Lecture 1 - 1/9/24

Books: Bisi will be using "Lie Algebras, Lie Groups, and Representations" by Hall, 2nd edition
Also "Lectures on Lie groups" by Adams
"Representations of compact Lie groups" by Bröcker
"Introduction to Lie algebras and representation theory"
"Lie Algebras and Lie Groups" by Bourbaki
"Lectures on Lie Groups and Lie algebras," Carter, Segal, et al.

Matrix Groups

Roughly speaking, a Lie group is a smooth manifold together with a smooth group structure.

$$\mu : G \times G \to G \text{ multiplication}$$
$$i : G \to G \text{ inverse}$$

These are smooth maps.

Example 0.1.

- $\mathbb{R}$ with addition
- S^1 under multiplication.
- If G_1, G_2 are Lie Groups, so is their product.
- Any torus is a Lie group. It will turn out that these are the only compact connected Abelian Lie groups.
- $\mathrm{GL}(n, \mathbb{R})$ is an open submanifold of $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$
- $G = \mathbb{R} \times \mathbb{R} \times S^1$ with the group structure

  $$(x_1, y_1, u_1) \cdot (x_2, y_2, u_2) = (x_1 + x_2, y_1 + y_2, e^{ix_1 y_2} u_1 u_2)$$

  We will see later that G has no faithful matrix representation, and so is not isomorphic to a matrix group.

Definition 0.1. A matrix group is a closed subgroup of $\mathrm{GL}(n, \mathbb{C})$
E.g $\mathrm{GL}(n, \mathbb{Q})$ is not closed in $\mathrm{GL}(n, \mathbb{C})$ and so is not a matrix group (although it is a group of matrices).

Definition 0.2. Recall that if $A \in M_n(\mathbb{C})$, then the adjoint of A, A^*, is the conjugate transpose of A, $\overline{A^T}$.
We say that $A \in M_n(\mathbb{C})$ is unitary if $A^*A = I$
This is equivalent to saying that A preserves the standard inner product $\langle x, y\rangle = \sum_i x_i \cdot \overline{y_i}$ on $\mathbb{C}^n$, i.e. $\langle x, y\rangle = \langle Ax, Ay\rangle$. As an exercise, show that $|\det(A)| = 1$
We say that $A \in M_n(\mathbb{R})$ is orthogonal if $A^T A = I$. This is equivalent to saying that A preserves the standard inner product on $\mathbb{R}^n$. Again, $|\det(A)| = 1$, and we obtain $O_n(\mathbb{R})$, $\mathrm{SO}_n(\mathbb{R})$
There are a plethora of examples:

Example 0.2.

- General and special linear groups $\mathrm{GL}_n, \mathrm{SL}_n$
- Unitary and Orthogonal groups $U_n(\mathbb{C}), \mathrm{SU}_n(\mathbb{C})$
- $O_n(\mathbb{R}), \mathrm{SO}_n(\mathbb{R})$
- $O_n(\mathbb{C}), \mathrm{SO}_n(\mathbb{C})$, which are NOT the same as the unitary groups, e.g. they preserve different bilinear forms, i.e. the extension of the standard product on $\mathbb{R}^n$ is NOT an inner product on $\mathbb{C}^n$

Remark
More generally $U(p, q)$ preserves the standard Hermitian form on $\mathbb{C}^n$ of signature (p, q), where $p + q = n$

Symplectic Groups

These are slightly confusing because there are three sets of them, $Sp_n(\mathbb{R}), Sp_n(\mathbb{C}), Sp_n$ and their definition involves skew symmetric rather than symmetric forms.
Suppose $\mathbb{F}$ is $\mathbb{C}$ or $\mathbb{R}$. Consider the following Skew-symmetric form on $\mathbb{F}^{2n}$ given by

$$\omega(x, y) = \sum_{j=1}^{n} (x_j y_{n+j} - x_{n+j} y_j)$$

Here is another way of writing this: if Ω is the block matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, then

$$\omega(x, y) = (x, \Omega y)$$

where $(x, y) = \sum_j x_j y_j$ is the standard inner product if $\mathbb{F} = \mathbb{R}$, and just a bilinear form if $\mathbb{F} = \mathbb{C}$.

Definition 0.3. The symplectic group over $\mathbb{F}$, $S_{p_n}(\mathbb{F})$, is defined by

$$Sp_n(\mathbb{F}) = \{A \in M_{2n}(\mathbb{F}) \mid \omega(Ax, Ay) = \omega(x, y)\} \underbrace{=}_{\text{exercise}} \{A \in M_{2n}(\mathbb{F}) \mid -\Omega A^t \Omega = A^{-1}\}$$

The compact symplectic group Sp_n is defined by

$$Sp_n \overset{\text{def}}{=} Sp_n(\mathbb{C}) \cap U_{2n}$$

The Heisenburg group is the set of all matrices A of the form

$$\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$

Lecture 2 - 1/16/24

Compactness and connectedness

Definition 0.4. A matrix group $G \subseteq \mathrm{GL}(n, \mathbb{C})$ is compact if it is compact as a subset of $M_n(\mathbb{C}) \cong \mathbb{C}^{n^2} \cong \mathbb{R}^{(2n)^2}$
If $G \subseteq M_n(\mathbb{C})$, this is compact if and only if G is closed and bounded.

Example 0.3. $O(n), \mathrm{SO}(n), U(n), \mathrm{SU}(n)$ are compact. The boundedness condition comes from the fact that the columns of any element of any of these things are unit vectors.
On the other hand, $\mathrm{SL}_n(\mathbb{R})(n \geq 2)$ is not compact, because it is unbounded, e.g. take $n = 2$ and consider the matrix

$$\begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$$

Definition 0.5. A matrix group G is path-connected if for any two $A, B \in G$, there is a continuous $\gamma : [0, 1] \to G$ such that $\gamma(0) = A$ and $\gamma(1) = B$ (continuous with respect to its topology as a subspace of $\mathrm{GL}(n, \mathbb{C})$)
Remark: We will see later that connectedness and path-connectedness for a matrix Lie group are equivalent. Henceforth we will assume this and drop the adjective "path."

Example 0.4.
If G is a matrix group, then the connected component containing the identity, called G_0, is a normal subgroup of G.
(Later, after we've learned about the exponential map we'll see that G_0 is closed in G and is itself a matrix group).

To see this, write down some obvious paths, e.g. and suppose that $A \in G_0$, $\Gamma : [0,1] \to G$ is a path with $\Gamma(0) = I$ and $\Gamma(1) = A$.
Then $\Gamma^{-1}[0,1] \to G$, given by composing Γ with the inverse map, $t \mapsto \Gamma(t)^{-1}$ is a continuous path in G connecting I to A^{-1}, and so A and A^{-1} are in the same connected component. We can do the same thing with products and conjugates.

Example 0.5. The group $\mathrm{GL}(n, \mathbb{C})$ is connected for all $n \geq 1$. It is clear for $n = 1$. Assume $n \geq 2$.
Suppose that $A, B \in \mathrm{GL}(n, \mathbb{C})$ and observe that there are only finitely many solutions of the equation $\det(\lambda A + (1 - \lambda)B) = 0$
This is a polynomial in λ which is not identically zero.
Let $\gamma : [0,1] \to \mathbb{C}$ with $\gamma(0) = 0$ and $\gamma(1) = 1$ be a continuous path in $\mathbb{C}$ which avoids the set of solutions of the above equation.
Then if we define $\Gamma(t) = \gamma(t)A + (1 - \gamma(t))B$, $0 \leq t \leq 1$, this is a path in $\mathrm{GL}(n, \mathbb{C})$ that connects B and A.

Example 0.6.
The groups $\mathrm{SL}_n(\mathbb{C})$, $U(n)$, and $\mathrm{SU}(n)$, these are all connected for $n \geq 1$, and the group $\mathrm{GL}(n, \mathbb{R})$ is not connected but has two components.

The Geometry of $\mathrm{SU}(2, \mathbb{C})$

Elements of $\mathrm{SU}(2, \mathbb{C})$ may be written in the form

$$\begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$$

with $|a|^2 + |b|^2 = 1$. Now recall that quaternions $q = t + xi + yj + zk$, with $t, x, y, z \in \mathbb{R}$ may also be identified with 2x2 matrices of the above shape by

$$i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Hence $\mathrm{SU}(2, \mathbb{C})$ is the same as the group of unit quaternions. Topologically, this is just a 3-sphere, and so is simply connected, so every loop in $\mathrm{SU}(2, \mathbb{C})$ may be continuously shrunk to a point.
Remark:
We will see later that if G is a simply connected Lie group, then there is a natural bijection between representations of G and representations of its Lie algebra.

The Matrix Exponential

Recall that there is a norm on $M_n(\mathbb{C})$ which is defined as follows:
If $X \in M_n(\mathbb{C})$, then

$$\|X\| \stackrel{\text{def}}{=} \left(\sum_{i,j} |X_{ij}|^2\right)^{\frac{1}{2}} = (Tr(XX^T))^{\frac{1}{2}}$$

Remark: $\|X^n\| \leq \|X\|^n$

Definition 0.6. If $X \in M_n(\mathbb{C})$, then

$$e^X \stackrel{\text{def}}{=} \sum_{m=0}^{\infty} \frac{X^m}{m!}$$

This converges for all X and is a continuous function of X.
Exercise: Prove this by using the M test and the above remark.

Proposition 1. *Suppose that $X, Y \in M_n(\mathbb{C})$. Then*

1. $e^0 = I$

2. *The matrix $e^X \in \mathrm{GL}(n, \mathbb{C})$ and $(e^X)^{-1} = e^{-X}$*

3. *If $\alpha, \beta \in \mathbb{C}$, then $e^{(\alpha+\beta)X} = e^{\alpha X} \cdot e^{\beta X}$*

4. *If $XY = YX$, then $e^{X+Y} = e^X \cdot e^Y$. Note that these equalities are NOT true in general!*

5. *If C is invertible, then $e^{CXC^{-1}} = Ce^X C^{-1}$*

6. $\|e^X\| \leq e^{\|X\|}$

Proposition 2. *If $X \in M_n(\mathbb{C})$, then e^{tX} is a smooth curve in $M_n(\mathbb{C})$, and $\frac{d}{dt}e^{tX} = Xe^{tX} = e^{tX}X$*

Proof. Differentiate the power series for e^{tX}. ∎

So if we have a first order ODE

$$\frac{dV}{dt} = XV$$

with $V(0) = V_0$, with $V(t) \in \mathbb{R}^n$ and $X \in M_n(\mathbb{C})$, then the unique solution is given by $V(t) = e^{tX}v_0$

Computation

To compute the matrix exponential, use the fact (Jordan canonical form) that every $X \in M_n(\mathbb{C})$ may be written $X = D + N$, where D is diagonal and N is nilpotent. Further, $DN = ND$, so we can use property 4 above.

The matrix logarithm

Recall the situation for the complex logarithm: the power series

$$\log(z) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(z-1)^n}{n}$$

is defined and is analytic for all z with $|z-1| < 1$. For all such z, we have $e^{\log(z)} = z$.
If $|w| < \log(2)$, then $|e^w - 1| < 1$, and $\log(e^w) = w$.
This motivates the following definition.

Definition 0.7. If $A \in M_n(\mathbb{C})$, then we define

$$\log(A) = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(A-I)^m}{m}$$

whenever the series converges.
$\log(A)$ is defined and continuous on $\{A \in M_n(\mathbb{C}) \mid \|A - I\| < 1\}$ because $\|(A-I)^m\| \leq \|A - I\|^m$, plus convergence properties for the series of the complex logarithm.

(a) On $\{A \in M_n(\mathbb{C}) \mid \|A - I\| < 1\}$, we have $e^{\log(A)} = A$

(b) If $\|A\| < \log(2)$, then $\|e^A - I\| < 1$ and we have $\log(e^A) = A$

The idea we use to prove this is to prove it for diagonal matrices, and use the fact that diagonalizable matrices are dense, and everything in sight is continuous, as well as the fact that the matrix exponential behaves well with respect to conjugation.

Lecture 3 - 1/23/24

Here are some more useful properties of the matrix logarithm.

1. There exists a constant c such that on the set $\{B \in M_n(C) \mid \|B\| < \frac{1}{2}\}$ we have

$$\|\log(I + B) - B\| < c\|B\|^2$$

2. $\log(I + B) = B + O(\|B\|^2)$

Proof. Observe that

$$\log(I + B) - B = B^2 \sum_{m=2}^{\infty} (-1)^{m+1} \frac{B^{m+2}}{m}$$

and this implies the result. ∎

To motivate the next result, recall that we have

$$e^x = \lim_{m\to\infty}\left(1+\frac{x}{n}\right)^m$$

We're going to use the following result to classify one-parameter subgroups of $\mathrm{GL}(n,\mathbb{C})$

Proposition 3. *Suppose that $X \in M_n(\mathbb{C})$ and that $\{C_m\} \in M_n(\mathbb{C})^{\mathbb{N}}$ is a sequence of matrices such that $\|C_m\| \leq \frac{constant}{m^2}$. Then*

$$\lim_{x\to\infty}(I+\frac{X}{m}+C_m)^m = e^X$$

Proof. When M is large, the left-hand side the above lies in the domain of log, and we have

$$\log(I+\frac{X}{m}+C_m) = \frac{X}{m}+C_m+E_m$$

where E_m is an error term, which satisfies the condition $\|E_m\| \leq c\|\frac{X}{m}+C_m\|^2$ Because of the conditions on C_m, we know that this is less than or equal to $\frac{\text{constant}}{m^2}$ Then

$$I+\frac{X}{m}+C_m = \exp\left(\frac{X}{m}+C_m+E_m\right)$$

So

$$\left(I+\frac{X}{m}+C_m\right)^m = \exp(X+mC_m+mE_m)$$

The result follows because $\|C_m\|, \|E_m\| = O(\frac{1}{m^2})$, and the exponential is continuous. ∎

Recall that in general $e^{X+Y} \neq e^X \cdot e^Y$ unless X and Y commute.

Theorem 0.1 (The Lie Product Formula)**.** *Suppose that $X, Y \in M_n(\mathbb{C})$. Then*

$$e^{X+Y} = \lim_{m\to\infty}\left(e^{\frac{X}{m}}\cdot e^{\frac{Y}{m}}\right)^m$$

Proof. We have $e^{\frac{X}{m}}\cdot e^{\frac{Y}{m}} = I+\frac{X}{m}+\frac{Y}{m}+O(\frac{1}{m^2})$. For m large, we may write

$$\begin{aligned}\log(e^{\frac{X}{m}}\cdot e^{\frac{Y}{m}}) &= \log(I+\frac{X}{m}+\frac{Y}{m}+O(\frac{1}{m^2}))\\ &= \frac{X}{m}+\frac{Y}{m}+O(\frac{1}{m^2})\end{aligned}$$

Therefroe, exponentiating both sides gives

$$e^{\frac{X}{m}} \cdot e^{\frac{Y}{m}} = \exp(\frac{X}{m} + \frac{Y}{m} + O(\frac{1}{m^2}))$$

whence

$$(e^{\frac{X}{m}} \cdot e^{\frac{Y}{m}})^m = \exp(X + Y + O(\frac{1}{m}))$$

and this implies the result.

Theorem 0.2. *If* $X \in M_n(\mathbb{C})$, *then* $\det(e^X) = e^{\mathrm{tr}(X)}$.

Proof. Use the same idea as before: prove it for diagonalizable matrices and then use the fact that diagonalizable matrices are dense.

Definition 0.8. A function $A : \mathbb{R} \to \mathrm{GL}(n, \mathbb{C})$ is called a <u>one-parameter subgroup of $\mathrm{GL}(n, \mathbb{C})$</u> if it satisfies the following conditions:

1. A is continuous

2. $A(t + s) = A(t)A(s)$ for all $t, s \in \mathbb{R}$

Note the second item implies that $A(0) = I$

Theorem 0.3. *If A is a one-parameter subgroup in* $\mathrm{GL}(n, \mathbb{C})$, *then there exists a unique* $X \in M_n(\mathbb{C})$ *such that*

$$A(t) = e^{tX}$$

Proof. We start by showing uniqueness. If such an X exists, then

$$X = \frac{d}{dt}|_{t=0} A(t)$$

and so X must be unique.
Now we show existence.

