Lecture 1

Lecture 2, 10/3/23

There are 3 key types of homology

1. Simplicial homology, denoted $H_n^{\triangle}(x)$. These are simple enough to compute, but hard to prove general statements.

2. Singular homology, denoted $H_n(x)$. This has the benefit of being great for theorem proving, but horrible to compute with.

3. Cellular homology, which has no special notation, because by the time we get here we'll know they're all the same anyways. This is the "professional computational tool."

So now, we begin with simplicial homology.

Simplicial Homology

We start with spaces X that have a simplicial structure (or something close to this that's slightly more complicated to define but easier to compute with).

Definition 0.1. An n-dimensional simplex, denoted $\triangle^n$, is the set

$$\triangle^n \overset{\text{def}}{=} \{(t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{t_i} = 1, t_i \geq 0\}$$

A general simplex in $\mathbb{R}^m$ is a set of $n+1$ points in $\mathbb{R}^m$ such that the smallest dimensional affine space containing all $n+1$ of these points is n-dimensional.
These simplices have a canonical linear ordering on the vertices, so this is really a (general) ordered simplex
A simplicial complex is a collection of subsets of some larger set which is closed under the operation of taking subsets.
Using tehse to compute homology of even some basic things can be a total pain.
One cannot put a simplicial structure on the torus without 14 vertices, 21 edges, and 7 faces, meaning our chain complex looks like

$$\mathbb{Z}^{14} \longrightarrow \mathbb{Z}^{21} \longrightarrow \mathbb{Z}^{7}$$

so we would have to think about a 14×21 and a 21×7 matrix, and calculate kernels and images, etc. etc.

Definition 0.2. A general (ordered) n-simplex is an ordered list $[v_0, v_1, \ldots, v_n]$ of points in some high enough dimensional Euclidean space in general position.
Given an n-simplex $\triangle^n = \{(t_0, t_1, \ldots, t_n) \mid \sum t_i = 1, t_i \geq 0\}$. The list $(t_0, t_1, \ldots, t_n)$ are sometimes called barycentric coordinates.
If we have a point $t_0\vec{v_1}+t_1\vec{v_1}+\cdots+t_n\vec{v_n} \in [v_0, \ldots, v_n]$, we can send it to $t_0\vec{u_0}+\cdots+t_n\vec{u_n}$.
The punchline of all of this is that if we have two n-simplices $[v_0, \ldots, v_n]$ and $[u_0, \ldots, u_n]$, there is a natural bijective correspondence given by barycentric coordinates.

Boundary maps

If we have a triangle with vertices v_0, v_1, v_2, we could naively define the "boundary" of the interior as the edges oriented counterclockwise.
But the edges all come with an inherited orientation from $[v_0, v_1, v_2]$, and one of these doesn't match the orientation we want the circle to have.
So we say

$$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

Notation: if we have a simplex $[v_0, \ldots, v_i, \ldots, v_n]$, and we want to denote the simplex obtained by removing the ith vertex from this list, we write $[v_0, \ldots, \hat{v}_i, \ldots, v_n]$.

Definition 0.3. We define the simplicial boundary map as

$$\partial[v_0, \ldots, v_n] = \sum_i (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_n]$$

What if we took the boundary of the boundary, $\partial(\partial[v_0, \ldots, v_n])$?
We define the boundary map to be a map between free $\mathbb{Z}$-modules, so we can define it to be $\mathbb{Z}$-linear.
Going back to the earlier example,

$$\begin{aligned}
\partial(\partial([v_0, v_1, v_2])) &= \partial([v_1, v_2] - [v_0, v_2] + [v_0, v_1]) \\
&= \partial[v_1, v_2] - \partial[v_0, v_2] + \partial[v_0, v_1] \\
&= ([v_2] - [v_1]) - ([v_2] - [v_0]) + ([v_1] - [v_0]) \\
&= 0
\end{aligned}$$

So the boundary of the boundary is trivial in this example.

