Lecture 1 - 3/5/24

A good starting point is Newton's equation $V(q_1, \ldots, q_n)$ for a particle:

$$m\ddot{q}_i = -\frac{\partial V}{\partial q_i}$$

The first observation is that if energy is

$$E = \frac{m}{2}\dot{q}^2 + V(q)$$

then E is constant along solution curves (take the t derivative).
A classic physics trick is to reduce nth order to first order by letting higher derivatives be introduced as new variables. Introduce $p_i = m\dot{q}_i$. We have the equations

$$\dot{q}_i = \frac{1}{m}p_i \qquad \dot{p}_i = -\frac{\partial V}{\partial q_i}$$

The energy becomes "Hamiltonian."

$$H(q,p) = \frac{1}{2m}\sum_{i=1}^{n} p_i^2 + V(q)$$

We can write these equations from earlier quite nicely in terms of the Hamiltonian (if you know the potential you know the Hamiltonian, and vice-versa) as

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Hamilton's equation. This looks similar to $\dot{X}_i = -\frac{\partial V}{\partial x_i}$, the equation of a gradient flow.
One advantage of these Hamiltonian equations is that we have lots of symmetry, i.e. for coordinate changes

$$\tilde{q}_i = f_i(q,p),\ \tilde{p}_i = g_i(q,p); \qquad \tilde{H}(\tilde{q},\tilde{p}) = H(q,p)$$

then in new coordinates, $\dot{\tilde{q}}_i = \frac{\partial \tilde{H}}{\partial \tilde{p}_i}, \dot{\tilde{p}}_i = -\frac{\partial \tilde{H}}{\partial \tilde{q}_i}$

Example 0.1. $\tilde{p}_i = -q_i, \tilde{q}_i = p_i$

Example 0.2. $\tilde{q}_i = q_i, \tilde{p}_i = p_i + \varphi_i(q_1, \ldots, q_n)$
We think of this Hamiltonian as having a very large infintie dimensional symmetry group, in contrast to the earlier graident flow, which has a very small symmetry group.

A Hamiltonian vector field

$$X_H = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} \right)$$

If we take the exterior derivative,

$$dH = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i \right)$$

We can write Hamilton's equation as $\iota(X_H)\omega = -dH$, with $\omega = \sum_{i=1}^{n} dq_i \wedge dp_i$ where ι means contraction.
Often we take this equation as the definition of the Hamiltonian vector field, which defines a differential equation, which defines a flow, et cetera.
Remark: The word "Symplectic" was introduced by Hermann Weyl in the theory of Lie groups.
Symplectic manifolds were introduced by Charles Ehrsmann and Paulette Libermann around 1948.
In the 60s and 70s, Souriau, Kostant (SP?), others did more work such as trying to phrase classical mechanics in this language. Many others, such as Arnold, Thurston, who showed that there symplectic and complex manifolds are not the same. Arnold initiated a program of symplectic topology in 74(?). Weinstein, Steinberg, Guillemain...

Part 1: Symplectic Linear Algebra

Definition 0.1. A symplectic structure on a finite dimensional (real for now) vector space E is a bilinear form

$$\omega : E \times E \to \mathbb{R}$$

which is

(i) Skew-symmetric, meaning $\omega(v, w) = \omega(w, v)$

(ii) Nondegenerate, meaning $\ker \omega \overset{\text{def}}{=} \{v \in E \mid \omega(v, w) = 0 \text{ for all } w \in E\}$ is trivial. (Every vector has a friend).

In terms of $\omega^\flat : E \to E^*$, $v \mapsto \omega(v, \cdot)$.
Skew-symmetry means $(\omega^\flat)* = -\omega^\flat$.
Non-degeneracy means $\ker(\omega^\flat) = 0$

Example 0.3.

1. "Standard symplectic structure" For $E = \mathbb{R}^{2n}$ with basis $e_1, \dots, e_n, f_1, \dots, f_n$, if we set

$$\omega(e_i, e_j) = 0, \omega(f_i, f_j) = 0, \omega(e_i, f_j) = \delta_{ij}$$

2. For V any finite dimensional vector space, setting $E = V \oplus V^*$, and

$$\omega((v_i, \alpha_i), (v_2, \alpha_2)) = \langle \alpha_1, v_2 \rangle - \langle \alpha_2, v_1 \rangle$$

3. If V is any finite-dimensional complex inner product space $h : V \times V \to \mathbb{C}$.
If we take $E = V$, $\omega(v, w) = \mathrm{Im}(h(v, w))$ is symplectic.

Note that these three are all actually the same example.

Definition 0.2. A symplectomorphism between symplectic vector spaces (E_i, ω_i), $(i = 1, 2)$ is a linear isomorphism $A : E_1 \to E_2$, such that

$$\omega_2(Av, Aw) = \omega_1(v, w)$$

for all $v, w \in E_1$ (I.e $\omega_1 = A^*\omega_2$).
Remark: stipulating that it is an isomorphism is a little overkill, because anything satisfying the second condition is injective.
Symplectomorphisms of (E, ω) to itself are denoted $\mathrm{Sp}(E, \omega)$, and is called the symplectic group.
In this sense, it is easy to see that those three examples are all symplectomorphic.

Lecture 2 - 9/10/24

Subspace of symplectic vector space

Let $F \subseteq E$. Define $F^\omega = \{v \in E \mid \omega(v, w) = 0 \text{ for all } w \in F\}$, the "$w$-orthogonal" space.
In terms of $\mathrm{ann}(F) = \{\alpha \in E^* \mid \alpha(v) = 0 \text{ for all } v \in F\}$
Note that $\omega^\flat : F^\omega \to \mathrm{ann}(F)$ is an isomorphism.

Proposition 1.

- $\dim F^\omega = \dim E - \dim F$
- $(F^\omega)^\omega = F$, $(F_1 \cap F_2)^\omega = F_1^\omega + F_2^\omega$, $(F_1 + F_2)^\omega = (F_1)^\omega \cap (F_2)^\omega$

Proof.

- $\dim F^\omega = \dim(\mathrm{ann}(F)) = \cdots$

- Since elements of F are orthogonal to elements of F^ω, we have $F \subseteq (F^\omega)^\omega$; by dimension count have equality. Etc

Definition 0.3. A subspace $F \subseteq E$ is called

- isotropic if $F \subseteq F^\omega$
- coisotropic if $F^\omega \subseteq F$
- Lagrangian if $F^\omega = F$.

Note F is isotropic if and only if $\omega|_{F\times F} = 0$
Note:
If F is isotropic, then $\dim F \leq \frac{1}{2}\dim E$
If F is coisotropic, then $\dim F \geq \frac{1}{2}\dim E$
In both cases, if equality holds, F is Lagrangian.
So if F is Lagrangian, then $\dim F = \frac{1}{2}\dim E$.