Claim. *A is smooth*

Proof. Let $f(s)$ be a smooth real-valued function supported on a small neighborhood of 0 with $f(s) \geq 0$ and $\int_{\mathbb{R}} f(s)\, ds = 1$(i.e. a "bump function").
Consider:

$$B(t) \stackrel{\text{def}}{=} \int_{\mathbb{R}} A(t + s) f(s)\, ds$$

By a change of variables, this is equal to

$$B(t) = \int_{\mathbb{R}} A(u) f(u - t)\, du$$

We can then differentiate under the integral by t, so then $B(t)$ is smooth because f is smooth.
Appealing to the fact that $A(t+s) = A(t)A(s)$, we obtain

$$B(t) = A(t) \int_{\mathbb{R}} A(s)f(s)\, ds$$

The condition on f and the continuity of A imply that the constant matrix $\int_{\mathbb{R}} A(s)f(s)\, ds$ is closed to $A(0) = I$, and so is invertible.
Hence we have

$$A(t) = B(t) \left(\int_{\mathbb{R}} A(s)f(s)\, ds \right)^{-1}$$

and so A is smooth as claimed.
Now we define

$$X = \frac{d}{dt}|_{t=0} A(t)$$

Claim. $A(t) = e^{tX}$

Proof. Because $A(t)$ is smooth, for small t we have

$$\|A(t) - A(0) - tA'(0)\| = \|A(t) - (I - tX)\| = O(t^2)$$

And so for each fixed t and sufficiently large m, we have

$$A(\frac{t}{m}) = I + (\frac{t}{m})X + O(\frac{1}{m^2})$$

Since A is a one-parameter subgroup, we have

$$\begin{aligned} A(t) &= (A(\frac{t}{m}))^m \\ &= (I + \frac{t}{m}X + O(\frac{1}{m^2}))^m \end{aligned}$$

and the result follows on letting m tend to ∞ and applying our earlier proposition. ■

The Lie Algebra of a matrix group

Let G be a matrix group. The Lie Algebra $\mathfrak{g}$ is the set of all matrices X such that $e^{tX} \in G$ for all $t \in \mathbb{R}$ (i.e. the set of all X such that the one-parameter subgroup corresponding to X lies in G).

Lecture 4 - 1/25/24

Last time we defined the Lie Algebra $\mathfrak{g}$ of a matrix group $G \subseteq M_n(\mathbb{C})$ as all matrices $X \in M_n(\mathbb{C})$ such that $e^{tX} \in G$ for all real numbers t.

Example 0.7. Let's examine general linear groups.
Suppose that $X \in M_n(\mathbb{C})$. Then $\det(e^{tX}) = e^{\operatorname{tr}(X)}$, which is never zero, so $e^{tX} \in \mathrm{GL}(n,\mathbb{C})$ for all t, so the Lie algebra $\mathfrak{g}l_n$ of $\mathrm{GL}(n,\mathbb{C})$ is just $M_n(\mathbb{C})$.
Similarly, if $X \in M_n(\mathbb{C})$, then $e^{tX} \in \mathrm{GL}(n,\mathbb{R})$ for all $t \in \mathbb{R}$. Also we have

$$X = \frac{d}{dt}(e^{tX})|_{t=0}$$

is real. So the Lie algebra $\mathfrak{g}l_n(\mathbb{R})$ of $\mathrm{GL}(n,\mathbb{R})$ is equal to $M_n(\mathbb{R})$
Consequence:
The above argument shows that if $G \subseteq \mathrm{GL}(n,\mathbb{R})$, then $\mathfrak{g} \subseteq M_n(\mathbb{R})$

Special linear groups

If $\operatorname{tr}(X) = 0$, then $\det(e^{tX}) = 1$ for all $t \in \mathbb{R}$.
Conversely, if $\det(e^{tX}) = 1$ for all $t \in \mathbb{R}$, then $e^{t\operatorname{tr}(X)} = 1$ for all $t \in \mathbb{R}$, and so $t \cdot \operatorname{tr}(X) = 2\pi i m, (m \in \mathbb{Z})$, so we must have $\operatorname{tr}(X) = 0$.
Thus $\mathfrak{s}l_n(\mathbb{C}) = \{X \in M_n(\mathbb{C}) \mid \operatorname{tr}(X) = 0\}$

Unitary groups

Recall that U is unitary if and only if $U^* = U^{-1}$. So e^{tX} is unitary if and only if

$$\begin{aligned}(e^{tX})^* &= (e^{tX})^{-1} \\ &= e^{-tX}\end{aligned}$$

and hence $(e^{tX})^* = e^{tX^*}$, we see that e^{tX} is unitary iff

$$e^{tX^*} = e^{-tX}$$

This holds if $X^* = -X$. Differentiating both sides of the above, and evaluate at $t = 0$ shows that the above implies $X^* = -X$.
So we have

$$\mathfrak{s}l_n(\mathbb{C}) = \{X|inM_n(\mathbb{C}) \mid X^* = -X\}$$

we further deduce that $\mathfrak{s}u_n(\mathbb{C}) = \{X \in M_n(\mathbb{C}) \mid X^* = -X, \operatorname{tr}(X) = 0\}$

Orthogonal groups

Recall that O_n has two connected components, and the one containing the identity is SO_n.
Since the exponential of a matrix in the Lie algebra is automatically in the identity component of the group, we see that the Lie Algebra of O_n, $\mathfrak{o}_n$, is equal to the Lie algebra of SO_n
Recall that $R \in M_n(\mathbb{R})$ is orthogonal if $R^T = R^{-1}$. So if $X \in M_n(\mathbb{R})$, then e^{tX} is orthogonal if and only if $(e^{tX})^T = (e^{tX})^{-1}$, i.e.

$$e^{(tX)^T} = e^{-tX}$$

This in turn happens iff $X^t = -X$.
So $\mathfrak{o}_n(\mathbb{R}) = \mathfrak{s}o_n(\mathbb{R}) = \{X \in M_n(\mathbb{R}) \mid X^T = -X\}$

Symplectic groups

Recall that

$$\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

Then

$$\mathfrak{s}p_n(\mathbb{R}) = \{X \in M_n(\mathbb{R}) \mid \Omega X^T \Omega = X\}$$
$$\mathfrak{s}p_n(\mathbb{C}) = \{X \in M_n(\mathbb{C}) \mid \Omega X^T \Omega = X\}$$

and $\mathfrak{s}p_n = \mathfrak{s}p_n(\mathbb{C}) \cap U_{2n}$.

Properties of Lie Algebras and matrix groups

Proposition 4. *Let G be a matrix group with Lie algebra $\mathfrak{g}$, and suppose that $X \in \mathfrak{g}$. Then $e^X \in G^\circ$ (meaning the connected component containing the identity).*

Proof. Observe that the continuous path $\Gamma : [0,1] \to G$ defined by $t \mapsto e^{tX}$ connects I to e^X

■

Proposition 5. *Let G be a matrix group with Lie Algebra $\mathfrak{g}$. Suppose that $X \in \mathfrak{g}$ and $A \in G$. Then $AXA^{-1} \in \mathfrak{g}$.*

Proof.
Observe that $e^{t(AXA^{-1})} = Ae^{tX}A^{-1}$, whence the result follows because $e^{tX} \in G$, and $A \in G$

■

Proposition 6. *Let G be a matrix group with Lie algebra $\mathfrak{g}$, and suppose $X, Y \in \mathfrak{g}$. Then*

1. $sX \in \mathfrak{g}$ *for all* $s \in \mathbb{R}$

2. $X + Y \in \mathfrak{g}$

3. $XY - YX \in \mathfrak{g}$

Proof.

1. Observe that $e^{t(sX)} = e^{(ts)X} \in G$ for all $t \in \mathbb{R}$.

2. If $XY = YX$, then $e^{t(X+Y)} = e^{tX}e^{tY} \in G$, and this holds for all $t \in \mathbb{R}$. In general, we have

$$e^{t(X+Y)} = \lim_{m\to\infty} (e^{t\frac{X}{m}} \cdot e^{t\frac{Y}{m}})^m$$

This is an invertible limit of elements of G, which therefore lies in G, as claimed.

Remark: We have no shown that $\mathfrak{g}$ is a real vector space.

3. Observe that

$$\begin{aligned}\frac{d}{dt}(e^{tX}Ye^{-tX})|_{t=0} &= (XY)e^0 + (e^0Y)(-X) \\ &= XY - YX\end{aligned}$$

Now $e^{tX}Ye^{-tX} \in \mathfrak{g}$ for all $t \in \mathbb{R}$. Since $\mathfrak{g}$ is a real vector space, the derivative of a smooth curve in $\mathfrak{g}$ also lies in $\mathfrak{g}$, and hence $XY - YX \in \mathfrak{g}$

■

Definition 0.9. If $A, B \in M_n(\mathbb{C})$, we define the Lie Bracket $[A, B]$ by

$$[A, B] \stackrel{\text{def}}{=} AB - BA$$

We have therefore shown that the Lie Algebra is closed under the Lie Bracket.

Theorem 0.4. *Suppose that G, H are matrix groups with corresponding Lie Algebras $\mathfrak{g}, \mathfrak{h}$. Let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a homomorphism of matrix groups (i.e. a continuous homomorphisms of the underlying groups).*
Then there exists a unique $\mathbb{R}$-linear map $\tilde{\phi} : \mathfrak{g} \to \mathfrak{h}$ such that the following square commutes:

$$\begin{tikzcd} G \arrow[r, "\phi"] & H \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\tilde{\phi}"] & \mathfrak{h} \arrow[u, "\exp"'] \end{tikzcd}$$

Furthermore,

1. *If $X \in \mathfrak{g}$ and $A \in G$, then*

$$\tilde{\phi}(AXA^{-1}) = \phi(A)\tilde{\phi}(X)\phi(A^{-1})$$

2. *For all $X, Y \in \mathfrak{g}$, we have*

$$\tilde{\phi}([X, Y]) = [\tilde{\phi}(X), \tilde{\phi}(Y)]$$

So $\tilde{\phi}$ is a Lie Algebra homomorphism

3. *If $X \in \mathfrak{g}$, then*

$$\tilde{\phi}(X) = \frac{d}{dt}\phi(e^{tX})|_{t=0}$$

Note this makes sense because $\phi(e^{tX})$ is a one parameter subgroup of H, so is smooth.

4. *If $\psi : H \to K$ is a homomorphism of matrix groups, then $\widetilde{\psi \circ \phi} = \tilde{\psi} \circ \tilde{\phi}$*

Proof. Observe that since ϕ is continuous, $\phi(e^{tX})$ is a one-parameter subgroup of $H \subseteq \mathrm{GL}_n(\mathbb{C})$. Hence there exists a unique $Z \in \mathrm{GL}_n(\mathbb{C})$ such that

$$\phi(e^{tX}) = e^{tZ}$$

for all $t \in \mathbb{R}$.
It follows also that $Z \in \mathfrak{h}$ because $e^{tZ} = \phi(e^{tX}) \in H$ for all $t \in \mathbb{R}$. Set $\tilde{\phi}(X) = Z$.
Then commutativity of the diagram follows on setting $t = 1$ in the above.
Let us now show that $\tilde{\phi}$ is $\mathbb{R}$-linear.
First observe that $\tilde{\phi}(sX) = s\tilde{\phi}(X)$ for all $s \in \mathbb{R}$, because the above equation implies

$$\phi(e^{tsX}) = e^{tsZ}$$

Next:

$$\begin{aligned}
e^{t\tilde{\phi}(X+Y)}B &= e^{\tilde{\phi}(t(X+Y))} \\
&= \phi(\lim_{m\to\infty}(e^{t\frac{X}{m}} \cdot e^{t\frac{Y}{m}})^m) \\
&= \lim_{m\to\infty}(\phi(e^{t\frac{X}{m}}) \cdot \phi(e^{t\frac{Y}{m}}))^m \\
&= \lim_{m\to\infty}(e^{t\tilde{\phi}(X)/m}e^{t\tilde{\phi}}(M)/m)^m \\
&= e^{t(\tilde{\phi}(X)+\tilde{\phi}(Y))}
\end{aligned}$$

Then we differentiate both sides and evaluate at $t = 0$ to obtain

$$\tilde{\phi}(X + Y) = \tilde{\phi}(X) + \tilde{\phi}(Y)$$

So now we have shown that $\tilde{\phi}$ is $\mathbb{R}$-linear.

Lecture 5 - 1/30/24

We continue the proof.

1. We have

$$\begin{aligned}\exp(t\tilde{\psi}(AXA^{-1}) &= \phi(\exp(tAXA^{-1}) \\ &\vdots \\ &= \phi(A)\exp(t\tilde{\phi}(X))\phi(A)^{-1}\end{aligned}$$

Now differentiate both sides at $t = 0:$

$$\tilde{\phi}(AXA^{-1}) = \phi(A)\tilde{\phi}(X)\phi(A)^{-1}$$

2. Recall that

$$[X, Y] = \frac{d}{dt}(e^{tX}Ye^{-tX})|_{t=0}$$

So

$$\begin{aligned}\tilde{\phi}[X, Y] &= \tilde{\phi}(\frac{d}{dt}(e^{tX}Ye^{-tX})|_{t=0} \\ &= \frac{d}{dt}\tilde{\phi}(e^{-tX}Ye^{-tX})|_{t=0} \\ &= \frac{d}{dt}e^{t\tilde{\phi}(X)}\tilde{\phi}(Y)e^{-t\tilde{\phi}(X)} \\ &= [\tilde{\phi}(X), \tilde{\phi}(Y)]\end{aligned}$$

3. Suppose that η is another sudch map. Then

$$\begin{aligned}e^{t\eta(X)} &= e^{\eta(tX)} \\ &= \phi(tX)\end{aligned}$$

So

$$\eta(X) = \frac{d}{dt}\phi e^{tX}|_{t=0}$$

Whence $\eta = \tilde{\phi}$

4. Exercise

■

Adjoint Maps

Definition 0.10. Let G be a matrix group with Lie Algebra $\mathfrak{g}$. For each element $A \in G$, we define its adjoint map as a linear map by

$$\mathrm{Ad}_A : \mathfrak{g} \to \mathfrak{g}$$

by

$$X \mapsto AXA^{-1}$$

We write Ad for the map $A \mapsto \mathrm{Ad}_A$.

Proposition 7. *For each $A \in G$, $\mathrm{Ad}_A \in \mathrm{GL}(\mathfrak{g})$, and $(\mathrm{Ad}_A)^{-1} = \mathrm{Ad}_{(A^{-1})}$. The map $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is a homorphism of matrix groups. Further,*

$$\mathrm{Ad}_A[X, Y] = [\mathrm{Ad}_A(X), \mathrm{Ad}_A(Y)]$$

for all $X, Y \in \mathfrak{g}$.

Proof. Exercise

∎

It follows that (viewing $\mathrm{GL}(\mathfrak{g})$ as a matrix group) there is a map

$$\tilde{\mathrm{Ad}} \stackrel{\text{def}}{=} \mathrm{ad} : \mathfrak{g} \to \mathrm{GL}(\mathfrak{g})$$

defined by

$$X \mapsto \mathrm{ad}_X$$

Such that

$$e^{\mathrm{ad}_X} = \mathrm{Ad}_{e^X}$$

for all $X \in G$

Proposition 8. *Let G be a matrix group with Lie Algebra $\mathfrak{g}$. Then for all $X, Y \in \mathfrak{g}$, we have*

$$\mathrm{ad}_X(Y) = [X, Y]$$

Proof. Recall from the contruction of $\tilde{\phi}$ given earlier that we have

$$\mathrm{ad}_X = \frac{d}{dt}(\mathrm{Ad}_{e^{tX}})|_{t=0}$$

Hence

$$\mathrm{ad}_X(Y) = \frac{d}{dt}(e^{tX} Y e^{-tX})|_{t=0} = [X, Y]$$

∎

The Exponential Mapping

Definition 0.11. If G is a matrix group with Lie Algebra $\mathfrak{g}$, then the exponential map for G is the map

$$\exp : \mathfrak{g} \to G$$

In general, this is neither injective nor surjective. However, locally it is a homeomorphism.

Theorem 0.5. *If G is a matrix group with Lie Algebra $\mathfrak{g}$, then there exists an open neighborhood U of zero in $\mathfrak{g}$ and V of I in G such that* $\exp$ *maps U homeomorphically onto V.*

Proof. We already know that this result holds for $\mathrm{GL}_n(\mathbb{C})$ because of what we've proven about the logarithm.
To prove the result in general, we need a preparatory lemma:

Lemma 1. *Suppose that $\{g_n\} \in G^{\mathbb{N}}$ and that $g_n \to I$. Let $y_n = \log(g_n)$ and note that this is defined for all sufficiently large n.*
Suppose that

$$y_n/\|y_n\| \to y \in \mathfrak{g}l_n(\mathbb{C})$$

Then $y \in \mathfrak{g}$

Proof.
We want to show that $\exp(ty) \in G$ for all $t \in \mathbb{R}$.
We first observe that $(t/\|y_n\|)y_n \to ty$ as $n \to \infty$. Since $g_n \to I$, it follows that $y_n \to 0$, and so $\|y_n\| \to 0$.
We define $m_n = [\frac{t}{\|y_n\|}]$, where $[-]$ means the integer part.
Then $m_n\|y_n\| \to t$ as $n \to \infty$
So

$$\exp(m_n y_n) = \exp(m_n\|y_n\|(\frac{y_n}{\|y_n\|}) \to \exp(ty)$$

as $n \to \infty$. However,

$$\begin{aligned} \exp(m_n y_n) &= \exp(y_n)^{m_n} \\ &= (g_n)^{m_n} \\ &\in G \end{aligned}$$

Since G is closed, we deduce that $\exp(ty) \in G$. t was arbitrary, so we are done. ∎

Back to the proof of the theorem. We have $\mathfrak{g}l_n(\mathbb{C}) \cong C^{n^2} = \mathbb{R}^{2n^2}$. We view $\mathfrak{g}$ as a subspace of $\mathbb{R}^{2n^2}$. Write $\mathcal{D}$ for the orthogonal complement of $\mathfrak{g}$ under the standard inner product on $\mathbb{R}^{2n^2}$. Consider the map $\Phi : \mathfrak{g} \oplus \mathcal{D} \to \mathrm{GL}_n(\mathbb{C})$ defined by $(X, Y) \mapsto e^X e^Y$.

