Lecture 1, 3/9/25 (Happy birthday to me)

Oh dear, we're starting with chapter 2 of Hartshorne...
Read chapter 1.1 of Hartshorne before friday
Test your understanding of the important bits against Exercise 1.4(Zariski vs product topology)
Following theorem is perhaps unconventional for an ag class.
We use the "Bourbaki conventions:"

Definition 0.1. A topological space X is said to be quasicompact if for every open cover $X = \bigcup_{i \in I} U_i$, there existss a finite subcover $I' \subset I$ such that $X = \bigcup_{i \in I'} U_i$.
This is usally called "compact".

Definition 0.2. A topological space is said to be compact if it is quasicompact and Hausdorff.

Recall: X is called Hausdorff if for all pairs (x, y) of distinct points there exist neighborhoods U_x, U_y of x, y, such that $U_x \cap U_y = \varnothing$.
In French, one uses the term "separated space."
These terms will reappear in algebraic geometry when studying separated schemes.

This property is equivalent to the following: For $(x, y) \in X \times X \setminus \triangle$ (the diagonal elements $\{(x, x) \mid x \in X\}$), there are neighborhoods $U \ni x, V \ni y$, with $U \times V \cap \triangle = \varnothing$. Hence $U \times V$ lies entirely within the complement of the diagonal. So (x, y) is in the interior of $X \times X \setminus \triangle$.
Using the definition of the product topology, one can show that X is a Hausdorff space if and only if $\triangle$ is closed in the product topology. This is the formulation which will be meaningful when we transport to algebraic geometry.

Theorem 0.1 (Gelfond-Naymark). *Roughly:*
A compact (quasicompact + hausdorff) topological space can be "recovered" from the ring $C(X) \stackrel{\text{def}}{=} C(X, \mathbb{R})$ *of continuous real-valued functions.*

Proof. This is a special case of what they proved. Will get into proof later ∎

In particular, we want it to be true that if X, Y are two compact spaces with abstractly isomorphic rings of functions, i.e. $C(X) \cong C(Y)$, then X, Y should be homeomorphic, $X \cong Y$.

From rings to spaces To fix conventions:

Definition 0.3. When we write "ring", we always mean a commutative, unital ring. So $C(X)$ is indeed always a ring (obviously).

First step:
Try to recover the underlying set of points.
Ideals: given $x \in X$, we obtain a ring homomorphism, called the evaluation index at x, $e_x : C(X) \to \mathbb{R}$ which takes a continuous real-valued function and evaluates it at x: $f \mapsto f(x)$.
Since $(f+g)(x) = f(x) + g(x)$, and similarly for multiplication, this really is a ring homomorphism.

Fact: This map (e_x) is surjective because of constant functions.
Thus we have the isomorphisms $\mathbb{R} \cong \frac{C(X)}{\ker(e_x)}$. We refer to the denominator as $\mathfrak{M}_x$, the ideal of functions vanishing at $x \in X$. Note that the quotient is a field, so $\mathfrak{M}_x$ is maximal.

Definition 0.4. Let R be a ring. We denote by $\operatorname{Spec}_{\max}(R)$ the set of maximal ideals in R.

Proposition 1. *Let X be compact. Then there exists a bijection of sets $X \cong \operatorname{Spec}_{\max} C(X)$. The precise claim may be summarized as follows:*

- *Every maximal ideal I of $C(X)$ is of the form $I = e_x$ for some $x \in X$.*
- *If x, y are points in X, and $\mathfrak{M}_x = \mathfrak{M}_y$, then $x = y$.*

Proof. ∎

What about the topology? Let R be an abstract ring with the additional property that for every maximal ideal $\mathfrak{M} \in \operatorname{Spec}_{\max} R$, the quotient $R/\mathfrak{M} \cong \mathbb{R}$. Then we can make the following construction: for every element of the ring, we can associate to every element $f \in R$ a function $f : \operatorname{Spec}_{\max} R \to \mathbb{R}$ in the following way: $\mathfrak{M} \mapsto \overline{f} \in \mathbb{R} \cong R/\mathfrak{M}$.

Aside: To an algebraist, we think of $\mathbb{R}|(\overline{\mathbb{Q}} \cap \mathbb{R})$ as a transcendental extension, $\mathbb{R} = (\overline{\mathbb{Q}} \cap \mathbb{R})(\alpha_0, \alpha_1, \dots)$. So, there are lots of field automorphisms on $\mathbb{R}$, none of which are continuous.
Aside over.

Now: Look at the coarsest topology on $\operatorname{Spec}_{\max} R$ such that all functions $\mathfrak{M} \mapsto f + \mathfrak{M} \in \mathbb{R}$ are continuous for each $f \in R$.
That is, the topology on $\operatorname{Spec}_{\max} R$ is generated by preimages $f^{-1}(U)$, where $f : \operatorname{Spec}_{\max} R \to \mathbb{R}$ denotes the map associated with $f \in R$.
Due to the existence of noncontinuous elements of $\operatorname{Aut}(\mathbb{R})$, it is problematic to work with the standard topology.
It is in some way “unnatural” to think of the topology of $\mathbb{R}$ analyticaly, if we want to do algebra.

Instead: We use the cofinite topology on $\mathbb{R}$ instead, i.e. the nonempty open subsets are the complements of finite sets.

Definition 0.5. Let R be a ring. Then the Zariski topology on $\operatorname{Spec}_{\max} R$ is the topology generated by "standard open subsets," which are defined as subsets of the form

$$U_f = \{\mathfrak{M} \in \operatorname{Spec}_{\max} R \mid f \notin \mathfrak{M}\}$$

It is in a certain way "algebraically robust".

Remark: The condition that $f \notin \mathfrak{M}$ has a very geometric meaning. If every maximal ideal is of the shape $\mathfrak{M}_x$, then this condition is equivalent to $\underbrace{f(x)}_{=f+\mathfrak{M}_x} \neq 0$.

So the Zariski topology is generated by non-vanishing loci.
Why (maximal) spectrum of a Ring? Let A be a normal (meaning commutes with its adjoint) matrix/operator. Look at the commutative ring R in End_{cts} generated by $A, A^\dagger$, take the closure $\overline{R}$. Then $\operatorname{Spec}_{\max} \overline{R} = \operatorname{Spec}(A)$, where the right hand side is the functional analysis spectrum of A.

Lecture 2, 5/9/25

Last time: Gelfand-Naymark
We had a "dictionary" relating compact spaces and their function rings. Given an abstract ring of functions, we can reconstruct a compact space. Points correspond to maximal ideals, with the topology generated by preimages $f^{-1}(U)$, where $f : \operatorname{Spec}_{\max} R \to \mathbb{R}$ is the map $\mathfrak{M} \mapsto \overline{f} \in R/\mathfrak{M} \cong \mathbb{R}$.

Today: Nullstellensatz (Hilbert zero theorem)
Aside on etymology: "Nullstellen" means "a zero of a function/polynomial", and "satz" means theorem.
Fix: A field k, assumed to be

- Algebraically closed
- (for simplicity) uncountable

Given a subset T of a polynomial ring over k, $T \subseteq R_n \overset{\text{def}}{=} k[X_1, \dots, X_n]$, we denote by $Z(T)$ the set of common zeroes in k^n:

$$Z(T) \overset{\text{def}}{=} \{(x = (x_1, \dots, x_n) \mid f(x) = 0 \, \forall f \in T\}$$

The collection of subsets obtained in this way are called "algebraic sets" by Hartshorne. In this class, we will call them affine algebraic varieties.

Claim. *Denoting by (T) the ideal in R_n generated by T, we have $Z\big((T)\big) = Z(T)$*

Proof. Think

■

<u>Conversely:</u> Given any subset $S \subseteq k^n$, we may consider the ideal of polynomials in k_n vanishing on S.

$$\mathcal{I}(S) = \{f \in R_n \mid f(z) = 0 \,\forall z \in S\}$$

$$\begin{tikzcd}
\text{alg subsets} \arrow[r, bend left, dotted, "\mathcal{I}"] & \text{ideals} \arrow[l, bend left, dotted, "Z"]
\end{tikzcd}$$

<u>Careful:</u>

- $\mathcal{I}\big(Z(I)\big) \supset I$

  $Z\big(\mathcal{I}(S)\big) \supset \overline{S}$ (we call $\overline{S}$ the <u>Zariski closure</u>, which just means the closure in the Zariski topology)

Definition 0.6. The <u>Zariski topology</u> is defined to be the topology on k^n with closed subsets being the algebraic subsets.
Reminder: we assume a field k is algebraically complete and uncountable.

Lemma 1. *Let L/k be a field extension with* $\dim_k(L) \leq |\mathbb{N}|$. *Then $L = k$.*

Proof. Assume by contradiction that there exists $x \in L \setminus k$. Consider the <u>uncountable</u> family given by

$$\{\frac{1}{(x-\lambda)} \mid lambda \in k\}$$

But $\dim_k L \leq |\mathbb{N}|$, so there is a k-linear relation. That is, there exists $\lambda_1, \ldots, \lambda_r \in k, \mu_1 \ldots, \mu_r \in k$ so that

$$\sum_{i=1}^{r} \frac{\mu_i}{x - \lambda_i} = 0$$

Clearing the denominators:

$$\sum_{i=1}^{r} \mu_i \prod_{s \neq r} (x - \lambda_s) = 0$$

This is $P(t)$ for some P in $k[t]$. But k is algebraically closed, so t is in k, contradiction.

■

Corollary 0.2 (Weak Nullstellensatz)**.** *Let $T \subset R_n$ such that $Z(T) = \varnothing$. Then $(T) = (1) = R_n$.*

Proof. Assume by contradiction that $(T) := I \neq R_n$. By Zorn's lemma, there exists a maximal ideal $\mathfrak{M} \supset I$. We look at the chain of quotient maps

$$R_n \to R_n/I \to \underbrace{R_n/\mathfrak{M}}_{\text{field}} = k$$

The composition sends $X_i \to x_i \in k$. So $\{R_n \to k\} \supset \{R_n/I \to k\}$. But the former is k^n, and the latter is $Z(I)$, which is nonzero, contradicting that $\mathfrak{m}$ is maximal. ■

<u>Now:</u> <u>Rabinowitsch trick</u>

Lemma 2. *Let $T = \{f_1, \dots, f_r\} \subset R_n$ and $f \in \mathcal{I}\big(Z(T)\big)$, i.e. if $f_i(x) = 0$ for all $i = 1, \dots, r$, then $f(x) = 0$.*
Then there is an $N \in \mathbb{N}$ such that $f^N \in (T)$.

Proof. Add an auxiliary variable t, work with the ring $R_n[t] \equiv R_{n+1}$.
By assumption, $\{(1 - tf), f_1, \dots, f_r\} = T'$ doesn't have a common zero, so by weak Nullstellensatz, $(T') = (1)$, so there exists $g_0, \dots, g_r \in R_n[t]$ so that $g_0(1 - tf) + g_1 f_1 + \cdots + g_r f_r = 1$.
Substitute $t = \frac{1}{f}$, and $g_1 f_1 + \cdots + g_r f_r = 1$ in a ring of rational functions: $R_n[\frac{1}{f}]$.
Clearing denominators by multiplying by a sufficiently high power of f, we get another expression

$$\tilde{g}_1 f_1 + \cdots + \tilde{g}_r f_r = f^N \in R_n$$

So $f^N \in (T)$. ■

Definition 0.7. Let $I \subset R$ be an ideal. We denote by $\sqrt{I} \subset R$ the <u>radical</u> of I, the set of all $x \in R$ so that $x^n \in I$ for some n.

Theorem 0.3 (Nullstellensatz)**.** *For an ideal $I \subset R$, we have $\mathcal{I}\big(Z(I)\big) = \sqrt{I}$*

Proof. Combine the lemma with the fact that R_n is a Noetherian ring (i.e. ideals are finitely generated). ■

Corollary 0.4. *There is a 1-1 correspondence between affine algebraic k-varieities (up to isomorphism) and finitely generated reduced k-algebras (up to isomorphism)*

Proof. $Z(\sqrt{I})$ corresponds to $R_n/\sqrt{I}$. An isomorphism between varieities is a pair of polynomial maps that map the varieties onto each other and are mutual inverses. ■

There is a stronger version, which gives an equivalence of categories. AffVar_k is the category whose objects are affine k-varieties, and whose morphisms are polynomial maps between ambient spaces preserving the varieties. The category $(\text{Alg}_k^{red})^{\text{op}}$ is the opposite category of reduced finitely generated k-algebras. The above furnishes an equivalence of these categories.

Lecture 3, 8/9/25

Today: Sheaves via Étalé Spaces

Most textbooks:

- Define presheaves first on a fixed space
- Then define gluing condition for sections of presheaves
- Sheaves are defined as presheaves satisfying the gluing condition

étaler is the French word for "to spread out."
Later on, we will encounter the word étale, which will appear in the notion of étale morphisms of schemes and étale cohomology.
Warning: don't drop the accent aigue

Definition 0.8. Let X be a topological space. A continuous map $\pi : \mathcal{S} \to X$ is called a local homeomorphism if the following are satisfied:

- π is an open map
- For every $x \in \mathcal{S}$, there is an open neighborhood $U \ni x$ such that $\pi|_U : U \to \pi(U)$ is a homeomorphism.

In this case, we will say that $\mathcal{S}$ is étalé above X, or call it an étalé space, or simply a sheaf on X.

Example 0.1.

1. $\varnothing \hookrightarrow X$

2. $\mathrm{Id}_X : X \to X$

3. Let I be a set with the discrete topology. Then $\mathrm{pr} : X \times I \to X$

4. Any covering space, e.g. the Möbius covering $\mathbb{S}^1 \to \mathbb{S}^1$ sending z to z^2, viewing $\mathbb{S}^1$ as a subset of $\mathbb{C}$.

5. The inclusion $\iota : U \hookrightarrow X$ for any open subset U.

6. For $x \in X$, build a new space by doubling x:

$$X \coprod_{X \setminus \{x\}} X = (X \coprod X)/\sim$$

There's a natural map $\bigtriangledown$ to X, the co-diagonal map.

7. Let $I \neq \varnothing$ be a set.

$$\nabla : S_{I,x} \overset{\text{def}}{=} \underbrace{X \coprod_{X\backslash\{x\}} \cdots \coprod_{X\backslash\{x\}} X}_{I \text{ times}} \to X$$

Non-example:
Take a non-open subset $M \subset X$. Then the inclusion $\iota : M \hookrightarrow X$ is not a local homeomorphism.

Definition 0.9. Let $U \subseteq X$ be open, $\mathcal{S}$ an étalé space above X. Then a section on U is a continuous map $s : U \to \mathcal{S}$ such that $\pi \circ s = \mathrm{Id}_U$. That is, the diagram commutes:

$$\begin{tikzcd} & \mathcal{S} \arrow[d, "\pi"] \\ U \arrow[ur, "s"] \arrow[r, hook, "\iota"] & X \end{tikzcd}$$

The set of all sections on U will be denoted by $\mathcal{S}(U)$ or $\Gamma(U, \mathcal{S})$.
If $U = X$, then s is called a global section, and we use the notation $\Gamma(\mathcal{S})$ or $\Gamma(X)$.
Let's revisit the examples above:

1.

$$\begin{tikzcd} & \varnothing \arrow[d, hook] \\ U \arrow[ur, dotted, "?"] \arrow[r, hook] & X \end{tikzcd}$$

If U is nonempty, $\mathcal{S}(U)$ will be empty, and it will be a singleton if U is empty (namely, $\mathrm{Id}_\varnothing : \varnothing \to \varnothing$)

2.

$$\begin{tikzcd} & X \arrow[d, "\mathrm{Id}"] \\ U \arrow[ur, hook, "\iota"] \arrow[r, hook] & X \end{tikzcd}$$

In this case, $\mathcal{S}(U) = \{\iota\}$, the inclusion.

3.

$$\begin{tikzcd} & X \times I \arrow[d, "pr"] \\ U \arrow[ur, "?"] \arrow[r, hook] & X \end{tikzcd}$$

In this case, $\mathcal{S}(U) = I$ if U is connected. Otherwise, it is the set of continuous maps from U to I, where I carries the discrete topology. We can also think of this as the set of ways to express U as a disjoint union of open subsets indexed by I.

4.

$$\begin{tikzcd} & \mathcal{S}^1 \arrow[d, "z^2"] \\ U \arrow[r, hook] \arrow[ur, dotted] & \mathcal{S}^1 \end{tikzcd}$$

$$\mathcal{S}(U) = \begin{cases} \varnothing & U = \mathcal{S}^1 \\ \{*\} & U = \varnothing \\ ? & U \text{ general} \end{cases}$$

For U general, $\mathcal{S}(U) = \{f : U \to \mathbb{C} \mid \forall z \in U, f(z)^2 = z\}$

5.

$$\begin{tikzcd} & U \arrow[d, hook, "\iota_U"] \\ V \arrow[r, hook, "\iota_V"] \arrow[ur, dotted] & X \end{tikzcd}$$

$\mathcal{S}(V) = \{*\}$ if $V \subset U$, $\varnothing$ otherwise.

6.

$$\begin{tikzcd} & S = X \coprod_{X\setminus\{x\}} X \arrow[d, "\triangledown"] \\ U \arrow[r, hook] \arrow[ur, dotted, "s"] & X \end{tikzcd}$$

$\mathcal{S}(U) = \{*\}$ if $U \not\ni x$, otherwise $\{1, 2\}$, depending on the choice of which of the two copies of the point x.

7.

$$\begin{tikzcd} & S_{I,x} \arrow[d, "\triangledown"] \\ U \arrow[r, hook] \arrow[ur, dotted, "s"] & X \end{tikzcd}$$

Again, $\mathcal{S}(U) = \{*\}$ if $U \not\ni x$, and I if $U \ni x$.

We call this example the "skyscraper sheaf at x"

There are many other examples, some even more interesting, which can be described using this theory.

Holomorphic functions as continuous sections

Let $X = \mathbb{C}$ with the standard topology.

Claim. *There exists a space $\mathscr{H}$ with a local homeomorphism $\pi : X \to \mathbb{C}$ such that continuous sections correspond to holomorphic functions on $\mathbb{C}$, i.e.*

$$\mathscr{H}(U) \cong \{f : U \to \mathbb{C} \mid f \text{ holomorphic } \}$$

compatible with restrictions to smaller open subsets.

Proof. As a set,

$$\mathscr{H} = \coprod_{z_0 \in \mathbb{C}} \{\sum_{n \in \mathbb{N}} c_n (z - z_0)^n \mid \exists r > 0 \text{ the series converges absolutely in a radius } r \text{ around } z_0 \}$$

The map from $\mathscr{H} \to \mathbb{C}$ is given by sending a power series which converges in a radius around z_0 to z_0.
To get the topology, we choose the strongest topology on $\mathscr{H}$ such that for every open subset U, and every holomorphic function $f : U \to \mathbb{C}$, the induced map $Xf : U \to \mathscr{H}$ given by $z_0 \mapsto \text{Taylor}(f, z = z_0)$ is continuous.
Exercise: Check that $\mathscr{H}(U) = \{f : U \to \mathbb{C} \text{ holomorphic } \}$ in a natural way.
Remark: This looks like a generalization of a phase space of $\mathbb{C}$ with a real topology:

$$\mathbb{R}^n \to \mathbb{R}^{2n}, x(t) \mapsto ((x(t), \dot{x}(t))$$

For this week, read Hartshorne section 2.1 (sheaves)

Lecture 4, 10/9/25

Today: Stalks
On Monday, we did sheaf theory via étalé spaces.
We define a sheaf as a continuous map $\pi : \mathcal{S} \to X$ which is a local homeomorphism. In this case we say $\mathcal{S}$ is an étalé space, or simply a sheaf on X.

Definition 0.10. Let X be a space, $\pi : \mathcal{S} \to X$ an étalé space over X. For every $x \in X$ we denote the preimae $\pi^{-1}(x)$ by $\mathcal{S}_x$ and call it the stalk of $\mathcal{S}$ at x.
Do NOT call it a fiber! (We will use this terminology for something different later)

Example 0.2.

1. When $\mathcal{S} = \varnothing \hookrightarrow X$, for all x, $\mathcal{S}_x = \varnothing$.

2. When $\mathcal{S} = X$, $\pi = \text{Id} : X \to X$, $\mathcal{S}_x = \{x\}$, a singleton.

3. When $\mathcal{S} = X \times I$, for a discrete space I, $pr : X \times I \to X$ the projection map, we have $\mathcal{S}_x \cong I$.

4. When $\mathcal{S} = S_{I,x}$, the skyscraper sheaf at x, $\bigtriangledown : S_{I,x} \to X$, we have $\mathcal{S}_y = \{*\}$ a singleton if $y \neq x$, and $\mathcal{S}_x = I$

5. Consider the space $\mathscr{H} \to \mathbb{C} = X$ defined last time, the sheaf of holomorphic functions. Then

$$\mathscr{H}_x = \{\sum_{k=0}^{\infty} c_k(z-z_0)^k \mid \exists \varepsilon > 0 \text{ the sum converges in a ball of radius } \varepsilon \text{ around } z_0\}$$

Lemma 3.

Existence: *Let $\pi : \mathcal{S} \to X$ be an étalé space over X, $x \in X$, and let $y \in \mathcal{S}_x$ be an element of the stalk. Then there exists an open neighborhood $U \ni x$ and section $s \in \mathcal{S}(U)$, $s : U \to \mathcal{S}$, such that $s(x) = y$.*

Uniqueness: *Further, given two pairs $(U_1, s_1), (U_2, s_2)$ with this property, then there exists $V \subset U_1 \cap U_2$ such that $s_1|_V = s_2|_V$.*

Proof. Left as an exercise. Hint: use that π is a local homeomorphism.
Categorical reformulation:
Consider the collection of all neighborhoods of x, ordered by inclusion, and take

$$ev_x : \mathrm{colim}_{U \ni x \text{ open}} \mathcal{S}(U) \to \mathcal{S}_x$$

Then this is a bijection.
In the case of sets, we can describe the right hand side as equivalence classes of pairs $\{(U, s)\}$, $U \ni x$ open, $s \in \mathcal{S}(U)$, where $(U, s) \sim (V, t)$ if there exists an open $W \subseteq U \cap V$ such that $s|_W = t|_W$.
This colimit corresponds to the set of germs of sections near x.

Lemma 4.

1. *Let $f : X \to Y, g : Y \to Z$ be composable continuous maps. Denote by h their composition, $h = g \circ f$. Then if f, g are local homeomorphisms, then h is a local homeomorphism.*

2. *If g and h are local homeomorphisms, then f is a local homeomorphism.*

Proof. Omitted

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Definition 0.11 (Category of sheaves)**.** Let X be a topological space. Then the Category of sheaves on X, $Sh(X)$, is defined to have as its objects the étalé spaces over X, and morphisms defined to be those $\varphi : \mathcal{S}_1 \to \mathcal{S}_2$ making the diagram commute:

$$\begin{tikzcd} \mathcal{S}_1 \arrow[rr, "\varphi"] \arrow[dr, "\pi_1"] & & \mathcal{S}_2 \arrow[dl, "\pi_2"'] \\ & X & \end{tikzcd}$$

Lemma 5 (Isomorphism criterion)**.** *Let $\phi : \mathcal{S}_1 \to \mathcal{S}_2$ be a morphism in $Sh(X)$. Then φ is an isomorphism if and only if $(\mathcal{S}_1)_x \to (\mathcal{S}_2)_x$ is bijective for all $x \in X$.*

Proof. Suppose $\varphi : \mathcal{S}_1 \to \mathcal{S}_2$ is a bijection of sets. Bijective continuous open maps are homeomorphisms, thus there is an inverse in $Sh(X)$. Other direction is clear.

■

Lemma 6 (Injectivity criterion)**.** *The above holds replacing bijection with injection.*

Proof. ■

We can restate in terms of sections.

Lemma 7. *Let $\varphi : \mathcal{S}_1 \to \mathcal{S}_2$ be a morphism in $Sh(X)$ such that for every $U \subseteq X$ open, the induced map $\mathcal{S}_1(U) \to \mathcal{S}_2(U)$ is a bijection. Then φ is an isomorphism.*

Proof. Apply the isomorphism criterion,

$$\begin{tikzcd} (\mathcal{S}_1)_x \arrow[r, "\cong"] \arrow[d, "\varphi", "\cong"'] & colim_{U \ni x} \mathcal{S}_1(U) \arrow[d, "\cong"] \\ (\mathcal{S}_2)_x \arrow[r, "\cong"] & colim_{U \ni x} \mathcal{S}_2(U) \end{tikzcd}$$

So the induced maps on every stalk is an iso, so φ is an isomorphism.
This also works with injection.

■

We expect the same to hold for surjections. That is, we would hope that if $\varphi : \mathcal{S}_1 \to \mathcal{S}_2$ is surjective, then for all $U \subset X$, the induced map $\mathcal{S}_1(U) \to \mathcal{S}_2(U)$ is surjective.
This is <u>false!</u>
<u>Counterexample:</u>
Let $X = \mathbb{S}^1$. We have the sheaf $\mathrm{Id}_X : X \to X$. It has the Möbius automorphism $z \mapsto z^2$, which is also a sheaf over X:

$$\begin{tikzcd} \mathcal{S} \arrow[r] \arrow[dr] & X \arrow[d, "\mathrm{Id}_X"] \\ & X \end{tikzcd}$$

If both unlabeled maps are $z \mapsto z^2$, then the upper map is a surjective map of étalé spaces, but $\mathcal{S}(X) = \varnothing$ does not surject onto $X(X) = \{*\}$.

Lemma 8 (Local lifts exist)**.** *Given a surjection $\mathcal{S}_1 \to \mathcal{S}_2$ in $Sh(X)$, and open $U \subseteq X$, and a section $s \in \mathcal{S}_2(U)$, there exists an open cover $U = \bigcup_{i \in I} U_i$ and sections $t_i \in \mathcal{S}_1(U_i)$ such that $\varphi(t_i) = s|_{u_i}$ for all i.*

Proof. Let φ be a surjective map of étalé spaces. For all $x \in X$, $\varphi : (\mathcal{S}_1)_x \to (\mathcal{S}_2)_x$ is surjective. We take the element $[(s, U)] \in (\mathcal{S}_2)_x$, which has a preimage $[(t, V)]$. We can repeat this for every $x \in U_i$ to obtain the collection of pairs (U_i, t_i).
Abstract perspective:
Consider the commutative triangle

$$\begin{tikzcd} \mathcal{S}_n \arrow[r, "\varphi"] \arrow[dr] & \mathcal{S}_2 \arrow[d] \\ & X \arrow[u, bend right, "s"'] \end{tikzcd}$$

With s a global section. Then $s \in \mathcal{S}_2$ can be lifted to $t \in \mathcal{S}_1(X)$ if and only if $s^{-1}\mathcal{S}_1$ has a global section

Lecture 4, 12/9/25

Today: Fiber products (of spaces), preimage and pushforward sheaves, presheaves.
Recall: a sheaf on X is the same thing as an étalé space over X, a topological space $\mathcal{S}$ with a local homeomorphism $\pi : \mathcal{S} \to X$, and a morphism of sheaves is a map $\varphi : \mathcal{S}_1 \to \mathcal{S}_2$ making the diagram commute:

$$\begin{tikzcd} \mathcal{S}_1 \arrow[rr, "\varphi"] \arrow[dr] & & \mathcal{S}_2 \arrow[dl] \\ & X & \end{tikzcd}$$

Note that φ must also be a local homeomorphism. We define the stalk at a point x, $\mathcal{S}_x$, as simply the preimage $\pi^{-1}(x) \subseteq \mathcal{S}$.

Fiber products

Given a diagram of continuous maps of topological spaces

$$\begin{tikzcd} X \times_Y Z \arrow[r] \arrow[d] & Z \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

The top left $X \times_Y Z \overset{\text{def}}{=} \{(x,z) \in X \times Z \mid f(x) = g(z)\}$ endowed with the subspace topology in $X \times Z$.
Universal property:

$$\begin{tikzcd}
T \arrow[dr, dotted, "\exists!"] \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, "\alpha"'] & & \\
& X \times_Y Z \arrow[r] \arrow[d] & Z \arrow[d, "g"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

Given an α, β as above making the diagram commute, there is a unique map from T to $X \times_Y Z$ making the diagram commute.

Example 0.3.

1. The usual product:

$$\begin{tikzcd}
X \times Z \arrow[r] \arrow[d] & Z \arrow[d, "g"] \\
X \arrow[r, "f"] & \{*\}
\end{tikzcd}$$

2. The fiber above a point y:

$$\begin{tikzcd}
f^{-1}(y) \arrow[r] \arrow[d] & \{*\} \arrow[d, "y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Preimage-sheaf:
Given a continuous $f : Y \to X$, we have a functor $f^{-1} : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$,

$$(\pi : \mathcal{S} \to X) \mapsto (\pi' : \mathcal{S} \times_X Y \to Y)$$

$$\begin{tikzcd}
\mathcal{S} \times_X Y \arrow[r] \arrow[d, "\pi'"] & \mathcal{S} \arrow[d, "\pi"] \\
Y \arrow[r, "f"] & X
\end{tikzcd}$$

Lemma 9. *π' in the above is indeed a sheaf*

Proof. Chase definitions ■

Remark: f^{-1} preserves stalks: that is, for all $y \in Y$, $(f^{-1}\mathcal{S})_y \cong \mathcal{S}_{f(y)}$
We also have a functor going the other direction, $f_* : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$. So given $f : Y \to X$, $\mathcal{S} \mapsto f_*\mathcal{S}$. This is called the pushforward.

Given a sheaf $\pi : \mathcal{S} \to Y$, we want a sheaf $\tilde{\mathcal{S}}$ and $\tilde{\pi} : \tilde{\mathcal{S}} \to X$, as well as a function $\tilde{f} : \mathcal{S} \to \tilde{\mathcal{S}}$ so that the diagram commutes:

$$\begin{tikzcd} \mathcal{S} \arrow[r, "\tilde{f}"] \arrow[d, "\pi"] & \tilde{\mathcal{S}} \arrow[d, "\tilde{\pi}"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

It is not immediately clear at all how to construct such a thing. Note that $\tilde{f}$ need not be a local homeomorphism here.

Presheaves

Let X be a space. Denote by $\mathrm{Open}(X)$ the category of open subsets of X, with inclusions as morphisms. That is $\mathrm{Hom}_{U,V} = \{*\}$ if $V \subseteq U$, and $\mathrm{Hom}_{U,V} = \varnothing$ otherwise.

Definition 0.12. A presheaf on X is defined to be a functor from

$$\mathcal{F} : \mathrm{Open}(X)^{op} \to \mathsf{Set}$$

Concretely:

For all $U \subset X$, specify a set $\mathcal{F}(U)$, such that for all $V \subset U$ inclusions, there is a restriction map $r^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ such that the following properties hold:

- $r^U_U = \mathrm{Id}_{\mathcal{F}(U)}$
- For $W \subseteq V \subseteq U$, we want
$$r^V_W \circ r^U_V = r^U_W$$
So restricting from U to V, and then from V to W, is the same as just restricting straight from U to W.