Definition 0.4. The set of Lagrangian subspaces is denoted $\mathrm{Lag}(E,\omega)$, called the "Lagrangian Grassmannian".

Proposition 2.

(a) $\mathrm{Lag}(E,\omega) \neq \varnothing$

(b) For every $M \in \mathrm{Lag}(E,\omega)$, there exists $L \in \mathrm{Lag}(E,\omega)$ with $L \cap M = \{0\}$ (i.e. $E = L \oplus M$).

Proof.

(a) By induction: Suppose $F \subseteq E$ is isotropic. If F is Lagrangian, we're done. Otherwise, $F \subset F^\omega$ is a proper subspace. Pick $v \in F^\omega \setminus F$. Then $F' = F + \mathrm{Span}\{v\}$ is again isotropic. The process ends when it becomes Lagrangian.

(b) By induction: suppose $F \subseteq E$ is isotropic, with $F \cap M = \{0\}$. If F is Lagrangian, we are done. Otherwise, $F + (F^\omega \cap M) \subseteq F^\omega$ is an isotropic subspace, hence is a proper subspace of F^ω. Pick $v \in F^\omega \setminus (F + (F^\omega \cap M))$; $F' = F + span\{v\}$

Claim. $F' \cap M = \{0\}$.

Proof. Indeed: if $y \in F' \cap M$, write $y = x + tv$, $x \in F, t \in \mathbb{R}$. Then

$$tv = y - x \in (F + M) \cap F^\omega = F + (F^\omega \cap F)$$

∎

■

<u>Exercise:</u> Given $L \in \mathrm{Lag}(E, \omega)$. Let $F \subseteq E$ be any complement, i.e. $E = L \oplus F$.

(i) Show that there exists a unique linear map $A : F \to L$ such that $F^\omega = \{v + Av \mid v \in F\}$

(ii) Show that all $F_t = \{v + tAv \mid v \in F\}$ is a complement to L.

(iii) $F_{\frac{1}{2}}$ is Lagrangian

Given (E, ω), we can choose a Lagrangian splitting $E = L \oplus M$.

Proposition 3. *The choice of splitting identifies $M \cong L^*$ and determines a symplectomorphism*

$$E \to L \oplus L^*$$

Proof. Every $w \in M$ defines a linear functional $\alpha_\omega \in L^*$, $\alpha_\omega(v) = \omega(v, w)$
The map $M \to L^*$, $w \mapsto \alpha_w$ is an isomorphism, using the non-degeneracy of the symplectic structure. The resulting map $E \cong L \oplus M \to L \oplus L^*$ is a symplectomorphism (by formula for sympletic structure on $L \oplus L^*$).

■

Proposition 4. *For every symplectic (E, ω), there exists a symplectomorphism $E \to \mathbb{R}^{2n}$, where $\mathbb{R}^{2n}$ has the standard symplectic structure.*

Proof. Choose a Lagrangian splitting $E = L \oplus L^*$. Now pick basis of L, dual basis of L^* to identify $E \cong \mathbb{R}^{2n}$
<u>Remark:</u>
$\mathrm{Lag}(\mathbb{R}^2) = \mathbb{RP}(1) \cong S^1$
$\mathrm{Lag}(\mathbb{R}^4) = ?$
<u>Exercise:</u>
Given $L \in \mathrm{Lag}(E, \omega)$, show that the set of all

- complement to L is an affine space with corresponding linear space $\mathrm{Hom}(E/L, L)$ (note if L is lagrangian, then E/L is naturally identified with L^*).
- Lagrangian complements to L is an affine space with corresponding linear space the self-adjoint maps $L^* \to L$.

In general, $\dim \mathrm{Lag}(\mathbb{R}^{2n}) = \frac{n(n+1)}{2}$

Linear Reduction:

Let (E,ω) be symplectic. $F \subseteq E$ is symplectic if $\omega|_{F\times F}$ is nondegenerate. Equivalently, $\underbrace{F\cap F^\omega}_{=\ker \omega|_{F\times F}} = \{0\}$

Note that F is symplectic if and only if F^ω, and $E = F \oplus F^\omega$.

In general, if F is not symplectic, we can make it symplectic by quotienting by $\ker(\omega|_{F\times F}) = F\cap F^\omega$.

Proposition 5. *For any subspace F, the quotient $E_F = F/(F\cap F^\omega)$ inherits a symplectic structure:*

$$\omega_F(\pi(v),\pi(w)) = \omega(v,w)$$

where π is the quotient map $\pi : F \to F/(F\cap F^\omega)$

Proof. It's well defined: E.g, if $\pi(v) = 0$, then $v \in F\cap F^\omega$, so $\omega(v,w) = 0$ for all $w \in F$.

It's non-degenerate: If $\pi(v) \in \ker(\omega_F)$, then $\omega(v,w) = 0$ for all $w \in F$, so $v \in F\cap F^\omega$, so $\pi(v) = 0$.

Note: For F coisotropic, $E_F = F/F^\omega$

Proposition 6. *For F coisotropic, $L \subseteq E$ Lagrangian, the subspace $\pi(L\cap F) = L_F$ is again Lagrangian.*

Proof. Clearly, L_F is isotropic. To show L_F is Lagrangian, count dimension: $(L_F) = (L\cap F)/(L\cap F^\omega)$.

$$\begin{aligned}
\dim(L\cap F^\omega) &= \dim E - \dim(L\cap F^\omega)^\omega \\
&= \dim E - \dim(L+F) \\
&= \underbrace{\dim E}_{2\dim L} - \dim L - \dim F + \dim(L\cap F) \\
&= \dim L - \dim F + \dim(L\cap F)
\end{aligned}$$

So

$$\begin{aligned}
\dim(L_F) &= \dim(L\cap F) - \dim(L\cap F^\omega) \\
&= \dim F - \dim L \\
\dim E_f &= \dim F - \dim F^\omega \\
&= 2\dim F - \dim E \\
&= 2(\dim F - \dim L)
\end{aligned}$$

∎

So, we have constructed a map $\mathrm{Lag}(E,\omega) \to \mathrm{Lag}(E_F,\omega_F), L \mapsto L_F$

Warning: This map is not continuous!

It is discontinuous at the set of L's where L, F are not transverse. Away from this set, it's smooth.
Exercise:
Let $E = \mathbb{R}^4$ with standard symplectic basis. Take

$$F = \operatorname{span}\{e_1, e_2, f_1\}$$

Then $F^\omega = \operatorname{span}\{e_2\}$.
$F/F^\omega \cong \mathbb{R}^2 = \operatorname{span}\{e_1, f_1\}$.
Let $L_t = \operatorname{span}\{e_1 + tf_2, e_2 + tf_1\}$.