We view Φ as a map from $\mathbb{R}^{2n^2} \to \mathbb{R}^{2n^2}$. Note that $\mathrm{GL}_n(\mathbb{C})$ may be viewed as an open subset of $\mathfrak{g}l_n(\mathbb{C}) \cong \mathbb{R}^{2n^2}$.
Now we show that Φ is locally invertible at the origin: we have

$$\frac{d}{dt}\Phi(tX,0)|_{t=0} = X$$

$$\frac{d}{dt}\Phi(0,tY)|_{t=0} = Y$$

So the derivative at the origin is the identity map, which is invertible. The Inverse Function Theorem now implies that Φ has a continuous local inverse defined in an open neighborhood of I.
Now let U be any open neighborhood of 0 in $\mathfrak{g}$.

Claim. $\exp(U)$ *contains an open neighborhood of I.*

Proof. Suppose not. Then we may find a sequence $\{g_n\} \in G^{\mathbb{N}}$ such that $g_n \to I$ and such that no g_n lies in $\exp(U)$. But since Φ is locally invertible at I, it follows that for large n we may write (uniquely)

$$g_n = \exp(X_n)\exp(Y_n)$$

where $X_n \in \mathfrak{g}, Y_n \in \mathcal{D}$. Now, $g_n \to I$ and Φ^{-1} is continuous, so $X_n, Y_n \to 0$. So for large n, $X_n \in U$, and so we must have $Y_n \neq 0$. Otherwise, we would have $g_n \in \exp(U)$.
Now set

$$\tilde{g_n} = \exp(Y_n) = \exp(-X_n)g_n$$

Thus $\tilde{g_n} \in G$ and $\tilde{g_n} \to I$
As the unit ball in $\mathcal{D}$ is compact, we may choose a subsequence of Y_n, which after relabeling we still call Y_n, such that $Y_n/\|Y_n\| \to Y$, whence $Y \in \mathcal{D}$ and $\|Y\| = 1$.
Now our preparatory lemma implies that $Y \in \mathfrak{g}$, contradicting the fact that $\mathfrak{g} \perp \mathcal{D}$.
This establishes the claim, namely that if U is an open neighborhood of 0 in $\mathfrak{g}$, then $\exp(U)$ contains an open neighborhood of I in G.
If U is sufficiently small, then exp is injective on $\overline{U}$ (the existence of the matrix logarithm implies that exp is injective near 0).
Let log denote the inverse map defined on $\exp(\overline{U})$. Then log is continuous because $\overline{U}$ is compact and exp is 1-1 and continuous. Now let $V \subseteq \exp(\overline{U})$ be an open neighborhood of I and set $U_1 = \log(V)$. Then U_1 is open and $\exp|_{U_1} \to V$ is a homeomorphism.
This completes the proof.

∎

Lie Algebras

Definition 0.12. A Lie Algebra over a field F is a vector space $\mathfrak{g}$ equipped with a product $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the following properties:

1. $[-,-]$ is bilinear

2. $[X,X] = 0$ (hence in particular $[X,Y] = -[Y,X]$).

3. The Jacobi Identity:

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$$

We've seen many examples. Here is another:

Definition 0.13. If A is an associative F-algebra, then define $[-,-] : A \times A \to A$ by

$$[X,Y] \stackrel{\text{def}}{=} XY - YX$$

This gives A the structure of a Lie Algebra.

Definition 0.14. If $\mathfrak{g}_1, \mathfrak{g}_2$ are Lie Algebras, then a Lie algebra homomorphism is an F-linear map $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ satisfying

$$\phi[X,Y] = [\phi(X), \phi(Y)]$$

Definition 0.15. Suppose that U_1, U_2 are subspaces of a Lie algebra $\mathfrak{g}$. Define

$$[U_1, U_2] = \text{span}\{[u_1, u_2] \mid u_i \in U_i, i = 1, 2\}$$

Lecture 6 - 2/1/24

It's time for more definitions and adjectives.

Definition 0.16. A subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a subspace of $\mathfrak{g}$ satisfying $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$.

Definition 0.17. An ideal $\mathfrak{h} \subseteq \mathfrak{g}$ is a subalgebra which satisfies $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$.
If $\mathfrak{h}_1, \mathfrak{h}_2$ are ideals in $\mathfrak{g}$, then so is $[\mathfrak{h}_1, \mathfrak{h}_2]$.

Definition 0.18. We say a Lie algebra $\mathfrak{g}$ is simple if the only ideals of $\mathfrak{g}$ are $\mathfrak{g}$ and 0 and $\dim \mathfrak{g} \geq 2$.

Definition 0.19. We say that $\mathfrak{g}$ is abelian if $[\mathfrak{g}, \mathfrak{g}] = 0$. So if $\mathfrak{g}$ is abelian and $\dim \mathfrak{g} \geq 2$, then $\mathfrak{g}$ is simple.

Definition 0.20. A lower central series is a sequence

$$\mathfrak{g} = \mathfrak{g}^1 \supseteq \mathfrak{g}^2 \supseteq \mathfrak{g}^3 \supseteq \cdots$$

of ideals of $\mathfrak{g}$, generated inductively by letting $\mathfrak{g}^{n+1} = [\mathfrak{g}, \mathfrak{g}^n]$.

Definition 0.21. We say $\mathfrak{g}$ is nilpotent if $\mathfrak{g}^i = 0$ for large i.
Every Abelian $\mathfrak{g}$ is nilpotent. So is

$$\mathfrak{g} = \{A \in M_n(\mathbb{F}) \mid A_{ij} = 0 \forall i \geq j\}$$

Definition 0.22. We define a descending chain by

$$\mathfrak{g}^{(0)} = \mathfrak{g} \supseteq \mathfrak{g}^{(1)} \supseteq \mathfrak{g}^{(2)} \supseteq \cdots$$

of ideals of $\mathfrak{g}$ inductively by $\mathfrak{g}^{(n+1)} = [g^{(n)}, \mathfrak{g}^{(n)}]$.

Definition 0.23. We say that $\mathfrak{g}$ is soluble if $\mathfrak{g}^{(i)} = 0$ for large i.

Proposition 9. *Every nilpotent Lie algebra is soluble.*

Proof. Exercise

Definition 0.24. We say that $\mathfrak{g}$ is semi-simple if it has no non-zero soluble ideals.

Representations

Suppose that $\mathfrak{g}$ is a Lie algebra over a field $\mathbb{F}$.

Definition 0.25. A representation of $\mathfrak{g}$ is a homomorphism

$$\rho : \mathfrak{g} \to \mathfrak{g}l_n(\mathbb{F}) = [\mathrm{End}(\mathbb{F}^n)]$$

for some n.
We say that two such representations ρ_1, ρ_2 are equivalent if there is $T \in \mathrm{GL}_n(\mathbb{F})$ such that

$$\rho_1(X) = T^{-1}\rho_2(X)T$$

Definition 0.26. A left $\mathfrak{g}$-module is an $\mathbb{F}$-vector space V equipped with a left action

$$\mathfrak{g} \times V \to V : (x, v) \mapsto x \cdot v$$

such that

1. The map $(x, v) \mapsto x \cdot v$ is bilinear

2. We have

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$$

for all $x, y \in \mathfrak{g}, v \in V$

Every finite-dimensional $\mathfrak{g}$-module yields a representation of $\mathfrak{g}$ in the usual way.

Definition 0.27. The adjoint module is $\mathfrak{g}$ viewed as a module over itself, and the corresponding representation is called the adjoint representation.
Notation
If U is a subspace of V, and $\mathfrak{h}$ is a subspace of $\mathfrak{g}$, then we write

$$[\mathfrak{g}, U] = \operatorname{Span}\{[X, u] \mid X \in \mathfrak{h}, u \in U)\}$$

Definition 0.28. We say that U is a submodule of V, $[\mathfrak{g}, U] \subseteq U$.
Say that V is irreducible if its only submodules are V and 0.

Representations of compact matrix groups

Theorem 0.6. *Suppose that G is a unitary group and that*

$$\Pi : G \to \mathrm{GL}(V)$$

is a finite dimensional unitary representation. Then Π is completely reducible.
If $\mathfrak{g}$ is a real Lie algebra and $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ is finite-dimensional and unitary (i.e. $\pi(X)^* = -\pi(X)$ for all $X \in \mathfrak{g}$), then π is completely reducible.

Proof. Let $\langle -, - \rangle$ be the inner product on V, and suppose that W is an invariant subspace for Π or π. Then

$$V = W \oplus W^{\perp}$$

and $W^{\perp}$ is an invariant subspace (Check this!). The result follows by induction on the dimension of V.
(Bisi is using unitary to mean "preserves an inner product")

■

Theorem 0.7. *Suppose that G is a compact matrix group. Then every finite dimensional representation of G*

$$\Pi : G \to \mathrm{GL}(V)$$

is unitary (and so by the previous theorem is completely reducible).

Proof. The key point is that there is a left-invariant Haar measure $d\mu$ on G (which here may be constructed explicitly using the existence of a volume form on G).
If $\langle -, - \rangle$ is any inner product on V define another inner product $\langle -, - \rangle_G$ on V by

$$\langle v, w \rangle_G = \int_G \langle \Pi(g)v, \Pi(g)w \rangle \, d\mu$$

This integral converges because of compactness.

Representations of $\mathfrak{sl}_2(\mathbb{C})$

Here is a basis of $\mathfrak{sl}_2(\mathbb{C})$:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Here are some relations:

$$[H, X] = 2X, [H, Y] = -2Y, [X, Y] = H$$

Suppose that

$$\pi : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V)$$

is an irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$
(recall $\mathfrak{gl}(V)$ is the Lie algebra of the matrix group $\mathrm{GL}(V)$)

Lemma 2. *Suppose that u is an eigenvalue of H with eigenvalue α. Then*

$$\pi(H)\pi(X)u = (\alpha + 2)\pi(X)u$$

Henve either $\pi(X)u = 0$ or $\pi(X)u$ is an eigenvalue of $\pi(H)$ with eigenvalue $\alpha + 2$. Similarly,

$$\pi(H)\pi(Y)u = (\alpha - 2)\pi(Y)u$$

so a similar statement holds.
Here we have not in fact assumed that V is irreducible.

Proof. We have *e.g.*

$$[\pi(H), \pi(X)] = \pi(([H, X])) = 2\pi(X)$$

and so

$$\begin{aligned} \pi(H)\pi(X)u &= \pi(X)\pi(H)u + 2\pi(X)u \\ &= \pi(X)(\alpha u) + 2\pi(X)u \\ &= \pi(\alpha + 2)\pi(X)u \end{aligned}$$

■

Theorem 0.8. *For each integer $m \geq 0$, there is an irreducible complex representation of $\mathfrak{sl}_2(\mathbb{C})$ of dimension $m + 1$, and any two such representations are isomorphic.*

Proof. Let u be an eigenvector for $\pi(H)$ with eigenvalue α.
Then for each $k \geq 0$, we have

$$\pi(H)\pi(X)^k u = (\alpha + 2k)\pi(X)^k u$$

Choose $N \geq 0$ to be the largest integer satisfying $\pi(X)^N u \neq 0$ (this exists as $\pi(H)$ has only finitely many eigenvalues)
Set $\lambda = \alpha + 2N$
For each $k \geq 0$, set

$$u_k = \pi(Y)^k u$$

Check that we have the following relations:

$$\pi(H)u_k = (\lambda - 2k)u_k$$

Set $u_0 = \pi(X)^N u$, so $\pi(X)u_0 = 0$. So by induction, $\pi(X)u_k = k[\lambda - (k-1)]u_{k-1}, (k \geq 1)$
Choose m maximal so that

$$u_k = \pi(Y)^k u_0 \neq 0$$

for all $k \leq m$ but $u_{m+1} = 0$.
Then $\pi(X)u_{m+1} = 0$ and so we have

$$0 = \pi(X)u_{m+1} = (m+1)(\lambda - m)u_m$$

But $m + 1 \neq 0, u_m \neq 0$, whence it follows that $\lambda = m$.
We deduce that $u_1, \dots, u_m$ are linearly independent and $V = \text{span}\{u_1, \dots, u_m\}$.

■

Lecture 7 - 2/6/24

Theorem 0.9. *Suppose that $\pi : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V)$ is a (not necessarily irreducible) finite dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$. Recall that we have a basis:*

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

of $\mathfrak{sl}_2(\mathbb{C})$. Then:

1. *Every eigenvalue of $\pi(H)$ is an integer*

2. *$\pi(X)$ and $\pi(Y)$ are nilpotent*

3. *Define $S \in \mathrm{GL}(V)$ by*

$$S = e^{\pi(X)} e^{-\pi(Y)} e^{\pi(X)}$$

then $S\pi(H)S^{-1} = -\pi(H)$.

4. *If k is an eigenvalue of $\pi(H)$, then the rest are*

$$\cdots, -|k|, -|k| + 2, \dots, |k| - 2, |k|, \cdots$$

Proof. Exercise, using what we did last time: for (1) and (4), look at what $\pi(X), \pi(Y)$ do to an eigenvector of $\pi(H)$.
For (2), take a basis of generalized eigenvectors of $\pi(H)$ and consider $\pi(X)v, \pi(Y)v$, for a basis vector v.
For (3), use

$$\begin{aligned} e^{\pi(X)}\pi(H)e^{\pi(X)} &= \mathrm{Ad}_{e^{\pi(X)}}(\pi(H)) \\ &= e^{\mathrm{ad}_{\pi(X)}}(\pi(H)) \\ &= \pi(H) + [\pi(X), \pi(H)] + \dots \\ &= \pi(H) - 2\pi(X) \end{aligned}$$

etc.

Prototypical example: $\mathfrak{sl}_n(\mathbb{C})$

Recall that $\mathfrak{sl}_n(\mathbb{C})$ consists of $n \times n$ complex matrices with trace 0.
If $X \in \mathfrak{sl}_2(\mathbb{C})$ and $A \in \mathfrak{gl}_n(\mathbb{C})$, then

$$\mathrm{tr}([X, A]) = \mathrm{tr}(XA - AX) = 0$$

and so $\mathfrak{sl}_n(\mathbb{C})$ is an ideal in $\mathfrak{gl}_n(\mathbb{C})$. In particular, $\mathfrak{gl}_n(\mathbb{C})$ is not simple.

Proposition 10. *$\mathfrak{sl}_n(\mathbb{C})$ is simple for $n \geq 2$.*

Proof. We sketch the main idea. Suppose that $\mathfrak{k}$ is a non-zero ideal of $\mathfrak{sl}_n(\mathbb{C})$. Write E_{ij} for the elementary matrix with 1 in the ij position and 0 elsewhere.
Suppose that $A \in \mathfrak{k}, A \neq 0$. Via considering $[A, E_{ij}] \in \mathfrak{k}$, who that $\mathfrak{k}$ contains an elementary matrix E_{ij}.

$$[A, E_{ij}]_{k\ell} = A_{ki}\delta_{kj} - \mathbb{A}_{jk}\delta_{ji}(\text{bisi thinks, etc})$$

Then via $[E_{ji}, E_{jk}] = E_{ik}$, etc, show that $\mathfrak{k} = \mathfrak{sl}_n(\mathbb{C})$ ∎

Set

$$\mathfrak{h} = \{h \in \mathfrak{sl}_n(\mathbb{C}) \mid h \text{ is diagonal } \}$$

Easy check - $[\mathfrak{h}, \mathfrak{h}] = 0$, and so $\mathfrak{h}$ is Abelian. (Note that $\mathfrak{h}$ is a subalgebra of $\mathfrak{sl}_n(\mathbb{C})$) of dimension $n - 1$).
Now view $\mathfrak{sl}_n(\mathbb{C})$ as being a left $\mathfrak{h}$-module.

