Ex 2.1. Direct Evaluation of the convolution sum

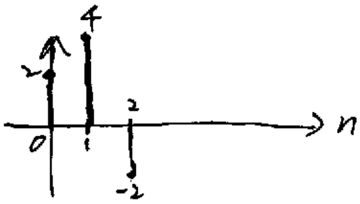
System : $y[n] = x[n] + \frac{1}{2} x[n-1]$.

Letting $x[n] = \delta[n]$

$\Rightarrow y[n] = h[n] = \delta[n] + \frac{1}{2}\delta[n-1]$

$$h[n] = \begin{cases} 1, & n=0 \\ \frac{1}{2}, & n=1 \\ 0, & \text{others} \end{cases} \quad \text{—— ②}$$

Now, let input be $x[n] = \begin{cases} 2, & n=0 \\ 4, & n=1 \\ -2, & n=2 \\ 0, & \text{others} \end{cases}$



$$x[n] = 2\delta[n] + 4\delta[n-1] - 2\delta[n-2] \quad \text{—— ①}$$

input = 0 for $n<0$ and $n>2$.

<Sol.>

Give ① and ② find ③

$$\underset{①}{x[n]} * \underset{②}{h[n]} = \underset{③}{y[n]}$$

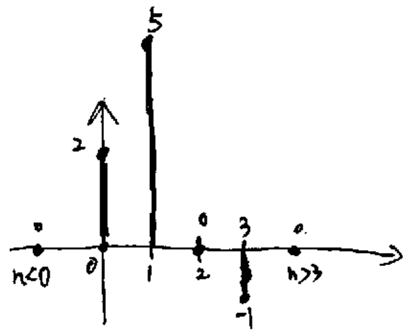
$$y[n] = 2h[n] + 4h[n-1] - 2h[n-2]$$

$$= 2(\delta[n] + \tfrac{1}{2}\delta[n-1]) + 4(\delta[n-1] + \tfrac{1}{2}\delta[n-2]) - 2(\delta[n-2] + \tfrac{1}{2}\delta[n-3])$$

$$= 2\delta[n] + \delta[n-1] + 4[n-1] + 2\delta[n-2] - 2\delta[n-2] - \delta[n-3]$$

$$= 2\delta[n] + 5\delta[n-1] - 0\delta[n-2] - \delta[n-3] \ \#$$

$$y[n] = \begin{cases} 0, & n<0 \\ 2, & n=0 \\ 5, & n=1 \\ 0, & n=2 \\ -1, & n=3 \\ 0, & n>3 \end{cases} \ \#$$



Ex 2.2. Convolution Sum Evaluation by using Intermediate Signal

System Impulse Response $h[n] = (\frac{3}{4})^n u[n]$.

$y[n] = \sum_{k=-\infty}^{\infty} w_n[k]$ (2.6)

$y[n] = \sum_{k=-\infty}^{k=\infty} x[k] \cdot h[n-k] = x[n] * h[n]$

Input = $x[n] = u[n]$

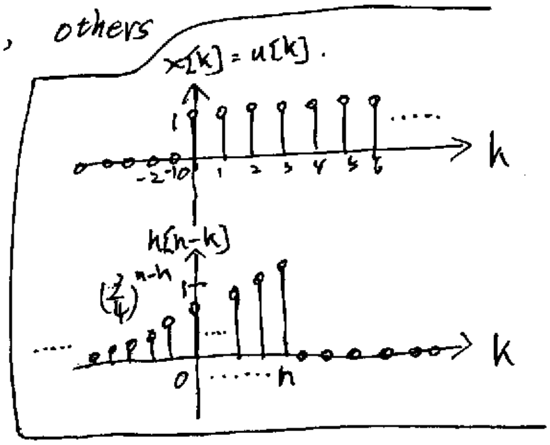
<Sol.>

$h[n] = (\frac{3}{4})^n u[n]$

$h[n-k] = (\frac{3}{4})^{n-k} u[n-k]$

$h[n-k] = \begin{cases} (\frac{3}{4})^{n-k}, & k \leq n \\ 0, & \text{others} \end{cases}$

$y[n] = \sum_{k=-\infty}^{\infty} u[k] (\frac{3}{4})^{n-k} u[n-k] = \sum_{k=0}^{n} (\frac{3}{4})^{n-k}$



① when $n < 0$:

$y[n] = 0$

② when $n \geq 0$.

