Lecture 1 - 3/5/24

A good starting point is Newton's equation $V(q_1, \ldots, q_n)$ for a particle:

$$m\ddot{q}_i = -\frac{\partial V}{\partial q_i}$$

The first observation is that if energy is

$$E = \frac{m}{2}\dot{q}^2 + V(q)$$

then E is constant along solution curves (take the t derivative).
A classic physics trick is to reduce nth order to first order by letting higher derivatives be introduced as new variables. Introduce $p_i = m\dot{q}_i$. We have the equations

$$\dot{q}_i = \frac{1}{m}p_i \qquad \dot{p}_i = -\frac{\partial V}{\partial q_i}$$

The energy becomes "Hamiltonian."

$$H(q,p) = \frac{1}{2m}\sum_{i=1}^{n} p_i^2 + V(q)$$

We can write these equations from earlier quite nicely in terms of the Hamiltonian (if you know the potential you know the Hamiltonian, and vice-versa) as

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Hamilton's equation. This looks similar to $\dot{X}_i = -\frac{\partial V}{\partial x_i}$, the equation of a gradient flow.
One advantage of these Hamiltonian equations is that we have lots of symmetry, i.e. for coordinate changes

$$\tilde{q}_i = f_i(q,p),\ \tilde{p}_i = g_i(q,p); \qquad \tilde{H}(\tilde{q},\tilde{p}) = H(q,p)$$

then in new coordinates, $\dot{\tilde{q}}_i = \frac{\partial \tilde{H}}{\partial \tilde{p}_i}, \dot{\tilde{p}}_i = -\frac{\partial \tilde{H}}{\partial \tilde{q}_i}$

Example 0.1. $\tilde{p}_i = -q_i, \tilde{q}_i = p_i$

Example 0.2. $\tilde{q}_i = q_i, \tilde{p}_i = p_i + \varphi_i(q_1, \ldots, q_n)$
We think of this Hamiltonian as having a very large infintie dimensional symmetry group, in contrast to the earlier graident flow, which has a very small symmetry group.

A Hamiltonian vector field

$$X_H = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} \right)$$

If we take the exterior derivative,

$$dH = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i \right)$$

We can write Hamilton's equation as $\iota(X_H)\omega = -dH$, with $\omega = \sum_{i=1}^{n} dq_i \wedge dp_i$ where ι means contraction.
Often we take this equation as the definition of the Hamiltonian vector field, which defines a differential equation, which defines a flow, et cetera.
Remark: The word "Symplectic" was introduced by Hermann Weyl in the theory of Lie groups.
Symplectic manifolds were introduced by Charles Ehrsmann and Paulette Libermann around 1948.
In the 60s and 70s, Souriau, Kostant (SP?), others did more work such as trying to phrase classical mechanics in this language. Many others, such as Arnold, Thurston, who showed that there symplectic and complex manifolds are not the same. Arnold initiated a program of symplectic topology in 74(?). Weinstein, Steinberg, Guillemain...

Part 1: Symplectic Linear Algebra

Definition 0.1. A symplectic structure on a finite dimensional (real for now) vector space E is a bilinear form

$$\omega : E \times E \to \mathbb{R}$$

which is

(i) Skew-symmetric, meaning $\omega(v, w) = \omega(w, v)$

(ii) Nondegenerate, meaning $\ker \omega \overset{\text{def}}{=} \{v \in E \mid \omega(v, w) = 0 \text{ for all } w \in E\}$ is trivial. (Every vector has a friend).

In terms of $\omega^\flat : E \to E^*$, $v \mapsto \omega(v, \cdot)$.
Skew-symmetry means $(\omega^\flat)* = -\omega^\flat$.
Non-degeneracy means $\ker(\omega^\flat) = 0$

Example 0.3.

1. "Standard symplectic structure" For $E = \mathbb{R}^{2n}$ with basis $e_1, \dots, e_n, f_1, \dots, f_n$, if we set

$$\omega(e_i, e_j) = 0, \omega(f_i, f_j) = 0, \omega(e_i, f_j) = \delta_{ij}$$

2. For V any finite dimensional vector space, setting $E = V \oplus V^*$, and

$$\omega((v_i, \alpha_i), (v_2, \alpha_2)) = \langle \alpha_1, v_2 \rangle - \langle \alpha_2, v_1 \rangle$$

3. If V is any finite-dimensional complex inner product space $h : V \times V \to \mathbb{C}$.
If we take $E = V$, $\omega(v, w) = \mathrm{Im}(h(v, w))$ is symplectic.