Claim. *$\mathbb{C} \cdot E_{ij}$ is a one-dimensional $\mathfrak{h}$-submodule of $\mathfrak{sl}_2(\mathbb{C})$*

Proof. ∎

If $h = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$, then $[h, E_{ij}] = (\lambda_i - \lambda_j)E_{ij}$.
So we have a decomposition

$$\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{h} \oplus \sum_{i \neq j} \mathbb{C} \cdot E_{ij}$$

Observe that each $\mathfrak{h}$-module $\mathbb{C} \cdot E_{ij}$ yields a 1-dimensional representation of $\mathfrak{h}$. Define $\varepsilon_{ij} : \mathfrak{h} \to \mathbb{C}$ by $\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \mapsto \lambda_i - \lambda_j$
The set Φ of $n(n-1)$ representations of $\mathfrak{h}$ that arise in this way are called the set of roots of $\mathfrak{sl}_n(\mathbb{C})$ with respect to $\mathfrak{h}$.
So $\Phi \subseteq \mathrm{Hom}(\mathfrak{h}, \mathbb{C})$
Some properties of Φ

1. If $\alpha \in \Phi$, then $-\alpha \in \Phi$ also
(For if $\alpha = \varepsilon_{ij}$, then $-\alpha = \varepsilon_{ji}$)

2. Define $\alpha_i \in \Phi$ by $\alpha_i = \varepsilon_{i,i+1}$. Then the set $\Pi = \{\alpha_1, \ldots, \alpha_{n-1}\}$ of fundamental roots is a basis of $\mathrm{Hom}(\mathfrak{h}, \mathbb{C})$

3. Observe that we have $\varepsilon_{ij} = \begin{cases} \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} & i < j \\ -(\alpha_j + \alpha_{j+1} + \cdots + \alpha_{i-1}) & i > j \end{cases}$

So we may write $\Phi = \Phi^+ \coprod \Phi^-$ where Φ^+ consists of positive integer combinations of elements of Π, and Φ^- consists of negative integer combinations of elements of Π

(Semi)Simple Lie Algebras

Definition 0.29. Suppose that $\mathfrak{g}$ is a Lie algebra over $\mathbb{C}$, and let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$
Define the idealizer $I(\mathfrak{h})$ of $\mathfrak{h}$ in $\mathfrak{g}$ by

$$I(\mathfrak{h}) = \{X \in \mathfrak{g} \mid [X, \mathfrak{h}] \subseteq \mathfrak{h}\}$$

Then $I(\mathfrak{h})$ is a subalgebra of $\mathfrak{g}$ and $\mathfrak{h}$ is an ideal of $I(\mathfrak{h})$. (In fact, $I(\mathfrak{h})$ is the largest subalgebra of $\mathfrak{g}$ in which $\mathfrak{h}$ is an ideal).
We say that $\mathfrak{h}$ satisfies the idealizer condition if $I(\mathfrak{h}) = \mathfrak{h}$ if $I(\mathfrak{h}) = \mathfrak{h}$.
Recall that a Lie algebra $\mathfrak{g}$ is nilpotent if $\mathfrak{g}^(i) = 0$ for large i, where $\mathfrak{g}^(n) = [\mathfrak{g}, \mathfrak{g}^{(n-1)}]$, and $\mathfrak{g}^{(0)} = \mathfrak{g}$.

Definition 0.30. Say that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ if $\mathfrak{h}$ is nilpotent and $\mathfrak{h}$ satisfies the idealizer condition.
With this definition, one has the following result.

Theorem 0.10. *Every finite-dimensional Lie algebra $\mathfrak{g}$ has a Cartan subalgebra, and any two Cartan subalgebras are conjugate via an automorphism of $\mathfrak{g}$.*

Proof. See section 15 and 16 of Humphreys.
The idea is that for each $X \in \mathfrak{g}, \lambda \in \mathbb{C}$, we define

$$L_{\lambda,X} \stackrel{\text{def}}{=} \{Y \in \mathfrak{g} \mid (\mathrm{ad}_X - \lambda I)^i Y = 0 \text{ for some integer } i \geq 1$$

i.e. this is the λ-generalized eigenspace in $\mathfrak{g}$ of ad_X.
Then (math 108A),

$$\mathfrak{g} = \oplus_\lambda L_{\lambda,X}$$

Say that X is regular if $\dim(L_{0,X})$ is minimal (as this varies over X)
Then if X is regular, $L_{0,X}$ is a Cartan subalgebra of $\mathfrak{g}$.

■

Definition 0.31. Say that a Lie algebra $\mathfrak{g}$ is semisimple if it has no nonzero soluble ideals.

Definition 0.32. Say that a Lie algebra $\mathfrak{g}$ is reductive if there exists a compact matrix group K with Lie algebra $\mathfrak{k}$ such that $\mathfrak{g}$ is isomorphic to $\mathfrak{k}_{\mathbb{C}}$, which is $\mathfrak{k}$ with the scalars extended, i.e. $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$

Example 0.8. We have $u_n \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{g}l_n(\mathbb{C})$
For if $X \in M_n(\mathbb{C})$, then we have

$$X = \frac{X - X^*}{2} + i\frac{X + X^*}{2} \in u + iu$$

But also note that

$$\mathfrak{g}l_n(\mathbb{C}) \cong \mathfrak{g}l_n(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$$

despite the fact that $\mathrm{GL}_n(\mathbb{R})$ is not compact.
If $\mathfrak{g}$ is reductive, then it carries an inner product with nice properties.

Lecture 8 - 2/8/24

Recall that $\mathfrak{g}$ is reductive if there exists a compact matrix group K with lie algebra $\mathfrak{k}$ such that

$$\mathfrak{g} = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{k}_{\mathbb{C}}$$

If $\mathfrak{g}$ is reductive, then it carries an inner product with nice properties:

Proposition 11. *Suppose that $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ is a reductive Lie algebra. Then there exists an inner product on $\mathfrak{g}$ that is real valued on $\mathfrak{k}$ and which satisfies the following:*

$$\langle \mathrm{ad}_X(Y), Z \rangle = -\langle Y, \mathrm{ad}_X(Z) \rangle$$

for all $X \in \mathfrak{k}, Y, Z \in \mathfrak{g}$. Further, if we define

$$(X_1 + iX_2)^* \stackrel{\text{def}}{=} -X_1 + iX_2$$

where $X_1, X_2 \in \mathfrak{k}$, then the above inner product also satisfies

$$\langle \mathrm{ad}_X(Y), Z \rangle = \langle Y, \mathrm{ad}_{X^*}(Z) \rangle$$

for all $X, Y, Z \in \mathfrak{g}$

Proof. The $\mathbb{R}$-valued K-invariant (under the adjoint action) inner product on $\mathfrak{k}$ extends to an inner product on $\mathfrak{g}$ for which the adjoint action of K is unitary.
The corresponding adjoint actions of $\mathfrak{k}$ is then also unitary. ■

Theorem 0.11. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{C}$. Then the following are equivalent:*

1. $\mathfrak{g}$ *is semi-simple*

2. $\mathfrak{g}$ *is a direct sum of simple Lie algebras*

3. $\mathfrak{g}$ *is reductive with trivial center.*

(In this case, we say that $\mathfrak{k}$ is a <u>*compact real form for* $\mathfrak{g}$</u>*)*

We use (3) as our definition of "semisimple."

Proof. ■

<u>Recall:</u> If $A \in M_n(\mathbb{C})$ with $AA^* = A^*A$, then A is diagonalizable (and $\mathbb{C}^n$ admits an orthonormal eigenbasis for A)

Theorem 0.12. *Suppose $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ is a finite-dimensional complex semisimple Lie algebra, and let $\mathfrak{h}$ be a Cartan subalgebra for $\mathfrak{g}$. Then:*

1. $\mathfrak{h}$ *is abelian*

2. *The centralizer of $\mathfrak{h}$ in $\mathfrak{g}$ is $\mathfrak{h}$*

3. *If $H \in \mathfrak{h}$, then $\mathrm{ad}_H : \mathfrak{g} \to \mathfrak{g}$, is diagonalizable. (In fact $\mathfrak{h}$ is a maximal subalgebra of $\mathfrak{g}$ consisting of semisimple (i.e. diagonalizable) elements)*

Theorem 0.13. *With notation as above, let $\mathfrak{t}$ be any maximal commutative subalgebra of $\mathfrak{k}$. Then*
$$\mathfrak{h} = \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{t}_{\mathbb{C}}$$
is a maximal Cartan subalgebra of $\mathfrak{g}$.

Proof.

Claim. $\mathfrak{h}$ *is a maximal commutative subalgebra of* $\mathfrak{g}$

Proof. Suppose $X \in \mathfrak{g}$. Then we may write (uniquely!)
$$X = X_1 + iX_2$$
with $X_1, X_2 \in \mathfrak{k}$. So if $[X, \mathfrak{h}] = 0$, then $[X_1, \mathfrak{h}] = 0, [X_2, \mathfrak{h}] = 0$. This can only happen if $X_1, X_2 \in \mathfrak{t}$, and so $X \in \mathfrak{h}$. This proves the claim

■

If $X \in \mathfrak{k}$, then $\mathrm{ad}_X : \mathfrak{k} \to \mathfrak{k}$ is skew self-adjoint (because $\mathfrak{k}$ admits a K-invariant inner product). This extends to an inner product on $\mathfrak{g}$ and so ad_X is diagonalizable.
This implies that the elements of $\{\mathrm{ad}_Y : \mathfrak{k} \to \mathfrak{k} \mid Y \in \mathfrak{t}\}$ are simultaneously diagonalizable (as these elements commute). Deduce from this that the elements of $|h$ are simultaneously diagonalizable as well.

■

Definition 0.33. The rank of a semisimple Lie algebra $\mathfrak{g}$ is the dimension of any Cartan subalgebra.
Observe that the restriction of the inner product $\langle -, - \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ to $\mathfrak{h}$ induces an identification of $\mathfrak{h}$ with its dual:
$$\begin{aligned} \mathfrak{h} &\to \mathrm{Hom}(\mathfrak{h}, \mathbb{C}) \\ \alpha &\mapsto \{H \mapsto \langle \alpha, H \rangle\} \end{aligned}$$

Definition 0.34. Say that $\alpha \in \mathfrak{h}$ ($\alpha \neq 0$) is a root (relative to $\mathfrak{h}$) if there exists $\mathfrak{g} \ni X \neq 0$ such that
$$[H, X] = \langle \alpha, H \rangle X$$
for all $H \in \mathfrak{h}$. We say that X is a root vector for α
We write R for the set of roots in $\mathfrak{g}$. Define the root space $\mathfrak{g}_\alpha$ to be the set of root vectors for α.

Proposition 12.

(a) Each root α lies in $i\mathfrak{t} \subseteq \mathfrak{h}$

(b) We have $\mathfrak{g}_0 = \mathfrak{h}$

Proof.

(a) Suppose that $H \in \mathfrak{t}$. Then ad_H is skew self-adjoint on $\mathfrak{h}$ and so has pure imaginary eigenvalues. So if $\alpha \in \mathfrak{h}$ is a root, then $\langle \alpha, H \rangle$ must be pure imaginary. Since $\langle -, - \rangle$ is real on $\mathfrak{t}$, it follows that $\alpha \in i\mathfrak{t}$.

(b) We have

$$\mathfrak{g}_0 = \{X \in \mathfrak{g} \mid \mathfrak{h}, X] = 0\} = \mathfrak{h}$$

because $\mathfrak{h}$ is a maximal commutative subalgebra of $\mathfrak{g}$.

■

Theorem 0.14. *The Lie algebra $\mathfrak{g}$ may be written as the following direct sum of vector spaces*

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \right)$$

Proof. Follows from the fact the elements of $\mathfrak{h}$ are simultaneously diagonalizable.

■

Proposition 13. *For any $\alpha, \beta \in R$, we have*

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$$

Proof. Suppose that $X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_\beta$. Then

$$\begin{aligned}
[H, [X, Y]] &= [[H, X], Y] + [X, [H, Y]] \\
&= [\langle \alpha, H \rangle X, Y] + [X, \langle \beta, H \rangle Y] \\
&= \langle \alpha + \beta, H \rangle [X, Y]
\end{aligned}$$

So $[X, Y] \in \mathfrak{g}_{\alpha+\beta}$.

■

Proposition 14.

1. *Suppose that $\alpha \in R$ and $X \in \mathfrak{g}_\alpha$. Then $X^* = \mathfrak{g}_{-\alpha}$ and so $-\alpha \in R$ also.*

2. *The roots of $\mathfrak{g}$ span $\mathfrak{h}$.*

Proof.

1. Suppose that $X \in \mathfrak{g}$ with $X = X_1 + iX_2$, $(X_1, X_2 \in \mathfrak{k})$ and define

$$\overline{X} = X_1 - iX_2 = -X^*$$

Then if $H \in \mathfrak{h}$, we have

$$\begin{aligned}\overline{[X,H]} &= [H,X_1] - i[H,X_2] \\ &= [H,\overline{X}]\end{aligned}$$

So if $X \in \mathfrak{g}$ with $\alpha \in i\mathfrak{t}$ and if $H \in \mathfrak{t}$, then

$$\begin{aligned}[H,\overline{X}] &= \overline{[H,X]} \\ &= \overline{\langle \alpha, H\rangle X} \\ &= -\langle \alpha, H\rangle \overline{X}\end{aligned}$$

because $\langle \alpha, H\rangle$ is pure imaginary. So, via linearity,

$$[H,\overline{X}] = -\langle \alpha, H\rangle \overline{X}$$

for all $H \in \mathfrak{h}$.

Hence $\overline{X} \in \mathfrak{g}_{-\alpha}$ and so $X^* \in \mathfrak{g}_{-\alpha}$.

2. Suppose not. Then there exists $\mathfrak{h} \ni H \neq 0$ such that $\langle \alpha, H\rangle = 0$ for all $\alpha \in R$. We then have

1. $[H,\mathfrak{h}] = 0$ and

2. If $\alpha \in R$ and $X \in \mathfrak{g}_\alpha$, then

$$\begin{aligned}[H,X] &= \langle \alpha, H\rangle X \\ &= 0\end{aligned}$$

BY these two facts, imply that H lives in the center of $\mathfrak{g}$, which is trivial.

Lecture 9 - 2/13/24

To continue our analysis, we are going to find lots of copies of $\mathfrak{s}l_2(\mathbb{C})$ in $\mathfrak{g}$. We make use of the following lemma:

Lemma 3. *Suppose that $X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_{-\alpha}$, and $H \in \mathfrak{h}$. Then*

(a) $[X,Y] \in \mathfrak{h}$

(b) $\langle [X,Y], H\rangle = \langle \alpha, H\rangle \langle Y, X^*\rangle$

Proof.

(a) We have $[X,Y] \in \mathfrak{g}_{\alpha-\alpha} = \mathfrak{g}_0 = \mathfrak{h}$

(b)

$$\begin{aligned}\langle [X,Y],H\rangle &= \langle \mathrm{ad}_X(Y),H\rangle\\ &= \langle Y,\mathrm{ad}_{X^*}(H)\rangle\\ &= -\langle Y,[H,X^*]\rangle\\ &= -\langle Y,\langle -\alpha,H\rangle X^*\rangle\\ &= \langle \alpha,H\rangle\langle Y,X^*\rangle\end{aligned}$$

■

Theorem 0.15. *Suppose that $\alpha \in R$. Then there exist linearly independent elements $X_\alpha \in \mathfrak{g}_\alpha, Y_\alpha = X_\alpha^* \in \mathfrak{g}_{-\alpha}$, $H_\alpha \in \mathfrak{h}$, such that H_α is a multiple of α and such that*

$$[H_\alpha,X_\alpha] = 2H_\alpha, [H_\alpha,Y_\alpha] = -2Y_\alpha, [X_\alpha,Y_\alpha] = H_\alpha$$

Proof. Let $X \in \mathfrak{g}_\alpha$ be nonzero. Then $X^* = -\overline{X} \in \mathfrak{g}_{-\alpha}$ is also nonzero.

Claim. *$[X,X^*] \in \mathfrak{h}$ is a multiple of α.*

Proof. Applying the immediately preceding lemma, with $Y = X^*$, gives the following:

$$\langle [X,X^*],H\rangle = \langle \alpha,H\rangle\langle X^*,X^*\rangle$$

So $\langle [X,X^*],H\rangle = 0 \iff \langle \alpha,H\rangle = 0$, and from this we deduce that $[X,X^*]$ is a multiple of α.
Now taking $H = [X,X^*]$ yields

$$\begin{aligned}\langle [X,X^*],[X,X^*]\rangle &= \langle \alpha,[X,X^*]\rangle\\ &= \langle X^*,X^*\rangle\end{aligned}$$

whence it follows that $\langle \alpha,[X,X^*]\rangle$ is real and positive.

■

Now we set $H = [X,X^*]$, and define $H_\alpha = \frac{2}{\langle \alpha,H\rangle}H$, $X_\alpha = \sqrt{\frac{2}{\langle \alpha,H\rangle}}X$, $Y_\alpha = \sqrt{\frac{2}{\langle \alpha,H\rangle}}$, and show that these work.

■

<u>Remark:</u> Since H_α is a multiple of α, we see that $H_\alpha = \frac{2}{\langle \alpha,\alpha\rangle}\alpha$, called the <u>co-root</u> of α. We write

$$\mathscr{S}^\alpha = \mathrm{Span}\langle X_\alpha,Y_\alpha,H_\alpha\rangle$$

This acts on $\mathfrak{g}$ via the adjoint representation.

Lemma 4. *If $\alpha, c\alpha \in R$ and $|c| > 1$, then $c = \pm 2$.*

Proof. If $\alpha \in R$ and $X \in \mathfrak{g}_{c\alpha}$ is non-zero, then

$$\begin{aligned}[H_\alpha, X] &= \langle c\alpha, H_\alpha \rangle X \\ &= \overline{c}\langle \alpha, H_\alpha \rangle X \\ &= 2\overline{c}X\end{aligned}$$

So from our earlier results on representations of $\mathfrak{sl}_2(\mathbb{C})$, $2\overline{c}$ has to be an integer, so $\overline{c}$ is an integer multiple of $\frac{1}{2}$.
Similarly, reversing the roles of $\alpha, c\alpha$ in the above argument yields that $\frac{1}{c}$ is also a half-integer. This can only happen if $c = \pm 2$.