There is a category of presheaves on X, which we denote by $\mathrm{Psh}(X)$, which we can define as

$$\mathrm{Psh}(X) \stackrel{\text{def}}{=} \mathrm{Fun}(\mathrm{Open}(X)^{op}, \mathsf{Set})$$

There is a functor $I : \mathrm{Sh}(X)$ to $\mathrm{Psh}(X)$ sending $\pi : \mathcal{S} \to X$ to the map sending an open $U \subseteq X$ to its set of sections, $\mathcal{S}(U)$.
One can verify this is indeed a presheaf.

Surprisingly: Of more interest to us is the existence of a functor $^+ : \mathrm{Psh}(X) \to \mathrm{Sh}(X)$, called the presheaf's associated sheaf, or its sheafification, which interacts nicely with I, in the sense that $^+ \circ I \simeq \mathrm{Id}_{\mathrm{Sh}(X)}$, and $^+ \circ ^+ \simeq ^+$
It takes a presheaf $\mathcal{F}$ and sends it to a sheaf $\mathcal{F}^+$. This implies that I is an emebedding of categories. So passing from the étalé space to the presheaf of sections loses no information.

Construction of the sheafification:

Let $\mathcal{F} \in \mathrm{Psh}(X)$. Let's first construct the set of points of an étalé space on X. We define the stalk of a presheaf as follows.

For any $x \in X$, we define

$$\mathcal{F}_x \overset{\text{def}}{=} \operatorname{colim}_{U \ni x \text{ open}} \mathcal{F}(U)$$

Note that for this to make sense we do need the functoriality of $\mathcal{F}$. We can define them as germs of sections in exactly the same way, where a "section" over U is just an element of $\mathcal{F}(U)$.
This also gives a clear morphism $\pi : \underbrace{\coprod_{x \in X} \mathcal{F}_x}_{\overset{\text{def}}{=} \mathcal{S}} \to X$.
We now topologize $\mathcal{S}$. For a topoloical space T, every map $f : \mathcal{S} \to T$ is continuous if and only if for all $U \subseteq X$, for all $s \in \mathcal{F}(U)$, the composition $f \circ s : U \to T$ is continuous:

$$\begin{tikzcd}
T & \mathcal{S} \arrow[l] \arrow[d, "\pi"] \\
U \arrow[u] \arrow[r, hook] \arrow[ur, dotted, "s"] & X
\end{tikzcd}$$

For any neighborhood V of x, and $s \in \mathcal{F}(V)$, we have a map $s : V \to \mathcal{S}$ given by $y \mapsto s_y$, the image of s in the colimit definition of the stalk at y. So we topologize $\mathcal{S}$ in the weakest way so that this is the case. Using this definition, we can check that π is continuous. All sections $s \in \mathcal{F}(U)$ give rise to continuous sections of the étalé space $\mathcal{S}$.
Remark: The topology on $\mathcal{S}$ is generated by sets of the form $s(U)$ for all $s \in \mathcal{F}(U)$ for all U open.
One can check that π is a local homeomorphism.

Claim. *If $\pi : \mathcal{S} \to X$ is an étalé space then the sheafification of the presheaf of sections of $\mathcal{S}$ agrees with $\mathcal{S}$.*

Proof. Recall the lemma that $\pi^{-1}(x)$ can be described as a colimit. So the map $\mathcal{S} \to \coprod_{x \in X} \mathcal{S}_x$ (where the $\mathcal{S}$ on the left-hand side is the presheaf associated to $\mathcal{S}$) is a continuous bijection.
Homeomorphisms are precisely the continuous maps which are bijective and open, and one can check that this map is open by construction of the presheaf associated to $\mathcal{S}$.

■

Pushforward

Given a continuous map $f : Y \to X$, we have the functor $f_* : \mathrm{Psh}(Y) \to \mathrm{Psh}(X)$, given by

$$\mathcal{F} \mapsto \Big((U \subseteq X) \mapsto (\mathcal{F}(f^{-1}(U))\Big)$$

So $f_*(\mathcal{F}) = F \circ f^{-1}$, where f^{-1} is the functor sending $\mathrm{Open}(X) \to \mathrm{Open}(Y)$.

Claim. $f^*(\mathrm{Sh}(Y)) \subset \mathrm{Sh}(X)$

Proof. Future assignment. ∎

Lecture 5, 15/9/25

Last time: presheaves and pushforwards
Today: Sheaves $\subset$ presheaves, locally ringed spaces.
Reading assignment for this week: Hartshorne section 2.2, up to and including example 2.3.3.

Proposition 2. *Let X be a space, and let $\mathcal{F} \in \mathrm{Psh}(X)$ be a presheaf on X. Then the canonical map $\mathcal{F} \to \mathcal{F}^+$ is an isomorphism if and only if for every open subset U, and every open cover $U = \bigcup_{i \in I} U_i$, the following is an equalizer:*

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] & \prod_{i\in I} \mathcal{F}(U_i) \arrow[r, shift left, bend left, "r^{U_i}_{U_{ij}}"] \arrow[r, shift right, bend right, "r^{U_j}_{U_{ij}}"'] & \prod_{(i,j)\in I^2} \mathcal{F}(U_{ij}) \end{tikzcd}$$

where $U_{ij} = U_i \cap U_j$.

Proof. In a minute ∎

Translation: Given a collection of sections $s_i \in \mathcal{F}(U_i)$ such that for all i, j we have $r^{U_i}_{U_{ij}}(s_i) = r^{U_j}_{U_{ij}}(s_j)$, then there is a unique section $s \in \mathcal{F}(U)$ such that $r^U_{U_{ij}} = s_i$ for all i.
Remark:

$$\begin{tikzcd} A \arrow[r] & B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C \end{tikzcd}$$

is called an equalizer in a category $\mathcal{C}$ if

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "f\times g"] \\ C \arrow[r, "\triangle"] & C^2 \end{tikzcd}$$

is a pullback, i.e. $A \simeq C \times_{C\times C} B$. So the map from A to B is a universal map making f, g equal in the composition.
Now for the proof of the proposition, which will require a lemma.

Lemma 10. *Let $\mathcal{F} \in \mathrm{Psh}(X)$ such that $\mathcal{F}$ the gluing condition. Let $s_{1,2} \in \mathcal{F}(U)$ be local sections such that for all $x \in U$ we have*

$$(s_1)_x = (s_2)_x \in \mathcal{F}_x \stackrel{\text{def}}{=} \mathrm{colim}_{U\ni x}\mathcal{F}(U)$$

Then $s_1 = s_2$.

Proof. For all x there is a neighborhood $x \in U_x \subset X$ such that $r^U_{U_x}(s_1) = r^U_{U_x}(s_2)$. Then s_1 and s_2 re glue to the local sections $s_x \stackrel{\text{def}}{=} r^U_{U_x}(s_1)$ or $r^U_{U_x}(s_2)$. By uniqueness, $s_1 = s_2$. ■

Now for the proof of the proposition.

Proof. Easier direction:
Sections of étalé spaces satisfy the gluing condition just because of their nature as functions.
Harder direction:
We want to show that if the gluing condition is satisfied, then $\mathcal{F} \simeq \mathcal{F}^+$.
We have a presheaf $\mathcal{F}$ and étalé space $\mathcal{S} \to X$, the presheaf of sections $\mathcal{F}^+$.
Take $s \in \mathcal{F}^+(U)$ an arbitrary section, by definition for all $x \in U$, $s(x) \in \mathcal{F}_x = \mathcal{S}_x$. Thus there exists an open neighborhood $U_x \ni x$ and a section $s_x : U_x \to \mathcal{S}$ such that $s_x(x) = s(x)$. Since $\pi : \mathcal{S} \to X$ is a local homeomorphism, we may further shrink the neighborhood U_x to ensure that $s_x|_{U_x} = s|_{U_x}$.
Now we apply the lemma to $U_x \cap U_y$ to obtain $s_x|_{U_{xy}} = s_y|_{U_{xy}}$ for all pairs of points (x, y). Because $\mathcal{F}$ is assumed to satisfy the gluing condition, this yields the existence of a globally defined section $t \in \mathcal{F}(U)$ such that $t|_{U_x} = s|_x$ for all x.
It remains to show that $s = t$. This follows from another application of the lemma: by construction, they $s_x = t_x$ for all $x \in U$, so by the lemma, $s = t$ as a section in $\mathcal{F}(U)$. ■

Recall from friday the claim:

Claim. *Let $f : Y \to X$ be continuous. Then $f_* : \mathrm{Psh}(Y) \to \mathrm{Psh}(X)$ sends $f_*(\mathrm{Sh}(Y)) \subset \mathrm{Sh}(X)$, where the inclusion means the essential image.*

Proof. We just have to check that the pushforward $f_*\mathcal{F}$ also satisfies the gluing condition.
By definition

$$f_*\mathcal{F} \overset{\text{def}}{=} \mathcal{F} \circ f^{-1}$$

This is the composition of $f^{-1} : \text{Open}(X)^{op} \to \text{Open}(Y)$ and $\mathcal{F} : \text{Open}(Y) \to \mathsf{Set}$.
And the preimage of an open cover is an open cover.

∎

Locally ringed spaces

Definition 0.13. Let X be a space. A ring object is an object $\mathcal{R}$ along with two "binary operations," $+, \cdot : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$, and maps (thought of as sections) $0, 1 : X \to \mathcal{R}$ (i.e. $0, 1 \in \mathcal{R}(X)$), such that the usual ring axioms, re-expressed by commutative diagrams, hold.
Commutativity of addition:

$$\begin{tikzcd}
\mathcal{R} \times \mathcal{R} \arrow[r, "swap"] \arrow[dr, "+"'] & \mathcal{R} \times \mathcal{R} \arrow[d, "+"] \\
 & \mathcal{R}
\end{tikzcd}$$

Existence of identity:

$$\begin{tikzcd}
\mathcal{R} \arrow[r, "\mathrm{Id}_R \times c_1"] \arrow[d, "\mathrm{Id}_R"] & \mathcal{R} \times \{1\} \arrow[d, "\mathrm{Id} \times c"] \\
\mathcal{R} & \mathcal{R} \times \mathcal{R} \arrow[l, "\cdot"']
\end{tikzcd}$$

et cetera.

Definition 0.14. Let X be a space. Then a ring object $\mathcal{R}$ in the category $\text{Sh}(X)$ is called a sheaf of rings on X. The pair $(X, \mathcal{R})$ is called a ringed space.
This is equivalent to a presheaf $\mathcal{R} : \text{Open}(X)^{op} \to \mathsf{Ring}$ with the gluing condition.

Lecture 6, 17/9/25

Today: Locally ringed spaces
Last time: Ringed spaces.
Recall a ring space is a pair $(X, \mathcal{R})$, consisting of a topological space X, and a ring object $\mathcal{R}$ in $\text{Sh}(X)$.

Example 0.4.

- Let $X = \{*\}$. Then $\mathcal{R}$ is just a ring. This is because $\mathrm{Sh}(X)$ in that case is equivalent to Set, and a ring object in Set is just a ring.
- Take X to be any space. We can define the sheaf $C(X)$ of continuous real-valued functions. So as a presheaf, $C(X)(U) = \mathrm{Hom}(U, \mathbb{R})$.

Definition 0.15. Given a ring object $\mathcal{R}$, we define $\mathcal{R}^*$ as the fiber product

$$\begin{tikzcd} \mathcal{R}^* \arrow[r] \arrow[d] & \{1\} \arrow[d] \\ \mathcal{R} \times \mathcal{R} \arrow[r, "\cdot"] & \mathcal{R} \end{tikzcd}$$

So $\mathcal{R}^* = (\mathcal{R} \times \mathcal{R}) \times_{\mathcal{R}} \{1\}$. This is the set of pairs of elements of $\mathcal{R}$ which multiply to 1. The projections being injective mean that we can view it as the set of invertible elements of $\mathcal{R}$.

Claim. *The projections from $\mathcal{R}^*$ to $\mathcal{R}$ are injections. That is, $\mathcal{R}^* \subset \mathcal{R}$ is a subsheaf. So the inclusion is an open map of étalé spaces.*

Definition 0.16. Given $(X, \mathcal{R})$, $U \subset X$ open, $f \in \mathcal{R}(U)$, we define U_f by

$$\begin{tikzcd} U_f \arrow[r] \arrow[d] & \mathcal{R}^* \arrow[d] \\ U \arrow[r, "f"] & \mathcal{R} \end{tikzcd}$$

So $U_f = f^{-1}(\mathcal{R}^*) = \{x \in U \mid f(x) \in \mathcal{R}^*\}$

Claim. *$\mathcal{R}^*$ is a group object in* $\mathrm{Sh}(X)$*.*

Proof. Volunteer

■

Definition 0.17. Let $\mathcal{R} \in \mathrm{Sh}(X)$ be a ring object. Then $\mathcal{R}$ is <u>local</u> if for all open $U \subset X$, for all $f \in \mathcal{R}(U)$, we have $U = U_f \cup U_{1-f}$.
This means that for every point $x \in U$, at least one of $f(x), 1 - f(x)$ are invertible.
In this case, we say that $(X, \mathcal{R})$ is a <u>Locally ringed space</u>.

Example 0.5.

- (X, C_X) is local
- $(\{*\}, \mathcal{R})$ is local if and only if there is a unique maximal ideal in $\mathcal{R}$.

Lemma 11. *Let $(X, \mathcal{R})$ be a locally ringed space, $U \subset X$ open, and $f_1, \ldots, f_r \in \mathcal{R}(U)$. Then*

$$f_1 + \cdots + f_r = 1 \implies \bigcup_{i=1}^{r} U_f = U$$

Proof. Volunteer

■

<u>Goal:</u> We want to associate to a ring R a <u>universal</u> locally ringed space $(X, \mathcal{R})$, such that there exists a ring homomorphism

$$R \to \mathcal{R}(X)$$

with the property that, given a locally ringed space $(Y, \mathcal{S})$, with a homomorphism of rings $R \to \mathcal{S}(Y)$, we want a unique map of locally ringed spaces from $(Y, \mathcal{S})$ to $(X, \mathcal{R})$, such that ...
The solution $(X, \mathcal{R})$ will be denoted by $(\operatorname{Spec} R, \mathcal{O})$
This is the same notation used for the set of prime ideals. <u>Why</u>?
Assume that such a universal locally ringed space exists.
$|X|$ denotes the underlying set or space of points of a locally ringd space $(X, \mathcal{R})$

Claim. *$|X|$ is in canonical bijection with the set of prime ideals of $\mathbb{R}$.*

$$|X| \cong \{\mathfrak{p} \subset R \mid \mathfrak{p} \text{ is prime}\}$$

Proof. To go from left to right, we take the ring R, and localize $R_{\mathfrak{p}}$, meaning we invert every element in the complement of $\mathfrak{p}$. This is by design a local ring, and we can think of this as being a locally ringed space $(\{*\}, R_{\mathfrak{p}})$. By universality we get a map into $(X, \mathcal{R})$, and we consider the image of the point $\{*\}$.
To go the other way, we consider $\mathcal{R}_x$. This will be a local ring, and there is a map of rings frome $R \to \mathcal{R}_x$. $\mathcal{R}_x$ being a local ring means there is a unique maximal ideal $\mathfrak{m}$. We take this maximal ideal, and take its preimage under this ring homomorphism. The preimage of a maximal ideal may not be maximal, but the preimage of a prime ideal is always prime, and maximal ideals are prime. So the preimage is prime.

■

<u>Candidate topology</u> on $\operatorname{Spec} \mathcal{R}$ is the smallest topology containing $U_f = \{\mathfrak{p} \in \operatorname{Spec} R \mid f \notin \mathfrak{p}\}$.
This is called the Zariski topology on $|\operatorname{Spec} R|$. They key thing is that these subsets are always open in ringed spaces.

Definition 0.18 (Structure Sheaf $\mathcal{O}$ on $|\operatorname{Spec} R|$)**.** Let $\mathcal{O} \stackrel{\text{def}}{=} \coprod_{\mathfrak{p} \in |\operatorname{Spec} R|} R_{\mathfrak{p}}$, with the topology generated by the sections $\frac{g}{f^n} : U_f \to \mathcal{O}$ for all $\frac{g}{f^n} \in R_f$.

Lecture 7, 19/9/25

First the volunteers prove the things promised last time.

Claim. *Let $(X, \mathcal{R})$ be a locally ringed space, $U \subseteq X$ open, $f_1, \ldots, f_n \in \mathcal{R}(U)$. Then*

$$\sum_{i=1}^{r} f_i = 1 \implies U = \bigcup_{i=1}^{r} U_{f_i}$$

Proof. We have a map $\mathcal{R}(U) \to \mathcal{R}_x$ by definition of a colimit. Because we are in a locally ringed space, it has a maximal ideal, $(\mathcal{R}_x, \mathfrak{m}_x)$. We can (equivalently to what we said earlier) define $U_{f_i} = \{x \in X \mid \overline{f} \not\equiv 0 \bmod \mathfrak{m}_x\}$.
Geometrically, this means $\{x \in U \mid \overline{f}(x) \neq 0\}$.
By assumption, at least one of the $f_i \in \mathfrak{m}_x$. Then $\sum_{i=1}^r f_i \in \mathfrak{m}_x$.
Thus one of the f_i is a unit. By ring theory, units map to units, so every x lies in a U_{f_i}.

■

Claim. *$\mathcal{R}^*$ is a group object in the category* $\mathrm{Sh}(X)$*.*

Proof. Recall
We have the diagram

$$\begin{tikzcd}
\mathcal{R}^* \arrow[r] \arrow[d] & \{1\} \arrow[d] \\
\mathcal{R} \times \mathcal{R} \arrow[r] & \mathcal{R}
\end{tikzcd}$$

Then we have $\pi_1 : \mathcal{R}^* \hookrightarrow \mathcal{R}$, this is injective on stalks $\mathcal{R}_x^* \to$ units in $\mathcal{R}_x$ (we are identifying $\mathcal{R}^*$ with its image).
So we have an injective mapping $\mathcal{R}^*(U) \hookrightarrow \mathcal{R}(U)^*$. This is onto since $r \in \mathcal{R}(U)^*$ implies $(r, r^{-1}) \in \mathcal{R}^*(U)$.
If $s, t \in \mathcal{R}^*(U_x)$, then $s_x, t_x \in \mathcal{R}_x^*$ for any $x \in U$, so we get $U_x \supset \{x\}$ and $s', t' \in \mathcal{R}(U_x)^*$, so $U = \bigcup_x U_x$.
So $\mathcal{R}(U)^*$ contains $s|_{U_x} = s_x$.
I didn't really follow this sorry, I should think more about this / ask the guy about his proof.

■

<u>Slogan:</u> If $(X, \mathcal{R})$, $f \in \mathcal{R}(U)$ is a locally ringed space, then for $x \in X$, $\overline{f}_x \in \mathcal{R}_x/\mathfrak{m}_x = \kappa(x)$ can be thought of as the value of the "generalized function" f with values $\overline{f}_x \in \kappa(x)$ at all points x.
Watch out: thinking of this as a function taking values is risky, because there is a choice involved.
<u>$\operatorname{Spec} R$</u>
Let R be a ring. Then we have a locally ringed space $(\operatorname{Spec} R, \mathcal{O})$
<u>Topological space:</u> We define $|\operatorname{Spec} R| = \{\mathfrak{p} \mid \mathfrak{p} \subset R \text{ is prime }\}$, endowed with the Zariski topology, i.e. the topology generated by open subsets $U_f \overset{\text{def}}{=} \{\mathfrak{p} \in |\operatorname{Spec} R| \mid f \notin \mathfrak{p}\}$, for any $f \in R$.
<u>Motivation:</u> Historically speaking, this is how we define the topology on $\operatorname{Spec}_{\max}$.
More importantly, we have this universality, since for $(X, \mathcal{R})$ locally ringed and $f \in \Gamma(U, \mathcal{R}), U_f$ is open.

Structure sheaf $\mathcal{O}$

As a set, the étalé space,

$$\mathcal{O} = \coprod_{\mathfrak{p} \in |\operatorname{Spec} R|} R_{\mathfrak{p}}$$

The étalé projection sends $R_{\mathfrak{p}}$ to $\mathfrak{p}$.
The topology is generated as follows. For any $\frac{g}{f^n} = s \in R_f = R[f^{-1}]$, and any point $\mathfrak{p}$ we get a map sending s to its image in $R_{\mathfrak{p}}$ (because f is not in $\mathfrak{p}$ by def of U_f). The topology is generated by the images of all of these.
The topology on $\mathcal{O}$ is the universal one, such that a map $g : \mathcal{O} \to Y$ is continuous if and only if $g \circ s$ is continuous for all f, for all s as above.
This is an étalé space by construction, such that the stalks are the local rings $R_{\mathfrak{p}}$. That is,

$$\mathcal{O}_{\mathfrak{p}} = R_{\mathfrak{p}}$$

are local rings.
Thus, $(|\operatorname{Spec} R|, \mathcal{O})$ is a locally ringed space, where $\mathcal{O}(U_f) = R_f$. The advantage over the typical definition of a sheaf is that we have only defined the sections on a subbasis, and we have to prove some things about how we can sheafify these into a sheaf.

Theorem 0.5. *Let $X = (|X|, \mathcal{R})$ be a locally ringed space, and $\varphi : R \to \mathcal{R}(X)$ be a ring homomorphism. Then*

(a) There exists a morphism of locally ringed spaces $X \to \operatorname{Spec} R$

(b) This map is universal with respect to this property.

Proof. Next time ■

Definition 0.19. A morphism of locally ringed spaces $f : Y \to X$ is a continuous map $|f| : |Y| \to |X|$, along with a morphism of sheaves of rings (a morphism of ring objects in $\operatorname{Sh}(X)$) $f^\sharp : f^{-1}(\mathcal{R}_X) \to \mathcal{R}_Y$ such that for all $y \in Y$, the map between stalks

$$\mathcal{R}_{X,|f|(y)} \to \mathcal{R}_{Y,y}$$

is a local ring map, i.e. $(f^\sharp_y)^{-1}(\mathfrak{m}_y) = \mathfrak{m}_{|f|(y)}$

Lecture 8, 22/9/25

Today: Prove the universal property of $\operatorname{Spec} R$, global sections on $\operatorname{Spec} R$ of the structure sheaf $\mathcal{O}$, (affine) schemes.

<u>Recall:</u> Given a ring R, there exists a topological space $|\operatorname{Spec} R|$, whose points are the prime ideals of R, and endowed with the Zariski topology, whose opens are generated by sets of the form $D_f = \{\mathfrak{p} \in \operatorname{Spec} R \mid f \notin \mathfrak{p}\}$
<u>Reading assignment:</u> Finish 2.2 in Hartshorne by the end of this week.

On the space $|\operatorname{Spec} R|$, we have the structure sheaf $\mathcal{O}_R$, constructed by putting a topology on $\coprod_{\mathfrak{p} \in |\operatorname{Spec} R|} R_{\mathfrak{p}}$, generated as follows:
For every $s \in R_f$, we have a function from $D(f)$ to $\mathcal{O}_R$ given by sending a prime ideal $\mathfrak{p}$ to the image of s in $R_{\mathfrak{p}}$, and by construction this is both well defined and a section of the map $\mathcal{O}_R \to |\operatorname{Spec} R|$.
We want the coarses topology making these maps continuous.
We refer to the pair $(|\operatorname{Spec} R|, \mathcal{O}_R)$ by simply $\operatorname{Spec} R$.

Lemma 12. *Let $X = (|X|, \mathcal{O}_X)$ be a locally ringed space and $\varphi : R \to \mathcal{O}_X(|X|)$ a ring homomorphism. Then the following map Φ from $|X|$ to $|\operatorname{Spec} R|$, given by $x \mapsto \varphi^{-1}(\mathfrak{m}_x) \in |\operatorname{Spec} R|$ is continuous.*

Proof. <u>Remark:</u> For all $x \in |X|$, we have a local ring $(\mathcal{O}_X)_x$, which contains a unique maximal ideal, which we denote by $\mathfrak{m}_x$ (or $\mathfrak{m}_{X,x}$), and a ring homomorphism $\varphi : R \to \mathcal{O}_{X,x}$, which we can pull back to get a prime ideal in R (this map is defined because the original map φ goes to $\mathcal{O}_X(|X|)$, which is a set of sections, and each of those sections has an image in $\mathcal{O}_{X,x}$). Now for the proof:

By definition, it suffices to check, for all $f \in R$, that $\Phi^{-1}\big(D(f)\big) \subset |X|$ is <u>open</u> in $|X|$.

Claim.

$$\Phi^{-1}\big(D(f)\big) = |X|_{\varphi(f)} = \{x \in X \mid \varphi(f)_x \in \mathcal{O}_{X,x}^{\times} = \mathcal{O}_{X,x} \setminus \mathfrak{m}_x\}$$

Proof. By definition of Φ, we have ring homomorphisms

$$\begin{tikzcd} R \arrow[r, "\phi"] \arrow[dr, "q"'] & \mathcal{O}_{X,x} \\ & R_{\mathfrak{p}} = \Phi(x) \arrow[u, dotted, "\exists!"] \end{tikzcd}$$

This is a local ring at $\mathfrak{p}$, and it detects and reflects invertibility.
So $\Phi^{-1}\big(D(f)\big) = |X|_{\varphi(f)}$

∎

Lemma 13. Φ *can be promoted to a morphism of locally ringed spaces:* $X \to \operatorname{Spec} R$.

Proof. We need to build a map of sheaves of rings $\Phi^{-1}\mathcal{O}_R \to \mathcal{O}_X$
We will do this stalk by stalk: For

$$\begin{tikzcd}
\coprod_{x\in|X|} R_{\mathfrak{p}=\Phi(x)} \arrow[r, "\varphi_x"] \arrow[d] & \coprod \mathcal{O}_{X,x} \arrow[d] \\
X & X
\end{tikzcd}$$

Where the top right is the topology given by using the sheafification viewing it as a presheaf of sections.
We have the map $\varphi_x : R_\mathfrak{p} \to \mathcal{O}_{X,x}$, $\mathfrak{p} = \varphi^{-1}(\mathfrak{m}_x)$ is a local ring map, so $(\Phi, (\varphi_x)_{x\in X})$ is a local ring map, granting continuity of $\to$.
This uses the definition of the topology on the étalé space of $\mathcal{O}_R$.

■

Lemma 14. *Given $(|X|, \mathcal{O}_X), (|Y|, \mathcal{O}_Y)$ with ring homomorphisms $\varphi : R \to \mathcal{O}_X(|X|), \psi : R \to \mathcal{O}_Y(|Y|)$ and a morphism of locally ringed spaces $g : Y \to X$ which is compatible with φ and ψ, then there is a commutative diagram in the category of locally ringed spaces*

$$\begin{tikzcd}
Y \arrow[r, "\Psi"] \arrow[dr, "g"'] & \operatorname{Spec} R \\
& X \arrow[u, "\Phi"']
\end{tikzcd}$$

where Ψ, Φ are the maps induced by ψ, φ, as in the previous lemma.

Proof. ■

This construction yields a functor Spec : $\mathsf{Ring}^{op} \to$ LocRingdSp

Theorem 0.6. *The functor* Spec *is an embedding and a (quasi)-inverse (on the essential image of* Spec*) is given by taking global sections $\Gamma(\mathcal{O})$:* $\mathrm{LocRingdSp}^{op} \to$ *Ring, $(|X|, \mathcal{O}_X) \mapsto \mathcal{O}_X(|X|)$.*

Proof. In class presentation.

■

The key lemma is the following:

Lemma 15. $\mathcal{O}_R(|\operatorname{Spec} R|) \cong R$

Proof. In class presentation

■

With all this, the following definitions finally make sense:

Definition 0.20.

1. A locally ringed space X, isomorphic to $\operatorname{Spec} R$ for some R, is called an affine scheme

2. A scheme is a locally ringed space $X = (|X|, \mathcal{O}_X)$ such that there exists an open cover $\bigcup |U_i| = |X|$, such that for all i, we have that $U_i = (|U_i|, \mathcal{O}_X|_{U_i})$ is an affine scheme.

Analogy:
Recall that a smooth manifold is a topological space with certain properties (2nd countable, hausdorff) with a set of smooth charts. If we forget the desired topological spaces, then this category also embeds into the category of locally ringed spaces, with the functor given by sending $M = (|M|, \mathcal{C}_M^\infty)$, where $\mathcal{U}$ is a (maximal) atlas to $(|M|, \mathcal{O}_M)$, where $\mathcal{O}_M(U)$ is the set of $\mathbb{R}$-valued smooth functions on U.
So $(|M|, \mathcal{C}_M^\infty)$ is covered by sets that look like $(\mathbb{R}^n, \mathcal{C}_{\mathbb{R}^n}^\infty)$

Lecture 9, 24/9/25

Summary: An object of LocRingdSpc is a pair $(|X|, \mathcal{O}) = X$, with $|X|$ a space, $\mathcal{O}$ a ring object in $\operatorname{Sh}(|X|)$ which is local, which means that for any u, and any section $s \in \mathcal{O}(U)$, $U_s \cup U_{1-s} = U$, where U_s is defined by the fiber product

$$\begin{tikzcd} U_s \arrow[r] \arrow[d] & \mathcal{O}^* \arrow[d] \\ U \arrow[r, "s"] & \mathcal{O} \end{tikzcd}$$

Equivalently, $U_s = \{x \in U \mid s(x) \in \mathcal{O}^*\}$. So, given any two sections s, t with $s + t = 1$, their loci of invertibility cover U, in analogy with the weak nullstellensatz. This implies that all stalks are local rings, that is $\mathcal{O}_x$ has a unique maximal ideal, $\mathfrak{m}_x$.
Last time: We have a bijection

$$\operatorname{Hom}_{\text{LocRingdSpc}}(X, \operatorname{Spec} R) = \operatorname{Hom}_{\mathsf{Ring}}(R, \underbrace{\mathcal{O}_X(|X|)}_{\text{ring of generalized functions on } X})$$

Stated: $\Gamma(|\operatorname{Spec} R|, \mathcal{O}_R) \cong R$
(to be shown on friday)
Today: Examples of schemes

Affine Line: $\mathbb{A}_{\mathbb{Z}}^n \overset{\text{def}}{=} \operatorname{Spec} \mathbb{Z}[T_1, \dots, T_n]$

Corollary 0.7. $\mathcal{O}_X(|X|) = \operatorname{Hom}_{\text{LocRingdSpc}}(X, \mathbb{A}_{\mathbb{Z}}^n)$

Proof.
Consider $\varphi \in \operatorname{Hom}_{\mathsf{Ring}}(\mathbb{Z}[T], \mathcal{O}_X(|X|))$, and map it to $\varphi(T) \in \mathcal{O}_X(|X|)$.