(a) Check L_t are Lagrangian

(b) Compute $(L_t)_F \subseteq \mathbb{R}^2$ and find it's discontinuous at $t = 0$.

Compatible complex structures

Recall:
Given a complex vector space V, we can always regard it as a real vector space of twice the dimension. "Multiplication by $\sqrt{-1}$" becomes a real linear transformation $\mathcal{J} \in \operatorname{Hom}_{\mathbb{R}}(V, V)$, $\mathcal{J}^2 = -I$.
Conversely, a real vector space with such a $\mathcal{J}$ is called a complex structure, and we can imbue it with complex multiplication by defining

$$(a + ib)v = av + b(\mathcal{J}v)$$

Definition 0.5. Let (E, ω) be a symplectic vector space. A complex structure $\mathcal{J}$ (meaning $\mathcal{J}^2 = -I$) is ω-compatible if

$$g(v, w) \stackrel{\text{def}}{=} \omega(v, \mathcal{J}w)$$

defines an inner product. Denote by $\mathcal{J}(E, \omega) \stackrel{\text{def}}{=} \{\omega\text{−compatible complex structure }\}$.
Given $j \in \mathcal{J}(E, \omega)$, we get a complex inner product by

$$h(v, w) \stackrel{\text{def}}{=} g(v, w) + \sqrt{-1}\omega(v, w)$$

Remark: $\mathcal{J} \in \mathcal{J}(E, \omega)$ is a symplectomorphism:

$$\begin{aligned}
\omega(\mathcal{J}v, \mathcal{J}w) &= g(\mathcal{J}v, w) \\
&= g(w, \mathcal{J}v) \\
&= \omega(w, \mathcal{J}^2 v) \\
&= -\omega(w, v) \\
&= \omega(v, w)
\end{aligned}$$

Remark: For $\mathbb{R}^{2n}$, there is a standard complex structure given by $\mathcal{J}(e_i) = f_i, \mathcal{J}(f_i) = -e_i$.
This identifies $\mathbb{R}^{2n} \cong \mathbb{C}^n$. We can come up with more complex structures by picking an $A \in \mathrm{Sp}(E, \omega)$ and considering $\mathcal{J} \mapsto A\mathcal{J}A^{-1}$.
We have a map $\mathcal{J}(E, \omega) \to \mathrm{Riem}(E)$ (real inner products) given by $\mathcal{J} \mapsto g$, where g is as above.
There is a canonical left inverse $\varphi : \mathrm{Riem}(E) \to \mathcal{J}(E, \omega)$ as follows:

Proposition 7. *There is a canonical retraction (in the sense of topology) from* $\mathrm{Riem}(E) \to \mathcal{J}(E, \omega)$.

Proof. Given $k \in \mathrm{Riem}(E)$, define $A \in \mathrm{GL}(E)$ by $k(v, w) = \omega(v, Aw)$.
A is not a complex structure in general, but it's skew-symmetric with respect to h:

$$A^T = -A$$

Define $|A| = (A^T A)^{\frac{1}{2}} = (-A^2)^{\frac{1}{2}}$. This commutes with A by functional calculus, and define $\mathcal{J} = A|A|^{-1}$. This will do the job.

■

Lecture 3 - 9/12/24

Let (E, ω) be a symplectic vector space. Let $\mathcal{J} \in \mathrm{Hom}(E, E), \mathcal{J}^2 = -I$. Y is ω-compatible if

$$g(v, w) \stackrel{\text{def}}{=} \omega(v, \mathcal{J}w)$$

is a (real) inner product.

Example 0.4. Let $E = \mathbb{R}^{2n} = \mathrm{span}\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$. Let $\mathcal{J}e_i = f_i, \mathcal{J}f_i = -e_i$.
Then g is the standard inner product.
Remark:

- Note: in the definition of ω-compatible, any two of $\omega, \mathcal{J}, g$ deetermines the third.
- $\mathcal{J} \in \mathrm{Sp}(E, \omega) \cap O(E, g)$
- $h(v, w) = g(v, w) + i\omega(v, w)$ is a complex inner product, with corresponding unitary group $U(n) = \underbrace{U(E, h)}_{\text{preserves } h} = \underbrace{\mathrm{Sp}(E, \omega)}_{\text{preserves } \omega} \cap \underbrace{O(E, g)}_{\text{preserves } g}$. Recall that $U(n)$ is compact and connected, and has $\pi_1 = \mathbb{Z}$

Let $\mathcal{J}(E, \omega) = \{\mathcal{J} \mid \omega\text{-compatible }\}$
We have a map $\psi : \mathcal{J}(E, \omega) \to \mathrm{Riem}(E) \subseteq \mathrm{Sym}^2(E)$, which is contractible.

Theorem 0.1. *There is a canonical retraction*

$$\phi : \mathrm{Riem}(E) \to \mathcal{J}(E, \omega)$$

such that $\phi \circ \psi = \mathrm{Id}$.

Corollary 0.2. $\mathcal{J}(E, \omega)$ *is contractible*

Proof. Send

$$\mathcal{J} \mapsto \varphi\left((1-t)\psi(\mathcal{J}) + tg_0\right)$$

∎

Proof. Let $k \in \mathrm{Riem}(E)$ be given. Define $A \in \mathrm{Hom}(E, E)$ by $k(v, w) = \omega(v, Aw)$. Then $A = -A^T$ (skew-adjoint with respect to k). Why? note that

$$\begin{aligned} k(v, A^{-1}y) &= \omega(v, y) \\ &= -\omega(y, v) \\ &= -k(y, A^{-1}v \\ &= -k(A^{-1}v, y) \end{aligned}$$

So $(A^{-1})^T = A^{-1}$, so $A^T = -A$.
Hence, we can define $|A| = \sqrt{A^T A} = \sqrt{-A^2}$. Put $\mathcal{J} = A|A|^{-1}$. Then $\mathcal{J}^2 = A|A|^{-1}A|A|^{-1} = A^2|A|^{-2} = -I$.
Now we check that it defines an inner product:

$$\begin{aligned} g(v, w) &= \omega(v, \mathcal{J}w) \\ &= \omega(v, A|A|^{-1}\omega) \\ &= k(v, |A|^{-1}w) \\ &= k(|A|^{-\frac{1}{2}}v, |A|^{-\frac{1}{2}}w) \end{aligned}$$

is an inner product.

