Lecture 1

Definition 0.1. A chart around a point $x \in X$, where X is a topological space, is a set (U, φ), where U is an open neighborhood of x, and $\varphi : U \to \mathbb{R}^n$ is a homeomorphism onto its image with $\varphi(x) = 0$.

Lecture 2, 10/3/23

Recall a topological manifold is a topological space which is

1. Hausdorff

2. 2nd countable

3. Locally Euclidean

Our goal is to move to smooth manifolds, on which we can do calculus.
This is done by picking a point on our smooth manifold, translating it into a linear space through the use of charts, doing the calculus, and then translating back.

Example 0.1. Let $M = \mathbb{R}$ with the chart $\varphi(x) = x^3 = y$. Then the function $f(x) = x^2$ is differentiable in the normal sense, but

$$f \circ \psi^{-1}(y) = y^{\frac{2}{3}}$$

is not differentiable at $y = 0$!

Definition 0.2. Two charts $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta)$ are C^∞ compatible if, whenever $U_\alpha \cap U_\beta \neq \varnothing$, then we have a smooth function

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

whose inverse is also smooth.
The maps $\varphi_\alpha \circ \varphi_\beta^{-1}, \varphi_\beta \circ \varphi_\alpha^{-1}$ are called transition maps, or sometimes coordinate changes.

Example 0.2. On S^2 we have the stereographic projections

$$\mathcal{U}_1 = S^2 \setminus \{S\} \xrightarrow{\varphi_1} \mathbb{R}^2$$

$$\mathcal{U}_2 = S^2 \setminus \{N\} \xrightarrow{\varphi_2} \mathbb{R}^2$$

Then

$$\varphi_2 \circ \varphi_1^{-1} = \left(\frac{4u}{u^2 + v^2}, \frac{4v}{u^2 + v^2} \right)$$

This is C^∞ on the domain $\varphi_1(\mathcal{U}_1 \cap \mathcal{U}_2)$, and similarly for the other way around. So these are C^∞ compatible charts.

Example 0.3. Let $M = \mathbb{R}$, with chart $(\mathcal{U} = \mathbb{R}, \varphi(x) = x)$, $(\mathcal{V} = \mathbb{R}, \psi(x) = x^3)$
These are two incompatible charts!

Definition 0.3. An atlas is a collection of charts $(U_\alpha, \varphi_\alpha)$ such that

$$\bigcup_\alpha U_\alpha = M$$

Definition 0.4. A smooth structure/ C^∞ structure / differentiable structure on a topological manifold is an atlas $\mathscr{U} = \{(U_\alpha, \varphi_\alpha)\}$ such that

1. All charts in $\mathscr{U}$ are pairwise C^∞ compatible.

2. The atlas is maximal in the sense that any chart (U, φ) which is C^∞ compatible with every element of $\mathscr{U}$ is contain in $\mathscr{U}$

Proposition 1 (1.17)**.** *Let M be a topological manifold. Then*

(a) Every smooth atlas is contained in a unique maximal smooth atlas, i.e. a smooth structure.

(b) Two smooth atlases determine the same smooth structure if and only if their union is a smooth atlas.

Proof. I omit it because i didn't really follow and don't think it's that important it's 1.17 in Lee sorrrryyyyyyy

■

The upshot of all of this is that we can specify a smooth structure by specifying a smooth atlas.

Example 0.4. $(\mathbb{R}^n, \varphi = \mathrm{Id}_{\mathbb{R}^n})$ is the standard smooth structure on $\mathbb{R}^n$

Example 0.5. $(\mathbb{R}, \varphi = \mathrm{Id}_{\mathbb{R}})$, $(\mathbb{R}, \psi(x) = x^3)$ are two different smooth structures on $\mathbb{R}$.
From the above example, it seems we are overcounting - there are many different distinct smooth structures on $\mathbb{R}$. Later, we will fix this by introducing the notion of diffeomorphism.
Every topological manifold with a single chart has a smooth structure.

Example 0.6. $\mathbb{R}^3$ minus a knot

Example 0.7. $\mathrm{GL}(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n^2}$ because it is the preimage of $\mathbb{R} \setminus \{0\}$ under the continuous map det

Definition 0.5. A smooth manifold is a topological manifold together with a smooth structure.

Example 0.8. S^2 needs two charts as a consequence of the hairy ball theorem.

Example 0.9. Here is a smooth structure on S^n.
S^n is defined as the solution locus of the polynomial $x_1^2 + \cdots + x_{n+1}^2 = 1$ in $\mathbb{R}^{n+1}$.
For each $i = 1, \ldots, n+1$, let $U_i^\pm$ be all the points on S^n such that x_i is positive or negative, and define $\varphi_i^\pm$ by just deleting the ith coordinate, so the sum of squares is strictly less than 1.
It is clear that this is indeed an atlas.
We now claim that $\{(U_i^\pm, \varphi_i^\pm)\}_{i=1}^{n+1}$ is a smooth atlas.

$$\begin{aligned}(\varphi_1^+ \circ (\varphi_2^-)^{-1})(y_1, \ldots, y_n) &= \varphi_1^+ \left(y_1, -\sqrt{1 - \sum_{i=1}^n y_i^2}, y_2, \ldots, y_n \right) \\ &= \left(-\sqrt{1 - \sum_{i=1}^n y_i^2}, y_2, \ldots, y_n \right)\end{aligned}$$

This is smooth because $\sum y_i^2 < 1$, so the square root will not cause any trouble.
The others can be checked similarly!
$\mathscr{U} = \{(U_i^\pm, \varphi_i^\pm)\}_{i=1}^{n+1}$ defines a smooth structure on S^n which is called the standard smooth structure on S^n

Remark

Stereographic chart and the standard charts given above are C^∞ compatible.

Lecture 3, 10/5/23

Our goal today is to describe $\mathbb{RP}^n$ as a smooth manifold.

Definition 0.6. We define $\mathbb{RP}^n$ as $\mathbb{R}^{n+1}$ quotiented by the action of $\mathbb{R}$ given by scaling. That is, $(x_1, \ldots, x_{n+1}) \sim (y_1, \ldots, y_{n+1})$ if one is a nonzero scalar multiple of the other.
We can also describe $\mathbb{RP}^n$ as a quotient of S^2 by the $\mathbb{Z}_2$-action given by the antipodal map.
We have a canonical projection $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$. We describe such points using so-called homogeneous coordinates. We write $\pi(x) = [x]$. This denotes the line passing through x.

Claim. $\mathbb{RP}^n$ *is Hausdorff, Second Countable, and admits a smooth structure.*

Lemma 1. π *is an open map. In particular,* $\mathbb{RP}^n$*, as a topological space, is second countable.*

Proof.
Let $U \subseteq \mathbb{R}^{n+1} \setminus \{0\}$ be open. We want to show that $\pi(U) \subseteq \mathbb{RP}^n$ is open, i.e. $\pi^{-1}(\pi(U))$ is open.
Now
$$\pi^{-1}(\pi(U)) = \bigcup_{x\in U}[x] = \bigcup_{x\in U}\bigcup_{\lambda\neq 0}\{\lambda x\}$$
For $\lambda \neq 0$, we define $\lambda(U)$ to be the image of U under the map $x \mapsto \lambda x$.
If $\lambda \neq 0$, the $\lambda(U)$ is open for U open.
Thus
$$\pi^{-1}(\pi(U)) = \bigcup_{\lambda\neq 0\in\mathbb{R}} \lambda(U)$$
is open.

∎

Lemma 2. *Let X be a topological space, $\sim$ an equivalence relation on X. Put the quotient topology on $X/\sim$. Assume that $\pi : X \to X/\sim$ is an open mapping. Then if X is second countable,*

1. *If X is 2nd countable, then so is $X/\sim$.*

2. *$X/\sim$ is Hausdorff if and only if $R = \{(x,y) \mid x \sim y\} \subset X \times X$ is closed.*

Proof. Shut up!

∎

So $\mathbb{RP}^n$ is Hausdorf, since
$$R = \{(x,y) \in \left(\mathbb{R}^{n+1}\setminus\{0\}\right) \times \left(\mathbb{R}^{n+1}\setminus\{0\}\right) \mid y = \lambda x, \lambda \neq 0\}$$
is closed.
Remark: $y = \lambda x$ for some nonzero λ is equivalent to the statement
$$\sum_{i,j=1}^{n+1}(x_i y_j - x_j y_i)^2 = 0$$
We can express this as a function $F(x,y)$, which is continuous, and this set is the preimage of 0, which is closed. So, this is closed.
Finally, for $\mathbb{RP}^n$ to be a smooth manifold, it needs a smooth atlas.

Notation

Homogeneous coordinates on $\mathbb{RP}^n$ work as follows. We denote $pi(x_1,\ldots,x_{n+1})$ by $[x+1;\cdots;x_{n+1}]$.
Note also, for any i, and any $[x] \in \mathbb{RP}^n$ with nonzero x_i'th coordinate, $[x]$ can be represented by a unique equivalence class such that the ith entry in the homogeneous coordinates is 1.