So by the theorem stated above, then $\mathcal{O}_X(|X|) = \mathrm{Hom}(X, \mathbb{A}^n_{\mathbb{Z}})$
So generalized functions on X are in 1-1 correspondence with morphisms $X \to \mathbb{A}^n_{\mathbb{Z}}$ ■

Remark: $\mathbb{A}^1_{\mathbb{Z}}$ is a ring object in LocRingdSpc. This is very deep, but also tautological. Abstract nonsense...

Definition 0.21. $\mathbb{G}_{m,\mathbb{Z}} \overset{\text{def}}{=} \operatorname{Spec} \mathbb{Z}[T, T^{-1}]$, often called the multiplicative group (scheme).

Corollary 0.8. *$\mathrm{Hom}(X, \mathbb{G}_{m,\mathbb{Z}} = \mathcal{O}_X(|X|)^*$.*

Definition 0.22. $\mathrm{GL}_{n,\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[T_{11}, \ldots, T_{nn}, \det\left((T_{ij})\right)^{-1}]$

Corollary 0.9. *$\mathrm{Hom}(X, \mathrm{GL}_{n,\mathbb{Z}}) = \{A = (s_{ij})_{i,j=n} \mid \det A$ multiple of $s_{ij} \in \mathcal{O}_X(|X|)\}$ (???)*

Proof. ■

Remark: $\mathbb{G}_{m,\mathbb{Z}}, \mathrm{GL}_{n,\mathbb{Z}}$ are group objects in LocRingdSpc.

Fiber Products I

Lemma 16. *Suppose we have ring homomorphisms*

$$\begin{tikzcd} B \arrow[r] \arrow[d] & C \\ A & \end{tikzcd}$$

Then $\operatorname{Spec}(A \otimes_B C)$ is the fiber product $\operatorname{Spec} A \times_{\operatorname{Spec} B} \operatorname{Spec} C$ in LocRingdSpc.

Proof. Consider the set of maps $\{A \otimes_B C \to \mathcal{O}_X(|X|)\}$, whic h we identify with $\mathrm{Hom}(X, \operatorname{Spec}(A \otimes_B C))$. But the first is equal to the fiber product

$$\begin{tikzcd} B \arrow[r] \arrow[d] & C \arrow[d, "\psi"] \\ A \arrow[r, "\varphi"] & \mathcal{O}_X(|X|) \end{tikzcd}$$

, where we send $a \otimes c$ to $\varphi(a) \cdot \psi(b)$. But by the same property this is the same as the coproduct

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \operatorname{Spec} C \arrow[d] \\ \operatorname{Spec} A \arrow[r] & \operatorname{Spec} B \end{tikzcd}$$

So this is establishes the universal property,

$$\begin{tikzcd}
X \arrow[dr, dotted, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \operatorname{Spec}(A \otimes_B C) \arrow[r] \arrow[d] & \operatorname{Spec} C \\
& \operatorname{Spec} A \arrow[r] & \operatorname{Spec} B
\end{tikzcd}$$

■

Remark:
We have the correspondence between

$$\begin{tikzcd}
X \arrow[r] \arrow[dr] & \operatorname{Spec} A \arrow[d] \\
& \operatorname{Spec} B
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \arrow[r] & \mathcal{O}_X(|X|) \\
B \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Concrete examples

1. $\varnothing = \bigcup \varnothing$, and it is a scheme, because $\varnothing = \operatorname{Spec} 0$, which is an affine scheme.

2. Let k be a field. Then $|\operatorname{Spec} k| = \{*\}$, the zero ideal in k. Then $\operatorname{Spec} k = (\{*\}, k)$, a point with a field attached to it.

3. $\operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$

4. The spectrum of any artinian local ring. In this and the above case, the set of prime ideals is a single point. Cor:

Take $R = \mathbb{Z}[S,T]/(S^2 - T^2 - 1)$, and $X = \operatorname{Spec} R$. Then

$$\operatorname{Hom}(\operatorname{Spec} \mathbb{R}, X) = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

and

$$\operatorname{Hom}(\operatorname{Spec} \mathbb{R}[\varepsilon]/(\varepsilon^2), X) = \{(z,v) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid z \in S^1, v \perp z\} = TS^1$$

More generally,

$$\operatorname{Hom}(\operatorname{Spec} k, X) = k\text{-solutions to a system of equations}$$

$$\operatorname{Hom}(\operatorname{Spec} k[\varepsilon]/(\varepsilon^2), X) = k\text{-tangent vectors at } k\text{-points of a variety}$$

5. $\operatorname{Spec}\mathbb{Z}$ is final in LocRingdSpc, i.e. for all X, there is a unique map $X \to \operatorname{Spec}\mathbb{Z}$, because such a map corresponds to a ring homomorphism $\mathbb{Z} \to \mathcal{O}_X(|X|)$, and $\mathbb{Z}$ is initial in Ring (because 1 must be sent to 1).

Observe $|\operatorname{Spec}\mathbb{Z}| = \{(0)\} \cup \{\text{prime numbers}\}$, with topology generated by $U_n = \{(0), p \mid p \nmid n\}$, $n \in \mathbb{Z}$. All non-empty opens contain (0). So $U_n \setminus \{(0)\} = \mathbb{P} \setminus (\text{finite subset})$. So it is basically the cofinite topology, except there is an additional point in every open subset. So $\overline{(O)} = |\operatorname{Spec}\mathbb{Z}|$. We will refer to such points as <u>generic points</u>.

If we consider $\mathcal{O}_{\mathbb{Z}}(|\operatorname{Spec}\mathbb{Z}|) = \mathbb{Z}$, then $\mathcal{O}_{\mathbb{Z},p} = \mathbb{Z}[q^{-1} \mid q \neq p]$, and $\mathcal{O}_{\mathbb{Z},(0)} = \mathbb{Z}_{(0)} = \mathbb{Q}$.

To see this, consider $s \in \Gamma(|\operatorname{Spec}\mathbb{Z}|)$. This is Zariski locally represented by a fraction $s_i \in U_{d_i}, s_i = \frac{n_i}{d_i}$. Then by locality $\cup_i U_{d_i} = |\operatorname{Spec}\mathbb{Z}|$ is equivalent to $\gcd(d_i) = 1$. Since $\gcd(d_i) = 1$ and $\frac{n_i}{d_i} = \frac{n_j}{d_j}$ for each i, j, this implies $\frac{n_i}{d_i} = \lambda \in \mathbb{Z}$. This works in any PID.

10, 26/9/25

<u>Presentation: Kareem</u>
Facts to take for granted: $V(I) = \{p \in |\operatorname{Spec} R \mid I \subseteq p\}$, $D(f)^c = V\left((f)\right)$, $V(I) = V(\sqrt{I}) \subset V(J) \subset V(\sqrt{j})$ iff $\sqrt{I} \supset \sqrt{j}$
$\bigcap V(I) = V(\sum I)$.

Claim. *$\varphi : R_f \to \mathcal{O}(D(f))$ is an isomorphism, with the map sending $\frac{a}{f^n}$ to $s(\mathfrak{p}) = \frac{a}{f^n}$ in $V_{\mathfrak{p}}$.*

Proof. Kareem went too fast. Sorry ∎

<u>Fact:</u> $|\operatorname{Spec} R|$ is always quasi-compact, no matter the ring, so the underlying space of an affine scheme is always quasi-compact.
So for any family of non-zero R_i, $\sqcup_{n\in\mathbb{Z}} |\operatorname{Spec} R_n|$ is a non-affine scheme. It is not affine because the underlying space is not quasicompact.
<u>Remark:</u> It is true that if you take the finite product of rings, then

$$\operatorname{Spec}\left(\prod_{i=1}^r R_i\right) = \sqcup_{i=1}^r \operatorname{Spec} R_i$$

But this is not true for infinite products, because $\operatorname{Spec}\left(\prod_{i\in I} R_i\right)$ is quasicompact. Contemplate $|\operatorname{Spec}\mathbb{F}_2^{\mathbb{N}}|$.

Criterion for affine-ness

Let $X = (|X|, \mathcal{O})$ be a locally ringed space. Then there is a unique homomorphism to an affine scheme $\operatorname{aff}_X : X \to \operatorname{Spec} \mathcal{O}(|X|)$ called the affinization of X, denoted aff_X
And:
X is affine if and only if aff_X is an isomorphism of locally ringed spaces.
$\mathbb{A}^n_{\mathbb{Z}}$ with a doubled origin

Recall: $\mathbb{A}^1_{\mathbb{Z}} \stackrel{\text{def}}{=} \operatorname{Spec} \mathbb{Z}[T]$. This contains $D(T) = |\operatorname{Spec} \mathbb{Z}[T, T^{-1}]| = |\mathbb{G}_{m,\mathbb{Z}}|$.

Definition 0.23. We denote by

$$\mathbb{A}^1_{\mathbb{Z},double-O} \stackrel{\text{def}}{=} \mathbb{A}^1_{\mathbb{Z}} \sqcup_{\mathbb{G}_{m,\mathbb{Z}}} \mathbb{A}^1_{\mathbb{Z}}$$

This pushout still exists in the category of locally ringed spaces, so this definition makes sense. Indeed, the underlying topological space is this pushout.
We define the structure sheaf to be the unique $\mathcal{O}$ such that $\mathcal{O}|_{\mathbb{A}^1_{\mathbb{Z}}} = \mathcal{O}_{\mathbb{A}^1_{\mathbb{Z}}}$, where those are the two copies of $\mathbb{A}^1_{\mathbb{Z}}$.
By construction, X is a scheme, because it can be covered by two affine scheme.

Claim. *X is not affine.*

Proof.

1. First, consider the equalizer of the following diagram:

$$\begin{tikzcd} \mathcal{O}_X(|X|) \arrow[r] & \mathcal{O}_{\mathbb{A}^n}(\mathbb{A}^n) \oplus \mathcal{O}(\mathbb{A}^n) \arrow[r, bend left] \arrow[r, bend right] & \mathcal{O}(\mathbb{G}_m) \end{tikzcd}$$

Where the top map is $T_1 \mapsto T$, and the bottom $T_2 \mapsto T$, where T_i are the two copies of T. Recall $\mathcal{O}_X(|X|) = \mathbb{Z}[T]$, so this diagram reads

$$\begin{tikzcd} \mathbb{Z}[T] \arrow[r] & \mathbb{Z}[T_1] \oplus \mathbb{Z}[T_2] \arrow[r, bend left, "T_1 \mapsto T"] \arrow[r, bend right, "T_2 \mapsto T"'] & \mathbb{Z}[T, T^{-1}] \end{tikzcd}$$

But $\operatorname{aff}_X : X \to \mathbb{A}^1_{\mathbb{Z}}$ on $|\mathbb{G}_{m,\mathbb{Z}}|$ $\operatorname{aff}_X|_{\mathbb{G}_m}$ is not an iso. Indeed,

$$|\operatorname{aff}_X^{-1}(\mathfrak{m}_0)| = 2$$

Because $\operatorname{aff}_X^{-1}(|\mathbb{G}|) = |\mathbb{G}|$, so $|X \setminus \operatorname{aff}_X^{-1}(|\mathbb{G}|)| = 2$, thus $|\operatorname{aff}_X|$ is not injective, so aff_X cannot be an isomorphism of locally ringed spaces.

■

Another example: $\mathbb{P}^1_{\mathbb{Z}}$.

Consider again the pushout of spaces

$$\begin{tikzcd} \mathbb{G}_{m,\mathbb{Z}} \arrow[r, hook] \arrow[d, hook] & \mathbb{A}^n_- \arrow[d] \\ \mathbb{A}^1_+ \arrow[r, hook] & \mathbb{P}^1_{\mathbb{Z}} \end{tikzcd}$$

Where $\mathbb{A}^1_\pm = \operatorname{Spec}\mathbb{Z}[T^\pm]$, and of course $\mathbb{G}_{m,\mathbb{Z}} = \operatorname{Spec}\mathbb{Z}[T, T^{-1}]$.
We define this by first taking the pushout of topological spaces to the topological space, and then glue the structure sheaves together on $|\mathbb{A}^1_+|, |\mathbb{A}^1_-|$ accordingly. We call this $\mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}$. This is by construction covered by two affine schemes and thus a scheme.
To comput $\mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(|\mathbb{P}^1_{\mathbb{Z}}|)$, we look at another equalizer

$$\begin{tikzcd} \mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(|\mathbb{P}^1_{\mathbb{Z}}|) \arrow[r] & \mathcal{O}_{\mathbb{A}^1_+}(|\mathbb{A}^1_+|) \oplus \mathcal{O}_{\mathbb{A}^1_-}(|\mathbb{A}^1_-|) \arrow[r, bend left] \arrow[r, bend right] & \Gamma(\mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}) \end{tikzcd}$$

Agai, this diagram reads

$$\begin{tikzcd} \mathbb{Z}[T] \cap \mathbb{Z}[T^{-1}] \arrow[r] & \mathbb{Z}[T] \oplus \mathbb{Z}[T^{-1}], \arrow[r, bend left] \arrow[r, bend right] & \mathbb{Z}[T, T^{-1}] \end{tikzcd}$$

So $\operatorname{aff}_{\mathbb{P}^1_{\mathbb{Z}}} : \mathbb{P}^1_{\mathbb{Z}} \to \operatorname{Spec}\mathbb{Z}$, but $|\mathbb{P}^1_{\mathbb{Z}}| \supset |\mathbb{A}^1_{\mathbb{Z}}|$ is sent to $|\operatorname{Spec}\mathbb{Z}|$, so aff cannot be an isomorphism here.

Lecture 11, 29/9/25

Reading assignment: Section 2.3 (start with definition of an open subscheme).
Next quiz: Friday
Today:

- Open immersions
- Fibre products

Definition 0.24. A morphism of locally ringed spaces $(|f|, f_*) = f : Y \to X$ is called an open immersion if

- $|f| : |Y| \to |X|$ is open and injective
- $f_* : f^{-1}\mathcal{O}_x \to \mathcal{O}_Y$ is an isomorphism of sheaves

Definition 0.25. Let X be a locally ringed space, $|U| \subseteq |X|$ an open subset. Then we define

$$U \stackrel{\text{def}}{=} (|U|, \mathcal{O}_X|_{|U|})$$

This is also a locally ringed space, and the map $\iota : U \hookrightarrow X$ (given by the inclusion, and the ring isomorphism $\iota_* : \mathcal{O}_X|_{|U|} \to \mathcal{O}_U$) is an open immersion
Fact:
For an open immersion $f : Y \to X$, there exists an open subset $|U| \subset |X|$ such that there is a commutative diagram

$$\begin{tikzcd} Y \arrow[dr, "f"] & \\ & X \\ U \arrow[uu, "\simeq"] \arrow[ur, "\iota_U"'] & \end{tikzcd}$$

Remark: A morphism of locally ringed spaces $f : Y \to X$ which is open and injective is an open immersion if and only if for all $y \in Y$, the map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,f(y)}$ is an isomorphism

Definition 0.26. If X is a scheme and $|U| \subset |X|$ is open, then U is called an open subscheme of X.

Lemma 17. *$U = (|U|, \mathcal{O}_X|_U)$ is a scheme if X is a scheme.*

Proof. It suffices to show this for $X = \operatorname{Spec} R$.
Let $|U| \subset |\operatorname{Spec} R|$ be open. Then $|U| = \bigcup_{\alpha \in A} D(f_\alpha)$. Note $D(f_\alpha) = |\operatorname{Spec} R_{f_\alpha}|$.
So U is covered by the affine schemes $\operatorname{Spec} R_{f_\alpha}$ ■

Warning: U is not necessarily affine.

Example 0.6. Recall $\mathbb{A}^2_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[S,T]$. Consider $|U| = D(S) \cup D(T)$
Note $|U| \neq |\mathbb{A}^2_{\mathbb{Z}}|$, and in fact the complement $|\mathbb{A}^2_{\mathbb{Z}}| \setminus |U|$ is generated by $V\big((S,T)\big) = |\operatorname{Spec} \mathbb{Z}|$. Then

$$\mathbb{Z}[S,T] = \mathcal{O}_U(|U|) = \mathcal{O}_{D(S)}(D(S)) \cap \mathcal{O}_{D(T)}(D(T)) = \mathbb{Z}[S,T]_S \cap \mathbb{Z}[S,T]_T$$

So $\operatorname{aff}_U : U \to \mathbb{A}^2_{\mathbb{Z}}$ (given by ι) is not surjective. So U is non-affine.

Lemma 18. *Let*

$$\begin{tikzcd} & Z \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be a diagram of locally ringed spaces, with f and open immersion.
Then $X \times_Y Z$ exists, and the morphism $f' : X \times_Y Z \to Z$ given by projection onto the second coordinate is an open immersion
So we can lift open immersions

Proof. Consider $\left(\underbrace{g^{-1}(f(|X|))}_{|U|}, \mathcal{O}_Z|_{|U|}\right)$, and show that it has the required properties and satisfies the universal property of the fiber product.
First, $(g^{-1}(|U|), \mathcal{O}_Z|_{g^{-1}(|U|)}) \to (|Z|, \mathcal{O}_Z)$ given by the inclusion of $|g^{-1}(U)|$ is an open immersion.
Let $W \in$ LocRingdSpc. Then consider the diagram

$$\begin{tikzcd}
W \arrow[dr, dotted, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& g^{-1}(|U|, \mathcal{O}_Z|_{g^{-1}(|U|)}) \arrow[r] \arrow[d] & Z \arrow[d, "g"] \\
& U \arrow[r, "f"] & Y
\end{tikzcd}$$

The dotted map exists and is unique just on the level of continuous maps automatically, $h : W \to Z$ corresponds to $h^{-1}\mathcal{O}_Z \to \mathcal{O}_W$, where $\mathcal{O}_Z = h^{-1}(\mathcal{O}_Z|_{g^{-1}(|U|)})$, recalling that f is an open immersion.

■

Example 0.7. For $U, V \subset X$ are subschemes, then for the cartesian diagram

$$\begin{tikzcd}
U \cap V \arrow[r, hook] \arrow[d, hook] & V \arrow[d, hook] \\
U \arrow[r, hook] & X
\end{tikzcd}$$

We haved $U \cap V = U \times_X V$
Pushouts along open immersions

Lemma 19. *Let*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"] & Y \\
Z &
\end{tikzcd}$$

be a diagram of schemes and open immersions. Then the pushout $X \sqcup_Y Z$ exists in the category of schemes.

Proof. Define $|Y \sqcup_X Z|$ by taking the pushout in the category of topological spaces, i.e. $\frac{|Y| \sqcup |Z|}{\sim}$, where $\sim$ identifies $f(x)$ and $g(x)$ for all $x \in |X|$.
Topology: $U \subset |Y| \sqcup_{|X|} |Z|$ is open if and only if $U \cap |Y|$ and $U \cap |Z|$ are open.
Glue the étalé spaces of $\mathcal{O}_Y, \mathcal{O}_Z$ along $\mathcal{O}_X$ the same way.
This shows its a locally ringed space because the stalks are local rings, and is also covered by affine schemes.

■

Lemma 20. *Assume that in the diagram below, both inner squares are Cartesian (meaning fibre products).*

$$\begin{tikzcd} W' \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & Z \arrow[d] \\ X' \arrow[r] & X \arrow[r] & Y \end{tikzcd}$$

Then the outer square is Cartesian.

Proof. Volunteer

■

Fibre products part 2

Proposition 3. *Let*

$$\begin{tikzcd} & Z \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be a diagram of schemes. Then the fibre product $X \times_Y Z$ exists in LocRingdSpc *and is a scheme.*

Proof. Recall: This is true if X, Y, Z are affine by the universal property of Spec.
Sketch of proof:

1. Choose open covers $(U_\alpha), (V_\alpha), (W_\alpha)$ of X, Y, Z by affine schemes, so that $f : U_\alpha \to V_\alpha$ and $g : W_\alpha \to V_\alpha$ via restriction, i.e. $|f|(|U_\alpha|) \subset |V_\alpha| \supset g(|W_\alpha|)$

2. Glue $U_\alpha \times_{V_\alpha} W_\alpha$ to a scheme we can call $X \times_Y Z$

3. Prove it satisfies the universal property.

Lecture 12, 1/10/25

Recall: Cartesian Squares = fibre product squares.
Cartesian squares can be juxtaposed as in the previous lemma.
Test on friday (next lecture)
We have shown that fibre prodcuts along open immersions exist, and are open immersions (I denote open immersions by $\hookrightarrow$ when the diagram is in the category of schemes/locally ringed spaces).
In particular, they exist in the category of schemes.

Lemma 21. *Let*

$$\begin{tikzcd} & Z \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

be a diagram in a category $\mathcal{C}$ and $Y' \to Y$ be a morphism. Assume that the fibre products $X' = X \times_Y Y$ and $Z' = Z \times_Y Y'$ exist, and that $X \times_Y Z$ and $X' \times_{Y'} Z'$ exist. Then

$$X' \times_{Y'} Z' \cong (X \times_Y Z) \times_Y Y'$$

As an exercise, complete this cube (imagine the things on the right are "folded" up)

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[d] & Z' \arrow[l] \arrow[d] \\ X \arrow[r] & Y & Z \arrow[l] \end{tikzcd}$$

Lemma 22. *Let $W, F \in$ LocRingdSpc. Then the following presheaf,* $\mathrm{Hom}(-, F)$: $\mathrm{Open}(|W|)^{op} \to \textsf{Set}$ *is a sheaf $|U| \mapsto \mathrm{Hom}_{\mathrm{LocRingdSpc}}(U, W)$, as in the commutative diagram*

$$\begin{tikzcd} \mathrm{Open}(|W|)^{op} \arrow[rr] \arrow[dr] & & \textsf{Set} \\ & \mathrm{LocRingdSpc}^{op} \arrow[ur, "Hom"'] & \end{tikzcd}$$

Where "Hom" means $\mathrm{Hom}(-, F)$ *(tikzcd freaks out if there are parens for some reason :( )*

Proof. Volunteer. ∎

Here is the main lemma of today:

Lemma 23. *Let*

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be a diagram of schemes. Assume that each scheme has an open cover $X = X_1 \cup X_2, Y = Y_1 \cup Y_2, Z = Z_1 \cup Z_2$, such that $f(|X_i|) \subset |Y_i| \supset g(|Z_i|)$, and $X_i \subset X, Y_i \subset Y, Z_i \subset Z$ are open subschemes for $i \in \{1, 2\}$.
If $X_i \times_{Y_i} Z_i$ exists and is a scheme for each i then $X \times_Y Z$ exists and is a scheme.

Proof. At first we need the following.

Claim. *$X_{12} \times_{Y_{12}} Z_{12}$ exists and is a scheme, where $X_{12} = X_1 \cap X_2$, and similarly for Y_{12}, Z_{12}.*

Proof. Consider the fibre product

$$\begin{tikzcd} X_1 \times_{Y_1} Z_1 \arrow[r] \arrow[d] & Z_1 \arrow[d] \\ X_1 \arrow[r] & Y_1 \end{tikzcd}$$

By the lemma we can take the fibre product

$$\begin{tikzcd} X_{12} \times_{Y_1} Z_1 \arrow[r, hook] \arrow[d] & X_1 \times_{Y_1} Z_1 \arrow[r] \arrow[d] & Z_1 \arrow[d] \\ X_{12} \arrow[r, hook] & X_1 \arrow[r] & Y_1 \end{tikzcd}$$

Then we can extend this to the diagram (all 4 squares are Cartesian)

$$\begin{tikzcd} X_{12} \times_{Y_1} Z_{12} \arrow[r] \arrow[d] & X_1 \times_{Y_1} Z_{12} \arrow[r] \arrow[d, hook] & Z_{12} \arrow[d, hook] \\ X_{12} \times_{Y_1} Z_1 \arrow[r, hook] \arrow[d] & X_1 \times_{Y_1} Z_1 \arrow[r] \arrow[d] & Z_1 \arrow[d] \\ X_{12} \arrow[r, hook] & X_1 \arrow[r] & Y_1 \end{tikzcd}$$

Then by the other lemma

$$(X_{12} \times_{Y_1} Z_{12}) \times_{Y_1} Y_{12} \cong X_{12} \times_{Y_{12}} Z_{12}$$

■

Now: Pushout of

$$\begin{tikzcd}
X_{12} \times_{Y_{12}} Z_{12} \arrow[r, hook] \arrow[d, hook] & X_2 \times_{Y_2} Z_2 \arrow[d, hook] \\
X_1 \times_{Y_1} Z_1 \arrow[r, hook] & X \times_Y Z = P
\end{tikzcd}$$

Claim. *P satisfies the universal product of the fibre product.*

Proof. This follows from the sheaf property: use the lemma for an arbitrary W, letting $F = P$.
Sketch:
Take the diagram

$$\begin{tikzcd}
W \arrow[dr, dotted, "\exists!"] \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, "\alpha"'] & & \\
& P \arrow[r] \arrow[d] & Z \arrow[d] \\
& X \arrow[r] & Y
\end{tikzcd}$$

Apply sheaf property to an appropriate open cover $W = \cup_{i \in I} U_i$ such that $\alpha(U_i) \subset X_1 \cup X_2$ and similarly for β and $Y_1 \cup Y_2$. ∎

This proves the main lemma ∎

Theorem 0.10. *Let*

$$\begin{tikzcd}
& Z \arrow[d] \\
X \arrow[r] & Y
\end{tikzcd}$$

be a diagram of schemes. Then $X \times_Y Z \in \mathrm{LocRingdSpc}$ exists and is a scheme.

Proof. Let $\mathcal{S}$ be the partially ordered set of open subdiagrams

$$\begin{tikzcd}
& W \arrow[d] \arrow[dr, phantom, "\subset" sloped] & \\
U \arrow[r] \arrow[dr, phantom, "\subset" sloped] & V \arrow[dr, phantom, "\subset" sloped] & Z \arrow[d] \\
& X \arrow[r] & Y
\end{tikzcd}$$

Then the fibre product $U \times_V W$ exists and is a scheme:

$$
\begin{tikzcd}
U \times_V W \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & W \arrow[d] \arrow[dr, phantom, sloped, "\subset"] & \\
U \arrow[r] \arrow[dr, phantom, sloped, "\subset"] & V \arrow[dr, phantom, sloped, "\subset"] & Z \arrow[d] \\
& X \arrow[r] & Y
\end{tikzcd}
$$

We know that $\mathcal{S} \neq \varnothing$, since X, Y, Z are covered by affine opens and for diagrams of affine schemes the fibre product exists and is a scheme.
we also have that for each chain of such diagram s

$$
\begin{tikzcd}
& W_i \arrow[d] & \\
U_i \arrow[r] & V_i \arrow[dr, phantom, sloped, "\subset"] & Z \arrow[d] \\
& X \arrow[r] & Y
\end{tikzcd}
$$

we can take the ascending union sheaf property to get an upper bound.
By Zorn's lemma, there is a maximal open subdiagram

$$
\begin{tikzcd}
& W_0 \arrow[d] \\
U_0 \arrow[r] & Z_0
\end{tikzcd}
$$

such that $U_0 \times_{V_0} W_0$ exists and is a scheme.
If $U_0 \neq X$ and $V_0 \neq Y$ and $W_0 \neq Z$, then we could find an open subdiagram of affine subschemes

$$
\begin{tikzcd}
& W_1 \arrow[d] \\
U_1 \arrow[r] & V_1
\end{tikzcd}
$$

such that $U_0 \cup U_1$ or $V_0 \cup V_1$ or $W_1 \cup W_0$ is strictly larger. Applying the main lemma above, we get a contradiction.

$\blacksquare$

Lecture 12, 10/3/25

The following are in class presentations

Lemma 24. *Let $W, F \in \mathrm{LocRingdSpc}$. Then $\mathcal{F} : \mathrm{Open}(|W|)^{op} \to \mathrm{LocRingdSpc}^{op} \to$ **Set**, given by the composition*

$$|U| \mapsto (|U|, \mathcal{O}_W|_{|U|}) = \mathcal{U} \mapsto \mathrm{Hom}(\mathcal{U}, F)$$

is a sheaf.

Proof. Let $U = \cup U_{i\in I}$ be a covering of U. Then

$$\mathrm{Hom}(U, F) \to \Pi_{i\in I}\mathrm{Hom}(U_i, F) \to \Pi_{i,j\in I}\mathrm{Hom}(U_{ij}, F)$$

is an equalizer, as in the def of sheaf condition above (pretend there are two arrows on the right). It is enough to show that the following is a coequalizer:

$$U \longleftarrow \coprod U_i \leftleftarrows \coprod U_{ij}$$

I missed it.

Lemma 25. *Consider the diagram*

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "D_1"] & B_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "D_2"] & C_1 \arrow[d] \\ A_2 \arrow[r] & B_2 \arrow[r] & C_2 \end{tikzcd}$$

Then if D_1, D_2 are cartesian, so is D_3, the following square:

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "D_3"] & C_1 \arrow[d] \\ A_2 \arrow[r] & C_2 \end{tikzcd}$$

Further, if D_2, D_3 are cartesian, so is D_1.

Proof. Consider an object X

$$\begin{tikzcd} X \arrow[rrrd, bend left] \arrow[rd, bend right] & & & \\ & A_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "D_1"] & B_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "D_2"] & C_1 \arrow[d] \\ & A_2 \arrow[r] & B_2 \arrow[r] & C_2 \end{tikzcd}$$

There exists a unique map $g_1 : X \to B_1$:

$$\begin{tikzcd}
X \arrow[drrr, bend left] \arrow[drr, dotted, "g_1"] \arrow[ddr, bend right] & & & \\
& A_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "D_1"] & B_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "D_2"] & C_1 \arrow[d] \\
& A_2 \arrow[r] & B_2 \arrow[r] & C_2
\end{tikzcd}$$

But then there is an f_1 :

$$\begin{tikzcd}
X \arrow[drrr, bend left] \arrow[drr, bend left, "g_1"] \arrow[dr, dotted, "f_1"'] \arrow[ddr, bend right] & & & \\
& A_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "D_1"] & B_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "D_2"] & C_1 \arrow[d] \\
& A_2 \arrow[r] & B_2 \arrow[r] & C_2
\end{tikzcd}$$

The proof of the second assertion proceeds similarly (I will add the diagrams there if I feel ike it)

■

Lemma 26. *This is the cube lemma from last time. The diagrams are too complex so I will have to do them later*

Proof. ■

Today:

- Closed immersions 1
- Bla bla on base change

$\mathbb{K}$-varieties are defined by systems of equations, for example

$$X^2Y - 2XZ + 1$$

They can be studed over any (algebraically closed) field $\mathbb{K}$.
The scheme given by $W = \operatorname{Spec} Z[X,Y,Z]/(X^2Y - 2XZ + 1)$ provides an integral model which allows us to study this equation over any field we like.
We have a "base change" to any field by fibre products:

$$W \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{K} = \operatorname{Spec}\mathbb{K}[X,Y,Z]/(X^2Y - 2XZ + 1)$$

We will replace base fields by "base schemes". One should not study properties of schemes/varieties per se, but properties of their morphisms. This is the philosophy of Grothendieck.

Definition 0.27. A morphism of schemes $i : Y \to X$ is called a closed immersion if $|i| : |Y| \to |X|$ is injective and closed, and the map of structure sheaves, $|i|^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ is surjective.

