Lecture 1

Definition 0.1. A chart around a point $x \in X$, where X is a topological space, is a set (U, φ), where U is an open neighborhood of x, and $\varphi : U \to \mathbb{R}^n$ is a homeomorphism onto its image with $\varphi(x) = 0$.

Lecture 2, 10/3/23

Recall a topological manifold is a topological space which is

1. Hausdorff

2. 2nd countable

3. Locally Euclidean

Our goal is to move to smooth manifolds, on which we can do calculus.
This is done by picking a point on our smooth manifold, translating it into a linear space through the use of charts, doing the calculus, and then translating back.

Example 0.1. Let $M = \mathbb{R}$ with the chart $\varphi(x) = x^3 = y$. Then the function $f(x) = x^2$ is differentiable in the normal sense, but

$$f \circ \psi^{-1}(y) = y^{\frac{2}{3}}$$

is not differentiable at $y = 0$!

Definition 0.2. Two charts $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta)$ are C^∞ compatible if, whenever $U_\alpha \cap U_\beta \neq \varnothing$, then we have a smooth function

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

whose inverse is also smooth.
The maps $\varphi_\alpha \circ \varphi_\beta^{-1}, \varphi_\beta \circ \varphi_\alpha^{-1}$ are called transition maps, or sometimes coordinate changes.

Example 0.2. On S^2 we have the stereographic projections

$$\mathcal{U}_1 = S^2 \setminus \{S\} \xrightarrow{\varphi_1} \mathbb{R}^2$$

$$\mathcal{U}_2 = S^2 \setminus \{N\} \xrightarrow{\varphi_2} \mathbb{R}^2$$

Then

$$\varphi_2 \circ \varphi_1^{-1} = \left(\frac{4u}{u^2 + v^2}, \frac{4v}{u^2 + v^2} \right)$$

This is C^∞ on the domain $\varphi_1(\mathcal{U}_1 \cap \mathcal{U}_2)$, and similarly for the other way around. So these are C^∞ compatible charts.

Example 0.3. Let $M = \mathbb{R}$, with chart $(\mathcal{U} = \mathbb{R}, \varphi(x) = x)$, $(\mathcal{V} = \mathbb{R}, \psi(x) = x^3)$ These are two incompatible charts!

Definition 0.3. An atlas is a collection of charts $(U_\alpha, \varphi_\alpha)$ such that

$$\bigcup_\alpha U_\alpha = M$$

Definition 0.4. A smooth structure/ C^∞ structure / differentiable structure on a topological manifold is an atlas $\mathscr{U} = \{(U_\alpha, \varphi_\alpha)\}$ such that

1. All charts in $\mathscr{U}$ are pairwise C^∞ compatible.

2. The atlas is maximal in the sense that any chart (U, φ) which is C^∞ compatible with every element of $\mathscr{U}$ is contain in $\mathscr{U}$

Proposition 1 (1.17). *Let M be a topological manifold. Then*

(a) Every smooth atlas is contained in a unique maximal smooth atlas, i.e. a smooth structure.

(b) Two smooth atlases determine the same smooth structure if and only if their union is a smooth atlas.

Proof. I omit it because i didn't really follow and don't think it's that important it's 1.17 in Lee sorrrryyyyyyy

∎

The upshot of all of this is that we can specify a smooth structure by specifying a smooth atlas.

Example 0.4. $(\mathbb{R}^n, \varphi = \mathrm{Id}_{\mathbb{R}^n})$ is the standard smooth structure on $\mathbb{R}^n$

Example 0.5. $(\mathbb{R}, \varphi = \mathrm{Id}_{\mathbb{R}})$, $(\mathbb{R}, \psi(x) = x^3)$ are two different smooth structures on $\mathbb{R}$. From the above example, it seems we are overcounting - there are many different distinct smooth structures on $\mathbb{R}$. Later, we will fix this by introducing the notion of diffeomorphism.
Every topological manifold with a single chart has a smooth structure.

Example 0.6. $\mathbb{R}^3$ minus a knot

Example 0.7. $\mathrm{GL}(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n^2}$ because it is the preimage of $\mathbb{R} \setminus \{0\}$ under the continuous map det

Definition 0.5. A smooth manifold is a topological manifold together with a smooth structure.

Example 0.8. S^2 needs two charts as a consequence of the hairy ball theorem.

