# Lecture 4 - 9/12/24

Recall: A space $X$ is a $K(A, n)$ space if its only nontrivial homotopy group is $\pi_n(X) = A$.

**Lemma 1.** *$K(A, n)$'s exist for all groups $A$ (abelian if $n \geq 2$) and all $n \geq 1$.*

*Proof.* Choose a generating set $\{g_\alpha\}$ for $A$ and a set of relations $\{r_\beta\}$ such that $A = \langle g_\alpha, r_\beta \rangle$.
We build a CW complex $X$ where $X^{(n)} = \bigvee_\alpha S^n_\alpha$. Now glue on an $(n+1)$-cell for every $\beta$ along a map $S^{n+1} \to \bigvee_\alpha S^n_\alpha$ representing $r_\beta$. We have an $(n+1)$-complex with $\pi_n$ what we want, but may have nontrivial $\pi_{n+1}$.
At stage $k$, we've constructed a complex such that $\pi_n(X^{(k)}) = A$ and for all $i \leq k, i \neq n$, $\pi_i(X^{(k)}) = 0$. Now we add $k+1$-cells whose boundary maps kill the elements of $\pi_k$. By cellular approximation, this doesn't affect $\pi_i$ for $i < k$.

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**Lemma 2.** *Any two CW complexes that are $K(A, n)$'s are homotopy equivalent.*

*Proof.* Let $X$ be the complex from the previous lemma, and let $Y$ be some other $K(A, n)$ complex.
We

**1.** Build a map $X \to Y$ which induces equivalences on all $\pi_i$.

**2.** Apply Whitehead theorem.

First, map basepoint to basepoint, send $n$-cells $S^n_\alpha$ to corresponding generators of $\pi_n(Y)$ via a map $f_n$. We know that $f_n \circ \partial_\beta$ is null homotopic, so we can extend the map to $e^n_\beta$. For higher cells, we can always extend in an arbitrary way.

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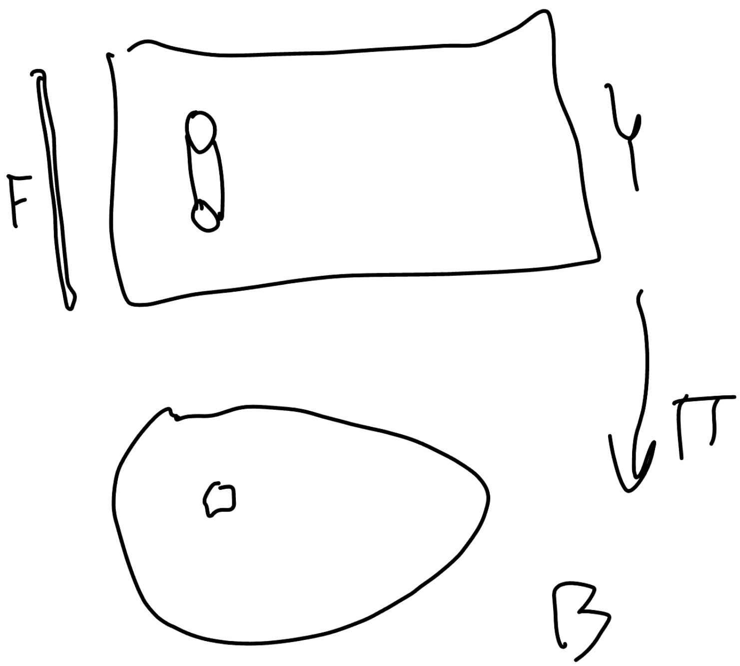
How to understand $H^*(K(A, n))$? We know $H^*(K(\mathbb{Z}, 1)) = H^*(S^1)$, so we can try to use the fibration

$$\begin{tikzcd} K(\mathbb{Z}, n-1) \cong \Omega(K(\mathbb{Z}, n) \arrow[r] & \mathcal{P}K(\mathbb{Z}, n) \cong * \arrow[d] \\ & K(\mathbb{Z}, n) \end{tikzcd}$$

## Serre spectral sequence

Say we have some base space $B$, with $\pi_1(B) = 0$, or, more generally, $\pi_1(B) \curvearrowright H_*(F)$ trivially.