Example 0.8. Given a surjection of rings $\varphi : A \to B$, one obtains a closed immersion

$$\operatorname{Spec}(\varphi) : \operatorname{Spec} B \to \operatorname{Spec} A$$

Let $I \overset{\text{def}}{=} \ker(\varphi)$. Then $B = A/I$. On assignment 3, it was shown that

$$|\operatorname{Spec} A/I| \to |\operatorname{Spec} A|$$

has closed image. From there, it is not hard to show this is a homeomorphism onto a closed subset, consisting of prime ideals containing $I : \{\mathfrak{p} \mid \mathfrak{p} \supset I\} = V(I)$, which is closed.

Warning: For every closed subset $|Y| \subset |X|$ there could be infinitely many closed immersions $Y \to X$ with that image.

Example 0.9. Consider $\operatorname{Spec} \mathbb{K}[T]/(T^n) \to \operatorname{Spec} \mathbb{K}[T]$

Lecture 13, 6/10/25

Definition 0.28. Today:

- Study closed immersions into affine schemes
- Maybe separated morphisms

Lemma 27. *Let R be a ring, and M and R-module such that for every prime ideal $\mathfrak{p} \in |\operatorname{Spec} R|$, we have $M_{\mathfrak{p}} = 0$. Then $M = 0$.*

Proof. We know that for every $m \in M$, there is some $\mathfrak{p}$ and $f_{\mathfrak{p}} \in R \setminus \mathfrak{p}$ such that $fm = 0$, meaning $\frac{m}{1} = 0$ in $M_{\mathfrak{p}}$ by assumption.
Since $\bigcup_{\mathfrak{p} \in \operatorname{Spec} R} D(f_{\mathfrak{p}}) = |\operatorname{Spec} R|$, there exists a finite subcover $f_1, \ldots, f_r$, with $(f_1, \ldots, f_r) = 1$ and $f_i \cdot m = 0$ for all i.
So then there are $g_1, \ldots, g_r \in R$ such that $g_1 f_1 + \cdots + g_r f_r = 1$, so hitting m with this gives 0, so for all $m \in M$ we have $m = 1 \cdot m = 0$.

■

Later we will see how to associate a sheaf $\mathcal{M}$ to an R-module M on $|\operatorname{Spec} R|$, such that $\mathcal{M}$ is actually an $\mathcal{O}$-module and the stalk of $\mathcal{M}$ at $\mathfrak{p}$ is isomorphic to $M_{\mathfrak{p}}$.
The sheaves $\mathcal{M}$ arising this way on $\operatorname{Spec} R$ are called quasi-coherent.

Lemma 28. *Let $i : Y \twoheadrightarrow X$ be a closed immersion of affine schemes: $X = \operatorname{Spec} R, Y = \operatorname{Spec} S$. Then there exists a surjective ring morphism $\varphi : R \twoheadrightarrow S$ such that $i = \operatorname{Spec}(\varphi)$.*

Proof. The universal property of Spec implies that there exists some ring homomorphism $\varphi : R \to S$ such that $i = \operatorname{Spec}(\varphi)$. So the goal is to show that φ is a surjection of $\operatorname{Spec}(\varphi)$ is a closed immersion.
Let $M = \operatorname{coker}(\varphi : R \to S)$, viewed as a morphism of R-modules, where R acts on S by $r \cdot s \overset{\text{def}}{=} \varphi(r) \cdot s$.
If φ is surjective, then $M = 0$. For every prime $\mathfrak{p}$ we have $M_{\mathfrak{p}} = \operatorname{coker}(R_{\mathfrak{p}} \to S_{\mathfrak{p}})$, and this map corresponds to $\operatorname{coker}(\mathcal{O}_{X,x} \to \mathcal{O}_{Y,x})$, which is 0.
Applying the lemma, we obtain that $M = 0$, thus φ is surjective. ∎

Proposition 4. *Let $i : Y \hookrightarrow X = \operatorname{Spec} R$ be a closed immersion. Then there exists and ideal $I \subset R$ and an isomorphism $Y \cong \operatorname{Spec} R/I$ such that the following diagram commutes:*

$$\begin{tikzcd}
Y \arrow[r, hook] \arrow[d, phantom, "\cong"] \arrow[dr, phantom, "\curvearrowright"] & X \arrow[d, phantom, "\cong"] \\
\operatorname{Spec} R/I \arrow[r, hook] & \operatorname{Spec} R
\end{tikzcd}$$

Proof. We require the following lemma:

Lemma 29. *Let Y be a scheme and $f_1, \ldots, f_r \in \mathcal{O}_Y(|Y|)$ such that $(f_1, \ldots, f_r) = 1$ in $\mathcal{O}_Y(|Y|)$. Let $Y_{f_i} \overset{\text{def}}{=} Y \times_{A^1_{\mathbb{Z}}} \mathbb{G}_{m,\mathbb{Z}}$ be an open subscheme where f_i is <u>invertible</u>. Assume that for all i, these loci Y_{f_i} are affine schemes. Then Y is an affine scheme.*

Proof. Let's skip the proof ∎

Corollary 0.11. *Given a closed immersion $i : Y \hookrightarrow \operatorname{Spec} R$, Y is an affine scheme.*

Proof. Since Y is a scheme, for every $y \in |Y|$, there exists an open affine neighborhood $U_y \subset |Y|$. Then $U_y = V_y \cap |Y|$, where $V_y \subset |\operatorname{Spec} R|$ is open.
By the definition of the Zariski topology, there is an element $f_y \in R$ such that $D(f_y) \cap |Y| \subset U_y$.
One then only need to note that $D(f_y) \cap |Y| = V_y \cap D(f_y)$.
Hence: we get a commutative diagram

$$\begin{tikzcd}
D_{U_y}(f_y) = D(f_y) \cap U_y \arrow[r] \arrow[d, phantom, "\cap"] & D(f_y) \arrow[d, phantom, "\cap"] \\
Y \arrow[r] & X
\end{tikzcd}$$

This is a Cartesian diagram, with the top arrow a closed immersion of affine schemes. So Y_{f_i} is affine. Since this works for every $y \in Y$, using quasi-compactness of $|Y| \subset |X|$ (closed subset of quasicompact is quasicompact), we have a cover $f_1, \dots, f_r$, as in the affineness criterion.

■

Definition 0.29. A scheme X is reduced if for every $x \in X$, the local ring $\mathcal{O}_{X,x}$ is reduced, i.e. there are no non-zero nilpotents, meaning that if $f^n = 0$ implies $f = 0$, which is equivalent to $\sqrt{0} = 0$.
Remark: Let $|U| \subset |X|$ be open, X reduced. Then $\mathcal{O}_X(|U|)$ is reduced.

Corollary 0.12. *Let X be a reduced scheme. Then a closed immersion $i : Y \hookrightarrow X$ with maximal image (meaning $|i| : (|Y|) = |X|$) is an isomorphism of schemes.*

Proof. Without loss of generality, assume $X = \operatorname{Spec} R$ is an affine scheme.
Then R is a reduced ring by the above remark.
Further, $Y = \operatorname{Spec} R/I \hookrightarrow \operatorname{Spec} R$ from the proposition.
Also, the image of $(|i|)$ is the set of all prime ideals containing I, but this is $|\operatorname{Spec} R|$, and hence $I \subset \bigcap_{\mathfrak{p}} \mathfrak{p} = \sqrt{0} = 0$

■

Corollary 0.13. *Let X be a scheme. Show that for any closed subset $A \subset |X|$, there exists a closed immersion which is unique up to unique isomorphism, from a reduced scheme $i : Y \to X$, with the image of $|i|$ exactly A.*

Proof. ■

Remark For $A = |X|$, this is called the underlying reduced scheme of X, X^{red}.

Lemma 30. *Let $i : Y \hookrightarrow X$ be a closed immersion and $X' \to X$ an arbitrary morphism. Then the map $Y' = Y \times_X X' \hookrightarrow X'$ is also a closed immersion:*

$$\begin{tikzcd}
Y' \arrow[r, hook] \arrow[d, "f'"'] & X' \arrow[d, "f"] \\
Y \arrow[r, hook] & X
\end{tikzcd}$$

Proof. ■

Definition 0.30. A morphism of schemes $f : Y \to X$ is separated if the diagonal, $\Delta_{Y/X} : Y \to Y \times_X Y$ is a closed immersion.

Example 0.10. Affine schemes $R \to S$ are separated, because the codiagonal map $\bigtriangledown : S \otimes_R S \to S$ which sends $s_1 \otimes s_2$ to $s_1 \cdot s_2$ is surjective, because we have $s \otimes 1 \mapsto 1$.

Lecture 14, 8/10/25

Reading assignment: Hartshorne 2.4.
Emmanuel:
Goal: Closed immersions are preserved by base change, i.e. $f : X \xrightarrow{/} Y$, $g : X' \to X$, then $Y' \xrightarrow{/} X'$, where $Y' = Y \times_X X'$.

1. We'll show

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "g"] \\ Y \arrow[r, "f", "/" marking] & S \end{tikzcd}$$

implies $Z \xrightarrow{/} Y$

2. Why is this enough? taking 1 for granted, we can see

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X' \arrow[d] \\ Y \arrow[r, "/" marking] & X \end{tikzcd}$$

implies $Y' \xrightarrow{/} X'$.

We have $Y \xrightarrow{f\,/} S$, $f^\sharp : \mathcal{O}_S \twoheadrightarrow \mathcal{O}_Y$, $I = \ker(f^\sharp) \subset \mathcal{O}_S$, so $I \hookrightarrow \mathcal{O}_Y$ (although really it is a quotient of I in $\mathcal{O}_Y$).
Consider $\mathrm{Im}(g^{-1}I \to \mathcal{O}_X) \subset \mathcal{O}_X$ and let $\tilde{Z}$ be the underlying set of this.
In LocRingdSpc, $\tilde{Z}$ is the fibre product, i.e. it satisfies the diagram in 1. So $\tilde{Z} = Z$ is a scheme, and $Z \xrightarrow{/} X$
Alex M:

Lemma 31. *Let X be a scheme, $A \subset |X|$ a closed set. There is a unique closed immersion $Y \overset{i}{\hookrightarrow} X$ such that $|i|(|Y|) = A$, and i is a closed immersion.*

Proof. We start with the affine case. Let $X = \operatorname{Spec} R$ be affine. From the last class, we know it uniquely factors through an ideal I:

$$\begin{tikzcd} Y \arrow[rr, hook] \arrow[dr, hook, "\sim"'] & & \operatorname{Spec} R \\ & \operatorname{Spec} R/I \arrow[ur, hook] & \end{tikzcd}$$

We can set $I = \bigcap_{\mathfrak{p} \in A} \mathfrak{p}$. If there is some other closed immersion of this form,

$$\begin{tikzcd}
\operatorname{Spec} R/I \arrow[dr, hook] \arrow[dd, phantom, "\wr\|"] & \\
& \operatorname{Spec} R \\
\operatorname{Spec} R/J \arrow[ur, hook] &
\end{tikzcd}$$

Then they have to have the same image because of something about radicals.
Now for the scheme case.
Let $X = \bigcup U_i$, $A \subset |X|$ closed. Then $A \cap U_i \subset U_i$, and the proof follows.

$\blacksquare$

Lemma 32. *Let* $Y \hookrightarrow X$ *be a closed immersion.*
Then the fiber product $Y \times_X Y \to Y$ *is an isomorphism:*

$$\begin{tikzcd}
Y \arrow[r, hook, "\mathrm{Id}"] \arrow[d, hook, "\mathrm{Id}"'] & Y \arrow[d, hook] \\
Y \arrow[r, hook] & X
\end{tikzcd}$$

Proof. Volunteer

$\blacksquare$

<u>Today:</u>

- Separated morphisms
- Density arguments

<u>Recall:</u> (From 1st lecture of this course)
A topological space X is Hausdorff if and only if the diagonal subset $\triangle \subset X \times X$ is closed.
This motivates the definition of a separated morphism:

Definition 0.31. A scheme X is <u>separated</u> if the diagonal map $\triangle_X : X \to X \times X$ is a closed immersion, where $\triangle_X$ is defined by

$$\begin{tikzcd}
X \arrow[dr, dotted, "\exists! \triangle_X"] \arrow[drr, bend left, "\mathrm{Id}_X"] \arrow[ddr, bend right, "\mathrm{Id}_X"'] & & \\
& X \times X \arrow[r] \arrow[d] & X \arrow[d] \\
& X \arrow[r] & \operatorname{Spec} \mathbb{Z}
\end{tikzcd}$$

where all of this is occuring in the category of schemes.

Definition 0.32. A morphism $f : Y \to X$ of schemes is separated if $\triangle_{Y/X} : Y \to Y \times_X X$ is a closed immersion:

$$\begin{tikzcd}
X \arrow[dr, dotted, "\exists! \triangle_{Y/X}"] \arrow[drr, bend left, "\mathrm{Id}_Y"] \arrow[ddr, bend right, "\mathrm{Id}_Y"'] & & \\
& Y \times_X Y \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\
& Y \arrow[r, "f"] & \operatorname{Spec} Z
\end{tikzcd}$$

Example 0.11. $\operatorname{Spec} R$ is separated:

$$\begin{tikzcd}
\operatorname{Spec} R \arrow[dr, dotted, "\exists! \triangle_{\operatorname{Spec} R}"] \arrow[drr, bend left, "\mathrm{Id}_{\operatorname{Spec} R}"] \arrow[ddr, bend right, "\mathrm{Id}_{\operatorname{Spec} R}"'] & & \\
& \operatorname{Spec} R \times \operatorname{Spec} R \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d] \\
& \operatorname{Spec} R \arrow[r] & \operatorname{Spec} \mathbb{Z}
\end{tikzcd}$$

is the dual of the diagram

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d] & R \arrow[d] \arrow[dd, bend left] \\
R \arrow[r] \arrow[dr] & R \otimes_{\mathbb{Z}} R \arrow[d, "m"] \\
& R
\end{tikzcd}$$

, where m is given by $r \otimes 1 \mapsto r$.

Remark: In EGA, schemes are called préschémes, and separated schemes are called schémes.

Proposition 5. *Let*

$$\begin{tikzcd}
Z \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] \arrow[dr] & & Y \arrow[dl, "h"] \\
& X &
\end{tikzcd}$$

be a commutative diagram of schemes such that

- *h is separated*

- *Z is reduced*
- $h \circ f = h \circ g$

Assume that there is an open dense $U \subset Z$ such that $f|_U = g|_U$. Then $f = g$.

Proof. Let E be the equalizer, with U a subset

$$\begin{tikzcd}
U \arrow[r, hook, "\circ"] \arrow[d, "\mathrm{Id}_U"] & E \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, hook, "\Delta_{Y/X}"] \\
U \arrow[r, hook, "\circ"] & Z \arrow[r, "{(f,g)}"] & Y \times_X Y
\end{tikzcd}$$

We know that $E \hookrightarrow Z$ is a closed immersion such that $E \cap U = U$. Thus $|E| = |Z|$. Since Z is reduced, the morphism from E to Z is an isomorphism. Since the equalizer of f, g is $E = Z$, we have $f = g$.

■

Definition 0.33. A scheme is <u>irreducible</u> if every nonempty open set is dense.

Definition 0.34. A scheme which is reduced and irreducible is <u>integral</u>

Example 0.12. $\operatorname{Spec} R$ is integral if and only if R doesn't have zero divisors, i.e. R is an integral domain.

Lemma 33. *Let X be integral. Then*

$$\bigcap_{U \subset |X| \text{ open }, U \neq \varnothing} U = \{\eta_X\}$$

for some point $\eta_X \in |X|$.

Example 0.13. $\operatorname{Spec} R \ni \eta_R = (0)$ if R is <u>integral</u>.
$D(f) = \{\mathfrak{p} \mid f \notin \mathfrak{p}\}$
A refined version of the earlier proposition:
Let

$$\begin{tikzcd}
Z \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] \arrow[dr] & & Y \arrow[dl, "h"] \\
& X &
\end{tikzcd}$$

be the same setup, but this time with Z integral. Then $f = g$ if and only if $f_\eta = g_\eta$, where f_η denotes the composition $\operatorname{Spec} \underbrace{k(\eta_Z)}_{\mathcal{O}_{Z,\eta}} \to Z \to Y$

<u>Remark:</u> $k(\eta)$ is the function field of Z,

$$k(\eta) = \operatorname{colim}_{U \subset |Z|, U \neq \varnothing} \Gamma(U, \mathcal{O}_U)$$

i.e. its field of generalized rational functions.

Example 0.14. There is an affine scheme which describes arbitrary matrices, referred to as

$$\mathbb{A}^{n\times n}_{\mathbb{Z}} \overset{\text{def}}{=} \operatorname{Spec} \mathbb{Z}[T_{11}, \dots, T_{nn}]$$

And any map $\operatorname{Spec} R \to \mathbb{A}^{n\times n}_{\mathbb{Z}}$ corrresponds to some element of $M_{n\times n}(R)$, the ring of $n \times n$ matrices with elements in R.
The formula which computes the characteristic polynomial is a map $\mathbb{A}^{n\times n}_{\mathbb{Z}} \to \mathbb{A}^n\mathbb{Z}$.
If we take the "universal matrix," i.e. the one with coefficients (T_{ij}), we get a map $\mathbb{A}^{n\times n}_{\mathbb{Z}} \to \mathbb{A}^{n\times n}_{\mathbb{Z}}$, $A \mapsto P_A(A)$.
By restricting to the generic point, we're just dealing with matrices over the function field, $\mathbb{Q}(T_{ij})$. There, we know Cayley-Hamilton is true just by diagonalizing the universal matrix over $\overline{\mathbb{K}}$. It suffices to prove Cayley-Hamilton theorem for fields and diagonalizable matrices.

Lecture 15, 10/10/25

Presentation: Nick from twitter

Lemma 34. *Let* $Y \overset{i}{\hookrightarrow} X$ *be a closed immersion.*
Then $Y \cong Y \times_X Y$.

Proof. Consider the diagram

$$\begin{tikzcd}
Z \arrow[rd, dotted, "h"] \arrow[rrd, bend left, "g"] \arrow[rdd, bend right, "f"'] & & \\
& Y \arrow[r, "\mathrm{Id}_Y"] \arrow[d, "\mathrm{Id}_Y"'] & Y \arrow[d, hook, "i"] \\
& Y \arrow[r, hook, "i"'] & X
\end{tikzcd}$$

By injectivity of i on the level of topological spaces, $i \circ f = i \circ g$ implies $f = g$.
So $h = f = g$.
On the level of stalks, $i^\sharp_p : \mathcal{O}_{X,i(p)} \to \mathcal{O}_{Y,p}$ is surjective for all $p \in Y$.
Further,

$$(f \circ i)^\sharp_Q = (g \circ i)^\sharp_Q : \mathcal{O}_{X,(f\circ i)(Q)} \to \mathcal{O}_{Z,Q}$$

for all $Q \in |Z|$
So $f^\sharp_p = g^\sharp_p$ for all $p \in |Y|$. Thus, we may define $h^\sharp \overset{\text{def}}{=} f^\sharp = g^\sharp$. This establishes the existence of h, and uniqueness is clear. By construction it satisfies the universal property. By universality, $Y \cong Y \times_X Y$.

■

Corollary 0.14. *Closed immersions are separated.*

Proof. This is really what the proof above is saying. ■

<u>Similarly:</u> Open immersions are separated.
<u>Today:</u> We will show that separatedness is closed under composition and base change.

Example 0.15. Here is a non-example: the scheme, constructed some time ago, $\mathbb{A}^1_{\mathbb{Z}\text{, doubled origin}}$, the affine line with two origins, defined as the pushout $\mathbb{A}^1\mathbb{Z} \sqcup_{\mathbb{G}_{m,\mathbb{Z}}} \mathbb{A}^1_{\mathbb{Z}}$ along the identity maps, is not separated.

To see this, there are two inclusions $\mathbb{A}^1_{\mathbb{Z}} \underset{i_2}{\overset{i_1}{\rightrightarrows}} \mathbb{A}^1_{\mathbb{Z}\text{, doubled origin}}$.

Then $i_1|_{\mathbb{G}_{m,\mathbb{Z}}} = i_2|_{\mathbb{G}_{m,\mathbb{Z}}}$, but $i_1 \neq i_2$ because they don't have the same set-theoretic image.
Some auxiliary lemmas first:

Lemma 35. *Given a diagram of schemes*

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & X \end{tikzcd}$$

If g is separated, then there is a natural morphism

$$Z \times_Y Z \xrightarrow{(\mathrm{Id}_Z, \mathrm{Id}_Z)} Z \times_X Z$$

which is a <u>closed immersion</u>.

Example 0.16. Let $X = \operatorname{Spec} \mathbb{Z}$, Y be separated. Then $Z \times_Y Z \hookrightarrow Z \times Z$.

Proof. (of lemma)
Consider the diagram

$$\begin{tikzcd} Z \times_Y Z \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Z \times_X Z \arrow[d] \\ Y \arrow[r, hook, "\triangle_{Y/X}"] & Y \times_X Y \end{tikzcd}$$

The required morphisms exists since a base change of a closed immersion is a closed immersion.

■(?)

Proposition 6. *Let*

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & X \end{tikzcd}$$

be a diagram of schemes as before. Then if f, g are separated, then $h = g \circ f$ is separated.

Proof. First look at the diagonal
$$\begin{tikzcd} Z \arrow[r, "\triangle_{X/Y}"] \arrow[rr, hook, bend right, "\triangle_{Z/X}"'] & Z \times_Y Z \arrow[r, hook] & Z \times_X Z \end{tikzcd}$$
∎

Proposition 7. *Let $f : Y \to X$ and $g : X' \to X$ be morphisms of schemes. Denote the base change $\underbrace{Y \times_X X'}_{=Y'} \to X'$” by f.*

Then if f is separated, then f' is separated:

$$\begin{tikzcd} Y' \arrow[r, "sep"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X' \arrow[d] \\ Y \arrow[r, "sep"] & X \end{tikzcd}$$

Proof. <u>Recall:</u> There is a canonical isomorphism

$$(Y \times_X Y) \times_X X' \simeq Y' \times_{X'} Y'$$

Claim. *This is compatible with diagonals morphisms:*

$$\begin{tikzcd} Y' \arrow[r, "\triangle_{Y'/X}"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Y' \times_{X'} Y' \arrow[d] & \simeq & (Y \times_X Y) \times_X X' \\ Y \arrow[r, hook, "\triangle_{Y'/Y}"] & Y \times_X Y & & \end{tikzcd}$$

Proof. (of the claim)

$$\begin{tikzcd} Y' \arrow[r, hook, "\triangle_{Y'/X'}"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Y' \times_{X'} Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X' \arrow[d] \\ Y \arrow[r, hook, "\triangle_{Y/X}"] & Y \times_X Y \arrow[r] & X \end{tikzcd}$$

Also use Yoneda somehow?

∎(?)

Separatedness is a property of morphisms of schemes, such that

1. Identity morphisms have it

2. Closed under composition

3. Preserved by base change

Remark: Injective maps of schemes don't satisfy (3).

$$\begin{tikzcd}
\operatorname{Spec}\mathbb{C} \sqcup \operatorname{Spec}\mathbb{C} \arrow[r, hook] \arrow[d] & \operatorname{Spec}\mathbb{C} \arrow[d] \\
\operatorname{Spec}\mathbb{C} \arrow[r] & \operatorname{Spec}\mathbb{R}
\end{tikzcd}$$

Yoneda's Lemma

Let $\mathcal{C}$ be a category. Then for all $Y \in \mathcal{C}$. Then there is a functor from $\mathcal{C}^{op} \to \mathsf{Set}$, which we denote by h_Y, given by $x \mapsto \operatorname{Hom}_{\mathcal{C}}(X, Y)$ (the functor represented by x).
This yields another functor $i : \mathcal{C} \to \operatorname{Psh}\mathcal{C}$, where $\operatorname{Psh}\mathcal{C}$ is the category of functors from $\mathcal{C}^{op}$ to Set.
Special case: (of Yoneda's Lemma)
The functor i is fully faithful, meaning it is bijective on Hom sets. This is called the Yoneda embedding.
So we can think of any category $\mathcal{C}$ as a subcategory of $\operatorname{Psh}\mathcal{C}$. If we want to check that a certain diagram is a cartesian diagram, we can test it on the level of functors represented by the diagram:

$$\begin{tikzcd}
S \arrow[dr, dotted, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& P \arrow[r] \arrow[d] & Z \arrow[d] \\
& X \arrow[r] & Y
\end{tikzcd}$$

To check this is cartesian, we can show that for every S, the diagram

$$\begin{tikzcd}
P(S) \arrow[r] \arrow[d] & Z(S) \arrow[d] \\
X(S) \arrow[r] & Y(S)
\end{tikzcd}$$

is cartesian, where $X(S) = h_X(S)$.
The full statement of Yoneda's lemma is that for all $X \in \mathcal{C}$ we have $\operatorname{Hom}_{\operatorname{Psh}\mathcal{C}}(h_X, F) \simeq F(X)$
To go from left to right, for any natural transformation from h_X to F, we can evaluate it at $\operatorname{Id}_X \in h_X(X) = \operatorname{Hom}(X, X) \to F(X)$

Lecture 16, 15/10/25

Reading: Section 2.4 still.

<u>Quiz:</u> Monday, Oct 20 (Ahhhhhhhh)

Lemma 36. *Let*

$$\begin{tikzcd} Z \arrow[rr, "g"] \arrow[dr] & & Y \arrow[dl, "f"] \\ & Z & \end{tikzcd}$$

be a diagram of schemes.
Then if f is separated, then Γ_g, the <u>graph of g</u> is a <u>closed immersion</u>, where $\Gamma_g \overset{\text{def}}{=} (\mathrm{Id}, g) : Z \to Z \times_X Y$, $z \mapsto (z, g(z))$ on the level of sets.

Proof. Volunteer ∎

Last time we stated that properties of morphisms of schemes should:

1. Include identity maps

2. Be closed under compositions

3. Preserved by base change (fibre products)

So if a morphism $Y \xrightarrow{f} X$ has property P, then

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X' \arrow[d] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

should also have property P.
These 3 hold for open immersions, closed immersions, and separated morphisms.
Injective morphisms don't satisfy property 3, i.e. are not preserved by base change.
For example, consider the fibre product

$$\begin{tikzcd} \operatorname{Spec}\mathbb{C} \sqcup \operatorname{Spec}\mathbb{C} \arrow[r] \arrow[d] & \operatorname{Spec}\mathbb{C} \arrow[d] \\ \operatorname{Spec}\mathbb{C} \arrow[r] & \operatorname{Spec}\mathbb{R} \end{tikzcd}$$

<u>Another non-example:</u>

Definition 0.35. A morphism of schemes $f : Y \to X$ is called <u>closed</u> if the underlying continuous map, $|f| : |Y| \to |X|$, is closed.

Lemma 37. *The class of closed morphisms is <u>not</u> closed under fibre products*

Proof. He drew a picture of the graph of $\frac{1}{x}$:
We have $\mathbb{A}^1_k = \operatorname{Spec} k[T] \to \operatorname{Spec} k$. On the right, the underlying space is a singleton, so any map to it is closed. Then we take the base change with the morphism itself:

$$\begin{tikzcd} H \arrow[dr, hook] & & \\ & \mathbb{A}^2_k \arrow[r] \arrow[d] & \mathbb{A}^1_k \arrow[d] \\ & \mathbb{A}^1_k \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

Recall $\mathbb{A}^2_k = \operatorname{Spec} k[S,T]$. The top map is just the projection map given by forgetting one of the variables. We can consider the closed subset of the affine plane given by $H = V\left((ST-1)\right)$. The image of this subset under the projection is $|\mathbb{G}_{m,k}| = \mathbb{A}^1_k \setminus \{0\}$. This is an open subset, but is <u>not</u> a closed subset.

■

This motivates the following definition:

Definition 0.36. A morphism $f : Y \to X$ is called <u>universally closed</u> if, for every other morphism $g : X' \to X$, the base change $Y' = Y \times_X X' \to X'$ is closed.
<u>Remark:</u> One can mimic this definition for any property that does not satisfy (3) above, eg universal injection, universal homeomorphism, etc.
<u>Intuitively speaking:</u> Universally closed morphisms are closely related to compat spaces and proper maps in topology.
Here is a property of quasi-compact spaces in topology:

Lemma 38. *If $f : Y \to X$ is a continuous map from a quasi-compact space Y to a Hausdorff space X, then $f(Y) \subset X$ is a closed subset.*

Proof. Standard point-set fare

■

For universally closed maps and separated morphisms, we have the following analogue:

Proposition 8. *Let*

$$\begin{tikzcd} Z \arrow[rr, "g"] \arrow[dr, "h"'] & & Y \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

be a commuting diagram of schemes, with f separated and h universally closed. Then the image $|g|(|Z|) \subset |Y|$ a closed subset.

Proof. The proof follows from the following consequence of lemma 36:
Under the same assumptions as that lemma, the image $\Gamma_g(|Z|)$ is a closed subset of $|Z \times_X Y|$.
We can now prove the proposition:
Observe that the image of $g(|Z|) = \pi_Y\left(\Gamma_g(|Z|)\right)$, where again Γ_g is given by

$$\begin{tikzcd} Z \arrow[r] \arrow[dr, bend right, "g"'] & Z \times_X Y \arrow[d, "\pi_Y"] \\ & Y \end{tikzcd}$$

(by virtue of definition).
By assumption, we have that the map $Z \to X$ is universally closed, and so its base change $Z \times_X Y \to X$ is a closed morphism. Thus, the image of the graph is also closed.

■

Remark: The same argument also works, mutatis mutandis, in the category of topological spaces.
So there is a definition of universally closed maps and so on, and a space is universally closed if the map $X \to \{*\}$ is universally closed. Using this line of argumentation shows that a continuous map from a universally closed space X to a Hausdorff space Y has closed image.
Using this: A universally closed space is necessarily quasi-compact.
This shows: Universally closed in topology implies quasi-compact topological space.

Example 0.17 (of universally closed morphisms)**.** They are actually not so easy to come by. One way to come up with examples is to consider affine schemes, whose morphisms are given by ring morphisms:

Proposition 9. *Let $f : A \to B$ be a ring homomorphism such that B is a finitely generated A-module (in this case we say that the ring homomorphism is finite).*
Then the map induced by f, $\operatorname{Spec} f : \operatorname{Spec} B \to \operatorname{Spec} A$, is universally closed.

Proof. Possible in class presentation (feel free to skip most of the commutative algebra).

■

Another example:

Lemma 39. *Consider the projective line over any base, $\mathbb{P}^1_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ (resp. $\mathbb{P}^1_{\mathbb{R}} \to \operatorname{Spec} \mathbb{R}$, etc.) is a universally closed morphism.*

Proof. Next time

■

Intuitively, this is a strong contender for universally closed maps because in topology, $\mathbb{P}^1_{\mathbb{C}}$ corresponds to the Riemann sphere $\mathbb{C} \cup \{\infty\}$, which is compact with respect to the standard topology.