∎

Lemma 5. *Let $\alpha \in R$ such that if $|c| < 1$ then $c\alpha \notin R$.*
Let V^α be the subspace spanned by

$$\{H_\alpha\} \cup \{\mathfrak{g}_\beta \mid \beta \textit{ is a multiple of } \alpha$$

Then $V^\alpha = \mathscr{S}^\alpha$

Proof.

Claim. *First, V^α is a subalgebra of $\mathfrak{g}$.*

Proof. **1.** Suppose that $\beta \in R$ is a multiple of α, and $X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_\beta$. Then for all $H \in \mathfrak{h}$, we have

$$\langle [X, Y], H \rangle = \langle \beta, H \rangle \langle Y, X^* \rangle$$

So (cf earlier argument!),

$$\langle [X, Y] \rangle^\perp = \langle \beta \rangle^\perp$$

in $\mathfrak{h}$, and so $[X, Y]$ is a multiple of β and so is a multiple of α.

2. If $X \in \mathfrak{g}_\beta$, then

$$[H_\alpha, X] = \langle \alpha, H_\alpha \rangle X \in \mathfrak{g}_\beta$$

3. $[\mathfrak{g}_{\beta_1}, \mathfrak{g}_{\beta_2}] \subseteq \mathfrak{g}_{\beta_1+\beta_2}$, and $\beta_1 + \beta_2$ is a multiple of α
Now consider the adjoint action of $\mathscr{S}^\alpha$ on V^α.
We have $\langle \alpha, H_\alpha \rangle = 2$, and $\beta \in \{\pm\alpha, \pm 2\alpha\}$
So the only possible eigenvalues for ad_{H_α} acting on V^α are $0, \pm 1, \pm 2, \pm 4$
Plainly, $\mathscr{S}^\alpha \subseteq V^\alpha$. Consider the orthogonal complement U^α of $\mathscr{S}^\alpha$ in V^α. Show that U^α is $\mathscr{S}^\alpha$-invariant (use the fact that $X \in \mathscr{S}^\alpha$implies $X^* \in \mathscr{S}^\alpha$).
Now if $U_\alpha \neq 0$, then our earlier results on the representations of $l_2(\mathbb{C})$ imply there exists $X \in U^\alpha$ with $X \neq 0$ and such that $\mathrm{ad}_{H_\alpha}(X) = [H_\alpha, X] = 0$.
This is a contradiction since the only eigenvector of ad_{H_α} in V^α with eigenvalue zero is H_α, which is orthogonal to U^α.

Hence $V^\alpha = \mathscr{S}^\alpha$.

∎

The upshot of this is that we have shown the subspace of $\mathfrak{g}$ spanned by $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$
The following result is now immediate:

Theorem 0.16.

1. *If $\alpha \in R$, then the only multiples of α that lie in R are $\pm\alpha$.*

2. *Each root space $\mathfrak{g}_\alpha$ has dimension 1.*

∎

Proposition 15. *If $\alpha, \beta \in R$, then*

$$2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} = \langle \beta, H_\alpha \rangle$$

is an integer.

Proof. If X is a root vector associaed to β, then by definition we have

$$[H_\alpha, X] = \langle \beta, H_\alpha \rangle X$$

and now the result follows, since every eigenvalue of H_α is an integer (see earlier results about representations of $\mathfrak{sl}_2(\mathbb{C})$)

∎

Theorem 0.17. *If $\alpha, \beta \in R$, then so is*

$$\beta - 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha\beta}\alpha$$

Proof. Consider the linear maps

$$s_\alpha : \mathfrak{h} \to \mathfrak{h}$$

defined by $s_\alpha(H) = H - 2\frac{\langle \alpha, H \rangle}{\langle \alpha, \alpha \rangle}\alpha$

Claim. *The map s_α restricts to a linear map $i\mathfrak{t} \to i\mathfrak{t}$.*

Proof. We know that the roots live in $i\mathfrak{t}$ and $\langle -, - \rangle$ is real on $\mathfrak{t}$. So if $H \in i\mathfrak{t}$, then $s_\alpha(H) \in \mathfrak{t}$.

∎

Next, notice that $\mathfrak{h} = \langle \alpha \rangle \oplus \langle \alpha \rangle^\perp$, and so $s_\alpha(\alpha) = -\alpha$ while $s_\alpha(H) = H$ if $H \in \langle \alpha \rangle^\perp$.
So we can see that s_α is the reflection in the hyperplane in $i\mathfrak{t}$ orthogonal to α.
In particular, s_α is an orthogonal linear transformation.

We want to show that $s_\alpha(\beta) \in R$. We do this by considering an associated root vector.
Consider the linear operator $S_\alpha : \mathfrak{g} \to \mathfrak{g}$ defined by

$$S_\alpha = e^{\operatorname{ad}_{X_\alpha}} e^{-\operatorname{ad}_{Y_\alpha}} e^{\operatorname{ad}_{X_\alpha}}$$

Now if $H \in \mathfrak{h}$, then

$$[H, X_\alpha] = \langle \alpha, H \rangle X_\alpha$$
$$[H, Y_\alpha] = \langle \alpha, H \rangle Y_\alpha$$

So if $\langle \alpha, H \rangle = 0$, then ad_H commutes with both $\operatorname{ad}_{X_\alpha}, \operatorname{ad}_{Y_\alpha}$
This in turn implies that $S_\alpha \operatorname{ad}_H S_\alpha^{-1} = \operatorname{ad}_H$ whenever $\langle \alpha, H \rangle = 0$.
Also, recall (basic facts about representations of $\mathfrak{s}l_2(\mathbb{C})$) then we have

$$S_\alpha \operatorname{ad}_H S_\alpha^{-1} = -\operatorname{ad}_{H_\alpha}$$

Putting all of this together gives $S_\alpha \operatorname{ad}_H S_\alpha^{-1} = \operatorname{ad}_{s_\alpha(H)}$ for all $H \in \mathfrak{h}$.
Now suppose that X is a root vector associated to β, and consider $S_\alpha^{-1}(X) \in \mathfrak{g}$.

Claim. *$S_\alpha^{-1}(X)$ is a root vector associated to $s_\alpha(\beta)$.*

Proof.

$$\begin{aligned}
&\operatorname{ad}_H(S_\alpha^{-1}(X)) \\
&\qquad = S_\alpha^{-1}(S_\alpha \operatorname{ad}_H S_\alpha^{-1})(X) \\
&\qquad = S_\alpha^{-1} \operatorname{ad}_{S_\alpha(H)}(X) \\
&\qquad = \langle \beta, s_\alpha(H) \beta S_\alpha^{-1}(X) \\
&\qquad = \langle S_\alpha^{-1}(\beta), H \rangle S_\alpha^{-1}(X) \\
&\qquad = \langle S_\alpha(\beta), H \rangle S_\alpha^{-1}(X)
\end{aligned}$$

This completes the proof

■

Definition 0.35. The Weyl group is the subgroup of $\mathrm{GL}(\mathfrak{h})$ generated by $\{s_\alpha \mid \alpha \in R\}$

Lecture 10 - 2/15/24

The following theorem summarizes the basic properties of roots we have established so far:

Theorem 0.18. *Let R be the set of roots of a semi-simple Lie algebra $\mathfrak{g}$. The set R is a finite set of nonzero vectors in a real inner product space E and*

(a) *The set R spans E.*

(b) *If $\alpha \in R$, then $-\alpha \in R$, and these are the only multiples of $\alpha \in R$.*

(c) *If $\alpha, \beta \in R$, then $S_\alpha(\beta) \in R$, the reflection of β across the hyperplane perpendicular to α*

(d) *If $\alpha, \beta \in R$, then $2\frac{\langle \alpha,\beta\rangle}{\langle \alpha,\alpha\rangle} \in \mathbb{Z}$*

These are the properties of a root system.

Proof. ■

Root systems

Definition 0.36. An abstract root system (E, R), with E an inner product space, R a finite subset, satisfies the four conclusions of the above theorem.

Definition 0.37. Suppose that $(E_1, R_2), (E_2, R_2)$ are root systems, an isomorphism $f : R_1 \to R_2$ is a linear isomorphism $E_1 \to E_2$ (not necessarily an isometry!), such that $f(R_1) = R_2$, and $f(S_\alpha(\beta)) = S_{f(\alpha)}(f(\beta))$ for all $\alpha, \beta \in R_1$

Definition 0.38. The Weyl group $W = W(R)$ of a root system (E, R) is the group generated by

$$\{S_\alpha \mid \alpha \in R\}$$

So W is a subgroup of $O(E) \subseteq \mathrm{GL}(E)$, so W permutes the elements of R.

Proposition 16. *Suppose that (E, R) is a root system and that $\alpha, \beta \in R$ with $\alpha \neq \pm\beta$. Write $p(\beta, \alpha) = 2\frac{\langle \beta,\alpha\rangle}{\langle \alpha,\alpha\rangle} \in \mathbb{Z}$, the length of the projection of β onto α.*

(a) *If θ is the angle between α and β, and $\|\beta\| \geq \|\alpha\|$ are the only possibilities:*

$p(\alpha, \beta)$	$p(\beta, \alpha)$	θ	$\left(\frac{\|\beta\|}{\|\alpha\|}\right)^2$
0	*0*	$\frac{\pi}{2}$	*undetermined*
1	*1*	$\frac{\pi}{3}$	*1*
-1	*-1*	$\frac{2\pi}{3}$	*1*
1	*2*	$\frac{\pi}{4}$	*2*
-1	*-2*	$\frac{3\pi}{4}$	*2*
1	*3*	$\frac{\pi}{6}$	*3*
-1	*-3*	$\frac{5\pi}{6}$	*3*

(b) *If $\langle \alpha, \beta\rangle > 0$, then $\alpha - \beta \in R$. If $\langle \alpha, \beta\rangle < 0$, then $\alpha + \beta \in R$*

Proof.

(a) Observe that

$$p(\beta, \alpha) = 2 \left(\frac{\|\beta\|}{\|\alpha\|} \right) \cos(\theta)$$

and similarly for $p(\alpha, \beta)$, so

$$p(\alpha, \beta) \cdot p(\beta, \alpha) = 4 \cos^2(\theta)$$

and this must be an integer.

Now $0 \leq 4\cos^2(\theta) \leq 4$, and $\cos(\theta) \neq \pm 1$ since $\alpha \neq \pm\beta$.

The entries in the table correspond to the integer solutions of $xy = n$, $n = 0, 1, 2, 3$.

(b) Notice that the second assertion follows from the fact that replacing β by $-\beta$. From the table in (a), it follows that if $\langle \alpha, \beta \rangle > 0$, then either $p(\beta, \alpha) = 1$, or $p(\alpha, \beta) = 1$.

If $p(\alpha, \beta) = 1$, then $S_\beta(\alpha) = \alpha - \beta \in R$. If $p(\beta, \alpha) = 1$, then $S_\alpha(-\beta) = -(\beta \cdot \alpha) \in R$, and so the result follows.

Bases and Weyl chambers

Definition 0.39. Say that $\Delta \subseteq R$ is a base of R if

(B1) Δ is a vector space basis of E.

(B2) Each $\beta \in R$ can be written in the form

$$\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$$

with integers k_α which are either all ≥ 0 or all ≤ 0. Elements of Δ are called simple roots. The expression $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$ is unique.

Say that β is positive if all $k_\alpha \geq 0$ and β is negative if all $k_\alpha \leq 0$.
We define the height of β to be

$$\sum_{\alpha \in \Delta} k_\alpha$$

We may partially order E by writing $\gamma \succcurlyeq \delta$ only if $\gamma = \delta$ or $\gamma - \delta$ is a sum of simple roots with real nonnegative coefficients.
We write R^+ and R^- for the sets of positive and negative roots, respectively.

Proposition 17. *If $\alpha, \beta \in \Delta$ with $\alpha \neq \beta$, then*

$$\langle \alpha, \beta \rangle < 0$$

Proof. Suppose $\langle \alpha, \beta \rangle > 0$. Then $\alpha - \beta \in R$, and this is impossible because the expansion of $\alpha - \beta$ in terms of a linear combination of simple roots has both positive and negative coefficients.
Notation:
Suppose that (E, R) is a root system.

(a) For any $\gamma \in E$, $V_\gamma \overset{\text{def}}{=} \{x \in E \mid \langle x, \gamma \rangle = 0\}$

(b) We say that γ is regular if $\gamma \notin \bigcup_{\alpha \in R} V_\alpha$ (show that regular elements exist as an exercise. A good argument will also show that even if R were countably infinite, regular elements would still exist).

(c) If γ is regular, set

$$R^+(\gamma) \overset{\text{def}}{=} \{\alpha \in R \mid \langle \alpha, \gamma \rangle > 0\}$$
$$R^-(\gamma) \overset{\text{def}}{=} \{\alpha \in R \mid \langle \alpha, \gamma \rangle < 0\}$$

We have $R = R^+(\gamma) \cup R^-(\gamma)$ because γ is regular.

Say that $\alpha \in R$ is decomposable if $\alpha = \alpha_1 + \alpha_2$, with $\alpha_1, \alpha_2 \in R^+$.

Lemma 6. *Suppose that V_λ is a hyperplane in E and $\{\lambda_1, \dots, \lambda_r\}$ is a subset of E, all of whose elements lie strictly on the same side of V_λ (e.g. $\langle \lambda, \lambda_i \rangle < 0$ for all i, or strictly greater, as long as the signs are the same)*
Suppose also that the elements in the set form pairwise obtuse angles (i.e. $\langle \lambda_i, \lambda_j \rangle \leq 0$ for $i \neq j$). Then $\{\lambda_1, \dots, \lambda_r\}$ are linearly independent.

Proof. Note that all the $\langle \lambda_i, \lambda \rangle$ have the same sign.
Suppose that

$$\sum_i r_i \lambda_i = \sum_j s_j \lambda_j = \mu$$

say with $r_i, s_j > 0$ and i, j running through disjoint subsets of $\{1, \dots, n\}$.
Then

$$\langle \mu, \mu \rangle = \sum_{i,j} r_i s_j \langle \lambda_i, \lambda_j \rangle \leq 0$$

Hence $\mu = 0$, hence

$$0 = \langle \mu, \lambda \rangle = \sum_i r_i \langle \lambda, \lambda_i \rangle$$

and so each $r_i = 0$, and similarly each $s_j = 0$.
Exercise:
Show that if $\mathcal{B}$ is any basis of E, then there exists $\gamma \in E$ such that $\langle \gamma, e \rangle > 0$ for all $e \in \mathcal{B}$

Theorem 0.19. *Suppose that $\gamma \in E$ is regular. Then the set $\Delta(\gamma)$ of all indecomposable roots of $R^+(\gamma)$ is a base for R. Moreover, any base for R arises in this way.*

Proof.

(a) We show that each $\alpha \in R^+(\gamma)$ is a nonnegative integral linear combination of elements of $\Delta(\gamma)$. Suppose that this is false. Then there is an $\alpha \in R^+(\gamma)$ that does not satisfy this property, and that $\langle \gamma, \alpha \rangle$ is minimal. Then $\alpha \notin \Delta(\gamma)$, and so we may write:

$$\alpha = \alpha_1 + \alpha_2$$

with $\alpha_1, \alpha_2 \in R^+(\gamma)$. Hence

$$\langle \alpha, \gamma \rangle = \langle \alpha_1, \gamma \rangle + \langle \alpha_2, \gamma \rangle$$

Each $\langle \alpha_i, \gamma \rangle > 0$, and so $\langle \alpha_i, \gamma \rangle < \langle \alpha, \gamma \rangle$. Hence by minimality we have that $\alpha_1 + \alpha_2$ are positive, and therefore so is α.

Lecture 11, 2/20/24

We now prove (b).
It now follows that $\Delta(\gamma)$ spans E. To show linear independence, it suffices to show that $\langle \alpha, \beta \rangle \leq 0$ for all distinct $\alpha, \beta \in \Delta(\gamma)$.
Suppose not. Then for some distinct $\alpha, \beta \in \Delta(\gamma)$ we have $\langle \alpha, \beta \rangle > 0$. So $\alpha - \beta \in R$. Hence either $\alpha - \beta \in R^+(\gamma)$ or $\beta - \alpha \in R^+(\gamma)$. If $\alpha - \beta \in R^+(\gamma)$, then since $\alpha = \beta + (\alpha - \beta)$, α is decomposable, a contradiction, and similarly if $\beta - \alpha \in R^+(\gamma)$.
Now, suppose that Δ is a base. Choose $\gamma \in E$ with $\langle \gamma, \alpha \rangle > 0$ for all $\alpha \in \Delta$ (see earlier exercise). Then γ is regular, because γ is not in the hyperplane orthogonal to any of these roots.

Claim. $\Delta = \Delta(\gamma)$

Proof.
It follows from property B2 of a base (on page 35) that $R^+ \leq R^+(\gamma)$, so $-R^+ \leq -R^+(\gamma)$.
Hence $R^+(\gamma) = R^+$.
Since Δ is a base every element of $R^+(\gamma)$ is a positive integral combination of elements of Δ, so

$$\Delta = \{\text{indecomposable elements}\} = \Delta(\gamma)$$

■

Proposition 18.