Example 0.9. Here is a smooth structure on S^n.
S^n is defined as the solution locus of the polynomial $x_1^2 + \cdots + x_{n+1}^2 = 1$ in $\mathbb{R}^{n+1}$.
For each $i = 1, \ldots, n+1$, let $U_i^{\pm}$ be all the points on S^n such that x_i is positive or negative, and define $\varphi_i^{\pm}$ by just deleting the ith coordinate, so the sum of squares is strictly less than 1.
It is clear that this is indeed an atlas.
We now claim that $\{(U_i^{\pm}, \varphi_i^{\pm})\}_{i=1}^{n+1}$ is a smooth atlas.

$$\begin{aligned}(\varphi_1^+ \circ (\varphi_2^-)^{-1})(y_1, \ldots, y_n) &= \varphi_1^+ \left(y_1, -\sqrt{1 - \sum_{i=1}^n y_i^2}, y_2, \ldots, y_n \right) \\ &= \left(-\sqrt{1 - \sum_{i=1}^n y_i^2}, y_2, \ldots, y_n \right)\end{aligned}$$

This is smooth because $\sum y_i^2 < 1$, so the square root will not cause any trouble.
The others can be checked similarly!
$\mathscr{U} = \{(U_i^{\pm}, \varphi_i^{\pm})\}_{i=1}^{n+1}$ defines a smooth structure on S^n which is called the <u>standard smooth structure on S^n</u>

<u>Remark</u>

Stereographic chart and the standard charts given above are C^∞ compatible.

Lecture 3, 10/5/23

Our goal today is to describe $\mathbb{RP}^n$ as a smooth manifold.

Definition 0.6. We define $\mathbb{RP}^n$ as $\mathbb{R}^{n+1}$ quotiented by the action of $\mathbb{R}$ given by scaling. That is, $(x_1, \ldots, x_{n+1}) \sim (y_1, \ldots, y_{n+1})$ if one is a nonzero scalar multiple of the other.
We can also describe $\mathbb{RP}^n$ as a quotient of S^2 by the $\mathbb{Z}_2$-action given by the antipodal map.
We have a canonical projection $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$. We describe such points using so-called <u>homogeneous coordinates</u>. We write $\pi(x) = [x]$. This denotes the line passing through x.

Claim. $\mathbb{RP}^n$ *is Hausdorff, Second Countable, and admits a smooth structure.*

Lemma 1. π *is an open map. In particular,* $\mathbb{RP}^n$*, as a topological space, is second countable.*

Proof.
Let $U \subseteq \mathbb{R}^{n+1} \setminus \{0\}$ be open. We want to show that $\pi(U) \subseteq \mathbb{RP}^n$ is open, i.e. $\pi^{-1}(\pi(U))$ is open.
Now
$$\pi^{-1}(\pi(U)) = \bigcup_{x \in U} [x] = \bigcup_{x \in U} \bigcup_{\lambda \neq 0} \{\lambda x\}$$
For $\lambda \neq 0$, we define $\lambda(U)$ to be the image of U under the map $x \mapsto \lambda x$.
If $\lambda \neq 0$, the $\lambda(U)$ is open for U open.
Thus
$$\pi^{-1}(\pi(U)) = \bigcup_{\lambda \neq 0 \in \mathbb{R}} \lambda(U)$$
is open.

■

Lemma 2. *Let X be a topological space, $\sim$ an equivalence relation on X. Put the quotient topology on $X/\sim$. Assume that $\pi : X \to X/\sim$ is an open mapping. Then if X is second countable,*

1. *If X is 2nd countable, then so is $X/\sim$.*

2. *$X/\sim$ is Hausdorff if and only if $R = \{(x, y) \mid x \sim y\} \subset X \times X$ is closed.*

Proof. Shut up!

■

So $\mathbb{RP}^n$ is Hausdorf, since
$$R = \{(x, y) \in \left(\mathbb{R}^{n+1} \setminus \{0\}\right) \times \left(\mathbb{R}^{n+1} \setminus \{0\}\right) \mid y = \lambda x, \lambda \neq 0\}$$
is closed.
Remark: $y = \lambda x$ for some nonzero λ is equivalent to the statement
$$\sum_{i,j=1}^{n+1} (x_i y_j - x_j y_i)^2 = 0$$
We can express this as a function $F(x, y)$, which is continuous, and this set is the preimage of 0, which is closed. So, this is closed.
Finally, for $\mathbb{RP}^n$ to be a smooth manifold, it needs a smooth atlas.

<u>Notation</u>

<u>Homogeneous coordinates</u> on $\mathbb{RP}^n$ work as follows. We denote $pi(x_1, \ldots, x_{n+1})$ by $[x + 1; \cdots ; x_{n+1}]$.
Note also, for any i, and any $[x] \in \mathbb{RP}^n$ with nonzero x_i'th coordinate, $[x]$ can be represented by a unique equivalence class such that the ith entry in the homogeneous coordinates is 1.