In the general case, we have

$$\partial\left(\sum_i(-1)^i[v_0,\ldots,\hat{v}_i,\ldots,v_n]\right)$$
$$=\sum_i(-1)^i\partial[v_0,\ldots,\hat{v}_i,\ldots,v_n]$$
$$=\sum_i(-1)^i\left(\sum_{j<i}(-1)^j[v_0,\ldots,\hat{v}_j,\ldots,\hat{v}_i,\ldots,v_n]+\sum_{j>i}(-1)^{j-1}[v_0,\ldots,\hat{v}_i,\ldots,\hat{v}_j,\ldots,v_n]\right)$$
$$=0$$

I don't feel like reformatting this lol

Definition 0.4. A delta-structure (Δ-structure) on a topological space X is a collection of continuous maps $\sigma_\alpha : \Delta^n \to X$ (where the n depends on α) such that

1. $\sigma_\alpha|_{(\Delta^n)^\circ}$ is injective, and in fact every $x \in X$ is the image of the interior of exactly one of these.

2. σ_α restricted to a face of Δ^n is "equal to" $\sigma_\beta : \Delta^{n-1}$ (once you apply the canonical barycentric coordinate-based identification of their domains)

Given $\sigma_\alpha : \Delta^n \to X$ in a Δ-structure, let $e^n_\alpha \overset{\text{def}}{=} \sigma_\alpha((\Delta^n)^\circ)$

3. For all $A \subseteq X$, A is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n

This third condition precludes silly things like letting every point be a vertex.

Definition 0.5. If X is space with a Δ-structure, the nth chain group $\Delta_n(x)$ is the set of finite formal linear combinations of the σ_α's with n-dimensional domain.

Definition 0.6. We have to define what the boundary map δ should do to the function σ_α. We define

$$\partial\sigma_\alpha \overset{\text{def}}{=} \sum_i(-1)^i\sigma_\alpha|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}$$

By the second axiom of a Δ-structure, this will be a σ_β for some β. That is, this is indeed a map from Δ_n to Δ_{n-1}

Lecture 3, 10/5/23

Reminder, we refer to the chain groups in simplicial homology as $\Delta_n(x)$, and $H^\Delta_n(x)$ refers to the homology group.
We refer to the chain groups in singular homology as $C_n(x)$, and the homology groups as $H_n(x)$.

2 key properties of $H_n^{\Delta}(x)$

- If $\dim(X) < \infty$, then $H_k^{\Delta}(X) = 0$ for all $k > \dim(X)$.
- If X has finitely many k-dimensional cells, then $H_k^{\Delta}(X)$ is finitely generated.

Example 0.1. Here are 3 quick examples

1. S^1 with the obvious cell structure, with a single vertex and a single edge.

Δ_1 and Δ_0 will both be just $\mathbb{Z}$. Call the edge e and the point x (of course, really we want to consider e and x as maps into X).

So we have $\partial e = e|_{[v_1]} - e|_{[v_0]} = x - x = 0$

So

$$H_n^{\Delta}(S^1) = \begin{cases} \mathbb{Z} & \text{if } n = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

A fact that we can take on faith for now is that the first homology group is the abelianization of the fundamental group.

2. We'll view the torus with the Delta structure given last time. It is a square with a line thru the diagonal, and we identify the top and bottom, and we identify the right and left. We call the top triangle U, the bottom one L.

$\Delta_2(T)$ will be $\mathbb{Z}^2$, $\Delta_1(T)$ will be $\mathbb{Z}^3$, and $\Delta_0(T)$ will be $\mathbb{Z}^1$.

We get that $\partial U = [v_1, v_2] - [v_0, v_2] + [v_0, v_1] = b - c + a$, and $\partial L = a - c + b$

Now, $a \mapsto 0x, v \mapsto 0x, c \mapsto 0x$ because all three are loops.

So the map from Δ_1 to Δ_0 is really the zero map.

Consider $\partial(mU+nL) = m\partial(U)+n\partial(L) = m(a+b-c)+n(a+b-c) = (m+n)(a+b-c)$. This can only be zero if $m + n = 0$, so the kernel is generated by $U + L$. So $H_2(T) = \mathbb{Z}$, $H_1(T) = Z^2, H_0^{\Delta}(T) = \mathbb{Z}$

3. A bunch of drawings about $\mathbb{RP}^2$. The calculation basically goes the same, but we flip some orientations in our original delta structure.