∎

If k was instead g with a compatible complex structure, then A must be that complex structure on the nose.
Now, $\mathrm{Sp}(E, \omega)$ acts on $\mathcal{J}(E, \omega)$ by

$$A \cdot \mathcal{J} = A\mathcal{J}A^{-1}$$

Proposition 8. *The action of* $\mathrm{Sp}(E, \omega)$ *on* $\mathcal{J}(E, \omega)$ *is transitive. That is, for any* $\mathcal{J}_1, \mathcal{J}_2 \in \mathcal{J}(E, \omega)$*, there is an* $A \in \mathrm{Sp}(E, \omega)$ *with* $\mathcal{J}_1 = A\mathcal{J}_2A^{-1}$*. It has stabilizers at* $\mathcal{J} \in \mathcal{J}(E, \omega)$ *the unitary group* $U(E)$*, with respect to* $\mathcal{J}$*. I.e.:*

$$\mathcal{J}(E, \omega) = \mathrm{Sp}(E, \omega)/U(E)$$

Corollary 0.3. $\mathrm{Sp}(E,\omega)$ *is connected.*

Proof. $\mathcal{J}(E,\omega)$ is connected, and $U(E)$ is connected, so $\mathrm{Sp}(E,\omega)$ is connected. ■

Proof. Given $\mathcal{J},\mathcal{J}' \in \mathcal{J}(E,\omega)$, let $e_1,\dots,e_n$ be an orthonormal basis for E a complex inner product space (with respect to $\mathcal{J}$).
Then $e_1,\dots,e_n, f_1 = \mathcal{J}e_1,\dots,f_n = \mathcal{J}e_n$ is a symplectic basis. Similarly, define $e_1',\dots,e_n',f_1',\dots,f_n'$ by $Ae_i = e_i', Af_i = f_i'$. This defines a symplectic transformation $A \in \mathrm{Sp}(E,\omega)$, with $A\mathcal{J}A^{-1} = \mathcal{J}'$. ■

More on $\mathrm{Sp}(E,\omega)$

Proposition 9. $\mathrm{Sp}(E,\omega)$ *is a connected Lie group of dimension $2n^2+n$, where* $\dim E = 2n$

Proof. By Cartan's theorem, every closed (in the sense of topology) subgroup of a Lie group is a Lie group.
This applies to $\mathrm{Sp}(E,\omega) \subseteq \mathrm{GL}(E)$ (invertible transformations). For connected, see above.
To get dimension, consider action of $\mathrm{GL}(E) \supseteq \mathcal{U} = \{\text{symplectic forms}\}$ on $\bigwedge^2 E^*$ the space of skew-symmetric bilinear forms. This action is transitive, with stabilizer at $\omega \in \mathcal{U}$ given by $\mathrm{Sp}(E,\omega)$.
Hence $\mathcal{U} = \mathrm{GL}(E)/\mathrm{Sp}(E,\omega)$, and using this we can count dimensions:

$$\underbrace{\dim \mathcal{U}}_{\dim = \dim \bigvee^2 E^* = \binom{2n}{2}} = \underbrace{\dim \mathrm{GL}(E)}_{\dim=(2n)^2} - \dim \mathrm{Sp}(E,\omega)$$

So $\dim \mathrm{Sp}(E,\omega) = (2n)^2 - \frac{2n(2n-1)}{2} = 2n^2+n$

Lecture 4 - 9/17/24

Geometry of $\mathrm{Sp}(E,\omega)$ and $\mathrm{Lag}(E,\omega)$

So far, we know $\mathrm{Sp}(E,\omega)$ is a Lie group. It is also connected, with dimension $2n^2+n$, where $\dim E = 2n$.
For any $\mathcal{J} \in \mathcal{J}(E,\omega)$, we have a real inner product $g(v,w) = \omega(v,\mathcal{J}w)$, and a complex one given by $h = g + \sqrt{-1}\omega$.
$U(E) \subseteq \mathrm{Sp}(E,\omega)$, $\mathcal{J}(E,\omega) = \mathrm{Sp}(E,\omega)/U(E)$
(Enough to consider $E = \mathbb{R}^{2n}$, $\mathcal{J}e_i = f_i, \mathcal{J}f_i = -e_i$).
Let $()^T$ be the transpose with respect to the metric g. In other words, $g(Av,w) = g(v,A^Tw)$.

Proposition 10. *$A \in \mathrm{GL}(E)$ is* <u>*symplectic*</u> *if and only if $A^T = \mathcal{J} A^{-1} \mathcal{J}^{-1}$.*

Proof. $A \in \mathrm{Sp}(E, \omega)$ if and only if for all $v, w, \omega(Av, Aw) = \omega(v, w)$. Note that $g(\mathcal{J} v, w) = \omega(v, w)$. So for all v, w,

$$\begin{aligned} g(\mathcal{J} Av, Aw) &= g(\mathcal{J} v, w) \\ &= g(A^T \mathcal{J} Av, w) \end{aligned}$$

so $A^T \mathcal{J} A = \mathcal{J}$.

∎

Consequence: if $A \in \mathrm{Sp}(E, \omega)$, then $\det(A) = 1$. This follows from

$$\begin{aligned} \det(A) &= \det(A^T) \\ &= \det(\mathcal{J} A^{-1} \mathcal{J}^{-1}) \\ &= \det(A)^{-1} \end{aligned}$$

So $\det(A)^2 = 1$, and since $\mathrm{Sp}(E, \omega)$ is connected, $\det(A) = 1$.
Now, in $\mathbb{R}^{2n}$, if we let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

$$A \in \mathrm{Sp}(E, \omega) \iff \begin{cases} a^T c = c^T a \\ b^T d = d^T b \\ a^T d - b^T c = I \end{cases}$$

Note: $\mathrm{Sp}(E, \omega)(\mathbb{R}^2, \omega) = \mathrm{SL}(2, \mathbb{R})$, the 2x2 matrices of determinant 1.
Another consequence is: A is symplectic implies that A^T is symplectic. This means we can use <u>polar decomposition</u>.
Recall: $A \in \mathrm{GL}(m, \mathbb{R})$ has a polar decomposition $A = U|A|$, where $|A|$ is positive definite, $|A| = \sqrt{A^T A}$, and $U \in O(n)$.
Can write $|A| = \exp(\xi)$, with $\xi^T = \xi$.
Thus, $\mathrm{GL}(n, \mathbb{R}) = O(n) \times \{\xi \mid \xi^T = \xi\}$ as a manifold.
For any $G \subseteq \mathrm{GL}(m, \mathbb{R})$, which is invariant under $A \mapsto A^T$, we get a polar decomposition

$$G = K \times p$$

where $K = G \cap O(m)$, $p = \mathfrak{g} \cap \{\xi \mid \xi^T = \xi\}$ where $\mathfrak{g} = \{\xi \mid \exp(t\xi) \in G \text{ for all } t\}$
In particular, $G = \mathrm{Sp}(E, \omega) \cong \mathrm{Sp}(2n, \omega)$, we get $K = \mathrm{Sp}(2n, \omega) \cap O(2n) = U(n)$ and $p = \{\xi \in \mathrm{Sp}(2n, \omega) \mid \xi = \xi^T\}$.
So we get that $\mathrm{Sp}(E, \omega) = U(e) \times p$.
Upshot: $\mathrm{Sp}(E, \omega)$ deformation retracts onto its maximal compact subgroup $U(E)$.