Lemma 3. $\mathbb{RP}^n$ *admits a smooth structure.*

Proof.
Let $U_i = \{[x_1; \cdots ; x_{n+1}] \in \mathbb{RP}^n \mid x_i \neq 0\}$. This is open for $i = 1, \dots, n+1$.
Note that $\bigcup_{i=1}^{n+1} U_i = \mathbb{RP}^n$, because not every coordinate can be zero.
If $x_i \neq 0$ (i.e. $[x] \in U_i$) then

$$[x_1; \cdots ; x_i; \cdots ; x_{n+1}] = [\frac{x_1}{x_i}; \cdots ; 1; \cdots ; \frac{x_{n+1}}{x_i}]$$

So, we can define a bijection by sending $[x]$ to the point $(\frac{x_1}{x_i}, \dots, \hat{1}, \dots, \frac{x_{n+1}}{x_i}) \in \mathbb{R}^n$.
Call this φ_i
This is a homeomorphism.

Claim. $\{(U_i, \varphi_i)\}_{i=1}^{n+1}$ *is a smooth atlas for* $\mathbb{RP}^n$

Proof. Let's check, on $\varphi_2(U_1 \cap U_2)$

$$\begin{aligned}
\varphi_1 \circ \varphi_2^{-1}(y_1, \dots, y_n) &= \varphi_1 \left([y_1; 1; y_2; \cdots ; y_n]\right) \\
&= \varphi_1 \left([1; \frac{1}{y_1}; \frac{y_2}{y_1}; \cdots ; \frac{y_n}{y_1}]\right) \\
&= \left(\frac{1}{y_1}, \frac{y_2}{y_1}, \dots, \frac{y_n}{y_1})\right)
\end{aligned}$$

In $\varphi_2(U_1 \cap U_2)$, neither the first nor the second coordinates are zero, so we don't run into any division by zero problems.
The rest can be checked similarly.

■

Hence, $\mathbb{RP}^n$ is a smooth manifold.

Example 0.10. $\mathbb{RP}^2 = S^2/\mathbb{Z}_2$, $x \mapsto -x$.

Lecture 4, 10/10/23

Chapter 2: Smooth maps

Smooth functions

Let M^n be a smooth manifold, and let $f : M \to \mathbb{R}$ be a function (not assumed to be continuous necessarily, though it will turn out to be). For any point p, there is a chart (U, φ) around that point. We can consider the composition

$$\varphi(U) \xrightarrow{\varphi^{-1}} M \xrightarrow{f} \mathbb{R}$$

which is a function from $\mathbb{R}^n \to \mathbb{R}$, to which we may apply the usual definition of smoothness at $\varphi(p)$.

Definition 0.7. A smooth function from a manifold M to $\mathbb{R}$ is a function which is smooth in the above sense at every point.

Example 0.11. Given a smooth $f : \mathbb{R}^{n+1} \to \mathbb{R}$, we can consider the restriction $f|_{S^n} : S^n \to \mathbb{R}$. This is also smooth.
To check, we consider the charts $(U_i^{\pm}, \varphi_i^{\pm})_{i=1}^{n+1}$ the "standard" smooth structure on S^n, as defined earlier.
We have

$$f \circ (\varphi_1^{+})^{-1}(y_1, \dots, y_n) = f\left(\sqrt{1 - \sum_{i=1}^{n} y_i^2}, y_1, \dots, y_n\right)$$

All of these things are smooth on their domains, so this is smooth. The rest can be checked similarly.

Remark

If $f : M \to \mathbb{R}$ is C^∞, then $f : M \to \mathbb{R}$ is continuous.
Note $f|_U = (f \circ \varphi^{-1}) \circ \varphi$. $f \circ \varphi^{-1}$ is C^∞ on $\mathbb{R}^n$, hence continuous, and φ is continuous.

Definition 0.8. Let M^n be a smooth manifold and (U, φ) a smooth chart. This sends U to $\varphi(U)$, and p to $(x_1(p), \dots, x_n(p))$.
Define $f_i : U \to \mathbb{R}$ by $p \mapsto x_i(p)$, that is $f_i = \pi_i \circ f$.
f_i is a smooth function defined on U.
We can see $f_i \circ \varphi^{-1}(x_1, \dots, x_n) = x_i \in C^\infty$
These are called local coordinates about a point p / on an open set U

Remark

If M^n is a smooth manifold and $U \subseteq M$, then U inherits the smooth structure from M. If the atlas for M is given by $\{(U_\alpha, \varphi_\alpha)\}$, then the atlas for U is simply given by $\{(U_\alpha \cap U, (\varphi_\alpha)|_{U_\alpha \cap U})\}$
In particular, if (U, φ) is a smooth chart, then U is a smooth manifold with a single chart.

Definition 0.9. We write $C^\infty(M)$ to mean the collection of smooth functions $f : M \to \mathbb{R}$
These come with a nice ring structure - we can add and multiply by scalars, making it into an infinite dimensional vector space. In fact, because you can multiply them, they form an $\mathbb{R}$-algebra.
These local coordinate functions can be extended to $C^\infty(M)$.

Smooth maps

Let M, N be smooth manifolds, and $F : M \to N$ a map (again, not assumed to be continuous).