Lecture 17, 17/10/25

Presentation: Coleton
<u>Want to show:</u>
If $f : B \to A$ is a ring morphism such that B is a finitely generated A-modules, then the induced map $\operatorname{Spec} f : \operatorname{Spec} A \to \operatorname{Spec} B$ is universally closed.
Recall that an integral map $B \to A$ is a map such that each $a \in A$ is the root of some monic polynomial in B.
If we have an integral map $B \hookrightarrow A$, by the lying over theorem we have $\operatorname{Spec} A \to \operatorname{Spec} B$ is a surjection
Texwriter was bieng weird so i missed the rest sorry coleton

■

Presentation:
Don't know this person's name i'm so sorrrrrry
Let

$$\begin{tikzcd} Z \arrow[r, "g"] & Y \arrow[d, "f"] \\ & X \end{tikzcd}$$

be a diagram of schemes. If f is separated, then $\Gamma_g = (\mathrm{Id}_Z, g)$ is closed.
We resort to a diagram chase:

$$\begin{tikzcd} Z \arrow[r, "\Gamma_g"] \arrow[d, "g"] \arrow[dr, phantom, "\ulcorner", very near start] & Z \times Y \arrow[r, "\Gamma'"] \arrow[d, "(g\circ pr_2, pr_1)"] \arrow[dr, phantom, "\ulcorner", very near start] & Z \arrow[d, "g"] \\ Y \arrow[r, hook, "\triangle_{Y/X}"] & Y \times_X Y \arrow[r] \arrow[d, "pr_2"] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, "f"] \\ & Y \arrow[r] & X \end{tikzcd}$$

■

<u>Reminder:</u> Quiz 3 on monday!

Lemma 40. *The composition of universally closed morphisms is universally closed*

Proof. It's clear that the composition of closed maps is closed, so we must show universality. We have a diagram

$$\begin{tikzcd} Z' \arrow[r, "f'"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Y' \arrow[r, "g'"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X' \arrow[d] \\ Z \arrow[r, "f"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

If f, g are universally closed, then f', g' are closed, so $(g \circ f)'$ is closed. This is true for all possible base changes, so the composition itself is universally closed.

■

Proposition 10. *Consider the diagram*

$$\begin{tikzcd} Z \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

be a diagram of schemes such that g is separated and h is universally closed. Then f is universally closed.

. Remark: This implies the statement we saw last time, but it is even better.

Proof. If g is separated, then $\Gamma_f : Z \hookrightarrow Z \times_X Y$ is a closed immersion.

Then

$$\begin{tikzcd} f = pr_2 = \Gamma_f : Z \arrow[r, hook] & Z \times_X Y \arrow[r, "pr_2"] \arrow[d] & Y \arrow[d] \\ & Z \arrow[r, "h"] & X \end{tikzcd}$$

Then f is a composition of the uniersally closed morphisms pr_2 and Γ_f (closed immersion).

■

Morphisms of finite type

Definition 0.37. A ring homomorphism $\varphi : A \to B$ is said to be of finite type if B is finitely generated as an A algebra.
That is, there are finitely many elements $b_1, \ldots, b_n$ such that $B = \varphi(A)[b_1, \ldots, b_n]$.
Equivalently, B is isomorphic to a quotient of $A[T_1, \ldots, T_n]$, and the map $\varphi : A \to B$ commutes with the embedding of A into the polynomial ring via constant polynomials.
Warning: Do not confuse this with being finite from last time.

Definition 0.38. A ring R is said to be <u>of finite type</u> if the canonical map $\mathbb{Z} \to \mathbb{R}$ (canonical because $\mathbb{Z}$ is initial in Ring) is of finite type.

Example 0.18. 1. $\mathbb{Z} \to \mathbb{Q}$ is not of finite type, intuitively because $\mathbb{Q} = \mathbb{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \dots]$

2. $\mathbb{R} \to \mathbb{C}[S, T, W]/(ST - W)$ is of finite type, because we can think of this as

$$\begin{tikzcd}
R \arrow[rr] \arrow[dr] & & \frac{\mathbb{C}[S,T,W]}{ST-W} \\
& \frac{\mathbb{R}[i,S,T,W]}{ST-W} \arrow[ur, phantom, "\cong" sloped] &
\end{tikzcd}$$

Definition 0.39. A morphism of schemes $f : Y \to X$ is said to be <u>locally of finite type</u> if there exists an affine cover $\{U_i\}_{i \in I}$ of X such that for every $i \in I$ and affine open cover $f^{-1}(U_i) = \bigcup_{j \in J_i} V_j^{(i)}$ such that all the morphisms $\operatorname{Spec} R_j^{(i)} = V_j^{(i)} \to U_i = \operatorname{Spec} R_i$ are of finite type.
In other words, the ring homomorphism $R_i \to S_j^{(i)}$ is a homomorphism of finite type.

Definition 0.40. $f : Y \to X$ is said to be <u>of finite type</u> if it is locally of finite type and quasi-compact.
This is equivalent to assuming the existence of an indexing set J_i which is finite for each i

$$\begin{tikzcd}
\{f^{-1}(U_i)\} \arrow[r] \arrow[d] & \{U_i\} \arrow[d, phantom, "\cap"] \\
Y \arrow[r] & X
\end{tikzcd}$$

Each $f^{-1}(U_i)$ will be covered by finitely many $V_j^{(i)}$. We do not necessarily assume that I is finite because we want it to be well behaved with respect to base change.
This is equivalent to (exercise on assignment 6) asserting that for any morphism from an affine scheme $\operatorname{Spec} R$ to X, the base change is a quasi-compact scheme of finite type over $\operatorname{Spec} R$:

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \operatorname{Spec} R \arrow[d] \\
Y \arrow[r, "f"] & X
\end{tikzcd}$$

Definition 0.41. A scheme X is called <u>locally Noetherian</u> if there exists a cover by affine open subschemes $U_i = \operatorname{Spec} R_i$ with R_i a Noetherian ring for all i.
If there is a finite cover with this same property, then we say that X is a <u>Noetherian scheme</u>, or we simply say that X is <u>Noetherian</u>.

Example 0.19. If R is a Noetherian ring, then $\mathbb{A}^n_{\mathbb{R}} = \mathbb{A}^n_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{R}$ is Noetherian. As a consequence, if we have a closed immersion $X \hookrightarrow Y$. into a Noetherian scheme, then X is Noetherian.
This implies that any scheme of finite type over a Noetherian scheme is Noetherian.

Proposition 11. *Let $f : Y \to X$ be a morphism of finite type of locally Noetherian schemes such that for every other morphism of finite type $g : X' \to X$, the base change*

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is a closed morphism. Then f is universally closed.

Remark: This implies that X' is locally Noetherian.

Proof Sketch:

- Reduce to $\underbrace{X' \to X}_{\text{arbitrary base change}}$ being a morphism of affine schemes

$$\begin{tikzcd} X' \arrow[d, "g"'] & = & \operatorname{Spec} S \arrow[d] \\ X & = & \operatorname{Spec} R \end{tikzcd}$$

  to $S = \bigcup_{S_i/R \text{ of finite type}} S_i$
- Reduce to $\operatorname{Spec} S =$inverse limit of $\operatorname{Spec} S_i$.
- Some other shit
- Then the same will hold for the base change.

Lecture 18, 20/10/25

<u>Last time:</u>

Claim. *For finite type morphisms between Noetherian schemes, $f : Y \to X$, it suffices to check universal closedness along $g : X' \to X$ of <u>finite type</u>.*

Proof. We want to show that for Z an arbitrary scheme, and for $g : Z \to X$ an arbitrary morphism, the base change $Y \times_X Z \to Z$ is <u>closed</u>, assuming this is true for all finite type maps.
Without loss of generality, let $X = \operatorname{Spec} R, Z = \operatorname{Spec} S$, $\varphi : R \to S$.

1. Let $S = \cup R'$, where R' is of finite type over R. S can be exhausted by subrings R' of finite type over R.

2. $|\operatorname{Spec} S| = \lim_{\leftarrow R'/Rf.t} \operatorname{Spec} R'$

We have a map $S \supset \mathfrak{p} \mapsto (\mathfrak{p}' = \mathfrak{p} \cap R')$, and we have $(\mathfrak{p}')_{R'} \mapsto \mathfrak{p} = \bigcup_{R'/R} \mathfrak{p}'$.

<u>Variant:</u> $|Y \times_{\operatorname{Spec} R} \operatorname{Spec} S| \cong \lim_{\leftarrow} |Y \times_{\operatorname{Spec} R} \operatorname{Spec} R'|$

3. $|Y \times_{\operatorname{Spec} R} \operatorname{Spec} R'| \to |\operatorname{Spec} R'|$ is closed for every $R' \subset S$ of finite type over R.

Taking the inverse limit, we get the closed map

$$\lim_{\leftarrow} |Y \times_{\operatorname{Spec} R} \operatorname{Spec} R'| \cong |Y \lim_{\leftarrow} |Y \times_{\operatorname{Spec} R} \operatorname{Spec} R'| \to \lim_{\leftarrow} |\operatorname{Spec} R'| \cong |\operatorname{Spec} S|$$

∎

Chevalley's theorem on constructibility

Theorem 0.15 (Chevalley)**.** *Let $f : Y \to X$ be a morphism of finite type between Noetherian schemes. Then $f(|Y|)$ is a <u>constructible set</u>, i.e. there exist finitely many subsets $Z_i \subset |X|$ such that $f(|Y|) = \bigcup Z_i$ and $Z_i \hookrightarrow\!\!\circ\!\!\to \overline{Z_i}$.*

Proof.

Definition 0.42. A subset Z_i is <u>locally closed</u> if it is the intersection of an open subset with a closed subset.

Example 0.20. Here is a constructible subset $C \subset \mathbb{A}_k^2$ which is neither open, closed, or locally closed.

Let $C = \underbrace{|\mathbb{A}_k^2 \setminus \mathbb{A}_k^1|}_{\text{open}} \cup \underbrace{\{0\}}_{\text{closed}}$.

Claim. *C is the image of a morphism $f : \mathbb{A}_k^2 \to \mathbb{A}_k^2$*

Proof. $f(x, y) = (x, xy)$

∎

Let's now prove the theorem:

Proof. The proof will use

- Spreading out strategy
- Noetherian induction

and is based (at least) on the following two lemmas:

Lemma 41. *Let $f : Y \to X$ be of finite type, with X (and thus Y) Noetherian. Assume that $f^{-1}(\eta) = \varnothing$ where $\eta \in |X|$ is a generic point. Then there is an open neighborhood of η such that $f^{-1}(U) = \varnothing$.*

Proof. $f^{-1}(\eta) = Y_\eta = Y \times_X \operatorname{Spec} \mathcal{O}_{X,\eta} = \varnothing$.
Without loss of generality, we may assume that Y and X are affine, meaning $X = \operatorname{Spec} R$ for some Noetherian ring, and $Y = \operatorname{Spec} S$. So $S/R = R[T_1, \dots, T_n]/I$ for some ideal $I = (f_1, \dots, f_m)$.
The assumption $Y \times_X \operatorname{Spec} \mathcal{O}_{X,\eta} = \varnothing$ is equivalent to $\big(R[T_1, \dots, T_n]/I\big) \otimes_R \mathcal{O}_\eta = 0$.
This is again equivalent to $\mathcal{O}_\eta I \subset \mathcal{O}_\eta[T_1, \dots T_n]$ is the unit ideal.
This is again equivalent to the existence of classes $g_1, \dots, g_n \in \mathcal{O}_\eta[T_1, \dots, T_n$ such that

$$\sum_{i=1}^{n} g_i f_i = 1 \in \mathcal{O}_\eta[T_1, \dots, T_n]$$

So there is some element $h \in R$ such that $g_i \in R_h$ and the above equation holds in R_h.
This implies the base change $R[T_1, \dots, T_n]/(R_i I) = 0$, and thus $f^{-1}\big(D(h)\big) = \varnothing$ ∎

Lemma 42. *If the generic point lies in the image, then there is an open subset which also lies in the image.*

Proof. Next time ∎

Lecture 19, 22/10/25

We continue the proof of the theorem.
Recall: Let X be a Noetherian scheme. A subset $C \subset |X|$ is called constructible if it is a finite union of locally closed subsets, where locally closed means the intersection of an open and a closed.
Remark: If we denote by $\operatorname{Const}(X)$ the constructible sets, then it is the smallest subset of $\mathcal{P}(X)$, the power set of X, with the properties that

- It contains the topology (ie all opens)
- It is closed under finite unions
- It is closed under finite intersections

Theorem 0.16 (Chevalley)**.** *Let $Y \to X$ be a morphism of finite type, Y and X Noetherian. Then $f(|Y|) \subset |X|$ is constructible.*

. Remark: Let $f : k^n \to k^n$ be a polynomial map.
Then the image of f, $f(k^n) = \{y \in k^n \mid \exists x \in k^n, f(x) = y\}$, by this theorem, can be described without using the existence quantifier.
That is, we can re-write

$$\{y \in k^n \mid \exists x \in k^n, f(x) = y\} = \{y \in k^n \mid g_i(y) = 0 \text{ and/or } g_i(y) \neq 0\}$$

where $g_1, \dots, g_r$ is some collection of polynomials depending on f. (I might have written this incorrectly).
This is called quantifier elimination.
Now for the proof of Chevalley

Proof. Last time, we proved the following lemma:
Under the standing assumptions about $f : Y \to X$, if the generic point $\eta_X \notin f(|Y|)$, then there exists an open $U \ni \eta$ such that $U \cap f(|Y|) = \varnothing$, which is equivalent to the assertion that $f^{-1}(U) = \varnothing$.
We now prove another lemma:

Lemma 43. *Assume $\eta_X \in f(|Y|)$. Then there exists an open. $U \ni \eta_X$ such that $U \subset f(|Y|)$.*

Proof. As noted before, the proof uses two strategies:

- Spreading-out
- Noetherian induction

Spreading-out We take a Noetherian ring R, with $R = \operatorname{colim} R_i = \lim_{\leftarrow, i \in I} R_i$, meaning R is assumed to be a colimit over some collection $\{R_i\}_{i \in I}$
Let $R \to S$ be an R-algebra of finite type, i.e. $S \cong R[T_1, \dots, T_n]/I$, a quotient of the polynomial ring with finitely many generators by some ideal.

Lemma 44 (A)**.** *There exists $i \in I$, and some ring S_i such that there exists a ring homomorphism $R_i \to S_i$, such that*

$$S \cong S_i \otimes_{R_i} R$$

Proof. Geometrically: Applying Spec, we can express $\operatorname{Spec} S$ as a base change

$$\begin{tikzcd} \operatorname{Spec} S \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \operatorname{Spec} R \arrow[d] & = & \lim_{\leftarrow} \operatorname{Spec} R_i \\ \operatorname{Spec} S_i \arrow[r] & \operatorname{Spec} R_i & & \end{tikzcd}$$

Proof. Volunteer

■

Lemma 45 (B). *Assume that $R \to S$ is a finite morphism (i.e. S is finitely generated as an R-module (meaning NOT necessarily as an algebra!)).*
Then our morphism $R_i \to S_i$ can be chosen to be finite

Proof. Volunteer

∎

We can now prove the earlier Lemma (Lemma 43 in these notes)

Proof. Assume without loss of generality that $X = \operatorname{Spec} R, Y = \operatorname{Spec} S$ are affine.
Let η_X be a generic point contained in $f(|Y|)$.
For our purposes, we start by assuming that X is integral.
Then $\mathcal{O}_{\eta_X} = \kappa(\eta_X)$, the residue field at η_X, the function field of X. We have the following:

$$\begin{tikzcd} Y_{\eta_X} \arrow[r, "f.t"] \arrow[d] & \operatorname{Spec} \kappa(\eta_X) \arrow[d] \\ Y \arrow[r, "f.t"] & X \end{tikzcd}$$

By assumption, $Y_{\eta_X} \neq \varnothing$, hence there exists a finite field extension $\kappa' \mid \kappa$ such that there is a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} \kappa' \arrow[r] \arrow[d] & \operatorname{Spec} \kappa \arrow[d] & = & \lim_{\leftarrow U \ni \eta_x} \mathcal{O}_X(U) \\ Y \arrow[r] & X \end{tikzcd}$$

Applying the spreading-out lemma B, we obtain some affine-open $U \ni \eta_X$ and finite morphism $\tilde{U} \to U$ such that there is a commutative diagram

$$\begin{tikzcd} \tilde{U} \arrow[r, "\text{finite}"] \arrow[d] & U \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

$\tilde{U} \to U$ being closed and dominant (meaning the generic point η_U is contained in its image) implies that $\tilde{U} \to U$ is surjective.
This shows that U is entirely contained in $f(|Y|)$.

∎

Now: Noetherian induction

Let $U_0 \subset |Y|$ be an open subset with the property that $f(|U_0|)$ is constructible.

Claim. *There exists a nonempty U_0 with this property*

Proof. There are two cases:

1. If η_X, the generic point in X, is contained $f(|Y|)$, then we may apply the second lemma to get a $U \subset f(|Y|)$, and then $f^{-1}(U) \subset Y$ works.

2. If $\eta_X \notin f(|Y|)$, then there exists a closed subset $|Z| \subset |X|$ such that $f(|Y|) \subset |Z|$ (as a consequence of the first lemma from last time).

It follows that there exists a closed subscheme Z with $f : Y \to Z$ being a well-defined morphism of schemes.

Eventually, you can assume that $\eta_Z \in f(|Y|)$ by the first case.

If $U_0 \neq Y$, then there is a $y \in Y \setminus U_0$ and an open neighborhood $V \ni y$ such that (V) is constructible (by the cases 1 and 2 applied to some open neighborhood).
We can now take $U_1 = U_0 \cup V$, and repeat this process: assume by contradiction that the process never stops. Then we get an infinite sequence

$$U_0 \subsetneq U_1 \subsetneq \cdots$$

of open sets. Let $U = \bigcup_{n \in \mathbb{N}} U_n$.
But $|X|$ is Noetherian, so U is quasicompact, so this is a contradiction. Hence, there are only finitely many.
Eventually we get a $U_m = |Y|$, which implies that $f(|Y|)$ is constructible.

$\blacksquare$

This is Noetherian induction.

Lecture 20, 24/10/25

Remark on a remark: Last time, we remarked on quantifier elimination as a consequence of Chevalley, but k being algebraically closed is an essential assumption for quantifier elimination to work, which was neglected last time.
Consider a polynomial map $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$. By definition this is a map from k^n to k^n:

$$\begin{tikzcd} \mathbb{A}^n_k \arrow[r] & \mathbb{A}^n_k \\ \| & \\ k^n \arrow[r, "f"] & k^n \end{tikzcd}$$

Let $k = \mathbb{R}$, and $f(x) = x^2$. Then the image of f is all nonnegative reals, $\mathbb{R}_{\geq 0}$.

The scheme-theoretic image is different: the map $\mathbb{A}^1_{\mathbb{R}} \xrightarrow{f} \mathbb{A}^1_{\mathbb{R}}$ is surjective: if we have a concrete $\mathbb{R}$-point given by the maximal ideal $(z - x)$ for a point $x \in \mathbb{R}$, we compute the base change

$$\begin{tikzcd} {} \arrow[r] \arrow[d] & \operatorname{Spec} \mathbb{R} \arrow[d, "x"] \\ \mathbb{A}^1_{\mathbb{R}} \arrow[r, "f(z)=z^2"] & \mathbb{A}^1_{\mathbb{R}} \end{tikzcd}$$

$$\mathbb{R}[z] \otimes_{\mathbb{R}[w]} \frac{\mathbb{R}[z]}{(z-x)} \cong \frac{\mathbb{R}[z]}{(z^2-x)}$$

Because the $\mathbb{R}$-point $(z^2 - x) = (z - \sqrt{x})(z + \sqrt{x})$ can be factored over the reals, so it breaks into two real points.
If $x < 0$, then $z^2 - x$ is prime: otherwise, it breaks into 2 real points.
Warning: In the above there has been some abuse of notation, namely conflation of zs...
The point is that no matter what x is it's a nonzero ring, so there's a prime ideal for every point, I guess?
This mape is finite and dominant, and thus surjective (dominant means the generic point is in the image).

Discrete valuation rings

What is a dvr?

Definition 0.43. A discrete valuation ring (or dvr) is a ring R together with a non-zero map $\nu : R \to \mathbb{N} \cup \{\infty\}$ (where here, $0 \in \mathbb{N}$), called a *valuation* such that the following holds:

1. $\nu(x) = \infty \iff x = 0$

2. $\nu(x) + \nu(y) = \nu(xy)$ (using the obvious extension of addition on $\mathbb{N} \cup \{\infty\}$)

3. $\nu(x + y) \geq \min(\nu(x), \nu(y))$

Example 0.21.

1. $\mathbb{Z}_{(p)}$, obtained by inverting every prime except p, is a discrete valuation ring. The valuation ν is given by $\nu_p : \mathbb{Z} \to \mathbb{N} \cup \{\infty\}$, defined by

$$\max\{k : p^k \mid n\}$$

It is a simple exercise that this evaluation can be extended to a valuation from $\mathbb{Z}_{(p)}$

2. Consider $\mathcal{O}^{hol}_{\mathbb{C},z_0}$, the ring of germs of holomorphic functions near z_0. This is a discrete valuation ring. The valuation is given as follows. Let $f : U \to \mathbb{C}$ be holomorphic, where $U \ni z_0$. Then near z_0, f can be written as $(z - z_0)^{\nu(f)} \cdot h$, where $h(z_0) \neq 0$. We define $\nu(f)$ to be the thing in the exponent that makes this true, i.e. the vanishing order of f at z_0.

We can do a similar procedure with the meromorphic functions, due to the general fact that any valuation ν will extend uniquely to a valuation from the fraction field of R to $\mathbb{Z} \cup \{\infty\}$ (where now we allow integral values and not just natural). Note the fraction field exists because the dvr condition forces R to be an integral domain.

3. For a field k, $m \in k[T]$

$$\mathcal{O}_{\mathbb{A}^1_k, m} = k[T]_m$$

In commutative algebra, one shows the following characterization of dvr's:

Theorem 0.17 (Characterization of discrete valuation rings)**.** *The ring R is a dvr if and only if it is a local Noetherian integral domain that is normal and of Krull dimension 1.*
Normal means that $R \subset \operatorname{Frac} R$ is integrally closed

Proof. ■

Corollary 0.18. *Let R be Noetherian, integral, and normal, and let $\mathfrak{p} \in \operatorname{Spec} R$ be a prime ideal of codimension 1 (meaning $R_{\mathfrak{p}}$ has Krull dimension 1).*
Then the localization $R_{\mathfrak{p}}$ is a dvr. This is exactly what has happened in example 3 above.

Proof. ■

Geometric interpretation: It is the "germ of a (smooth) curve."
Facts from commutative algebra: If R is a dvr, then

- R is a PID
- A generator π of the maximal ideal $\mathfrak{m}_R \subset R$ is called the uniformizer
- Every ideal $I \subset R$ is of the shape $\mathfrak{m}^n$ for some $n \in \mathbb{N}$.
- For any $x \in R \setminus \{0\}$, we have

  $$\underbrace{\nu(x)}_{\text{normalized valuation}} = \min\{n \in \mathbb{N} \mid x \notin \mathfrak{m}^{n+1}\}$$

  In other words, x is the ν-th power of π times c, where $c \in R^* \iff \nu(c) = 0$.
- $\nu : K = \operatorname{Frac} R \to \mathbb{Z} \cup \{\infty\}$ can be recovered by taking the preimage $R = \nu^{-1}(\mathbb{N} \cup \{\infty\})$

Proposition 12 (Valuation criterion for closedness)**.** *Let X be a Noetherian scheme and $C \subset |X|$ a constructible subset. Then C is closed if and only if for every dvr R and every scheme morphism $f : \operatorname{Spec} R \to X$ such that $f(|\operatorname{Spec} \operatorname{Frac} R|) \subset C$, we have that $f(|\operatorname{Spec} R|) \subset C$*

Proof. Next time

Lecture 21, 27/10/25

Correction:
A ring R is called a dvr if it hasa valuation $\nu : R \to \mathbb{N} \cup \{\infty\}$ with the properties defined last time, and $x \in R^\times \iff \nu(x) = 0$.
Remark This last property implies that R is local with the maximal ideal

$$\mathfrak{m} = \nu^{-1}(\{x \in \mathbb{N} \mid x \geq 1\})$$

Valuation criterion (for closedness)

Proposition 13. *Let X be a Noetherian scheme, with $C \subset |X|$ constructible. Then $C = \overline{C}$ if and only if for every dvr R, with fraction field K and every morphism $f : \operatorname{Spec} R \to X$ with $f(|\operatorname{Spec} K|) \subset C$, we have $f(|\operatorname{Spec} R|) \subset C$.*

Proof. Without loss of generality, we may assume that C is dense in X, by replacing X with $\overline{C}$ and taking the underlying reduced subscheme.

First, assume that the valuative criterion holds, i.e. assume that for every dvr R and any scheme morphism $f : \operatorname{Spec} R \to X$, with $f(|\operatorname{Spec} K|) \subset C$, we have $f(|\operatorname{Spec} R|) \subset C$. Our goal is to show that $C = |X|$.

By constructibility and density, C contains an open $U \subset |X|$. Since we may assume without loss of generality that $X = \operatorname{Spec} A$ is affine, and we may also assume that U is standard-open, i.e. $U = D(\lambda) \subset C$ for some $\lambda \in A = \mathcal{O}_X(|X|)$.
Consider a prime ideal $\mathfrak{p} \in |\operatorname{Spec} A|$ containing the element λ, which is minimal with respect to this property.
By construction, $\mathfrak{p} \notin D(\lambda)$. By some standard commutative algebra, $\dim A_\mathfrak{p} = 1$, and hence

$$D(\lambda) = |X| \setminus Z(\lambda)$$

The geometric explanation is that a hypersurface has codimension 1.

After replacing A by its normalization, we obtain that $A_\mathfrak{p}$ is a dvr (integral, normal, and local ring of krull dimension 1).
Since $\operatorname{Frac} A_\mathfrak{p} = \operatorname{Frac} A$, and $\operatorname{Frac} A_\mathfrak{p} \subset D(\lambda) \subset C$, we get a contradiction with the hypothesis of the valuative criterion:

$$\begin{tikzcd}
\operatorname{Spec} \operatorname{Frac} A_\mathfrak{p} \arrow[r] \arrow[d, phantom, "\cap"] & C \arrow[d] & \\
\operatorname{Spec} A_\mathfrak{p} \arrow[r] \arrow[ur, dotted] & X \arrow[r, phantom, "="] & \operatorname{Spec} A
\end{tikzcd}$$

So $\mathfrak{p}$ is also in C. This contradicts that $\mathfrak{p} \notin D(\lambda) \subset C$
Actually this is not enough: <u>Let's get back to this later:</u>
"Trust me it's correct, it's in $\overline{\text{EGA}}$" - Michael

Now let's prove the other direc tion.
Let's prove that $C \subset |X|$ is closed implies that the conclusion of the valuation criterion holds.

For a Zariski-continuous $f : \overbrace{|\operatorname{Spec} R|}^{f^{-1}(C)\subset} \to \overbrace{|X|}^{C\subset}$, we have $|\operatorname{Spec} R| = \{\mathfrak{m}, (0)\}$, $\underline{\{\mathfrak{m}\}}$ is <u>closed</u> and (0) is open, so the topology τ is given by $\tau = \{(0), |\operatorname{Spec} R|\}$, so $\overline{(0)} = \operatorname{Spec} R$ (general point property).
$\{(0)\} = D(\pi) = |\operatorname{Spec} R_\pi| = \operatorname{Spec} K$. Since $f(|\operatorname{Spec} K|) \subset C \implies (0) \in f^{-1}(C)$.
$f^{-1}(C)$ is closed implies $f^{-1}(C) = |\operatorname{Spec} R| \implies f(|\operatorname{Spec} R|) \subset C$.

■?

Theorem 0.19. 1. *Let $f : Y \to X$ be a morphism of Noetherian schemes of finite type. Then f is <u>universally closed</u> if for every dvr R and every diagram*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & Y \arrow[d] \\ \operatorname{Spec} R \arrow[r] & X \end{tikzcd}$$

there exists a lift

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & Y \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dotted] & X \end{tikzcd}$$

so that the diagram commutes.

2. *Let $f : Y \to X$ be as before. Then f is separated if and only if for every dvr R and every diagram*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & Y \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, shift left, "h"] \arrow[ur, shift right, "g"'] & X \end{tikzcd}$$

we have $g = h$. That is, if a dashed arrow exists, it is unique.

Proof. We will prove (1) $\implies$ (2) in the following way, by applying statement 1 to $\triangle_{Y/X}$. Recall $\triangle_{Y/X} : Y \to Y \times_X Y$. This is always a locally closed immersion, i.e. it factors as the composition of an open immersion with a closed immersion, or equivalently for every point $z \in Y \times_X Y$, there exists an open neighborhood $Y \times_X Y \supset U \ni z$ such that the base change $\triangle_{Y/X}(U) \hookrightarrow\!\!\!\!\!/\; U$ is a closed immersion.

This follows from the fact that morphisms of affine schemes are separated. If we take $z \in \triangle_{Y/X}(|Y|) = |Y|$ and choose an affine-open neighborhood $\operatorname{Spec} R \ni z$ mapping to $\operatorname{Spec} S \subset X$,

Then We can look at the diagram

$$\begin{tikzcd}
\operatorname{Spec} R \arrow[r, hook, "\triangle_{\operatorname{Spec} R/\operatorname{Spec} S}"] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & \operatorname{Spec}(R \otimes_S R) \arrow[d, hook] \\
Y \arrow[r] & Y \times_X Y
\end{tikzcd}$$

Remark: Note that $f : Y \to X$ being of finite type gives that $\triangle_{Y/X}$ is of finite type. Now, in one direction, if f is separated, and we have two maps h, g

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & Y \arrow[d, hook, "\triangle_{Y/X}"] \\
\operatorname{Spec} R \arrow[r] \arrow[ur, dotted, shift left, "h"] \arrow[ur, dotted, shift right, "g"'] & Y \times_X Y
\end{tikzcd}$$

Continued next time

Lecture 22, 29/10/25

Let everything in sight be a Noetherian scheme or a finite type morphism, or a dvr.

$$\operatorname{Frac} R \quad = \quad \begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\
\operatorname{Spec} R \arrow[r] \arrow[ur, dotted] & X
\end{tikzcd}$$

Valuation criterion

1. The existence of the dotted morphism is equivalent to f being universally closed

2. Uniqueness of the dotted morphism is equivalent to f being separated

3. The unique existence of the dotte morphism is equivalent to f being proper (which means to be of finite type, separated, and universally closed).

Clearly (3) follows just from combining (1) and (2).
Last time, we showed that (2) is a consequence of (1) (applied to $\triangle_{Y/X}$ rather than f).
Also last time, it was explained that if $f : Y \to X$ is of finite type, then $\triangle_{Y/X}$ is of finite type as well.