So $\Delta_2(\mathbb{RP}^2) = \mathbb{Z}^2, \Delta_1(\mathbb{RP}^2) = \mathbb{Z}^3, \Delta_0(\mathbb{RP}^2) = \mathbb{Z}^2$

We have $a \mapsto x - y, b \mapsto x - y, c \mapsto y - y = 0$,

We get $H_0(\mathbb{RP}^2) = \mathbb{Z}$, $H_2(\mathbb{RP}^2) = 0, H_1^{\Delta}(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$

Singular Homology

Definition 0.7. A singular n-simplex is any (continuous) map $\sigma : \Delta^n \to X$.

Definition 0.8. The nth singular chain group $C_n(X)$ is the free abelian group with the singular n-chains on X as a basis. We consider 2 maps the same if they are the same function when we reparamaterize both triangles by barycentric coordinates.
We define

$$\partial\sigma = \sum_i (-1)^i \sigma|_{[v_0 \cdots \hat{v}_i \cdots v_n]}$$

Definition 0.9. Elements in $C_n(X)$ are called n-chains
Elements in $\ker \partial_n$ are n-cycles
Elements in $\operatorname{Im} \partial_{n+1}$ are n-boundaries
Elements of $H_n(X)$ are homology classes
So $H_n(X)$ is the n-cycles modulo the n-boundaries.
An advantage of all this is that it's very easy to see that the singular homology is a homeomorphism invariant, something which is not at all obvious for singular homology, as that uses the information of the Δ-structure.
If $X = \coprod_\alpha X_\alpha$ where each X_α is a path component of X, then, at the level of the chain groups, there is a natural decomposition of $C_n(X)$ into $\bigoplus_\alpha C_n(X_\alpha)$, as each n-simplex is path-connected. Further, the boundary map ∂ is in fact a direct sum $\oplus_\alpha \partial_\alpha$.
So, the homology $H.(X)$ splits as a direct product

$$H_n(X) = \bigoplus_\alpha H_n(X_\alpha)$$

What is the singular homology of a point?

For each n, $C_n(\cdot)$ is just $\mathbb{Z}$. We have $\partial_2 : C_2 \to C_1$ is given by just the identity map, because a triangle has 3 sides.
So ∂_n is 1 if n is odd, 0 if it's even (maybe other way around whatever)
But this sequence is exact except at C_0, where the kernel is everything and the image is nothing, so

$$H_n(\cdot) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Let's calculate $H_0(X)$ for an arbitrary space X.

Proposition 1. *If X is path connected (and nonempty) then $H_0(X) = \mathbb{Z}$.*

Proof. For the proof, let $\varepsilon : C_0(X) \to \mathbb{Z}$ defined by $\sum n_\alpha x_\alpha$. There is a natural map from these elements to $\mathbb{Z}$, called the augmentation map, given by just adding up the coefficients.
If you put this at the end of the chain complex, you get something called reduced homology

Lecture 4, 10/10/23

Definition 0.10. For a topological space X, we have the chain complex

$$\cdots \longrightarrow C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0$$

We can also consider the slight modification of it

$$\cdots \longrightarrow C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\varepsilon} \mathbb{Z}$$

Here, ε is the augmentation map, which sends $\sum n_i x_i$ to $\sum n_i$. This is still a chain complex.
The homology of this chain complex is the reduced singular homology of X.
We denote these by $\tilde{H}_n(X)$.
Note that for $n > 0$, $\tilde{H}_n(X) \cong H_n(X)$.
We have the relationship $\tilde{H}_0(X) \oplus \mathbb{Z} = H_0(X)$
Recall

$$H_n(\cdot) = \begin{cases} 0 & n > 0 \\ \mathbb{Z} & n = 0 \end{cases}$$

For reduced homology, we have

$$\tilde{H}_n(\cdot) = 0$$

always.

Homotopy Invariance

Given a map $f : X \to Y$, we have induced maps on chain complexes

$$\begin{tikzcd} C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "f_\sharp"] & C_n(X) \arrow[r, "\partial"] \arrow[d, "f_\sharp"] & C_{n-1}(X) \arrow[d, "f_\sharp"] \\ C_{n+1}(Y) \arrow[r, "\partial"] & C_n(Y) \arrow[r, "\partial"] & C_{n-1}(Y) \end{tikzcd}$$

We also have induce maps on the sequence of abelian groups

$$\begin{tikzcd} H_{n+1}(X) \arrow[d, "f_*"] & H_n(X) \arrow[d, "f_*"] & H_{n-1}(X) \arrow[d, "f_*"] \\ H_{n+1}(Y) & H_n(Y) & H_{n-1}(Y) \end{tikzcd}$$

