Lecture 1 - 3/5/24

A good starting point is Newton's equation $V(q_1, \ldots, q_n)$ for a particle:

$$m\ddot{q}_i = -\frac{\partial V}{\partial q_i}$$

The first observation is that if energy is

$$E = \frac{m}{2}\dot{q}^2 + V(q)$$

then E is constant along solution curves (take the t derivative).
A classic physics trick is to reduce nth order to first order by letting higher derivatives be introduced as new variables. Introduce $p_i = m\dot{q}_i$. We have the equations

$$\dot{q}_i = \frac{1}{m}p_i \qquad \dot{p}_i = -\frac{\partial V}{\partial q_i}$$

The energy becomes "Hamiltonian."

$$H(q,p) = \frac{1}{2m}\sum_{i=1}^{n} p_i^2 + V(q)$$

We can write these equations from earlier quite nicely in terms of the Hamiltonian (if you know the potential you know the Hamiltonian, and vice-versa) as

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Hamilton's equation. This looks similar to $\dot{X}_i = -\frac{\partial V}{\partial x_i}$, the equation of a gradient flow.
One advantage of these Hamiltonian equations is that we have <u>lots</u> of symmetry, i.e. for coordinate changes

$$\tilde{q}_i = f_i(q,p),\ \tilde{p}_i = g_i(q,p); \qquad \tilde{H}(\tilde{q},\tilde{p}) = H(q,p)$$

then in new coordinates, $\dot{\tilde{q}}_i = \frac{\partial \tilde{H}}{\partial \tilde{p}_i}, \dot{\tilde{p}}_i = -\frac{\partial \tilde{H}}{\partial \tilde{q}_i}$

Example 0.1. $\tilde{p}_i = -q_i, \tilde{q}_i = p_i$

Example 0.2. $\tilde{q}_i = q_i, \tilde{p}_i = p_i + \varphi_i(q_1, \ldots, q_n)$
We think of this Hamiltonian as having a very large infintie dimensional symmetry group, in contrast to the earlier graident flow, which has a very small symmetry group.

A Hamiltonian vector field

$$X_H = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} \right)$$

If we take the exterior derivative,

$$dH = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i \right)$$

We can write Hamilton's equation as $\iota(X_H)\omega = -dH$, with $\omega = \sum_{i=1}^{n} dq_i \wedge dp_i$ where ι means contraction.
Often we take this equation as the definition of the Hamiltonian vector field, which defines a differential equation, which defines a flow, et cetera.
Remark: The word "Symplectic" was introduced by Hermann Weyl in the theory of Lie groups.
Symplectic manifolds were introduced by Charles Ehrsmann and Paulette Libermann around 1948.
In the 60s and 70s, Souriau, Kostant (SP?), others did more work such as trying to phrase classical mechanics in this language. Many others, such as Arnold, Thurston, who showed that there symplectic and complex manifolds are not the same. Arnold initiated a program of symplectic topology in 74(?). Weinstein, Steinberg, Guillemain...

Part 1: Symplectic Linear Algebra

Definition 0.1. A symplectic structure on a finite dimensional (real for now) vector space E is a bilinear form

$$\omega : E \times E \to \mathbb{R}$$

which is

(i) Skew-symmetric, meaning $\omega(v, w) = \omega(w, v)$

(ii) Nondegenerate, meaning $\ker \omega \overset{\text{def}}{=} \{v \in E \mid \omega(v, w) = 0 \text{ for all } w \in E\}$ is trivial. (Every vector has a friend).

In terms of $\omega^\flat : E \to E^*$, $v \mapsto \omega(v, \cdot)$.
Skew-symmetry means $(\omega^\flat)* = -\omega^\flat$.
Non-degeneracy means $\ker(\omega^\flat) = 0$

Example 0.3.

1. "Standard symplectic structure" For $E = \mathbb{R}^{2n}$ with basis $e_1, \dots, e_n, f_1, \dots, f_n$, if we set

$$\omega(e_i, e_j) = 0, \omega(f_i, f_j) = 0, \omega(e_i, f_j) = \delta_{ij}$$

2. For V any finite dimensional vector space, setting $E = V \oplus V^*$, and

$$\omega((v_i, \alpha_i), (v_2, \alpha_2)) = \langle \alpha_1, v_2 \rangle - \langle \alpha_2, v_1 \rangle$$

3. If V is any finite-dimensional complex inner product space $h : V \times V \to \mathbb{C}$.

If we take $E = V$, $\omega(v, w) = \mathrm{Im}(h(v, w))$ is symplectic.

Note that these three are all actually the same example.

Definition 0.2. A symplectomorphism between symplectic vector spaces (E_i, ω_i), $(i = 1, 2)$ is a linear isomorphism $A : E_1 \to E_2$, such that

$$\omega_2(Av, Aw) = \omega_1(v, w)$$

for all $v, w \in E_1$ (I.e $\omega_1 = A^*\omega_2$).
Remark: stipulating that it is an isomorphism is a little overkill, because anything satisfying the second condition is injective.
Symplectomorphisms of (E, ω) to itself are denoted $\mathrm{Sp}(E, \omega)$, and is called the symplectic group.
In this sense, it is easy to see that those three examples are all symplectomorphic.

Lecture 2 - 9/10/24

Subspace of symplectic vector space

Let $F \subseteq E$. Define $F^\omega = \{v \in E \mid \omega(v, w) = 0 \text{ for all } w \in F\}$, the "$w$-orthogonal" space.
In terms of $\mathrm{ann}(F) = \{\alpha \in E^* \mid \alpha(v) = 0 \text{ for all } v \in F\}$
Note that $\omega^\flat : F^\omega \to \mathrm{ann}(F)$ is an isomorphism.

Proposition 1.

- $\dim F^\omega = \dim E - \dim F$
- $(F^\omega)^\omega = F$, $(F_1 \cap F_2)^\omega = F_1^\omega + F_2^\omega$, $(F_1 + F_2)^\omega = (F_1)^\omega \cap (F_2)^\omega$

Proof.

- $\dim F^\omega = \dim(\mathrm{ann}(F)) = \cdots$

- Since elements of F are orthogonal to elements of F^ω, we have $F \subseteq (F^\omega)^\omega$; by dimension count have equality. Etc

Definition 0.3. A subspace $F \subseteq E$ is called

- isotropic if $F \subseteq F^\omega$
- coisotropic if $F^\omega \subseteq F$
- Lagrangian if $F^\omega = F$.

Note F is isotropic if and only if $\omega|_{F\times F} = 0$
Note:
If F is isotropic, then $\dim F \leq \frac{1}{2}\dim E$
If F is coisotropic, then $\dim F \geq \frac{1}{2}\dim E$
In both cases, if equality holds, F is Lagrangian.
So if F is Lagrangian, then $\dim F = \frac{1}{2}\dim E$.

Definition 0.4. The set of Lagrangian subspaces is denoted $\mathrm{Lag}(E, \omega)$, called the "Lagrangian Grassmannian".

Proposition 2.

(a) $\mathrm{Lag}(E, \omega) \neq \varnothing$

(b) For every $M \in \mathrm{Lag}(E, \omega)$, *there exists* $L \in \mathrm{Lag}(E, \omega)$ *with* $L \cap M = \{0\}$ *(i.e.* $E = L \oplus M$*).*

Proof.

(a) By induction: Suppose $F \subseteq E$ is isotropic. If F is Lagrangian, we're done. Otherwise, $F \subset F^\omega$ is a proper subspace. Pick $v \in F^\omega \setminus F$. Then $F' = F + \mathrm{Span}\{v\}$ is again isotropic. The process ends when it becomes Lagrangian.

(b) By induction: suppose $F \subseteq E$ is isotropic, with $F \cap M = \{0\}$. If F is Lagrangian, we are done. Otherwise, $F + (F^\omega \cap M) \subseteq F^\omega$ is an isotropic subspace, hence is a proper subspace of F^ω. Pick $v \in F^\omega \setminus (F + (F^\omega \cap M))$; $F' = F + span\{v\}$

Claim. $F' \cap M = \{0\}$.

Proof. Indeed: if $y \in F' \cap M$, write $y = x + tv$, $x \in F, t \in \mathbb{R}$. Then

$$tv = y - x \in (F + M) \cap F^\omega = F + (F^\omega \cap F)$$

∎

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<u>Exercise:</u> Given $L \in \operatorname{Lag}(E, \omega)$. Let $F \subseteq E$ be any complement, i.e. $E = L \oplus F$.

(i) Show that there exists a unique linear map $A : F \to L$ such that $F^\omega = \{v + Av \mid v \in F\}$

(ii) Show that all $F_t = \{v + tAv \mid v \in F\}$ is a complement to L.

(iii) $F_{\frac{1}{2}}$ is Lagrangian

Given (E, ω), we can choose a Lagrangian splitting $E = L \oplus M$.

Proposition 3. *The choice of splitting identifies $M \cong L^*$ and determines a symplectomorphism*

$$E \to L \oplus L^*$$

Proof. Every $w \in M$ defines a linear functional $\alpha_\omega \in L^*$, $\alpha_\omega(v) = \omega(v, w)$
The map $M \to L^*$, $w \mapsto \alpha_w$ is an isomorphism, using the non-degeneracy of the symplectic structure. The resulting map $E \cong L \oplus M \to L \oplus L^*$ is a symplectomorphism (by formula for sympletic structure on $L \oplus L^*$).

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Proposition 4. *For every symplectic (E, ω), there exists a symplectomorphism $E \to \mathbb{R}^{2n}$, where $\mathbb{R}^{2n}$ has the standard symplectic structure.*

Proof. Choose a Lagrangian splitting $E = L \oplus L^*$. Now pick basis of L, dual basis of L^* to identify $E \cong \mathbb{R}^{2n}$
<u>Remark:</u>
$\operatorname{Lag}(\mathbb{R}^2) = \mathbb{RP}(1) \cong S^1$
$\operatorname{Lag}(\mathbb{R}^4) = ?$
<u>Exercise:</u>
Given $L \in \operatorname{Lag}(E, \omega)$, show that the set of all

- complement to L is an affine space with corresponding linear space $\operatorname{Hom}(E/L, L)$ (note if L is lagrangian, then E/L is naturally identified with L^*).
- Lagrangian complements to L is an affine space with corresponding linear space the self-adjoint maps $L^* \to L$.

In general, $\dim \operatorname{Lag}(\mathbb{R}^{2n}) = \frac{n(n+1)}{2}$

Linear Reduction:

Let (E,ω) be symplectic. $F \subseteq E$ is symplectic if $\omega|_{F\times F}$ is nondegenerate. Equivalently, $\underbrace{F \cap F^\omega}_{=\ker \omega|_{F\times F}} = \{0\}$

Note that F is symplectic if and only if F^ω, and $E = F \oplus F^\omega$.
In general, if F is not symplectic, we can make it symplectic by quotienting by $\ker(\omega|_{F\times F}) = F \cap F^\omega$.

Proposition 5. *For any subspace F, the quotient $E_F = F/(F\cap F^\omega)$ inherits a symplectic structure:*

$$\omega_F(\pi(v),\pi(w)) = \omega(v,w)$$

where π is the quotient map $\pi : F \to F/(F\cap F^\omega)$

Proof. It's well defined: E.g, if $\pi(v) = 0$, then $v \in F\cap F^\omega$, so $\omega(v,w) = 0$ for all $w \in F$.
It's non-degenerate: If $\pi(v) \in \ker(\omega_F)$, then $\omega(v,w) = 0$ for all $w \in F$, so $v \in F\cap F^\omega$, so $\pi(v) = 0$.
Note: For F coisotropic, $E_F = F/F^\omega$

Proposition 6. *For F coisotropic, $L \subseteq E$ Lagrangian, the subspace $\pi(L\cap F) = L_F$ is again Lagrangian.*

Proof. Clearly, L_F is isotropic. To show L_F is Lagrangian, count dimension: $(L_F) = (L\cap F)/(L\cap F^\omega)$.

$$\begin{aligned}
\dim(L\cap F^\omega) &= \dim E - \dim(L\cap F^\omega)^\omega \\
&= \dim E - \dim(L+F) \\
&= \underbrace{\dim E}_{2\dim L} - \dim L - \dim F + \dim(L\cap F) \\
&= \dim L - \dim F + \dim(L\cap F)
\end{aligned}$$

So

$$\begin{aligned}
\dim(L_F) &= \dim(L\cap F) - \dim(L\cap F^\omega) \\
&= \dim F - \dim L \\
\dim E_f &= \dim F - \dim F^\omega \\
&= 2\dim F - \dim E \\
&= 2(\dim F - \dim L)
\end{aligned}$$

∎

So, we have constructed a map $\mathrm{Lag}(E,\omega) \to \mathrm{Lag}(E_F,\omega_F), L \mapsto L_F$
Warning: This map is not continuous!

It is discontinuous at the set of L's where L, F are not transverse. Away from this set, it's smooth.
Exercise:
Let $E = \mathbb{R}^4$ with standard symplectic basis. Take

$$F = \text{span}\{e_1, e_2, f_1\}$$

Then $F^\omega = \text{span}\{e_2\}$.
$F/F^\omega \cong \mathbb{R}^2 = \text{span}\{e_1, f_1\}$.
Let $L_t = \text{span}\{e_1 + tf_2, e_2 + tf_1\}$.

(a) Check L_t are Lagrangian

(b) Compute $(L_t)_F \subseteq \mathbb{R}^2$ and find it's discontinuous at $t = 0$.

Compatible complex structures

Recall:
Given a complex vector space V, we can always regard it as a real vector space of twice the dimension. "Multiplication by $\sqrt{-1}$" becomes a real linear transformation $\mathcal{J} \in \text{Hom}_{\mathbb{R}}(V, V)$, $\mathcal{J}^2 = -I$.
Conversely, a real vector space with such a $\mathcal{J}$ is called a complex structure, and we can imbue it with complex multiplication by defining

$$(a + ib)v = av + b(\mathcal{J}v)$$

Definition 0.5. Let (E, ω) be a symplectic vector space. A complex structure $\mathcal{J}$ (meaning $\mathcal{J}^2 = -I$) is ω-compatible if

$$g(v, w) \overset{\text{def}}{=} \omega(v, \mathcal{J}w)$$

defines an inner product. Denote by $\mathcal{J}(E, \omega) \overset{\text{def}}{=} \{\omega\text{−compatible complex structure }\}$.
Given $j \in \mathcal{J}(E, \omega)$, we get a complex inner product by

$$h(v, w) \overset{\text{def}}{=} g(v, w) + \sqrt{-1}\omega(v, w)$$

Remark: $\mathcal{J} \in \mathcal{J}(E, \omega)$ is a symplectomorphism:

$$\begin{aligned}
\omega(\mathcal{J}v, \mathcal{J}w) &= g(\mathcal{J}v, w) \\
&= g(w, \mathcal{J}v) \\
&= \omega(w, \mathcal{J}^2 v) \\
&= -\omega(w, v) \\
&= \omega(v, w)
\end{aligned}$$

Remark: For $\mathbb{R}^{2n}$, there is a standard complex structure given by $\mathcal{J}(e_i) = f_i, \mathcal{J}(f_i) = -e_i$.
This identifies $\mathbb{R}^{2n} \cong \mathbb{C}^n$. We can come up with more complex structures by picking an $A \in \mathrm{Sp}(E, \omega)$ and considering $\mathcal{J} \mapsto A\mathcal{J}A^{-1}$.
We have a map $\mathcal{J}(E, \omega) \to \mathrm{Riem}(E)$ (real inner products) given by $\mathcal{J} \mapsto g$, where g is as above.
There is a canonical left inverse $\varphi : \mathrm{Riem}(E) \to \mathcal{J}(E, \omega)$ as follows:

Proposition 7. *There is a canonical retraction (in the sense of topology) from* $\mathrm{Riem}(E) \to \mathcal{J}(E, \omega)$.

Proof. Given $k \in \mathrm{Riem}(E)$, define $A \in \mathrm{GL}(E)$ by $k(v, w) = \omega(v, Aw)$.
A is not a complex structure in general, but it's skew-symmetric with respect to h:

$$A^T = -A$$

Define $|A| = (A^T A)^{\frac{1}{2}} = (-A^2)^{\frac{1}{2}}$. This commutes with A by functional calculus, and define $\mathcal{J} = A|A|^{-1}$. This will do the job.

■

Lecture 3 - 9/12/24

Let (E, ω) be a symplectic vector space. Let $\mathcal{J} \in \mathrm{Hom}(E, E), \mathcal{J}^2 = -I$. Y is ω-compatible if

$$g(v, w) \stackrel{\text{def}}{=} \omega(v, \mathcal{J}w)$$

is a (real) inner product.

Example 0.4. Let $E = \mathbb{R}^{2n} = \mathrm{span}\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$. Let $\mathcal{J}e_i = f_i, \mathcal{J}f_i = -e_i$. Then g is the standard inner product.
Remark:

- Note: in the definition of ω-compatible, any two of $\omega, \mathcal{J}, g$ deetermines the third.
- $\mathcal{J} \in \mathrm{Sp}(E, \omega) \cap O(E, g)$
- $h(v, w) = g(v, w) + i\omega(v, w)$ is a complex inner product, with corresponding unitary group $U(n) = \underbrace{U(E, h)}_{\text{preserves } h} = \underbrace{\mathrm{Sp}(E, \omega)}_{\text{preserves } \omega} \cap \underbrace{O(E, g)}_{\text{preserves } g}$. Recall that $U(n)$ is compact and connected, and has $\pi_1 = \mathbb{Z}$

Let $\mathcal{J}(E, \omega) = \{\mathcal{J} \mid \omega\text{-compatible }\}$
We have a map $\psi : \mathcal{J}(E, \omega) \to \mathrm{Riem}(E) \subseteq \mathrm{Sym}^2(E)$, which is contractible.

Theorem 0.1. *There is a canonical retraction*

$$\phi : \mathrm{Riem}(E) \to \mathcal{J}(E, \omega)$$

such that $\phi \circ \psi = \mathrm{Id}$.