$\triangle_{Y/X}$ is a locally closed immersion. That is, we can write $\triangle_{Y/X}$ as a composition of an open immersion with a closed immersion:

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[dr, "\triangle_{Y/X}"] & U \arrow[d, hook] \\ & Y \times_X Y \end{tikzcd}$$

Theorem 0.20. *$\triangle_{Y/X}$ is a closed immersion if and only if $\triangle_{Y/X}$ is universally closed.*

Proof. Consider the commuting square

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & Y \arrow[d, "\triangle_{Y/X}"] \\ \operatorname{Spec} R \arrow[r, "{(g,h)}"] \arrow[ur, dotted] & Y \times_X Y \end{tikzcd}$$

The existence of the dotted arrow is equivalent to $g = h$:

$$\begin{tikzcd} \operatorname{Spec} R \arrow[dr, "{(g,h)}"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "g"'] & & \\ & Y \times_X Y \arrow[r] \arrow[d] & Y \arrow[d] \\ & Y \arrow[r] & X \end{tikzcd}$$

Since $\triangle_{Y/X}$ is separated, there can be at most 1 dotted arrow

$$\begin{tikzcd} \operatorname{Spec} R \arrow[r, dotted] \arrow[dr] & Y \arrow[d, "\triangle_{Y/X}"] \\ & Y \times_X Y \end{tikzcd}$$

This is how (2) follows from (1), so clearly the hard work happens in the first part:
First, assume that f is universally closed.

Goal: The existence of a dotted line for a given diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & Y \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dotted] & X \end{tikzcd}$$

Replacing X with the fibre product $Y \times_X \operatorname{Spec} R$, we may assume without loss of generality the base $X = \operatorname{Spec} R$.
So we have a universally closed morphism f:

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r, phantom, "="] \arrow[ur, dotted] & \operatorname{Spec} R \end{tikzcd}$$

Replace Y by $\overline{g(|\operatorname{Spec} K|)} = Z$
Now we have

$$\begin{tikzcd} Z \arrow[r, hook, "/" marking] \arrow[d] & Y \arrow[dl, "\text{universally closed}"] \\ \operatorname{Spec} R & \end{tikzcd}$$

Now: $f : Z \to \operatorname{Spec} R$ is universally closed and sominant (meaning $\eta_{\operatorname{Spec} R}$ is in the image).
So Z is irreducible and hence integral.
This implies f is surjective, so there is a point $x \in Z$ such that $f(x) = \mathfrak{m}_R \in |\operatorname{Spec} R|$
Now replace Z by $\operatorname{Spec}(\underbrace{\mathcal{O}_{Z,x}^{norm}}_{A})$, where we take the normalizer, which is the integral closer in the fraction field, which is just the fraction field of Z.
By construction, A is a subring of the fraction field of its fraction field, which is also K. By commutativity of its diagram, $A \supset R$, but $A \neq K$.
This implies $A = R$ since R is a dvr.
Now for the other direction:
Assume the existence of the dotted arrow. We want to show this implies that f is universally closed.

First step: We show that f is closed.

$$\begin{tikzcd} Y \arrow[d, "f"] \\ X \end{tikzcd}$$

By Chevalley, $f(|X|) = C \subset Y$ is constructible.
Assume for contradiction that $C \neq \overline{C}$.
Then by the valuation criterion for closedness, there exists a dvr R and a morphism $g : \operatorname{Spec} R \to X$ such that $g(|\operatorname{Spec} K|) \subset C$ but $g(|\operatorname{Spec} R|) \subsetneq C$.
So the crossed out map might not a priori exist:

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r, "\times" marking] & Y \arrow[d, "f"] \\
\operatorname{Spec} R \arrow[r] & X
\end{tikzcd}$$

Replace K by K'/K finite, R by R', the integral closure of R in K'.
Since R'/R is integral, by coleton's lemma this shows that $\operatorname{Spec} R' \to \operatorname{Spec} R$ is dominant, hence closed and surjective.
By taking the maximal ideal in R, replace R' by $(R'_{\mathfrak{p}})^{norm}$.
This is a dvr.
<u>Now:</u>

$$\begin{tikzcd}
\operatorname{Spec} K' \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\
\operatorname{Spec} R' \arrow[r] \arrow[ur, dotted] & X
\end{tikzcd}$$

We get a lift

$$\begin{tikzcd}
\operatorname{Spec} R' \arrow[r] \arrow[d, two heads] & Y \arrow[d] \\
\operatorname{Spec} R \arrow[r] & X
\end{tikzcd}$$

So $C = g(|\operatorname{Spec} R|)$, a contradiction.

∎

On the valuative criterion for closed subsets

Proposition 14. *Let A be a local and integral Noetherian ring (which is not a field). Then there exists a dvr R such that $\operatorname{Frac} R \cong \operatorname{Frac} A$ and a morphism of rings $A \to R$ such that the maximal ideal $\mathfrak{m}_A$ lies in the image $\operatorname{Spec} R \to \operatorname{Spec} A$.*

Proof. We will give this later. First, a Lemma:

Lemma 46. *Let A be as before. There exists $a \in A$ such that $a^n \notin \mathfrak{m}^{n+1}$ for all $n \in \mathbb{N}$.*

Example 0.22. Assume that A is a polynomial ring with two generators, $A = k[x,y]_{(x,y)} \supset (x,y) = \mathfrak{m}$.
It is clear (arguing with degrees) that the $x^n \notin \mathfrak{m}_A^{n+1} = (x^{n+1}, y^{n+1}, xy^{n+1}, \dots)$
Now let's prove the lemma:

Proof. The ideal $\mathfrak{m}_A = (x_1, \dots, x_r)$ is finitely generated (since A is Noetherian).
Assume by contradiction that for every generator i there exists n_i such that $x_i^{n_i} \in \mathfrak{m}_A^{n+i+1}$.
Then: $\mathfrak{m}_A^n = \mathfrak{m}_A^{n+1}$, where $n = \sum n_i$.
Why?

$$\mathfrak{m}_A^n = \left(\prod_{i=1}^{r} x_i^{e_i}, \sum e_i = n \right)$$

So there exists an index i such that $e_i \geq n_i$ (otherwise we would not have this property), and thus $x_i^{e_i} \in \mathfrak{m}^{e_i+1}$.
Thus $\prod_{i=1}^r x_i^{e_i} \in \mathfrak{m}_A^{n+1}$
By Nakayama's lemma, $\mathfrak{m} = 0$, a contradiction.

■

We can now prove the proposition:

Proof of prop: Let A and a be as before. Define

$$B \stackrel{\text{def}}{=} A \cup a^{-1}\mathfrak{m} \cup a^{-2}\mathfrak{m} \cup \cdots \subset \operatorname{Frac} A$$

Claim. *B is a Noetherian Ring*

Proof. The ring property is clear. Noetherianness is a consequence of A being Noetherian, which implies that $\mathfrak{m}$ is finitely generated, which implies that B is of finite type over A.

■

Claim. $a \notin B^\times$

Proof of claim: Assume by contradiction that there exists $x \in B$ such that $ax = 1$.
Then x would be of the shape

$$x = \frac{y}{a^n}$$

where $y \in \mathfrak{m}^n$.
So the equation

$$a\left(\frac{y}{a^n}\right) = 1$$

just impjlies that $ay = a^n$, and so the left hand side is in $\mathfrak{m}^{n+1}$, implying $a^n \in \mathfrak{m}^{n+1}$, contradicting our choice of a.

■

Claim. $B\mathfrak{m} = Ba$

Proof of claim:

$$B\mathfrak{m} = \underbrace{A\mathfrak{m}}_{\mathfrak{m}} \cup a^{-1}\mathfrak{m}^2 \cup a^{-2}\mathfrak{m}^3 \cup \cdots$$

So

$$Ba = \underbrace{A \cdot a}_{\in\mathfrak{m}} \cup \mathfrak{m} \cup a^{-1}\mathfrak{m}^2 \cup a^{-2}\mathfrak{m}^3 \cup \cdots$$

■

Now: Let $\mathfrak{q} \subset B$ be a prime ideal containing a which is minimal with respect to this property.
By the first claim, $\dim B\mathfrak{q} = 1$.
Thus $B\mathfrak{q}^{norm}$ is a dvr
It only remains to be shown that

$$\mathfrak{q} \cap A = \mathfrak{m}$$

But $\mathfrak{q} \cap A \supset \mathfrak{m}$ which is maximal, so $\mathfrak{q} \cap A = \mathfrak{m}$, so this is our desired dvr.

■

Lecture 23, 31/10/25

Presentation: Jo(h?)nathan
This is a lemma from a while ago

Proposition 15. *Let*

$$R = \lim_{\rightarrow} R_i$$

be a noetherian ring expressed as a filtered colimit, and S a finitely presented R-algebra, meaning the quotient of some polynomial algebra by an ideal which is finitely generated. This is automatic for Noetherian rings, but this proof does give a slightly more general argument.
Then there is some $N \in \mathbb{N}$, some S_N, and some morphism $R_N \to S_N$ such that $S_N \otimes_{R_N} R \cong S$

Proof. Consider the diagram

$$\begin{tikzcd}
R_1 \arrow[r] \arrow[d, "f_1"'] & R_2 \arrow[r] \arrow[dl, "f_2"] & R_3 \arrow[r] \arrow[dll, "f_3"] & \cdots \\
R
\end{tikzcd}$$

$R = \lim_{\rightarrow} R_i = R_1 \subset R_2 \subset R_3 \subset \cdots$ We want to show that for some I, $S \cong R[x_1, \ldots, x_r]/I$.
But R is Noethereian, so any $I = \langle f_1, \ldots, f_\ell \rangle \subseteq R[x_1, \ldots, x_r]$ is finitely generated.

Take N large enough so that the set of coefficients of all f_1 through f_N are in R_N.
We can always do this by Noetherianness and other stuff.
Take $S_N = R_N[x_1, \dots, x_N]/\langle f_1, \dots, f_\ell\rangle$
Then $S_N \otimes_{R_N} R = R[x_1, \dots, x_r]/\langle f_1, \dots, f_\ell\rangle$.

■

This is another example of "spreading out."
Now, lets finish the last steps of the proof of the valuation criterion of closedness:

Proposition 16. *For every local, Noetherian, and integral ring A there exists a dvr R and a ring homomorphism $A \to R$ such that $\operatorname{Spec} R \to \operatorname{Spec} A$ is dominant and $\mathfrak{m}_A$ lies in its image.*

Remark on the proof: The algebra B used in the proof,

$$B = A \cup a^{-1}\mathfrak{p} \cup a^{-2}\mathfrak{p}^2 \cup \cdots \subset \operatorname{Frac} A$$

where $a \in \mathfrak{m}_A$ satisfies $a^n \notin \mathfrak{m}_A^{n+1}$, is related to the blowup of $\operatorname{Spec} A$ in $\mathfrak{m}_A \in \operatorname{Spec} A$.
Picture: $A = K[X,Y]_{(X,Y)}$. Then the blowup, $BL_{\mathfrak{m}_A} \to \operatorname{Spec} A$, is a particular proper morphism which embeds into $\mathbb{P}^1_A = \mathbb{P}^1_K \times_{\operatorname{Spec} K} \operatorname{Spec} A \to \operatorname{Spec} A$.

$$\begin{tikzcd}
 & \mathbb{P}^1_K \times_{\operatorname{Spec} K} \operatorname{Spec} A \arrow[d, "(pr\times pr)"] & = & \mathbb{P}^1_A \\
BL_{\mathfrak{m}_A} \arrow[ur, hook, "|" marking] \arrow[r] & \operatorname{Spec} A & \subset & \mathbb{A}^2_K \\
BL_{\mathfrak{m}_A} \times_{prA} \operatorname{Spec} A \arrow[u, hook, "|" marking] \arrow[r, "\cong"] & \operatorname{Spec} A \setminus \mathfrak{m}_A \arrow[u, hook, "|" marking] & &
\end{tikzcd}$$

The above diagram is highly probable to be nonsense.
Anyway, the heuristic is that we can think of the blow up of a point as being all the points passing through a line, or something? Blowups will be defined formally later, don't worry about it I guess.

Proof of the valuation criterion: Let $C \subset |X|$ be constructible. We assume that for every dvr R and for every morphism $f : \operatorname{Spec} R \to X$ with $|f|(|\operatorname{Spec} K|) \subset C$, we have $f(|\operatorname{Spec} R|) \subset C$.
As explained previously, we assume wlog that $\overline{C} = X$.

By contradiction, we let $x \in |X| \setminus C$, and replace X by $\underbrace{\operatorname{Spec} \mathcal{O}_{X,x}}_{A}$.

We may then assume $\mathfrak{m}_A \notin C \subset |\operatorname{Spec} A|$, but contains a generic point corresponding to the zero ideal.
The lemma then yields precisely the dvr and morphism $\operatorname{Spec} R \to \operatorname{Spec} A$, so that $f(|\operatorname{Spec} K|)$ =generic point of $\operatorname{Spec} A$, and $f(\mathfrak{m}_K) = \mathfrak{m}_A = x$, a contradiction.

■

Applications of the evaluative criterion

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dotted, "\exists!"] & X \end{tikzcd}$$

Assume that X is noetherian, f is of finite type, R a dvr.
Then the unique existence of the dotted is equivalent to f being proper.

Corollary 0.21. *This also holds for R being a Dedekind domain with fraction field K.*

Example 0.23.

- $R = \mathbb{Z} \subset \mathbb{Q}$,
- $R = K[T] \subset K(T)$

Definition 0.44. A ring R is called a Dedekind domain if it is

- Normal
- Noetherian
- Integral
- of Krull dimension 1

This is equivalent to being Noetherian and integral, and all local rings at non-zero prime ideals are dvrs.
A bunch of stuff I missed because I am tired I'm sorry i'm so fucking sorry I'm sorry
I guess the aboe as all proof of this equivalence?

■??

Applications:
Arithmetic:
Let $X \to \operatorname{Spec} \mathbb{Z}$ be a proper morphism (meaning of finite type, separated, universally closed).
Then: $X(\mathbb{Q}) = X(\mathbb{Z})$.
Recall that $X(\mathbb{Q})$ is defined as the collection of pooints

$$\begin{tikzcd} & \operatorname{Spec} X \arrow[d] \\ \operatorname{Spec} \mathbb{Q} \arrow[r] \arrow[ur] & \operatorname{Spec} \mathbb{Z} \end{tikzcd}$$

$X(\mathbb{Q}) = \operatorname{Hom}(\operatorname{Spec} \mathbb{Q}, X)$, and $X(\mathbb{Z}) = \operatorname{Hom}(\operatorname{Spec} \mathbb{Z}, X)$.
Geometric side: Let $X \to \operatorname{Spec} K$ be proper. Then every $f \in \mathcal{O}_X(|X|)$ is locally constant.

Sketch of proof: $f : X \to A_k^1$ corresponds to a morphism of schemes relative to $\operatorname{Spec} k$:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\text{proper}"'] & & \mathbb{A}_k^1 \arrow[dl, "\text{separated}"] \\ & \operatorname{Spec} k & \end{tikzcd}$$

Then f is proper.
So $f(|X|)$ is closed, so its image is a finite set of closed points, or $|\mathbb{A}_k^1|$. The result of the assertion is equivalent to f being non-surjective.
Assume by contradiction that f is surjective. Then

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr] \arrow[ddr, bend right] & \mathbb{A}_k^1 \arrow[d, phantom, "\cap"] \\ & \mathbb{P}_k^1 \arrow[d, "\text{proper, to be shown next week}"] \\ & \operatorname{Spec} k \end{tikzcd}$$

But $f : X \to \mathbb{P}_k^1$ is also proper, thus has closed image. But the image (???) $|\mathbb{A}_k^1| \subset \mathbb{P}_k^1$ by construction, which isnt closed, a ccontradiction.

■

Lecture 24, 3/11/25

Today: More applications of valuative criterion, and flat morphisms.
Next quiz on friday
Reading assignment: Sheaves and modules (he forgot which sections) (maybe 2.5?)
Recall the valuative criterion for properness: given a diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dotted] & X \end{tikzcd}$$

With R a dvr (or more generally a Dedekind domain), $K = \operatorname{Frac} R$, f of finite type, Y, X Noetherian, then the existence of the dotted line is equivalent to f being a proper morphism.
<u>Last time:</u> We proved completeley that if $X \to \operatorname{Spec} \mathbb{Z}$ is proper, then rational points are integral points, $X(\mathbb{Q}) = X(\mathbb{Z})$, that is, $\operatorname{Hom}(\operatorname{Spec} \mathbb{Q}, X) = \operatorname{Hom}(\operatorname{Spec} \mathbb{Z}, X)$: for

every map $f : \operatorname{Spec}\mathbb{Z}$ to X, we can get an extension

$$\begin{tikzcd} \operatorname{Spec}\mathbb{Q} \arrow[r, "f'"] \arrow[d] & Y \arrow[d] \\ \operatorname{Spec}\mathbb{Z} \arrow[ur, "f"] \arrow[r, equal] & \operatorname{Spec}\mathbb{Z} \end{tikzcd}$$

We also showed that if $X \to \operatorname{Spec} k$ is proper, and $f \in \mathcal{O}_X(|X|)$, then f is <u>locally constant</u>.
$f \in \mathcal{O}_X(|X|)$ corresponds to a morphism of schemes $f : X \to \mathbb{A}^1_k = \operatorname{Spec} k[T]$ over $\operatorname{Spec} k$ by universal property of spec, because a section on the left corresponds to a map of polynomial rings sending the generator of $k[T]$ to f.
Now suppose the lower right morphism is separated and the lower left is proper: then the top is proper

$$\begin{tikzcd} X \arrow[rr, "\text{proper}"] \arrow[dr, "\text{proper}"'] & & \mathbb{A}^1_k \arrow[dl, "\text{separated}"] \\ & \operatorname{Spec} k & \end{tikzcd}$$

<u>Word I can't read:</u>
Now it is to be shown that if we have the setup

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left, "\text{universally closed}"] \arrow[dr] & \mathbb{A}^1_k \arrow[r, hook, "\circ"] & \mathbb{P}^1_k \arrow[dl, "g"] \\ & \operatorname{Spec} k & \end{tikzcd}$$

then g is proper.
Assume by contradiction that $f(|X|) = |\mathbb{A}^1_k|$, then $g(|X|) \subset |\mathbb{P}^1_k|$ is not closed, in contradiction to g being universally closed.
Since $f : X \to \mathbb{A}^1_k$ is universally closed, we have $f(|X|)$ is either finite or $|\mathbb{A}^1_k|$.
The second option is excluded, thus $f(|X|) \subset |\mathbb{A}^1_k|$ is a finite set of closed points.
So f is locally constant.
<u>More generally</u> If $\mathcal{F}$ is a coherent sheaf over X, then $\Gamma(X, \mathcal{F})$ forms a finite-dimensional vector space.
The most general form of this remark is that if $f : Y \to X$ is proper, then $R^i f_* \mathcal{F}$ is coherent. This is something like relative sheaf cohomology.
End remark

Proposition 17. $\mathbb{P}^1_{\mathbb{Z}} \to \operatorname{Spec}\mathbb{Z}$ *is proper.*

Proof. We will use the valuative criterion.

Proof Sketch: We want to show that for any dvr R, $\mathbb{P}^1_{\mathbb{Z}}(K) = \mathbb{P}^1_{\mathbb{Z}}(R)$ for $R = \operatorname{Frac} K$.

Claim.

- $\mathbb{P}^1_{\mathbb{Z}}(K) = \frac{K^2 \setminus \{(0,0)\}}{K^\times}$
- $\mathbb{P}^1_{\mathbb{Z}}(R) = \{\underbrace{(\lambda\ \mu)}_{matrix} \in R^2 \mid R^2 \to R \textit{ is surjective } \}/R^\times$

Assuming the claim: Assuming the claim holds, let $[(x, y)] = (x : y) \in \mathbb{P}^1_{\mathbb{Z}}(K)$. Then we can express $x = \pi^{\nu(x)}u$, $y = \pi^{\nu(y)}v$, where π is the uniformizer of R, u, v units, and letting $m = \min(\nu(x), \nu(y))$, we know

$$(x : y) = (\pi^{\nu(x) \leq m} u : \pi^{\nu(y) \leq m})$$

And one of those is invertible, giving a surjection $R^2 \to R$ (?) showing that every K-point stems from a unique α, so $\mathbb{P}^1_{\mathbb{Z}}(K) = \mathbb{P}^1_{\mathbb{Z}}(R)$.

■

The claim holds more generally for local rings.

Lemma 47. *Let R be a local ring. Then*

$$\operatorname{Hom}(\operatorname{Spec} R, \mathbb{P}^1_{\mathbb{Z}}) = \{(\lambda\ \mu) \in R^2 \mid surj\}/R^2$$

Proof. Recall $\mathbb{G}_{m,k} = \operatorname{Spec} k[T, T^{-1}]$
Recall $\mathbb{P}^1_{\mathbb{Z}} = \mathbb{A}^1_{\mathbb{Z}} \sqcup_{\mathbb{G}_{m,\mathbb{Z}}} \mathbb{A}^1_{\mathbb{Z}}$, gluing them together over the morphism inverting coordinates. Then

$$\begin{tikzcd}
 & \mathbb{A}^2_{\mathbb{Z}} \setminus \operatorname{Spec} \mathbb{Z} \arrow[d] & \\
\operatorname{Spec} R \arrow[ur, dotted, "(\lambda\ \mu)"] \arrow[r] & \mathbb{P}^1_{\mathbb{Z}} \arrow[r, phantom, "="] & \mathbb{A}^1_{\mathbb{Z}} \sqcup_{\mathbb{G}_{m,\mathbb{Z}}} \mathbb{A}^1_{\mathbb{Z}}
\end{tikzcd}$$

Compare with

$$\begin{tikzcd}
\mathbb{C}^2 \setminus \{(0,0)\} \arrow[d] \\
\mathbb{CP}^n \ni [z : w]
\end{tikzcd}$$

where the downward arrow is $\mathbb{C}^\times$-invariant.

■

■

One more application: now fixed points.

Proposition 18. *Let $X/\operatorname{Spec} k$ be proper and $\mathbb{G}_{m,k} \times_{\operatorname{Spec} k} X \to X$ be a group action. Let's assume $k = \overline{k}$.*
<u>Then:</u> $X^{\mathbb{G}_{m,k}}(k) \neq \varnothing$
(Special case of Borel's theorem)

Proof. Let $x \in X(k)$ be any k-point. Let $\mathrm{orb}_x : \mathbb{G}_{m,k} \to X$ be the orbit of x, i.e.

$$\begin{tikzcd} \mathbb{G}_{m,k} \times_{\operatorname{Spec} k} X \arrow[r] & X \\ \mathbb{G}_{m,k} \times \{x\} \arrow[u] \arrow[ur, "\mathrm{orb}_x"'] & \end{tikzcd}$$

Applying the valuative criterion for Dedekind rings, recalling $\mathbb{G}_{m,k} = \operatorname{Spec} k[T, T^{-1}]$, we see that there is an extension $\overline{\mathrm{orb}_x} : \mathbb{A}^1_k \to X$.
For every $\lambda \in k^\times$, the rep orb (?) is λ-equivalent:

$$\begin{tikzcd} \mathbb{G}_{m,k} \arrow[r, "\mathrm{orb}_y"] \arrow[d, "m_y"] & X \arrow[d, "m_x"] \\ \mathbb{G}_m \arrow[r, "\mathrm{orb}_x"] & X \end{tikzcd}$$

Thus
The image point

$$y \overset{\text{def}}{=} \overline{\mathrm{orb}_x}(\underbrace{0}_{\in \mathbb{A}^1_k})$$

is fixed by all $\lambda \in k^\times$.
Since $k^\times \subset \mathbb{G}_{m,k}$ is Zariski-dense, y is a $\mathbb{G}_{m,k}$-fixed point. ■

Remark: This only works algebraically, i.e. not in the complex analytic category

Example 0.24. Consider $\mathbb{C}^\times / q^{\mathbb{Z}}$, $|q| < 1$. This is a complex manifold, diffeomorphic to an elliptic curve $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, which is an elliptic curve over $\mathbb{C}$.
$\mathbb{C}^\times$ acts by translating on this elliptic curve E, thus there is no fixed point.

Lecture 25, 5/11/25

Reminder: Quiz 4 on Friday
Today:

- Flat morphisms
- Quasicoherent sheaves (if time permits)

Definition 0.45. Let M be an R-module. Then M is called flat if for every injection of R modules $\iota : A \hookrightarrow B$, the tensor product $\iota \otimes \mathrm{Id}_M : A \otimes M \to B \otimes M$ is also injective.
Remark:

Given any short exact sequence of R-modules,

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

the following shorter sequence

$$A \otimes M \longrightarrow B \otimes M \longrightarrow C \otimes M \longrightarrow 0$$

is also exact for any R-module M. If the short sequence

$$0 \longrightarrow A \otimes M \longrightarrow B \otimes M \longrightarrow C \otimes M \longrightarrow 0$$

is also exact, then M is flat. That is, M is flat if the functor $- \otimes M$ is <u>left exact</u>.

Example 0.25. Here is a non-example.
Take $R = \mathbb{Z}, M = \mathbb{Z}/2$. Hit the SES

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

with $\otimes\mathbb{Z}/2$. Then the sequence

$$0 \longrightarrow \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

is <u>not</u> exact, because obviously the 0 map fails injectivity.
<u>To be explained:</u>
For a PID, flatness is equivalent to being torsion free.

Example 0.26. If $R = \mathbb{Z}$, $M = \begin{cases} \mathbb{Z}^{\oplus I} \\ \mathbb{Q} \end{cases}$ are <u>flat modules</u>

Definition 0.46. A ring homomorphism $R \to S$ is called <u>flat</u> if S is flat as an <u>R-module</u>.
Of course if we have such a notion for rings we might as well generalize to schemes:

Definition 0.47. A morphism of schemes $f : Y \to X$ is said to be <u>flat</u> if for every $y \in |Y|$, the homomorphism $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is <u>flat</u>.
<u>Remark:</u> If R is any ring, $S \subset R$ a multiplicative subset, the localization $R \to R[S^{-1}]$ is <u>flat</u>, because $M \otimes_R R[S^{-1}] = M[S^{-1}]$.
<u>Remark:</u> A morphism $f : Y \to X$ which is surjective and flat is called <u>faithfully flat</u>.
Michael claims these are "simply the best class of morphisms to work with."
fppf means "Faithfully flat, finite presentation."

Example 0.27.

- $\operatorname{Spec}\mathbb{Q} \to \operatorname{Spec}\mathbb{Z}$ is flat, but not faithfully flat.

- $\operatorname{Spec}\mathbb{Z}[\frac{1}{2}] \cup \operatorname{Spec}\mathbb{Z}[\frac{1}{3}] \to \operatorname{Spec}\mathbb{Z}$ is faithfully flat.
- $\operatorname{Spec} k[T] \to \operatorname{Spec} k[T]$ induced by $T \mapsto T^n$ is faithfully flat so long as $n \geq 1$.

We have a valuative criterion:

Theorem 0.22. *Let $f : Y \to X$ be flat and of finite presentation (meaning f is of finite type and X is Noetherian).*
Then f is universally open

Proof. We will show that f is open if it is flat, of finite type, and X is Noetherian.

Idea: We will apply the valuative criterion for closedness to the complement of the image of the morphism, $|X| \setminus f(|Y|)$, which is constructible by Chevalley.
The proof relies on Baer's criterion for flatness:

Lemma 48. *Let M be an R-module such that for any ideal $I \subset R$, the map $I \otimes_R M \to R \otimes_R M \cong M$ is injective. Then M is flat.*

Sketch of proof. This is a statement from homological algebra. We will prove it with an application of Zorn's lemma (fun fact, zorn means "wrath" in German).
Let $A \hookrightarrow B$ be injective.
Consider the collection of all submodules A' such that $A \subset A' \subset B$ and $A \otimes_R M \hookrightarrow A' \otimes_R M$ is injective.
This collection is nonempty because $A' = A$ works.
It is closed under taking increasing unions. So by Zorn's lemma there is some $\tilde{A} \subset B$ which is maximal with respect to this property.
Assume by contradiction there is some $b \in B \setminus \tilde{A}$. Consider $A' = \tilde{A} + Rb$.
Use

$$\begin{tikzcd} I \arrow[r] \arrow[d] & R \arrow[d] \\ \tilde{A} \arrow[r] \arrow[ur, phantom, "\lrcorner"] & A'' \end{tikzcd}$$

and the assumption to conclude that if R is a PID, then $I = (\lambda) \subset R$, and as R-module, thus flat (??)

∎

Corollary 0.23. *DVR's are PIDs.*

Proof. ∎
Assume that R is a dvr, $K = \operatorname{Frac} R$, and $\alpha : \operatorname{Spec} R \to X$ such that

$$\alpha(|\operatorname{Spec} K|) \subset |X| \setminus f(|Y|)$$

This yields $Y \times_X \operatorname{Spec} K = \varnothing$.

<u>Consider:</u> $Y_R \stackrel{\text{def}}{=} Y \times_X \operatorname{Spec} R$. <u>Want:</u> Y_R is <u>empty</u>.
Let $y \in Y_R$, then by assumption $R \to \mathcal{O}_{Y_r,y}$ is flat, and $\mathcal{O}_{Y_R,y} \otimes_R K = 0$.
Thus $\mathcal{O}_{Y_R,y}$ is Torsion, and therefore not <u>flat</u>.
We are implictly using that the base change of a flat module is also flat.
If the image is not open, the morphism cannot be flat.

■

Now to talk about sheaves of modules.

<u>Quasi-coherent sheaves</u>

Let X be a scheme, $R \stackrel{\text{def}}{=} \mathcal{O}_X(|X|)$, let M be an R-module. We then have a well-defined sheaf on X

$$\mathcal{O}_X \otimes_R M$$

To get this, first take the pre-sheaf sending an open $|U|$ to $\mathcal{O}_X(|U|) \otimes_R M$, (viewing $\mathcal{O}_X(|U|)$ as an R-module via the restriction morphism) and then sheafify. By definition,

$$(\mathcal{O}_X \otimes_R M)_x = \mathcal{O}_{X,x} \otimes_R M$$

because tensor products preserve colimits. This is interesting if there are a lot of global sections.

Definition 0.48. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on X (recall this is a module object in $\operatorname{Sh}(|X|)$ over the ring object $R \stackrel{\text{def}}{=} \mathcal{O}_X$) is called <u>quasi-coherent</u> if there is an open cover $\{U_i\}_{i \in I}$ by affines such that $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i} \otimes M_i$ for an $\mathcal{O}_{U_i}(|U_i)$-module M_i.
We will prove this theorem eventually:

Definition 0.49. $\operatorname{QCoh}(X)$ is the category of quasicoherent sheaves

Theorem 0.24.

$$\operatorname{QCoh}(\operatorname{Spec} R) \cong \operatorname{Mod}(R)$$

This is an equivalence not just of categories, but of symmetric monoidal categories!