Any element of $C_n(X)$ is some simplex map σ, so for $\sigma \in C_n(X)$, we define $f_\sharp(\sigma) = f \circ \sigma$. Then

$$f_\sharp(\partial\sigma) = f_\sharp\left(\sum_i (-1)^i \sigma|_{[v_0 \cdots \hat{v}_i \cdots v_n]}\right) = \sum_i (-1)^i (f \circ \sigma|_{[v_0 \cdots \hat{v}_i \cdots v_n]})$$

This is equal to $\partial(f \circ \sigma) = \partial f_\sharp(\sigma)$.
So $\partial \circ f_\sharp = f_\sharp \circ \partial$. So $f_\sharp$ gives a chain map.
If α is a cycle in $C_n(X)$, then $\partial\alpha = 0$, so $f_\sharp \partial\alpha = 0$. But this has to be equal to $\partial f_\sharp \alpha$, so $\partial(f_\sharp\alpha) = 0$, so $f_\sharp(\alpha)$ is a cycle.
A similar diagram chase will show that $f_\sharp$ sends boundaries to boundaries.
This is where the map f_* comes from.
Suppose we have homotopic maps

$$\begin{tikzcd}
X \arrow[d, bend right, "g \cong"'] \arrow[d, bend left, "f"] \\
Y
\end{tikzcd}$$

then $f_*, g_* : H_n(X) \to H_n(Y)$, are equal.
We have maps

$$\begin{tikzcd}
C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, bend right, "f_\sharp"'] \arrow[d, bend left, "g_\sharp"] & C_n(X) \arrow[r, "\partial"] \arrow[d, bend right, "f_\sharp"'] \arrow[d, bend left, "g_\sharp"] & C_{n-1}(X) \arrow[d, bend right, "f_\sharp"'] \arrow[d, bend left, "g_\sharp"] \\
C_{n+1}(Y) \arrow[r, "\partial"] & C_n(Y) \arrow[r, "\partial"] & C_{n-1}(Y)
\end{tikzcd}$$

Our goal is to define maps p to make a chain homotopy between g and f

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, bend right, "f_\sharp"'] \arrow[d, bend left, "g_\sharp"] & C_n(X) \arrow[r, "\partial"] \arrow[d, bend right, "f_\sharp"'] \arrow[d, bend left, "g_\sharp"] \arrow[dl, "P"'] & C_{n-1}(X) \arrow[r] \arrow[d, bend right, "f_\sharp"'] \arrow[d, bend left, "g_\sharp"] \arrow[dl, "P"'] & \cdots \\
\cdots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial"] & C_n(Y) \arrow[r, "\partial"] & C_{n-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

That is, we want $\partial P + P\partial = g_\sharp - f_\sharp$. Suppose for a moment that we have such a P.
Recall that if α is a cycle, then $(\partial P + P\partial)\alpha = (g_\sharp - f_\sharp)(\alpha)$. But because α is a cycle, $P \circ \partial\alpha = 0$, this means $g_\sharp(\alpha) - f_\sharp(\alpha) = \partial P(\alpha)$, meaning that $f_*([\alpha]) = [f_\sharp(\alpha)] = [g_\sharp(\alpha)] = f_*([\alpha])$.
We have in effect just shown that if we have such a relation between chain maps f, g, then they induce the same map on homology.

Definition 0.11. A map such as P above is called a <u>chain homotopy</u>
We now construct the operator in this setting.

If we have a homotopy, then we have commutative diagrams

$$\begin{tikzcd} \Delta^n \arrow[d, "\sigma"] \arrow[dr, "f_t \circ \sigma"] & \\ X \arrow[r, "f_t"] & Y \end{tikzcd}$$

This gives us a map $\Delta^n \times I \to Y$, which sends $(\underbrace{[t_0, \ldots, t_n]}_{\text{barycentric coordinates}}, t) \mapsto F(\sigma([t_0, \ldots, t_n]), t)$

But $\Delta^n = [v_0, \ldots, v_n]$ is an ordered n-simplex, and so is the unit interval $I = [0, 1]$. The idea is we want to somehow view $\Delta^n \times I$ as an $n + 1$ simplex.