$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$

$= \sum_{k=0}^{n} (\frac{3}{4})^{n-k}$

$= (\frac{3}{4})^n \sum_{n=0}^{n} (\frac{4}{3})^k = (\frac{3}{4})^n \frac{[(\frac{4}{3})^{n+1} - 1]}{\frac{4}{3} - 1}$

$= 4 - 3(\frac{3}{4})^n$

$\therefore y[n] = \begin{cases} 0, & n < 0 \\ 4 - 3(\frac{3}{4})^n, & n \geq 0 \end{cases}$ #

For $n = -5$: $y[-5] = 0$ #

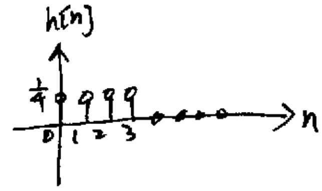
For $n = 5$: $4 - 3(\frac{3}{4})^5 = 3.288$ #

For $n = 10$: $4 - 3(\frac{3}{4})^{10} = 3.831$ #

Moving-Average System:

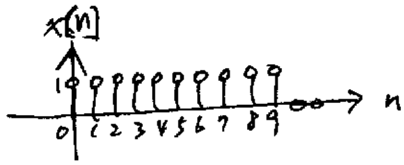
Ex 2.3 Reflect-and-shift Convolution Sum Evaluation

$$y[n] = \frac{1}{4}\sum_{k=0}^{3} x[n-k]$$



impluse response $h[n]$, $x[n] = \delta[n]$

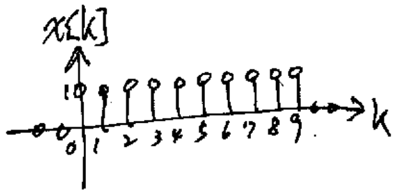
$$h[n] = \frac{1}{4}\left(u[n] - u[n-4]\right) \text{ —— } ②$$

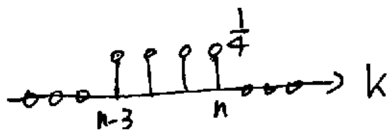


input : $x[n] = u[n] - u[n-10]$ —— ①

$$\underset{③}{y[n]} = \underset{①}{x[n]} * \underset{②}{h[n]}$$

$$y[n] = \frac{1}{4}\sum_{k=-\infty}^{\infty}\left(u[k]-u[k-10]\right)\left(u[n-k]-u[n-k-4]\right) \text{ —— } ③$$





<Sol.>

1st : interval : $w_n[k] = 0$.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = \sum_{k=-\infty}^{\infty} w_n[k] = 0$$

2nd interval : $0 \le n \le 3$

$$y[n] = \sum_{k=0}^{n} \frac{1}{4} = \frac{n+1}{4}$$

3rd interval : $3 < n \le 9$

$$y[n] = \sum_{k=n-3}^{n} \frac{1}{4} = 1$$

4th interval : $9 < n \le 12$

$$y[n] = \sum_{k=n-3}^{9} \frac{1}{4} = \frac{13-n}{4}$$

5th interval : $n > 12 = 0$.

五種情況

$n < 0$

$0 \le n \le 3$

$3 < n \le 9$

$9 < n \le 12$

$n > 12$

$$y[n] = \begin{cases} \frac{n+1}{4} & 0 \le n \le 3, \\ 1 & 3 < n \le 9, \\ \frac{13-n}{4} & 9 < n \le 12, \\ 0 & \text{others}, \end{cases}$$

#

Ex 2.4: First-order Recursive System: Reflect-and-shift Convolution Sum Evaluation

given: $y[n] - \rho y[n-1] = x[n]$

Let the input $x[n] = b^n u[n+4]$

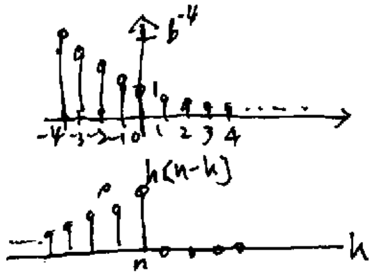
$x[n] = \delta[n] \quad \overset{3}{y[n]} = \overset{2}{h[n]} * \overset{1}{x[n]} = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$