Lemma 3. $\mathbb{RP}^n$ *admits a smooth structure.*

Proof.
Let $U_i = \{[x_1; \cdots ; x_{n+1}] \in \mathbb{RP}^n \mid x_i \neq 0\}$. This is open for $i = 1, \ldots, n+1$.
Note that $\bigcup_{i=1}^{n+1} U_i = \mathbb{RP}^n$, because not every coordinate can be zero.
If $x_i \neq 0$ (i.e. $[x] \in U_i$) then

$$[x_1; \cdots ; x_i; \cdots ; x_{n+1}] = [\frac{x_1}{x_i}; \cdots ; 1; \cdots ; \frac{x_{n+1}}{x_i}]$$

So, we can define a bijection by sending $[x]$ to the point $(\frac{x_1}{x_i}, \ldots, \hat{1}, \ldots, \frac{x_{n+1}}{x_i}) \in \mathbb{R}^n$.
Call this φ_i
This is a homeomorphism.

Claim. $\{(U_i, \varphi_i)\}_{i=1}^{n+1}$ *is a smooth atlas for* $\mathbb{RP}^n$

Proof. Let's check, on $\varphi_2(U_1 \cap U_2)$

$$\begin{aligned}
\varphi_1 \circ \varphi_2^{-1}(y_1, \ldots, y_n) &= \varphi_1\left([y_1; 1; y_2; \cdots ; y_n]\right) \\
&= \varphi_1\left([1; \frac{1}{y_1}; \frac{y_2}{y_1}; \cdots ; \frac{y_n}{y_1}]\right) \\
&= \left(\frac{1}{y_1}, \frac{y_2}{y_1}, \ldots, \frac{y_n}{y_1})\right)
\end{aligned}$$

In $\varphi_2(U_1 \cap U_2)$, neither the first nor the second coordinates are zero, so we don't run into any division by zero problems.
The rest can be checked similarly.

■

Hence, $\mathbb{RP}^n$ is a smooth manifold.

Example 0.10. $\mathbb{RP}^2 = S^2/\mathbb{Z}_2$, $x \mapsto -x$.

Lecture 4, 10/10/23

Chapter 2: Smooth maps

Smooth functions

Let M^n be a smooth manifold, and let $f : M \to \mathbb{R}$ be a function (not assumed to be continuous necessarily, though it will turn out to be). For any point p, there is a chart (U, φ) around that point. We can consider the composition

$$\varphi(U) \xrightarrow{\varphi^{-1}} M \xrightarrow{f} \mathbb{R}$$

which is a function from $\mathbb{R}^n \to \mathbb{R}$, to which we may apply the usual definition of smoothness at $\varphi(p)$.

Definition 0.7. A smooth function from a manifold M to $\mathbb{R}$ is a function which is smooth in the above sense at every point.

Example 0.11. Given a smooth $f : \mathbb{R}^{n+1} \to \mathbb{R}$, we can consider the restriction $f|_{S^n} : S^n \to \mathbb{R}$. This is also smooth.
To check, we consider the charts $(U_i^\pm, \varphi_i^\pm)_{i=1}^{n+1}$ the "standard" smooth structure on S^n, as defined earlier.
We have

$$f \circ (\varphi_1^+)^{-1}(y_1, \dots, y_n) = f\left(\sqrt{1 - \sum_{i=1}^n y_i^2}, y_1, \dots, y_n\right)$$

All of these things are smooth on their domains, so this is smooth. The rest can be checked similarly.

Remark

If $f : M \to \mathbb{R}$ is C^∞, then $f : M \to \mathbb{R}$ is continuous.
Note $f|_U = (f \circ \varphi^{-1}) \circ \varphi$. $f \circ \varphi^{-1}$ is C^∞ on $\mathbb{R}^n$, hence continuous, and φ is continuous.

Definition 0.8. Let M^n be a smooth manifold and (U, φ) a smooth chart. This sends U to $\varphi(U)$, and p to $(x_1(p), \dots, x_n(p))$.
Define $f_i : U \to \mathbb{R}$ by $p \mapsto x_i(p)$, that is $f_i = \pi_i \circ f$.
f_i is a smooth function defined on U.
We can see $f_i \circ \varphi^{-1}(x_1, \dots, x_n) = x_i \in C^\infty$
These are called local coordinates about a point p / on an open set U

Remark

If M^n is a smooth manifold and $U \subseteq M$, then U inherits the smooth structure from M. If the atlas for M is given by $\{(U_\alpha, \varphi_\alpha)\}$, then the atlas for U is simply given by $\{(U_\alpha \cap U, (\varphi_\alpha)|_{U_\alpha \cap U})\}$
In particular, if (U, φ) is a smooth chart, then U is a smooth manifold with a single chart.

Definition 0.9. We write $C^\infty(M)$ to mean the collection of smooth functions $f : M \to \mathbb{R}$
These come with a nice ring structure - we can add and multiply by scalars, making it into an infinite dimensional vector space. In fact, because you can multiply them, they form an $\mathbb{R}$-algebra.
These local coordinate functions can be extended to $C^\infty(M)$.

Smooth maps

Let M, N be smooth manifolds, and $F : M \to N$ a map (again, not assumed to be continuous).

Definition 0.10. At any point $p \in M$, there is a chart (U, φ) with $p \in U$. Similarly, there is a chart (V, ψ) on N such that $F(U) \subseteq V$.
We can consider the composition

$$\varphi(U) \xrightarrow{\varphi^{-1}} U \xrightarrow{F} F(U) \xrightarrow{\psi} \psi(F(U))$$

The composition $\psi \circ F \circ \varphi^{-1}$, defined on $\varphi(U \cap F^{-1}(V))$, is a function from $\mathbb{R}^m$ to $\mathbb{R}^n$, where m and n are the dimensions of M, N respectively. We may then check if this function is smooth. If it is, we say that F is smooth at the point p.
A map $F : M \to N$ is a smooth map if F is smooth in the above sense at every point p.

Lemma 4. *If $F : M \to N$ is C^∞, then F is continuous.*

Proof. **Remark**

Without the additional acquirement that $F(U) \subseteq V$ in the definition, this is false.
It will suffcie to check that for all $p \in M$, there exists an open U containing p such that $f|_U : U \to N$ is continuous.
Since F is smooth, there exist smooth charts (U, φ) of M at p, (V, ψ) of N at $F(p)$ such that

$$\hat{F} = \psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$$

is C^∞.
Therefore, $F|_U = \psi^{-1} \circ \hat{F} \circ \varphi$ is C^0, because everything in sight is also C^0. ■

Lecture 5, 10/12/23

Hint for homework problem:
If $F^*(C^\infty(N)) \subset C^\infty(M)$, then F is C^∞.

Example 0.12. Consider $F : (\mathbb{R}, \varphi = \mathrm{Id}) \to (\mathbb{R}, \psi(x) = x^5)$ given by $x \mapsto x^{\frac{1}{5}}$. This is C^∞.
Check: $\psi \circ F \circ \psi^{-1}(x) = \psi \circ F(x) = \psi(x^{\frac{1}{5}}) = x$.
In fact, F^{-1} is C^∞! i.e. F is a diffeomorphism.

Definition 0.11. A function $f : M \to N$ between two smooth manifolds is a diffeomorphism if F is a bijection, and both F and F^{-1} are C^∞.
In other words, it is an isomorphism in the category of smooth manifolds.

Question:

how many smooth structures on $\mathbb{R}^n$ are there *up to diffeomorphism?*
If $n \leq 3$, then there is a unique smooth structure *up to diffeomorphism*, a result due to Moise for $n = 3$, and Radon for $n \leq 2$.
When $n \geq 5$, there is a unique smooth structure *up to diffeomorphism*, a result due to Stalling.
For $n = 4$, there are uncountably many smooth structures *even up to diffeomorphism*! This is due to a result by Donaldson.
These are the so-called "exotic" $\mathbb{R}^4$s.

What about compact spaces?

The simplest example is S^n. This question can be put into a much bigger context.
The smooth category is a subcategory of the topological category, this question is getting at the difference between them. The homotopy category is a subcategory of Top.
The "holy grail" would be a complete classification of spaces up to homotopy equivalence/homeomorphism/diffeomorphism.

Poincaré conjecture:

Every closed n-dimensional topological manifold homotopic to S^n is actually homeomorphic to the sphere.
This is true for $n \geq 5$, famous work of Smale in 1966.
For $n = 4$, the answer is also yes, due to Freedman in 1986 (he used to be here!!!!!!!)
For $n = 3$ the answer is also yes, due to Perelman in 2006
For $n \leq 2$, the answer is yes.
Every one of these earned a fields medal!!!
What about in the smooth category?

Smooth Poincaré conjecture:

Every smooth manifold homeomorphic to S^n is actually diffeomorphic to S^n
Milnor, 1962: There exists a smooth structure on S^7 which is not diffeomorphic to the standard one. In fact, there are 28 of them up to diffeomorphism.
For each dimension n, we have a resolution to the smooth conjecture, but for $n = 4$ it is still open.
Remark
The way we defined a C^∞ structure, we can also define a C^k structure, where $k =\in \mathbb{N}$.
We may also define C^ω, which is real analytic, which is a stronger condition than smoothness.