I didn't quite follow the drawings Jon drew, but the punchline is that the product of simplices has a canonical decomposition as a Δ-structure itself. The idea is we decompose $\Delta^n \times I$ as these $n + 1$ simplices, and restrict the homotopy to those $n + 1$ simplices.

We want to think of the boundary of the prism operator as $g_\sharp - f_\sharp - P\partial$

Define $\phi_i([t_0, t_1, \ldots, t_n]) = t_{i+1} + \cdots + t_n$ (dunno what this does but he wrote it down)

The exact definition can be found in Hatcher, but it will be a chain homotopy.

Consequences

Suppose X, Y are homotopy equivalent. That is, there are maps $f : X \to Y$, $g : Y \to X$ such that $f \circ g, g \circ f$ are homotopic to the identities on their respective domains. But this means that $f \circ g$ is homotopic to the identity, which is the identity on homology, and similarly to $g \circ f$. So the chain complexes for X and Y are isomorphic, as f_*, g_* are mutual inverses.

So, any contractible topological space has reduced homology groups which are all trivial.

Lecture 5, 10/12/23

Exact sequences of excision

Theorem 0.1 (Big Theorem)**.** *Consider the sequence*

$$A \xhookrightarrow{i} X \xrightarrow{q} X/A$$

If X is a topological space and $A \subseteq X$ is a subspace such that A has a neighborhood

$N(A)$ which deformation retracts to A then there exists a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r, "i_*"] & \tilde{H}_{n+1}(A) \arrow[r, "q_*"] & \tilde{H}_{n+1}(A) \arrow[r, "\partial"] & \tilde{H}_{n+1}(X/A) \arrow[r, "i_*"] & \tilde{H}_n(A) \arrow[dllll, "q_*"] & \\
\tilde{H}_n(X) \arrow[r, "\partial"] & \tilde{H}_n(X/A) \arrow[r, "\partial"] & \tilde{H}_{n-1}(A) \arrow[r, "i_*"] & \tilde{H}_{n-1}(X) \arrow[r, "q_*"] & \cdots
\end{tikzcd}$$

Proof. We will prove this later ■

We will define the connecting map ∂ later.
If X is contractible, then $\tilde{H}_n = 0$ for all n.

Remark

Consider the sequences

$$0 \longrightarrow A \xrightarrow{\alpha} B$$

$$A \xrightarrow{\alpha} B \longrightarrow 0$$

If the first sequence is exact, then α is injective. If the second is exact, then α is surjective. So in the long exact sequence for S^n, because $\tilde{H}_n(S^n) = 0$,

$$\tilde{H}_{i+1}(S^n) \cong \tilde{H}_i(S^{n-1}) \cong \cdots \cong \tilde{H}_j(S^0)$$

We have

$$H_i(S^0) = \begin{cases} Z \oplus \mathbb{Z} & i = 0 \\ 0 & i > 0 \end{cases}$$

and

$$H_i(S^0) = \begin{cases} Z & i = 0 \\ 0 & i > 0 \end{cases}$$

So $\mathbb{Z} = \tilde{H}_0(S^0) \cong \tilde{H}_1(S^1) \cong \cdots$.

Corollary 0.2.

$$\tilde{H}_i(S^n) = \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n \end{cases}$$

Proof. ■

Corollary 0.3.

$$H_i(S^n) = \begin{cases} \mathbb{Z} & i \in \{n, 0\} \\ 0 & \textit{otherwise} \end{cases}$$

Proof. ■

So, with all of these applications, we now begin building up the tools to prove the big theorem.

Definition 0.12. Consider the sequence of abelian groups

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

If this is exact, this is called a short exact sequence (SES) of abelian groups.
Notice that $H_n(X/A) \neq H_n(X)/H_n(A)$!!!
If $A \hookrightarrow X$, we have the chain map

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
C_{n+1}(A) \arrow[r, hook, "i_*"] \arrow[d, "\partial"] & C_{n+1}(X) \arrow[d, "\partial"] \\
C_n(A) \arrow[r, hook, "i_*"] \arrow[d, "\partial"] & C_n(X) \arrow[d, "\partial"] \\
C_{n-1}(A) \arrow[r, hook, "i_*"] \arrow[d] & C_{n-1}(X) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

i_* really is an inclusion, because every singular chain on A is also a singular chain on X.
If we have $A \hookrightarrow B$ as abelian groups, we can uniquely extend this to a short exact sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{q} \frac{B}{\operatorname{Im}\alpha} \longrightarrow 0$$