Corollary 0.2. $\mathcal{J}(E, \omega)$ *is contractible*

Proof. Send

$$\mathcal{J} \mapsto \varphi\left((1-t)\psi(\mathcal{J}) + tg_0\right)$$

■

Proof. Let $k \in \mathrm{Riem}(E)$ be given. Define $A \in \mathrm{Hom}(E, E)$ by $k(v, w) = \omega(v, Aw)$. Then $A = -A^T$ (skew-adjoint with respect to k). Why? note that

$$\begin{aligned} k(v, A^{-1}y) &= \omega(v, y) \\ &= -\omega(y, v) \\ &= -k(y, A^{-1}v \\ &= -k(A^{-1}v, y) \end{aligned}$$

So $(A^{-1})^T = A^{-1}$, so $A^T = -A$.
Hence, we can define $|A| = \sqrt{A^T A} = \sqrt{-A^2}$. Put $\mathcal{J} = A|A|^{-1}$. Then $\mathcal{J}^2 = A|A|^{-1}A|A|^{-1} = A^2|A|^{-2} = -I$.
Now we check that it defines an inner product:

$$\begin{aligned} g(v, w) &= \omega(v, \mathcal{J}w) \\ &= \omega(v, A|A|^{-1}\omega) \\ &= k(v, |A|^{-1}w) \\ &= k(|A|^{-\frac{1}{2}}v, |A|^{-\frac{1}{2}}w) \end{aligned}$$

is an inner product.

■

If k was instead g with a compatible complex structure, then A must be that complex structure on the nose.
Now, $\mathrm{Sp}(E, \omega)$ acts on $\mathcal{J}(E, \omega)$ by

$$A \cdot \mathcal{J} = A\mathcal{J}A^{-1}$$

Proposition 8. *The action of* $\mathrm{Sp}(E, \omega)$ *on* $\mathcal{J}(E, \omega)$ *is transitive. That is, for any* $\mathcal{J}_1, \mathcal{J}_2 \in \mathcal{J}(E, \omega)$, *there is an* $A \in \mathrm{Sp}(E, \omega)$ *with* $\mathcal{J}_1 = A\mathcal{J}_2A^{-1}$. *It has stabilizers at* $\mathcal{J} \in \mathcal{J}(E, \omega)$ *the unitary group* $U(E)$, *with respect to* $\mathcal{J}$. *I.e.:*

$$\mathcal{J}(E, \omega) = \mathrm{Sp}(E, \omega)/U(E)$$

Corollary 0.3. $\mathrm{Sp}(E,\omega)$ *is connected.*

Proof. $\mathcal{J}(E,\omega)$ is connected, and $U(E)$ is connected, so $\mathrm{Sp}(E,\omega)$ is connected. ■

Proof. Given $\mathcal{J},\mathcal{J}' \in \mathcal{J}(E,\omega)$, let $e_1,\dots,e_n$ be an orthonormal basis for E a complex inner product space (with respect to $\mathcal{J}$).
Then $e_1,\dots,e_n, f_1 = \mathcal{J}e_1,\dots,f_n = \mathcal{J}e_n$ is a symplectic basis. Similarly, define $e_1',\dots,e_n',f_1',\dots,f_n'$ by $Ae_i = e_i', Af_i = f_i'$. This defines a symplectic transformation $A \in \mathrm{Sp}(E,\omega)$, with $A\mathcal{J}A^{-1} = \mathcal{J}'$. ■

More on $\mathrm{Sp}(E,\omega)$

Proposition 9. $\mathrm{Sp}(E,\omega)$ *is a connected Lie group of dimension* $2n^2+n$*, where* $\dim E = 2n$

Proof. By Cartan's theorem, every closed (in the sense of topology) subgroup of a Lie group is a Lie group.
This applies to $\mathrm{Sp}(E,\omega) \subseteq \mathrm{GL}(E)$ (invertible transformations). For connected, see above.
To get dimension, consider action of $\mathrm{GL}(E) \supseteq \mathcal{U} = \{\text{symplectic forms}\}$ on $\bigwedge^2 E^*$ the space of skew-symmetric bilinear forms. This action is transitive, with stabilizer at $\omega \in \mathcal{U}$ given by $\mathrm{Sp}(E,\omega)$.
Hence $\mathcal{U} = \mathrm{GL}(E)/\mathrm{Sp}(E,\omega)$, and using this we can count dimensions:

$$\underbrace{\dim \mathcal{U}}_{\dim = \dim \bigvee^2 E^* = \binom{2n}{2}} = \underbrace{\dim \mathrm{GL}(E)}_{\dim = (2n)^2} - \dim \mathrm{Sp}(E,\omega)$$

So $\dim \mathrm{Sp}(E,\omega) = (2n)^2 - \frac{2n(2n-1)}{2} = 2n^2+n$

Lecture 4 - 9/17/24

Geometry of $\mathrm{Sp}(E,\omega)$ and $\mathrm{Lag}(E,\omega)$

So far, we know $\mathrm{Sp}(E,\omega)$ is a Lie group. It is also connected, with dimension $2n^2+n$, where $\dim E = 2n$.
For any $\mathcal{J} \in \mathcal{J}(E,\omega)$, we have a real inner product $g(v,w) = \omega(v,\mathcal{J}w)$, and a complex one given by $h = g + \sqrt{-1}\omega$.
$U(E) \subseteq \mathrm{Sp}(E,\omega)$, $\mathcal{J}(E,\omega) = \mathrm{Sp}(E,\omega)/U(E)$
(Enough to consider $E = \mathbb{R}^{2n}$, $\mathcal{J}e_i = f_i, \mathcal{J}f_i = -e_i$).
Let $()^T$ be the transpose with respect to the metric g. In other words, $g(Av,w) = g(v,A^Tw)$.

Proposition 10. *$A \in \mathrm{GL}(E)$ is <u>symplectic</u> if and only if $A^T = \mathcal{J} A^{-1} \mathcal{J}^{-1}$.*

Proof. $A \in \mathrm{Sp}(E, \omega)$ if and only if for all $v, w, \omega(Av, Aw) = \omega(v, w)$. Note that $g(\mathcal{J}v, w) = \omega(v, w)$. So for all v, w,

$$\begin{aligned} g(\mathcal{J}Av, Aw) &= g(\mathcal{J}v, w) \\ &= g(A^T \mathcal{J} A v, w) \end{aligned}$$

so $A^T \mathcal{J} A = \mathcal{J}$.

∎

Consequence: if $A \in \mathrm{Sp}(E, \omega)$, then $\det(A) = 1$. This follows from

$$\begin{aligned} \det(A) &= \det(A^T) \\ &= \det(\mathcal{J} A^{-1} \mathcal{J}^{-1}) \\ &= \det(A)^{-1} \end{aligned}$$

So $\det(A)^2 = 1$, and since $\mathrm{Sp}(E, \omega)$ is connected, $\det(A) = 1$.
Now, in $\mathbb{R}^{2n}$, if we let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

$$A \in \mathrm{Sp}(E, \omega) \iff \begin{cases} a^T c = c^T a \\ b^T d = d^T b \\ a^T d - b^T c = I \end{cases}$$

Note: $\mathrm{Sp}(E, \omega)(\mathbb{R}^2, \omega) = \mathrm{SL}(2, \mathbb{R})$, the 2x2 matrices of determinant 1.
Another consequence is: A is symplectic implies that A^T is symplectic. This means we can use <u>polar decomposition</u>.
Recall: $A \in \mathrm{GL}(m, \mathbb{R})$ has a polar decomposition $A = U|A|$, where $|A|$ is positive definite, $|A| = \sqrt{A^T A}$, and $U \in O(n)$.
Can write $|A| = \exp(\xi)$, with $\xi^T = \xi$.
Thus, $\mathrm{GL}(n, \mathbb{R}) = O(n) \times \{\xi \mid \xi^T = \xi\}$ as a manifold.
For any $G \subseteq \mathrm{GL}(m, \mathbb{R})$, which is invariant under $A \mapsto A^T$, we get a polar decomposition

$$G = K \times p$$

where $K = G \cap O(m)$, $p = \mathfrak{g} \cap \{\xi \mid \xi^T = \xi\}$ where $\mathfrak{g} = \{\xi \mid \exp(t\xi) \in G \text{ for all } t\}$
In particular, $G = \mathrm{Sp}(E, \omega) \cong \mathrm{Sp}(2n, \omega)$, we get $K = \mathrm{Sp}(2n, \omega) \cap O(2n) = U(n)$ and $p = \{\xi \in \mathrm{Sp}(2n, \omega) \mid \xi = \xi^T\}$.
So we get that $\mathrm{Sp}(E, \omega) = U(e) \times p$.
Upshot: $\mathrm{Sp}(E, \omega)$ deformation retracts onto its maximal compact subgroup $U(E)$.

Corollary 0.4. *There is a canonical isomorphism*

$$\mu : \pi_1(\mathrm{Sp}(E, \omega)) \to \mathbb{Z}$$

Proof. $\pi_1(\text{Sp}(E,\omega)) \cong \pi_1(U(E))$. Now, det : $U(E) \to \pi_1(U(1))$, and this map is an isomorphism.
This is the most primitive version of a "Maslov Index." It is a "Maslov index of loop of symplectomorphisms.
Exercise: If $A, B : S^1 \to \text{Sp}(E,\omega)$, then $\mu(AB) = \mu(A) + \mu(B)$.
Remark: We discussed the noncompact group $S_p(E,\omega) = \text{Sp}(2n,\mathbb{R})$. There is a compact "symplectic group" denoted $\text{Sp}(n)$. Both are "real forms" of the complex symplectic group $\text{Sp}(2n,\mathbb{C})$.

$$\text{Sp}(2n,\mathbb{R}) \subseteq \text{Sp}(2n,\mathbb{C}) \supseteq \text{Sp}(n)$$

but $\text{Sp}(2n,\mathbb{R}) \neq \text{Sp}(n)$. We have a similar situation for $\text{SL}(n,\mathbb{C})$:

$$\text{SL}(n,\mathbb{R}) \subseteq \text{SL}(n,\mathbb{C}) \supseteq \text{SU}(n)$$

These two on the left and right have the same complexification.

$$\mathbb{R}^* \subseteq \mathbb{C}^* \supseteq U(1)$$

Recall: The Lagrangian Grassmannian, $\text{Lag}(E) = \{L \subseteq E \mid L^\omega = L\}$. $\text{Lag}(E) \subseteq GR_n(E) = \{n\text{-dimensional subspaces of } E\}$, so it is a topological space in this way. Recall that $Gr_k(E)$ can be seen as a manifold in 2 ways:

(i) View it as a homogeneous space

(ii) Construct charts

(i) Pick any $G \subseteq \text{GL}(E)$ such that G acts transitively on $Gr_k(E)$ (e.g. $G = \text{GL}(E), G = O(E)$ for some inner product, $G = \text{SO}(E)$).

Let $H \subseteq G$ be a stabilizer of some fixed k-dim subspace. So $Gr_k(E) = G/H$

(ii) For any subspace $M \subseteq E$ of codimension k, the set $\{L \in Gr_k(E) \mid E = L \oplus M\}$ is canonically an affine space. It is isomorphic to $\{j : E/M \to E \mid \pi \circ j = \text{Id}\}$, $\pi : E \to E/M$, an affine space under $\text{Hom}(E/M, E)$

The punchline is that for any fixed L, we get a vector space, and we use this as a chart.

Now, we want to do the same thing with $\text{Lag}(E)$.

Proposition 11. *The group $U(E)$ acts transitively on $\text{Lag}(E)$ with stabilizers at given $L \in \text{Lag}(E)$ equal to $O(L)$. $U(E)$ is a Lie group, so $\text{Lag}(E) = U(E)/O(L)$ is a manifold of dimension* $\frac{n(n+1)}{2}$

Proof. Let $h(v,w) = g(v,w) + \sqrt{-1}\omega(v,w)$.
Note: On any $L \in \mathrm{Lag}(E)$, get $h|_{L\times L} = g|_{L\times L}$. A g-orthonormal basis $e_1, \dot{;} e_n$ of L is an h-orthonormal basis of E, given symplectic basis $e_1, \ldots, e_n, f_i, \ldots, f_n$, where $f_i = \mathcal{J}e_i$.
Given another L', choose g-orthonormal basis $e_1', \ldots, e_n'$ of L'. It's h-orthonormal basis of E.
The transformation taking $e_1, \ldots, e_n$ to $e_1', \ldots, e_n'$ is in $U(E) \subseteq \mathrm{Sp}(E,\omega)$ taking L to L'. The stabilizers of L are transformations for which $L = L'$, so they're in $O(L)$. Then

$$\begin{aligned}\dim \mathrm{Lag}(E) &= \dim U(n) - \dim O(n)\\ &= n^2 - \frac{n(n-1)}{2}\\ &= \frac{n(n+1)}{2}\end{aligned}$$

∎

Alternatively, pick $\mathcal{J} \in \mathcal{J}(E,\omega)$, then $\mathrm{Lag}(E) = \mathrm{Sp}(E)/\mathrm{Sp}(E)_L$ (?)
We get again a version of the Maslov index by Arnold.
The map $\det^2 : U(n) \to U(1), A \mapsto (\det(A))^2$ descends to a map $\mathrm{Lag}(\mathbb{R}^{2n}) \to U(1)$, hence gives a map on fundamental groups

$$\mu : \pi_1(\mathrm{Lag}(E)) \to \pi_1(U(1)) = \mathbb{Z}$$

Proposition 12. *(Arnold)*
This map is again an isomorphism

Proof. ∎
This is the maslov index of loop of Lagrangian subspaces.
Special Case $n = 1$
$\mathrm{Lag}(\mathbb{R}^2) = U(1)/O(1)$. $O(1) = \{\pm 1\}$, so this is a circle under polar identifications, so we get $\mathbb{RP}^1$, which is again S^1.
Given $M \in \mathrm{Lag}(E)$, let $\mathrm{Lag}(E; M) = \{L \in \mathrm{Lag}(E) \mid E = L \oplus M\}$

Proposition 13. $\mathrm{Lag}(E; M)$ *is canonically an affine space, with corresponding linear spaces* $\mathrm{Sym}^2(M) = \{$ *symmetric bilinear forms on* $M^*\} \cong$ *self adjoint maps* $M^* \to M$.

Proof. Let $\pi : E \to M^*$ where $\pi(E) =$ restriction of $\omega^\flat(v) \in E^*$ to M.
$\pi(v)(w) = \omega(v,w)$ for $w \in M$.
This projection map has kernel $M \subseteq E$ since M is Lagrangian, so gives isomorphisms $E/M \to M^*$.
$\mathrm{Lag}(E; M) = \{L \in \mathrm{Lag}(E) \mid L \oplus M = E\} \cong \{j : E/M \to E \mid j(M^*)$ is isotropic $, \pi \circ j = \mathrm{Id}\}$.

Given any such j, any other splitting j' is of the form $j'(m) = j(m) + \psi(m)$ for some $\psi : E/M = M^* \to M$.
For all $\mu_1, \mu_2 \in M^*$,

$$\begin{aligned}
0 &= \omega(j'(\mu_1), j'(\mu_2)) \\
&= \omega(j(\mu_1) + \psi(\mu_1), j(\mu_2) + \psi(mu_2) \\
&= \underbrace{\omega(j(\mu_1), j(\mu_2))}_{=0 \text{ since } j \text{ isotropic}} + \underbrace{\omega(\psi(\mu_1), \psi(\mu_2))}_{=0 \text{ since } M \text{ isotropic}} \\
&+ \omega(j(\mu_1), \psi(\mu_2)) + \omega(\psi(\mu_1), j(\mu_2)) \\
&= \langle \mu_1, \psi(\mu_2)\rangle - \langle \mu_2, \psi(\mu_1)\rangle
\end{aligned}$$

So ψ is self-adjoint, $\beta(\mu_1, \mu_2) = \langle \mu, \psi\mu\rangle$.
Again we see $\dim = \frac{n(n+1)}{2}$.
Down-to-earth version
Let $E = \mathbb{R}^{2n}$, $M = 0 \oplus \mathbb{R}^n = \operatorname{span}\{f_1, \ldots, f_n\}$.
$\mathbb{R}^{2n} = L \oplus M$ means L is graph of linear map $S : \mathbb{R}^n \to \mathbb{R}^n$.
L has basis

$$g_i = e_i + \sum_{j=1}^{n} S_{ij} f_j$$

is Lagrangian if and only if for all i, k,

$$\begin{aligned}
0 = \omega(g_i, g_k) &= \omega(e_i + \sum S_{ij} f_j, e_k + S_{kl} f_l \\
&= \cdots \\
&= S_{ki} - S_{ik}
\end{aligned}$$

Lecture 5 - 9/19/24

<u>Maslov Indices</u>

Let (E, ω) be a symplectic vector space, $\dim E = 2n$. We consider $\operatorname{Lag}(E)$, the Lagrangian Grassmannian, the set of all Lagrangian subspaces. We know this is homeomorphic to $U(n)/O(n)$, once you choose a symplectic basis. It is a manifold, of dimension $fracn(n+1)2$.
We have $\pi_1(\operatorname{Lag}(E)) \cong \pi_1(U(n)/O(n))$. The function $\det^2$ descends to a function on this space, and gives a morphism from $\pi_1(U(n)/O(n))$ to $\pi_1(U(1)) \cong \mathbb{Z}$.
So we have a canonical $\mu : \pi_1(\operatorname{Lag}(E)) \to \mathbb{Z}$ is called the <u>Maslov Index</u>. It is somewhat akin to winding number.