Corollary 0.4. *There is a canonical isomorphism*

$$\mu : \pi_1(\mathrm{Sp}(E, \omega)) \to \mathbb{Z}$$

Proof. $\pi_1(\text{Sp}(E,\omega)) \cong \pi_1(U(E))$. Now, $\det : U(E) \to \pi_1(U(1))$, and this map is an isomorphism.
This is the most primitive version of a "Maslov Index." It is a "Maslov index of loop of symplectomorphisms.
<u>Exercise:</u> If $A, B : S^1 \to \text{Sp}(E,\omega)$, then $\mu(AB) = \mu(A) + \mu(B)$.
<u>Remark:</u> We discussed the noncompact group $S_p(E,\omega) = \text{Sp}(2n,\mathbb{R})$. There is a compact "symplectic group" denoted $\text{Sp}(n)$. Both are "real forms" of the complex symplectic group $\text{Sp}(2n,\mathbb{C})$.

$$\text{Sp}(2n,\mathbb{R}) \subseteq \text{Sp}(2n,\mathbb{C}) \supseteq \text{Sp}(n)$$

but $\text{Sp}(2n,\mathbb{R}) \neq \text{Sp}(n)$. We have a similar situation for $\text{SL}(n,\mathbb{C})$:

$$\text{SL}(n,\mathbb{R}) \subseteq \text{SL}(n,\mathbb{C}) \supseteq \text{SU}(n)$$

These two on the left and right have the same complexification.

$$\mathbb{R}^* \subseteq \mathbb{C}^* \supseteq U(1)$$

Recall: The Lagrangian Grassmannian, $\text{Lag}(E) = \{L \subseteq E \mid L^\omega = L\}$. $\text{Lag}(E) \subseteq GR_n(E) = \{n\text{-dimensional subspaces of } E\}$, so it is a topological space in this way.
Recall that $Gr_k(E)$ can be seen as a manifold in 2 ways:

(i) View it as a homogeneous space

(ii) Construct charts

(i) Pick any $G \subseteq \text{GL}(E)$ such that G acts transitively on $Gr_k(E)$ (e.g. $G = \text{GL}(E), G = O(E)$ for some inner product, $G = \text{SO}(E)$).

  Let $H \subseteq G$ be a stabilizer of some <u>fixed</u> k-dim subspace. So $Gr_k(E) = G/H$

(ii) For any subspace $M \subseteq E$ of codimension k, the set $\{L \in Gr_k(E) \mid E = L \oplus M\}$ is canonically an affine space. It is isomorphic to $\{j : E/M \to E \mid \pi \circ j = \text{Id}\}$, $\pi : E \to E/M$, an affine space under $\text{Hom}(E/M, E)$

  The punchline is that for any fixed L, we get a vector space, and we use this as a chart.

Now, we want to do the same thing with $\text{Lag}(E)$.

Proposition 11. *The group $U(E)$ acts transitively on $\text{Lag}(E)$ with stabilizers at given $L \in \text{Lag}(E)$ equal to $O(L)$. $U(E)$ is a Lie group, so $\text{Lag}(E) = U(E)/O(L)$ is a manifold of dimension $\frac{n(n+1)}{2}$*

Proof. Let $h(v,w) = g(v,w) + \sqrt{-1}\omega(v,w)$.
Note: On any $L \in \text{Lag}(E)$, get $h|_{L\times L} = g|_{L\times L}$. A g-orthonormal basis $e_1, \dot{;} e_n$ of L is an h-orthonormal basis of E, given symplectic basis $e_1, \dots, e_n, f_i, \dots, f_n$, where $f_i = \mathcal{J} e_i$.
Given another L', choose g-orthonormal basis $e_1', \dots, e_n'$ of L'. It's h-orthonormal basis of E.
The transformation taking $e_1, \dots, e_n$ to $e_1', \dots, e_n'$ is in $U(E) \subseteq \text{Sp}(E,\omega)$ taking L to L'. The stabilizers of L are transformations for which $L = L'$, so they're in $O(L)$. Then

$$\begin{aligned}\dim \text{Lag}(E) &= \dim U(n) - \dim O(n)\\ &= n^2 - \frac{n(n-1)}{2}\\ &= \frac{n(n+1)}{2}\end{aligned}$$

∎

Alternatively, pick $\mathcal{J} \in \mathcal{J}(E,\omega)$, then $\text{Lag}(E) = \text{Sp}(E)/\text{Sp}(E)_L$ (?)
We get again a version of the Maslov index by Arnold.
The map $\det^2 : U(n) \to U(1), A \mapsto (\det(A))^2$ descends to a map $\text{Lag}(\mathbb{R}^{2n}) \to U(1)$, hence gives a map on fundamental groups

$$\mu : \pi_1(\text{Lag}(E)) \to \pi_1(U(1)) = \mathbb{Z}$$

Proposition 12. *(Arnold)*
This map is again an isomorphism

Proof. ∎
This is the maslov index of loop of Lagrangian subspaces.
Special Case $n = 1$
$\text{Lag}(\mathbb{R}^2) = U(1)/O(1)$. $O(1) = \{\pm 1\}$, so this is a circle under polar identifications, so we get $\mathbb{RP}^1$, which is again S^1.
Given $M \in \text{Lag}(E)$, let $\text{Lag}(E;M) = \{L \in \text{Lag}(E) \mid E = L \oplus M\}$

Proposition 13. $\text{Lag}(E;M)$ *is canonically an affine space, with corresponding linear spaces* $\text{Sym}^2(M) = \{$ *symmetric bilinear forms on* $M^*\} \cong$ *self adjoint maps* $M^* \to M$.

Proof. Let $\pi : E \to M^*$ where $\pi(E) =$ restriction of $\omega^\flat(v) \in E^*$ to M.
$\pi(v)(w) = \omega(v,w)$ for $w \in M$.
This projection map has kernel $M \subseteq E$ since M is Lagrangian, so gives isomorphisms $E/M \to M^*$.
$\text{Lag}(E;M) = \{L \in \text{Lag}(E) \mid L \oplus M = E\} \cong \{j : E/M \to E \mid j(M^*)$ is isotropic $, \pi \circ j = \text{Id}\}$.