Proof. Later

■

So when $X = \operatorname{Spec} R$, locally $\mathcal{F} = \mathcal{O}_U \otimes M$, so $\mathcal{F}|_{U_f} = (\mathcal{O}_U)_f \otimes M = M_f$.
We know this holds locally, but we will see it also holds globally.
So for any $\mathcal{F} \in \operatorname{QCoh}(\operatorname{Spec} R)$,

$$\mathcal{F}(D(A)) \cong \mathcal{F}(|X|)_f$$

Lecture 26, 7/11/25

Adjunctions

Just take the definition of adjunctions for a scalar product and replace it with Hom.

Definition 0.50. Two functors $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$ are called adjoint if, (in particular F is left adjoint to G) for any pair of objects $X \in \mathcal{C}, Y \in \mathcal{D}$, there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

This encodes a universal property.

Remark:

For every morphism $F(X) \to Y$, there is a unique morphism $X \to G(Y)$.

For instance, if $G : \mathcal{D} \to \mathcal{C}$ is an embedding of categories, $F(X) \in \mathcal{D}, Y \in \mathcal{D}$, we have some factorization

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \\ & F(X) \arrow[ur, dotted, "\exists!"] & \end{tikzcd}$$

Example 0.28. Sheafification is adjoint to the inclusion into presheaves.

This follows from the universal property of the sheafification: if $\mathcal{F}$ is a presheaf, $\mathcal{G}$ a sheaf,

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[dr] & \mathcal{F}^+ \arrow[d, dotted, "\exists!"] \\ & \mathcal{G} \end{tikzcd}$$

Thus:

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(\mathcal{F}^+, \mathcal{G}) \cong \mathrm{Hom}_{\mathrm{Psh}(X)}(\mathcal{F}, I(\mathcal{G}))$$

Theorem 0.25. *For $f : Y \to X$ continuous, we have an adjunction between f^{-1} : $\mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ and $f_* : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$.*

$$\mathrm{Hom}(-, \{_*) = \mathrm{Hom}(f^{-1}(-), -)$$

Proof. I will present this proof later in the case of an open inclusion! ∎

Lemma 49. *If F is left adjoint to G, then F preserves colimits. That is, if $X = \mathrm{colim}_{i \in I} X_i$ in $\mathcal{C}$, then $F(X)$ is the colimit of $\mathrm{colim}_{i \in I} F(X_i)$*

Proof. We need to show that for every $Y \in \mathcal{D}$ and for every morphism of diagrams $\{F(X_i) \to Y\}$, we want to show that there is some unique map from $F(X)$ to $F(Y)$

so we get a factorization

$$\begin{tikzcd}
F(X_i) \arrow[rr] \arrow[dr] & & Y \\
& F(X) \arrow[ur, dotted, "\exists!"] &
\end{tikzcd}$$

By the universal property of the colimit, we apply the adjunction relation to

$$\begin{tikzcd}
F(X_i) \arrow[rr] \arrow[dr] & & Y \arrow[r, "\text{adj}"] & \{X_i \arrow[rr] \arrow[dr] & & G(Y)\} \\
& F(X) \arrow[ur, "\exists!"] & & & X \arrow[ur, dotted, "\exists!"] \arrow[lll, "\text{adjunction}"'] &
\end{tikzcd}$$

■

Corollary 0.26. *Sheafification preserves colimits*

Proof. $(\operatorname{colim}_{i\in I} \mathcal{F}_i)^+ = \operatorname{colim}_{i\in I} \mathcal{F}_i^+$

■

Recall the def of quasi-coh sheaves:

Definition 0.51. Given X a scheme, $\mathcal{F} \in \operatorname{mod}(\mathcal{O}_X) \subset \operatorname{Sh}(X)$, we say that $\mathcal{F}$ is quasi-coherent if there exists an affine open cover of $|X| = \bigcup_{i\in I} U_i$ and $\mathcal{O}(|U_i|)$-modules M_i such that $\mathcal{F}|_{|U_i|} \cong \mathcal{O}_{U_i} \otimes_{\mathcal{O}_{U_i}} M_i$.
This construction also involves sheafification of the tensored presheaves.
There is a functor from $\operatorname{mod}(\mathcal{O}(U_i)) \to \operatorname{mod}(\mathcal{O}_{U_i}) \to \operatorname{Sh}(|U_i|)$. This functor is adjoint to the global sections functor, or, equivaneltly, to $\operatorname{Hom}(\mathcal{O}_{U_i}, -)$
Goal: Prove the following theorem:

Theorem 0.27. *If $X = \operatorname{Spec} R$, then*

$$QCoh(X) \cong \textsf{-}\textsf{\textit{Mod}}(R)$$

Proof. Here is the key lemma:

Lemma 50. *Let $\mathcal{F} \in QCoh(X), X = \operatorname{Spec} R$. Then $\Gamma(D(f), \mathcal{F}) \cong \Gamma(|X|, \mathcal{F}|_f)$ for all $f \in R$.*

Proof. Remark:
We know that this holds for $\mathcal{F} = \mathcal{O}_X^{\oplus I}$.
The lemma follows from the following local statement:

$$i : D(f) \hookrightarrow |X|$$

Claim. *$F \in QCoh(X)$ implies we can express f as a colimit*

$$\mathcal{F} \xrightarrow{f} \mathcal{F} \xrightarrow{f} \mathcal{F} \xrightarrow{f} \cdots$$

Proof. When we restrict the tower to $\mathcal{F}|_{D)(f)}$, then multiplication by f is invertible, so these are all isos, so the colimit pulled back along i^{-1} is the same as the colimit of the pullbacks along i^{-1}, which is isomorphic to $\mathcal{F}$ because we are taking a directed colimit along isomorphisms.
If we call the colimit of this $\mathcal{G}$, then using that $i^{-1}(\mathcal{G}) \cong i^{-1}(\mathcal{F})$, we see we get this diagram from the adjunction relation, implying we get a commutative diagram

$$\begin{tikzcd} F \arrow[r] \arrow[dr] & i^{-1}\mathcal{F} \arrow[d] \\ & \mathcal{G} \end{tikzcd}$$

From adjunction between i^{-1} and i_*. ∎

This proves the claim. The lemma follows from some formal considerations, discussed on monday. ∎

Lecture 27, 10/11/25

<u>Recall:</u> Let $X = \operatorname{Spec} R$, $i : D(f) \hookrightarrow X$, $\mathcal{F} \in QCoh(X)$

Lemma 51. $i_* i^{-1}\mathcal{F} \cong \operatorname{colim}[\ F \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} \cdots\ ]$

<u>*Motivation:*</u>. For $M \in Mod(R)$, we have

$$M_f \cong \operatorname{colim}[\ M \xrightarrow{f} M \xrightarrow{f} M \xrightarrow{f} \cdots\ ]$$

∎

This yields:

Proposition 19. *Let $\mathcal{F} \in QCoh(\operatorname{Spec} R)$. We have*

$$\Gamma(D(f), \mathcal{F}) \cong \Gamma(\operatorname{Spec} R, \mathcal{F})_f$$

for all $f \in R$

Proof. First, we know $|X| = |\operatorname{Spec} R|$ is a quasicompact space. This implies that Γ preserves directed colimits (see model solutions).
So $\Gamma(X, \operatorname{colim} \mathcal{F}) \cong \operatorname{colim} \Gamma(X, \mathcal{F}_i)$.

■

Applying this to the lemma:

$$\Gamma(X, i_*i^{-1}\mathcal{F}) \cong \Gamma(X, \mathrm{colim}[\ \mathcal{F}\mathcal{F} \xrightarrow{f} \mathcal{F} \xrightarrow{f} \cdots\ ])$$

In turn this is $\cong$ to

$$\cong \mathrm{colim}[\ \Gamma(X, \mathcal{F}) \xrightarrow{f} \Gamma(X, \mathcal{F}) \xrightarrow{f} \cdots\ ]$$

$$\cong \Gamma(X, \mathcal{F})$$

Where $\Gamma(X, \mathcal{F})_f$ is an R-module, and of course the f in the subscript denotes localization of an R-module.
Since $\mathcal{O}_R(|\operatorname{Spec} R|) \cong R$, use $\Gamma(X, i_*i^{-1}\mathcal{F}) \cong \Gamma(D(f), \mathcal{F})$

■

<u>Conclusion</u>
$\mathcal{F}(|X|)$ controlls all $\mathcal{F}(D(f))$, thus $\mathcal{F}(U)$, thus all <u>stalks</u>.

$$\Gamma : QCoh(\operatorname{Spec} R) \to Mod(R)$$

$$\mathcal{O}\otimes_R : Mod(R) \to QCoh(\operatorname{Spec} R)$$

$$M \mapsto \mathcal{O} \otimes_R M$$

$\mathcal{O} \otimes_R -$ is left adjoint to Γ, i.e.

$$\mathrm{Hom}_{QCoh(X)}(\mathcal{O} \otimes_R M, \mathcal{F}) \cong \mathrm{Hom}(M, \Gamma(\mathcal{F}))$$

Corollary 0.28. Γ *is <u>conservative</u>, i.e.* $\Gamma(\mathcal{F}) = 0$ *if and only if* $\mathcal{F} = 0$

Proof. One of these directions is easy.
For the other, let $\Gamma(\mathcal{F}) = 0$. Then by the proposition, one has $\Gamma(\mathcal{F})_f = 0$ for all $f \in R$, thus $\Gamma(D(f), \mathcal{F}) = 0$ for all standard affine open subsets, thus implying all stalks are zero.
So the map $0 \to \mathcal{F}$ is an isomorphism on stalks, thus is a sheaf isomorphism.

■

Lemma 52.

1. $\mathcal{O} \otimes_R -$ *is exact*

2. Γ *is exact. ($X = \operatorname{Spec} R$ is essential)*

Proof.

1. This boils down to exactness of localizations. Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be a short exact sequence of R-modules. Then for all $\mathfrak{p} \in |\operatorname{Spec} R|$, the localizations also fit into a short exact sequence:

$$0 \longrightarrow \underbrace{M'_{\mathfrak{p}}}_{(\mathcal{O}\otimes_R M')_{mkp}} \longrightarrow \underbrace{M_{\mathfrak{p}}}_{(\mathcal{O}\otimes_R M)_{\mathfrak{p}}} \longrightarrow \underbrace{M''_{\mathfrak{p}}}_{(\mathcal{O}\otimes_R M'')_{\mathfrak{p}}} \longrightarrow 0$$

This yields exactness of

$$0 \longrightarrow M' \otimes_R \mathcal{O} \longrightarrow M \otimes_R \mathcal{O} \longrightarrow M'' \otimes_R \mathcal{O} \longrightarrow 0$$

2. Since Γ is right adjoint, it is <u>limit preserving</u>, thus <u>kernel-preserving</u>, so one has a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

So a SES of modules

$$0 \longrightarrow \mathcal{F}'(|X|) \longrightarrow \mathcal{F}(|X|) \xrightarrow{\alpha} \mathcal{F}''(|X|) \longrightarrow 0$$

And $\mathcal{F}'(|X|) \cong \ker(\alpha)$

■

Claim. *If $\mathcal{F} \to \mathcal{G}$ is a surjective morphism in $QCoh(\operatorname{Spec} R)$, then $\mathcal{F}(|X|) \to \mathcal{G}(|X|)$ is surjective.*

Proof. $\mathcal{F}(|X|) = \operatorname{Hom}_{QCoh(X)}(\mathcal{O}, \mathcal{F})$

$$\begin{tikzcd}
 & \mathcal{O} \arrow[d] \arrow[dl, dotted, "\exists"] & \\
\mathcal{F} \arrow[r] & \mathcal{G} \arrow[r] & 0
\end{tikzcd}$$

<u>Remark:</u> The claim is equivalent to $\mathcal{O} \in QCoh(\operatorname{Spec} R)$ being <u>projective</u>
Now, consider the diagram

$$\begin{tikzcd}
\mathcal{P} \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & \mathcal{O} \arrow[r] \arrow[d] \arrow[l, dotted, bend right] \arrow[dl, dotted] & 0 \\
\mathcal{F} \arrow[r, two heads] & \mathcal{G} \arrow[r] & 0
\end{tikzcd}$$

Need to establish that for $\mathcal{P} \twoheadrightarrow \mathcal{O}$, the map $\mathcal{P}(|X|) \to \mathcal{O}(|X|)$ is also surjective.
Using the proposition again, $\mathcal{O} \otimes_R \Gamma(\mathcal{P}) \to \mathcal{P}$ is an iso on stalks (by virtue of the prop).
So $\mathcal{P} \cong \mathcal{O} \otimes_R \mathcal{P}$
The morphism $\mathcal{P} \to \mathcal{O}$ corresponds to a map $\Gamma(\mathcal{P}) \to R$, where $\Gamma(\mathcal{P}) = M$.
So $\mathcal{P} = \mathcal{O} \otimes_R M$, so we have an exact sequence

$$M \xrightarrow{\beta} R \longrightarrow \text{coker} \longrightarrow 0$$

Assume for contradiction that $\text{coker}(\beta) \neq 0$.
Then there is a prime ideal $\surd \subset R$ such that

$$\begin{tikzcd} 0 \\ (\text{coker}(\beta))_{\mathfrak{p}} \neq 0 \arrow[u] \\ R_{\mathfrak{p}} \arrow[u] \\ M_{\mathfrak{p}} \arrow[u] \end{tikzcd}$$

where the bottom arrow is surjective by assumption. Now, use the following result in category theory:

Lemma 53. *Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ be adjoint and exact functors between two abelian categories. Then, if F is fully faithful and G is conservative, then both F and G are mutually inverse equivalences.*

Proof. Paper by a guy named Bernstein and another guy, kicking off the field of geometric representation theory?
Maybe volunteer proof? $\blacksquare$

Now $\mathcal{O}\otimes$ is fully faithful by the proposition, so it follows, and all other conditions have been discussed. So, we may apply the lemma. $\blacksquare$

Here are some non-examples of quasicoherent sheaves:

- Let $X = \mathbb{A}^1_{\mathbb{C}}$. Then we define $\mathcal{F}^{ind}$ by $\mathbb{A}^1_{\mathbb{C}} \setminus Z = U \mapsto \mathcal{O}^{ind}(\mathbb{C} \setminus Z(\mathbb{C}))$

  Then
$$\mathcal{O}^{ind}(\mathbb{G}_{m,\mathbb{C}}) \ncong \mathcal{O}^{ind}(\mathbb{C})[\frac{1}{2}]$$

where RHS is basically the meromorphic functions. Now f with essential singulatiries.

- Let $\mathcal{F} \in QCoh(\mathbb{A}^1_{\mathbb{C}})$. Then we can associate to it the product of all stalks

$$\prod_{x \in |\mathbb{A}^1_{\mathbb{C}}|} \mathcal{F}_x \in ModO(\mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}})$$

is not quasicoherent. (?? resolution)

The exactness of Γ is related to the vanishing of cohomology on quasi-coherent sheaves on $\operatorname{Spec} r$.

Lecture 28, 10/13/25

My presentation of proof of (special case of) earlier theorem:

Theorem 0.29. *Let $\iota : U \hookrightarrow X$ be an inclusion map. Then there is an adjunction between $\iota^{-1} : \mathrm{Sh}(X) \to \mathrm{Sh}(U)$ and $\iota_* : \mathrm{Sh}(U) \to \mathrm{Sh}(X)$;*

$$\mathrm{Hom}(\mathcal{G}, \mathcal{F}_*) \cong \mathrm{Hom}(f^{-1}\mathcal{G}, \mathcal{F})$$

Proof. Let $V \subseteq U \subset |X|$ be open. Let $\mathcal{G} \in \mathrm{Sh}(X)$. Then $\iota^{-1}\mathcal{G}(V) = \mathcal{G}(V)$. This is because we are working in the special case where $\iota(V) = V$. Let $\mathcal{F} \in \mathrm{Sh}(U)$. Then, similarly, we have $\iota_*\mathcal{F}(V) = \mathcal{F}(V)$.
On the level of presheaves, a morphism is determined by morphisms of all the sections. Let $W \subset X, V \subset U$ be open. Then for any $F \in \mathrm{Hom}(\mathcal{G}, \mathcal{F}_*)$, there is a corresponding morphism

$$F(W) \in \mathrm{Hom}(\mathcal{G}(W), \mathcal{F}_*(W)) = \mathrm{Hom}(\mathcal{G}(W), \mathcal{F}(\iota^{-1}(W)) = \mathrm{Hom}(\mathcal{G}(W), \mathcal{F}(W \cap U))$$

. Further, for any $G \in \mathrm{Hom}(\iota^{-1}\mathcal{G}, \mathcal{F})$, there is a corresponding

$$G(V) \in \mathrm{Hom}(\iota^{-1}\mathcal{G}(V), \mathcal{F}(V)) = \mathrm{Hom}(\mathcal{G}(\iota(V)), \mathcal{F}(V)) = \mathrm{Hom}(\mathcal{G}(V), \mathcal{F}(V))$$

Lecture 29, 12/10/25

Presentation by Jason:

Lemma 54 (Category Theory Lemma)**.** *Let $L : \mathcal{C} \to \mathcal{D}, R : \mathcal{D} \to \mathcal{C}$ be adjoint pair between abelian categories, with L fully faithful, R exact, and R conservative, i.e. $Rd = 0 \implies d = 0$.*
Then L, R are mutually inverse equivalences.

Proof. Hom(C, C') is sent bijectively to Hom(LC, LC'), which is send isomorphically to Hom(C, RLC').
By Yoneda, C' is isomorphically sent to RLC' $\eta_{\mathcal{C}} \to BL$ is natural isomorphism.
We want to show $R \to \eta_{\mathcal{D}}$ is a natural isomorphism. This follows from the commutativity of the first diagram, which the second is send to:

$$\begin{tikzcd} Rd \arrow[r, "eta"] \arrow[dr] & RLRd \arrow[d, "R_\varepsilon"] \\ & RD \end{tikzcd}$$

$$\begin{tikzcd} LR \arrow[r, "\mathrm{Id}"] \arrow[dr, "R"] & LR \arrow[d, "\varepsilon"] \\ & 1 \end{tikzcd}$$

■[?]

Proposition 20. *Let $\mathcal{F} \in Mod(\mathcal{O}_X), X$ a scheme. Then the following are equivalent:*

1. *$\mathcal{F}$ is quasicoherent*

2. *For every $U \subset X$ affine open, for every $f \in \mathcal{O}_X(U)$, we have*

$$\Gamma(U_f, \mathcal{F}) = \Gamma(U, \mathcal{F})_f$$

3. *For all $U \subset X$ quasi-compact and open, for any $f \in \mathcal{O}_X(U)$, we have*

$$\Gamma(U_f, \mathcal{F}) = \Gamma(U, \mathcal{F})_f$$

4. *There exists an affine open cover $\{U_i\}$ such that $\mathcal{F}|_{U_i}$ is quasi-coherent.*

Assertion (4) tells us quasi-coherence is a local property.

Proof. We've already seen $1 \implies 2$. $1 \implies 3$ is almost the same proof.
$3 \implies 2$, (because 2 is just a special case of 3), $4 \implies 1$ are easy.
The hard one is $3 \implies 4$.

■

Corollary 0.30. *Let $g : Y \to X$ be a quasi-compact morphism, i.e. for all affine open $U \subset X$ the preimage $g^{-1}(U)$ is quasi-compact.*
Then $f_*(QCoh(Y)) \subset QCoh(X)$ preserves quasi-coherence.

Proof. Let $U \subset X$ be affine, open. Then $g_*\mathcal{F}(U) = \mathcal{F}(g^{-1}(U))$.
Since $g^{-1}(U)$ is quasi-compact, we have for any $f \in \mathcal{O}_X(U)$, $g_*\mathcal{F}(U_f) = \mathcal{F}(g^{-1}(U_f)) = \mathcal{F}(g^{-1}(U_{g^\sharp(f)}))$, where for $Y \to X$, $g^\sharp : \mathcal{O}_X \to g_*\mathcal{O}_Y$.
By quasi-compactness and (3) above, this is the same as $\mathcal{F}(g^{-1}(U))_{g^\sharp(f)} = (g_*\mathcal{F})(U)_f$.
Thus $g_*\mathcal{F}$ is quasi-coherent.

■

Proof. Some questions:

1. What about the converse?

No. If we let $X = \operatorname{Spec}\mathbb{C}$, $Y = \mathbb{A}^1_{\mathbb{C}}$, then every sheaf over X is quasi-coherent, but not so for Y. $Mod(\mathcal{O}_{\operatorname{Spec}\mathbb{C}}) \cong \vec{(\mathbb{C})} \cong QCoh(\operatorname{Spec}\mathbb{C})$.

All pushforwards along this map are quasi-coherent, but non-quasicoherent sheaves exist.

2. Do we really need quasi-compactness? Is there a non example of a morphism $g : Y \to X$ and $\mathcal{F} \in QCoh(Y)$ such that $g_*\mathcal{F} \notin QCoh(X)$? If we take X to be any scheme and take a countable union, then the projection is not quasi-compact. For most X and quasi-coherent sheaves in $QCoh(\coprod_{\mathbb{N}} X)$, the pushforward g_* is not quasi-coherent:

$$g_*\mathcal{F} = \prod_{i\in\mathbb{N}} \mathcal{F}|_{X\times\{i\}} = \prod_{i\in I} s_i^{-1}\mathcal{F}$$

For example:

$$(\prod_{i\in\mathbb{N}} M_i)_f \neq (\prod_{i\in\mathbb{N}} (M_i)_f)$$

Because for example $\mathbb{Z}[[t]][\frac{1}{2}] \neq (\mathbb{Z}[\frac{1}{2}])[[t]]$. Because an element of the left hand side is of the form $(\frac{1}{2})^k f$, where $f \in \mathbb{Z}[[t]]$. But on the right hand side, we could have something like $\sum_{i\in\mathbb{N}}(\frac{1}{2})^i f_i$

Now consider $f^{-1} : \operatorname{Sh}(X) \to \operatorname{Sh}(Y)$, for f a morphism of schemes.
Q: Does $f^{-1}(QCoh(X)) \subset QCoh(Y)$?
The answer is immediately no, for the trivial reason that it doesn't even preserve $\mathcal{O}$-modules.
Solution: Build a new functor, $f^* : Mod(\mathcal{O}_X) \to Mod(\mathcal{O}_Y)$.
Recall: f^{-1} is left-adjoint to f_*. We will simply ask this new functor, called pullback, to satisfy the same adjunction on $Mod(\mathcal{O}_X)$: for any $f : Y \to X$,

$$f_* : Mod(\mathcal{O}_Y) \to Mod(\mathcal{O}_X)$$

is a left adjoint $f^* : Mod(\mathcal{O}_X) \to Mod(\mathcal{O}_Y)$:

$$\operatorname{Hom}(f^*\mathcal{F}, \mathcal{G}) \cong \operatorname{Hom}(\mathcal{F}, f_*\mathcal{G})$$

Let $\mathcal{F} \in Mod(\mathcal{O}_X)$, then we deinfe $f^*(\mathcal{F}) \in Mod(f^{-1}\mathcal{O}_X)$, and by adjunction we have a ring homomorphism $f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ (in correspondence with $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$).
Simply taking tensor products in $\operatorname{Sh}(Y)$, we define $f^*\mathcal{F} \stackrel{\text{def}}{=} f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_Y$.
Now combine the usual adjunction for tensor products to f^{-1} and f_*, and tensor product to prove the desired adjunction.

Lemma 55. *Pullbacks always preserve quasi-coherent sheaves.*

Proof. Exercise.
Hint: use the criterion that f^* is a left adjoint and thus preserves colimits (and thus cokernels).

■

Lecture 30, 14/10/25

Now for my presentation: I didn't give the last one because I went stupid idiot mode and said some wrong stuff so now I am presenting the following which is hopefull ymarginally less wrong. I am so sorry:

Proposition 21. *For $\iota : U \hookrightarrow X$ an open inclusion, the functor $\iota^{-1} : \mathrm{Sh}(X) \to \mathrm{Sh}(U)$ is a left adjoint to $\iota_* : \mathrm{Sh}(U) \to \mathrm{Sh}(X)$.*

Proof. Let $\mathcal{F} \in \mathrm{Sh}(U), \mathcal{G} \in \mathrm{Sh}(X)$, and let $W \subseteq X$ be open, and $V \subseteq U$ be open. Because ι is an open inclusion, on the level of open sets we have

$$\iota^{-1}\mathcal{G}(V) = \mathcal{G}(V)$$

and

$$\iota_*\mathcal{F}(W) = \mathcal{F}(\iota^{-1}(W)) = \mathcal{F}(W \cap U)$$

This yields

$$\begin{aligned}\left((\iota^{-1} \circ \iota_* \circ \iota^{-1})\mathcal{G}\right)(V) &= (\iota^{-1} \circ \iota_*)(\iota^{-1}\mathcal{G})(V) \\ &= \iota^{-1}\mathcal{G}(V \cap U) \\ &= \iota^{-1}\mathcal{G}(V) \\ &= \iota^{-1}\mathcal{G}(V)\end{aligned}$$

and

$$\begin{aligned}\left((\iota_* \circ \iota^{-1} \circ \iota_*)\mathcal{F}\right)(W) &= (\iota_* \circ \iota^{-1})\mathcal{F}(W \cap U) \\ &= \iota_*\mathcal{F}(W \cap U) \\ &= \mathcal{F}(W \cap U \cap U) \\ &= \mathcal{F}(W \cap U) \\ &= \iota_*\mathcal{F}(W)\end{aligned}$$

We want natural transformations $\varepsilon : \iota^{-1}\iota_* \to \mathrm{Id}_{\mathrm{Sh}(U)}$ and $\eta : \mathrm{Id}_{\mathrm{Sh}(X)} \to \iota_*\iota^{-1}$ satisfying some properties.

For every sheaf $\mathcal{F} \in \mathrm{Sh}(U)$, we want a morphism $\varepsilon_{\mathcal{F}} : \iota^{-1}\iota_*\mathcal{F} \to \mathcal{F}$, and for every sheaf $\mathcal{G} \in \mathrm{Sh}(X)$, a morphism $\eta_{\mathcal{G}} : \mathcal{G} \to \iota_*\iota^{-1}\mathcal{G}$.
Consider ε defined by the collection of arrows $\varepsilon_{\mathcal{F}} : \iota^{-1}\iota_*\mathcal{F} = \mathcal{F} \to \mathcal{F}$ given by $\mathrm{Id}_{\mathcal{F}}$, and η defined by the collection of arrows $\eta_{\mathcal{G}} : \mathcal{G} \to \iota_*\iota^{-1}\mathcal{G}$ given by sending the sheaf $W \mapsto \mathcal{G}(W)$ to the sheaf $W \mapsto \mathcal{G}(W \cap U)$, with the map $\eta(\mathcal{G})_W : \mathcal{G}(W) \to \mathcal{G}(W \cap U)$ given by just the restriction. Consider the natural transformation

$$\varepsilon\iota^{-1} \circ \iota^{-1}\eta : \iota^{-1} \to \iota^{-1}$$

Unwrapping definitions, $\iota^{-1}\eta$ sends $\iota^{-1}\mathcal{G} \to \iota^{-1}\iota_*\iota^{-1}\mathcal{G} =$ by $(\eta\iota^{-1})_{\mathcal{G}} = \eta_{\iota^{-1}\mathcal{G}}$, sending $W \mapsto \mathcal{G}(W \cap U)$ to $W \mapsto \mathcal{G}(W \cap U)$. Also, $\varepsilon\iota^{-1}$ sends $\iota^{-1}\iota_*\iota^{-1}\mathcal{G} \to \iota^{-1}\mathcal{G}$, by $(\varepsilon\iota^{-1})_{\iota^{-1}\iota_*\iota^{-1}\mathcal{G}} : \iota^{-1}\iota_*\iota^{-1}\mathcal{G} \to \iota^{-1}\mathcal{G}$ given by sending the sheaf $W \mapsto \mathcal{G}(W \cap U)$ to the sheaf $W \mapsto \mathcal{G}(W \cap U)$ (i.e. the identity).
The composition of these maps sends the sheaf $W \mapsto \mathcal{G}(W \cap U)$ to the sheaf $W \mapsto \mathcal{G}(W \cap U \cap U) = \mathcal{G}(W \cap u)$ via the identity. A similar argument shows $\eta\iota_* \circ \iota_*\varepsilon = \mathrm{Id}_{\iota_*}$ ■

Back to quasicoherent sheaves
<u>Remark:</u> $\mathrm{QCoh}(X)$, $Mod(X)$ are abelian categories. $\mathrm{QCoh}(X)$ is an $\mathcal{O}_X$-module object in $\mathrm{Sh}(X)$

Proposition 22. $\mathrm{QCoh}(X) \subset Mod(\mathcal{O}_X)$ *is closed under kernels, cokernes, and extensions. That is, given a short exact sequence of $\mathcal{O}$-modules*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

With 2 out of 3 objects in $\mathrm{QCoh}(X)$, *then alos the third sheaf is in* $\mathrm{QCoh}(X)$

Proof. This is a purely <u>local statement</u>, where quasi-coherences is a local property.
<u>Thus:</u> We may assume without loss of generality that 2 out of 3 of the objects are affine.
<u>Use:</u>

$$\begin{tikzcd} \mathrm{QCoh}(X) \arrow[r, shift left, "\Gamma"] \arrow[d, phantom, "\cap"] & Mod(R) \arrow[l, shift left] \arrow[dl, "\mathcal{O}_X \otimes_R"] \\ Mod(\mathcal{O}_X) & \end{tikzcd}$$

Is an equivalence of categories such that both functors respect exactness.

■

Corollary 0.31. $\mathrm{QCoh}(X)$ *is <u>abelian</u>.*

Proof. ■
<u>Note:</u> There exists a tensor product $\otimes_{\mathcal{O}_X}$ on $\mathrm{QCoh}(X)$, together with a natural iso

$$\mathcal{F} \otimes \mathcal{G} \to \mathcal{G} \otimes \mathcal{F}$$

for all pairs of quasi-coherent sheaves, so this satisfies the usual axioms of the symmetric monoidal category.

Definition 0.52. A scheme is quasi-separated if the intersection of any two quasi-compact subsets is quasi-compact.

Theorem 0.32 (Gabriel)**.** *For X, Y being qcqs (quasi-compact, quasi-separated), one has* $\mathrm{QCoh}(X) \cong \mathrm{QCoh}(X)$ *if and only if* $X \cong Y$*.*

Proof. We will prove it for $X = \operatorname{Spec} R, Y = \operatorname{Spec} S$.
Recover: R from $Mod(R)$? In this case, $R \cong Z(Mod(R))$, where Z denotes the center of the category, which is $End(1_{Mod(R)}, 1_{Mod(R)})$.
We want a morphism of modules $f_M : M \to M$ such that for any other modules N there is some commutative diagram

$$\begin{tikzcd} M \arrow[r, "f_M"] \arrow[d] & M \arrow[d] \\ N \arrow[r, "f_N"] & N \end{tikzcd}$$

We can recover elements of R from $Z(Mod(R))$ by using the fact that R acts on any module.
Gabriel found a categorical description of $|\operatorname{Spec} R|$.
He realized that subcategories wrt one can localize $\mathrm{QCoh}(X)$ are in bijection with closed subsets of $|\operatorname{Spec} R|$.
In the Noetherian case, points of $|\operatorname{Spec} R|$ are irreducible closed subets.
Anyways, then we apply the affine case locally, plus sheafification, to reconstruct X from $\mathrm{QCoh}(X)$.