Impulse response: $h[n] = \rho h[n-1] + \delta[n]$

Since the system is causal, we have $h[n] = 0$ for $n < 0$. For $n = 0, 1, 2 \cdots$, we find that $h[0] = 1$, $h[1] = \rho$, $h[2] = \rho^2, \cdots$, or

$h[n] = \rho^n u[n]$ —— 2

$x[k] = \begin{cases} b^k, & -4 \le k \\ 0, & \text{others} \end{cases}$ and $h[n-k] = \begin{cases} \rho^{n-k}, & k \le n \\ 0, & \text{others} \end{cases}$



$y[n] = \sum_{k=-\infty}^{\infty} b^k u[k+4]\, \rho^{n-k} u[n-k]$

interval $\begin{cases} n < -4, & \underline{\ ?\ } \\ n \ge 4, & \underline{\ ?\ } \end{cases}$

when $n < -4$:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = 0.$$

when $n \ge 4$:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \sum_{k=-4}^{n} b^k \cdot \rho^{n-k}$$

$$= \rho^n \sum_{k=-4}^{n} \left(\frac{b}{\rho}\right)^k = \rho^n \cdot \frac{\left(\frac{b}{\rho}\right)^{-4} \cdot \left[1 - \left(\frac{b}{\rho}\right)^{n+5}\right]}{1 - \frac{b}{\rho}}$$

$$y[n] = \begin{cases} 0, & n < -4 \\ b^{-4}\left(\dfrac{\rho^{n+5} - b^{n+5}}{\rho - b}\right), & n \ge -4 \end{cases}$$

#

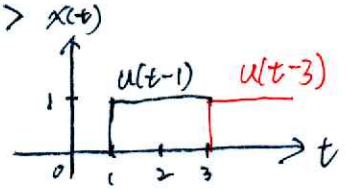
Ex 2.6 Reflect-and-shift Convolution Evaluation

Given $x(t) = u(t-1) - u(t-3)$ —— ①

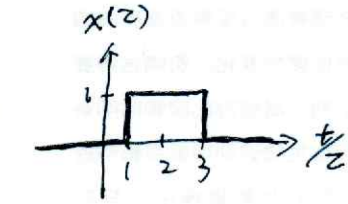
$h(t) = u(t) - u(t-2)$ —— ②

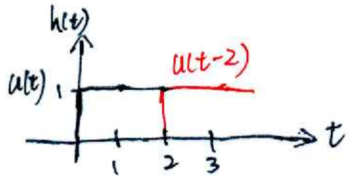
Evaluate the convolution intergral $y(t) = x(t) * h(t)$

<Sol.>

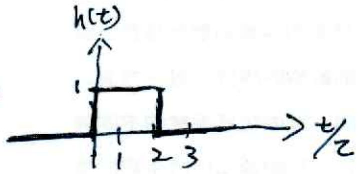


$\Rightarrow$



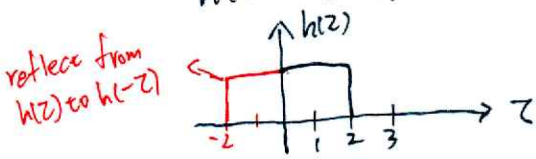


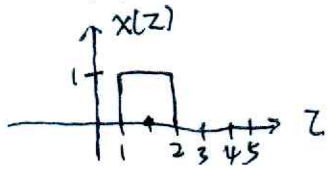
$\Rightarrow$



$$y(t) = x(t) * h(t) = \int_{\tau=-\infty}^{\tau=\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$h(t-\tau) = h(-\tau + t)$





$\Rightarrow$ Interval
$$\begin{cases} t < 1 \rightarrow y(t) = 0 \\ 1 \le t \le 3 \quad \underline{\ \ ?\ \ } \\ 3 \le t \le 5 \quad \underline{\ \ ?\ \ } \\ t \ge 5 \rightarrow y(t) = 0 \end{cases}$$