C^0 is just topological manifolds.
However, every object of C^k is homoemorphic to an object of C^∞ for $k > 0$. But for C^0, there are topological manifolds which cannot be smoothed, and there are different smooth structures.
So, we sometimes refer to a smooth structure as a differentiable structure. So as long as we have a first derivative, we can somehow achieve a result like the Weirstrass approximation theorem.
The (somewhat more) precise way to say this is that we can topologize the set of C^1 maps between two C^1 manifolds, and the smooth maps are dense in this set.
This is all theorems of Whitney and Nash

Partitions of unity (P.O.U)

From now on, when we say "manifold," it is understood to mean a smooth manifold with some smooth structure.
The idea is that a manifold is locally Euclidean, but not globally. We would like a way to reduce global information to sums of local information, and partitions of unity allow us to do that.

Goal:

We want to write $1 = \sum_\alpha f_\alpha, f_\alpha \in C^\infty(M)$, and nonzero only in some neighborhood.
We will use so-called "bump functions."
We have two issues.

1. We need to establish the existence of bump functions

2. We need to make sense of this summation

We now address 1 by showing the existence of smooth bump functions in $\mathbb{R}^n$.
Consider $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(t) = \begin{cases} e^{-\frac{1}{t}}, & t > 0 \\ 0 & t \leq 0 \end{cases}$$

This is C^∞
We have a "cutoff" function $h : \mathbb{R} \to [0, 1]$, which is identically 1 on $(-\infty, 1]$, decreasing on $[1, 2]$, and identically zero on $[2, \infty)$.
We can define it by

$$h(t) = \frac{f(2-t)}{f(2-t) + f(t-1)}$$

We pass to $\mathbb{R}^n$, i.e. a $H : \mathbb{R}^n \to [0, 1] \in C^\infty(\mathbb{R}^n)$ such that

1. $H \equiv 1$ on $\overline{B_1(0)}$

2. $H \equiv 0$ outside $B_2(0)$.

We can simply define $H = h(|x|)$
The idea is that for any manifold M, and any point $p \in M$, there is a chart (U, φ) about p. We can normalize so that $\varphi(U) \supset B_2(0)$ and $\varphi(p) = 0$.
We can define $H \circ \varphi \in C^\infty(U)$. We want to upgrade it to something in $C^\infty(M)$. We can do this easily by setting it to be identically zero outside of U.

Corollary 0.1. *For any $p \in M$, for any open $U \ni p$, and for any $f \in C^\infty(U)$, there exists an open $p \in V \subseteq U$, and a smooth function $\tilde{f} \in C^\infty(M)$, such that $\tilde{f}|_V = f|_V$.*

Proof. Take $\tilde{f}$ to be $(H \circ \varphi)f$. $H \circ \varphi$ is identically 1 on $\varphi^{-1}(B_1(0))$

■

Lecture 6, 10/17/23

Notation: for $f \in C^0(M)$, we define the support of f as

$$\operatorname{supp} f \stackrel{\text{def}}{=} \overline{\{x \in M \mid f(x) \neq 0\}}$$

Theorem 0.2. *Let M^n be a smooth manifold, with $F \subset M$ closed, and $K \subset\subset M$, meaning a compact subset. Suppose that $K \cap F = \varnothing$. Then there is a $f \in C^\infty(M)$ such that $0 \leq f \leq 1$, $f \equiv 1$ on K, and $f \equiv 0$ on F.*

Proof. By assumption, $\underbrace{K}_{\text{compact}} \subset \overbrace{M \setminus F}^{\text{open}}$.
For all $p \in K$, there is a chart (U_p, φ_p) such that $\varphi_p(p) = 0$, $U_p \subset M \setminus F$, and $\varphi_p(U_p) \supset \overline{B_2(0)}$
Consider the open cover of K given by $\{\varphi_p^{-1}(B_1(0))\}_{p \in K}$. By compactness, this admits a finite subcover

$$\{\varphi_{p_1}^{-1}(B_1(0)), \varphi_{p_2}^{-1}(B_1(0)), \ldots, \varphi_{p_k}^{-1}(B_1(0))\}$$

Set $\mathcal{U}_i = U_{p_i}$, $\varphi_i = \varphi_{p_i}$, and

$$f(x) = 1 - \prod_{i=1}^{k} \big(1 - (H \circ \varphi_i)(x)\big) \in C^\infty(M)$$

For any $p \in K$, $p \in \varphi_j^{-1}(B_1(0))$ for some j, so $H \circ \varphi_j(p) = 1$, so $f(p) = 1 - 0 = 1$.
Finally, we must verify that $f \equiv 0$ on F. This happens because $U_i \subset M \setminus F$, so $H \circ \varphi_i$ vanishes on F.

■

Now it is time for partitions of unity. These are a tool to piece together local information to get global information.
We want $1 \equiv \sum_\alpha f_\alpha$ for $f_\alpha \in C^\infty(M)$, $\operatorname{supp} f_\alpha \subset U_\alpha$, and $\{U_\alpha\}_\alpha$ is an open cover of M. We need to worry about sumability.
We insist that for any point p, there are only finitely many α such that $f_\alpha(p) \neq 0$.

Definition 0.12. Let X be a topological space. A collection of subsets $\{S_\alpha\}$ is called locally finite if for any $p \in X$, there is a neighborhood $U \ni p$ such that $U \cap S_\alpha = \varnothing$ for all but finitely many α.

Example 0.13. If $\{\operatorname{supp} f_\alpha\}$ is locally finite, then for all p, there is a neighborhood $U \ni p$ such that U intersects only finitely many $\operatorname{supp} f_\alpha$. In other words, there are only finitely many f_α such that $f_\alpha(p) \neq 0$

Definition 0.13. Let M be a smooth manifold with open cover $\mathscr{U} = \{U_\alpha\}$. A partition of unity subordinate to $\mathscr{U}$ is a collection $\{\psi_\alpha \in C^\infty(M)\}$ such that

(i) $0 \leq \psi_\alpha \leq 1$

(ii) $\operatorname{supp} \psi_\alpha \subset U_\alpha$

(iii) $\{\operatorname{supp} \psi_\alpha\}$ is locally finite

(iv) $\sum_\alpha \psi_\alpha \equiv 1$

Theorem 0.3 (Existence of P.O.U)**.** *Given a manifold M and open cover $\mathscr{U} = \{U_\alpha\}$, there is a partition of unity subordinate to $\mathscr{U}$.*

Example 0.14 (Separation property)**.** For all $p \neq q$, there exists $f \in C^\infty(M)$ such that $f(p) = 1, f(q) = 0$.
Let $\mathscr{U} = \{U, V\}$, with $U = M \setminus \{p\}, V = M \setminus \{q\}$. If there are ψ_1, ψ_2 satisfying $i - iv$, then $f = \psi_2$ will do.

Proof. We now prove the theorem.
We will only do the case when M is compact. For all $p \in M$, there exists a chart (V_p, φ_p) at p such that $\varphi_p(p) = 0$ and $\varphi_p(V_p) \supset \overline{B_2(0)}$ and $V_p \subset U_{\alpha(p)}$.
Let $W_p = \varphi_p^{-1}(B_1(0)) \ni p$. Then $\{W_p\}_{p \in M}$ is an open cover of M. By compactness, there is a finite subcover $W_1 = W_{p_1}, \cdots, W_k = W_{p_k}$
Set $\varphi_i = \varphi_{p_i}$ and $f_i = H \circ \varphi_i \in C^\infty(M)$.
Then $f_i \equiv 1$ on W_i, $\operatorname{supp} f_i \subset V_i = V_{p_i} \subset U_{\alpha(p_i)}$.
Since $\{W_1, \ldots, W_k\}$ covers M, $\sum_{i=1}^k f_i \geq 1$.
So we can simply take $g_i = \frac{f_i}{\sum_{i=1}^k f_i} \in C^\infty(M)$, so $\sum_{i=1}^k g_i \equiv 1$.
Recall that $V_i \subset U_{\alpha(i)}$, $i = 1, \ldots, k$. Put $\psi_\alpha = \sum_{\alpha(i)=\alpha} g_i$.
This is in $C^\infty(M)$, and $\operatorname{supp} \psi_\alpha \subset U_{\alpha(i)=\alpha} \operatorname{supp} g_i \subset U_\alpha$

and $\sum_\alpha \psi_\alpha = \sum_i g_i \equiv 1$

■

If M is not compact, we have the following theorem.

Theorem 0.4. *A manifold is always paracompact, i.e. any open cover has a locally finite refinement. In fact, for any open cover $\mathscr{U} = \{U_\alpha\}$, there ix a countable open cover $\{V_i\}$ such that*

(i) $\{V_i\}$ *is locally finite*

(ii) $\{V_i\}$ *is a refinement of* $\mathscr{U}$*, i.e.* $V_i \subset U_{\alpha(i)}$

(iii) Each V_i *is a domain of a normalized chart.*

Proof. This follows from second countability.