Note that these three are all actually the same example.

Definition 0.2. A symplectomorphism between symplectic vector spaces (E_i, ω_i), $(i = 1, 2)$ is a linear isomorphism $A : E_1 \to E_2$, such that

$$\omega_2(Av, Aw) = \omega_1(v, w)$$

for all $v, w \in E_1$ (I.e $\omega_1 = A^* \omega_2$).
Remark: stipulating that it is an isomorphism is a little overkill, because anything satisfying the second condition is injective.
Symplectomorphisms of (E, ω) to itself are denoted $\mathrm{Sp}(E, \omega)$, and is called the symplectic group.
In this sense, it is easy to see that those three examples are all symplectomorphic.

Lecture 2 - 9/10/24

Subspace of symplectic vector space

Let $F \subseteq E$. Define $F^\omega = \{v \in E \mid \omega(v, w) = 0 \text{ for all } w \in F\}$, the "$w$-orthogonal" space.
In terms of $\mathrm{ann}(F) = \{\alpha \in E^* \mid \alpha(v) = 0 \text{ for all } v \in F\}$
Note that $\omega^\flat : F^\omega \to \mathrm{ann}(F)$ is an isomorphism.

Proposition 1.

- $\dim F^\omega = \dim E - \dim F$
- $(F^\omega)^\omega = F$, $(F_1 \cap F_2)^\omega = F_1^\omega + F_2^\omega$, $(F_1 + F_2)^\omega = (F_1)^\omega \cap (F_2)^\omega$

Proof.

- $\dim F^\omega = \dim(\mathrm{ann}(F)) = \cdots$

- Since elements of F are orthogonal to elements of F^ω, we have $F \subseteq (F^\omega)^\omega$; by dimension count have equality. Etc

Definition 0.3. A subspace $F \subseteq E$ is called

- isotropic if $F \subseteq F^\omega$
- coisotropic if $F^\omega \subseteq F$
- Lagrangian if $F^\omega = F$.

Note F is isotropic if and only if $\omega|_{F\times F} = 0$
Note:
If F is isotropic, then $\dim F \leq \frac{1}{2}\dim E$
If F is coisotropic, then $\dim F \geq \frac{1}{2}\dim E$
In both cases, if equality holds, F is Lagrangian.
So if F is Lagrangian, then $\dim F = \frac{1}{2}\dim E$.

Definition 0.4. The set of Lagrangian subspaces is denoted $\mathrm{Lag}(E, \omega)$, called the "Lagrangian Grassmannian".

Proposition 2.

(a) $\mathrm{Lag}(E, \omega) \neq \varnothing$

(b) For every $M \in \mathrm{Lag}(E, \omega)$, there exists $L \in \mathrm{Lag}(E, \omega)$ with $L \cap M = \{0\}$ (i.e. $E = L \oplus M$).

Proof.

(a) By induction: Suppose $F \subseteq E$ is isotropic. If F is Lagrangian, we're done. Otherwise, $F \subset F^\omega$ is a proper subspace. Pick $v \in F^\omega \setminus F$. Then $F' = F + \mathrm{Span}\{v\}$ is again isotropic. The process ends when it becomes Lagrangian.

(b) By induction: suppose $F \subseteq E$ is isotropic, with $F \cap M = \{0\}$. If F is Lagrangian, we are done. Otherwise, $F + (F^\omega \cap M) \subseteq F^\omega$ is an isotropic subspace, hence is a proper subspace of F^ω. Pick $v \in F^\omega \setminus (F + (F^\omega \cap M))$; $F' = F + \mathit{span}\{v\}$

Claim. $F' \cap M = \{0\}$.