Given any such j, any other splitting j' is of the form $j'(m) = j(m) + \psi(m)$ for some $\psi : E/M = M^* \to M$.
For all $\mu_1, \mu_2 \in M^*$,

$$\begin{aligned} 0 &= \omega(j'(\mu_1), j'(\mu_2)) \\ &= \omega(j(\mu_1) + \psi(\mu_1), j(\mu_2) + \psi(mu_2) \\ &= \underbrace{\omega(j(\mu_1), j(\mu_2))}_{=0 \text{ since } j \text{ isotropic}} + \underbrace{\omega(\psi(\mu_1), \psi(\mu_2))}_{=0 \text{ since } M \text{ isotropic}} \\ &\quad + \omega(j(\mu_1), \psi(\mu_2)) + \omega(\psi(\mu_1), j(\mu_2)) \\ &= \langle \mu_1, \psi(\mu_2) \rangle - \langle \mu_2, \psi(\mu_1) \rangle \end{aligned}$$

So ψ is self-adjoint, $\beta(\mu_1, \mu_2) = \langle \mu, \psi\mu \rangle$.
Again we see $\dim = \frac{n(n+1)}{2}$.
Down-to-earth version
Let $E = \mathbb{R}^{2n}$, $M = 0 \oplus \mathbb{R}^n = \operatorname{span}\{f_1, \ldots, f_n\}$.
$\mathbb{R}^{2n} = L \oplus M$ means L is graph of linear map $S : \mathbb{R}^n \to \mathbb{R}^n$.
L has basis

$$g_i = e_i + \sum_{j=1}^{n} S_{ij} f_j$$

is Lagrangian if and only if for all i, k,

$$\begin{aligned} 0 = \omega(g_i, g_k) &= \omega(e_i + \sum S_{ij} f_j, e_k + S_{kl} f_l \\ &= \cdots \\ &= S_{ki} - S_{ik} \end{aligned}$$

Lecture 5 - 9/19/24

Maslov Indices

Let (E, ω) be a symplectic vector space, $\dim E = 2n$. We consider $\operatorname{Lag}(E)$, the Lagrangian Grassmannian, the set of all Lagrangian subspaces. We know this is homeomorphic to $U(n)/O(n)$, once you choose a symplectic basis. It is a manifold, of dimension $fracn(n+1)2$.
We have $\pi_1(\operatorname{Lag}(E)) \cong \pi_1(U(n)/O(n))$. The function $\det^2$ descends to a function on this space, and gives a morphism from $\pi_1(U(n)/O(n))$ to $\pi_1(U(1)) \cong \mathbb{Z}$.
So we have a canonical $\mu : \pi_1(\operatorname{Lag}(E)) \to \mathbb{Z}$ is called the Maslov Index. It is somewhat akin to winding number.

We want to generalize to paths of Lagrangians. Fix a Lagrangian subspace $M \in \mathrm{Lag}(E)$, and define

$$\mathrm{Lag}(E, M) = \{L \in \mathrm{Lag}(E) \mid L \cap M = 0\}$$

This is canonically an affine space, and so is contractible. Let $\sum_M = \mathrm{Lag}(E) \setminus \mathrm{Lag}(E, M) = \{L \mid L \cap M \neq 0\}$. This is some kind of singular space.
Consider a path $L : [a, b] \to \mathrm{Lag}(E), t \mapsto L(t)$ with $L(a), L(b) \notin \sum_M$
Define the Maslov Index $[L : M] \stackrel{\text{def}}{=}$ Maslov index of loop obtained by concatenating $L(t)$ with a path in $\mathrm{Lag}(E, M)$ to make a loop. The contractibility of this space means the choice of path doesn't matter.
This is Maslov's "original" index as intersection number with the sincular cycle $\sum_M$.
More generally, we want to find $[L_1 : L_2]$ for two arbitrary Lagrangian paths, a kind of signed number of nonzero intersections $L_1(t) \cap L_2(t)$ (remember that $L_i(t)$ is a vector space!).
Let $L_1, L_2 \in \mathrm{Lag}(E, M)$ related by some $\beta_{12} \in \mathrm{Sym}^2(M)$ (symmetric bilinear form on $M^* \cong L_1$. Recall from last time we have $\beta_{12} \cdot L_1 = L_2$, $\beta_{21} \cdot L_2 = L_1$, $\beta_{12} = \beta_{21}$. To this setting we can attach an invariant.
Does there exist a symplectomorphism A such that $(L_1, L_2, M) \mapsto_A (L_1, L_2', M)$, with L_2' a Lagrangian transverse to all the others.

Signature of symmetric bilinear form

For a symmetric matrix B, we say the Signature, $\mathrm{Sig}(B)$, is the number of positive eigenvalues minus the number of negative eigenvalues.
For a symmetric bilinear form β, $\mathrm{Sig}(\beta) = \mathrm{Sig}(B)$, for B the matrix of β in terms of a basis (which does not affect the eigenvalues).
The number $\mathrm{Sig}(\beta_{12})$ depends on M.

Proposition 14. *If $L_1, L_2, L_3 \in \mathrm{Lag}(E; M)$. Then the number $s(L_1, L_2, L_3) = \mathrm{Sig}(\beta_{21}) + \mathrm{Sig}(\beta_{32}) + \mathrm{Sig}(\beta_{13})$ is actually independent of M. This is also called a Maslov index.*

Proof.

Proposition 15.

1. $s(L_1, L_2, L_3) = s(L_2, L_3, L_1)$

2. $S(L_1, L_2, L_3) = -S(L_2, L_1, L_3)$

3. *Cocycle identity: for all Lagrangians L_1, L_2, L_3, L_4,*

$$S(L_2, L_3, L_4) - s(L_1, L_3, L_4) + s(L_1, L_2, L_4) - s(L_1, L_2, L_3) = 0$$

4. *If $M(t)$ is always transverse to L_1, L_2, then $s(L_1, L_2, M)$ doesn't depend on t.*

5. *Up to symplectomorphism, L_1, L_2, L_3 is uniquely determined by* $\dim(L_1 \cap L_2), \dim(L_2 \cap L_3), \dim(L_1 \cap L_3), \dim(L_1 \cap L_2 \cap L_3),\ s(L_1, L_2, L_3)$

Proposition 16. *Suppose $[a, b] \to \mathrm{Lag}(E), t \mapsto L_i(t)$, $i = 1, 2$ are paths, and that there exists some $M \in \mathrm{Lag}(E)$ such that $L_i(t) \cap M = 0$ for all $i = 1, 2, t \in [a, b]$. Then*

$$[L_1; L_2] = \frac{1}{2}(s(L_1(a), L_2(a), M) - s(L_1(b), L_2(b), M))$$

Proof. Suppose M' is another choice. First term changes by

$$\begin{aligned} s(L_1(a), L_2(a), M') - s(L_1(a), L_2(a), M') &= s(L_1(a), M, M') - s(L_2(a), M, M') \\ &= s(L_1(b), M, M') - s(L_2(b), M, M') \end{aligned}$$