Similarly, if we have a surjection $B \twoheadrightarrow C$, we can uniquely extend this to a short exact sequence

$$0 \longrightarrow \ker\beta \xrightarrow{\iota} B \xrightarrow{\beta} C \longrightarrow 0$$

Suppose that $A. \hookrightarrow B.$, meaning we have complexes

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
A_{n+1} \arrow[r, hook, "i"] \arrow[d] & B_{n+1} \arrow[d] \\
A_n \arrow[r, hook, "i"] \arrow[d] & B_n \arrow[d] \\
A_{n-1} \arrow[r, hook, "i"] \arrow[d] & B_n \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

then we can extend this to a larger diagram

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
0 \arrow[r] \arrow[d] & A_{n+1} \arrow[r, hook, "i"] \arrow[d] & B_{n+1} \arrow[r, two heads] \arrow[d] & C_{n+1} \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & A_n \arrow[r, hook, "i"] \arrow[d] & B_n \arrow[r, two heads] \arrow[d] & C_n \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & A_{n-1} \arrow[r, hook, "i"] \arrow[d] & B_n \arrow[r, two heads] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \arrow[d] \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{tikzcd}$$

If we do a big diagram chase, we can see that the maps between the C_i are unique and we can find them using just what we already have.
The punchline is that if we have a chain complex $A. \hookrightarrow B.$, we can extend this to a short exact sequence of chain complexes
Similarly, if $B. \twoheadrightarrow C.$, we can extend it.

So if we have

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
B_{n+1} \arrow[r, two heads, "i"] \arrow[d] & C_{n+1} \arrow[d] \\
B_n \arrow[r, two heads, "i"] \arrow[d] & C_n \arrow[d] \\
B_{n-1} \arrow[r, two heads, "i"] \arrow[d] & C_n \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

we can play the exact same game with a slightly different diagram chase.
So we can extend our earlier inclusion of $C.(A) \hookrightarrow C.(X)$ to

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \\
C_{n+1}(A) \arrow[r, hook, "i_*"] \arrow[d, "\partial"] & C_{n+1}(X) \arrow[r, two heads] \arrow[d, "\partial"] & {} \\
C_n(A) \arrow[r, hook, "i_*"] \arrow[d, "\partial"] & C_n(X) \arrow[r, two heads] \arrow[d, "\partial"] & C_n(X)/C_n(A) \\
C_{n-1}(A) \arrow[r, hook, "i_*"] \arrow[d] & C_{n-1}(X) \arrow[r, two heads] \arrow[d] & {} \\
\vdots & \vdots &
\end{tikzcd}$$

Definition 0.13. We define the relative singular n-chains in (X, A) $C_n(X, A)$ to be $C_n(X)/C_n(A)$.
The relative homology $H_n(X, A)$ is the homology of the relative sequence.
If we have a short exact sequence of chain complexes

$$0 \longrightarrow A. \xrightarrow{i} B. \xrightarrow{q} C. \longrightarrow 0$$

we have maps

$$\begin{tikzcd}
H_{n+1}(A.) \arrow[r, "i_*"] & H_{n+1}(B.) \arrow[r, "q_*"] & H_{n+1}(C.) \\
H_n(A.) \arrow[r, "i_*"] & H_n(B.) \arrow[r, "q_*"] & H_n(C.) \\
H_{n-1}(A.) \arrow[r, "i_*"] & H_{n-1}(B.) \arrow[r, "q_*"] & H_{n-1}(C.)
\end{tikzcd}$$

we wish to define connecting maps

$$\begin{tikzcd}
H_{n+1}(A.) \arrow[r, "i_*"] & H_{n+1}(B.) \arrow[r, "q_*"] & H_{n+1}(C.) \arrow[dll] \\
H_n(A.) \arrow[r, "i_*"] & H_n(B.) \arrow[r, "q_*"] & H_n(C.) \arrow[dll] \\
H_{n-1}(A.) \arrow[r, "i_*"] & H_{n-1}(B.) \arrow[r, "q_*"] & H_{n-1}(C.)
\end{tikzcd}$$

to make this a long exact sequence. To do this we will use the snake lemma.