■

Lecture 7, 10/19/23

Chapter 3: Tangent vectors and tangent spaces

Theree are two views of tangent spaces of $\mathbb{R}^n$: geometric or abstract.
For an $a \in \mathbb{R}^n$, we denote the tangent space at a as $T_a\mathbb{R}^n$. For $\mathbb{R}^n$, we identify it with its tangent space at any point.
Indeed, a tangent vector at a is a vector "based" at a, a pair (v, a), denoted v_a.
We can't do this on a manifold!
We will take a different viewpoint. Instead of thinking about directions, which represent tangent vectors, we're gonna look at directional derivatives.
Instead of a $v_a \in T_a\mathbb{R}^n$, we're gonna consider the directional derivative

$$D_{v_a} : C^\infty(\mathbb{R}^n) \to \mathbb{R}$$

This will act by

$$f \mapsto \frac{d}{dt}|_{t=0} f(a + tv)$$

This is the abstract viewpoint.
Now, we can't replicate the geometric viewpoint on a manifold, but we can replicate the abstract viewpoint.
The directional derivative satisfies the following:

- D_{v_a} is linear
- D_{v_a} satisfies the Leibniz rule. That is, given two smooth functions $f, g \in C^\infty(\mathbb{R}^n)$,

$$D_{v_a}(fg) = f(a)D_{v_a}g + g(a)D_{v_a}f$$

Definition 0.14. A <u>derivation at a</u> is a linear map $X : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ satisfying the Leibniz rule:

$$X(fg) = f(a)(Xg) + g(a)(Xf)$$

Example 0.15. D_{v_a} is a derivation at a

<u>**Notation**</u>

$$\widetilde{T_a\mathbb{R}} = \{X \mid X \text{ derivation at } a\}$$

Note that $\widetilde{T_a\mathbb{R}}$ is a vector space

Example 0.16. Let $v_a = (e_i, a)$. Then $D_{v_a}(f) = \frac{d}{dt}|_{t=0} f(a + te_i) = \dfrac{\partial f}{\partial x_i}$.

So we write $D_{v_a} = \dfrac{\partial}{\partial x_i}|_a$ for $v_a = (e_i, a)$

In general $v = (v_1, \dots, v_n)$,

$$D_{v_a} f = \frac{d}{dt}|_{t=0} f(a + tv) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a) v_i$$

Proposition 2. *The map $T_a\mathbb{R}^n \to \widetilde{T_a\mathbb{R}^n}$ given by $v_a \mapsto D_{v_a}$ is a linear isomorphism.*

Proof. <u>Remark:</u>
We can identify the geometric tangent space $T_a\mathbb{R}^n$ and the abstract tangent space $\widetilde{T_a\mathbb{R}^n}$. We will do that in this class. Now for the proof

(i) The map is linear: $(v_a + w_a) = (v + w)_a$, and $D_{v_a} + D_{w_a} = D_{(v+w)_a}$

(ii) Injective: Let $v_a \in T_a\mathbb{R}^n$ be such that $D_{v_a} = 0$, i.e. $D_{v_a} f = 0$. But if $v_a = (v_1, \dots, v_n)_a$, then $D_{v_a} f = \sum_{i=1}^n \dfrac{\partial f}{\partial x_i}(a) v_i$ for all $f \in C^\infty(\mathbb{R}^n)$. So setting $f(x) = x_i$, we get that $v_i = 0$ so $v = 0$, i.e. v_a is the zero vector.

(iii) Surjective: for any $X : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ derivation at a, there exists some $v_a \in T_a\mathbb{R}^n$ so that $X = D_{v_a}$, i.e. $Xf = D_{v_a} f$ for all $f \in C^\infty(\mathbb{R}^n)$. We will need two lemmas

Lemma 5. 1. *If f is a constant function, then $Xf = 0$.*

2. *If $f, g \in C^\infty(\mathbb{R}^n)$ with $f(a) = g(a) = 0$, then $X(fg) = 0$*

Proof. **1.** First let $f \equiv 1$.

$$\begin{aligned} X(1) &= X(1 \cdot 1) \\ &= 1 \cdot X(1) + 1 \cdot X(1) \\ &= 2X(1) \end{aligned}$$

So $X(1)$ must be zero.

In general if $f \equiv c$ for some constant c, because X is linear we have $X(c) = c \cdot X(1) = 0$.

2. This is a simple application of the Leibniz rule

■

Lemma 6 (Taylor expansion with remainder)**.** *For all $f \in C^\infty(\mathbb{R}^n)$, $f(x) = f(a) + \sum_{i=1}^n g_i(x)(x_i - a_i)$ where $g_i \in C^\infty(\mathbb{R}^n)$ and $g_i(a) = \dfrac{\partial f}{\partial x_i}(a)$*

Proof. By FTC,

$$\begin{aligned} f(x) - f(a) &= \int_0^1 \frac{d}{dt}(f(a + t(x-a)))\, dta \\ &= \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(a + t(x-a)) \cdot (x_i - a_i) \\ &= \sum_{i=1}^n (x_i - a_i) \underbrace{\int_0^1 \frac{\partial f}{\partial x_i}(a + t(x-a))\, dt}_{g_i(x)} \end{aligned}$$

■

Back to the proof of surjectivity:
Given $X \in T_a\mathbb{R}^n$ we want to find $v_a \in T_a\mathbb{R}^n$ such that $X = D_{v_a}$, i.e. $Xf = D_{v_a}f$ for all $f \in C^\infty(\mathbb{R})$
But the second lemma reduces arbitrary smooth function to

$$f = \underbrace{f(a)}_{\text{constant}} + \sum_{i=1}^n g_i(x)(x_i - a_i)$$

By linearity,

$$\begin{aligned} Xf &= \underbrace{X(f(a))}_{=0} + \sum_{i=1}^{n} X(g_i(x_i - a_i)) \\ &= \sum_{i=1}^{n} \left(X(g_i) \underbrace{(x_i - a_i)|_a}_{=0} + g_i(a) X(x_i - a_i) \right) \\ &= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a) \underbrace{X(x_i - a_i)}_{\text{scalar}} \end{aligned}$$

The equation $Xf = D_{v_a} f$ thus becomes

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a) X(x_i - a_i) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} a(v_i)$$

To make this true, we can simply pick $v_i = X(x_i - a_i)$, and the equation will always hold, no matter what f is.

■

Upshot

We have $T_a\mathbb{R}^n \cong \widetilde{T_a\mathbb{R}^n}$
We identify $(e_i)_a$ with $\frac{\partial}{\partial x_i}|_a (= D_{(e_i)_a})$
So $(v_1, \ldots, v_n)$ is identified with $\sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}|_a$
We will now define the latter on manifolds.

Lecture 8, 10/24/23

Definition 0.15. Let M^n be a smooth manifold. For $p \in M$, a derivation at p is a linear map

$$X : C^\infty(M) \to \mathbb{R}$$

satisfying the Leibniz rule at p, meaning that

$$X(fg) = f(p)X(g) + g(p)X(f)$$

Definition 0.16. The tangent space of M at p is the vector space of all derivations at p, denoted T_pM. Members of this space will be called tangent vectors.

Example 0.17. Let $\varepsilon > 0$, and consider $c : (-\varepsilon, \varepsilon) \to M^n$. If this is C^∞ it's called a curve.
Let $p = c(0)$.

Definition 0.17. We define $\dot{c}(0) \in T_pM$ by defining how it acts on $C^\infty(M)$ as follows.

$$\dot{c}(0)(f) = \frac{d}{dt}|_{t=0} f(c(t))$$

Remark
T_pM is a vector space over the reals.
It is really easy from here to define the pushforward/differential of a map.
Let M, N be smooth manifolds, and let $F : M \to N$ be smooth. Let $p \in M$. Then F "naturally" induces a linear map $F_* : T_pM \to T_{F(p)}N$ as follows.
Let $X \in T_pM$. Then $F_*(X) \in T_{F(p)}N$ acts on $C^\infty(N)$ by

$$F_*(X)(f) = X(f \circ F)$$

(check that this defines a derivation).
We also denote this as dF.

Proposition 3 (Basic Properties)**.**

1. *F_* is linear*

2. *If $F : M \to N$ and $G : N \to P$, then $(G \circ F)_* = G_* \circ F_*$. That is, $_*$ is **functorial!***

3. $\mathrm{Id}_M : M \to M$ *gives* $(\mathrm{Id}_M)_{*,p} = \mathrm{Id}_{T_pM}$

4. *If F is a diffeomorphism, then $F_* : T_pM \to T_{F(p)}N$ is a linear isomorphism.*

Proof.
4 follows from 1-3. If $F^{-1} \circ F = \mathrm{Id}_M$, then $(F^{-1} \circ F)_* = \mathrm{Id}_{T_pM}$. But this is also $F_*^{-1} \circ F_*$, so for their composition to be the identity (and the same argument shows the reverse composition is the identity on N), both have to be isomorphisms.
We now prove 1. Think
We now prove 2. We claim that.

$$(G \circ F)_*(X) = (G_* \circ F_*)(X)$$

We check that

$$(G \circ F)_*(X)f = X(f \circ (G \circ F))$$

and

$$(G_* \circ F_*)(X)f = G_*(F_*(X))f = F_*(X)(f \circ G) = X((f \circ G) \circ F)$$

∎

But what is T_pM? In particular, is it local? That is, does it depend only on info in a neighborhood of p?

