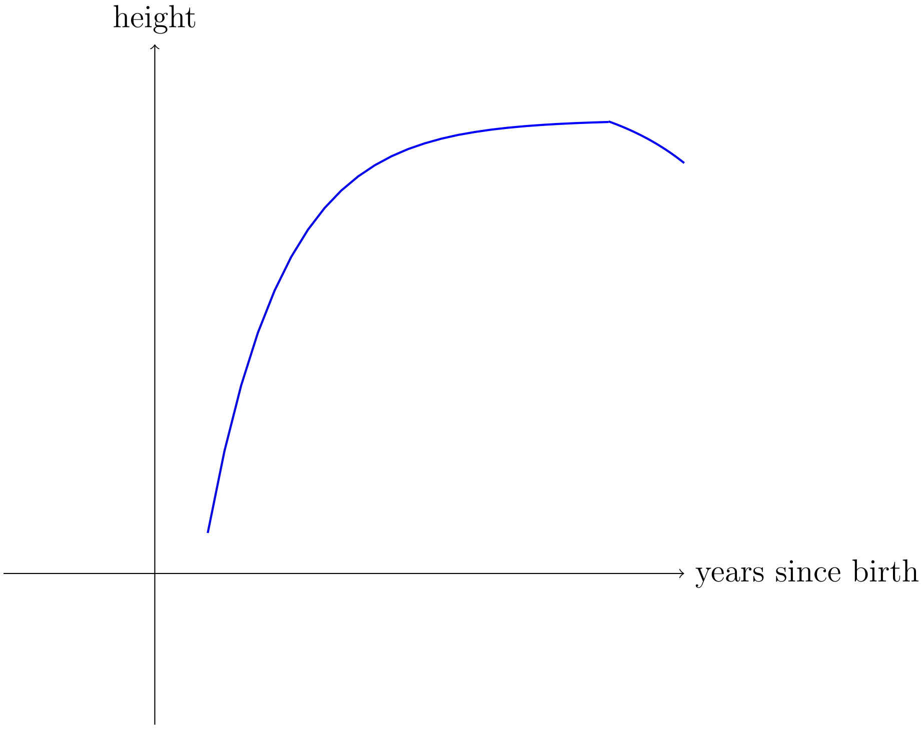
x	$g(x)$
0	$3^0 = 1$
1	$3^{-1} = \frac{1}{3}$
2	$3^{-2} = \frac{1}{9}$
3	$3^{-3} = \frac{1}{27}$
-1	$3^1 = 3$

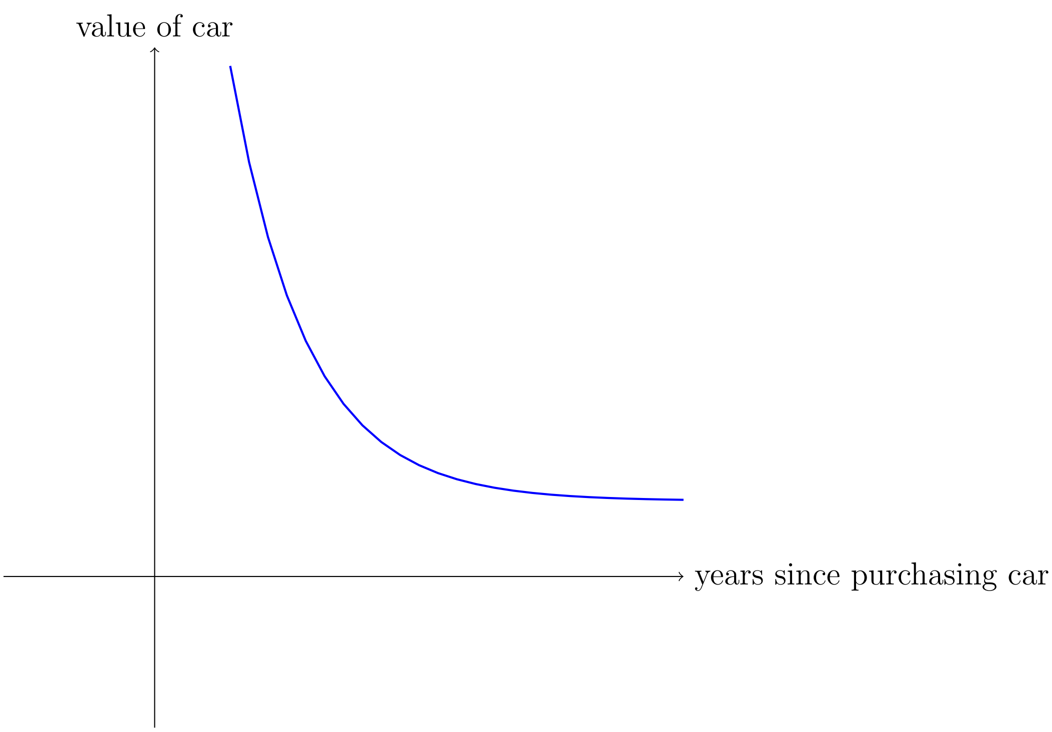
x	$h(x)$
0	$0^2 = 0$
1	$1^2 = 1$
2	$2^2 = 4$
3	$3^2 = 9$
-1	$(-1)^2 = 1$