We want to generalize to paths of Lagrangians. Fix a Lagrangian subspace $M \in \mathrm{Lag}(E)$, and define

$$\mathrm{Lag}(E,M) = \{L \in \mathrm{Lag}(E) \mid L \cap M = 0\}$$

This is canonically an affine space, and so is contractible. Let $\sum_M = \mathrm{Lag}(E) \setminus \mathrm{Lag}(E,M) = \{L \mid L \cap M \neq 0\}$. This is some kind of singular space.
Consider a path $L : [a,b] \to \mathrm{Lag}(E), t \mapsto L(t)$ with $L(a), L(b) \notin \sum_M$
Define the Maslov Index $[L : M] \stackrel{\text{def}}{=}$ Maslov index of loop obtained by concatenating $L(t)$ with a path in $\mathrm{Lag}(E,M)$ to make a loop. The contractibility of this space means the choice of path doesn't matter.
This is Maslov's "original" index as intersection number with the sincular cycle $\sum_M$.
More generally, we want to find $[L_1 : L_2]$ for two arbitrary Lagrangian paths, a kind of signed number of nonzero intersections $L_1(t) \cap L_2(t)$ (remember that $L_i(t)$ is a vector space!).
Let $L_1, L_2 \in \mathrm{Lag}(E,M)$ related by some $\beta_{12} \in \mathrm{Sym}^2(M)$ (symmetric bilinear form on $M^* \cong L_1$. Recall from last time we have $\beta_{12} \cdot L_1 = L_2$, $\beta_{21} \cdot L_2 = L_1$, $\beta_{12} = \beta_{21}$. To this setting we can attach an invariant.
Does there exist a symplectomorphism A such that $(L_1, L_2, M) \mapsto_A (L_1, L_2', M)$, with L_2' a Lagrangian transverse to all the others.

Signature of symmetric bilinear form

For a symmetric matrix B, we say the Signature, $\mathrm{Sig}(B)$, is the number of positive eigenvalues minus the number of negative eigenvalues.
For a symmetric bilinear form β, $\mathrm{Sig}(\beta) = \mathrm{Sig}(B)$, for B the matrix of β in terms of a basis (which does not affect the eigenvalues).
The number $\mathrm{Sig}(\beta_{12})$ depends on M.

Proposition 14. *If $L_1, L_2, L_3 \in \mathrm{Lag}(E; M)$. Then the number $s(L_1, L_2, L_3) = \mathrm{Sig}(\beta_{21}) + \mathrm{Sig}(\beta_{32}) + \mathrm{Sig}(\beta_{13})$ is actually independent of M. This is also called a Maslov index.*

Proof.

Proposition 15.

1. $s(L_1, L_2, L_3) = s(L_2, L_3, L_1)$

2. $S(L_1, L_2, L_3) = -S(L_2, L_1, L_3)$

3. *Cocycle identity: for all Lagrangians L_1, L_2, L_3, L_4,*

$$S(L_2, L_3, L_4) - s(L_1, L_3, L_4) + s(L_1, L_2, L_4) - s(L_1, L_2, L_3) = 0$$

4. *If $M(t)$ is always transverse to L_1, L_2, then $s(L_1, L_2, M)$ doesn't depend on t.*

5. *Up to symplectomorphism, L_1, L_2, L_3 is uniquely determined by* $\dim(L_1 \cap L_2), \dim(L_2 \cap L_3), \dim(L_1 \cap L_3), \dim(L_1 \cap L_2 \cap L_3),\ s(L_1, L_2, L_3)$

Proposition 16. *Suppose $[a, b] \to \mathrm{Lag}(E), t \mapsto L_i(t)$, $i = 1, 2$ are paths, and that there exists some $M \in \mathrm{Lag}(E)$ such that $L_i(t) \cap M = 0$ for all $i = 1, 2, t \in [a, b]$. Then*

$$[L_1; L_2] = \frac{1}{2}(s(L_1(a), L_2(a), M) - s(L_1(b), L_2(b), M))$$

Proof. Suppose M' is another choice. First term changes by

$$\begin{aligned} s(L_1(a), L_2(a), M') - s(L_1(a), L_2(a), M') &= s(L_1(a), M, M') - s(L_2(a), M, M') \\ &= s(L_1(b), M, M') - s(L_2(b), M, M') \end{aligned}$$

This is the change in the second term, so they cancel out.
General definition:
Consider a partition $a = t_0 < t_1 < \cdots < t_k = b$ such that for all $[t_{j-1}, t_j] \in M_j \in \mathrm{Lag}(E)$ with $L_i(t) \cap M_j = 0$ for all $t \in [t_{j-1}, t_j]$
Then

$$[L_1; L_2] = \frac{1}{2}\sum_{j=1}^{k}\big(s(L_1(t_{j-1}), L_2(t_{j-1}), M_j) - s(L_1(t_j), L_2(t_j), M_j)\big)$$

For $A \in \mathrm{Sp}(E, \omega)$, we have $\mathrm{Graph}(A) = \{(Av, v)\} \subseteq E \times \overline{E}$ (where $\overline{E}$ has the same symplectic form but with an opposite sign) is Lagrangian.
Define, for any path $A(t)$, $\mu(A) = [\mathrm{Graph}(A), \triangle]$

Lecture 6, 9/24/24

Part 2: Symplectic Manifolds

Recall that the Lie derivative of a vector field, $\mathscr{L}_X$, is given by $\frac{d}{dt}|_{t=0}(F_{-t})^*$, where F_t is a flow along X. Note $X(f) = \mathscr{L}_X f = \frac{d}{dt}|_{t=0}(F_{-t})^* f$
Differential of a map $F : M_1 \to M_2$ is a map $TF : TM_1 \to TM_2$.
For $f : M \to R$, $Tf : TM \to T\mathbb{R}$, while $df \in \Omega^1(M)$.
We will introduce symplectic manifolds by analogy to a complex manifold.

Complex manifolds

A complex manifold comes with a family of linear transformations $\mathcal{J}_m : T_mM \to T_mM$, with $\mathcal{J}_m^2 = -\operatorname{Id}_{T_mM}$, which depends smoothly on $m \in M$.
This is typically called an "almost complex structure".
A complex manifold is the same as a real manifold, but charts go to $\mathbb{C}^n$, and we want transition functions to be holomorphic. Any complex manifold has an almost complex structure on its tangent spaces, but a manifold with an almost complex structure is not necessarily a complex manifold.
It is some kind of integrability condition on $\mathcal{J} = \{\mathcal{J}_m\}$, and if this condition vanishes, then the almost complex structure comes from an honest complex manifold.

Symplectic manifolds

A symplectic manifold is equipped with a family of functions $\omega_m : T_mM \times T_mM \to \mathbb{R}$ which is symplectic, depending smoothly on m. This is called an "almost symplectic structure." There is again an integrability condition we can impose. We want to stipulate that ω_m arises from an $\omega = \{\omega_m\} \in \Omega^2 M$. The integrability condition is that $d\omega = 0$.

Definition 0.6. A symplectic structure on a manifold M is a non-degenerate 2-form $\omega \in \Omega^2(M)$ with $d\omega = 0$.
Non-degenerate just means that each $\omega|_{T_mM \times T_mM}$ is non-degenerate.

Proposition 17. *Any symplectic 2-form Ω, for $\dim M = 2n$, is non-degenerate if and only if* $\underbrace{\overbrace{\omega^n}}_{=\underbrace{\omega \wedge \cdots \wedge \omega}_{n\ times}} \neq 0$ *everywhere.*

Proof. Check at $m \in M$.
In one direction, suppose $(\omega_m)^n \neq 0$. We want to show $\ker \omega_m = 0$. Since $(\omega_m)^n \neq 0$, we have $\iota_V(\omega_m^n)$, where we define $\iota_v : \Omega^k(M) \to \Omega^{k-1}(M)$ by $\iota_V(\alpha) = \alpha(V, \cdots)$.
Anyways, $\iota_V(\omega^n)$ is nonzero for all $v \in T_m$.
But $\iota_V(\underbrace{\omega_m \wedge \cdots \wedge \omega_m}_{n}) = n(\iota_v\omega_m)\omega_m^{n-1}$, so $\iota_V\omega_m \neq 0$.

In the other direction, suppose $\ker(\omega_m) = 0$, so ω_m is symplectic. Let $e_1, \ldots, e_n, f_1, \ldots, f_n$ be a symplectic basis for T_mM with respect to ω_m.
Consider

$$\begin{aligned}
\iota_{e_n}\iota_{e_{n-1}}\cdots\iota_{e_1}(\omega_m^n) &= n(\iota_{e_n}\cdots\iota_{e_2})((\iota_{e_1}\omega_m) \wedge \omega_m^{n-1}) \\
&= n(n-1)(\iota_{e_n}\cdots\iota_{e_3})((\iota_{e_2}\omega_m)(\iota_{e_1}\omega_m)\omega_m^{n-2} \\
&\vdots \\
&= n!(\iota_{e_n}\omega_m) \wedge \cdots \wedge (\iota_{e_1}\omega_m)
\end{aligned}$$

So $\iota_{f_n} \cdots \iota_{e_1}(\omega_m^n) = \pm n! \neq 0$

Definition 0.7. Let (M, ω) be an (almost) symplectic manifold. The volume form

$$\bigwedge = \frac{\omega^n}{n!} = (\exp(\omega))_{\dim M}$$

is called the Liousville volume form on M.

Definition 0.8. Let (M, ω) be a symplectic manifold.

(a) A symplectomorphism is a diffeomorphism $F \in \mathrm{Diff}(M)$ preserving ω, i.e. $F^*\omega = \omega$. The group of symplectomorphisms of (M, ω) is denoted $\mathsf{Diff}(M, \omega)$

(b) A symplectic vector field on M is a vector field $X \in \mathscr{X}(M)$ preserving ω, i.e. $\mathscr{L}_X \omega = 0$.

The Lie algebra of symplectic vector fields is denoted $\mathscr{X}(M, \omega)$, i.e. the Local flow is symplectic.

Definition 0.9. Let (M, ω) be a symplectic manifold, let $H \in C^\infty(M)$.
The Hamiltonian vector field $X_H \in \mathscr{X}(M)$ is the unique vector field such that

$$\iota(X_H)\omega = -dH$$

The space of Hamiltonian vector fields is denoted $\mathscr{X}_{Ham}(M, \omega)$

Proposition 18. *Indeed, $\mathscr{X}_{Ham}(M, \omega) \subseteq \mathscr{X}(M, \omega)$.*

Proof. Let $X = X_H$ be Hamiltonian. We check

$$\begin{aligned}
\mathscr{L}_X \omega &= (d\iota_X + \iota_X d)\omega \\
&= d\iota_X \omega \\
&= -ddH \\
&= 0
\end{aligned}$$

by the Cartan Formula ∎

It turns out that ω is symplectic if and only if $\iota_X \omega$ is closed.
X is Hamiltonian if and only if ι_X is exact.
Basic examples:
Open subsets $U \subseteq \mathbb{R}^n$. Let $e_1, \ldots, e_n, f_1, \ldots, f_n$ be the standard symplectic basis.
Let $q_1, p_1, q_2, p_2, \ldots$ be corresponding coordinates. Take

$$\omega = \sum_{j=1}^{n} dq_i \wedge dp_i$$

Liousville Volume form: $\bigwedge = \frac{1}{n!}\omega \wedge \cdots \wedge \omega$,

$$\begin{aligned}\bigwedge &= \frac{1}{n!}(dq_1 \wedge dp_1 + \cdots) \wedge (dq_1 + dp_1 + \cdots) \wedge \cdots \\ &= dq_1 \wedge dp_1 \wedge dq_2 \wedge dp_2 \wedge \cdots \wedge dq_n \wedge dp_n\end{aligned}$$

which is the standard volume form.
Given $H \in C^\infty(U)$, $dH = \sum(\frac{\partial H}{\partial q_i}dq_i + \frac{\partial H}{\partial p_i}dp_i)$.
$\iota(X_H)\omega = -dH$ implies $X_H = \sum(\frac{\partial H}{\partial q_j}\frac{\partial}{\partial p_j} - \frac{\partial H}{\partial p_j}\frac{\partial}{\partial q_j})$, with corresponding ODE $\dot{q}_j = \frac{\partial H}{\partial p_j}, \dot{p}_j = -\frac{\partial H}{\partial q_j}$, which are Hamilton's equations.
Example: Cotangent bundles
Let $M = T^*Q$ (dual of TQ), with Q any manifold. Let $\pi : T^*Q \to Q$ be the projection.
There's a distinguished 1-form $\theta \in \Omega^1(T^*Q)$

Definition 0.10. The canonical 1-form $\theta \in \Omega^1(T^*Q)$ is defined in terms of its contractions with $v \in T_\mu(T^*Q)$, $\mu \in T^*Q$, where

$$\langle \theta_\mu, v\rangle = \langle \underbrace{\mu}_{\in T^*_{\pi(\mu)}Q}, \underbrace{T\pi(v)}_{\in T_{\pi(\mu)}Q}\rangle$$

Another perspective:
$T\pi : T_\mu(T^*Q) \to T_{\pi(\mu)}Q$ has a dual map $(T_\mu\pi)^* : \underbrace{T^*_{\pi(\mu)}Q}_{\ni\mu} \to T^*_\mu(T^*Q)$. Then

$(T_\mu\pi)^*(\mu) = \theta_|mu$
Another perspective:
For $\alpha \in \Omega^1(Q)$, let $\sigma_\alpha : Q \to T^*Q$ be the corresponding section.

Proposition 19. *$\theta \in \Omega^1(T^*Q)$ is the unique 1-form such that for all $\alpha \in \Omega^1(Q)$, $\sigma_\alpha^*\theta = \alpha$.*

Proof. Let $\omega \in T_qQ$, $q \in Q$. Let $\mu = \sigma_\alpha(q) = \alpha|_q$. Then

$$\begin{aligned}\langle \sigma_\alpha^*\theta|_q, \omega\rangle &= \langle \theta_\mu, (T_q\sigma_\alpha)(\omega)\rangle \\ &= \langle \mu, (T_\mu\pi)(T_q\sigma_\alpha)(\omega)\rangle \\ &= \langle \mu, \omega\rangle \\ &= \langle \alpha_q, \omega\rangle\end{aligned}$$

∎

In coordinates: Let $q_1, \ldots, q_n$ be local coordinates on $U \subseteq Q$. Then $dq_1, \ldots, dq_n \in \Gamma(T^*Q|_\mu)$ are (pointwise) basis of 1-forms on U.
This gives a basis of T_q^*Q, all q. Let $p_1, \ldots, p_n$ be fiber coordinates, with cotangent coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$.

Lemma 1. *In cotangent coordinates,*

$$\theta = \sum_{i=1}^{n} p_i dq_i$$

Proof. Let $\alpha = \sum_{i=1}^{n} \alpha_i(q_1, \dots, q_n) dq_i \in \Omega^1(Q)$.
Then

$$\sigma_\alpha : U \to T^*Q|_U, (q_1, \dots, q_n) \mapsto (\alpha_1(q_1, \dots, q_n), \alpha_2(q_1, \dots, q_n), \cdots)$$

i.e. $\sigma_\alpha^* p_i = \alpha_i, \sigma_\alpha^* q_i = q_i$.

$$\sigma_\alpha^*(\sum p_i dq_i) = \sum \alpha_i dq_i = \alpha$$

So $\theta = \sum p_i dq_i$

■

Theorem 0.5. *The 2-form*

$$\omega = -d\theta \in \Omega^2(T^*Q)$$

is symplectic.
This is the canonical symplectic form on T^*Q.

Proof. In local coordinates,

$$\begin{aligned} -d\theta &= -d\left(\sum p_i dq_i\right) \\ &= \sum dq_i \wedge dp_i \end{aligned}$$

■

We'll describe symplectomorphisms etc. of T^*Q, taking into account $\pi : T^*Q \to Q$ is a fibration.
Terminology: For any surjective submersion $\pi : P \to Q$, we say $F \in \mathsf{Diff}(P)$ "lifts" $f \in \mathsf{Diff}(Q)$ if $\pi \circ F = f \circ \pi$. Denote by $\mathsf{Diff}(P, \pi)$ the "fibration-preserving" diffeomorphisms.
We'll find all of $\mathsf{Diff}(T^*Q, \omega) \cap \mathsf{Diff}(T^*Q, \omega)$
Example: If $P = TQ$, $f \in \mathsf{Diff}(Q)$, then $f_T = Tf \in \mathsf{Diff}(TQ)$ is a lift.
If $P = T^*Q$, $f \in \mathsf{Diff}(Q)$, $f_{T^*} = (Tf^{-1})^* \in \mathsf{Diff}(T^*Q, \pi)$
The upshot is that all f_{T^*} are symplectic.
Other lifts: given $\alpha \in \Omega^1(Q)$ we get $G_\alpha \in \mathsf{Diff}(T^*Q, \pi)$ by adding α fiberwise.
We'll see that G_α is symplectic if and only if $d\alpha = 0$.

Lecture 7, 9/26/24

Let $\pi : T^*Q \to Q$ be the projection of the cotangent bundled, and let $\theta \in \Omega^1(T^*Q)$ be the canonical 1-form, characterized by $\sigma_\alpha^*\theta = \alpha$ (that is, if we view σ as a map from Q to T^*Q, then the pullback of the form θ along this map is α.
In coordinates, $\theta = \sum p_j dq_j$, $\omega = -2\theta$ is symplectic.
Let $\pi : P \to Q$ be a surjective submersion. Recall that $\mathsf{Diff}(P, \pi)$ is the set of diffeomorphisms of P which preserve the fibration, i.e. $F : P \to P$ such that $F \circ \pi = \pi \circ f$ for some $f \in \mathsf{Diff}(Q)$

Lemma 2. *Suppose that π has connected fibers. Then $\beta \in \Omega^*(P)$ is of the form $\beta = \pi^*(\alpha)$ if and only if for all vertical vector fields $Z \in \mathscr{X}(p)$, $\iota_Z(\beta) = 0$, and $\mathscr{L}_Z\beta = 0$.*

Proof. Exercise. Use coordinates adapted to submersion.
E.g. for $\beta \in \Omega^1(P)$, $\beta = \sum f_i(q, p)dq_i + \sum g_i(q, p)dp_i$
Given $f \in \mathsf{Diff}(Q)$, we have the cotangent lift $f_{T^*} \in \mathsf{Diff}(T^*Q)$, $f_{T^*} = (Tf^{-1})^*$.