Definition 0.10. At any point $p \in M$, there is a chart (U, φ) with $p \in U$. Similarly, there is a chart (V, ψ) on N such that $F(U) \subseteq V$.
We can consider the composition

$$\varphi(U) \xrightarrow{\varphi^{-1}} U \xrightarrow{F} F(U) \xrightarrow{\psi} \psi(F(U))$$

The composition $\psi \circ F \circ \varphi^{-1}$, defined on $\varphi(U \cap F^{-1}(V))$, is a function from $\mathbb{R}^m$ to $\mathbb{R}^n$, where m and n are the dimensions of M, N respectively. We may then check if this function is smooth. If it is, we say that F is smooth at the point p.
A map $F : M \to N$ is a smooth map if F is smooth in the above sense at every point p.

Lemma 4. *If $F : M \to N$ is C^∞, then F is continuous.*

Proof. **Remark**

Without the additional acquirement that $F(U) \subseteq V$ in the definition, this is false.
It will suffcie to check that for all $p \in M$, there exists an open U containing p such that $f|_U : U \to N$ is continuous.
Since F is smooth, there exist smooth charts (U, φ) of M at p, (V, ψ) of N at $F(p)$ such that

$$\hat{F} = \psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$$

is C^∞.
Therefore, $F|_U = \psi^{-1} \circ \hat{F} \circ \varphi$ is C^0, because everything in sight is also C^0.

∎

Lecture 5, 10/12/23

Hint for homework problem:
If $F^*(C^\infty(N)) \subset C^\infty(M)$, then F is C^∞.

Example 0.12. Consider $F : (\mathbb{R}, \varphi = \mathrm{Id}) \to (\mathbb{R}, \psi(x) = x^5)$ given by $x \mapsto x^{\frac{1}{5}}$. This is C^∞.
Check: $\psi \circ F \circ \psi^{-1}(x) = \psi \circ F(x) = \psi(x^{\frac{1}{5}}) = x$.
In fact, F^{-1} is C^∞! i.e. F is a diffeomorphism.

Definition 0.11. A function $f : M \to N$ between two smooth manifolds is a diffeomorphism if F is a bijection, and both F and F^{-1} are C^∞.
In other words, it is an isomorphism in the category of smooth manifolds.

Question:

how many smooth structures on $\mathbb{R}^n$ are there *up to diffeomorphism?*
If $n \leq 3$, then there is a unique smooth structure *up to diffeomorphism*, a result due to Moise for $n = 3$, and Radon for $n \leq 2$.
When $n \geq 5$, there is a unique smooth structure *up to diffeomorphism*, a result due to Stalling.
For $n = 4$, there are uncountably many smooth structures *even up to diffeomorphism*! This is due to a result by Donaldson.
These are the so-called "exotic" $\mathbb{R}^4$s.

What about compact spaces?

The simplest example is S^n. This question can be put into a much bigger context. The smooth category is a subcategory of the topological category, this question is getting at the difference between them. The homotopy category is a subcategory of Top.
The "holy grail" would be a complete classification of spaces up to homotopy equivalence/homeomorphism/diffeomorphism.

Poincaré conjecture:

Every closed n-dimensional topological manifold homotopic to S^n is actually homeomorphic to the sphere.
This is true for $n \geq 5$, famous work of Smale in 1966.
For $n = 4$, the answer is also yes, due to Freedman in 1986 (he used to be here!!!!!!!)
For $n = 3$ the answer is also yes, due to Perelman in 2006
For $n \leq 2$, the answer is yes.
Every one of these earned a fields medal!!!
What about in the smooth category?

Smooth Poincaré conjecture:

Every smooth manifold homeomorphic to S^n is actually diffeomorphic to S^n
Milnor, 1962: There exists a smooth structure on S^7 which is not diffeomorphic to the standard one. In fact, there are 28 of them up to diffeomorphism.
For each dimension n, we have a resolution to the smooth conjecture, but for $n = 4$ it is still open.
Remark
The way we defined a C^∞ structure, we can also define a C^k structure, where $k =\in \mathbb{N}$.
We may also define C^ω, which is real analytic, which is a stronger condition than smoothness.

C^0 is just topological manifolds.
However, every object of C^k is homoemorphic to an object of C^∞ for $k > 0$. But for C^0, there are topological manifolds which cannot be smoothed, and there are different smooth structures.
So, we sometimes refer to a smooth structure as a differentiable structure. So as long as we have a first derivative, we can somehow achieve a result like the Weirstrass approximation theorem.
The (somewhat more) precise way to say this is that we can topologize the set of C^1 maps between two C^1 manifolds, and the smooth maps are dense in this set.
This is all theorems of Whitney and Nash

Partitions of unity (P.O.U)

From now on, when we say "manifold," it is understood to mean a smooth manifold with some smooth structure.
The idea is that a manifold is locally Euclidean, but not globally. We would like a way to reduce global information to sums of local information, and partitions of unity allow us to do that.