Proof. Indeed: if $y \in F' \cap M$, write $y = x + tv$, $x \in F, t \in \mathbb{R}$. Then

$$tv = y - x \in (F + M) \cap F^\omega = F + (F^\omega \cap F)$$

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Exercise: Given $L \in \text{Lag}(E, \omega)$. Let $F \subseteq E$ be any complement, i.e. $E = L \oplus F$.

(i) Show that there exists a unique linear map $A : F \to L$ such that $F^\omega = \{v + Av \mid v \in F\}$

(ii) Show that all $F_t = \{v + tAv \mid v \in F\}$ is a complement to L.

(iii) $F_{\frac{1}{2}}$ is Lagrangian

Given (E, ω), we can choose a Lagrangian splitting $E = L \oplus M$.

Proposition 3. *The choice of splitting identifies $M \cong L^*$ and determines a symplectomorphism*

$$E \to L \oplus L^*$$

Proof. Every $w \in M$ defines a linear functional $\alpha_\omega \in L^*$, $\alpha_\omega(v) = \omega(v, w)$
The map $M \to L^*$, $w \mapsto \alpha_w$ is an isomorphism, using the non-degeneracy of the symplectic structure. The resulting map $E \cong L \oplus M \to L \oplus L^*$ is a symplectomorphism (by formula for sympletic structure on $L \oplus L^*$).

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Proposition 4. *For every symplectic (E, ω), there exists a symplectomorphism $E \to \mathbb{R}^{2n}$, where $\mathbb{R}^{2n}$ has the standard symplectic structure.*

Proof. Choose a Lagrangian splitting $E = L \oplus L^*$. Now pick basis of L, dual basis of L^* to identify $E \cong \mathbb{R}^{2n}$
Remark:
$\text{Lag}(\mathbb{R}^2) = \mathbb{RP}(1) \cong S^1$
$\text{Lag}(\mathbb{R}^4) = ?$
Exercise:
Given $L \in \text{Lag}(E, \omega)$, show that the set of all

- complement to L is an affine space with corresponding linear space $\text{Hom}(E/L, L)$ (note if L is lagrangian, then E/L is naturally identified with L^*).

- Lagrangian complements to L is an affine space with corresponding linear space the self-adjoint maps $L^* \to L$.

In general, $\dim \text{Lag}(\mathbb{R}^{2n}) = \frac{n(n+1)}{2}$

Linear Reduction:

Let (E, ω) be symplectic. $F \subseteq E$ is symplectic if $\omega|_{F\times F}$ is nondegenerate. Equivalently, $\underbrace{F \cap F^\omega}_{=\ker \omega|_{F\times F}} = \{0\}$

Note that F is symplectic if and only if F^ω, and $E = F \oplus F^\omega$.
In general, if F is not symplectic, we can make it symplectic by quotienting by $\ker(\omega|_{F\times F}) = F \cap F^\omega$.

Proposition 5. *For any subspace F, the quotient $E_F = F/(F\cap F^\omega)$ inherits a symplectic structure:*

$$\omega_F(\pi(v), \pi(w)) = \omega(v, w)$$

where π is the quotient map $\pi : F \to F/(F \cap F^\omega)$

Proof. It's well defined: E.g, if $\pi(v) = 0$, then $v \in F \cap F^\omega$, so $\omega(v, w) = 0$ for all $w \in F$.
It's non-degenerate: If $\pi(v) \in \ker(\omega_F)$, then $\omega(v, w) = 0$ for all $w \in F$, so $v \in F \cap F^\omega$, so $\pi(v) = 0$.
Note: For F coisotropic, $E_F = F/F^\omega$

Proposition 6. *For F coisotropic, $L \subseteq E$ Lagrangian, the subspace $\pi(L \cap F) = L_F$ is again Lagrangian.*

Proof. Clearly, L_F is isotropic. To show L_F is Lagrangian, count dimension: $(L_F) = (L\cap F)/(L \cap F^\omega)$.

$$\begin{aligned}
\dim(L \cap F^\omega) &= \dim E - \dim(L \cap F^\omega)^\omega \\
&= \dim E - \dim(L + F) \\
&= \underbrace{\dim E}_{2\dim L} - \dim L - \dim F + \dim(L \cap F) \\
&= \dim L - \dim F + \dim(L \cap F)
\end{aligned}$$

So

$$\begin{aligned}
\dim(L_F) &= \dim(L \cap F) - \dim(L \cap F^\omega) \\
&= \dim F - \dim L \\
\dim E_f &= \dim F - \dim F^\omega \\
&= 2\dim F - \dim E \\
&= 2(\dim F - \dim L)
\end{aligned}$$

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So, we have constructed a map $\mathrm{Lag}(E, \omega) \to \mathrm{Lag}(E_F, \omega_F), L \mapsto L_F$
Warning: This map is not continuous!