■?

Line bundles and locally free sheaves

Definition 0.53.

- A quasi-coherent sheaf $\mathcal{F}$ is said to be locally free if there exists an open cover $\{U_i\}_{i \in I}$ such that
$$\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus I_i}$$
- This is invertible if there is an open cover such that $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}$

Why invertible?
Because if $\mathcal{F}$ is invertible, there exists another quasi-coherent sheaf $\mathcal{G}$ such that $\mathcal{F} \otimes \mathcal{G} \cong \mathcal{O}_X$.

Proof. Define $\mathcal{F}^\vee$ as $\mathrm{Hom}(\mathcal{F}, \mathcal{O}_X)$. Then $\mathcal{F} \otimes \mathcal{F}^\vee \to \mathcal{O}_X$, by the Hom-tensor adjunction, is the same as an element of $\mathcal{F} \otimes \mathrm{Hom}(\mathcal{F}^\vee, \mathcal{O}_X)$. Since $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}$, ???

$$\mathcal{O}_{U_i} \otimes \mathcal{O}_{U_i}^\vee \to \mathcal{O}_{U_i}$$

is an iso. ∎

<u>Next week:</u> We will show that locally free sheaves on X up to isomorphisms correspond 1-1 with vector bundles on X, up to isomorphism.

Lecture 31, 17/10/25

Today we are going to talk about relative spec, affine morphisms, and vector bundles, which are the most important kind of quasi-coherent sheaves.

<u>Relative spec</u>

<u>Recall:</u> Spec : $\mathsf{Ring}^{op} \to$ LocRingdSpc is a functor satisfying some universal property: it is a (contravariant) right adjoint to the total sections functor. That is,

$$\mathrm{Hom}_{\mathrm{LocRingdSpc}}(X, \mathrm{Spec}\, R) \cong \mathrm{Hom}_{\mathsf{Ring}}(R, \Gamma(\mathcal{O}_X(|X|)))$$

In the relative setting, we fix a scheme X, called the <u>base scheme</u>.

Definition 0.54. A quasi-coherent sheaf of $\mathcal{O}_X$-algebras $\mathscr{A}$ is a commutative and unital algebra object in $(\mathrm{QCoh}(X), \otimes_{\mathcal{O}_X})$.
That is, $\mathscr{A} \in \mathrm{QCoh}(X)$, together with a morphism $\cdot : \mathscr{A} \otimes_{\mathcal{O}_X} \mathscr{A}$, (or alternatively, an $\mathcal{O}_X$-bilinear morphism from $\mathscr{A}$ to itself) satisfying the usual axioms of a commutative and unital ring, e.g. for commutativity, we have

$$\begin{tikzcd}
\mathscr{A} \otimes_{\mathcal{O}_X} \mathscr{A} \arrow[r, "\mathrm{swap}"] \arrow[dr, "\cdot"'] & \mathscr{A} \otimes_{\mathcal{O}_X} \mathscr{A} \arrow[d, "\cdot"] \\
& \mathscr{A}
\end{tikzcd}$$

For the unit, we have a morphism $\mathcal{O}_X \to \mathscr{A}$ satisfying

$$\begin{tikzcd}
\mathscr{A} \arrow[r] \arrow[drr, "\mathrm{Id}_{\mathscr{A}}"'] & \mathscr{A} \otimes_{\mathcal{O}_X} \mathscr{A} \arrow[r, "\mathrm{Id} \otimes 1"] & \mathscr{A} \otimes_{\mathcal{O}_X} \mathscr{A} \arrow[d, "\cdot"] \\
& & \mathscr{A}
\end{tikzcd}$$

Usually though, we will call our base scheme S, not X.

<u>In terms of sections</u>
$\mathscr{A}(U)$ is an $\mathcal{O}_{S(|U|)}$ algebra which is commutative, unital, so that the restriction maps are algebra morphisms, compatible with the restriction maps on $\mathcal{O}_S$.
We will denote the category of $\mathcal{O}_X$ algebras by $Alg(\mathcal{O}_X)$

Proposition 23. *There exists an adjunction*

$$\begin{tikzcd} \mathrm{Sh}_S^{qcqs} \arrow[r, shift right, "f\mapsto f_*"'] & (Alg(\mathcal{O}_S)) \arrow[l, shift right, "\mathrm{Spec}_S"'] \end{tikzcd}$$

where f_* is the pushforward of the structure sheaf $f_*\mathcal{O}_Y$ along a continuous map $f : Y \to S$.

Proof. This definition is local in S, hence we may assume without loss of generality that S is an affine scheme. We define $\mathrm{Spec}_S(\mathscr{A})$ to be the map $\mathrm{Spec}\,\mathscr{A}(|S|) \to S$ corresponding to the ring homomorphism $\mathcal{O}_S(|S|) \to \mathscr{A}(|S|)$ which is part of the $\mathcal{O}_S$-algebra structure on $\mathscr{A}$.

■

Definition 0.55. A quasi-compact, quasi-separated morphism of schemes $f : Y \to S$ is called <u>affine</u> if there exists an affine cover $\{U_i\}$ of S such that $|f|^{-1}(U_i) \subset Y$ is affine.
<u>Remark:</u> By definition, every affine morphism is quasi-compact and separated. This is a consequence of the fact that any morphism between affine schemes is a separated morphism.
Equivalently, affine morphisms are precisely those that arise by applying Spec_S:

Proposition 24. *A morphism $f : Y \to S$ is affine if and only if the natural map*

$$\begin{tikzcd} Y \arrow[r] \arrow[dr] & \mathrm{Spec}_S\, f_*\mathcal{O}_X \arrow[d] \\ & S \end{tikzcd}$$

is an <u>isomorphism</u>.

Proof. Volunteers?

■

Vector bundles and locally free sheaves

Last time we announced that isomorphisms classes of locally free sheaves of rank r on S are in 1-1 correspondence with isomorphism classes of rank r vector bundles on S, which can be defined as in other areas.

The relative spec construction can be thought of as being related to the total space construction of a locally free sheaf.
That is, the correspondence one way sends a locally free sheaf $\mathcal{F}$ to the affine morphism $Tot_S(\mathcal{F}) \to S$. In the other direction, when we take the pullback of this locally free sheaf we get $\mathbb{A}^r_{U_i}$, where

$$\mathbb{A}^r_{U_i} = \mathbb{A}^r_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} U_i$$

We know that

$$\mathbb{A}^r_{U_i} = \operatorname{Spec} \mathcal{O}_{U_i}(|U_i|)[T_1, \ldots, T_r]$$

if U_i is affine.

Definition 0.56. The total space of $\mathcal{F}$ relative to S, $\mathrm{Tot}_S(\mathcal{F})$, is defined to be

$$\mathrm{Tot}_S(\mathcal{F}) \overset{\mathrm{def}}{=} \underline{\operatorname{Spec}}_S \operatorname{Sym}^{\cdot}_{\mathcal{O}_S}(\mathcal{F}^\vee)$$

where $\operatorname{Sym}_{\mathcal{O}_S}$ is the functor left adjoint to the forgetful functor $Alg(\mathcal{O}_S) \to \mathrm{QCoh}(S)$. So

$$\operatorname{Hom}_{Alg(\mathcal{O}_S)}(\operatorname{Sym}_{\mathcal{O}_S} \mathcal{F}^\vee, \mathcal{A}) \cong \operatorname{Hom}_{\mathrm{QCoh}}(\mathcal{F}^\vee, \mathscr{A})$$

So we can think of a vector space as a scheme by taking polynomials in it, and taking spec of that.
Locally, $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus r}$, so $\mathcal{F}|_{U_i}^\vee \cong \mathcal{O}_{U_i}^{\oplus r}$, so $\operatorname{Sym}_{\mathcal{O}_S}(\mathcal{F}^\vee)|_{U_i} \cong \mathcal{O}_{U_i}[T_1, \ldots, T_r]$, so

$$(\operatorname{Spec}_S \operatorname{Sym} \mathcal{F}^\vee) \times_S U_i \cong \mathbb{A}^r_{U_i}$$

<u>Why $\mathcal{F}^\vee$?</u>

Lemma 56. *$\mathcal{F}(U) \cong Sect_U(\mathrm{Tot}(\mathcal{F}))$ for all opens $U \subset S$, where the sections are sections in the category of schemes:*

$$\begin{tikzcd}
 & \mathrm{Tot}(\mathcal{F}) \arrow[d] \\
U \arrow[ur, dotted, "s"] \arrow[r, hook, "\circ" description] & S
\end{tikzcd}$$

Proof. Next time.

Lecture 31, 19/10/25

<u>Today:</u> Serré's theorem, and more on vector bundles.
<u>Recall:</u> (Exam timeslot signup!)
There is a bijection between the concepts of vector bundles of rank r and locally free sheaves of rank r.

Lemma 57. *Let $\mathcal{F}$ be a locally free sheaf on X. Then the functor given by $\mathcal{F} \otimes - : \mathrm{QCoh}(X) \to \mathrm{QCoh}(X)$ is exact.*

Proof. Remark: The quasi-coherent sheaves with this property are called flat sheaves, in analogy with flat modules.
Now for the proof.
This is a local property, since $\otimes_{\mathcal{O}}$ is a sheaf-theoretic function, and exactness is determined locally on stalk. So, we may assume without loss of generality that $\mathcal{F}$ is free, i.e. $\mathcal{F} = \mathcal{O}_X^{\oplus I}$.

Then: $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} = \mathcal{G}^{\oplus I}$, which is clearly exact. ∎

Now we have shown that locally free sheaves are flat.

Definition 0.57. An R-module P is called projective if $\mathrm{Hom}(P, -)$ is exact. Equivalently, if for every surjection $M \twoheadrightarrow N$ and map $P \to N$, there is a lift:

$$\begin{tikzcd} & P \arrow[d] \arrow[dl, dotted, "\exists"'] & \\ M \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

Nonexistence of this lift gives the failure of $\mathrm{Hom}(P, -)$ to preserve cokernels. Note there is not necessarily any uniqueness assumption; this is not a universal property.

Lemma 58. *Let $X = \operatorname{Spec} R$ be an affine scheme, and $\mathcal{F} \in \mathrm{QCoh}(X)$ which is locally free. Then $\mathcal{F}(|X|)$ is a projective R-module.*

Proof. First, $\mathrm{QCoh}(X)_{\otimes_{\mathcal{O}}} \cong Mod(R)_{\otimes_R}$ as modular tensor categories. One direction is $\mathcal{F} \mapsto \mathcal{F}(|X|) \overset{\text{def}}{=} P$.

Then by virtue of the above equivalence we have $\mathrm{Hom}(\mathcal{F}, -) \cong \mathrm{Hom}(P, -)$, and by the tensor hom adjunction,

$$\begin{array}{ccc} \mathrm{Hom}(\mathcal{F}, -) & \cong & \mathrm{Hom}(P, -) \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ \Gamma(\mathcal{F}^\vee \otimes_{\mathcal{O}} -) & \cong & P^\vee \otimes - \end{array}$$

Since $\mathcal{F}$ is locally free of finite rank, so is $\mathcal{F}^\vee$. Thus $\mathcal{F}^\vee$ is flat. hence $\mathcal{F}^\vee \otimes = \mathrm{Hom}(\mathcal{F}, -) = \mathrm{Hom}(P, -)$ is exact. Thus, $P \in Proj(R)$. ∎

Lemma 59. *Let R be a local ring, P a finitely generated projective R-module. Then, there is an isomorphism $P \cong R^{\oplus n}$ for some $n \in \mathbb{N}$. (In other words, P is free of rank n).*

Proof. First, $\overline{P} \stackrel{\text{def}}{=} P \otimes_R R/\mathfrak{m} \in Vect(k)^{fg}$, which is a vector space over the residue field $R/\mathfrak{m} \stackrel{\text{def}}{=} k$.
So $\overline{P} \cong k^n$ for $m = \dim_k \overline{P}$ just by choosing a basis $\overline{x}_1, \ldots, \overline{x}_i$. There exist lifts $x_1, \ldots, x_i \in P$. This allows us to define a morphism $\alpha : R^n \to P$, and by Nakayama's lemma this is surjective.
Applying projectivity of P, there exists a section

$$\begin{tikzcd} R^n \arrow[r, two heads, "\alpha"'] & P \arrow[l, bend right, dotted, "\beta"'] \end{tikzcd}$$

where $\alpha \circ \beta = \mathrm{Id}_P$.
Since $\overline{\alpha}$ is an isomorphism, so is $\overline{\beta}$. Thus, $\overline{\beta}$ is surjective.
By another application of Nakayama, β is surjective.
Since B is a right-inverse, we have <u>injectivity</u>.
So $R^n \cong P$. ■

<u>Remark:</u> In the above proof, if we define $p \stackrel{\text{def}}{=} \beta \circ \alpha$, then p is idempotent, so $R^n \cong P \oplus Q$, where Q is the image of p.

Lemma 60. *Let R be a ring and $P \in Proj^{fg}(R)$ be a finitely generated projective R-module. Then the corresponding quasi-coherent sheaf is locally free of finite rank, i.e. $\mathcal{F} =$ "$\mathcal{O}_R \otimes_R P$"*

Proof. P is <u>finitely presented</u>, <u>since</u>:

$$R^n \twoheadrightarrow Q \hookrightarrow R^n \twoheadrightarrow P$$

surjects onto

$$R^n \longrightarrow R^n \longrightarrow P \longrightarrow 0$$

<u>Thus:</u> P is finitely presented.
<u>Then:</u> Apply spreading-out to the previous lemma using that the localization $P_\mathfrak{p}$ is a projective $R_\mathfrak{p}$-module.
In fact, for any ring homomorphism $R \to S$, we have that $P \otimes_R S$ is still a projective S-module.
Thus by the lemma, $P_\mathfrak{p} = R_\mathfrak{p}^n$, by spreading-out germs of sections to actual sections of $\mathcal{F} = \mathcal{O} \otimes_R P$.
By finite presentation of the module P, if these germs of sections give you a basis at the point $\mathfrak{p}$, it will do so on a small neighborhood. The property of being a basis will hold in a small neighborhood by <u>finite presentation</u>. ■

Theorem 0.33 (Serre). *On $X = \operatorname{Spec} R$ there exists an equivalence of categories $Proj^{fg}(R) \cong LocFreeSh^{fg}_{\mathcal{O}}(X)$, the category of locally free $\mathcal{O}$-modules of finite rank.*

Proof. We basically proved all of it today, modulo the following exercise:

Exercise Show that for $\mathcal{F}$ locally free of rank r, the resulting module $\mathcal{F}(|\operatorname{Spec} R|)$ is finitely generated. Note the assumption of affineness is necessary, and the proof uses quasi-compactness.

■

Theorem 0.34 (Swan/Serre-Swan). *Let X be a compact topological space (meaning quasi-compact and Hausdorff). Then there exists an equivalence of categories between topological vector bundles on X, $Vect(X)$, and projective R-modules which are finitely generated, where $R = C(X)$, the ring of continuous functions on X.*
We can think of this as an extension/addendum to the Gelfand-Neymark theorem.

Proof. ■

Lecture 32, 21/11/25

Presentation by: ROBERT ROBERT ROBBBBEERRTT
<u>Reminder:</u> On Monday, we established an adjunction between quasi-coherent quasi-separated schemes over S, Sh_S^{qcqs}, and quasi-coherent $\mathcal{O}_S$-algebras, $\text{-}\mathsf{Alg}(\mathcal{O}_S)$.
The left adjoint $-_* : \mathrm{Sh}_S^{qcqs} \to \text{-}\mathsf{Alg}(\mathcal{O}_S)$ is given by sending $f : Y \to S$ to f_*, the pullback sheaf.
For the other direction, we have the following construction:
Let A be a quasi-cohderent $\mathcal{O}_S$ module, $S = \bigcup \operatorname{Spec} B_i$, $A(U_i)$ is a B_i-algebra.
We have maps $B_i \to A(U_i)$, and gluing along the maps $\operatorname{Spec} A(U_i) \to \operatorname{Spec} B_i$ gives us a map $\pi : \operatorname{Spec}_S(A) \to S$.
The unit of this adjunction is given by

$$\begin{tikzcd} Y \arrow[r, "\eta"] \arrow[dr] & \operatorname{Spec}_S f_*\mathcal{O}_Y \arrow[d] \\ & S \end{tikzcd}$$

Proposition 25. *η is an isomorphism if and only if f is affine.*

Proof. First, assume η is an isomorphism. Then by construction $S = \bigcup_i \operatorname{Spec} B_i$, and we can take $\pi^{-1} \operatorname{Spec} B_i = \operatorname{Spec} A_i$.
Going the other way, we can use the trick that was shown in the proof on monday, where we go locally.

Assume that f is affine. Then if we prove it is isomorphic on local affines in Y, then it is an isomorphism everywhere.
We break this diagram into the pullback:

$$\begin{tikzcd} f^{-1}(\operatorname{Spec} B_i) = \operatorname{Spec} A_i \arrow[r] \arrow[dr] & \operatorname{Spec}_S f_* B_i \arrow[d] & = & \operatorname{Spec} A_i(|A_i|) & = & \operatorname{Spec} A_i \\ & \operatorname{Spec} B_i & & & & \end{tikzcd}$$

■

Ask big Rob for more details if you need them.
Today: Kähler differentials
We all know that differentiation is an extremely useful process in algebra, although the definition is initially not very algebraic.
We have a map $\mathbb{Z}[T] \to \mathbb{Z}[T]$ sending f to f', its formal derivative. This construction is $\mathbb{Z}$-linear, but not a ring map: $(fg)' \neq f'g'$. Instead, it satisfies the familiar Leibniz rule:

$$(fg)' = f'g + fg'$$

Maps of this type are known as derivations.
More generally, we have:

Definition 0.58. Let $R \to S$ be a ring homomorphism and M an S-module.
Then an R-derivation $\delta : S \to M$ is an R-linear map $\delta : S \to M$, satisfying the Leibniz rule:

$$\delta(\lambda x) = \lambda \cdot \delta(x) + \delta(\lambda) \cdot x$$

for $\lambda, x \in S$, where $\cdot$ denotes the S-module structure (which is both a left and a right structure).

Example 0.29.

(a) $\delta : \mathbb{Z}[T] \to \mathbb{Z}[T]$ given by the "derivative of polynomials" is a $\mathbb{Z}$-derivation

(b) Given any $R \to S$, we define $I \overset{\text{def}}{=} \ker S \otimes_R S \twoheadrightarrow S$ (where $\twoheadrightarrow$ is the codiagonal map), and let $\Omega^1_{S/R} \overset{\text{def}}{=} I/I^2$.

Claim. *Given $x \in S$, then $d(x) = 1 \otimes x - x \otimes 1 \in \Omega^1_{S/R} = I/I^2$ is an R-derivation.*

Proof.

- If we take $x \in R$, then $x \otimes 1 = 1 \otimes x$ (more generally, $(xy) \otimes z = y \otimes (xz)$ for all $x \in R$, $y, z. \in S$ (the usual relation of the tensor product).
  This implies $dx = 1 \otimes x - x \otimes 1 = 0$.

- For the next part, we want to show the Leibniz rule. This will show the necessity of quotienting out by I^2. We have the computation:
- For all $x, y \in S$, we have

$$I^2 \ni (1 \otimes x - x \otimes 1)(1 \otimes y - y \otimes 1) = 1 \otimes (xy) - x \otimes y - y \otimes x + (xy) \otimes 1$$

In I/I^2, we have the relation $1 \otimes (xy) + (xy) \otimes 1 = x \otimes y + y \otimes x$
Now consider

$$\begin{aligned} d(xy) &= 1 \otimes (xy) - (xy) \otimes 1 \\ xdy + (dx)y &\overset{\text{def}}{=} x(1 \otimes y - y \otimes 1) + y(1 \otimes x - x \otimes 1) \\ &= x \otimes y - xy \otimes 1 + y \otimes x - xy \otimes 1 \\ &= 1 \otimes xy + xy \otimes 1 - xy \otimes 1 - xy \otimes 1 \\ &= d(xy) \end{aligned}$$

Where the last line follows from the previous computation. This shows it is a derivation.

■

We call the space $\Omega^1_{S/R}$ the Kähler differentials.

Proposition 26. *$d : S \to \Omega^1_{S/R}$ is the universal R-derivation. That is, for all R-derivations $\delta : S \to M$, there is a unique $D : \Omega^1_{S/R} \to m$ which is S-linear such that*

$$\begin{tikzcd} S \arrow[r, "d"] \arrow[dr, "\delta"'] & \Omega^1_{S/R} \arrow[d, dotted, "\exists! D"] \\ & M \end{tikzcd}$$

of Existence. Uniqueness is left as an exercise.
For existence, we prove it as follows.
Construction:
Consider the direct sum $S \oplus M$. This itself has a natural ring structure, given by S-multiplication (on S and M) and the rule $M \cdot M = 0$:

$$(\underbrace{x}_{\in S}, \underbrace{m}_{\in M}) \cdot (y, m) \overset{\text{def}}{=} (xy, ym + xm)$$

There is a natural surjection $S \oplus M \twoheadrightarrow S$ given by $(s, m) \mapsto S$, and a canonical section s_1 given by $s_1(s) = (s, 0)$. This section has the advantage that it is a ring homomorphism.

For any R-derivation $\delta : S \to M$, we can define $s_2 \overset{\text{def}}{=} s_0 + (0, \delta) : S \to S \oplus M$, which is also a ring homomorphism and section, and satisfies $s_2|_R = \iota_{R \hookrightarrow S}$

Claim. *This is a 1-1 correspondence between sections satisfying these properties, and R-derivations.*

Proof. Consider the map $S \otimes_R S \to S \oplus M$ using the map $x \otimes y \mapsto (s_1(x), s_2(x))$ ∎

We then have the commutative diagram

$$\begin{tikzcd}
(S \otimes_R S)/I^2 \arrow[r] & S \oplus M \arrow[d, two heads] \\
I/I^2 \arrow[u, hook] \arrow[r, dotted] & M
\end{tikzcd}$$

Claim. *The dotted map D is the required map.*

Proof. We compute: if $dx = 1 \otimes x - x \otimes 1$, this is sent to

$$1 \otimes s_2(x) - s_1(x) \otimes 1 = \delta(x)$$

∎

Remark:
At a point x in a locally ringed space, we usually identify $\mathfrak{m}_x/(\mathfrak{m}_x)^2 = T_x^*X$ (usual definition of cotangent spaces in differential geometry), also works in classical algebraic geometry, where we call it the Zariski cotangent space.

Lecture 33, 24/11/25

Today: Smooth and étale morphisms and $\mathrm{SL}_{Y/X}$ Reminder: quiz on wednesday which I am going to fail.
This will be an overview because time is running out.
Recall: Last time, for a ring homomorphism $R \to A$, we defined the space of Kähler differentials, $\Omega^1_{A/R} = I/I^2$, where I is the kernel of the multiplication map $(A \otimes_R A) \to A$.
We have a morphism $A \to \Omega^1_{A/R}$, $f \mapsto f \otimes 1 - 1 \otimes f$.
We also have a universal morphism of derivations with values in an A-module.
Consequently, one can redefine $\Omega^1_{A/R}$ as the A-module generated by symbols $df, f \in A$, such that the Leibniz rule holds:

$$d(fg) = fdg + gdf$$

and for any $\lambda \in R$,

$$d\lambda = 0$$

That is, if we call the map $R \to A$ φ,

$$d(\varphi(r)) = 0$$

for any $r \in R$.
Schemes:
Let $f : Y \to X$ be a morphism of schemes.

1. If f is separated, then $\triangle_{Y/X} : Y \to Y \hookrightarrow Y \times_X Y$ is a closed immersion, thus we have a sheaf of ideals $\mathscr{I} = \ker\big(\mathcal{O}_{Y\times_X Y} \to ((\triangle_{Y/X})_* \mathcal{O}_Y)\big)$

Definition 0.59. We can define $\Omega^1_{Y/X} \overset{\text{def}}{=} \mathscr{I}/\mathscr{I}^2 \in \mathrm{QCoh}(Y)$
Note: For $\operatorname{Spec} A \to \operatorname{Spec} R$, we get $\Omega^1_{\operatorname{Spec} A/\operatorname{Spec} R}(|\operatorname{Spec} A|) = \Omega^1_{A/R}$

Lemma 61. *Let R be a ring with ideal I. Then I/I^2 is an R/I-module.*

Proof. Some commutative algebra ∎

In general, $\triangle_{Y/X} : Y \to Y \times_X Y$ is not a closed immersion, but it is locally closed, meaning we have a commutative triangle

$$\begin{tikzcd} Y \arrow[r, "\triangle_{Y/X}"] \arrow[dr, hook] & Y \times_X Y \\ & U \arrow[u, hook] \end{tikzcd}$$

So we can then define $\mathscr{I} = \ker(\mathcal{O}_U \to \triangle_* \mathcal{O}_Y)$.

Definition 0.60. A morphism of schemes $f : Y \to X$ is said to be smooth if:

- It is locally of finite presentation,
- It is flat,
- $\Omega^1_{Y/X}$ is locally free

This is related to a similar definition:

Definition 0.61. A morphism of schemes $f : Y \to X$ is said to be étale if it is

- It is locally of finite presentation
- it is flat
- $\Omega^1_{Y/X} = 0$

The last assumption can be replaced by the assumption that for all for all geometric points $\operatorname{Spec} \overline{k} \in X$, the base change $Y_{\overline{k}} = Y \times_X \operatorname{Spec} \overline{k}$ is a smooth $\overline{k}$-scheme.
Remark: For $k = \overline{k}, X \to \operatorname{Spec} \overline{k}$, smoothness (of X) is equivalent to $\dim_{k/x}(\mathfrak{m}_x/\mathfrak{m}_x^2)$ being locally constant if X is reduced.
$\mathfrak{m}_x/\mathfrak{m}_x^2 \cong (\Omega^1_{X/Y}) \otimes_{\mathcal{O}_{Y/X}} k$

Example 0.30.

- $\mathbb{A}^1_k$ is smooth
- $\mathbb{A}^n_k$ is smooth
- $\mathbb{P}^1_k$ is smooth

Non-example:
$\{xy = 0\} \subset \mathbb{A}^2_k$
If we look at the quasicoherent sheaf $\Omega^1_{X/k}$, it is not locally free at the origin, because the ranks of the fibers are not constant.

Definition 0.62. A scheme will be étale if it is smooth and has zero-dimensional fibres.

Example 0.31. We can take a circle, $\mathbb{G}_{m,\mathbb{C}} \to \mathbb{G}_{m,\mathbb{C}} = \operatorname{Spec} \mathbb{C}[T, T^{-1}]$, given by spec of the map $T \mapsto T^n$. For $n \neq 0$, this is an étale morphism.

Example 0.32. Let $L|k$ be a separable field extension of finite degree. Then $\operatorname{Spec} L \to \operatorname{Spec} k$ is étale.
Recall: We have computed the fibre product $\operatorname{Spec} L \times_{\operatorname{Spec} k} \operatorname{Spec} L \cong \operatorname{Spec} L \times Gal(L|k)$ if the extension is Galois, which gives us $\operatorname{Spec} L \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k} = \operatorname{Spec} \overline{k} \times Gal(L|K)$, which is finitely many $\overline{k}$-points, smooth $\overline{k}$-varieties of dimension 0.
Let $f : Y \to X \to \operatorname{Spec} k$, assume Y/k and X/k are smooth.
Then: f is smooth if and only if the induced map $f^*\Omega^1_{X/k} \to \Omega^1_{Y/k}$ is a surjection.
And f is étale if and onlfy if $f^*\Omega^1_{Y/k} \to \Omega^1_{X/k}$ is an isomorphism.
Remark:
This is exactly like in differential geometry, where surjection of this map corresponds to a submersion in differential geometry, and isomorphism of this map corresponds to being a local diffeomorphism.

Étale coordinates

Up to a certain level, every theorem in differential geometry is proven by using coordinates. We can do a similar thing in algebraic geometry:

Theorem 0.35. *Let X/k be smooth.*
Then, for every $x \in X$, there exists an open neighborhood $U \ni x$ and an étale morphism $f : U \to \mathbb{A}_k^{\dim_k X}$.
We can then use the coordinate functions, as in affine space, and pull them back:

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[dr, "x_i \circ f"'] & \mathbb{A}_k^{\dim_k X} \arrow[d, "x_i"] \\ & \mathbb{A}_k^1 \end{tikzcd}$$

Proof Sketch: For simplicity, let x be a closed point in $X(k)$, then $(\Omega^1_{X/k}) \otimes k \cong \mathfrak{m}_x/\mathfrak{m}_x^2$.
This is a finite-dimensional k-vector space, so we can pick a basis locally represented by $[f_1], \ldots, [f_n]$, represented by elements $\overline{f_i} \in \mathfrak{m}k_x \subset \mathcal{O}_{X,x}$, which are generalized functions.
We can think of those being by definition $df_1(x), \ldots, df_n(x)$.
Taking representatives of these elements, there exists a sufficiently small open neighborhood $U \ni x$ such that $f_i : U \to \mathbb{A}_k^1$ is well-defined for all i.
Then we simply define $f : U \to \mathbb{A}_k^n$.
<u>By construction:</u>

$$df : \left(\Omega^1_{X/k}\right) \otimes_{\mathcal{O}_{X,x}} k \to \underbrace{(\Omega^1_{\mathbb{A}_k^n/k}) \otimes k}_{kdt_1 \oplus \cdots \oplus kdt_n}$$

Thus df is an iso on a small neighobrhood of X.
So f is étale locally near x.

■

Theorem 0.36 (Infinitesimal criterion)**.** *Let $f : Y \to X$ be locally of finite presentation, let $A \twoheadrightarrow B$ be a surjection of rings of finite presentation, and let $I = \ker(A \to B)$ and $I^2 = 0$.*
<u>Then:</u>

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dotted] & X \end{tikzcd}$$

The existence of the dashed arrow for all such diagrams is equivalent to f being smooth, and the unique existence is equivalent to f being étale.

Proof. ■

If Y and X are smooth over k, then we can take $A = k[\varepsilon]/(\varepsilon^2)$, $B = k$, then we are dealing with the diagram

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[r] \arrow[d] & Y \arrow[d] \\
\operatorname{Spec} k[\varepsilon]/(\varepsilon^2) \arrow[r] \arrow[ur, dotted] & X
\end{tikzcd}$$

The existence of a dashed arrow amounts to the surjectivity of the differential between tangent spaces. The unique existence amounts to the differential being an iso, and thus a local diffeomorphism.