Proposition 4. *Let $X : C^\infty(M) \to \mathbb{R}$ be a derivation at p, and $f, g \in C^\infty(M)$. If there is an open $U \ni p$ such that $f|_U = g|_U$ then $Xf = Xg$.*

Proof. We want to show $X(f-g) = 0$.
Let $h = f - g \in C^\infty(M)$. Then $h|_U \equiv 0$.
$p \in U$, so there is a $\chi \in C^\infty(M)$ such that $\chi(p) = 0$ and $\chi \equiv 1$ outside U. Note that $h = \chi h$.
This implies $X(h) = X(\chi h) = \chi(p)Xh + h(p)X\chi$. $\chi(p) = 0$, and h vanishes at p, so $X(h) = 0$.

■

Remark
Another way to get the "locality" is to define a tangent vector at p to be a derivation on the space of "germs" of C^∞ functions defined only near p.

Corollary 0.5 (Locality)**.** *Let $U \subseteq M$ be open. Then for any $p \in U$, $T_pM \cong T_pU$.*

Proof. Note that we have the canonical inclusion $i : U \to M$ which is a smooth map.
Therefore we have a linear map $i_* : T_pU \to T_pM$
To show this is a linear isomorphism, we have to construct the inverse by hand.
We define

$$\sigma : T_pM \to T_pU$$

as follows. For $X \in T_pM$, $f \in C^\infty(U)$, $\sigma(X)f = X(\chi f)$, where χ is a bump function at p. σ is well-defined by previous lemma.
We continue the proof next time.

■

Lecture 9, 10/26/23

Proposition 5. *Let M^n be a smooth $n-manifold$. Then for any $p \in M$,* $\dim T_pM = \dim M = n$
In fact, if (U, φ) is a smooth chart, then

$$E_i = \varphi_*^{-1}\left(\frac{\partial}{\partial x_i}\right), i = 1, \ldots, n$$

is a basis of T_pM.

Proof. By locality, for any open $U \ni p$, $T_pM \cong T_pU$. By pushforward via φ, this is isomorphic to $T_{\varphi(p)}\varphi(U)$. But $\varphi(U) \subseteq \mathbb{R}^n$ is open, so again by locality $T_{\varphi(p)}\varphi(U) \cong T_{\varphi(p)}\mathbb{R}^n$.
Now, recall that a basis for $T_{\varphi(p)}\mathbb{R}^n$ consists of

$$\frac{\partial}{\partial x_i}|_{\varphi(p)}, i = 1, \ldots, n$$

■

Now, let $F : M^n \to N^m$ be smooth.
If (U, φ) is a chart at p, we have a basis for T_pM consisting of

$$E_i = (\varphi^{-1})_* \left(\frac{\partial}{\partial x_i}\right), \, i = 1, \dots, n$$

Similarly, if (V, ψ) is a chart at $F(p)$, we have a basis for $T_{F(p)}N$ consisting of

$$\tilde{E}_j = (\psi^{-1})_* \left(\frac{\partial}{\partial y_j}\right), \, i = 1, \dots, m$$

Proposition 6. *With respect to these bases, the matrix of $F_* : T_pM \to T_{F(p)}N$ is given by the jacobian matrix*

$$\left(\frac{\partial \hat{F}_j}{\partial x_i}\right), \, \hat{F} = \psi \circ F \circ \varphi^{-1}$$

Proof. We need to express $F_*(E_i)$ as a linear combination of the $\tilde{E}_j$s. But by definition,

$$\begin{aligned} F_*(E_i) &= F_*\left((\varphi^{-1})_*\left(\frac{\partial}{\partial x_i}\right)\right) \\ &= (F \circ \varphi^{-1})_* \left(\frac{\partial}{\partial x_j}\right) \\ &= (\psi^{-1} \circ \psi)_* \circ (F \circ \varphi^{-1})_* \\ &= (\psi^{-1})_* \circ (\overbrace{(\psi \circ F \circ \varphi^{-1})}^{\hat{F}})_* \left(\frac{\partial}{\partial x_i}\right) \end{aligned}$$

Claim.

$$\hat{F}_* \left(\frac{\partial}{\partial x_i}\right) = \sum_{j=1}^{m} \frac{\partial \hat{F}_j}{\partial x_i} \frac{\partial}{\partial y_j}$$

Proof. We put it off for a little bit, but, granted the claim, we have that

$$
\begin{aligned}
F_*(E_i) &= (\psi^{-1})_* \left(\sum_{j=1}^{m} \frac{\partial \hat{F}_j}{\partial x_i} \frac{\partial}{\partial y_j} \right) \\
&= \sum_{j=1}^{M} \frac{\partial \hat{F}_j}{\partial x_i} \underbrace{(\psi^{-1})_* \left(\frac{\partial}{\partial y_j} \right)}_{\tilde{E}_j}
\end{aligned}
$$

To establish the claim, we compute, for all $f \in C^\infty(\psi(V))$,

$$
\begin{aligned}
\hat{F}_* \left(\frac{\partial}{\partial x_i} \right) f &= \frac{\partial}{\partial x_i} (f \circ \hat{F}) \\
&= \frac{\partial}{\partial x_i} f(\hat{F}_1(x_1, \dots, x_n), \cdots, \hat{F}_m(x_1, \dots, x_n)) \\
&= \sum_{j=1}^{m} \frac{\partial f}{\partial y_j} \frac{\partial \hat{F}_j}{\partial x_i} \\
&= \left(\sum_{j=1}^{m} \frac{\partial \hat{F}_j}{\partial x_i} \frac{\partial}{\partial y_j} \right) f
\end{aligned}
$$

■

Corollary 0.6. *Suppose $p \in M$ and $(U, \varphi), (V, \psi)$ smooth charts at p. We have two bases for T_pM,*

$$
\begin{aligned}
E_i &= (\varphi^{-1})_* \left(\frac{\partial}{\partial x_i} \right)_* \\
\tilde{E}_i &= (\psi^{-1})_* \left(\frac{\partial}{\partial x_i} \right)
\end{aligned}
$$

These are related by

$$
E_i = \sum_{j=1}^{n} \frac{\partial (\psi \circ \varphi^{-1})_j}{\partial x_i} \tilde{E}_j
$$

Proof. Apply $F = \mathrm{Id}_M$ for

$$
E_i = \sum_{j=1}^{n} \frac{\partial (\psi \circ F \circ \varphi^{-1})_j}{\partial x_i} \tilde{E}_j
$$

■

Going back to curves, let $c : (-\varepsilon, \varepsilon_{\to} M^n$ a curve at p, i.e. $c(0) = p$.
Recall $\dot{c}(0) \in T_pM$, $\dot{c}(0) = c_*(\frac{d}{dt}|_{t=0}) \in T_pM$
Fact All tangent vectors are of this form.
In the homework, we want to show that T_pS^n is naturally isomorphic to the hyperplane perpindicular to S^n at p.
Use the canonical inclusion $i : S^n \to \mathbb{R}^n$, giving us an induced map $i_*(T_pS^n) \to T_p\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$.

Claim. *i_* is injective. Therefore we can identify T_pS^n with its image $i_*(T_pS^n) \subset T_p\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$.*

Proof. It is tricky to do by definition. Alternatively, note that $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to S^n$ given by $x \mapsto \frac{x}{|x|}$ is a smooth, $\pi \circ i = \mathrm{Id}$, and once we have that, $\pi_* \circ i_* = \mathrm{Id}$, so i_* is injective.

■

To see that $i_*(T_pS^n) \cong$hyperplane perpendicular to p, we have

$$i_*(\dot{c}(0)) = (i \dot{\circ} c)(0)$$

$$\tilde{c} = i \circ c(t) = (c_1(t), \dots, c_{n+1}(t))$$

$\dot{\tilde{c}}(0) = ?$

Tangent bundles

Let M be a manifold.
We have the tangent bundle $TM = \coprod_{p \in M} T_pM$ as a set. Elements have the form (p, v), with $v \in T_pM$
Goal: We want to put a topology and smooth structure on TM such that $\pi : TM \to M$, sending $(p, v) \to p$, is smooth.
We will first put a chart on it.
Let $\{(U_\alpha, \varphi_\alpha)\}$ be a smooth atlas of M. We will construct a smooth atlas $\{(\tilde{U}_\alpha = \pi^{-1}(U_\alpha), \tilde{\varphi}_\alpha)\}$ for TM.
We define $\tilde{\varphi}_\alpha : \pi^{-1}(U_\alpha) \to \varphi_\alpha(U_\alpha) \times \mathbb{R}^n$,

$$(p, v) \mapsto (\varphi_\alpha(p), (v^1, \dots, v^n))$$

where $v = \sum_{i=1}^n v^i E_i$, with E_i the standard basis, $E_i = (\varphi_\alpha^{-1})_* \left(\frac{\partial}{\partial x_i}\right)$
We have to check smooth compatibility. We have

$$\begin{aligned}(\tilde{\varphi}_\beta \circ \tilde{\varphi}_\alpha{}^{-1})(x_1, \dots, x_n, v^1, \dots, v^n) &= \tilde{\varphi}_\beta(\varphi_\alpha^{-1}(x), \sum_{i=1}^n v^i E_i \\ &= ((\varphi_\beta \circ \varphi_\alpha^{-1})(x), ?)\end{aligned}$$