It is discontinuous at the set of L's where L, F are not transverse. Away from this set, it's smooth.
Exercise:
Let $E = \mathbb{R}^4$ with standard symplectic basis. Take

$$F = \text{span}\{e_1, e_2, f_1\}$$

Then $F^\omega = \text{span}\{e_2\}$.
$F/F^\omega \cong \mathbb{R}^2 = \text{span}\{e_1, f_1\}$.
Let $L_t = \text{span}\{e_1 + tf_2, e_2 + tf_1\}$.

(a) Check L_t are Lagrangian

(b) Compute $(L_t)_F \subseteq \mathbb{R}^2$ and find it's discontinuous at $t = 0$.

Compatible complex structures

Recall:
Given a complex vector space V, we can always regard it as a real vector space of twice the dimension. "Multiplication by $\sqrt{-1}$" becomes a real linear transformation $\mathcal{J} \in \text{Hom}_\mathbb{R}(V, V)$, $\mathcal{J}^2 = -I$.
Conversely, a real vector space with such a $\mathcal{J}$ is called a complex structure, and we can imbue it with complex multiplication by defining

$$(a + ib)v = av + b(\mathcal{J}v)$$

Definition 0.5. Let (E, ω) be a symplectic vector space. A complex structure $\mathcal{J}$ (meaning $\mathcal{J}^2 = -I$) is ω-compatible if

$$g(v, w) \overset{\text{def}}{=} \omega(v, \mathcal{J}w)$$

defines an inner product. Denote by $\mathcal{J}(E, \omega) \overset{\text{def}}{=} \{\omega\text{−compatible complex structure }\}$.
Given $j \in \mathcal{J}(E, \omega)$, we get a complex inner product by

$$h(v, w) \overset{\text{def}}{=} g(v, w) + \sqrt{-1}\omega(v, w)$$

Remark: $\mathcal{J} \in \mathcal{J}(E, \omega)$ is a symplectomorphism:

$$\begin{aligned}
\omega(\mathcal{J}v, \mathcal{J}w) &= g(\mathcal{J}v, w) \\
&= g(w, \mathcal{J}v) \\
&= \omega(w, \mathcal{J}^2 v) \\
&= -\omega(w, v) \\
&= \omega(v, w)
\end{aligned}$$

<u>Remark:</u> For $\mathbb{R}^{2n}$, there is a standard complex structure given by $\mathcal{J}(e_i) = f_i, \mathcal{J}(f_i) = -e_i$.
This identifies $\mathbb{R}^{2n} \cong \mathbb{C}^n$. We can come up with more complex structures by picking an $A \in \mathrm{Sp}(E, \omega)$ and considering $\mathcal{J} \mapsto A\mathcal{J}A^{-1}$.
We have a map $\mathcal{J}(E, \omega) \to \mathrm{Riem}(E)$ (real inner products) given by $\mathcal{J} \mapsto g$, where g is as above.
There is a canonical left inverse $\varphi : \mathrm{Riem}(E) \to \mathcal{J}(E, \omega)$ as follows:

Proposition 7. *There is a canonical retraction (in the sense of topology) from* $\mathrm{Riem}(E) \to \mathcal{J}(E, \omega)$.

Proof. Given $k \in \mathrm{Riem}(E)$, define $A \in \mathrm{GL}(E)$ by $k(v, w) = \omega(v, Aw)$.
A is not a complex structure in general, but it's skew-symmetric with respect to h:

$$A^T = -A$$

Define $|A| = (A^T A)^{\frac{1}{2}} = (-A^2)^{\frac{1}{2}}$. This commutes with A by functional calculus, and define $\mathcal{J} = A|A|^{-1}$. This will do the job. ■