This is the change in the second term, so they cancel out.
General definition:
Consider a partition $a = t_0 < t_1 < \cdots < t_k = b$ such that for all $[t_{j-1}, t_j] \in M_j \in \mathrm{Lag}(E)$ with $L_i(t) \cap M_j = 0$ for all $t \in [t_{j-1}, t_j]$
Then

$$[L_1; L_2] = \frac{1}{2} \sum_{j=1}^{k} \big(s(L_1(t_{j-1}), L_2(t_{j-1}), M_j) - s(L_1(t_j), L_2(t_j), M_j)\big)$$

For $A \in \mathrm{Sp}(E, \omega)$, we have $\mathrm{Graph}(A) = \{(Av, v)\} \subseteq E \times \overline{E}$ (where $\overline{E}$ has the same symplectic form but with an opposite sign) is Lagrangian.
Define, for any path $A(t)$, $\mu(A) = [\mathrm{Graph}(A), \triangle]$

Lecture 6, 9/24/24

Part 2: Symplectic Manifolds

Recall that the Lie derivative of a vector field, $\mathscr{L}_X$, is given by $\frac{d}{dt}|_{t=0}(F_{-t})^*$, where F_t is a flow along X. Note $X(f) = \mathscr{L}_X f = \frac{d}{dt}|_{t=0}(F_{-t})^* f$
Differential of a map $F : M_1 \to M_2$ is a map $TF : TM_1 \to TM_2$.
For $f : M \to R$, $Tf : TM \to T\mathbb{R}$, while $df \in \Omega^1(M)$.
We will introduce symplectic manifolds by analogy to a complex manifold.

Complex manifolds

A complex manifold comes with a family of linear transformations $\mathcal{J}_m : T_mM \to T_mM$, with $\mathcal{J}_m^2 = -\operatorname{Id}_{T_mM}$, which depends smoothly on $m \in M$.
This is typically called an "almost complex structure".
A complex manifold is the same as a real manifold, but charts go to $\mathbb{C}^n$, and we want transition functions to be holomorphic. Any complex manifold has an almost complex structure on its tangent spaces, but a manifold with an almost complex structure is not necessarily a complex manifold.
It is some kind of integrability condition on $\mathcal{J} = \{\mathcal{J}_m\}$, and if this condition vanishes, then the almost complex structure comes from an honest complex manifold.

Symplectic manifolds

A symplectic manifold is equipped with a family of functions $\omega_m : T_mM \times T_mM \to \mathbb{R}$ which is symplectic, depending smoothly on m. This is called an "almost symplectic structure." There is again an integrability condition we can impose. We want to stipulate that ω_m arises from an $\omega = \{\omega_m\} \in \Omega^2 M$. The integrability condition is that $d\omega = 0$.

Definition 0.6. A symplectic structure on a manifold M is a non-degenerate 2-form $\omega \in \Omega^2(M)$ with $d\omega = 0$.
Non-degenerate just means that each $\omega|_{T_mM\times T_mM}$ is non-degenerate.

Proposition 17. *Any symplectic 2-form Ω, for $\dim M = 2n$, is non-degenerate if and only if* $\underbrace{\overbrace{\omega^n}}_{=\underbrace{\omega\wedge\cdots\wedge\omega}_{n\ times}} \neq 0$ *everywhere.*

Proof. Check at $m \in M$.
In one direction, suppose $(\omega_m)^n \neq 0$. We want to show $\ker \omega_m = 0$. Since $(\omega_m)^n \neq 0$, we have $\iota_V(\omega_m^n)$, where we define $\iota_v : \Omega^k(M) \to \Omega^{k-1}(M)$ by $\iota_V(\alpha) = \alpha(V, \cdots)$.
Anyways, $\iota_V(\omega^n)$ is nonzero for all $v \in T_m$.
But $\iota_V(\underbrace{\omega_m \wedge \cdots \wedge \omega_m}_{n}) = n(\iota_v\omega_m)\omega_m^{n-1}$, so $\iota_V\omega_m \neq 0$.

In the other direction, suppose $\ker(\omega_m) = 0$, so ω_m is symplectic. Let $e_1, \ldots, e_n, f_1, \ldots, f_n$ be a symplectic basis for T_mM with respect to ω_m.
Consider

$$\begin{aligned}
\iota_{e_n}\iota_{e_{n-1}}\cdots\iota_{e_1}(\omega_m^n) &= n(\iota_{e_n}\cdots\iota_{e_2})((\iota_{e_1}\omega_m)\wedge\omega_m^{n-1}) \\
&= n(n-1)(\iota_{e_n}\cdots\iota_{e_3})((\iota_{e_2}\omega_m)(\iota_{e_1}\omega_m)\omega_m^{n-2} \\
&\vdots \\
&= n!(\iota_{e_n}\omega_m)\wedge\cdots\wedge(\iota_{e_1}\omega_m)
\end{aligned}$$

So $\iota_{f_n} \cdots \iota_{e_1}(\omega_m^n) = \pm n! \neq 0$

Definition 0.7. Let (M, ω) be an (almost) symplectic manifold. The volume form

$$\bigwedge = \frac{\omega^n}{n!} = (\exp(\omega))_{\dim M}$$

is called the Liousville volume form on M.

Definition 0.8. Let (M, ω) be a symplectic manifold.

(a) A symplectomorphism is a diffeomorphism $F \in \mathrm{Diff}(M)$ preserving ω, i.e. $F^*\omega = \omega$. The group of symplectomorphisms of (M, ω) is denoted $\mathsf{Diff}(M, \omega)$

(b) A symplectic vector field on M is a vector field $X \in \mathscr{X}(M)$ preserving ω, i.e. $\mathscr{L}_X \omega = 0$.

The Lie algebra of symplectic vector fields is denoted $\mathscr{X}(M, \omega)$, i.e. the Local flow is symplectic.

Definition 0.9. Let (M, ω) be a symplectic manifold, let $H \in C^\infty(M)$.
The Hamiltonian vector field $X_H \in \mathscr{X}(M)$ is the unique vector field such that

$$\iota(X_H)\omega = -dH$$

The space of Hamiltonian vector fields is denoted $\mathscr{X}_{Ham}(M, \omega)$

Proposition 18. *Indeed,* $\mathscr{X}_{Ham}(M, \omega) \subseteq \mathscr{X}(M, \omega)$.