Goal:

We want to write $1 = \sum_\alpha f_\alpha, f_\alpha \in C^\infty(M)$, and nonzero only in some neighborhood.
We will use so-called "bump functions."
We have two issues.

1. We need to establish the existence of bump functions

2. We need to make sense of this summation

We now address 1 by showing the existence of smooth bump functions in $\mathbb{R}^n$.
Consider $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(t) = \begin{cases} e^{-\frac{1}{t}}, & t > 0 \\ 0 & t \leq 0 \end{cases}$$

This is C^∞
We have a "cutoff" function $h : \mathbb{R} \to [0, 1]$, which is identically 1 on $(-\infty, 1]$, decreasing on $[1, 2]$, and identically zero on $[2, \infty)$.
We can define it by

$$h(t) = \frac{f(2-t)}{f(2-t) + f(t-1)}$$

We pass to $\mathbb{R}^n$, i.e. a $H : \mathbb{R}^n \to [0, 1] \in C^\infty(\mathbb{R}^n)$ such that

1. $H \equiv 1$ on $\overline{B_1(0)}$

2. $H \equiv 0$ outside $B_2(0)$.

We can simply define $H = h(|x|)$
The idea is that for any manifold M, and any point $p \in M$, there is a chart (U, φ) about p. We can normalize so that $\varphi(U) \supset B_2(0)$ and $\varphi(p) = 0$.
We can define $H \circ \varphi \in C^\infty(U)$. We want to upgrade it to something in $C^\infty(M)$. We can do this easily by setting it to be identically zero outside of U.

Corollary 0.1. *For any $p \in M$, for any open $U \ni p$, and for any $f \in C^\infty(U)$, there exists an open $p \in V \subseteq U$, and a smooth function $\tilde{f} \in C^\infty(M)$, such that $\tilde{f}|_V = f|_V$.*

Proof. Take $\tilde{f}$ to be $(H \circ \varphi)f$. $H \circ \varphi$ is identically 1 on $\varphi^{-1}(B_1(0))$ ∎

Lecture 6, 10/17/23

Notation: for $f \in C^0(M)$, we define the support of f as

$$\operatorname{supp} f \stackrel{\text{def}}{=} \overline{\{x \in M \mid f(x) \neq 0\}}$$

Theorem 0.2. *Let M^n be a smooth manifold, with $F \subset M$ closed, and $K \subset\subset M$, meaning a compact subset. Suppose that $K \cap F = \varnothing$. Then there is a $f \in C^\infty(M)$ such that $0 \leq f \leq 1$, $f \equiv 1$ on K, and $f \equiv 0$ on F.*

Proof. By assumption, $\underbrace{K}_{\text{compact}} \subset \overbrace{M \setminus F}^{\text{open}}$.
For all $p \in K$, there is a chart (U_p, φ_p) such that $\varphi_p(p) = 0$, $U_p \subset M \setminus F$, and $\varphi_p(U_p) \supset \overline{B_2(0)}$
Consider the open cover of K given by $\{\varphi_p^{-1}(B_1(0))\}_{p \in K}$. By compactness, this admits a finite subcover

$$\{\varphi_{p_1}^{-1}(B_1(0)), \varphi_{p_2}^{-1}(B_1(0)), \ldots, \varphi_{p_k}^{-1}(B_1(0))\}$$

Set $\mathcal{U}_i = U_{p_i}$, $\varphi_i = \varphi_{p_i}$, and

$$f(x) = 1 - \prod_{i=1}^{k} \big(1 - (H \circ \varphi_i)(x)\big) \in C^\infty(M)$$

For any $p \in K$, $p \in \varphi_j^{-1}(B_1(0))$ for some j, so $H \circ \varphi_j(p) = 1$, so $f(p) = 1 - 0 = 1$.
Finally, we must verify that $f \equiv 0$ on F. This happens because $U_i \subset M \setminus F$, so $H \circ \varphi_i$ vanishes on F.

$\blacksquare$

Now it is time for partitions of unity. These are a tool to piece together local information to get global information.
We want $1 \equiv \sum_\alpha f_\alpha$ for $f_\alpha \in C^\infty(M)$, $\operatorname{supp} f_\alpha \subset U_\alpha$, and $\{U_\alpha\}_\alpha$ is an open cover of M. We need to worry about sumability.
We insist that for any point p, there are only finitely many α such that $f_\alpha(p) \neq 0$.

Definition 0.12. Let X be a topological space. A collection of subsets $\{S_\alpha\}$ is called locally finite if for any $p \in X$, there is a neighborhood $U \ni p$ such that $U \cap S_\alpha = \varnothing$ for all but finitely many α.

