Lecture 3 - 9/10/24

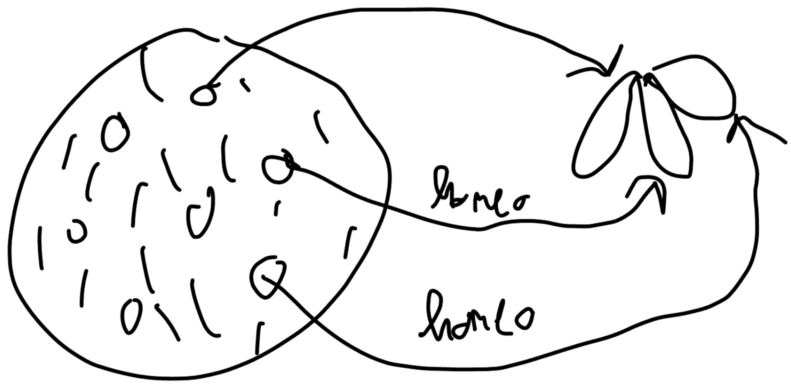
Recall

Theorem 0.1. *(Hurewicz)*
Let X be a connected CW complex, $n \geq 2$. Then if $\pi_k(X) = 0$ for $k < n$, then $H_k(X) = 0$ for $k < n$, and $H : \pi_n(X) \to H_n(X)$ is an isomorphism.

Proof. Last time, modulo the fact that H is injective.
We can assume $X^{(n)} = \bigvee_i S_i^n$. Suppose we have a map $f : S^n \to X$. Then we can

1. Push it into $X^{(n)}$ (by CW approximation theorem)

2. Make it smooth



3. Give it the structure of

Then $H([f]) = \sum_i a_i[S_i^n]$ and $\sum_i a_i[S_i^n] = \partial$(some set of $(n+1)$-cells) is by definition nullhomotopic, and $f + g$ has total degree 0 on S_i^n.
So $f + g$ is nullhomotopic by the π_n argument, so $f \simeq 0$.

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There is a relative version if one replaces X by (X, A).

Corollary 0.2. *Let X, Y be two simply connected spaces. If a map $f : X \to Y$ induces an isomorphism on H_n for all n, then $X \simeq Y$.*

Example 0.1. $S^2 \times S^2$ and $S^2 \vee S^2 \vee S^4$ are not homotopy equivalent, so there is no map between them preserving H_*.

Proof. Apply relative Hurewicz to the mapping cylinder. We can assume $X \hookrightarrow Y$, then use naturality of H.

$$\begin{tikzcd}
H_i(X) \arrow[r] & H_i(Y) \arrow[r] & H_i(Y,X) \arrow[r] & H_{i-1}(X) \arrow[r] & H_{i-1}(Y) \\
\pi_i(X) \arrow[r] \arrow[u] & \pi_i(Y) \arrow[r] \arrow[u] & \pi_i(Y,X) \arrow[r] \arrow[u] & \pi_{i-1}(X) \arrow[r] \arrow[u] & \pi_{i-1}(Y) \arrow[u]
\end{tikzcd}$$

The top left and top right morphisms are isomorphisms, $H_i(Y,X)=0$, and we know $\pi_1(X)\cong\pi_1(Y)\cong\pi_1(Y,X)\cong 0$.
By Hurewicz, $\pi_2(Y,X)\cong H_2(Y,X)\cong 0$.
So for every k, $\pi_k(Y,X)\cong H_k(Y,X)\cong 0$.
By the Whitehead theorem, $Y\cong X$.

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Corollary 0.3. *For any CW complex with $\pi_1(X)\cong 0$, there exists a CW complex* $X(n)\xrightarrow{a_n} X$ *such that*

(i) a_n *induces an isomorphism on* H_k *for* $k\le n$

(ii) $H_k(X(n))\cong 0$ *for* $k>n$.

Proof. Take $X^{(n)}$ and add some $(n+1)$-cells: we have the long exact sequence

$$\cdots\longrightarrow H_{n+1}(X,X^{(n)})\longrightarrow H_n(X^{(n)})\longrightarrow H_n(X)\longrightarrow\cdots$$

By relative Hurewicz, the first thing is isomorphic to $\pi_{n+1}(X,X^{(n)})$. Take a basis for the kernel of the second map, and take preimages in $H_{n+1}(X,X^{(n)})$ and corresponding maps.

$$f_i:(D^{n+1},\partial D^{n+1})\to(X,X^{(n)})$$

Use those to attach cells, completing the construction.