Proof. Let $X = X_H$ be Hamiltonian. We check

$$\begin{aligned}
\mathscr{L}_X \omega &= (d\iota_X + \iota_X d)\omega \\
&= d\iota_X \omega \\
&= -ddH \\
&= 0
\end{aligned}$$

by the Cartan Formula ∎

It turns out that ω is symplectic if and only if $\iota_X \omega$ is closed.
X is Hamiltonian if and only if ι_X is exact.
Basic examples:
Open subsets $U \subseteq \mathbb{R}^n$. Let $e_1, \ldots, e_n, f_1, \ldots, f_n$ be the standard symplectic basis.
Let $q_1, p_1, q_2, p_2, \ldots$ be corresponding coordinates. Take

$$\omega = \sum_{j=1}^{n} dq_i \wedge dp_i$$

Liousville Volume form: $\bigwedge = \frac{1}{n!}\omega \wedge \cdots \wedge \omega$,

$$\begin{aligned}\bigwedge &= \frac{1}{n!}(dq_1 \wedge dp_1 + \cdots) \wedge (dq_1 + dp_1 + \cdots) \wedge \cdots \\ &= dq_1 \wedge dp_1 \wedge dq_2 \wedge dp_2 \wedge \cdots \wedge dq_n \wedge dp_n\end{aligned}$$

which is the standard volume form.
Given $H \in C^\infty(U)$, $dH = \sum(\frac{\partial H}{\partial q_i}dq_i + \frac{\partial H}{\partial p_i}dp_i)$.
$\iota(X_H)\omega = -dH$ implies $X_H = \sum(\frac{\partial H}{\partial q_j}\frac{\partial}{\partial p_j} - \frac{\partial H}{\partial p_j}\frac{\partial}{\partial q_j})$, with corresponding ODE $\dot{q}_j = \frac{\partial H}{\partial p_j}, \dot{p}_j = -\frac{\partial H}{\partial q_j}$, which are Hamilton's equations.
Example: Cotangent bundles
Let $M = T^*Q$ (dual of TQ), with Q any manifold. Let $\pi : T^*Q \to Q$ be the projection.
There's a distinguished 1-form $\theta \in \Omega^1(T^*Q)$

Definition 0.10. The canonical 1-form $\theta \in \Omega^1(T^*Q)$ is defined in terms of its contractions with $v \in T_\mu(T^*Q)$, $\mu \in T^*Q$, where

$$\langle \theta_\mu, v\rangle = \langle \underbrace{\mu}_{\in T^*_{\pi(\mu)}Q}, \underbrace{T\pi(v)}_{\in T_{\pi(\mu)}Q}\rangle$$

Another perspective:
$T\pi : T_\mu(T^*Q) \to T_{\pi(\mu)}Q$ has a dual map $(T_\mu\pi)^* : \underbrace{T^*_{\pi(\mu)}Q}_{\ni\mu} \to T^*_\mu(T^*Q)$. Then

$(T_\mu\pi)^*(\mu) = \theta_| mu$
Another perspective:
For $\alpha \in \Omega^1(Q)$, let $\sigma_\alpha : Q \to T^*Q$ be the corresponding section.

Proposition 19. *$\theta \in \Omega^1(T^*Q)$ is the unique 1-form such that for all $\alpha \in \Omega^1(Q)$, $\sigma_\alpha^*\theta = \alpha$.*

Proof. Let $\omega \in T_qQ$, $q \in Q$. Let $\mu = \sigma_\alpha(q) = \alpha|_q$. Then

$$\begin{aligned}\langle \sigma_\alpha^*\theta|_q, \omega\rangle &= \langle \theta_\mu, (T_q\sigma_\alpha)(\omega)\rangle \\ &= \langle \mu, (T_\mu\pi)(T_q\sigma_\alpha)(\omega)\rangle \\ &= \langle \mu, \omega\rangle \\ &= \langle \alpha_q, \omega\rangle\end{aligned}$$

■

In coordinates: Let $q_1, \ldots, q_n$ be local coordinates on $U \subseteq Q$. Then $dq_1, \ldots, dq_n \in \Gamma(T^*Q|_\mu)$ are (pointwise) basis of 1-forms on U.
This gives a basis of T_q^*Q, all q. Let $p_1, \ldots, p_n$ be fiber coordinates, with cotangent coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$.

Lemma 1. *In cotangent coordinates,*

$$\theta = \sum_{i=1}^{n} p_i dq_i$$

Proof. Let $\alpha = \sum_{i=1}^{n} \alpha_i(q_1, \ldots, q_n) dq_i \in \Omega^1(Q)$.
Then

$$\sigma_\alpha : U \to T^*Q|_U, (q_1, \ldots, q_n) \mapsto (\alpha_1(q_1, \ldots, q_n), \alpha_2(q_1, \ldots, q_n), \cdots)$$

i.e. $\sigma_\alpha^* p_i = \alpha_i, \sigma_\alpha^* q_i = q_i$.

$$\sigma_\alpha^*(\sum p_i dq_i) = \sum \alpha_i dq_i = \alpha$$

So $\theta = \sum p_i dq_i$

■

Theorem 0.5. *The 2-form*

$$\omega = -d\theta \in \Omega^2(T^*Q)$$

is symplectic.
This is the canonical symplectic form on T^*Q.

Proof. In local coordinates,

$$\begin{aligned} -d\theta &= -d\left(\sum p_i dq_i\right) \\ &= \sum dq_i \wedge dp_i \end{aligned}$$

■

We'll describe symplectomorphisms etc. of T^*Q, taking into account $\pi : T^*Q \to Q$ is a fibration.
Terminology: For any surjective submersion $\pi : P \to Q$, we say $F \in \mathsf{Diff}(P)$ "lifts" $f \in \mathsf{Diff}(Q)$ if $\pi \circ F = f \circ \pi$. Denote by $\mathsf{Diff}(P, \pi)$ the "fibration-preserving" diffeomorphisms.
We'll find all of $\mathsf{Diff}(T^*Q, \omega) \cap \mathsf{Diff}(T^*Q, \omega)$
Example: If $P = TQ$, $f \in \mathsf{Diff}(Q)$, then $f_T = Tf \in \mathsf{Diff}(TQ)$ is a lift.
If $P = T^*Q$, $f \in \mathsf{Diff}(Q)$, $f_{T^*} = (Tf^{-1})^* \in \mathsf{Diff}(T^*Q, \pi)$
The upshot is that all f_{T^*} are symplectic.
Other lifts: given $\alpha \in \Omega^1(Q)$ we get $G_\alpha \in \mathsf{Diff}(T^*Q, \pi)$ by adding α fiberwise.
We'll see that G_α is symplectic if and only if $d\alpha = 0$.