Example 0.13. If $\{\operatorname{supp} f_\alpha\}$ is locally finite, then for all p, there is a neighborhood $U \ni p$ such that U intersects only finitely many $\operatorname{supp} f_\alpha$. In other words, there are only finitely many f_α such that $f_\alpha(p) \neq 0$

Definition 0.13. Let M be a smooth manifold with open cover $\mathscr{U} = \{U_\alpha\}$. A partition of unity subordinate to $\mathscr{U}$ is a collection $\{\psi_\alpha \in C^\infty(M)\}$ such that

(i) $0 \leq \psi_\alpha \leq 1$

(ii) $\operatorname{supp} \psi_\alpha \subset U_\alpha$

(iii) $\{\operatorname{supp} \psi_\alpha\}$ is locally finite

(iv) $\sum_\alpha \psi_\alpha \equiv 1$

Theorem 0.3 (Existence of P.O.U)**.** *Given a manifold M and open cover $\mathscr{U} = \{U_\alpha\}$, there is a partition of unity subordinate to $\mathscr{U}$.*

Example 0.14 (Separation property)**.** For all $p \neq q$, there exists $f \in C^\infty(M)$ such that $f(p) = 1, f(q) = 0$.
Let $\mathscr{U} = \{U, V\}$, with $U = M \setminus \{p\}, V = M \setminus \{q\}$. If there are ψ_1, ψ_2 satisfying $i - iv$, then $f = \psi_2$ will do.

Proof. We now prove the theorem.
We will only do the case when M is compact. For all $p \in M$, there exists a chart (V_p, φ_p) at p such that $\varphi_p(p) = 0$ and $\varphi_p(V_p) \supset \overline{B_2(0)}$ and $V_p \subset U_{\alpha(p)}$.
Let $W_p = \varphi_p^{-1}(B_1(0)) \ni p$. Then $\{W_p\}_{p \in M}$ is an open cover of M. By compactness, there is a finite subcover $W_1 = W_{p_1}, \cdots, W_k = W_{p_k}$
Set $\varphi_i = \varphi_{p_i}$ and $f_i = H \circ \varphi_i \in C^\infty(M)$.
Then $f_i \equiv 1$ on W_i, $\operatorname{supp} f_i \subset V_i = V_{p_i} \subset U_{\alpha(p_i)}$.
Since $\{W_1, \ldots, W_k\}$ covers M, $\sum_{i=1}^k f_i \geq 1$.
So we can simply take $g_i = \frac{f_i}{\sum_{i=1}^k f_i} \in C^\infty(M)$, so $\sum_{i=1}^k g_i \equiv 1$.
Recall that $V_i \subset U_{\alpha(i)}$, $i = 1, \ldots, k$. Put $\psi_\alpha = \sum_{\alpha(i)=\alpha} g_i$.
This is in $C^\infty(M)$, and $\operatorname{supp} \psi_\alpha \subset U_{\alpha(i)=\alpha} \operatorname{supp} g_i \subset U_\alpha$

and $\sum_\alpha \psi_\alpha = \sum_i g_i \equiv 1$

■

If M is not compact, we have the following theorem.

Theorem 0.4. *A manifold is always paracompact, i.e. any open cover has a locally finite refinement. In fact, for any open cover $\mathscr{U} = \{U_\alpha\}$, there ix a countable open cover $\{V_i\}$ such that*

(i) $\{V_i\}$ is locally finite

(ii) $\{V_i\}$ is a refinement of $\mathscr{U}$, i.e. $V_i \subset U_{\alpha(i)}$

(iii) Each V_i is a domain of a normalized chart.

Proof. This follows from second countability.

■

Lecture 7, 10/19/23

Chapter 3: Tangent vectors and tangent spaces

Theree are two views of tangent spaces of $\mathbb{R}^n$: geometric or abstract.
For an $a \in \mathbb{R}^n$, we denote the tangent space at a as $T_a\mathbb{R}^n$. For $\mathbb{R}^n$, we identify it with its tangent space at any point.
Indeed, a tangent vector at a is a vector "based" at a, a pair (v, a), denoted v_a.
We can't do this on a manifold!
We will take a different viewpoint. Instead of thinking about directions, which represent tangent vectors, we're gonna look at directional derivatives.
Instead of a $v_a \in T_a\mathbb{R}^n$, we're gonna consider the directional derivative

$$D_{v_a} : C^\infty(\mathbb{R}^n) \to \mathbb{R}$$

This will act by

$$f \mapsto \frac{d}{dt}|_{t=0} f(a + tv)$$

This is the abstract viewpoint.
Now, we can't replicate the geometric viewpoint on a manifold, but we can replicate the abstract viewpoint.
The directional derivative satisfies the following:

- D_{v_a} is linear
- D_{v_a} satisfies the Leibniz rule. That is, given two smooth functions $f, g \in C^\infty(\mathbb{R}^n)$,
$$D_{v_a}(fg) = f(a)D_{v_a}g + g(a)D_{v_a}f$$