$1 \le t \le 3$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} w_t(\tau)\, d\tau$$

$$w_t(\tau) = \begin{cases} 1, & 1 < \tau < t \\ 0, & \text{others} \end{cases}$$

$$y(t) = \int_{\tau=1}^{\tau=t \rightarrow 1 \le t \le 3} 1 \cdot 1 \cdot d\tau = \tau \Big|_{\tau=1}^{\tau=t} = t - 1 \ \#$$

$$y(t) = \begin{cases} 0, & t < 1 \\ t-1, & 1 \le t \le 3 \\ 5-t, & 3 \le t \le 5 \\ 0, & t > 5 \end{cases} \ \#$$

$3 \le t \le 5$

$$w_t(\tau) = \begin{cases} 1, & t-2 \le \tau \le 3 \\ 0, & \text{others} \end{cases}$$

$$y(t) = \int_{\tau=t-2}^{\tau=3} \cdot 1 \cdot 1 \cdot d\tau$$

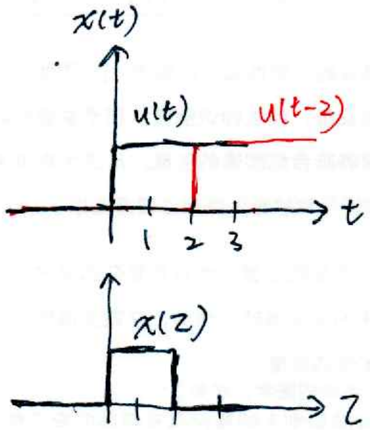
$$= \tau \Big|_{\tau=t-2}^{\tau=3} = 3 - (t-2) = 5 - t \ \#$$

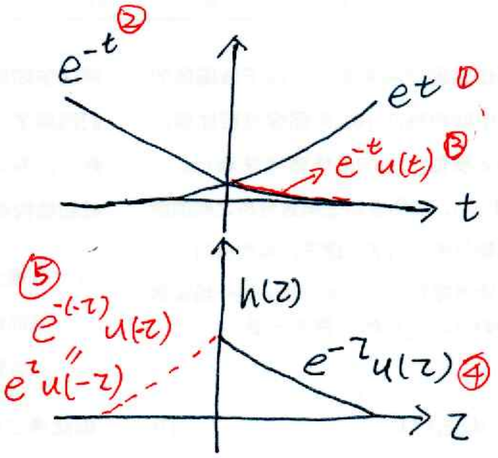
Ex 2.7. RC Circuit Output

impluse response : $h(t) = e^{-t} u(t)$

input : $x(t) = u(t) - u(t-2)$

$$y(t) = x(t) * h(t) = \int_{\tau=-\infty}^{\tau=\infty} x(\tau)\, h(t-\tau)\, d\tau$$





⑤ $h(-\tau) = e^{-(-\tau)} u(-\tau)$

$= e^{\tau} u(-\tau)$

Interval $\begin{cases} t<0, & y(t)=0 \\ 0 \le t \le 2, & \underline{\quad ? \quad} \\ t \ge 2, & \underline{\quad ? \quad} \end{cases}$

$\int e^t dt = e^t$

$\int e^{at} dt = \frac{1}{a} e^{at}$

$(e^{at})' = a e^{at}$

$0 \le t \le 2$:

$$\int_{\tau=0}^{\tau=t} 1 \cdot e^{-t+\tau} d\tau$$

$$= (e^{-t}) \cdot \int_{\tau=0}^{\tau=t} e^{\tau} dt$$

$$= e^{-t} \cdot e^{\tau} \Big|_{\tau=0}^{\tau=t}$$

$$= e^{-t} \cdot e^{t} - e^{0}$$

$$= e^{-t}(e^{t} - 1)$$

$$= e^{0} - e^{t} = 1 - e^{-t} \; \#$$

$t \ge 2$:

$$\int_{\tau=0}^{\tau=2} 1 \cdot e^{-t+\tau} d\tau$$

$$= (e^{-t}) \cdot \int_{\tau=0}^{\tau=2} e^{\tau} d\tau = (e^{-t}) \cdot e^{\tau} \Big|_{\tau=0}^{\tau=2}$$

$$= (e^{-t}) \cdot (e^{2} - e^{0}) = e^{-t} \cdot (e^{2} - 1) \; \#$$

$$y(t) = \begin{cases} 0, & t<0 \\ 1-e^{-t}, & 0 \le t \le 2 \\ (e^{2}-1) \cdot e^{-t}, & t \ge 2 \end{cases} \; \#$$

