

Lecture 1 - 3/5/24

A good starting point is Newton's equation $V(q_1, \dots, q_n)$ for a particle:

$$m\ddot{q}_i = -\frac{\partial V}{\partial q_i}$$

The first observation is that if energy is

$$E = \frac{m}{2}\dot{q}^2 + V(q)$$

then E is constant along solution curves (take the t derivative).

A classic physics trick is to reduce n th order to first order by letting higher derivatives be introduced as new variables. Introduce $p_i = m\dot{q}_i$. We have the equations

$$\dot{q}_i = \frac{1}{m}p_i \qquad \dot{p}_i = -\frac{\partial V}{\partial q_i}$$

The energy becomes “Hamiltonian.”

$$H(q, p) = \frac{1}{2m} \sum_{i=1}^n p_i^2 + V(q)$$

We can write these equations from earlier quite nicely in terms of the Hamiltonian (if you know the potential you know the Hamiltonian, and vice-versa) as

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Hamilton's equation. This looks similar to $\dot{X}_i = -\frac{\partial V}{\partial x_i}$, the equation of a gradient flow.

One advantage of these Hamiltonian equations is that we have lots of symmetry, i.e. for coordinate changes

$$\tilde{q}_i = f_i(q, p), \quad \tilde{p}_i = g_i(q, p); \qquad \tilde{H}(\tilde{q}, \tilde{p}) = H(q, p)$$

then in new coordinates, $\dot{\tilde{q}}_i = \frac{\partial \tilde{H}}{\partial \tilde{p}_i}, \dot{\tilde{p}}_i = -\frac{\partial \tilde{H}}{\partial \tilde{q}_i}$

Example 0.1. $\tilde{p}_i = -q_i, \tilde{q}_i = p_i$

Example 0.2. $\tilde{q}_i = q_i, \tilde{p}_i = p_i + \varphi_i(q_1, \dots, q_n)$

We think of this Hamiltonian as having a very large infinite dimensional symmetry group, in contrast to the earlier gradient flow, which has a very small symmetry group.

A Hamiltonian vector field

$$X_H = \sum_{i=1}^n \left(\frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} \right)$$

If we take the exterior derivative,

$$dH = \sum_{i=1}^n \left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i \right)$$

We can write Hamilton's equation as $\iota(X_H)\omega = -dH$, with $\omega = \sum_{i=1}^n dq_i \wedge dp_i$ where ι means contraction.

Often we take this equation as the definition of the Hamiltonian vector field, which defines a differential equation, which defines a flow, et cetera.

Remark: The word "Symplectic" was introduced by Hermann Weyl in the theory of Lie groups.

Symplectic manifolds were introduced by Charles Ehresmann and Paulette Libermann around 1948.

In the 60s and 70s, Souriau, Kostant (SP?), others did more work such as trying to phrase classical mechanics in this language. Many others, such as Arnold, Thurston, who showed that there symplectic and complex manifolds are not the same. Arnold initiated a program of symplectic topology in 74(?). Weinstein, Steinberg, Guillemain...

Part 1: Symplectic Linear Algebra

Definition 0.1. A symplectic structure on a finite dimensional (real for now) vector space E is a bilinear form

$$\omega : E \times E \rightarrow \mathbb{R}$$

which is

- (i) Skew-symmetric, meaning $\omega(v, w) = -\omega(w, v)$
- (ii) Nondegenerate, meaning $\ker \omega \stackrel{\text{def}}{=} \{v \in E \mid \omega(v, w) = 0 \text{ for all } w \in E\}$ is trivial. (Every vector has a friend).

In terms of $\omega^b : E \rightarrow E^*$, $v \mapsto \omega(v, \cdot)$.

Skew-symmetry means $(\omega^b)^* = -\omega^b$.

Non-degeneracy means $\ker(\omega^b) = 0$

Example 0.3.

1. “Standard symplectic structure” For $E = \mathbb{R}^{2n}$ with basis $e_1, \dots, e_n, f_1, \dots, f_n$, if we set

$$\omega(e_i, e_j) = 0, \omega(f_i, f_j) = 0, \omega(e_i, f_j) = \delta_{ij}$$

2. For V any finite dimensional vector space, setting $E = V \oplus V^*$, and

$$\omega((v_i, \alpha_i), (v_j, \alpha_j)) = \langle \alpha_i, v_j \rangle - \langle \alpha_j, v_i \rangle$$

3. If V is any finite-dimensional complex inner product space $h : V \times V \rightarrow \mathbb{C}$.

If we take $E = V$, $\omega(v, w) = \text{Im}(h(v, w))$ is symplectic.

Note that these three are all actually the same example.

Definition 0.2. A symplectomorphism between symplectic vector spaces (E_i, ω_i) , $(i = 1, 2)$ is a linear isomorphism $A : E_1 \rightarrow E_2$, such that

$$\omega_2(Av, Aw) = \omega_1(v, w)$$

for all $v, w \in E_1$ (I.e $\omega_1 = A^*\omega_2$).

Remark: stipulating that it is an isomorphism is a little overkill, because anything satisfying the second condition is injective.

Symplectomorphisms of (E, ω) to itself are denoted $\text{Sp}(E, \omega)$, and is called the symplectic group. In this sense, it is easy to see that those three examples are all symplectomorphic.