Over this, we have a fibration $Y$ with fiber $F$. Maybe we understand the homology of $B$ and the homology of $F$, and we want to understand the homology of $Y$. Because $B$ is a CW complex, we can look at its skeleta.




**Lemma 3.** *Let $Y_p = \pi^{-1}(B^{(p)}$. Then $H_n(Y_p, Y_{p-1}) \cong C_p(B; H_{n-p}(F))$.*

*Proof.* Let $\phi_\alpha : (D^p, \partial D^p) \to (B^{(p)}, B^{(p-1)})$ be the inclusion maps of cells $e^p_\alpha$. By Excision,

$$\begin{aligned} H_n(Y_p, Y_{p-1}) &\cong H_n\left(\coprod_\alpha F \times e^p_\alpha, \coprod_\alpha F \times \partial e^p_\alpha\right) \\ &\cong \oplus_\alpha H_n(F \times e^p, F \times \partial e^p) \\ &\cong \oplus_\alpha H_{n-p}(F) \cong C_p(B, H_{n-p}(F)) \end{aligned}$$

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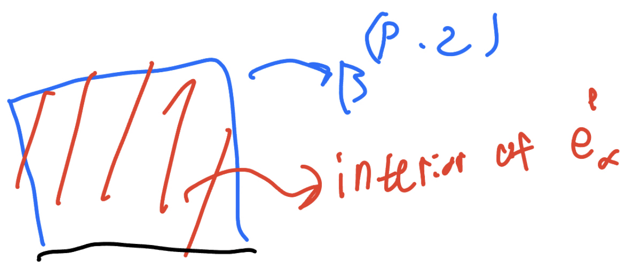
What parts of $H_n(Y_p, Y_{p-1})$ survive to $H_n(Y_{p+1}, Y_{p-2})$? We have the exact sequence

$$H_n(Y_p, Y_{p-1}) \longrightarrow H_n(Y_p, Y_{p-1}) \xrightarrow{\partial} H_{n-1}(Y_{p-1}, Y_{p-2})$$

$$Z_p(B; H_{n-p}(F)) \qquad C_p(B; H_{n-p}(F)) \xrightarrow{\partial} C_{p-1}(B; H_{n-p}(F))$$

To see the second $\partial$, consider a map $I^{p-1} \times F \to Y$ which lifts

$$\begin{tikzcd}
I^{p-1} \times F \arrow[r] & I^{p-1} \arrow[d, "\partial_\alpha"] \\
 & B
\end{tikzcd}$$




Use HLP to get a map $I^p \times F \to Y$ where $B^{(p-2)} \times F \to Y_{p-2}$, and thi s is the map we want.
By the same token, the stuff that survives into $H_n(Y_{p+1}, Y_{p-1})$.
Once we let $n$-cells in $(Y_p, Y_{p-1})$ see $n$-cells in $(Y_{p+1}, Y_p)$ and $(Y_{p-1}, Y_{p-2})$. What's left is $H_p(B; H_{n-p}(F))$
Next, there's some cancellation between the $(p, n-p)$ entry of this table and the $(p-2, n-p+1)$ entry, then the $(p-3, n-p+2)$, and so on, etc.
In the end, we should be left with "the part of $H_n(Y)$ that comes from $n$-cells that are $p$-cells of $B$ times $n-p$-cells of $F$."
Now arrange it all into a grid: indices $p, q = n-p$.
I'm not tikzing this.
I need to figure out all this stuff
Let's do some computation:
Some facts:

- You can do the same thing thing for $H^*$.
- Moreover, in this case cup products behave nicely with respect to the various pages.

**Example 0.1.** We want to compute $H^*(\mathbb{CP}^n)$ based on $S^1 \to S^{2n+1} \to \mathbb{CP}^n$. The $E_\infty$ page (with rational coefficients) should be $\mathbb{Q}$'s at the 0th and $(2n+1)$st diagonal and 0s elsewhere. Note that with $\mathbb{Q}$ coefficients, $H_n(B; H_q(F)) \cong H_n(B) \otimes H_q(F)$.