Proposition 20. *We have $(f_{T^*})^*\theta = \theta$. In particular, $f_{T^*} \in \mathsf{Diff}(T^*Q, \omega) \cap \mathsf{Diff}(T^*Q, \pi)$. Conversely, all $F \in \mathsf{Diff}(T^*Q, \pi)$ with $F^*\theta = \theta$ are of this form.*

Proof. Note this diagram commutes:

$$\begin{tikzcd}
T^*Q \arrow[r, "f_{T^*}"] & T^*Q \\
Q \arrow[u, "\sigma_\alpha"] \arrow[r, "f"] & Q \arrow[u, "\sigma_{(f^{-1})^*\alpha}"']
\end{tikzcd}$$

For any $\alpha \in \Omega^1(Q)$,

$$\begin{aligned}
\sigma_\alpha^*(f_T)^*\theta &= (f_T \circ \sigma_\alpha)^*\theta \\
&= (\sigma_{(f^{-1})^*\alpha} \circ f)^*\theta \\
&= f^* \circ ((\sigma_{(f^{-1})^*\alpha})^*\theta \\
&= f^*(f^{-1})^*\alpha \\
&= \alpha
\end{aligned}$$

Now for uniqueness?
The corresponding cotangent coordinates are $\theta = \sum p_j dq_j$, and we have $F^*q_j = q_j$, since $f = \mathrm{Id}_Q$. Hence $F^*\theta = \theta$ gives $F^*p_j = p_j$.
So $F = \mathrm{Id}_{T^*Q}$.

■?

Given $\alpha \in \Omega^1(Q)$, let $G_\alpha \in \mathsf{Diff}(T^*Q, \pi)$, $G_\alpha = \mu + \alpha|_{\pi(\mu)}$, i.e. $G_\alpha \circ \sigma_\beta = \sigma_{\alpha+\beta}$.

Proposition 21. *For $\alpha \in \Omega^1(Q)$,*

$$G_\alpha^*\theta = \theta + \pi^*\alpha$$

So $G_\alpha \in \mathsf{Diff}(T^*Q, \omega)$ if and only if $\alpha \in \Omega^1_{Cl}(Q)$, where Ω^1_{Cl} denotes the closed forms. Every $F \in \mathsf{Diff}(T^*Q, \omega) \cap \mathsf{Diff}(T^*Q, \pi)$ with trivial base map, then it is of the form $F = Q_\alpha$.

Proof. For $\beta \in \Omega^1(Q)$, we have

$$\begin{aligned}
\sigma_\beta^*(G_\alpha^*\theta - \pi^*\alpha) &= (G_\alpha \circ \sigma_\beta)^*\theta - (\pi \circ \sigma_\beta)^*\alpha \\
&= \sigma_{\alpha+\beta}^*\theta \\
&= \alpha + \beta - \alpha \\
&= \beta
\end{aligned}$$

For uniqueness part, suppose we have $F \in \mathsf{Diff}(T^*Q, \omega) \cap \mathsf{Diff}(T^*Q, \pi)$ induces $f = \mathrm{Id}_Q$.
Let $\tilde{\alpha} = F^*\theta - \theta$.
For $Z \in \mathscr{X}(T^*Q)$ vertical (i.e. at every point it is in the kernel of $d\pi$), we get

$$\iota_Z\tilde{\alpha} - \underbrace{\iota_Z\theta}_{=0} = f^*\iota_{(F_*Z)}\theta = 0$$

$$\mathscr{L}_Z\tilde{\alpha} = \iota_Z d\tilde{\alpha} + \underbrace{d\iota_Z\tilde{\alpha}}_{=0} = \iota_Z\underbrace{(f^*d\theta - d\theta)}_{=0} = 0$$

Hence $\tilde{\alpha} = \pi^*\alpha$.
$F^*\theta - \theta = \pi^*\alpha$.
So $F = G_\alpha$.
Recall for a representation of a group G on a vector space V, $V \rtimes G$ is the group on the set $V \times G$, with

$$(v, g)(v', g') = (v + g' \cdot v', gg')$$

In our case, $G = \mathsf{Diff}(Q)$, which acts on $\Omega^1_\alpha(Q)$.

Theorem 0.6.

$$\mathsf{Diff}(T^*Q, \omega) \cap \mathsf{Diff}(T^*Q, \pi) = \underbrace{\Omega^1_{Cl}(Q)}_{G_\alpha} \rtimes \underbrace{\mathsf{Diff}(Q)}_{f_{T^*}}$$

Proof. ∎

There is an infinitesimal counterpart

$$\mathscr{X}(T^*Q, \omega) \cap \mathscr{X}(T^*Q, \pi) = \Omega^1_{Cl}(Q) \rtimes \mathscr{X}(Q)$$

where $\mathscr{X}(Q)$ acts on $\Omega^1_{Cl}(Q)$ by the Lie derivative.
In particular, for $Y \in \mathscr{X}(Q)$, the cotangent lift $Y_{T^*} \in \mathscr{X}(T^*Q, \omega)$

Proposition 22. *We have $Y_{T^*} \in \mathscr{X}_{Ham}(T^*Q, \omega)$ with the Hamiltonian*

$$H = -\iota(Y_{T^*})\theta = -\langle \theta, Y_{T^*} \rangle$$

Proof. Let $X = Y_{T^*}$. Then $dH = -d\iota_X\theta = -\mathscr{L}_X\theta + \iota_X d\theta = -\iota_X\omega$
$(f_{T^*})^*\theta = \theta$, so $\mathscr{L}_{Y_{T^*}}\theta = 0$
So $X = X_H$

■

In coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$, $Y = \sum Y_j \frac{\partial}{\partial q_j}$
Note: for lifts X of Y, $\iota_X\theta$ doesn't depend on choice of lift !
So in coordinates, we can take $X = \sum Y_j \frac{\partial}{\partial q_j}$

$$\iota_X(\theta) = \sum Y_j \iota(\frac{\partial}{\partial q_j} + \sum p_k dq_k = \sum Y_j(q)P_j$$

$$H = -\sum Y_j(g)P_j$$

$$dH = \sum \frac{\partial Y_j}{\partial q_k} p_j dq_k + \sum Y_j dp_j$$

So with $\omega = \sum dq_i \wedge dp$,

$$X_H = Y_{T^*} = \sum Y_j \frac{\partial}{\partial q_j} - \sum p_j \frac{\partial Y_j}{\partial q_k} \frac{\partial}{\partial p_k}$$

Lecture 8, 10/1/24

Kähler Manifolds

An almost complex structure on a manifold M is a collection $\{\mathcal{J}_m\}$, $\mathcal{J}_m \in \text{End}(T_mM), \mathcal{J}_m^2 = -I$ depending smoothly on m, i.e. the map $M \mapsto \text{End}(TM) = \coprod \text{End}(T_mM)$, $m \mapsto \mathcal{J}_m$, has to be smooth.
A complex manifold has $\mathbb{C}^n \cong \mathbb{R}^{2n}$-valued charts with holomorphic transition functions. If we have a complex manifold, we have a complex structure on the tangent spaces.

Theorem 0.7. *(Newland-Nirenberg)*
$\mathcal{J}$ comes from a complex structure if and only if $(N_{ij})_{\mathcal{J}} = 0$

$$(N_{ij})_{\mathcal{J}} = [\mathcal{J}X, \mathcal{J}Y] - [X, Y] - \mathcal{J}[\mathcal{J}X, Y] + \mathcal{J}[\mathcal{J}Y, X]$$

In this case, the complex structure is unique

Proof. ■

Corollary 0.8. *If M is a complex manifold, $N \subseteq M$ a real submanifold, sutch that $\mathcal{J}(TN) \subseteq TN$, then N is a complex submanifold.*

Proof. ∎

Let (M, ω) be a symplectic manifold. $\mathcal{J}$ is ω-compatible if for all $m \in M$, $\mathcal{J}_m$ is ω_m-compatible.
Equivalently, for all vector fields $g(X, Y) = \omega(X, \mathcal{J}Y)$, $X, Y \in \mathscr{X}(M)$ is a Riemannian metric.
Let $\mathcal{J}(M, \omega) =$ the set of all ω-compatible almost complex structures.
From discussion for symplectic vector spaces, the map $\mathcal{J}(M, \omega) \to \mathrm{Riem}(M)$, $\omega \mapsto g$, has a left inverse $\mathrm{Riem}(M) \to \mathcal{J}(M, \omega)$. In particular, $\mathcal{J}(M, \omega) \neq \varnothing$.
Given $\mathcal{J}_0, \mathcal{J}_1 \in \mathcal{J}(M, \omega)$, there exists a smooth homotopy $\mathcal{J}_t \in \mathcal{J}(M, \omega)$

Definition 0.11. A Kähler manifold is a triple $(M, \omega, \mathcal{J})$ where M is a manifold, $\omega \in \Omega^2(M)$ symplectic, and $\mathcal{J} \in \mathcal{J}(M, \omega)$ is a complex (i.e integrable) structure.

Example 0.5. $M = \mathbb{C}^n$, standard complex and symplectic structure, and open subsets thereof.

Example 0.6. $M = \Sigma$ a 2-dimensional manifold, $\omega \in \Omega^2(\Sigma)$ the volume form (so $d\omega = 0$ automatically), any $\mathcal{J} \in \mathcal{J}(M, \omega)$ is automatically integrable.

Proposition 23. *If $(M, \omega, \mathcal{J})$ is Kähler, and $\iota : N \hookrightarrow M$ is a complex submanifold (i.e. $\mathcal{J}(TN) \subseteq TN$), then $(N, \omega_N, \mathcal{J}_M)$, with $\omega_n = \iota^*\omega$, is Kähler.*

Proof. All we have to show ω_N is non-degenerate, i.e. that every nonzero vector $v \in TN$ has a friend $\omega \in TN$ with $\omega(v, w) \neq 0$. Consider $w = \mathcal{J}v$. We have

$$\omega(v, w) = \omega(v, \mathcal{J}v) = g(v, v) > 0$$

∎

A consequence is that any complex submanifold of $\mathbb{C}^n$ is Kähler and in particular symplectic.
Consider $\mathbb{CP}(n) = \frac{\mathbb{C}^{n+1} \setminus \{0\}}{\mathbb{C} \setminus \{0\}} = S^{2n+1}/U(1)$
There is a complex structure such that $q : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}(n)$ is holomorphic.
Let $S^{2n+1} \subset \mathbb{C}^n$ be the unit sphere.
Then

$$\begin{tikzcd}
S^{2n+1} \arrow[r, hook, "\iota"] \arrow[d, "\pi"] & \mathbb{C}^{n+1} \setminus \{0\} \\
\mathbb{CP}(n) &
\end{tikzcd}$$

Theorem 0.9. $\mathbb{CP}(n)$ *has a unique symplectic structure, usually denoted ω_{FS} (for Fubini-Study), such that $\pi^*\omega_{FS} = \iota^*\omega$. With this symplectic structure it's a Kähler manifold.*

Proof. To show that $\iota^*\omega \in \Omega^2(S^2)$ descends under π, we have to show

$$\iota_z(\iota^*\omega) = 0, \mathscr{L}_Z(\iota^*\omega) = 0$$

for all vertical vector fields. By Cartan, since $\iota^*\omega$ is closed, it is enough to check the 1st condition.
Since the fibers of π are 1-dimensional, it is enough to show $\ker(T_z\pi) \subseteq \ker(\iota^*\omega)$
For $z \in S^{2n+1}$, $T_z(\mathbb{C}^{n+1} \setminus \{0\}) = \mathbb{C}^{n+1}$. $\ker(T_z q) = \mathrm{span}_{\mathbb{C}}(z) = \mathbb{C} \cdot z$.
This contains $\mathrm{span}_{\mathbb{R}}(z) = \mathbb{R} \cdot z = (T_z S^{2n+1})^\perp$, hence also $\mathcal{J}(\mathbb{R}z) \subseteq T_z S^{2n+1}$
We get

$$T_z(\mathbb{C}^{n+1} \setminus \{0\}) = \mathbb{R}z \oplus \underbrace{\mathcal{J}(\mathbb{R}z) \oplus \overbrace{\left(T_z S^{2n+1} \cap \mathcal{J}(T_z S^{2n+1}\right)}^{\text{complex structure(?)}}}_{T_z(S^{2n+1})}$$

and $\mathcal{J}(\mathbb{R}z) = \ker(T_z\pi)$, $\mathbb{R}z \oplus \mathcal{J}(\mathbb{R}z)$ is complex.
...

■

Basics of symplectic manifolds

Let $\mathfrak{X}(M, \omega) = \{X \mid \mathscr{L}_X\omega = 0\}$, and $\mathfrak{X}_{\mathrm{Hom}}(M, \omega) = \{X \mid \exists H, \iota_X\omega = -dH\}$
For $H \in C^\infty(M)$, we call X_H the vector field such that $\iota(X_H)\omega = -dH$.

Proposition 24. *There is an exact sequence*

$$0 \longrightarrow \mathfrak{X}_{Ham}(M, \omega) \longrightarrow \mathfrak{X}(M, \omega) \longrightarrow H^1(M) \longrightarrow 0$$

where $H^1(M)$ is the de Rham cohomology.

Proof. The map $\omega^\flat : TM \to T^*M$ given by $v \mapsto \iota_v\omega = \omega(v, \cdot)$, gives an isomorphism $\omega^\flat : \mathfrak{X}(M) \to \Omega^1(M)$.
We have $L_X\omega = 0 \iff d\iota_X\omega = 0 \iff \iota_X\omega \in \Omega^1_{Cl}(M)$
We see $\omega^\flat : \mathfrak{X}(M, \omega) \to \Omega^1_{Cl}(M), \omega^\flat : \mathfrak{X}_{Ham}(M, \omega) \to \Omega^1_{ex}(M)$

■

Proposition 25. *We have $[\mathfrak{X}(M, \omega), \mathfrak{X}(M, \omega)] \subseteq \mathfrak{X}_{Ham}(M, \omega)$*

Proof. Exercise

■

Proposition 26. *In fact, for $Y_1, Y_2 \in \mathfrak{X}(M, \omega)$*

$$[Y_1, Y_2] = X_{\omega(Y_1, Y_2)}$$

Proof. This is because

$$\begin{aligned} d\omega(Y_1, Y_2) &= d\iota_{Y_2}\iota_{Y_1}\omega \\ &= \mathscr{L}_{Y_2}(\iota_{Y_1}\omega) - \iota_{Y_2}d\iota_{Y_1}\omega \\ &= \iota(\mathscr{L}_{Y_2}Y_1)\omega + \iota_{Y_1}\mathscr{L}_{Y_2}\omega - \iota Y_2\mathscr{L}_{Y_1}\omega + \iota_{Y_1}\iota_{Y_2}\omega \\ &= -\iota([Y_1, Y_2])\omega \end{aligned}$$

Various things in the third line dissapear because of various things being symplectic. ∎

Consider next the map $C^\infty(M) \to \mathfrak{X}_{Ham}(M, \omega)$, $H \mapsto X_H$.
We have an exact sequence

$$0 \longrightarrow H^0(M) \longrightarrow C^\infty(M) \longrightarrow \mathfrak{X}_{Ham}(M, \omega) \longrightarrow 0$$

$H^0(M)$ is the locally constant functions. We want to define a Lie algebra on these things to make this a short exact sequence of Lie algebras.

Definition 0.12. The Poisson bracket of $F, G \in C^\infty(M)$ is defined by

$$\{F, G\} = \omega(X_F, X_G)$$

It can be shown that $\{F, G\} = -\{G, F\}$.

Proposition 27. $\{\cdot, \cdot\}$ *is a Lie bracket on* $C^\infty(M)$. *That is,*

1. $\{F, G\} = -\{G, F\}$

2. $\{F, \{G, H\}\} + \{G, \{H, F, \}\} + \{H, \{F, G, \}\} = 0$

The map $C^\infty(M) \to \mathfrak{X}_{Ham}(M, \omega), F \mapsto X_F$, *is a Lie algebra homomorphism.*

Proof.
We start by proving (2).
We want to show that $[X_F, X_G] = X_{\{F,G\}} = X_{\omega(X_F, X_G)}$, which follows from the previous proposition, $(*)$.
For (1), note first that

$$\{F, G\} = \omega(X_F, X_G) = \iota(X_G)\underbrace{\omega(X_f, \cdot)}_{-dF} = -\mathscr{L}_{X_G}F = \mathscr{L}_{X_F}G$$

Now

$$\begin{aligned}\{F,\{G,H\}\} &= \mathscr{L}_{X_F}\{G,H\} \\ &= \mathscr{L}_{X_F}\omega(X_G,X_F) \\ &= \underbrace{(\mathscr{L}_{X_F}\omega)}_{=0}(X_G,X_H)+\omega(\mathscr{L}_{X_F}X_G,X_H)+\omega(X_G,\mathscr{L}_{X_F}X_H \\ &= \omega([X_F,X_G],X_4)+\omega(X_G,[X_F,X_H]) \\ &= \omega(X_{\{F,G\}},X_H)+\omega(X_G,X_{\{F,G\}}) \\ &= \{\{F,G\},H\}+\end{aligned}$$

He erased it :(
Note: we also have

$$\{F,G\cdot H\}=\{F,G\}H+G\{F,H\}(*)$$

since $\{F,\cdot\}=\mathscr{L}_{X_F}$.
Remark: A Poisson structure on a manifold M is a Lie bracket $\{\cdot,\cdot\}$ on $C^\infty(M)$, satisfying the above equation.
(word I can't read) For classical mechanics: note for algebra $\mathscr{A}$, the commutator $[a,b]=ab-ba$ has a property similar to $(*)$: $[a,bc]=[a,b]c+b[a,c]$

Proposition 28. *If $\{F,G\}=0$, then*

(a) G is constant along the integral curves of X_F

(b) The flows of Hamiltonian vector fields X_F,X_G, commute.

Proof.

(a) From $\mathscr{L}_{X_F}G=\{F,G\}=0$.