1. *Suppose that $\alpha, \beta \in \Delta$ are distinct. Then $\langle \alpha, \beta \rangle \leq 0$, and $\alpha - \beta \notin R$.*

2. *Suppose that $\alpha \in R^+(\gamma)$ and $\beta \notin \Delta$. Then $\beta - \alpha \in R^+$ for some $\alpha \in \Delta$*

3. *Each $\beta \in R^+$ may be written*

$$\beta = \alpha_1 + \cdots + \alpha_n$$

with $\alpha_i \in \Delta$ not necessarily distinct, such that each partial sum $\alpha_1 + \cdots + \alpha_r \in R^+$

4. *If α is a simple root, then S_α permutes $R^+ \setminus \{\alpha\}$. So if $\delta = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$, then $S_\alpha(\delta) = \delta - \alpha$*

Proof.

1. If $\alpha - \beta \in R$, then this contradicts property B2 of a base. Now $\langle \alpha, \beta \rangle \leq 0$ follows from the table in the last lecture.

2. We know $\Delta = \Delta(\gamma_1)$ for some $\gamma_1 \in E$. Now $\Delta \cup \{\beta\}$ lie on the same side of V_{γ_1} and Δ spans E. Hence there exists $\alpha \in \Delta$ such that $\langle \alpha, \beta \rangle > 0$, and so $\beta - \alpha \in R$. Now since $\beta \in R^+$, we have

$$\beta = \sum_{\gamma \in \Delta} k_\gamma \gamma$$

with all $k_\gamma \geq 0$. Since $\beta \notin \Delta$, we have $k_\gamma \geq 0$ for at least two values of γ, and so at least 1 such γ is not equal to α.

The coefficient of this γ in $\beta - \alpha$ is equal to $k_\gamma > 0$, so $\beta - \alpha \in R^+$.

3. Follows from (b)

4. Suppose that $\beta \in R^+ \setminus \{\alpha\}$, with

$$\beta = \sum_{\gamma \in \Delta} k_\gamma \gamma$$

, say. Then there exists $\gamma \neq \alpha$ such that $k_\gamma > 0$. Now observe that the coefficient of this γ in

$$S_\alpha(\beta) = \beta - 2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

is k_γ, and so $S_\alpha(\beta) \in R^+$.

∎

Definition 0.40. Suppose that (E, R) is a root system. The connected components of

$$E \setminus \left(\bigcup_{\alpha \in R} V_\alpha \right)$$

are called the (open) Weyl chambers of E

So each regular element γ of E belongs to exactly one Weyl chamber, $C(\gamma)$, say.

We have $C(\gamma_1) = C(\gamma_2)$ if and only if γ_1, γ_2 lie on the same side of each hyperplane V_α.

So in this case, $R^+(\gamma_1) = R^+(\gamma_2)$, and if $R^+(\gamma_1) = R^+(\gamma_2)$, then $\Delta(\gamma_1) = \Delta(\gamma_2)$. So there is a natural bijection between Weyl chambers and bases.

We write $C(\Delta)$ for $C(\gamma)$ if $\Delta = \Delta(\gamma)$, and call this the fundamental Weyl chamber relative to Δ.

The goal: Show that the Weyl group W permutes the bases of R simply transitively, and is generated by the S_α for $\alpha \in \Delta$, where Δ is any base.

Here is a lemma about reflections:

Lemma 7. *Suppose that Φ is any finite set of vectors with spans E, and that $S_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$.*

Suppose that $\sigma \in \mathrm{GL}(E)$ satisfies:

- $\sigma(\Phi) = \Phi$
- *σ fixes some hyperplane V of E pointwise*
- *$\sigma(\alpha) = -\alpha$ for some $\alpha \in \Phi$.*

Then $\sigma = S_\alpha$

Proof. Observe that

$$E = V \oplus \langle\alpha\rangle = V_\alpha \oplus \langle\alpha\rangle$$

Hence $\sigma : E/\langle\alpha\rangle \to V$ and $S_\alpha E/\langle\alpha\rangle \to V_\alpha$ are isomorphisms. Set $\tau = \sigma \circ S_\alpha$.

Then τ fixes both $E/\langle\alpha\rangle$ and $\langle\alpha\rangle$ poitnwise, and so $(\tau - 1)E \leq \langle\alpha\rangle$ and $(\tau - 1)^2 E = 0$.

Next, observe that τ permutes Φ and so some τ^n fixes each element of Φ aned so fixes E because Φ spans E. Hence $(\tau^n - 1)E = 0$.

So the minimum power of τ divides both $(X - 1)^2$ and $X^n - 1$, and so $\tau = 1$???? ■

Recall that earlier we defind $p(\beta, \alpha) = 2\frac{\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$.

Proposition 19. *Suppose (E, R) is a root system.*

(a) The group W permutes R faithfully, i.e. if $\sigma \in W$ and $\sigma \neq \mathrm{Id}$, then for some $\alpha \in R$, $\sigma(\alpha) \neq \alpha$. Hence W is finite

(b) Suppose that $\sigma \in \mathrm{GL}(E)$ satisfies $\sigma(R) = R$. Then

$$\sigma S_\alpha \sigma^{-1} = S_{\sigma(\alpha)}$$

for all $\alpha \in R$, and furthermore

$$p(\sigma(\beta), \sigma(\alpha)) = p(\sigma, \alpha)$$

for all $\alpha, \beta \in R$

(c) *Suppose that* $\sigma \in \mathrm{GL}(E)$. *Then* $\sigma \in \mathrm{Aut}(R)$ *if and only if* $\sigma(R) = R$.

(d) *Suppose that* $f : (E, R) \to (E_1, R_1)$ *is an isomorphism of root systems. Then*

$$S_{f(\alpha)}(f(\beta)) = f(S_\alpha(\beta))$$

for all $\alpha, \beta \in R$. *Also,* f *induces an isomorphism* $W(R) \to W(R_1)$ *given by*

$$\sigma \mapsto f\sigma f^{-1}$$

Proof.

Lecture 12, 2/22/24

Recall: If $\alpha, \beta \in R$, then we define $p(\alpha, \beta) = 2\frac{\langle \beta,\alpha\rangle}{\langle \alpha,\alpha\rangle}$

Definition 0.41. An isomorphism $f : (E_1, R_1) \to (E_2, R_2)$ between two root systems is a vector space isomorphism $f : E_1 \to E_2$, (not necessarily an isometry!) such that $f(R_1) = R_2$, and

$$p(f(\alpha), f(\beta)) = p(\alpha, \beta)$$

for all $\alpha, \beta \in R_1$

Proposition 20. *Suppose* (E, R) *is a root system.*

(a) *The group* W *permutes* R *faithfully, i.e. if* $\sigma \in W$ *and* $\sigma \neq \mathrm{Id}$, *then for some* $\alpha \in R$, $\sigma(\alpha) \neq \alpha$. *Hence* W *is finite*

(b) *Suppose that* $\sigma \in \mathrm{GL}(E)$ *satisfies* $\sigma(R) = R$. *Then*

$$\sigma S_\alpha \sigma^{-1} = S_{\sigma(\alpha)}$$

for all $\alpha \in R$, *and furthermore*

$$p(\sigma(\beta), \sigma(\alpha)) = p(\sigma, \alpha)$$

for all $\alpha, \beta \in R$

(c) *Suppose that* $\sigma \in \mathrm{GL}(E)$. *Then* $\sigma \in \mathrm{Aut}(R)$ *if and only if* $\sigma(R) = R$.

(d) *Suppose that* $f : (E, R) \to (E_1, R_1)$ *is an isomorphism of root systems. Then*

$$S_{f(\alpha)}(f(\beta)) = f(S_\alpha(\beta))$$

for all $\alpha, \beta \in R$. *Also,* f *induces an isomorphism* $W(R) \to W(R_1)$ *given by*

$$\sigma \mapsto f\sigma f^{-1}$$

Proof.

(a) Immediate, since R is a finite set, and R spans E.

(b) Suppose $\alpha, \beta \in R$. First observe that

$$\sigma S_\alpha \sigma^{-1}(\sigma_\beta) = \sigma S_\alpha(\beta) \in R$$

because $S_\alpha(\beta) \in R$.

Next, note that

$$\begin{aligned}\sigma(S_\alpha(\beta)) &= \sigma(\beta - p(\beta,\alpha)\alpha)\\ &= \sigma(\beta) - p(\beta,\alpha)\sigma(\alpha)\end{aligned}$$

So we see that $\sigma S_\alpha \sigma^{-1}$

(i) Sends R to R

(ii) Fixes the hyperplane $V_{\sigma(\alpha)}$ pointwise

(iii) Sends $\sigma(\alpha)$ to $-\sigma(\alpha)$

We therefore see that

$$\sigma S_\alpha \sigma^{-1} = S_{\sigma(\alpha)}$$

Finally, observe that

$$S_{\sigma(\alpha)}(\sigma(\beta)) = \sigma(\beta) - p(\sigma(\beta), \sigma(\alpha))\sigma(\alpha)$$

and now (b) follows from this and the second equation from the top of this page.

(c) Follows immediately from (b)

(d) Observe that we have:

$$\begin{aligned}S_{f(\alpha)}(f(\beta)) &= f(\beta) - p(f(\beta), f(\alpha))f(\alpha)\\ &= f(\beta) - p(\beta,\alpha)f(\alpha)\\ &= f(S_\alpha(\beta))\end{aligned}$$

Now

$$\begin{aligned}(f \circ S_\alpha \circ f^{-1})(f(\beta)) &= f(S_\alpha(\beta))\\ &= S_{f(\alpha)}(f(\beta)) \in R_1\end{aligned}$$

So we see that $f \circ S_\alpha \circ f^{-1} \in W(R_1)$, and the result follows.

■

Lemma 8. *Suppose that (E, R) is a root system with base Δ.*

(a) *Let $\alpha_1, \dots, \alpha_t \in \Delta$ (not necessarily distinct) and write S_i for S_{α_i}.*
Suppose that $S_1, \dots, S_{t-1}(\alpha_t) < 0$. Then for some i with $1 \leq i \leq t-1$ we have

$$S_1 \cdots S_{i-1} S_{i+1} \cdots S_{t-1} = S_1 \cdots S_t$$

(b) *If $\sigma = S_1 \cdots S_t$ is an expression $\sigma \in W$ in terms of simple reflections corresponding to $\alpha_1, \dots, \alpha_t$, with t as small as possible, then $\sigma(\alpha_t) < 0$*

Proof.

(a) Define
$$\beta_i = \begin{cases} S_{i+1} \cdots S_{t-1}(\alpha_t) & 0 \leq i \leq t-2 \\ \alpha_t & i = t-1 \end{cases}$$
Then $\beta_0 < 0$ and $\beta_{t-1} \succcurlyeq 0$ and so we may choose j minimal such that $\beta_j \succcurlyeq 0$.
Then
$$S_j(\beta_j) = \beta_{j-1} \preccurlyeq 0$$
and this forces $\beta_j = \alpha_j$ because S_j permutes the positive roots not equal to α_j.
Now if $\sigma \in W$, then
$$S_{\sigma(\alpha)} = \sigma S_\alpha \sigma^{-1}$$
for all $\alpha \in R$. So in particular if we set $\sigma = S_2 \cdots S_{j-1}$, then we have
$$\begin{aligned} S_j &= S_{\alpha_j} \\ &= S_{\sigma(\alpha_1)} \\ &= \sigma^{-1} S_1 \sigma \end{aligned}$$
whence
$$\sigma S_j = S_1 \sigma$$
and this implies the result.

(b) Suppose not. Then $\sigma(\alpha_t) \succcurlyeq 0$. Observe that
$$S_t(\alpha_t) = -\alpha_t \preccurlyeq 0$$
aned so
$$S_1 \cdots S_{t-1}(\alpha_t) = -\sigma(S_t(\alpha_t)) \preccurlyeq 0$$

■

Theorem 0.20. *Let Δ be a base of R.*

(a) If $\gamma \in E$ is regular, then there exists $\sigma \in W$ with

$$\langle \alpha, \sigma(\gamma) \rangle > 0$$

for all $\alpha \in \Delta$

(b) If $\alpha \in R$, then there exists $\sigma \in W$ such that $\sigma(\alpha) \in W$. Hence W permute the roots of R transitively.

(c) If $\alpha \in R$, then there exists $\sigma \in W$ such that $\sigma(\alpha) \in \Delta$

(d) If $\sigma \in W$ and $\sigma(\Delta) = \Delta$, then $\sigma = 1$

Proof. Our strategy will be to cheat: set

$$W_1 = \langle S_\alpha \mid \alpha \in \Delta \rangle$$

We prove (a) and (b) for W_1, and then deduce that $W_1 = W$

(a) Write $\delta = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$. Choose $\sigma \in W_1$ with $\langle \delta, \sigma(\gamma) \rangle$ maximal. Observe that for all $\alpha \in \Delta$, we have $S_\alpha(\sigma) \in W_1$. Hence

$$\begin{aligned} \langle \delta, \sigma(\gamma) \rangle &\geq \langle \delta, S_\alpha \sigma(\gamma) \rangle \\ &= \langle S_\alpha \delta, \sigma(\gamma) \rangle \\ &= \langle \delta - \alpha, \sigma(\gamma) \rangle \\ &= \langle \delta, \sigma(\gamma) \rangle - \langle \alpha, \sigma(\gamma) \rangle \end{aligned}$$

so $\langle \alpha, sigma(\gamma) \rangle \geq 0$

Now if $\langle \alpha, \sigma(\gamma) \rangle = 0$, then $\gamma \in V_{\sigma(\alpha)}$, a contradiction since γ is regular. So we see that $\sigma^{-1}(\Delta)$ is a base with $\langle \gamma, \alpha \rangle > 0$ for all $\alpha_i \in \sigma^{-1}(\Delta)$. So we see that $\sigma^{-1}(\Delta) = \Delta(\gamma)$, and now transitivity of W_1 on bases follows as every base has the form $\Delta(\gamma)$ for some γ.

(b) From part (a), it suffices to show that each $\alpha \in R$ belongs to some base. Choose $\gamma_1 \in V_\alpha \setminus \bigcup_{\beta \in R, \beta \neq \pm\alpha} V_\beta$. Let

$$2\varepsilon = \min\{|\langle \gamma_1, \beta \rangle| \mid \beta \neq \pm\alpha\}$$

Now choose γ_2 with

$$0 < \langle \gamma_2, \alpha \rangle < \varepsilon, \qquad |\langle \gamma_2, \beta \rangle| < \varepsilon \, \forall \beta \neq \pm\alpha$$

(We may do this by choosing some γ_2 appropriately to satisfy the first inequality, and then adjusting its length to satisfy the others). Now define $\gamma = \gamma_1 + \gamma_2$. Then $0 < \langle \gamma, \alpha \rangle < \varepsilon$, $|\langle \gamma, \beta \rangle| > \varepsilon$ for all $\beta \neq \pm\alpha$.

Hence α is an indecomposable element of $R^+(\gamma)$, and so $\alpha \in \Delta(\gamma)$

(c) It is enough to show that if $\alpha \in R$, then $S_\alpha \in W_1$. From part (b), there exists $\sigma \in W_1$ such that $\sigma(\alpha) \in \Delta$. Hence $S_{\sigma(\alpha)} \in W_1$. Now observe that

$$S_{\sigma(\alpha)} = \sigma S_\alpha \sigma^{-1} \in W_1$$

and so $S_\alpha \in W_1$, as required.

(d) Suppose that this is false. Write σ as a product $S_1 \cdots S_t$ of simple reflections in such a way that t is the smallest possible. Now if $\sigma(\Delta) = \Delta$, then $\sigma(\alpha_t) \succcurlyeq 0$, a contradiction of Lemma 8.

■

Lecture 13, 2/27/24

Recall:
If R is a root system and $\alpha, \beta \in R$, we define

$$p(\alpha, \beta) \overset{\text{def}}{=} 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$$

Definition 0.42. The Cartan matrix of R (with respect to a base Δ) is the matrix

$$(p(\alpha, \beta))_{\alpha, \beta \in \Delta}$$

Recall: $p(\alpha, \alpha) = 2$ and if $\alpha \neq \beta$ then $p(\alpha, \beta) \leq 0$. Also $p(\alpha, \beta) \in \{0, -1, -2, -3\}$ and $0 \leq p(\alpha, \beta)p(\beta, \alpha) \leq 3$. The Cartan matrix determines R up to isomorphism.

Proposition 21. *Suppose that (E, R) and (E_1, R_1) are root systems with bases Δ, Δ_1, respectively.*
Let $\phi : \Delta \to \Delta_1$ be a bijection such that

$$p(\phi(\alpha), \phi(\beta)) = p(\alpha, \beta)$$

for all $\alpha, \beta \in \Delta$. Then there is a unique $f : E \to E_1$ which extends ϕ and sends R to R_1.