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Fibrations

Definition 0.1. A fiber bundle $p:Y\to B$ with fiber F is a map such that

- For all $x\in B$, $p^{-1}(X)\cong F$
- For all $x\in B$, there is a neighborhood $U\ni x$ such that

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\cong"] \arrow[d, "p"] & U\times F \arrow[dl, "p_1"] \\ U & \end{tikzcd}$$

  commutes.

  Example 0.2.

  1. $F\times B$

2. Any covering space (F) is discrete in this case

3. The Möbius strip locally looks like $\underbrace{S^1}_{=B} \times \underbrace{[0,1]}_{=F}$

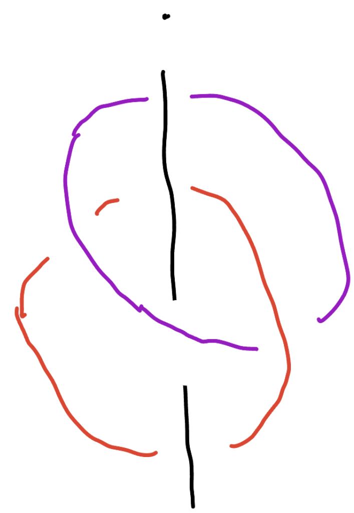
4. The Hop fibration, the quotient map $\mathbb{C}^2 \supseteq S^3 \to S^2 \cong \mathbb{C}P^1$ which sends

$$\begin{aligned} h(z_1, z_2) &= (z_1 : z_2) \\ &= \frac{z_1}{z_2} \end{aligned}$$

where in the last line $S^2 = \mathbb{C} \cup \{\infty\}$

preimages of points are S^1.

If we think of $S^3 = \mathbb{R}^3 \cup \{\infty\}$, then any two fibers link with each other. This is

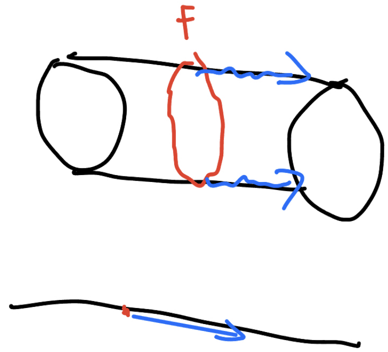


how you know the fibration is nontrivial.

More generally, $\mathbb{C}^{n+1} \supseteq S^{2n+1} \to \mathbb{C}P^n$, sending $(z_0, \dots, z_n) \to (z_0 : z_1 : \cdots : z_n)$.

Definition 0.2. A (Serre) fibration $p : Y \to B$ is a map which satisfies the homotopy lifting p for any CW complex X and any map $f : X \to Y$, there is a homotopy $F : X \times [0,1] \to B$ such that $F|_{t=0} = p \circ f$. So there is a lift

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "F"'] \arrow[ur, dotted] & B \end{tikzcd}$$

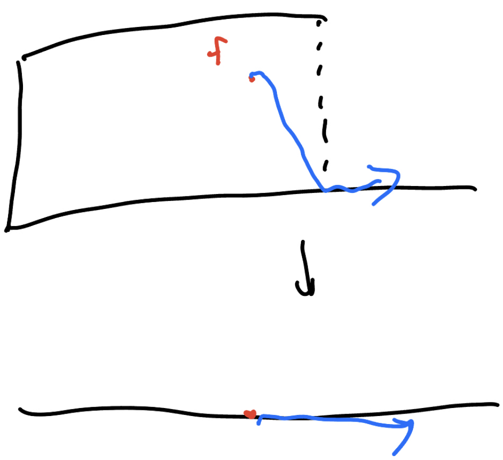


making the diagram commute.

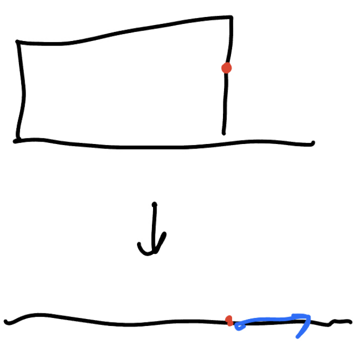
Exercise:

Show that fiber bundles are fibrations. Hint: enough to do it for $X = D^n$, and even for $F : D^n \to X$ stopping in a small region.

Example 0.3. 1. This is a fibration but not a fiber bundle:



This is not:



2. Given (Y, y_0) path-connected, let the path space

$$\mathcal{P}Y = \{\gamma : [0,1] \to Y \text{cts} \mid \gamma(0) = y_0\}$$

with the compact-open topology (i.e. the topology on $\mathrm{Map}(X, Y)$ such that for all Z, $f : Z \to \mathrm{Map}(X, Y)$ is continuous if and only if $Z \times X \to Y$, $(z, x) \mapsto f(z)(x)$ is continuous (hom-tensor adjunction))

Facts:

1. $\mathcal{P}Y$ is contractible via the contration

$$H(\gamma, t)(s) = \gamma((1-t)s)$$

2. $\mathcal{P}Y \to Y$ is a fibration sending $\gamma \mapsto \gamma(1)$ where the fiber is the loop space,

$$\Omega Y = \{\gamma : [0,1] \to Y \mid \gamma(0) = \gamma(1) = y_0\}$$

Exercise: Show that this is indeed the case

Exdercise: Use the Hom-tensor adjunction to show that it has HLP.