(b) $[X_F,X_G]=X_{\{F,G\}}=0$

■

Proposition 29. *For (M,ω) compact connected, the Lie algebra map $C^\infty(M)\to\mathfrak{X}_{Ham}(M,\omega)$ has a canonical splitting, i.e. a right inverse.*

Proof. Given $X\in\mathfrak{X}_{Ham}(M,\omega)$, let $H\in C^\infty(M)$ be the unique function such that $X=X_H$, and $\int_M H\frac{\omega^n}{n!}=0$
If F,G are normalized in this way, we get

$$\int_M\{F,G\}\frac{\omega^n}{n!}=\int_M(\mathscr{L}_{X_F}G)\frac{\omega^n}{n!}=\int_M\mathscr{L}_{X_F}\left(G\frac{\omega^n}{n!}\right)=0$$

In coordinates, $q_1,p_1,\ldots,q_n,p_n$,

$$\omega=\sum dq_i\wedge dp_i$$

$$X_F = \pm \sum \left(\frac{\partial F}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial}{\partial p_i} \right)$$

and

$$\{F, G\} = \pm \sum \left(\frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} \right)$$

Lecture 9, 10/3/24

Lagrangian Submanifolds

Let (M, ω) be a symplectic manifold.

Definition 0.13. A submanifold $N \subseteq M$ is

- Isotropic
- Coisotropic
- Lagrangian

If, for all $m \in N$, the tangent space $T_m N \subseteq T_m M$ has that property.
Remark: Letting $\iota : N \hookrightarrow M$, we have that N is isotropic if and only if $\iota^* \omega = 0$ and Lagrangian if also $\dim N = \frac{1}{2} \dim M$.
Remark: Every 1-dimensional submanifold is isotropic, and every codimension-1 submanifold is coisotropic.
Weinstein: "Everything is a Lagrangian submanifold"

Example 0.7.

1. Let $\pi : M = T^*Q \to Q$, and $N = \pi^{-1}(q)$ are Lagrangian for any $q \in Q$. This comes from $\omega = \sum dq_i \wedge dp_i$, looking at a fiber is setting $dq_i = 0$.

2. $M = T^*Q, N = Q$(as zero-section). More generally, if $\alpha \in \Omega^1(Q)$, the range of $\sigma_\alpha : Q \to T^*Q$ is Lagrangian if and only if $d\alpha = 0$.

proof

$$\sigma_\alpha^* \omega = -\sigma_\alpha^* d\theta = -d\sigma_\alpha^* \theta = -d\alpha$$

3. $\mathbb{R}^n \subseteq \mathbb{C}^n \cong \mathbb{R}^{2n}$, or likewise $\mathbb{RP}^n \subseteq \mathbb{CP}^n$ are Lagrangian submanifolds.

More generally, if $(M, \omega, \mathcal{J})$ is a Kähler manifold with a "complex conjugation" $F \in \mathsf{Diff}(M)$ (i.e. $F \circ F = \mathrm{Id}_M$, $F^* \circ \mathcal{J} = -\mathcal{J}$, $F^* g = g$) then the fixed set of F is a Lagrangian submanifold.

4. More generally, if (M, ω) is a symplectic manifold with a symplectic involution $F \in \mathsf{Diff}(M)$ (i.e. $F \circ F = \mathrm{Id}_M, F^*\omega = -\omega$), then the fixed point set is a Lagrangian submanifold.

proof:

Fixed point set of any involution is a submanifold, $N \subseteq M$.

This reduces the problem to tangent spaces $T_mN \subseteq T_mM$ (fixed point set of T_mF).

It is enough to look at symplectic vector space (V, ω) with $A \in \mathrm{Symp}(V, \omega)$, $A^2 = I, A^*\omega = -\omega$. Write $V = V_{+1} \oplus V_{-1}$, the eigenspaces of A.

<u>Claim:</u>

ω restricts to 0 on both $V_{\pm}$.

proof:

For $v, w \in V_{-1}$,

$$\omega(v, w) = -\omega(Av, Aw) = -\omega(-v, -w) = -\omega(v, w)$$

since $A^*\omega = -\omega$.

5. For (M, ω) symplectic, denote by $\overline{M}$ the same manifold but with the symplectic form negated. Then $\triangle_M \subseteq M \times \overline{M}$ is a Lagrangian submanifold.

More generally, if $F \in \mathsf{Diff}(M)$, then $gr(F) = \{(F(m), m) \mid m \in M\} \subseteq M \times \overline{M}$ is a Lagrangian submanifold if and only if $F \in \mathsf{Diff}(M, \omega)$.

6. Let $F : Q_1 \to Q_2$ be a smooth map. Then $gr(T^*F) \subseteq T^*Q_2 \times \overline{T^*Q_1}$ is Lagrangian, where $gr(T^*F) = \{(\mu_2, \mu_1) \mid \mu_1 = (T_q^*(F))(\mu_2), q = \pi(\mu_1)\}$

proof

Exercise (understand the linear case)

7. Suppose $S \subseteq Q$ is a submanifold. The conormal bundle $\nu^*(Q, S)$ which we define as follows. First, let

$$\nu(Q, S) \stackrel{\text{def}}{=} TQ|_S/TS$$

$(V/W)^* = \mathrm{ann}(W) \subseteq V^*$, i.e. the elements of V^* which kill W. So we define

$$\nu^*(Q, S) = \mathrm{ann}(TS) \subseteq T^*Q|_S$$

is a Lagrangian submanifold.

proof

Near any given point of S, choose coordinates $q_1, \ldots, q_n$ such that S is given by $q_{k+1} = \cdots = q_n = 0$.

Let $q_1, p_1, \dots, q_n, p_n$ be the corresponding coordinates on the cotangent bundle.

Then $\nu^*(Q, S)$ given by $q_{k+1} = \cdots = q_n = 0$

$TS = \text{span}\{\frac{\partial}{\partial q_1}, \cdots \frac{\partial}{\partial q_n}\}$

8. Prop: If $j : S \hookrightarrow Q$ is a submanifold, and $\alpha \in \Omega^1(S)$ is closed, then $N = \{\mu \in T^*Q|_S, j^*\mu = \alpha|_{\pi(\mu)}\} \subseteq T^*Q$ is again a lagrangian submanifold.

Rem: In particular, every function $f \in C^\infty(S)$ determines a Lagrangian submanifold of T^*Q : take $\alpha = df$.

Proof: Again use coordinates. This time, if $\alpha = \sum_{i=1}^k \alpha_i dq_i$, equations for N are $q_{k+1} = \cdots = q_n = 0$, $p_1 = \alpha_1, \dots, p_k = \alpha_k$

$$\sum_{i=1}^n dq_i \wedge dp_i = \sum_{i,j=1}^k \frac{\partial \alpha_i}{\partial q_j} dq_i \wedge dq_j = d\alpha$$

so this will be zero if and only if α is closed.

Lecture 10, 10/8/24

Lagrangian submanifolds (continued)

Recall: Let (M, ω) be a symplectic manifold, $F \in \mathsf{Diff}(M_1, M_2)$ is symplectic if and only if $gr(F)$, the graph of F, is a Lagrangian submanifold of $M_2 \times \overline{M}_1$.
More generally:

Definition 0.14. A Lagrangian relation from M_1 to M_2 is a Lagrangian submanifold of $M_2 \times \overline{M_1}$.
Write $R : M_1 \dashrightarrow M_2$.

Example 0.8.

- $R = gr(F)$, $F \in \mathsf{Diff}(M, \omega)$
- Any Lagrangian submanifold $N \subseteq M$ is a Lagrangian relation $pt \dashrightarrow M$, or $M \dashrightarrow pt$
- For $f \in C^\infty(Q_1, Q_2)$, the cotangent relation $T^*f : T^*Q_1 \dashrightarrow T^*Q_2$, that is $\underbrace{\mu_1}_{\in T_xQ_1} \sim \underbrace{\mu_2}_{\in T_{f(x)}Q_2} \iff \mu_1 = (T_xf)^*\mu_2$
- Given another Lagrangian relation $R' \subseteq M_3 \times \overline{M_2}$, i.e. $R' : M_2 \dashrightarrow M_3$, we can compose

$$R' \circ R \stackrel{\text{def}}{=} \{(m_3, m_1) \mid \exists m_2 \in M_2 : (m_3, m_2) \in R', (m_2, m_1) \in R\} \subseteq M_3 \times \overline{M_1}$$

Under suitable transversality conditions (e.g. $(R' \times R)$ needs to be transverse to $M_3 \times \triangle_{M_2} \times M_1$), $R' \circ R$ is a submanifold, and is again Lagrangian.

Weinstein calls this the Symplectic "category".

- Let $j : S \hookrightarrow Q$ be a submanifold, $\alpha \in \Omega^1_{Cl}(S)$. From j we get the cotangent relation

$$T^*j : T^*S \dashrightarrow T^*Q$$

From α, get Lagrangian submanifold $N_\alpha \subseteq T^*S$, which we think of as a relation $pt \dashrightarrow T^*S$.

By composition, we get $pt \dashrightarrow T^*Q$,i.e. a Lagrangian submanifold of T^*Q.

Applications:

Theorem 0.10. *(Tulczyjew)*
Let $E \to B$ be a vector bundle. Then there is a canonical symplectic isomorphism;

$$T^*E \to T^*E^*$$

Proof. Note that T^*E is not just a vector bundle over E, but over E^*, in some kind of "double bundle."

$$\begin{tikzcd} T^*E \arrow[r] \arrow[d] & E^* \arrow[d] \\ E \arrow[r] & B \end{tikzcd}$$

we also have

$$\begin{tikzcd} TE \arrow[r] \arrow[d] & TB \arrow[d] \\ E \arrow[r] & B \end{tikzcd}$$

We consider the pullback

$$\begin{tikzcd} \pi^*E \arrow[r] & TE \arrow[r] \arrow[d] & TB \arrow[d] \\ & E \arrow[r] & B \end{tikzcd}$$

dualizing, we have

$$\begin{tikzcd} T^*E^* \arrow[r] \arrow[d] & E \arrow[d] \\ E^* \arrow[r] & B \end{tikzcd}$$

So this is an isomorphism of double bundles, not just of vector bundles.

For the proof, we'll describe it in terms of its graph. Consider $\underbrace{E^* \oplus E}_{S}$ (meaning pairs (u, w) where u is a tangent vector and w is a cotangent vector on the same base point) as a submanifold of $\underbrace{E^* \times E}_{Q}$

We have a function $f : E^* \oplus E \to \mathbb{R}$ given by pairing. Take $\alpha = df$.
Then we get a Lagrangian submanifold of $T^*Q = T^*E^* \times T^*E$
After sign change in last factor, we get Lagrangian submanifold in $T^*E^* \times \overline{T^*E}$, i.e. a Lagrangian relation $T^*E \dashrightarrow T^*E^*$ one checks that this is the graph of a symplectomorphism.
One checks this by assuming $E = B \times V$, then $T^*E = T^*B \times T^*V, T^*E^* = T^*B \times T^*V^*, E^* = B \times V^*$. One finds: the map

$$T^*V = V \oplus V^* \to T^*V^* = V^* \oplus V$$

is $(\nu, \mu) \mapsto (-\mu, \nu)$

∎

Special case: $T^*(TQ) \cong T^*(T^*Q)$, the Legendre transform.

Coisotropic submanifolds

Recall that if (V, ω) is a symplectic vector space, then $F^\omega = \{v \in V \mid \omega(v, w) = 0 \text{for all } w \in F\}$
Recall $\omega^\flat : V \to V^*$: we have $\omega^\flat : F^\omega \to \operatorname{ann}(F)$
Recall: $N \subseteq M$ is coisotropic if $(TN)^\omega \subseteq TN$.
In general, consider for $N \subseteq M$,

$$I_N = C^\infty(M)_N = \{f \in C^\infty(M) \mid f|_N = 0\}$$

the vanishing ideal.

- $X \in \mathscr{X}(M)$ is tangent to $N \iff X$ preserves $C^\infty(M)_N$
- $v \in TM$ lies in $TN \iff v(f) = 0$ for all $f \in C^\infty(M)_N$.
- $T^*M|_N \supseteq \operatorname{ann}(TN) = \operatorname{span}\{df|_N \mid f \in C^\infty(M)_N\}$

Lemma 3. *For any submanifold N of (M, ω), the bundle TN^ω is spanned by X_f with $f \in C^\infty(M)_N$*

Proof. $\omega^\flat$ restricts to an isomorphism $TN^\omega \to \operatorname{ann}(TN)$, under this isomorphism $X_f|_N \mapsto df|_N$, and $\operatorname{ann}(TN)$ is spanned by all $df|_N, f \in C^\infty(M)_N$.

∎

Theorem 0.11. *The following are equivalent:*

1. *For all $f \in C^\infty(M)_N$, X_f is tangent to N*

2. *$C^\infty(M)_N$ is closed under $\{\cdot,\cdot\}$*

3. *$N \subseteq M$ is a coisotropic submanifold*

Proof. For the first, it follows because TN^ω is spanned by $X_f, f \in C^\infty(M)_N$.
For the second, for all $f \in C^\infty(M)_N$, X_f is tangent to N means that for all $g \in C^\infty(M)_N$, $\underbrace{X_f(g)}_{=\{f,g\}} \in C^\infty(M)_N$

■

If we want to check the second condition (that is, that $C^\infty(M)_N$ is closed under $\{\cdot,\cdot\}$), it's enough to check $\{f_i, f_j\} = 0$ for any collection $f_1, \ldots, f_k \in C^\infty(M)_N$ such that $df_i|_N$ span $\mathrm{ann}(TN)$

Proposition 30. *Let (M,ω) be symplectic, and $\pi : M \to Q$ a submersion. Then the fibers of π are all coisotropic $\iff \pi^*C^\infty(Q) \subseteq C^\infty(M)$*

Proof. In the left direction, suppose all the fibers are coisotropic. Given $q \in Q$, $N = \pi^{-1}(q)$ is coisotropic. For $f \in C^\infty(Q)$, then $\pi^*f - f(q) \in C^\infty(M)_N$. Given $f, g \in C^\infty(Q)$, get $\{\pi^*f, \pi^*g\} = \{\pi^*f - f(q), \pi^*g - g(q)\}$ vanishes on $N = \pi^{-1}(q)$. Since q was arbitrary, this bracket vanishes everywhere.
In the right direction, suppose that $\pi^*C^\infty(Q)$ has zero Poisson bracket.
Given $q \in Q$, we want to sho $N = \pi^{-1}(q)$ is coisotropic. Choose $f_1, \ldots, f_n$ with $f_i(q) = 0$ such that $df_i|_q$ span T_qQ.
Then their pullbacks π^*f_i are such that $d(\pi^*f_i)$ span $\mathrm{ann}(\pi^{-1}(q))$, and by assumption $\{\pi^*f_i, \pi^*f_j\} = 0$

■

Note for $\dim Q = \frac{1}{2}\dim M$, we get a "Lagrangian Fibration."

Constant rank submanifolds

Definition 0.15. A 2-form $\sigma \in \Omega^2(Q)$ has constant rank if the rank of $\sigma^\flat|_q : T_qQ \to T_q^*Q$ is constant.
This is equivalent to saying that the dimension of $\ker(\sigma) = \{v \mid \sigma(v,\cdot) = 0\}$ is constant (which is by definition the kernel of $\sigma^\flat$).

Definition 0.16. For (M,ω) a symplectic manifold with submanifold $j : N \hookrightarrow M$ has constant rank if $j^*\omega$ has constant rank.
E.g: isotropic, coisotropic, Lagrangian, symplectic

Proposition 31. *If $\sigma \in \Omega^2_{Cl}(Q)$ is a closed 2-form of constant rank, then $\ker(\sigma) \subseteq TQ$ is integrable, i.e. corresponds to some fibration.*

Proof. Use Fubini's theorem: $E \subseteq TQ$ is integrable if for all $X, Y \in \Gamma(E) \subseteq \Gamma(TQ) = \mathscr{X}(Q)$, we have $[X, Y] \in \Gamma(E)$
<u>Check:</u> Suppose $X, Y \in \ker(\sigma)$. Then

$$\begin{aligned}\iota([X,Y])\sigma &= \iota(L_X Y)\sigma \\ &= L_X(\underbrace{\iota(Y)\sigma}_{=0}) - \iota(Y)L_X\sigma \\ &= -\iota(Y)(\iota(X)d + d\iota(X))\sigma \\ &= 0\end{aligned}$$

So, if $N \subseteq M$ is constant rank, we get a fibration of N.
In particular, for coisotropic.

For $\sigma \in \Omega^2_{Cl}(Q)$ closed, constant rank, this fibration is called the <u>null fibration</u>.
<u>If</u> this foliation is "fibrating," i.e. comes from a submersion $\pi : Q \to B$ (i.e. the space of leaves of foliations is a manifold).
In that case, σ descends to a 2-form $\omega_B \in \Omega^2(B)$, $\pi^*\omega_B = \sigma$.
Furthermore, ω_B is symplectic (closed: $\pi^*\omega_B = d\pi^*\omega_B - dt = 0$, nondegenerate...)
So if $N \subseteq M$ (M, ω) is constant rank submanifold, and if (big if!) the nullfibration of $j^*\omega$ is fibrating, then $B = N/\sim$ (the space of leaves?) inherits a symplectic form.
This is the general case of "symplectic reduction."
<u>Moser argument</u>

Definition 0.17. The flow of a time-dependent vector field $X_t \in \mathscr{X}(Q)$, $t \in \mathbb{R}$, is a smooth family of diffeomorphisms $\varphi_t \in \mathsf{Diff}(Q)$ such that $\phi_0 = \mathrm{Id}$, $\phi_t^*(\mathscr{L}_{X_t} f) = -\frac{d}{dt}\phi_t^*(f)$
In coordinates, if $X_t = \sum_{i=1}^n a_i(x,t)\frac{\partial}{\partial x_i}$, this corresponds to the ODE

$$\frac{dx_i}{dt} = a_i(x(t), t)$$

Theorem 0.12. *(Moser stability for volume forms)*
Let Q be a compact oriented manifold, $\Lambda_0, \Lambda_1 \in \Omega^{\dim Q}(Q)$ are volume forms, with $\int_Q \wedge_0 = \int_Q \wedge_1$. Then there exists a diffeomorphism $\varphi \in \mathsf{Diff}(Q)$ such that $\varphi^*\Lambda^1 = \Lambda^0$.