To figure out what ? should be, we need an expression of $\sum_{i=1}^{n} v^i E_i$ in the basis $\tilde{E}_i = (\varphi_\beta^{-1})_* \left(\dfrac{\partial}{\partial x_i}\right)$.
But by an earlier proposition,

$$E_i = \sum_{j=1}^{n} \frac{\partial(\varphi_\beta \circ \varphi_\alpha^{-1})_j}{\partial x_i} \tilde{E}_j = \sum_{i=1}^{n}\sum_{j=1}^{n} v^i \frac{\partial(\varphi_\beta \circ \varphi_\alpha^{-1})_j}{\partial x_j} \tilde{E}_j = \sum_{j=1}^{n} \underbrace{\left(\sum_{i=1}^{n} v^i \frac{\partial(\varphi_\beta \circ \varphi_\alpha^{-1})_j}{\partial x_i}\right)}_{\text{use as?}} \tilde{E}_j$$

Lecture 10, 10/31/23

Chapter 4: Submersions, Immersions, & embeddings

Let $F : M^n \to N^m$ be a smooth map of smooth manifolds. For $p \in M$, we have the differential $dF_p = F_{*,p} : T_pM \to T_{F(p)}N$
The idea is that this is supposed to be a linear approximation to nonlinear objects. We explore in what sense this is true.

Definition 0.18.

(a) We say F is an immersion at p if $F_{*,p} : T_pM \to T_{F(p)}N$ is injective. F is an immersion if it is an immersion at all points.

(b) We say F is a submersion at p if $F_{*,p} : T_pM \to T_{F(p)}N$ is surjective. F is a submersion if it is a submersion at all points.

Remark
If F is an immersion/submersion at p, then F is so in a neighborhood of p.
The matrix of $F_{*,p}$ is the Jacobian of $\hat{F} = \psi \circ F \circ \varphi^{-1}$. , which is $\left(\dfrac{\partial \hat{F}_j}{\partial x_i}\right)_{n\times m}$.
If $F_{*,p}$ is injective, then the det of $n \times n$ submatrix is nonzero. If $F_{*,p}$ is surjective, then the det of $m \times m$ submatrix is nonzero. The reverse direction is true for both. Both of these determinants vary continuously, so the remark is proven.

Definition 0.19. $\operatorname{rank}_p F = \operatorname{rank}(F_{*,p} : T_pM \to T_{F(p)}N) = \dim \operatorname{Im} F_{*,p}$. This is less than or equal to $\min(n, m)$, which is the rank of the Jacobian of $\hat{F}$ at p.
Remark
F an immersion at $p \implies \operatorname{rank}_p F = n \leq m$
F a submersion at $p \implies m \leq n$

Example 0.18. Consider $c : (-1, 1) \to \mathbb{R}^3$ given by $c(t) = (c_1(t), c_2(t), c_3(t))$
This is an immersion if $\dot{c} \neq 0$. We can think of $\dot{c}(t)$ as $c_* \left(\frac{d}{dt}\right)$.
If c is an immersion, we call it a regular curve. (The image of c looks "smooth" in $\mathbb{R}^3$)
Consider

$$c(t) = \begin{cases} (e^{-\frac{1}{t}}, e^{-\frac{1}{t}}, 0) & t > 0 \\ 0 & t = 0 \\ (e^{\frac{1}{t}}, -e^{\frac{1}{t}}, 0) & t < 0 \end{cases}$$

This is C^∞. But $\dot{c}(0) = 0$, so this is not an immersion.

Example 0.19. If we have $\sigma : U \to \mathbb{R}^3$ where $U \subseteq \mathbb{R}^2$ is C^∞, this is the same as σ being a regular surface.

Theorem 0.7 (Rank Theorem)**.** *Let $F : M^n \to N^m$ be a C^∞ function with constant rank, i.e.* $\operatorname{rank}_p F = k$ *for all $p \in M$.*
Then for all $p \in M$, there are charts (U, φ) at p and (V, ψ) at $F(p)$ such that $F(U) \subseteq V$, and

$$\hat{F} = \psi \circ F \circ \varphi^{-1}$$

sends $(x_1, \ldots, x_n) \in \varphi(U) \subseteq \mathbb{R}^n$ to $(x_1, \ldots, x_k, 0, \ldots, 0) \in \psi(V) \subseteq \mathbb{R}^m$

Example 0.20. Consider $F : \mathbb{R}^2 \to \mathbb{R}^3$ given by $(x, y) \mapsto (x, y, f(x, y))$, with $f \in C^\infty(\mathbb{R}^2)$.

Then $\operatorname{rank} F = \operatorname{rank} dF = \operatorname{rank} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} = 2$, which is constant.

Put $\psi : \mathbb{R}^3 \to \mathbb{R}^3$ by $(x, y, z) \mapsto (x, y, z - f(x, y)) = (u, v, w)$. This is a diffeomorphism!
Then (u, v, w) will be our new coordinate system.
$\varphi = \mathrm{Id}$ for $\mathbb{R}^2$, so

$$\psi \circ F(x, y) = \psi(x, y, f(x, y)) = (x, y, 0)$$

Exactly as the rank theorem predicted.
In general, we'll use local diffeo's to modify our charts. This is easier to check than being a diffeomorphism.

Theorem 0.8 (Inverse Function Theorem)**.** *Let $U \subseteq \mathbb{R}^n$, and $G : U \to \mathbb{R}^n$ be C^∞. Let $p \in U$ such that $dG_p = \left(\frac{\partial G}{\partial x_i}\right)_{n \times n}(p)$ is invertible. Then G is a local diffeomorphism, i.e. there is an open neighborhood $p \in V \subseteq U$ such that $\tilde{V} = G(V) \subseteq \mathbb{R}^n$ is open and $G : V \to \tilde{V}$ is a diffeomorphism.*

Proof. We first prove the rank theorem.
Because the function is C^∞, there are charts $(\tilde{U}, \tilde{\varphi})$ at p, $(\tilde{V}, \tilde{\psi})$ at $F(p)$, $F(\tilde{U}) \subseteq \tilde{V}$, such that

$$\tilde{F} = \tilde{\psi} \circ F \circ \tilde{\varphi}^{-1} : \varphi(\tilde{U}) \to \psi(\tilde{V})$$

with $\operatorname{rank} \tilde{F} = k$.
For simplicity, assume $n = m = 2$ and $k = 1$.
$\tilde{F} : (x, y) \mapsto \tilde{F}(x, y) = (\tilde{F}^1(x, y), \tilde{F}^2(x, y))$.

$$1 \equiv \operatorname{rank} \tilde{F} = \operatorname{rank} \begin{pmatrix} \dfrac{\partial \tilde{F}^1}{\partial x} & \dfrac{\partial \tilde{F}^1}{\partial y} \\ \dfrac{\partial \tilde{F}^2}{\partial x} & \dfrac{\partial \tilde{F}^2}{\partial y} \end{pmatrix}$$

<u>Goal:</u>
We want to modify $(\tilde{U}, \tilde{\varphi})$ by diffeos so that

$$\psi \circ F \circ \varphi^{-1}(x, y) = (x, 0)$$

Where $\psi = \tilde{\psi} \circ G, \varphi = \tilde{\varphi} \circ H$, with G, H local diffeomorphisms.

<u>Step 1</u>

$G : \mathbb{R}^2 \to \mathbb{R}^2$ given by $(x, y) \mapsto (\tilde{F}^1(x, y), y)$
Without loss of generality, $\dfrac{\partial \tilde{F}^1}{\partial x}(0) \neq 0$.
To check G is a local diffeomorphism at 0, by the inverse function theorem, we just need to show that

$$dG_0 = \begin{pmatrix} \dfrac{\partial \tilde{F}^1}{\partial x} & \dfrac{\partial \tilde{F}^1}{\partial y} \\ 0 & 1 \end{pmatrix}$$

is indeed invertible, so G is a local diffeomorphism.
Recall $G(x, y) = (x^1, x^2) = (\tilde{F}^1(x, y), y)$
Check: $\tilde{F} \circ G^{-1} = \tilde{\psi} \circ F \circ \underbrace{\tilde{\varphi}^{-1} \circ G^{-1}}_{(G \circ \tilde{\varphi})^{-1}}$
Simplifies, i.e. $\tilde{F} \circ G^{-1}(x^1, x^2) = \tilde{F}(x, y) = (\tilde{F}^1(x, y), \tilde{F}^2(x, y)) = (x^1, \overline{\tilde{F}^2}(x^1, x^2))$

<u>Step 2</u>

jby6,m $\operatorname{rank} \tilde{F} \equiv 1$ so $\overline{\tilde{F}}(x^1, x^2)$ will depend only on the first variable.

Step 3

$H : (x^1, x^2) \to (x^1, x^2 - f(x'))$.

Lecture 11, 11/2/23

Consequences of rank theorem

- If F is an immersion, then locally, $\hat{F}(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0)$, so it is locally injective.
- If F is a submersion, then locally $\hat{F}(x_1, \ldots, x_n) = (x_1, \ldots, x_m)$, so it is locally surjective.