Definition 0.14. A <u>derivation at a</u> is a linear map $X : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ satisfying the Leibniz rule:

$$X(fg) = f(a)(Xg) + g(a)(Xf)$$

Example 0.15. D_{v_a} is a derivation at a

<u>Notation</u>

$$\widetilde{T_a\mathbb{R}} = \{X \mid X \text{ derivation at } a\}$$

Note that $\widetilde{T_a\mathbb{R}}$ is a vector space

Example 0.16. Let $v_a = (e_i, a)$. Then $D_{v_a}(f) = \frac{d}{dt}|_{t=0} f(a + te_i) = \frac{\partial f}{\partial x_i}$.
So we write $D_{v_a} = \frac{\partial}{\partial x_i}|_a$ for $v_a = (e_i, a)$
In general $v = (v_1, \dots, v_n)$,

$$D_{v_a} f = \frac{d}{dt}|_{t=0} f(a + tv) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(a) v_i$$

Proposition 2. *The map $T_a\mathbb{R}^n \to \widetilde{T_a\mathbb{R}^n}$ given by $v_a \mapsto D_{v_a}$ is a linear isomorphism.*

Proof. <u>Remark:</u>
We can identify the geometric tangent space $T_a\mathbb{R}^n$ and the abstract tangent space $\widetilde{T_a\mathbb{R}^n}$. We will do that in this class. Now for the proof

(i) The map is linear: $(v_a + w_a) = (v + w)_a$, and $D_{v_a} + D_{w_a} = D_{(v+w)_a}$

(ii) Injective: Let $v_a \in T_a\mathbb{R}^n$ be such that $D_{v_a} = 0$, i.e. $D_{v_a} f = 0$. But if $v_a = (v_1, \dots, v_n)_a$, then $D_{v_a} f = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(a) v_i$ for all $f \in C^\infty(\mathbb{R}^n)$. So setting $f(x) = x_i$, we get that $v_i = 0$ so $v = 0$, i.e. v_a is the zero vector.

(iii) Surjective: for any $X : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ derivation at a, there exists some $v_a \in T_a\mathbb{R}^n$ so that $X = D_{v_a}$, i.e. $Xf = D_{v_a} f$ for all $f \in C^\infty(\mathbb{R}^n)$. We will need two lemmas

Lemma 5. 1. *If f is a constant function, then $Xf = 0$.*

2. *If $f, g \in C^\infty(\mathbb{R}^n)$ with $f(a) = g(a) = 0$, then $X(fg) = 0$*

Proof. **1.** First let $f \equiv 1$.

$$\begin{aligned} X(1) &= X(1 \cdot 1) \\ &= 1 \cdot X(1) + 1 \cdot X(1) \\ &= 2X(1) \end{aligned}$$

So $X(1)$ must be zero.

In general if $f \equiv c$ for some constant c, because X is linear we have $X(c) = c \cdot X(1) = 0$.

2. This is a simple application of the Leibniz rule

■

Lemma 6 (Taylor expansion with remainder)**.** *For all $f \in C^\infty(\mathbb{R}^n)$, $f(x) = f(a) + \sum_{i=1}^n g_i(x)(x_i - a_i)$ where $g_i \in C^\infty(\mathbb{R}^n)$ and $g_i(a) = \frac{\partial f}{\partial x_i}(a)$*

Proof. By FTC,

$$\begin{aligned} f(x) - f(a) &= \int_0^1 \frac{d}{dt}(f(a + t(x-a)))\, dta \\ &= \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(a + t(x-a)) \cdot (x_i - a_i) \\ &= \sum_{i=1}^n (x_i - a_i) \underbrace{\int_0^1 \frac{\partial f}{\partial x_i}(a + t(x-a))\, dt}_{g_i(x)} \end{aligned}$$

■

Back to the proof of surjectivity:
Given $X \in T_a\mathbb{R}^n$ we want to find $v_a \in T_a\mathbb{R}^n$ such that $X = D_{v_a}$, i.e. $Xf = D_{v_a}f$ for all $f \in C^\infty(\mathbb{R})$
But the second lemma reduces arbitrary smooth function to

$$f = \underbrace{f(a)}_{\text{constant}} + \sum_{i=1}^n g_i(x)(x_i - a_i)$$

By linearity,

$$\begin{aligned} Xf &= \underbrace{X(f(a))}_{=0} + \sum_{i=1}^n X(g_i(x_i - a_i)) \\ &= \sum_{i=1}^n \left(X(g_i) \underbrace{(x_i - a_i)|_a}_{=0} + g_i(a) X(x_i - a_i) \right) \\ &= \sum_{i=1}^n \frac{\partial f}{\partial x_i}(a) \underbrace{X(x_i - a_i)}_{\text{scalar}} \end{aligned}$$

The equation $Xf = D_{v_a} f$ thus becomes

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a) X(x_i - a_i) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} a(v_i)$$

To make this true, we can simply pick $v_i = X(x_i - a_i)$, and the equation will always hold, no matter what f is.