Proof. Let $\Lambda_t = t\Lambda_1 + (1-t)\Lambda_0$. These are all volume forms, same volume.
We want a family of diffeomorphisms φ_t such that $\varphi_t^*\Lambda_t = \Lambda_0$.

Let X_t corresponding time-dependent vector field having φ_t as its flow. Then $\varphi_t^*\Lambda_t = \Lambda_0 \iff \frac{d}{dt}(\varphi_t^*\Lambda_t) = 0$. We calculate

$$\begin{aligned}\frac{d}{dt}(\varphi_t^*\Lambda_t) &= -\varphi_t^*(-\mathscr{L}_{X_t}\Lambda_t + \Lambda_1 - \Lambda_0)\\ &= \varphi_t^*(-d\iota(X_t)\Lambda_t + d\beta)\\ &= d\varphi_t^*(-\iota(X_t)\Lambda_t + \beta)\end{aligned}$$

Now define X_t by $\iota(X_t)\Lambda_t = \beta$ (where β is a primitive of $\Lambda_1 - \Lambda_0$ (because they are by definition cohomologous)) (this definition also uniquely determines X_t).
Then we are done.

■

Lecture 11, 10/10/24

Moser-Weinstein theorems

Background: Homotopy operators

Definition 0.18. Given smooth maps $F_0, F_1 \in C^\infty(Q_1, Q_2)$, a smooth homotopy between them is a map $F \in C^\infty([0,1] \times Q_1, Q_2), (t, q) \mapsto F_t(q)$, such that the values at $t = 0$ and $t = 1$ are the given ones.
In this case, $F_0^* = F_1^*$ in de Rham cohomology. But how can we see this?
Define a homotopy operator

$$\begin{tikzcd} \Omega^k(Q_2) \arrow[r, "F^*"] \arrow[rr, bend right, "h"'] & \Omega^k([0,1] \times Q_1) \arrow[r, "\int_{[0,1]}"] & \Omega^{k-1}(Q_1) \end{tikzcd}$$

Here

$$\int_{[0,1]} (ds \wedge \underbrace{\beta_s}_{\in\Omega^{k-1}(Q_1)} + \underbrace{\gamma_s}_{\in\Omega^k(Q_1)}) = \int_0^1 \beta_s |ds|$$

Exercise:
For $\alpha \in \Omega^k([0,1] \times Q_2)$, $\int_{[0,1]} d\alpha + d\left(\int_0^1 \alpha\right) = \iota_1^*\alpha - \iota_2^*\alpha$
One finds: $d \circ h + h \circ d = F_1^* - F_0^*$ as maps $\Omega^k(Q_2) \to \Omega^k(Q_1)$.
So h is a chain homotopy between F_1^*, F_0^*, so by basic homological algebra, $F_1^* = F_0^*$.

Example 0.9. Consider $\mathbb{R}^n$, let $F_1 = \mathrm{Id}, F_0(x) = 0$. Get homotopy by $F_t = tF_1$ for $t \in [0,1]$. So we get a homotopy operator $h : \Omega^k(\mathbb{R}^n) \to \Omega^{k-1}(\mathbb{R}^n)$.
The point is these homotopy operators give canonical primitives to closed forms.

Theorem 0.13. *(Moser stability theorem for symplectic structures)*
Let M be a compact manifold, $\omega_t \in \Omega^2(M)$ a family of symplectic forms with

$$\frac{d\omega_t}{dt} = d\beta_t$$

for some family of 1-forms β_t. Then there exists a family $\varphi_t \in \mathsf{Diff}(M)$, with $\varphi_0 = \mathrm{Id}_M$, such that

$$\varphi_t^*\omega_t = \omega_0$$

Proof. Recall: $X_t \in \mathscr{X}(M)$ time dependent vector field, has flow φ_t ($\varphi_0 = \mathrm{Id}_M$)

$$\frac{d}{dt}\varphi_t^*\alpha = -\varphi_t^*\left(\mathscr{L}_{X_t}\alpha\right)$$

for $\alpha \in \Omega^k(M)$.
We want $0 = -\frac{d}{dt}(\varphi_t^*\omega_t)$. We have

$$\begin{aligned} -\frac{d}{dt}(\varphi_t^*\omega_t) &= \varphi_t^*\left(\mathscr{L}_{X_t}\omega_t - \frac{d\omega_t}{dt}\right) \\ &= \varphi_t^* d(\iota_{X_t}\omega_t - \beta_t) \end{aligned}$$

(recall that by hypothesis $\frac{d\omega_t}{dt} = d\beta_t$, which means $[\frac{d\omega_t}{dt}] = 0$ in $H^2(M)$, which implies $[\omega_t]$ in $H^2(M)$ doesnt depent on t. The converse is also true, but is more difficult).
We use compactness to guarantee the existence of flows (this is also what we use it for in the Moser argument from last time).
Define X_t by $\iota_{X_t}\omega_t = \beta_t$

■

Theorem 0.14. *(Darboux' theorem (Libermann, 1948))*
Let (M, ω) be a symplectic manifold, $m \in M$. There exists a local coordinate chart (U, φ) around m, with coordinates $q_1, p_1, \ldots, q_n, p_n$, such that $\omega|_U = \varphi^*(\sum_{i=1}^n dq_i \wedge dp_i)$. Coordinates of this type are called "Darboux coordinates".

Proof. Start by choosing any coordinates around m to reduce to the case that $U \subseteq \mathbb{R}^{2n}$, with m the origin.
Let $\omega_0 = \sum_{i=1}^n dq_i \wedge dp_i$. Using linear change of coordinates, we may assume $\omega|_{T_0\mathbb{R}^{2n} \times T_0\mathbb{R}^{2n}}$ is the standard one.
Let $\omega_t = (1-t)\omega_0 + t\omega_1$, where ω_1 is the ω we defined above.
These are all standard on $T_0\mathbb{R}^{2n}$, hence are all symplectic on the origin, hence also on some open neighborhood of 0. By shrinking U, we may assume symplectic on all of U, and may also assume U is an open ball around 0. Now, by de Rham homotopy operator for $U = \mathbb{R}^{2n}$,

$$\frac{d\omega_t}{dt} = \omega_1 = \omega_0 = d\beta$$

Define X_t by $\iota(X_t) = \beta$. Let φ_t be its flow. But X_t vanishes at $0 \in U$ since $\beta_t|_0 = 0$. Hence close to 0, X_t is "small." Hence, the time-1-flow exists in some neighborhood $U' \subseteq U$.
Then define $\varphi_t : U' \to U$ for $t \in [0, 1]$.
Now we use Moser's argument.

■

Remark: If a compact Lie group G acts on (M, ω) by symplectic diffeomorphisms, and if $m \in M$ is G-fixed, then we get an equivariant Darboux theorem, i.e. with U G-invariant, and φ G-equivariant for some linear symplectic G-action on $\mathbb{R}^{2n}$.
(Indeed, can make all choices G-invariant by using averaging (because the Lie group is compact))

Lecture 12, 10/15/24

Normal Form Theorems

Last time, we did Darboux' theorem. Today, we consider generalizations along submanifolds.
Given a symplectic manifold (M, ω) and a submanifold $N \subseteq M$, the symplectic structure is uniquely determined on some neighborhood U of N by $\omega|_N \in \Gamma(\wedge^2 T^* M|_N)$.
More precisely:

Theorem 0.15. *("Master Theorem")*
Let (M_i, ω_i), $i \in \{0, 1\}$ be symplectic manifolds and $N_i \subseteq M_i$ submanifolds. Suppose there is an isomorphism of symplectic vector bundles (i.e. a vector bundle whose fibers have symplectic structures)

$$\begin{tikzcd} TM_0|_{N_0} \arrow[r, "\hat{\psi}"] \arrow[d] & TM_1|_{N_1} \arrow[d] \\ N_0 \arrow[r, "\psi"] & N_1 \end{tikzcd}$$

such that $\hat{\psi}$ extends $T\psi : TN_0 \to TN_1$. Then $\psi : N_0 \to N_1$ extends to a symplectomorphism $\varphi : U_0 \to U_1$ of open neighborhoods U_i of N_i such that $\tilde{\psi} = T\varphi|_{N_0}$

Proof. Later

■

Review: Normal bundles.
Given a submanifold $N \subseteq M$, then the normal bundle $\nu(M, N)$ is defined by $TM|_N/TN$, a vector bundle over N. Then:

- ν is functorial: Given $\varphi : M_0 \to M_1$ with $\varphi(N_0) \subseteq N_1$, we get $\nu(\varphi) : \nu(M_0, N_0) \to \nu(M_1, N_1)$.

- If $M = E \to N$ is a vector bundle, then $TE|_N = E \oplus TN$, so $\nu(E, N) = E$.

Definition 0.19. Given (M, N), a tubular neighborhood embedding $\varphi : U \to M$ of some open neighborhood $U \subseteq \nu(M, N)$ of N is an embedding such that $\varphi(N) \subseteq N$ and $\nu(\varphi)$ =identity of $\nu(M, V)$.

- Tubular neighborhood embeddings exist. For ex, can take U to be a bundle of open balls (ε-neighborhoods of N).

Now we prove the theorem.

Proof. Using tubular neighborhood embeddings, we may assume $M_i = \nu(M_i, N_i)$. The map $\hat{\psi}$ gives an isomorphism $\nu(M_0, N_0) \to \nu(M_1, N_1)$, because it takes the tangent bundle to the tangent bundle, and the tangent of the submanifold to the tangent of the submanifold.
Using this, we may assume $M_0 = M_1 = M$ is a vector bundle $\pi : M \to N = N_0 = N_1$ equipped with two symplectic forms, ω_0, ω_1, so that $\omega_0|_N = \omega_1|_N$.
In particular, $\iota^*\omega_0 = \iota^*\omega_1$, hence $[\omega_0] = [\omega_1]$.
The homotopy operator $h : \Omega^k(M) \to \Omega^{k-1}(M)$ for the vector bundle $\pi : M \to N$, gives $\beta \in \Omega^1(M)$, $\beta = h(\omega_1 - \omega_0)$ such that $d\beta = \omega_1 - \omega_0$.
We get a family of cohomologous symplectic forms on smaller neighborhood of $N \subseteq M$, $\omega_t = (1-t)\omega_0 + t\omega_1 = \omega_0 + td\beta$.
Now, use the Moser argument. ∎

Special case: Lagrangian submanifolds
Recall that $N \subseteq M$ is Lagrangian $\iff TN^\omega = TN \iff \iota^*\omega = 0, \dim N = \frac{1}{2}\dim M$

Lemma 4. *For N Lagrangian, we have canonical isomorphism of vector bundles $\nu(M, N) = T^*N$*

Proof. Recall the isomorphism $\omega^\flat : TM|_N \to T^*M|_N$ restricts to an isomorphism $TN^\omega \to \operatorname{ann}(TN)$
But for a Lagrangian submanifold, $TN^\omega = TN$.
So, $\nu(M, N) = TM/TN \to T^*M|_N / \operatorname{ann}(TN) = T^*N$ ∎

Theorem 0.16. *(Weinstein)*
Let (M, ω) be symplectic, and $N \subseteq M$ Lagrangian. Then there exists a tubular neighborhood embedding

$$\psi : U \to M$$

with $U \subseteq \nu(M, N) = T^*N$ which is symplectic (for the standard symplectic structure on T^*N).

Proof. We need to find $\hat{\psi}$ so that

$$\begin{tikzcd} T\nu(M,N)|_N \arrow[r, "\hat{\psi}"] \arrow[d] & TM|_N \arrow[d] \\ N \arrow[r, "\cong"] & N \end{tikzcd}$$

commutes. Now, $TM|_N$ has a Lagrangian tubbundle TN. Choose complex Lagrangian subbundle $L \subseteq TM|_N$ (e.g. Take $L = \mathcal{J}(TN)$ for compatible complex structure $\mathcal{J}$), symplectic structure identifies $L \cong T^*N$ so, $TM|_N = TN \oplus T^N$ with fiberwise symplectic structure given by pairing.
Do the same for $T(T^*M)|_N = TN \oplus T^*N \cong TM|_N$

■

For constant rank submanifolds $N \subseteq M$, define the <u>symplectic normal bundle</u> as

$$F = TN^\omega / (TN \cap TN^\omega)$$

Note $F \subseteq TM|_N/TN$

Theorem 0.17. *(Constant rank embedding theorem)*
Let (M_i, ω_i), $i \in 2$, $N_i \subseteq M_i$ constant rank submanifolds, with symplectic noraml bundles $F_i = TN_i^\omega/(TN_i \cap TN_i^\omega)$. Suppose

$$\begin{tikzcd} F_0 \arrow[r, "\hat{\psi}"] \arrow[d] & F_1 \arrow[d] \\ N_0 \arrow[r, "\psi"'] & N_1 \end{tikzcd}$$

is an isomorphism of symplectic vector bundles, and such that $\psi^*\iota_1^*\omega_1 = \iota_0^*\omega_0$. Then ψ extends to symplectomorphism of neighborhoods of N_0, N_1, inducing $\hat{\psi}$.

Proof. We want to reduce to the master theorem. Given constant rank submanifold $N \subseteq M$, define three symplectic vector bundles

$$\begin{aligned} E &= TN/(TN \cap TN^\omega) \\ F &= TN^\omega/(TN \cap TN^\omega) \\ G &= (TN \cap TN^\omega) \oplus (TN \cap TN^\omega)^* \end{aligned}$$

with E, G symplectidc subbundles.
Choosing splittings for E, F, we get

$$\begin{aligned} TN &\cong E \oplus (TN \cap (TN)^\omega) \\ TN^\omega &\cong F \oplus (TN \cap (TN)^\omega) \end{aligned}$$

One has

$$TM|_N/(TN + TN^\omega) \cong (TN \cap (TN^\omega))^*$$

choosing splittings get

$$TM|_N = (TN + TN^\omega) \oplus (TN \cap TN^\omega)^*$$

We get

$$TM|_N = E \oplus F \oplus (TN \cap TN^\omega) \oplus (TN \cap TN^\omega)^*$$

So $TM|_N$ is sum of three symplectic vector bundles.
Note $E \oplus (TN \cap TN^\omega) = TN$, $F \oplus (TN \cap TN^\omega) = TN^\omega$, and $(TN \cap TN^\omega) \oplus (TN \cap TN^\omega)^* = G$.
Do this for both $N_i \subseteq M_i, i \in 2$,

$$\begin{aligned} TM_0|_{N_0} &= E_0 \oplus F_0 \oplus G_0 \\ TM_1|_{N_1} &= E_1 \oplus F_1 \oplus G_1 \end{aligned}$$

The map ψ gives isomorphisms $E_0 \to E_1, G_0 \to G_1$, and $\hat{\psi}$ gives an isomorphism $F_0 \to F_1$. So by the master theorem we get $TM_0|_{N_0} \to TM_1|_{N_1}$ is an isomorphism. ■

Special cases:

1. N_i is coisotropic, i.e. $TN_i^\omega \subseteq TN_i, F_i = 0$.

Get symplectomorphism of neighborhoods provided that

$$\psi^*(\iota_1^*\omega_1) = \iota_0^*\omega_0$$

2. If N_i is isotropic, i.e. $TN_i^\omega \supseteq TN$, $F_i = TN_i^\omega/TN_i$. Need $F_0 \simeq F_1$.

3. If N_i is symplectic, we have $F_i = (TN_i)^\omega$. Need $N_0 \to N_1$ to be symplectic and an isomorphism of symplectic vector bundles.

Lagrangian fibrations

(Here, "fibration" is used as a synonym for "fiber bundle," and does NOT refer to Serre fibrations)

Let $\begin{array}{c} M \\ \downarrow{\scriptstyle \pi} \\ B \end{array}$ be a fiber bundle, i.e. surjective submersion such that all $\pi^{-1}(b) \cong F$, with local triviality, meaning that any point admits a neighborhood U with $\pi^{-1}(U) \cong U \times F$.

Definition 0.20. Let (M, ω) be symplectic. A Lagrangian fibration of M is a fibration $\begin{array}{c} M \\ \downarrow \pi \\ B \end{array}$ such that fibers are Lagrangian submanifolds.

Recall: These are the fibrations such that $\{\pi^* f, \pi^* g\} = 0$ for all $f, g \in C^\infty(B)$ and $\dim B = \frac{1}{2} \dim M$

Example 0.10.

1. $\pi : T^*Q \to Q$ cotangent bundle.

2. Suppose $Q = \mathbb{R}^n/\mathbb{Z}^n = (S^1)^n$. Then $TQ = Q \times \mathbb{R}^n$, $\pi : T^*Q = Q \times (\mathbb{R}^n)^* \to (\mathbb{R}^n)^*$ is also a Lagrangian fibration, with fibers the torus.

We'll see that fibers of Lagrangian fibration are products of vector spaces and tori (I notice these are exactly the homeomorphism types of connected Abelian Lie groups, but I don't know the significance of this).

Theorem 0.18. *Let (M, ω) be a symplectic manifold, and* $\begin{array}{c} M \\ \downarrow \pi \\ B \end{array}$ *a Lagrangian fibration with compact connected fibers. Then there is a canonical, fiberwise transitive action of the vector bundle* $\begin{array}{c} T^*B \\ \downarrow \\ B \end{array}$ *on* $\begin{array}{c} M \\ \downarrow \\ B \end{array}$.

Then M has structure of an "affine torus bundle."

Proof. Later ∎

Background:

A G-action on manifold M is when a Lie group G acts on a manifold M. Recall this is a function $G \times M \to M$, $(g, m) \mapsto g \cdot m$, so that $g \cdot (g' \cdot m) = (gg') \cdot m$, and $e \cdot m = m$.
The orbits through m are $\{g \cdot m \mid g \in G\} = G \cdot m$.
The stabilizer of m is $G_m = \{g \in G \mid g \cdot m = m\}$. Have $G \circ m = G/G_M$ is a manifold
If the action is transitive, meaning for any m, m', there is a g with $g \cdot m = m'$, then $G \circ m = M$. We say the action is free if all $G_m = \{e\}$, i.e. all stabilizers are trivial.
A vector space is a group. If it has a free and transitive action on manifold M, then M is an affine space.
If we have a free and transitive action of $G = \mathbb{R}^n/\mathbb{Z}^n$, then M is an affine torus.