  In fact, F must be an open map. In particular, if N is connected, then F is onto.

Embeddings

Definition 0.20. A C^∞ map $F : M \to N$ is an embedding if

1. F is a 1-1 immersion

2. If we give $F(M) \subset N$ the subspace topology, then $F : M \to F(M)$ is a homeomorphism, i.e. $F^{-1} : F(M) \to M$ is continuous.

Question: when will a 1-1 immersion be an embedding?

Proposition 7. *Let $F : M \to N$ be a 1-1 immersion. Then F is an embedding if any of the following holds:*

(a) F is an open or closed map

(b) F is a proper map, meaning the preimage of any compact set is compact.

(c) M is compact.

Proof. Clearly, c) follows from b).
It will suffice to show $F^{-1} : F(M) \to M$ is continuous.
a) is trivial by definition.
So we wil show b implies a. We will show $F : M \to N$ is a closed map.
Let $C \subseteq M$ be a closed set. We want to show $F(C)$ is closed. Let q be a limit point of $F(C)$
$q \in N$, and there is $U \ni q$ such that $\overline{U}$ is compact.
Because F i proper, $F^{-1}(\overline{U})$ is compact, so $C \cap F^{-1}(U)$ is also compact.

So $F(C \cap F^{-1}(U))$ is also compact. But this is equal to $F(C) \cap \overline{U}$, so $Q \in F(C) \cap \overline{U}$. So $q \in F(C)$.

∎

Remark
If $F : M \to N$ is a 1-1 immersion and M is compact, then F is an embedding.

Example 0.21. $i : S^n \hookrightarrow \mathbb{R}^{n+1}$ is an embedding

Theorem 0.9. *Let $F : M \to N$ be an immersion.*
Then F is locally an embedding, i.e. for all $p \in M$, there is an open $U \ni p$ such that $F|_U : U \to N$ is an embedding.

Example 0.22. Consider $\gamma : (-\pi, \pi) \to \mathbb{R}^2$ given by $\gamma(t) = (\sin(2t), \sin(t))$. This is a 1-1 immersion but not an embedding.
However, it is locally an embedding.

Proof. By the rank theorem, there are charts (U, φ) at p and (V, ψ) at $F(P)$ such that $F(U) \subset V$ and $\hat{F} = \psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is given by $x \mapsto (x, 0)$, which is a 1-1 immersion.
By shrinking U if necessary, we can assume that $\overline{U}$ is compact so $F : U \to F(U)$ is a closed map.

∎

Example 0.23. Consider $F : \mathbb{R} \to T^2$, the torus, where we think of T^2 as $\mathbb{R}^2/\mathbb{Z}^2$

Lecture 12, 11/7/23

Chapter 5: (Embedded) Submanifolds

One way manifolds arise is as the images of "nice" maps.

Definition 0.21. Given a smooth manifold M^m, a subset $S \subseteq M$ is an embedded/regular submanifold of dimension k if for all $p \in S$, there exists a chart (U, φ) of M at p, such that

1. $\varphi(p) = 0 \in \mathbb{R}^m$

2. $\varphi(U \cap S) = \varphi(U) \cap \{\mathbb{R}^k \times \{0\}\}$, where we view 0 as an element of $\mathbb{R}^{m-k}$

Question: If S a manifold?
Yes.

1. S has subspace topology, so is automatically Hausdorff and second countable.

2. S has a smooth atlas given by $\{(U \cap S, \pi \circ \varphi|_S)\}$, e.g. $(\pi \circ \psi) \circ (\pi \circ \varphi)^{-1} = \pi \circ (\psi \circ \varphi^{-1}) \circ i$ which is smooth.

The upshot is that S inherits both the topology and the smooth structure of the ambient manifold, which makes itself a smooth manifold.
Remark:
If $S \subseteq M$ is an embedded submanifold, then $i : S \hookrightarrow M$ is an embedding.

First Goal:

Show that the image of an embedding is an embedded submanifold.

Proposition 8. *If $F : N^n \to M^m$ is an embedding, then $S = F(N)$ is an embedded submanifold of M, and $F : N \to S$ is a diffeomorphism.*

Proof. F is an immersion, so for all $p \in S = F(N)$, there are charts (U, φ) of N at p and (V, ψ) of M at $F(p)$. Since F is an embedding, $F(U)$ is open in the subspace topology.
But $F(U) = F(N) \cap W$, $W \subseteq M$ open, so without loss of generality, $W \subseteq V$, so $(F(U) \subseteq V)$.
We claim that $(W, \psi|_W)$ is the desired ambient chart.
i.e. $\psi(S \cap W) = \psi(W) \cap (\mathbb{R}^k \times \{0\})$
Indeed, for all $q \in S \cap W = F(N) \cap W = F(U)$, $q = F(p)$ for some $p \in U$. So

$$\begin{aligned} \psi(q) &= \psi(F(p)) \\ &= \hat{F}(\varphi(p)) \in \mathbb{R}^k \times \{0\} \\ &= \psi \circ F \circ \varphi^{-1} \end{aligned}$$

■

So embedded submanifolds are exactly the images of embeddings.
Alternatively, we can view them as being defined by level sets of nice maps.

Theorem 0.10 (Constant Rank Level Set Theorem)**.** *Let $F : N^n \to M^m$ be C^∞ with rank $F \equiv k$. Then for all $q \in M$, $F^{-1}(q)$ is am embedded submanifold of N with dimension $n - k$.*

Proof. Another application of Rank theorem. Let $S = F^{-1}(q) \subseteq N$. For all $p \in S$, there are charts (U, φ) of N at p, (V, ψ) of M at $F(p)$ such that $F(U) \subseteq V$ and $\hat{F} = \psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ sends $(x_1, \ldots, x_n)$ to $(x_1, \ldots, x_k, 0, \ldots, 0)$.
We claim (U, φ) is an ambient chart, i.e. that $\varphi(S \cap U) = \varphi(U) \cap$slice.
Indeed, for any $\tilde{p} \in S \cap U$, $\varphi(\tilde{p}) = (\underbrace{0, \ldots, 0}_{k}, *, \ldots, *)$

■

Lecture 13

Missed it

Lecture 14, 11/14/23

Definition 0.22. A vector field on an open set $U \subseteq M$ is a smooth map $X : U \to TU$ such that $\pi \circ X = \mathrm{Id}_U$. I.e. $X(p) = (p, X_p)$ with $X_p \in T_pM$.
If (U, φ) is a chart, a basis for T_pU is given by $E_i = (\varphi^{-1})_* \left(\frac{\partial}{\partial x_i}\right)$. We will abuse notation by referring to E_i as simply $\frac{\partial}{\partial x_i}$.
We can write

$$X_p = \sum_{i=1}^{n} X_i(p) \frac{\partial}{\partial x_i}$$

Then X is smooth if and only if $X_i \in C^\infty(U)$ for each i.

Proposition 9. *X is a vector field on M if and only if $X : C^\infty(M) \to C^\infty(M)$ given by $f \mapsto Xf$, where Xf is given by $Xf(P) = X_p f$, is a derivation at p, i.e.*

1. *Linear:* $X(c_1 f + c_2 g) = c_1 Xf + c_2 Xg$

2. *Leibniz:* $X(fg) = (Xf)g + f(Xg)$

Proof. We start with " $\implies$ ". Given X a vector field on M, we need to check that

1. $(Xf)(p) = X_p f$ defines a smooth function. Check in a coordinate chart (U, φ). If X is a vector field, then

$$X_p = \sum_{i=1}^{n} X_i(p) \frac{\partial}{\partial x_i}$$

with $X_i(p) \in C^\infty(U)$. So $X_p f = \sum_{i=1}^{n} X_i(p) \frac{\partial f}{\partial x_i}$, which is the sum of products of smooth things and hence smooth.

2. We need to show X is a derivation at p, which is obviously true.

Now for the opposite direction. That is, we want to show that if $X : C^\infty(M) \to C^\infty(M)$ is a derivation, then it defines a vector field.
I wasn't paying attention, sorry.

∎

We denote the st of all vector fields X on M as $\mathfrak{X}(M)$

Let $F : M \to N$ be smooth, and let $X \in \mathfrak{X}(M)$. Then for each $p \in M$, we have a map $F_{*,p} : T_pM \to T_{F(p)}N$ which sends X_p to $F_{*,p}(X_p)$.
Question: Will we get a vector field on N?
No, because there are various issues:

1. F may not be surjective

2. F may not be injective

3. Even if F is bijective, the assignment might not be smooth.

Instead, we will assume there is a vector field on N to start with.

Definition 0.23. Let $F : M \to N$ be C^∞, $X \in \mathfrak{X}(M), Y \in \mathfrak{X}(N)$.
We say X, Y are <u>F-related</u> if $F_*(X_p) = Y_{F(p)}$ for all $p \in M$. We abuse notation by saying that $Y = F_*(X)$.
This means the diagram commutes:

$$\begin{tikzcd} C^\infty(N) \arrow[r, "Y"] \arrow[d, "F^*"] & C^\infty(N) \arrow[d, "F^*"] \\ C^\infty(M) \arrow[r, "X"] & C^\infty(M) \end{tikzcd}$$

For any $f \in C^\infty(N), F_*(X_p)f = Y_{F(p)}f$
$Yf \circ F = X(f \circ F)$
Even though we cannot push forward a vector field in general, there is one important special situation where we can.