■

Upshot

We have $T_a\mathbb{R}^n \cong \widetilde{T_a\mathbb{R}^n}$
We identify $(e_i)_a$ with $\frac{\partial}{\partial x_i}|_a (= D_{(e_i)_a})$
So $(v_1, \ldots, v_n)$ is identified with $\sum_{i=1}^n v_i \frac{\partial}{\partial x_i}|_a$
We will now define the latter on manifolds.

Lecture 8, 10/24/23

Definition 0.15. Let M^n be a smooth manifold. For $p \in M$, a derivation at p is a linear map

$$X : C^\infty(M) \to \mathbb{R}$$

satisfying the Leibniz rule at p, meaning that

$$X(fg) = f(p)X(g) + g(p)X(f)$$

Definition 0.16. The tangent space of M at p is the vector space of all derivations at p, denoted T_pM. Members of this space will be called tangent vectors.

Example 0.17. Let $\varepsilon > 0$, and consider $c : (-\varepsilon, \varepsilon) \to M^n$. If this is C^∞ it's called a curve.
Let $p = c(0)$.

Definition 0.17. We define $\dot{c}(0) \in T_pM$ by defining how it acts on $C^\infty(M)$ as follows.

$$\dot{c}(0)(f) = \frac{d}{dt}|_{t=0} f(c(t))$$

Remark
T_pM is a vector space over the reals.
It is really easy from here to define the pushforward/differential of a map.
Let M, N be smooth manifolds, and let $F : M \to N$ be smooth. Let $p \in M$. Then F "naturally" induces a linear map $F_* : T_pM \to T_{F(p)}N$ as follows.

Let $X \in T_pM$. Then $F_*(X) \in T_{F(p)}N$ acts on $C^\infty(N)$ by

$$F_*(X)(f) = X(f \circ F)$$

(check that this defines a derivation).
We also denote this as dF.

Proposition 3 (Basic Properties)**.**

1. *F_* is linear*

2. *If $F : M \to N$ and $G : N \to P$, then $(G \circ F)_* = G_* \circ F_*$. That is, $_*$ is **functorial!***

3. $\mathrm{Id}_M : M \to M$ *gives* $(\mathrm{Id}_M)_{*,p} = \mathrm{Id}_{T_pM}$

4. *If F is a diffeomorphism, then $F_* : T_pM \to T_{F(p)}N$ is a linear isomorphism.*

Proof.
4 follows from 1-3. If $F^{-1} \circ F = \mathrm{Id}_M$, then $(F^{-1} \circ F)_* = \mathrm{Id}_{T_pM}$. But this is also $F_*^{-1} \circ F_*$, so for their composition to be the identity (and the same argument shows the reverse composition is the identity on N), both have to be isomorphisms.
We now prove 1. Think
We now prove 2. We claim that.

$$(G \circ F)_*(X) = (G_* \circ F_*)(X)$$

We check that

$$(G \circ F)_*(X)f = X(f \circ (G \circ F))$$

and

$$(G_* \circ F_*)(X)f = G_*(F_*(X))f = F_*(X)(f \circ G) = X((f \circ G) \circ F)$$

■

But what is T_pM? In particular, is it local? That is, does it depend only on info in a neighborhood of p?

Proposition 4. *Let $X : C^\infty(M) \to \mathbb{R}$ be a derivation at p, and $f, g \in C^\infty(M)$. If there is an open $U \ni p$ such that $f|_U = g|_U$ then $Xf = Xg$.*

Proof. We want to show $X(f - g) = 0$.
Let $h = f - g \in C^\infty(M)$. Then $h|_U \equiv 0$.
$p \in U$, so there is a $\chi \in C^\infty(M)$ such that $\chi(p) = 0$ and $\chi \equiv 1$ outside U. Note that $h = \chi h$.
This implies $X(h) = X(\chi h) = \chi(p)Xh + h(p)X\chi$. $\chi(p) = 0$, and h vanishes at p, so $X(h) = 0$.

■

<u>Remark</u>
Another way to get the "locality" is to define a tangent vector at p to be a derivation on the space of "germs" of C^∞ functions defined only near p.

Corollary 0.5 (Locality)**.** *Let $U \subseteq M$ be open. Then for any $p \in U$, $T_pM \cong T_pU$.*

Proof. Note that we have the canonical inclusion $i : U \to M$ which is a smooth map.
Therefore we have a linear map $i_* : T_pU \to T_pM$
To show this is a linear isomorphism, we have to construct the inverse by hand.
We define

$$\sigma : T_pM \to T_pU$$

as follows. For $X \in T_pM$, $f \in C^\infty(U)$, $\sigma(X)f = X(\chi f)$, where χ is a bump function at p. σ is well-defined by previous lemma.
We continue the proof next time. ∎