Now, a vector bundle action of $\begin{array}{c} T^*B \\ \downarrow \\ B \end{array}$ means each $T^*B|_b$ acts on $\pi^{-1}(b)$.

Proof. Let $VM = \ker(T\pi) \subseteq TM$ the vertical subbundle. This is a Lagrangian subbundle. For any $\alpha \in \Gamma(T^*B) = \Omega^1(B)$, we have $\pi^*\alpha \in \Omega^1(M)$
Let $X_\alpha \in \mathscr{X}(M)$ defined by $\iota(X_\alpha)\omega = -\pi^*\alpha$
Now, we claim that $X_\alpha \in \Gamma(VM)$, i.e. it is a vertical vector field. For all $Y \in \Gamma(VM)$, we have $\omega(X_\alpha, Y) = \iota(Y)\iota(X_\alpha)\omega = -\iota(Y)\pi^*\alpha = 0$.
So X is vertical.
Note: Restriction of X_α to $\pi^{-1}(m)$ depends only on $\alpha|_m$.
This gives a map $\pi^*(T^*B) \to VM$.
Let $F_\alpha^t : M \to M$ be the flow of that vector field. Since X_α is vertical it takes fibers to fibers, and exists for all time because the fibers are compact. Action of T^*B is the trace 1-form F_α^1 (etc.)

Lecture 13, 10/17/24

Lagrangian fibrations and a dim-angle coordinates(?)

Recall; $\{H, F\} = 0 \iff L_{X_H}F = 0$, which implies that the integral curves of X_H are constrained by level sets of F.
More generally, consider $F_1, \ldots, F_k$ such that $\{F_i, F_j\} = 0$, $\{H, F_i\} = 0$, then we can try to build coordinate systems to "solve" X_H.
Recall: Regular level sets of $F = (F_1, \ldots, F_k)$ are coisotropic submanifolds of codimension k.
For $k = n$ these will be Lagrangian submanifolds.

Definition 0.21. A <u>Lagrangian submersion</u> $\pi : M \to B$ is a submersion with Lagrangian fibers.
We'll see: All compact fibers of a Lagrangian submersion of a Lagrangian submersion are tori.
For any submersion $\pi : M \to B$, we have an exact sequence of vector bundles over M

$$0 \longrightarrow \ker(T\pi) \longrightarrow TM \xrightarrow{q} \pi^*(TB) \longrightarrow 0$$

where $\pi^*(TB)$ is the pullback bundle, where $\pi^*(TB)|_m \subseteq T_{\pi(m)}B$. So, $TM/\ker(T\pi) = \pi^*(B)$.
For a Lagrangian submersion, ω gives a nondegenerate pairing between $\ker(T\pi)$ and $TM/\ker(T\pi)$
(Recall: For symplectic vector space V, and Lagrangian subspace $L \subseteq V$, we have a pairing between $L, V/L$, i.e. we have an identification $L \cong (V/L)^*$)
Thus $\ker(T\pi) \cong (TM/\ker(T\pi))^* = \pi^*(T^*B)$
Thus $\ker(T_m\pi) \cong T^*_{\pi(M)}B$ for all m.
So, for any fiber $\pi^{-1}(b)$, $\ker(T\pi)|_{\pi^{-1}(b)} = T(\pi^{-1}(b)) \cong \pi^{-1}(b) \times T_b^*B$

Proposition 32. *For a Lagrangian submersion $\pi : M \to B$ there is a canonical isomorphism*

$$\underbrace{\ker(T\pi)}_{\text{tangent bundle to fibers}} \cong \pi^*(T^*B)$$

Proof. ■
Taking $v \in \ker(T_m\pi)$ to $\mu \in T^*_{\pi(m)}B$ defined by $\iota(v)\omega_m = -(T_m\pi)^*\mu$
For $\mu \in T^*_bB$, we get $X_\mu \in \mathscr{X}(\pi^{-1}(b))$ by this isomorphism.
Equivalently, for $\alpha \in \Omega^1(B)$, we get $X_\alpha \in \Gamma(\ker(T\pi)) \subseteq \mathscr{X}(M)$

Proposition 33. *For $\mu_1, \mu_2 \in T^*_bB$, we have $[X_{\mu_1}, X_{\mu_2}] = 0$.*

Proof. Extend μ_i to $\alpha_i \in \Omega^1(B)$. Let $X_i = X_{\alpha_i}$. By def, $\iota(X_i)\omega = -\pi\alpha_i$. Now,

$$\begin{aligned}\iota(\underbrace{[X_1, X_2]}_{L_{X_1}X_2})\omega &= L_{X_1}(\iota_{X_2}\omega) - \iota_{X_2}L_{X_1}\omega \\ &= -L_{X_1}\pi^*\alpha_2 - \iota_{X_2}\iota_{X_1}\underbrace{d\omega}_{=0} - \iota_{X_2}d\underbrace{\iota_{X_1}}_{-\pi^*\alpha_1}\omega \\ &= -L_{X_1}\pi^*\alpha + L_{X_2}\pi^*\alpha_1 - d\iota_{X_2}\pi^*\alpha_1\end{aligned}$$

Because X_i are vertical vector fields (in the kernel of $T\pi$), all three of these terms above disappear.

■

<u>Remark:</u> Choose a basis of $T^*_bB, \mu_1, \ldots, \mu_n$. Get corrresponding vector fields $X_{\mu_1}, \ldots, X_{\mu_n}$.
We can use these to build coordinate systems.
For $m \in \pi^{-1}(b), (t_1, \ldots, t_n) \mapsto F_{t_1}^{X_{\mu_1}} \circ \cdots \circ F_{t_n}^{F_{X_{\mu_n}}}(m)$ is a coordinate system. In fact, this is canonical up to affine transformation.
So $\pi^{-1}(b)$ has an affine structure.

Definition 0.22. Call a Lagrangian submersion $\pi : M \to B$ <u>complete</u> if for all $X_\alpha, \alpha \in \Omega^1(M)$ are complete, i.e. flow exists for all t.
E.g. if fibers $\pi^{-1}(b)$ are compact, then it is complete.
Let $F_\alpha(t) : M \to M$ be the flow of X_α. ($F^t_\alpha \circ F^s_\beta = F^s_\beta \circ F^t_\alpha$. Let $F_\alpha = F^1_\alpha$. We have $F_\alpha \circ F_\beta = F_{\alpha+\beta}$ (we are using that $F^t_\alpha \circ F^t_\beta = F^t_{\alpha+\beta}$ when they commute).
On each $\pi^{-1}(b)$, get action of T^*_bB (viewed as an Abelian group).
Then $\mu \circ m = F_\mu(m)$. Note $(\mu_1 + \mu_2) \circ m = \mu_1 \circ (\mu_2 \circ m)$, and $0 \circ m = m$.

Proposition 34. *For a complete Lagrangian submersion, all fibers $\pi^{-1}(b)$ have locally free, transitive action of T^*_bB, meaning that there is only one orbit, and discrete stabilizers.*

Proof. Since $X_\mu, \mu \in T_b^*B$ spans $\ker(T\pi)$, the orbits (which are fibers) are n-dimensional, hence are open, hence all of $\pi^{-1}(b)$.
So $\pi^{-1}(b) = T_b^*B/\Lambda_b$, where Λ_b is the stabilizer of b.

■

We've seen that $\pi^{-1}(b) = T_b^*B/\Lambda_b$, where Λ_b is discrete. Choose a basis of $e_1, \ldots, e_n\ell$ of Λ_b, extend to basis $e_1, \ldots, e_n$ of T_b^*B
Then $\pi^{-1}(b) = \mathbb{R}^n/\mathbb{Z}^k = (\mathbb{R}/\mathbb{Z})^k \times \mathbb{R}^{n-k}$. In particular, if this is compact, it must be a torus.

Lecture 14, 10/22/24

Missed

Lecture 15, 10/24/24

Hamiltonian group actions

Review of Lie theory:

- A Lie group G is a manifold, also a group, such $Mult_G : G \times G \to G$ is C^∞.
- Cartan's theorem: every closed subgroup of a Lie group is a Lie group.
- A Lie algebra $\mathfrak{g} = T_eG$ with bracket from left invariant vector fields, the set of which we denote by $\mathscr{X}^L(G)$ (i.e. invariant under all $L_a : G \to G, g \mapsto ag$). For matrix Lie groups, the bracket is the commutator.
- There is a map called the exponential map $\exp : \mathfrak{g} \to G$ which sends $\xi \mapsto \exp(\xi) = \gamma_\xi(1)$, where $\gamma_\xi : \mathbb{R} \to G$ is a one-parameter subgroup such that $\frac{d}{dt}|_{t=0}\gamma_\xi(t) = \xi$. The assignment of $\mathfrak{g} = T_eG$ to G is functorial. For matrix Lie groups, it is the matrix exponential
- Adjoint action: For every $a \in G$, we have the action $Ad_a : G \to G$, $g \mapsto aga^{-1}$. At the tangent space we have the derivative T_eAd_a, and the assignment $a \mapsto T_eAd_a$ is a Lie algebra morphism $G \mapsto \mathrm{Aut}(\mathfrak{g}) \subseteq \mathrm{GL}(\mathfrak{g})$.

  This induces a Lie algebra morphism $\mathfrak{g} \to \mathrm{Aut}(\mathfrak{g})$, which is the Lie algebra of $\mathrm{GL}(\mathfrak{g})$. THe image of ξ under this is called ad_ξ. It turns out that

$$ad_\xi(y) = \frac{d}{dt}Ad_{\exp(t\xi)}(y)$$

The assignment of G to $T_eG = \mathfrak{g}$ is functorial. That is, if we have $\varphi : G_1 \to G_2$, then the a map $T_e\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ makes the diagram commute:

$$\begin{tikzcd} \mathfrak{g}_1 \arrow[r, "\exp"'] \arrow[d, "T_e\varphi"] & G_1 \arrow[d, "\varphi"] \\ \mathfrak{g}_2 \arrow[r, "\exp"] & G_2 \end{tikzcd}$$

A consequence is that $\exp(ad_\xi) = \mathrm{Ad}(\exp(\xi))$

Definition 0.23. Let G be a Lie group. A <u>G-action</u> on a manifold Q is a group homomorphism $\mathscr{A} : G \to \mathsf{Diff}(Q)$, $g \mapsto \mathscr{A}_g$, such that the map $\mathscr{A} : G \times Q \to Q, (g, q) \mapsto \mathscr{A}_g(q)$ is smooth.

Definition 0.24. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. A $\mathfrak{g}$-action on Q is a Lie algebra homomorphism $\mathfrak{g} \to \mathscr{X}(Q)$, $\xi \mapsto \xi_Q$ such that the map $\mathfrak{g} \times Q \to TQ, (\xi, q) \mapsto \xi_Q(q)$ is smooth.

Example 0.11.

1. For $Q = G$, we have the G-actions

- Adjoint action: $g \mapsto Ad_g, a \mapsto gag^{-1}$
- Left multiplication: $g \mapsto L_g, a \mapsto ga$
- Right multiplication: $g \mapsto R_{g^{-1}} : a \mapsto ag^{-1}$

2. Any representation of G, i.e. a group homomorphism $G \mapsto \mathrm{GL}(V)$ is a G-action

3. Given a G-action on Q, we get actions on TQ, T^*Q induced by $g \mapsto \mathscr{A}_g$, $G \curvearrowright TQ$, $g \mapsto T\mathscr{A}_g$, $G \curvearrowright T^*Q, g \mapsto (T\mathscr{A}_{g^{-1}})^*$

<u>Notation:</u> For $G \curvearrowright Q$, instead of $g \mapsto \mathscr{A}_g$, we write $\mathscr{A}_g(q) = g \cdot q$. Note

$$g_1 \cdot (g_2 \cdot q) = (g_1 \cdot g_2) \cdot q$$

Given a G-action $\mathscr{A} : G \mapsto \mathsf{Diff}(G)$ $g \mapsto \mathscr{A}_g$, we get a Lie algebra $\mathfrak{g}$-action $\mathfrak{g} \mapsto \mathscr{X}(Q), \xi \mapsto \xi_Q$ as follows.
From $\mathscr{A}$ we get a G-representation on functions:

$$(g \cdot f)(q) = f(g^{-1} \cdot q)$$

Define ξ_Q by derivative:

$$(\mathscr{L}_{\xi_q}(f))(q) = \frac{d}{dt}|_{t=0} f(\exp(-t\xi) \cdot q)$$

I.e ξ_Q is the vector field having flow $g \mapsto (\exp t\xi) \cdot q$

Proposition 35. *The map $\mathfrak{g} \to \mathscr{X}(Q)$, $\xi \mapsto \xi_Q$ is a Lie algebra action, i.e.*

$$[\xi_Q, \eta_Q] = [\xi, \eta]_Q$$

We have

$$(Ad_g \xi)_Q = (\mathscr{A}_g)_* \xi_Q$$

Proof. ■

For any $\xi \in G$, let $\xi^L \in \mathscr{X}^L(G)$ be the corresponding left-invariant vector field, and $\xi^R \in \mathscr{X}^R(G)$ the right invariant vector field with the property that $\xi^R|_e = \xi^L|_e = \xi$

Example 0.12. Consider left multiplication of G on itself $g \cdot a = ga$. Then $\xi_Q =?$ Since this commutes with right multiplication, ξ_Q is a right-invariant vector field. At e, we have $(\mathscr{L}_{\xi_Q} f)(e) = \frac{d}{dt}|_{t=0} f(\exp(-t\xi)) = -\underbrace{\xi}_{\in T_e G}(f)$

So $\xi_Q = -\xi^R$. So

$$[-\xi^R, -\eta^R] = -[\xi, \eta]^R$$

similarly for the action $g \cdot a = ag^{-1}$, we get the left invariant vector field $\xi_Q = \xi^L$. So

$$[\xi^L, \eta^L] = [\xi, \eta]$$

For action $g \cdot a = gag^{-1}$, $\xi_Q = \xi^L - \xi^R$

Lecture 16, 1/12/24

Hamiltonian action and moment maps

Recall: Given a group action $\mathscr{A} : G \to \mathsf{Diff}(M)$, we define generating vector fields by

$$\mathfrak{g} \to \mathscr{X}(M), \xi \mapsto \xi_M$$

$$(\mathscr{L}_{\xi_M} f)(m) = \frac{d}{dt}|_{t=0} f(\exp(-t\xi) \cdot m)$$

where $\exp : \mathfrak{g} \to G$ is the Lie exponential.

Example 0.13. Let $M = \mathbb{R}^n, G = \mathrm{GL}(n, \mathbb{R})$, $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R}) = \mathrm{End}(R^n)$. What is the formula for the generating vector fields?

For $A \in \mathfrak{gl}(n, \mathbb{R})$,

$$\begin{aligned}(\mathscr{L}_{A_{\mathbb{R}^n}} f)(x) &= \frac{d}{dt}|_{t=0} f(\exp(-tA)\cdot x)\\ &= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x)(Ax)_i\\ &= \underbrace{-\sum_{i,j=1}^{n} A_{ij}x_j \frac{\partial}{\partial x_i}}_{A_{\mathbb{R}^n}} f\end{aligned}$$

If we take the commutator,

$$[A_{\mathbb{R}^n}, B_{\mathbb{R}^n}] = [A, B]_{\mathbb{R}^n}$$

In general,

$$[\xi_M, \zeta_M] = [\xi, \zeta]_M$$

Example 0.14. Take $M = \mathbb{R}^n, G = \mathbb{R}^n$ as an additive Lie group. Then $\mathfrak{g} = \mathbb{R}^n$ (with zero Lie bracket), and $\exp : \mathfrak{g} \to G$ is given by the identity: $\exp(b) = b$. For any $b \in \mathfrak{g} = \mathbb{R}^n$ gives rise to a 1-parameter subgroup $\gamma_b(t) = tb$, and the derivative of this is again b.
Let G act on $\mathbb{R}^n$ by translations, namely $x \mapsto x - b$ (we use a minus sign so formulas turn out nicer, but can put a plus sign).
Then the generating vector field is

$$\begin{aligned}(\mathscr{L}_{B_{\mathbb{R}^n}} f)(x) &= \frac{d}{dt}|_{t=0} f(\exp(-tb)\cdot x)\\ &= \frac{d}{dt}|_{t=0} f(x + tb)\\ &= \sum_i \frac{\partial f}{\partial x_i} b_i\end{aligned}$$

so $b_{\mathbb{R}^n} = \sum b_i \frac{\partial}{\partial x_i}$
For a real vector space V, let $G = \mathrm{GL}(V)$, $T_v V = V$. Then

$$\xi_V|_v = -\xi \cdot v, \mathfrak{g} = \mathfrak{gl}(V) = \mathrm{End}(V)$$

Let (M, ω) be a (connected) symplectic manifold, $\mathscr{A} : G \to \mathsf{Diff}(M)$, which has generating vector field $\mathfrak{g} \to \mathscr{X}(M, \omega) = \{X \mid \mathscr{L}_X \omega = 0\}$
Recall we have $C^\infty(M) \to \mathscr{X}(M)$ given by $f \mapsto X_f$, where X_f is determined by. $\iota(X_f)\omega = -df$. Such a vector field is called Hamiltonian.