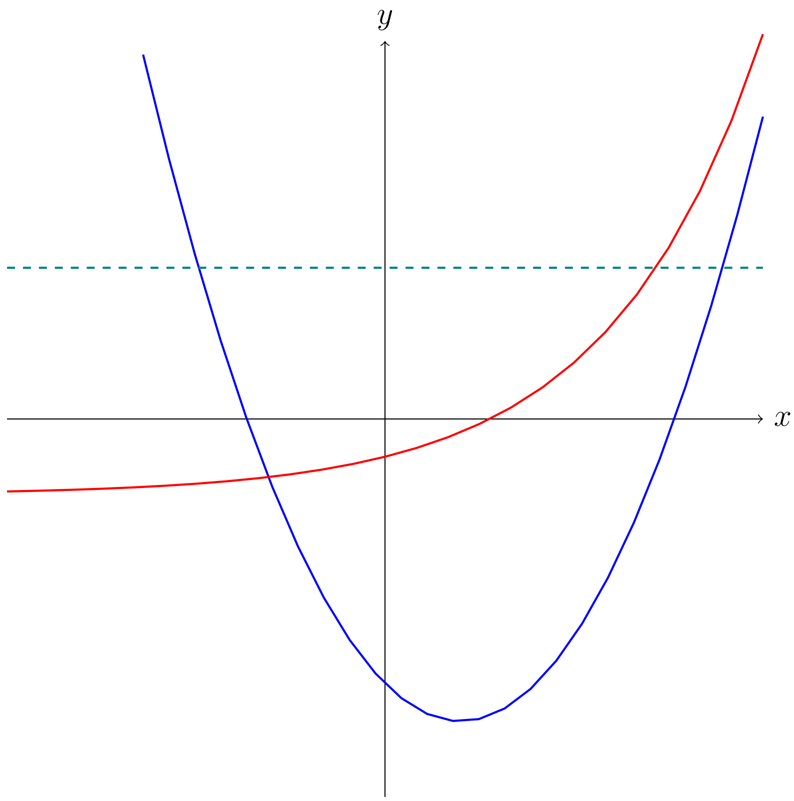
- f is strictly increasing
- g is strictly decreasing
- h is neither
  - h is strictly increasing on $[0, \infty)$
  - h is strictly decreasing on $(-\infty, 0]$

3 "Real-world" examples





4 Horizontal line test



A function is strictly increasing or strictly decreasing if a horizontal line crosses it only once.

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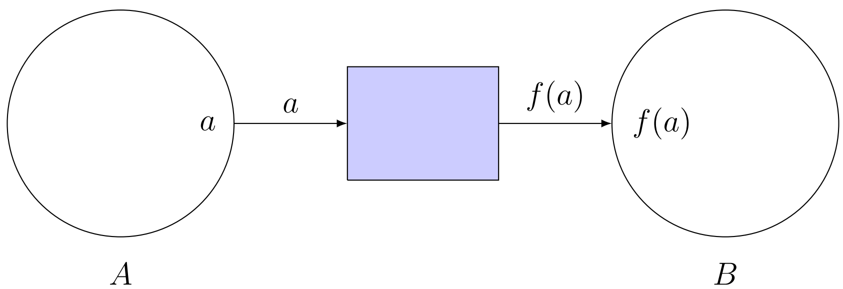
Functions: Mapping from Sets to Sets

Video companion

1 Function as a machine



A function $f : A \to B$ is a rule/formula/machine that transforms each element $a \in A$ into $f(a) \in B$.

$$\begin{aligned} a &: \text{input} \\ f(a) &: \text{output} \end{aligned}$$

2 Examples

Abstract example:

$$\begin{aligned} A &= \{1, 2, 10\} \qquad B = \{\text{apple}, \text{DE}, \text{monkey}\} \\ f &: A \to B \\ f(1) &= \text{apple} \\ f(2) &= \text{apple} \\ f(10) &= \text{monkey} \end{aligned}$$

Study participants test positive or negative:

$$X = \{\text{all people in VBS study}\} \qquad Y = \{+, -\}$$
$$\text{Test} : X \to Y$$
$$\text{Test(person)} = +$$
$$\text{Test(person)} = -$$

Profit by year:

$$Y = \{...2010, 2011, 2012, ...\} \qquad \mathbb{R}$$
$$\text{Profit} : Y \to \mathbb{R}$$
$$\text{Profit(year)} = \text{profit/loss in year}$$
$$\text{Profit(2011)} = 1,007$$
$$\text{Profit(2012)} = -10,000$$

3 Supervised learning

Given: some examples of inputs $a \in A$ and outputs $f(a) \in B$
Mission: figure out $f : A \to B$