Proof. Define f by extending ϕ from Δ to E by linearity. So if $\alpha, \beta \in \Delta$, then

$$\begin{aligned} S_{\phi(\alpha)} \circ f(\beta) &= S_{\phi(\alpha)}(\phi(\beta)) \\ &= \phi(\beta) - p(\phi(\beta), \phi(\alpha))\phi(\alpha) \end{aligned}$$

and also

$$\begin{aligned} f \circ S_\alpha(\beta) &= f(\beta - p(\beta,\alpha)\alpha) \\ &= \phi(\beta) - p(\beta,\alpha)\phi(\alpha) \end{aligned}$$

so we see $S_{\phi(\alpha)} \circ f = f \circ S_\alpha$ for all $\alpha \in \Delta$.
So $W_1 = fWf^{-1}$, and this is enough because $R = W(\Delta)$ and $R_1 = W_1(\Delta)$.

■

Definition 0.43. A Coxeter graph is a finite graph in which each pair of vertices is joined by $0, 1, 2$, or 3 edges.
If R is a root system with base Δ, then the coxeter graph of R (with respect to Δ) is the graph whose vertices are elements of Δ, with distinct vertices α, β being joined by $p(\alpha,\beta)p(\beta,\alpha)$ edges.
It will suffice to consider irreducible root systems.

Proposition 22. *Suppose E_1, E_2 are subspaces of E, and suppose that $E = E_1 \oplus E_2$. Suppose also that (E, R) is a root system, with $R \subseteq E_1 \cup E_2$. Set $R_i = R \cap E_i$, $i = 1, 2$. Then*

(a) The subspaces E_1, E_2 are orthogonal.

(b) The set R_i is a root system in E_i.

Proof.

(a) Suppose that $\alpha \in R_1$ and $\beta \in R_2$. Then $\alpha - \beta \notin R_1 \cup R_2$, and so is not a root. Hence $\langle \alpha, \beta \rangle \leq 0$. Similarly $\langle \alpha, \beta \rangle \geq 0$, and so $\langle \alpha, \beta \rangle = 0$.

Since R_i spans E_i, part (a) follows.

(b) Observe eg that if $\alpha \in R_1$, then S_α preserves E_2 and therefore also preserves R.

■

We say that R is the sum of R_1 and R_2.

Definition 0.44. We say that R is irreducible if it cannot be written as a sum in a nontrivial way.

Proposition 23. *A root system (E, R) is irreducible if and only if its Coxeter graph is connected (and non-empty)*

Proof. Exercise

■

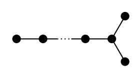
Theorem 0.21. *Every connected, non-empty coxeter graph associated to a root system is isomorphic to one of the following:*

A_n

B_n

C_n

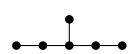
D_n



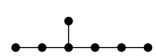
G_2

F_4

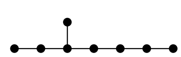
E_6



E_7



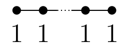
E_8



Proof. ■

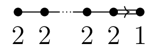
The Coxeter graph does not determine the Cartan matrix because it only gives angles between parts of roots in the base without indicating which of the roots is longer.
In order to determine the Cartan matrix, we need to specfy the ratios of the lengths of the roots.
So, if α_r, say is the shortest root, we label each root α_i in the Coxeter graph with a weight $w_i = \frac{\|\alpha_i\|^2}{\|\alpha_r\|^2}$. This labelled Coxeter graph is called a dynkin diagram. The Dynkin diagrams for the above are

A_n



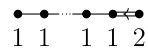
This is $\mathfrak{sl}_{n+1}$

B_n



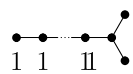
This is $\mathfrak{so}_{2n+1}$

C_n



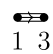
This is $\mathfrak{sl}_{2n}$

D_n

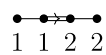


This is $\mathfrak{so}_{2n}$. The rest are exceptional Lie algebras.

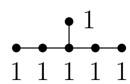
G_2



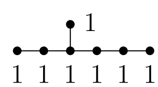
F_4



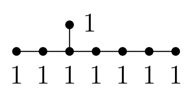
E_6



E_7



E_8



Here is how to recover the Cartan matrix from the Dynkin diagram:
If $\alpha = \beta$ then $p(\alpha, \beta) = 2$
If $\alpha \neq \beta$ and if α, β are not joined by an edge, then $p(\alpha, \beta) = 0$
If $\alpha \neq \beta$ and if α, β are joined by at least one edge, and if the coefficient of α is less than the coefficient of β, then $p(\alpha, \beta) = -1$.
If $\alpha \neq \beta$ and if α, β are joined by at least i edges ($1 \leq i \leq 3$) and if the coefficient of α is greater than or equal to that of β, then $p(\alpha, \beta) = -i$

The classification theorem

Suppose that $\mathfrak{g}$ is a finite dimensional semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$. Recall that we showed

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \right)$$

We also showed that there exists $H_\alpha \in \mathfrak{h}$, $X_\alpha \in \mathfrak{g}_\alpha$, $Y_\alpha \in \mathfrak{g}_{-\alpha}$ such that

$$[H_\alpha, X_\alpha] = 2X_\alpha,\ [H_\alpha, Y_\alpha] = -2Y_\alpha$$

Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be a base of R. To ease notation, write H_i instead of H_{α_i}, etc. Also, set $p(i, j) = p(\alpha_i, \alpha_j)$.

Theorem 0.22. *The algebra $\mathfrak{g}$ is generated by the elements*

$$\{X_i, Y_i, H_i \mid 1 \leq i \leq n\}$$

satisfying the following relations:

$(S1)$ $[H_i, H_j] = 0,\ 1 \leq i, j \leq n$

$(S2)$ $[X_i, Y_i] = H_i, [X_i, Y_j] = 0$ *if* $i \neq j$

$(S3)$ $[H_i, X_j] = p(j, i)X_j,\ [H_i, Y_j] = -p(i, j)Y_i$

(S_{ij}^+) $(\mathrm{ad}_{X_i})^{p(j,i)+1}(X_j) = 0$ *if* $i \neq j$

(S_{ij}^-) $(\mathrm{ad}_{Y_i})^{-p(j,i)+1}(Y_j) = 0$ *if* $i \neq j$

Proof.

$(S1)$ Suppose that $\alpha \in R^+$ and then recall that we may write $\alpha = \alpha_{i_1} + \cdots + \alpha_{i_k}$ in such a way that all th epartial sums $\alpha_{i_1} + \cdots + \alpha_{i_h}$, $h < k$, are in R^+. Then the elements

$$[X_{i_k}, [X_{i_{k-1}}, [\ldots, [X_{i_2}, X_{i_1}] \ldots]]$$

is a nonzero element of $\mathfrak{g}_\alpha$

$(S2)$ Observe that if $i \neq j$, then $[X_i, Y_j] \in \mathfrak{g}_{\alpha_i - \alpha_j}$ and $\alpha_i - \alpha_j \in R^+$

$(S3)$ "It just works, look at the Cartan matrix" - Bisi

Lecture 14, 2/29/24

We finish the proof:

(S_{ij}^+) Observe that the weight of $(\mathrm{ad}_{X_i})^{-p(i,j)+1}(X_j)$ is equal to $-p(j,i)\alpha_i + \alpha_i + \alpha_j = \alpha_j - p(j,i)\alpha_i + \alpha_i = S_j(\alpha_i - \alpha_j)$. $\alpha_i - \alpha_j \notin R$, so $\mathfrak{g}_{\alpha_i - \alpha_j} = 0$

(S_{ij}^-) Similarly.

■

Theorem 0.23 (Serre)**.** *Suppose that R is a root system with base $\Delta = \{\alpha_1, \ldots, \alpha_n\}$. Let $\mathfrak{g}$ be the Lie algebra generated by the elements*

$$\{X_i, Y_i, H_i \mid 1 \leq i \leq n\}$$

satisfying the relations listed in the previous theorem. Then $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra, with Cartan subalgebra spanned by the H_i, with corresponding root system equal to R.

Proof. Omitted - one can look in Humphries, for example.

■

Abstract Theory of Weights

Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra, and $\mathfrak{h}$ be a Cartan subalgebra. Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be a base of the root system R of $\mathfrak{g}$. Write H_i for H_{α_i}, etc.

Definition 0.45. An abstract (or integral) weight is a linear function $\lambda : \mathfrak{g} \to \mathbb{C}$ such that $\lambda(H_i) \in \mathbb{Z}$ for all i. Say that λ is dominant if $\lambda(H_i) \geq 0$ for all i. The fundamental weights λ_i are those defined by

$$\lambda_i(H_j) = \delta_{ij}$$

i.e. given by the dual basis of H_i.

Example 0.9. If $\mathfrak{g} = \mathfrak{s}l_2(\mathbb{C})$, then the only fundamental weight is $\lambda_1 = \frac{\alpha}{2}$

$$\lambda_1(H_1) = \frac{1}{2}\alpha(H_\alpha) = \frac{2}{2} = 1$$

Let $\wedge^+$ denote the set of dominant weights. Every dominant weight λ may be written

$$\lambda = \sum_i n_i \lambda_i$$

with $\mathbb{Z} \ni n_i \geq 0$. Say that λ is strongly dominant if $n_i > 0$.

The universal enveloping algebra of a Lie algebra

Recall that there is a functor

$$\mathscr{L} : \{\text{Associative Algebras}\} \to \{\text{Lie Algebras}\}$$

sending A to $\mathscr{L}(A)$. If $a, b \in A$, then $[a, b] = ab - ba$.
Suppose that $\mathfrak{g}$ is a Lie algebra.

Definition 0.46. A universal enveloping algebra of $\mathfrak{g}$ is any pair (U, ι) consisting of an associative algebra U and a Lie algebra homomorphism $\iota : \mathfrak{g} \to \mathscr{L}(U)$, satisfying the following universal property:
If A is an associative algebra and $\pi : \mathfrak{g} \to \mathscr{L}(A)$ is a Lie algebra homomorphism, then there exists a unique $\tilde{\pi} : U \to A$ of associative algebras such that the following triangle commutes

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota"] \arrow[dr, "\pi"'] & U \arrow[d, dotted, "\exists ! \tilde{\pi}"] \\ & A \end{tikzcd}$$

$\mathrm{Hom}(\mathfrak{g}, \mathscr{L}(A)) = \mathrm{Hom}(U(\mathfrak{g}), A)$, so U is a left adjoint to $\mathscr{L}$
Suppose that (U_1, ι_1) is another universal enveloping algebra for $\mathfrak{g}$. We have

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\iota"] \arrow[dr, "\iota_1"'] & U \arrow[d, dotted, "\exists ! \tilde{\iota_1}"] \\ & U_1 \end{tikzcd}$$

Similarly we have $\tilde{\iota} : U_1 \to U$. Check that $\tilde{\iota} \circ \tilde{\iota_1} = \mathrm{Id}$, $\tilde{\iota_1} \circ \tilde{\iota} = \mathrm{Id}$

Theorem 0.24. *If $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, then $U(\mathfrak{g}) = U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2)$.*

Proof. Set $U = U(\mathfrak{g})$, $U_i = U(\mathfrak{g}_i)$, $i = 1, 2$. Define:

$$\begin{aligned} f : \mathfrak{g} &\to U_1 \otimes U_2 \\ x &\mapsto \iota_1(x_1) \otimes 1 + 1 \otimes \iota_2(x_2) \end{aligned}$$

Check that f is a Lie algebra homomorphism of associative algebras, so induces a homomorphism $\tilde{f} : U \to U_1 \otimes U_2$ of associative algebras.
Next consider the homomorphisms

$$\phi_i : \mathfrak{g}_i \hookrightarrow \mathfrak{g} \to U$$

these induce homomorphisms $\tilde{\phi}_i : U_i \to U$. Now define

$$\begin{aligned} \phi : U_1 \otimes U_2 &\to U \\ x_1 \otimes x_2 &\mapsto \tilde{\phi}_1(x_1) \cdot \tilde{\phi}_2(x_2) \end{aligned}$$

Check that $\phi \circ \tilde{f} = \text{Id}$ and $\tilde{f} \circ \phi = \text{Id}$
Alternatively, this is just because U is left adjoint to $\mathscr{L}$, and so preserves colimits, and so preserves coproducts.

■

Theorem 0.25. *If (U, ι) is a universal enveloping algebra of $\mathfrak{g}$ then the correspondence $\pi \mapsto \tilde{\pi}$ induces a bijection between representations of $\mathfrak{g}$ and left U-modules.*

Proof. If $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of $\mathfrak{g}$ in a vector space V, then V may be viewed as being a left U module via $u \cdot v = \tilde{\pi}(u)v$ for all $u \in U, v \in V$.
Suppose conversely that V is a left U-module. Then V is a $\mathbb{C}$-vector space and defining

$$\pi(X)u = (\iota(X))v$$

$(v \in V, X \in \mathfrak{g})$ yields a representation of $\mathfrak{g}$. These two constructions are inverse to each other because $\tilde{\pi} \circ \iota = \pi$

■

<u>Remark:</u> The map $[-1] : \mathfrak{g} \to \mathfrak{g}$ (multiplication by -1) is an involution on $\mathfrak{g}$, so induces an involution $u \mapsto u'$ on U such that $\iota(X)' = -\iota(X)$. This enables us to view left U modules as right U modules, via

$$v \cdot u = u' \cdot v$$

To show the universal enveloping algebra exists, consider the tensor algebra

$$T(\mathfrak{g}) = \bigoplus_{k=0}^{\infty} \mathfrak{g}^{\otimes k}$$

together with the natural map $\iota : \mathfrak{g} \to T(\mathfrak{g})$. Notice that $T(\mathfrak{g})$ is generated by 1 and $\iota(\mathfrak{g})$. (If A is any associative algebra) then any linear map $\mathfrak{g} \to A$ extends uniquely to a homomorphism $T(\mathfrak{g}) \to A$ of associative algebras).
Let J be the two-sided ideal in $T(\mathfrak{g})$ generated by all elements

$$X \otimes Y - Y \otimes X - [X, Y]$$

with $X, Y \in \mathfrak{g}$. Now we set $U(\mathfrak{g}) = T(\mathfrak{g})/J$ and let $\iota : \mathfrak{g} \to U(\mathfrak{g})$ denote the obvious map.

Theorem 0.26. *The pair $(U(\mathfrak{g}), \iota)$ is a universal enveloping algebra of $\mathfrak{g}$.*

Proof. Suppose that A is an associative algebra and $\pi : \mathfrak{g} \to \mathscr{L}(A)$ is a Lie algebra homomorphism.
Uniqueness of $\tilde{\pi}$: follows from the fact that the images of 1 and $\iota(\mathfrak{g})$ generate $U(\mathfrak{g})$.
Existence of $\tilde{\pi}$: Let $\pi_1 : T(\mathfrak{g}) \to A$ be given by the universal property of $T(\mathfrak{g})$. Show that π_1 kills J.

■

Lecture 15, 3/7/24

Symmetric Algebras

Suppose that E is a (finite-dimensional) vector space over a field $\mathbb{F}$ of characteristic 0. If we define a Lie bracket on E by setting $[v,w]=0$ for all $v,w\in E$, then $U(E)$ is the symmetric algebra $S(E)$ of E.
Recall the definition of the symmetric algebra $S(E)$ of E: the algebra $S(E)$ is the largest Abelian quotient of the tensor algebra $T(E)$ of E.
So e.g. if we let I_n be the ideal of $E^{\otimes n}$ generated by

$$\{\alpha-\sigma(\alpha)\mid \alpha\in E^{\otimes n},\sigma\in S_n\}$$

Then $S^n(E)$ is $E^{\otimes n}/I_n$.
Then $S(E)=\bigoplus_{n=0}^{\infty} S^n(E)$
In concrete terms, if $\{v_1,\dots,v_m\}$ is a basis of E, and

$$\iota : E\to \mathbb{F}[x_1,\dots,x_n]$$

is given by

$$v_i\mapsto x_i$$

Then one may check (easy exercise) that the pair $(\mathbb{F}[x_1,\dots,x_n],\iota)$ satisfies the appropriate universal property of $(U(E),\iota)$.
We can identify $S(E)$ with the polynomial algebra $\mathbb{F}[x_1,\dots,x_n]$
A basis for $S(E)$ is then given by the set of monomials of the form $x_{i_1}\cdots x_{i_r}$, where $i_1\le i_2\le\cdots\le i_r$

Symmetric tensors

Definition 0.47. A symmetric tensor of $T(E)\stackrel{\text{def}}{=}\bigoplus_{i=0}^{\infty}E^{\otimes i}$ is a of degree n is an element of $E^{\otimes n}$ that is invariant under the action of the symmetric group S_n. These form a vector space $\mathrm{Sym}^n(E)\subseteq E^{\otimes n}$ and we set

$$\mathrm{Sym}(E)=\bigoplus_{n=0}^{\infty}\mathrm{Sym}^n(E)$$

Note that this is a graded vector space and is not an algebra since the product of two symmetric tensors is not necessarily symmetric.
The natural map

$$\mathrm{Sym}^n(E)\hookrightarrow E^{\otimes n}\to S^n(E)$$

is an isomorphism of vector spaces with inverse given by

$$x_{i_1}\cdots x_{i_n}\mapsto \frac{1}{n!}\sum_{\sigma\in S_n} v_{i_{\sigma(1)}}\otimes\cdots\otimes v_{i_{\sigma(n)}}$$

note if we are not in characteristic zero, this need not be an isomorphism. As an exercise, write down an example of this in characteristic 2.
Upshot: $S(E)$ and $\mathrm{Sym}(E)$ are isomorphic as graded vector spaces.