3. Any map $f : X \to Y$ can be replaced by a homotopy equivalent fibration. Let $\mathcal{P}_f = \{(\gamma : [0,1] \to Y, x \in X) \mid \gamma(0) = f(x)\} \subseteq \underbrace{\tilde{\mathcal{P}}}_{\text{unbased paths}} \quad Y \times X$

Then we have

$$\begin{tikzcd} \mathcal{P}_f(\gamma, x) \arrow[d] \arrow[dr] & \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

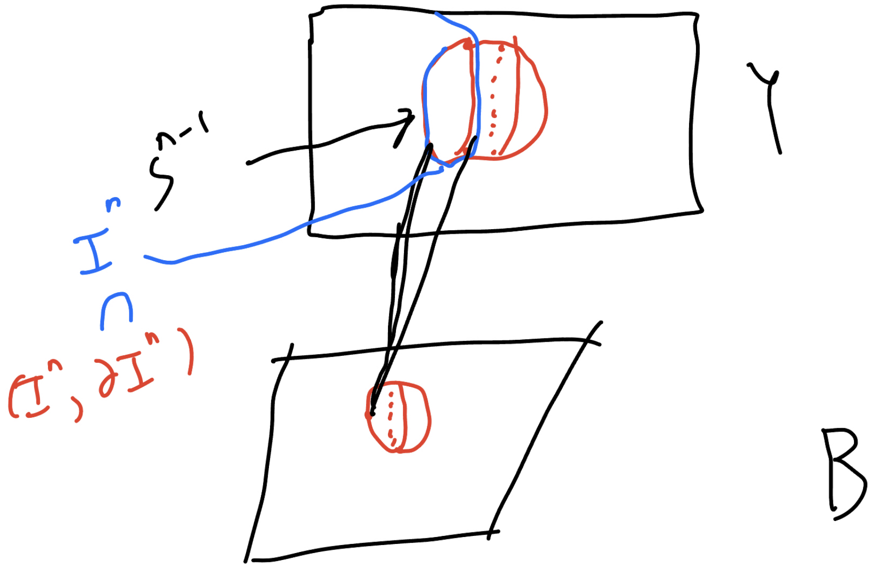
THe vertical map sends $(\gamma, x) \mapsto x$, the diagonal sends $(\gamma, x) \mapsto \gamma(1)$, which is a fibration, and this diagram commutes up to homotopy.

4.
- A homotopy of maps $X \to Y$ can be thought of as a map $X \times [0,1] \to Y$ but we can also think of it as a map $X \to \tilde{\mathcal{P}}Y$
- A nullhomotopy of a map $X \to Y$ is a map from $C(X) \to Y$, or $X \to \mathcal{P}(Y)$

Theorem 0.4. *(Long exact sequence of a fibration)*
For a fibration $F \hookrightarrow Y \twoheadrightarrow B$, there is a long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_n(F) \xrightarrow{i_*} \pi_n(Y) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \longrightarrow \cdots$$

Proof. (sketch)
We want to think of maps of S^n as being a map of $(I^n, \partial I^n)$. Here is a picture (don't really get?):



<u>Exercise:</u> Show that all fibers of a fibration are (w.) homotopy equivalent (use HLP)

Corollary 0.5. 1.

$$\pi_n(\Omega X) \cong \pi_{n+1}(X)$$

2. *For $n > 1$, $\pi_n(S^1) \cong \pi_n(\mathbb{R}) \cong 0$, so by the Hopf fibration $\pi_n(S^3) \cong \pi_n(S^2)$ for $n \geq 3$, and* $\pi_n(\mathbb{C}P^2) = \begin{cases} \mathbb{Z} & n = 2 \\ \pi_n(S^{2n+1}) & \text{otherwise} \end{cases}$

$$\lim_{k\to\infty} \pi_n(\mathbb{C}P^k) = \pi_n(\mathbb{C}P^\infty) = \begin{cases} \mathbb{Z} & n = 2 \\ 0 & n \neq 2 \end{cases}$$

Definition 0.3. An <u>Eilenberg-Maclane space</u> $K(A, n)$ is a space with

$$\pi_k(A) = \begin{cases} A & k = n \\ 0 & k \neq n \end{cases}$$

So $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$-space.