Proposition 10. *If $F : M \to N$ is a diffeomorphism, and $X \in \mathfrak{X}(M)$ then there is a unique $Y \in \mathfrak{X}(N)$ such that $Y = F_*(X)$.*

Proof. If Y exists then $Y_{F(p)} = F_*(X_p)$, so it is unique.
This also defines Y but we need to check smooth dependence on p.
We check on a function $f \in C^\infty(N)$. Because $Yf \circ F = X(f \circ F)$, $Yf = [X(f \circ F)] \circ F^{-1} \in C^\infty(M)$

∎

Definition 0.24. Let $F : M \to M$ be a diffeomorphism, $X \in \mathfrak{X}(M)$. Say X is <u>invariant under F</u> if $F_*(X) = X$, i.e. $F_*(X_p) = X_{F(p)}$ for all p.

Lecture ??, 11/28/23

I missed the last few lectures, cry about it!

Chapter 9: Integral curves and flows

Definition 0.25. Let $X \in \mathfrak{X}(M)$. An integral curve of X is a curve $\gamma : (-\varepsilon, \varepsilon) \to M$ such that $\dot{\gamma}(t) = X(\gamma(t))$.

Example 0.24. Let $M = \mathbb{R}^2$, and $X = -y\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial y}$

Let $\gamma(t) = (x(t), y(t))$. For this to be an integral curve, we need $\dot{\gamma}(t) = X(\gamma(t))$. We have

$$\dot{\gamma}(t) = x'(t)\frac{\partial}{\partial x} + y'(t)\frac{\partial}{\partial y}$$

and

$$X(\gamma(t)) = -y(t)\frac{\partial}{\partial x} + x(t)\frac{\partial}{\partial y}$$

so for $X(\gamma(t)) = \dot{\gamma}(t)$, we need $x'(t) = -y(t)$ and $y'(t) = x(t)$. This is a system of first order ODEs.

If we impose the initial condition $\gamma(0) = (a, b)$, we get a unique solution,

$$\begin{cases} x(t) = a\cos(t) - b\sin(t) \\ y(t) = b\cos(t) + a\sin(t) \end{cases}$$

In general, an integral curve satisfies a system of first order ODEs.

If (U, φ) is a chart of M, we can write

$$X = \sum_{i=1}^{n} X^i(x_1, \ldots, x_n) \overbrace{\frac{\partial}{\partial x_i}}^{=(\varphi^{-1})_*\left(\frac{\partial}{\partial x_i}\right)}$$

If $\gamma : (-\varepsilon, \varepsilon) \to M$, we can consider the composition $\varphi \circ \gamma$, a curve on $\mathbb{R}^n$. If we abuse notation somewhat, and write $\gamma = \varphi \circ \gamma$, we see

$$\dot{\gamma} = \sum_{i=1}^{n} x_i'(t).\frac{\partial}{\partial x_i}$$

where $\varphi \circ \gamma = (x_1, \ldots, x_n)$

For γ to be an integral curve of X, we need $\dot{\gamma}(t) = X(\gamma(t))$, so

$$\begin{cases} x_1'(t) = X^1(x_1, \ldots, x_n) \\ \vdots \\ x_n'(t) = X^n(x_1, \ldots, x_n) \end{cases}$$

This is an autonomous system. We get existenece, uniqueness, stability from standard ODE theory.

Proposition 11. *Let $X \in \mathfrak{X}(M)$. For all $p \in M$, there is an open neighborhood $U \ni p$ and $\varepsilon > 0$ so that for all $g \in U$, there is an integral curve $\gamma_g(t) : (-\varepsilon, \varepsilon) \to M$ such that $\gamma_g(0) = g$. Moreover, such $\gamma_g(t)$ is unique, and γ_q depends smoothly on q.*

Proof. Standard ODE shit in local coordinates

■

Definition 0.26. $X \in \mathfrak{X}(M)$ is called complete if for all $p \in M$, the integral curve $\gamma(t)$ with $\gamma(0) = p$ is defined for all $t \in \mathbb{R}$

Proposition 12. *Let $F : M \to N$ be C^∞, $X \in \mathfrak{X}(M), Y \in \mathfrak{X}(N)$ be F-related. Then any integral curve $\gamma(t)$ of X will be mapped to an integral curve of Y under F. Conversely, if F takes integral curves of X to integral curves of Y, then X, Y are F-related.*

Proof. Wasn't paying attention, don't care

Lecture something, 11/30/23

Lemma 7 (Uniform Time Lemma)**.** *If X is a vector field such that there is some $\varepsilon_0 > 0$ such that for any $t \leq \varepsilon_0$, the integral curve at any point exists for t, then X is complete.*
The contrapositive says that if X is not complete, then for any $\varepsilon > 0$ there is a point p such that there is no integral curve from $(-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = p$.

Proof. ■

Recall that if G is a Lie group, and $X \in \mathfrak{g}$, then X is complete.

Definition 0.27. A vector field X is called compactly supported if it is identically zero outside of a compact set.

Theorem 0.11. *Every compactly supported vector field is complete.*

Corollary 0.12. *Every vector field on a compact manifold is complete.*

Proof. For simplicity, assume M is a compact manifold.
For each point p, there is a neighborhood U_p and a $\varepsilon_p > 0$ such that for any point $p' \in U_p$, an integral curve $\gamma : (-\varepsilon_p, \varepsilon_p) \to M$ exists such that $\gamma(0) = p'$ (this exists because of standard ODE shit)
So $\{U_p\}$ is an open cover, so by compactness has a finite subcover.
Let $U_1 = U_{p_1}, \ldots, U_k = U_{p_k}$, $\varepsilon_i = \varepsilon_{p_i} > 0$.
Set $\varepsilon_0 = \min\{\varepsilon_1, \ldots, \varepsilon_m\} > 0$, and by the uniform time of existence lemma it is complete.

■

Let $X \in \mathfrak{X}(M)$ be complete. For all $p \in M$, there is an integral curve $\gamma_p(t), t \in \mathbb{R}, \gamma_p(0) = p$.

Define $\theta : \mathbb{R} \times M \to M$ by $(t, p) \mapsto \gamma_p(t)$. This is C^∞. Observe

1. $\theta(0, p) = p$ for all $p \in M$

2. $\theta(t, \theta(s, p)) = \theta(t + s, p)$

Definition 0.28. θ is called the flow generated by X (one-parameter group action).
Another view:
Fix $t \in \mathbb{R}$, then $\theta_t : M \to M$ is a diffeomorphism, $p \mapsto \theta(t, p)$. We have

$$\begin{cases} \theta_{-t} \circ \theta_t = \mathrm{Id} \\ \theta_0 = \mathrm{Id} \end{cases}$$

this is a one parameter family of diffeomorphisms of M.

Example 0.25. Let $G = \mathrm{GL}(n, \mathbb{R})$, $A \in T_{I_n} G \simeq gl(n, \mathbb{R})$, then X_A is the left invariant vector field given by A. For all $g \in G$, $X_A(g) = gA$. What's the flow generated by X_A?
We need to compute integral curves $\gamma_g(t)$

$$\begin{cases} \dot{\gamma}_g(t) = X_A(\gamma_g(t)) = \gamma_g(t) A \\ \gamma_g(0) = 0 \end{cases}$$

So we have $x' = xa, x(0) = c$, so $x = ce^{at}$
We get $\gamma_g(t) = ge^{tA}$
The flow $\theta(t, g) = ge^{tA}$
Upshot:
If $X \in \mathfrak{X}(M)$ is complete, it generates a flow by integration. If We have a one-parameter family of diffeomorphisms $\theta : \mathbb{R} \times M \to M$, by differentiation we get a complete vector field.
What if X is not complete? In this case, we have something called the local flow.

Proposition 13 (Fundamental Theorem on Flow)**.** *Let $X \in \mathfrak{X}(M)$. Then this generated a local flow on M, i.e for all $p \in M$, there is some $\varepsilon > 0$ and open $U \ni p$ such that there is $\theta : (-\varepsilon, \varepsilon) \times U \to M$ defined by $(t, p) \mapsto \gamma_p(t)$ which satisfies*

1. *$\theta(0, q) = q$ for all $q \in U$.*

2. $\theta(t, \theta(s, q)) = \theta(t + s, q)$

Definition 0.29. For $X, Y \in \mathfrak{X}(M)$, we define the Lie Derivative of Y with respect to X, $L_X Y \in \mathfrak{X}(M)$, as follows. Associated to X are integral curves, even if they are only locally defined. We have vectors $Y(\theta(t, p))$. We have a local diffeomorphism θ_{-t} which we can use to pushforward $Y(theta(t, p))$. Then we can take the difference because we are in the same vector space:

$$(L_X Y)(p) = \lim_{t \to 0} \frac{1}{t} \left((\theta_{-t})_* (Y(\theta(t, p))) - Y(p) \right)$$

Theorem 0.13.

$$L_X Y = [X, Y]$$