Ex 2.8 Another Reflect-and-shift Convolution Evaluation

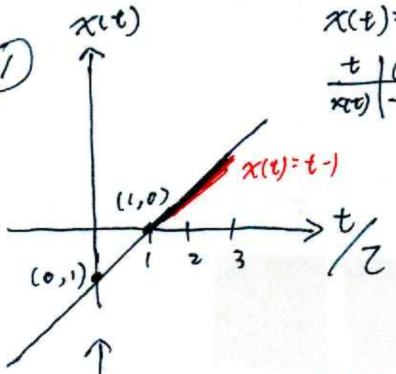
$$x(t) = (t-1)[u(t-1) - u(t-3)] \quad \text{——} ①$$

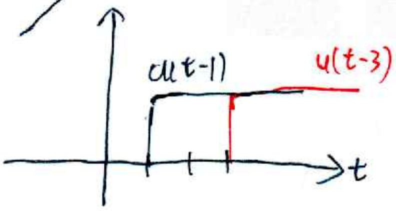
$$h(t) = u(t+1) - 2u(t-2) \quad \text{——} ②$$

①

$x(t) = t-1$

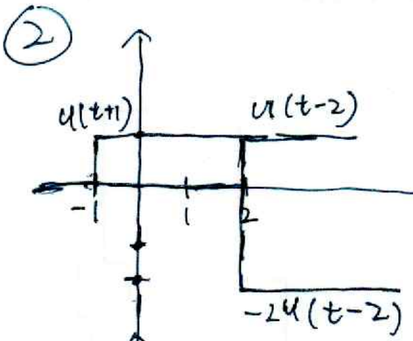
t	0	1
$x(t)$	-1	0



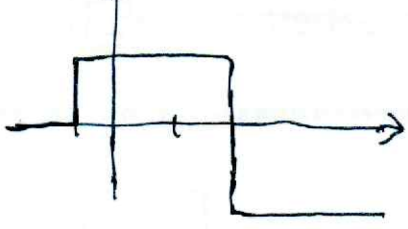


$$x(t) = x(\tau) = (\tau-1)[u(\tau-1) - u(\tau-3)]$$

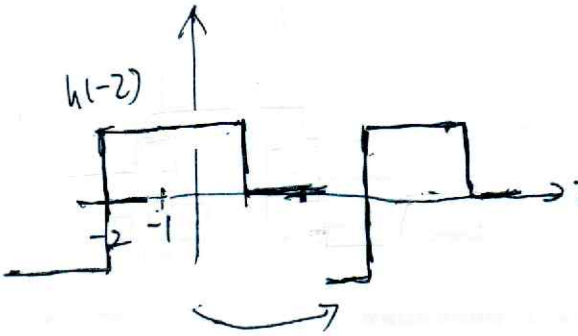
②



$\to t \Rightarrow h(t) = u(t+1) - 2u(t-2)$



$\to \tau \Rightarrow h(\tau) = u(\tau+1) - 2u(\tau-2)$



$\Rightarrow$

$$h(-\tau) = u(-\tau+1) - 2u(-\tau-2)$$

$$h(-\tau+t) = u(-\tau+1+t) - 2u(-\tau-2+t)$$

from 0 to t

討論

$$y(t)=\begin{cases} 0, & t\le 0 \\ \underline{?}, & 0<t\le 2 \\ \underline{?}, & 2<t\le 3 \\ \underline{?}, & 3<t\le 5 \\ \underline{?}, & t>5 . \end{cases}$$

when $0<t\le 2$:

$$y(t)=\int_{\tau=1}^{\tau=3} x(\tau)h(t-\tau)d\tau=\int_{\tau=1}^{\tau=t+1}(\tau-1)\cdot 1\cdot d\tau$$