Definition 0.25. The G-action is called weakly Hamiltonian if all generating vector fields are Hamiltonian vector fields, i.e. $\xi_M \in \mathscr{X}_{Ham}(M,\omega)$.
We have the short exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow C^\infty(M) \longrightarrow \mathscr{X}_{Ham}(M,\omega) \longrightarrow 0$$

where the kernel of $C^\infty \to \mathscr{X}_{Ham}(M,\omega)$ is the constant functions. But because the action is weakly hamiltonian, we have a map from $\mathfrak{g} \to \mathscr{X}_{Ham}(M,\omega)$, which we can lift:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r] & C^\infty(M) \arrow[r] & \mathscr{X}_{Ham}(M,\omega) \arrow[r] & 0 \\
& & & \mathfrak{g} \arrow[u] \arrow[ul, dotted, "\tilde{\Phi}"] &
\end{tikzcd}$$

The choice of a linear map $\tilde{\Phi} : \mathfrak{g} \to C^\infty(M)$ (lifting $\xi \to \xi_M$) is called a weak moment map, or sometimes a comoment map.
In general, this won't be a Lie algebra homomorphism.
The comoment map can be regarded as a map $\Phi : M \to \mathfrak{g}^*$, $\langle \Phi(m), \xi \rangle = \tilde{\Phi}(\xi)(m)$
This is called the (weak) moment map.

Definition 0.26. A symplectic G-action on a symplectic manifold (M,ω) is called Hamiltonian if there exists a G-equivariant $\Phi : M \to \mathfrak{g}^*$ such that $\iota(\xi_M)\omega = -d\langle \Phi, \xi \rangle$
The map $\tilde{\Phi} : G \to C^\infty(M)$, $\tilde{\Phi}(\xi)(m) = \langle \Phi(m), \xi \rangle$ is the comoment map.
Remarks:

1. G acts on $\mathfrak{g}$ by $g \cdot \xi = Ad_g \xi$, hence on $\mathfrak{g}^*$ by $g \cdot \mu = (Ad_{g^{-1}})^* \mu$. So equivariance means $\Phi(g \cdot m) = g \cdot \Phi(m)$.

Alternatively, $\tilde{\Phi}(Ad_g \xi) = g \cdot \tilde{\Phi}(\xi)$, so $(g \cdot f)(m) = f(g^{-1} \cdot m)$

Infinitesimally, $\mathfrak{g}$ acts on $\mathfrak{g}^*$ by $\xi \cdot \mu = -(ad_\xi)^* \mu$

Now, $\Phi(g \cdot m) = g \cdot \Phi(m)$ implie $\mathscr{L}_{\zeta_M} \Phi = -(ad_\zeta)^* \Phi$

2. Name comes from French "application moment" (Souriaux (sp?)), but "Moment map" is an incorrect translation. Correct is "momentum map".

3. If G is abelian (e.g. $\mathbb{R}^n$, torii, or products thereof), equivariance means invariance.

Proposition 36. *For Hamiltonian G-action, the comoment map*

$$\tilde{\Phi} : \mathfrak{g} \to C^\infty(M)$$

is a Lie algebra homomorphism.

Proof. Denote $\Phi^\xi = \tilde{\Phi}(\xi) = \langle \Phi, \xi \rangle$. Then $\{\Phi^\xi, \Phi^\eta\} = \underbrace{\mathscr{L}_{\xi_M}\Phi^\eta}_{\mathscr{L}_{\xi_M}\langle \Phi, \eta \rangle} = \langle -ad^*_\xi \Phi, g \rangle = -\langle \Phi, [\xi, \eta] \rangle = -\Phi^{[\xi,\eta]}$ (there is a sign error in here somewhere...)

■

Proposition 37. *A weakly Hamiltonian G-action on (M, ω) is Hamiltonian in following cases:*

(a) G compact

(b) M compact

Proof. Φ is G-equivariant if and only if $(g \cdot \Phi)(m) = \Phi(g \cdot m)$. Given a weak moment map, define

$$(g \cdot \Phi)(m) = g \cdot (\Phi(g^{-1} \cdot m))$$

This is again a weak moment map. If G is compact, can average

$$\tilde{\Phi}(m) = \int_G (g \cdot \Phi)(m)|dg|$$

This is a G-equivariant map. This proves (a).
Now, assume M is compact. Note: weak moment maps $\Phi : M \to \mathfrak{g}^*$ are unique up to a constant. If M is compact, we can fix a normalization by

$$\int_M \Phi \frac{\omega^n}{n!} = 0$$

This uniquely determines Φ. If Φ is normalized then $g \cdot \Phi$ is again normalized, hence $g \cdot \Phi = \Phi$.

■

Remark:
If we have a lie algebra homomorphism $\mathfrak{g} \to \mathscr{X}_{Ham}(M, \omega)$, we want to lift to a Lie algebra homomorphism $\tilde{\Phi}$. We have the central extension $\hat{\mathfrak{g}}$

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r] & C^\infty(M) \arrow[r] & \mathscr{X}_{Ham}(M,\omega) \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r] & \hat{\mathfrak{g}} \arrow[r] \arrow[u] & \mathfrak{g} \arrow[r] \arrow[u] \arrow[l, bend left, dotted] & 0
\end{tikzcd}$$

So the weak Hamiltonian action is Hamiltonian if G is connected and the central extension $\hat{\mathfrak{g}}$ is "trivial" (splits), E.G. $G = \mathrm{SL}(n, \mathbb{R})$.

Example 0.15. Recall: Cotangent bundles $M = T^*Q, \omega = -d\theta$ (in coordinates $\theta = \sum p_i dq_i$).
For $Y \in \mathscr{X}(Q)$, the cotangent lift $Y_{T^*} \in \mathscr{X}(T^*Q)$ is Hamiltonian vector field, with function

$$H = \langle \theta, Y_{T^*} = \iota(Y_{T^*})\theta \in C^\infty(T^*Q)$$

This gives a Lie algebra morphism (exercise)

$$\mathscr{X}(Q) \to C^\infty(T^*Q), Y \mapsto \iota(Y_{T^*})\theta$$

This is $\mathsf{Diff}(Q)$-equivariant (Exercise).
If Q has a G-action, $\mathscr{A} : G \to \mathsf{Diff}(Q) \curvearrowright T^*Q$
we gete a G-action on T^*Q by composition, with comoment map

$$\tilde{\Phi} : \mathfrak{g} \to C^\infty(T^*Q), \xi \mapsto \iota(\xi_{T^*Q})\theta$$

In coordinates, if

$$Y = \sum Y_j(q)\frac{\partial}{\partial q_j} \implies H = \sum Y_j(q)p_j$$

Special case: $Q = \mathbb{R}^n, G = \mathrm{GL}(n, \mathbb{R})$, $G \curvearrowright T^*Q = T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$
Generating vector fields are given by the formula: for $A \in \mathfrak{gl}(n, \mathbb{R}) = Mat_{\mathbb{R}}(n)$,

$$A_{\mathbb{R}^n}|_x = -\sum_{i,j} A_{ij}x_j\frac{\partial}{\partial x_i}$$

Comoment map: $\mathfrak{g} \to C^\infty(T^*\mathbb{R}^n), A \mapsto -\sum A_{ij}q_jp_i$.
To get moment map, first identify $\mathfrak{gl}(n, \mathbb{R}) \simeq \mathfrak{gl}(n, \mathbb{R})^*$ by the bilinear form

$$\langle A, B\rangle = \mathrm{tr}(AB)$$

This form is Ad-invariant:

$$\langle Ad_gA, Ad_gB\rangle = \langle A, B\rangle$$

In terms of this identity (?) $\Phi : T^*\mathbb{R}^n \to \mathfrak{gl}(n, \mathbb{R})$, $(q, p) \mapsto \Phi(q, p)_{ij} = -q_jp_i$
Same kind of calculation for any $G \subseteq \mathrm{GL}(n, \mathbb{R})$ such that $\mathfrak{g}$ invariant under $A \mapsto A^T$.
E.G $G = \mathrm{SO}(n)$ rotation group, with $\mathfrak{g} = so(n) = \{A \mid A^T = -A\}$, the Skew-adjoint matrices.
The moment map becomes

$$\Phi : T^*\mathbb{R}^n \to so(n)^* = so(n)$$

$$\begin{aligned}(q, p) \mapsto \Phi(q, p)_{ij} &= -\frac{1}{2}(q_jp_i - q_ip_j)\\ &= \frac{1}{2}(q_ip_j - q_jp_i)\end{aligned}$$

If $h = 3$ this is $\vec{q} \times \vec{p}$ (up to factor). In physics this is angular momentum, hence momentum map.

Example 0.16. Special case: $Q = \mathbb{R}^n$, $M = T^*\mathbb{R}^n$, $G = \mathbb{R}^n$ acting by $x \mapsto x - b$.
Generating vector field: $b_{\mathbb{R}^n} = \sum b_i \frac{\partial}{\partial q_i}$
Hamiltonian: $\sum b_i p_i$
Comoment map: $\tilde{\Phi} : \mathbb{R}^n \to C^\infty(T^*\mathbb{R}^n), b \mapsto \tilde{\Phi}(q,p) = b \cdot p$.
$\Phi : T^*\mathbb{R}^n \to (\mathbb{R}^n)^* \cong \mathbb{R}^n$ (by dot product), $(q,p) \mapsto p$ (linear momentum).
These calculations generalize to "exact symplectic manifolds," which are symplectic manifolds whose symplectic form is also exact, i.e. $\omega = -d\theta$ (for some $\theta \in \Omega^1(M)$)
Whenever $\mathscr{A} : G \to \mathsf{Diff}(M)$ preserves θ, (i.e. $\mathscr{A}_g^*\theta = \theta$), then the generating vector field ξ_M are Hamiltonian, with obvious candidate for the moment map

$$\langle \Phi, \xi \rangle = \iota(\xi_M)\theta$$

Check:

$$\begin{aligned} d\langle \Phi, \xi \rangle &= d\iota(\xi_M)\theta \\ &= (\mathscr{L}(\xi_M) - \iota(\xi_M)d)\theta \\ &= \iota(\xi_M)\omega \end{aligned}$$

There is a sign error here somewhere as well...

Example 0.17. : Symplectic representation
Let (E, ω) be a symplectic vector space. Then the symplectic group $G = \mathrm{Sp}(E, \omega)$ acts on E
Generating vector fields for this action: $\xi_E|_V = -\xi \cdot v$ (identifying $T_vE \cong E$)
$\Phi : E \to \mathfrak{g}^*$, $\langle \Phi(v), \xi \rangle = (\cdots)\frac{1}{2}\omega(\xi v, v)$. Taking the exterior derivative,

$$\begin{aligned} \langle d\langle \Phi, \xi \rangle|_V, w \rangle &= \frac{d}{dt}|_{t=0}\langle \Phi(v + tw), \xi \rangle \\ &= \frac{d}{dt}|_{t=0}\omega(\xi \cdot (v + tw), v + tw) \\ &= \omega(\xi \cdot w, v) + \omega(\xi \cdot v, w) \\ &= \omega(\xi \cdot v, w) \\ &= -\langle \iota(\xi_E|_V)w, w \rangle \end{aligned}$$

Remark: If $G \curvearrowright (M, \omega)$ is a Hamiltonian action, and $\varphi : H \to G$ is a Lie group homomorphism, then we get an action $H \curvearrowright (M, \omega)$ just by composition. Then the action of H is again Hamiltonian, with moment map

$$\mathfrak{h} \to \mathfrak{g} \to C^\infty(M)$$

and moment map

$$M \to \mathfrak{g}^* \to \mathfrak{h}^*$$

Lecture 17, 11/14/24

Suppose $G \curvearrowright (M, \omega)$ is a Hamiltonian G-action. Then there exists a G-equivariant $\Phi : M \to \mathfrak{g}^*$ such that

$$\iota(\zeta_M)\omega = -d\langle \Phi, \zeta \rangle$$

(i.e. $\langle \Phi, \zeta \rangle$ are Hamiltonian for ζ_M).
Then $\mathfrak{g} \to C^\infty(M)$ given by $\zeta \mapsto \langle \Phi, \zeta \rangle$ is a Lie algebra map.
Exact Case:
Let $\omega = -d\theta$. If G preserves θ then

$$\langle \Phi, \zeta \rangle = -\iota(\zeta_M)\theta$$

Example 0.18. Let (E, ω) be a symplectic vector space, and let $G = \mathrm{Sp}(E, \omega) \curvearrowright E$ is Hamiltonian with moment map

$$\langle \Phi(v), \zeta \rangle = -\frac{1}{2}\omega(v, \zeta v)$$

Example 0.19. Pick compatible complex structure $\mathcal{J}$ on (E, ω). Then E becomes a complex vector space with Hamiltonian metric (inner prod) given by

$$h(v, w) = \underbrace{g(v, w)}_{=\omega(v, \mathcal{J}w)} + i\omega(v, w)$$

Now, $U(E) \subseteq \mathrm{Sp}(E, \omega)$ preserves ω. If $\varphi : H \to G$ is a group homomorphism then

$$\Phi_H = \varphi^* \circ \Phi$$

where $\varphi^* : \mathfrak{g}^* \to \mathfrak{h}^*$ is the dual map.
Notation: For inner product spaces, and linear map $A : E \to F$, let $A^\dagger : F \to E$ be the adjoint. Then

$$U(E) = \{A : F \to E \mid A^\dagger = A^{-1}\}$$
$$u(E) = \{\xi : E \to E \mid \xi^\dagger = -\xi\}$$

So

$$h(v, \xi v) = -h(\xi v, v) = -\overline{h(v, \xi v)}$$

So $h(v, \xi v) = i\omega(v, \xi v)$, and so

$$\langle \Phi(v), \xi \rangle = \frac{i}{2}h(v, \xi v)$$

$u(E)$ has positive definite, $U(E)$-invariant metric given by

$$(\xi, \eta) = \operatorname{tr}(\xi^\dagger \eta) = -\operatorname{tr}(\xi, \eta)$$

So we identify $u(E) \cong u(E)^*$ by this metric.
Write

$$\begin{aligned}
\langle \Phi(v), \xi \rangle &= \frac{i}{2} h(v, \xi v) \\
&= \frac{i}{2} v^\dagger \xi v \\
&= \frac{i}{2} \operatorname{tr}(v^\dagger \xi v) \\
&= \frac{i}{2} \operatorname{tr}(v v^\dagger \xi) \\
&= \operatorname{tr}(\frac{i}{2} v v^\dagger \xi)
\end{aligned}$$

where we view v as a linear map $\mathbb{C} \to E$. So

$$\Phi(v) = -\frac{i}{2} v v^\dagger$$

For sepcial case $E = \mathbb{C}^n$, $v = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$, $v^\dagger = (\overline{z_1}, \cdots, \overline{z_n})$, so

$$\Phi(z) = \frac{1}{2i} z z^\dagger$$

$$(\Phi(z))_{ij} = \frac{1}{2\sqrt{-1}} z_i \overline{z_j}$$

Example 0.20. Let $M = \mathbb{CP}(n)$, with Fubini-Study form ω_{FS} and with action of $U(n+1)$, $[z] = [z_0 : \cdots : z_n]$, $A \cdot [z] = [Az]$.
This action is Hamiltonian, with moment map

$$\Phi : \mathbb{CP}(n) \to u(n+1)^* \geq u(n+1)$$

$$[z_0 : \cdots : z_n] \mapsto \frac{1}{2\sqrt{-1}} \frac{z z^\dagger}{\|z\|^2}$$

We will see a proof of this later.

Coadjoint orbits

Recall: A homogeneous G-space is a manifold M with <u>transitive</u> G-actio n (i.e. only one orbit).
Given $m \in M$, with stabilizer $H = G_m$, get $M = G/H$.
Consider homogeneous Hamiltonian G-space $(M, \omega), \Phi : M \to \mathfrak{g}^*$.
Since the generating vector fields ξ_M span tangent space to M everywhere, the 2-form ω is determined entirely by moment map condition !

$$\omega(\xi_M, \cdot) = -d\langle \Phi, \xi \rangle$$

$$\begin{aligned}
\omega(\xi_M, \eta_M) &= -\iota(\eta_M) d\langle \Phi, \xi \rangle \\
&= -\mathscr{L}(\eta_M)\langle \Phi, \xi \rangle \\
&= -\langle \mathscr{L}(\eta_M)\Phi, \xi \rangle \\
&= \langle \eta \cdot \Phi, \xi \rangle \\
&= -\langle \Phi, \eta \cdot \xi \rangle \\
&= \langle \Phi, [\xi, \eta] \rangle
\end{aligned}$$

where the $\cdot$ is the coadjoint action.
So $\omega(\xi_M, \zeta_M) = \langle \Phi, [\xi, \eta] \rangle$
Note also: $\Phi(M) \subseteq \mathfrak{g}^*$ is a single coadjoint orbit.

Theorem 0.19 (Kostant, Kirillov, Souriau (Sp? all)). *Let $\mathscr{O} \subseteq \mathfrak{g}^*$ be an orbit under the coadjoint action. Then $\mathscr{O}$ has a unique symplectic 2-form ω such that the G-action is Hamiltonian, with $\Phi : \mathscr{O} \to \mathfrak{g}^*$ the inclusion.*

Proof. Recall generating vector field for G-action on $\mathfrak{g}^*$ are $\xi_{\mathfrak{g}^*}|_\mu = -\xi \cdot \mu$ (coadjoint representation).
This is also formula for $\xi_{\mathscr{O}}|_M$. Only candidate for ω (taking care of uniqueness) is

$$\begin{aligned}
\omega|_\mu(\xi_{\mathscr{O}}|_M, \eta_{\mathscr{O}}|_M) &= \langle \mu, [\xi, \eta] \rangle \\
&= -\langle \xi \cdot \mu, \eta \rangle \\
&= \langle \eta \cdot \mu, \xi \rangle
\end{aligned}$$

This is well-defined. Now, $\xi_G|_M \in \ker \omega|_M \iff \omega_M(\xi_{\mathscr{O}}|_M, \eta_{\mathscr{O}}|_M) = 0$ for all $\eta \iff \langle \xi \cdot \mu, \eta \rangle = 0$ for all $\eta \iff \xi \cdot \mu = 0$, and gives nondegenerate.
If ω closed:

$$\begin{aligned}
\iota(\xi_{\mathscr{O}}) d\omega &= \underbrace{\mathscr{L}(\xi_{\mathscr{O}})}_{=0}\omega - d\iota(\xi_{\mathscr{O}})\omega \\
&= dd\langle \Phi, \xi \rangle \\
&= 0
\end{aligned}$$

∎