$$=\left(\frac{1}{2}\tau^2-\tau\right)\Big|_{\tau=1}^{\tau=t+1}$$

$$=\left[\frac{1}{2}(t+1)^2-(t+1)\right]-\left[\frac{1}{2}(1)^2-1\right]$$

$$=\left[\frac{1}{2}(t^2+2t+1)-t-1\right]-\left(-\frac{1}{2}\right)$$

$$=\frac{1}{2}t^2\cancel{+t}\cancel{+\frac{1}{2}}\cancel{-t}\cancel{-1}\cancel{+\frac{1}{2}}$$

$$=\frac{1}{2}t^2 \quad \#$$

when $2\le t\le 3$:

$$y(t)=\int_{\tau=1}^{\tau=3}(\tau-1)\cdot 1\cdot d\tau$$

$$=\left(\frac{1}{2}\tau^2-\tau\right)\Big|_{\tau=1}^{\tau=3}$$

$$=\left[\left(\frac{1}{2}\cdot 9-3\right)-\left(\frac{1}{2}\cdot 1-1\right)\right]$$

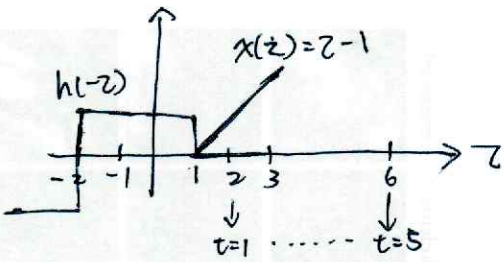
$$=\left[\frac{9}{2}-3+\frac{1}{2}\right]$$

$$=\frac{10-6}{2}$$

$$=2 \quad \#$$

when $3 \leq t \leq 5$:

$$y(t) = \int_{\tau=1}^{\tau=3} x(\tau)\, h(t-\tau)\, d\tau$$



$$y(t) = \int_{\tau=1}^{\overset{\rightarrow t=5,\ \tau=3}{\tau=t-2}} (\tau-1)\cdot -1 \cdot d\tau$$

$$+ \int_{\tau=t-2}^{\tau=3} (\tau-1)+1\, d\tau$$

$$= -\left(\frac{1}{2}\tau^2 - \tau\right)\Big|_{\tau=1}^{\tau=t-2} + \left[\left(\frac{1}{2}\tau^2 - \tau\right)(-1)\right]\Big|_{\tau=t-2}^{\tau=3}$$

$$= -\left[\frac{1}{2}(t-2)^2 - (t-2) - \left(\frac{1}{2} - 1\right)\right] + \left[\frac{1}{2}\cdot 9 - 3 - \left(\frac{1}{2}(t-2)^2 - (t-2)\right)\right]$$

$$= -\left[\frac{1}{2}t^2 - \frac{1}{2}\cdot 4t + \frac{1}{2}\cdot 4 - t + 2 + \frac{1}{2}\right] + \left[\frac{9}{2} - \frac{6}{2} - \left(\frac{1}{2}t^2 - 2t + 2 - t + 2\right)\right]$$

$$= -\left[\frac{t^2}{2} - 2t - t + 4\frac{1}{2}\right] + \left[-\frac{t^2}{2} + 3t - \frac{5}{2}\right]$$

$$= -t^2 + 3t + 3t - \frac{9}{2} - \frac{5}{2}$$

$$= -t^2 + 6t - 7 \ \#$$

when $t > 5$

$$y(t) = \int_{\tau=1}^{\tau=3} (\tau-1)(-1)\, d\tau$$

$$= -\left(\frac{1}{2}\tau^2 - \tau\right)\Big|_{\tau=1}^{\tau=3}$$

$$= -\left[\left(\frac{1}{2}9 - 3\right) - \left(\frac{1}{2}\cdot 1 - 1\right)\right]$$

$$= -\left[\frac{9}{2} - \frac{6}{2} + \frac{1}{2}\right] = -2 \ \#$$

$$y(t) = \begin{cases} 0, & t < 0 \\ \frac{t^2}{2}, & 0 \leq t < 2 \\ 2, & 2 \leq t < 3 \\ -t^2 + 6t - 7, & 3 \leq t < 5 \\ -2, & t \geq 5 \end{cases}$$

#