The Filtration on $U(\mathfrak{g})$

Now suppose that $\mathfrak{g}$ is a Lie algebra with universal enveloping algebra $U(\mathfrak{g})$. Define $U_n(\mathfrak{g})$ to be the subspace of $U(\mathfrak{g})$ generated by elements of the form

$$\iota(X_1)\cdots\iota(X_n)$$

where $X_i\in\mathfrak{g}$ and $n\le m$.
So e.g. $U_0(\mathfrak{g})=\mathbb{F}, U_1(\mathfrak{g})=\mathbb{F}\otimes\iota(\mathfrak{g})$ and

$$0=U_{-1}(\mathfrak{g})\subseteq U_0(\mathfrak{g})\subseteq U_1(\mathfrak{g})\subseteq\cdots\subseteq U_n(\mathfrak{g})\subseteq\cdots$$

We define

$$\mathrm{gr}_n(U(\mathfrak{g}))\overset{\text{def}}{=}\frac{U_n(\mathfrak{g})}{U_{n-1}(\mathfrak{g})}$$

and we set

$$\mathrm{gr}\,U(\mathfrak{g})=\bigoplus_{n=0}^{\infty}\mathrm{gr}_m\,U(\mathfrak{g})$$

This is a graded algebra because the map

$$U_p(\mathfrak{g})\times U_q(\mathfrak{g})\to U_{p+q}(\mathfrak{g})$$

given by $(a,b)\mapsto ab$. induces a bilinear map

$$\mathrm{gr}_p(U(\mathfrak{g}))\times\mathrm{gr}_q(U(\mathfrak{g}))\to\mathrm{gr}_{p+q}(U(\mathfrak{g}))$$

Proposition 24. *The algebra* $\mathrm{gr}(U(\mathfrak{g}))$ *is generated by the image of* $\iota(\mathfrak{g})$*.*

Proof. This follows from the fact that each $U_n(\mathfrak{g})$ is generated by $\iota(\mathfrak{g})$ ■

Theorem 0.27. *The algebra* $\mathrm{gr}(U(\mathfrak{g}))$ *is commutative.*

Proof. It suffices to show that $\overline{\iota(X)}, \overline{\iota(Y)}$ commute in $\mathrm{gr}_2(U(\mathfrak{g}))$ for all $X, Y \in \mathfrak{g}$. This holds because

$$\iota(X)\iota(Y) - \iota(Y)\iota(X) = \iota([X,Y]) = 0$$

because $\iota([X,Y])$ is one degree lower than $\iota(X)\iota(Y) - \iota(Y)\iota(X)$. $\iota([X,Y]) \in U_2(\mathfrak{g})$. ■

Corollary 0.28. *The family of monomials $\iota(X_{i_1}) \cdots \iota(X_{i_m})$ with $i_1 \leq i_2 \leq \cdots \leq i_m$, and $m \leq n$, generated $U_n(\mathfrak{g})$.*

Proof. The proof is via induction on n. The case $n = 0$ is trivial.
Suppose that $a \in U_n(\mathfrak{g})$, and write $\overline{a}$ for its image in $\mathrm{gr}_n(U(\mathfrak{g}))$, as before. Then $\overline{a}$ is a polynomial of degree n in the image $\overline{\iota(X_1)}$, and so we may write

$$a = A + B$$

where A is a linear combination of products of monomials $\iota(X_{i_1}) \cdots \iota(X_{i_m})$ with $i_1 \leq \cdots \leq \iota_m$, $m \leq n$, and $B \in U_{n-1}(\mathfrak{g})$
The result follows by induction. ■

As $\mathrm{gr}(U(\mathfrak{g}))$ is commutative, it follows from the universal property of $S(\mathfrak{g})$ that the natural map

$$\mathfrak{g} \to \mathrm{gr}(U(\mathfrak{g}))$$

extends to a homomorphism

$$\Phi : S(\mathfrak{g}) \to \mathrm{gr}(U(\mathfrak{g}))$$

and this homomorphism is surjective because $\mathrm{gr}(U(\mathfrak{g}))$ is generated by the image of $\iota(\mathfrak{g})$.

Theorem 0.29 (Poincaré-Birkhoff-Witt)**.** *The homomorphism Φ is an isomorphism.*

Proof. Later ■

Let's see some consequences.

Theorem 0.30. *A basis of $U(\mathfrak{g})$ is given by the family of monomials of the form $\iota(X_{i_1}) \cdots \iota(X_{i_m})$, $i_1 \leq \cdots \leq i_m$, $m \geq 0$.*

Proof. To ease notation, for $\mathbb{M} = (i_1, \ldots i_m)$, for $i_1 \leq \cdots \leq i_m$, write

$$X_{\mathbb{M}} = \iota(X_{i_1}) \cdots \iota(X_{i_m})$$

and set $\ell(\mathbb{M}) = m$
Observe that for each $m > 0$, the elements $X_{\mathbb{M}}$ with $\ell(\mathbb{M}) = n$ lie in $U_n(\mathfrak{g})$ and the images of these elements in $\mathrm{gr}_n(U(\mathfrak{g}))$ are the images of basis elements of $S^n(\mathfrak{g})$ we considered earlier.

Now if the set $\{X_{\mathbb{M}} \mid \ell(\mathbb{M}) \geq 1\}$ is not a basis of $U(\mathfrak{g})$, then there exists a linear relation of the form

$$\sum_{\ell(\mathbb{M})=n} c_{\mathbb{M}} X_{\mathbb{M}} = \sum_{\ell(\mathbb{M})<m} c_{\mathbb{M}} X_{\mathbb{M}}$$

with some $c_{\mathbb{M}}$ on the left-hand side nonzero.
Hence

$$\sum_{\ell(\mathbb{M})=n} c_{\mathbb{M}} X_{\mathbb{M}} \equiv 0 \pmod{()} U_{n-1}(\mathfrak{g}))$$

and the PBW Theorem implies that this is impossible.
(In fact, we have shown that this theorem is equivalent to the PBW Theorem)

■

Theorem 0.31. *If $\mathfrak{g}$ is a Lie algebra and $\mathfrak{h}$ is a subalgebra, then the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ extends to an inclusion of universal algebras $U(\mathfrak{h}) \hookrightarrow U(\mathfrak{g})$*

Proof. Take an ordered basis of $\mathfrak{h}$ and extend it to one of $\mathfrak{g}$.

■

Lecture 16, 3/12/24

Let $\mathfrak{g}$ be a finite dimensional Lie algebra. We have a universal enveloping algebra $(U(\mathfrak{g}), \iota)$, $\iota : \mathfrak{g} \to U(\mathfrak{g})$. We defined a filtration on $U(\mathfrak{g})$, $U_0(\mathfrak{g}) \subseteq U_1(\mathfrak{g}) \subseteq \cdots \subseteq U_n(\mathfrak{g}) \subseteq \cdots$, with

$$U_n(\mathfrak{g}) = \{\iota(X_1) \cdots \iota(X_n) \mid X_i \in \mathfrak{g}\}$$

We set

$$\operatorname{gr} U(\mathfrak{g}) = \bigoplus_{i=0}^{\infty} \frac{U_n(\mathfrak{g})}{U_{n-1}(\mathfrak{g})}$$

and noted that $\operatorname{gr} U(\mathfrak{g})$ is a commutative graded algebra.
Recall we had the PBW Theorem and its (equivalent) corollary:

Theorem 0.32 (PBW)**.** *The natural map*

$$\Phi : S(\mathfrak{g}) \to \operatorname{gr} U(\mathfrak{g})$$

is an isomorphism.

Corollary 0.33. *The family of monomials $\iota(X_{i_1}) \cdots \iota(X_{i_n})$ with $i_1 \leq i_2 \leq \cdots \leq i_n$,$n \geq 1$ is a basis of $U(\mathfrak{g})$*

Proof. ■

Recall that $\mathrm{Sym}^n(\mathfrak{g})$ is the vector subspace of $\mathfrak{g}^{\otimes n}$ consisting of symmetric tensors. We have an isomorphism ι of vector spaces

$$\begin{tikzcd} \mathrm{Sym}^n(\mathfrak{g}) \arrow[d, "\iota"] \\ S^n(\mathfrak{g}) \end{tikzcd}$$

Theorem 0.34 (PBW Theorem, Poor man's version)**.** *Suppose that $\mathfrak{g}$ is a finite dimensional Lie algebra which admits a faithful representation $\mathfrak{g} \to \mathfrak{gl}(V)$. Then the map:*

$$\mathrm{Sym}(\mathfrak{g}) \to U(\mathfrak{g})$$

is an isomorphism of graded vector spaces.

Proof due to Noah Snyder.

For each $n \geq 0$, we have the following diagram

$$\begin{tikzcd} \mathrm{Sym}^n(\mathfrak{g}) \arrow[r, "\tilde{\Phi}_n"] \arrow[d, "q_n"] & U_n(\mathfrak{g}) \arrow[d] \\ S^n(\mathfrak{g}) \arrow[r, "\Phi_n"] & \mathrm{gr}_n(U(\mathfrak{g})) \end{tikzcd}$$

where q_n is an isomorphism of vector spaces which henceforth will be viewed as an identification, and Φ_n is surjective

Claim. *: $\tilde{\Phi_n}$ is surjective.*

Proof. By induction. Suppose that $\alpha \in U_n(\mathfrak{g})$, with image $\overline{\alpha} \in \mathrm{gr}_n U(\mathfrak{g})$. Then we may write

$$\Phi_n(t) = \overline{\alpha}$$

for some $t \in S^n(\mathfrak{g})$, whence $\alpha - \tilde{\Phi}_n(t) \in U_n(\mathfrak{g}) \subseteq \mathrm{im}\,\tilde{\Phi}_{n-1}$. It follows by induction that $\tilde{\Phi}_n$ is surjective for all n, and so $\tilde{\Phi}$ is surjective. We must now show that $\tilde{\Phi}$ is injective.

The Lie algebra homomorphism $\mathfrak{g} \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$,

$$\begin{aligned} \mathfrak{g} &\to U(\mathfrak{g}) \otimes U(\mathfrak{g}) \\ X &\mapsto X \otimes 1 + 1 \otimes X \end{aligned}$$

induces a homomorphism (coproduct) $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$. This is also induced by the homomorphism

$$\tilde{\Delta} : T(\mathfrak{g}) \to T(\mathfrak{g}) \otimes T(\mathfrak{g})$$

on tensor algebras given by $X \mapsto X \otimes 1 + 1 \otimes X$ for $X \in \mathfrak{g}$.
Now suppose that $\tilde{\Phi}$ is not injective, and that $f \in \ker(\tilde{\Phi})$ with $f \neq 0$.
Suppose also that $f \in \mathrm{Sym}^{\leq n}(\mathfrak{g}) \stackrel{\text{def}}{=} \bigoplus_{m=0}^{n} \mathrm{Sym}^m(\mathfrak{g})$.
Consider $g = \tilde{\Delta}(f) - 1 \otimes f - f \otimes 1$.
Then we see that $g \in \mathrm{Sym}^{\leq n-1} 9\mathfrak{g}) \otimes \mathrm{Sym}^{\leq n-1}(\mathfrak{g})$ (the top degree terms cancel out).
Now we show, by considering the image of g in the symmetric algebra, we show that if $n \geq 2$, then $g \neq 0$: in $S(\mathfrak{g})$, the map $\tilde{\Delta}$ may be viewed as a map

$$\tilde{\Delta} : F[\{x_i\}_i] \to F[\{x_i\}_i, \{y_i\}_i]$$

where on the right hand side we are writing x_i for $x_i \otimes 1$ and y_i for $1 \otimes x_i$, with $\tilde{\Delta}(f)(x_i, y_i) = f(x_i + y_i)$.
So if $g = 0$ then $f(x_i + y_i) = f(x_i) + f(y_i)$, i.e. f is an additive polynomial. There are no non-linear additive polynomials in characteristic 0, e.g. if f is homogenous of degree $n \geq 2$, then

$$2^n f(x_i) = f(2x_i) = 2f(x_i)$$

whence $(2^n - 2)f = g$. It now follows by induction that if $f \in \ker \tilde{\Phi}$, then f is linear. Since $\mathfrak{g}$ admits a faithful representation $\mathfrak{g} \to \mathfrak{gl}(V)$, it follows that $\iota : \mathfrak{g} \to U(\mathfrak{g})$ is injective, and so $f = 0$.

■

Representation Theory

Usual setup and notation: $\mathfrak{g}$ is a finite dimensional semisimple Lie algebra, with fixed Cartan subalgebra $\mathfrak{h}$, with Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be a base of R.
We often identify $\mathfrak{h}$ with its dual via the inner product $\langle -, - \rangle$.
Set:

$$\mathfrak{n} \stackrel{\text{def}}{=} \sum_{\alpha \in R^+} \mathfrak{g}_\alpha$$

$$\mathfrak{n}^- \stackrel{\text{def}}{=} \sum_{\alpha \in R^-} \mathfrak{g}_\alpha$$

$$\mathfrak{b} \stackrel{\text{def}}{=} \mathfrak{h} \otimes \mathfrak{n}$$

($\mathfrak{n}$ is for nilpotent and $\mathfrak{b}$ is for Borel)

Theorem 0.35.

(a) *We have*

$$\begin{aligned}\mathfrak{g} &= \mathfrak{n}^- \oplus \mathfrak{g} \oplus \mathfrak{n} \\ &= \mathfrak{n}^- \oplus \mathfrak{b}\end{aligned}$$

(b) *The algebra $\mathfrak{n}$ and $\mathfrak{n}^-$ are algebras consisting of nilpotent elements (and are nilpotent)*

(c) *The algebra $\mathfrak{b}$ is solvable.*

Proof. Exercise ■

Suppose that $\pi : \mathfrak{g} \to \mathfrak{g}l(V)$ is a finite-dimensional representation.

Definition 0.48. Say that $\omega \in \mathfrak{h}$ is a <u>weight</u> of V if the space

$$V_\omega \overset{\text{def}}{=} \{v \in V : \pi(H)v = \omega(H)v\text{for all } H \in \mathfrak{h}\}$$

is nonzero.
Call the dimension of V_ω the multiplicity of ω in V, and say that the non-zero elements of V_ω have weight ω.

Proposition 25.

(a) $\pi(\mathfrak{g}_\alpha) \cdot V_\omega \subseteq V_{\alpha+\omega}$ *if* $\alpha \in \mathbb{R}$.

(b) *The sum* $U = \sum_\omega V_\omega$ *is direct, and* U *is a submodule of* V.

Proof.

(a) If $X \in \mathfrak{g}_\alpha$, $v \in V_\omega$, and $H \in \mathfrak{h}$, then

$$\begin{aligned}\pi(H)(\pi(X)v) &= \pi(X)(\pi(H)v) + \pi([H,X])v \\ &= (\omega(H) + \alpha(H))\pi(X)v\end{aligned}$$

and so $\pi(X)v$ has weight $\omega + \alpha$.

(b) Clear.

■

Definition 0.49. Say that a vector $v \in V$ is a <u>primitive element of weight ω</u> if:

(a) $v \neq 0$ and v has weight ω

(b) We have $\pi(X_\alpha)v = 0$ for all $\alpha \in R^+$.

(So the primitive elements are the eigenvectors of the Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$).

Lecture 17, 3/14/24

Let $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$.

Definition 0.50. Say that $v \in V$ is primitive of weight ω if

(i) v has weight ω

(ii) $X_\alpha \cdot v = 0$ for all $\alpha \in R^+$

Recall: $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}$.
Suppose that v^+ is an eigenvector of $\mathfrak{h}$
Set $W = \mathbb{C} \cdot v^+$. Then W is a representation of $\mathfrak{b}$ and so is a representation of $\mathfrak{h}$.
So v^+ is an eigenvector of $\mathfrak{h}$, and so there is a linear map $\lambda : \mathfrak{h} \to \mathbb{C}$ such that $H \cdot v^+ = \lambda(H)v^+$ for all $H \in \mathfrak{h}$.
So $W = W_\lambda$.
Also if $\alpha \in R^+$ and $X_\alpha \in \mathfrak{b}$, then

$$X_\alpha \cdot v^+ \in W_{\lambda+\alpha} = 0$$

Hence v^+ is primitive.
The converse is clear.

Proposition 26. *Suppose that V is a primitive element of weight ω, and let $U = \mathfrak{g} \cdot V$. Then*

1. *If $R^+ = \{\beta_1, \cdots, \beta_k\}$, then U is spanned (as a vector space) by elements of the form*

$$X_{\beta_1}^{m_1} \cdots X_{\beta_k}^{m_k} \cdot v$$

with $m_i \in \mathbb{N}$ (according to Bisi, any civilized human being considers 0 to be a natural number).

2. *The weights of U are of the form*

$$\omega - \sum_{i=1}^{n} p_i \alpha_i$$

with $p_i \in \mathbb{N}$. These weights have finite multiplicity.

3. *The weight ω has multiplicity 1.*

4. *The module U is an indecomposable $\mathfrak{g}$-module.*

Proof. Set $A = U(\mathfrak{g}), B = U(\mathfrak{b}), C = U(\mathfrak{n}^-)$, and view V as an A-module.

1. Then since $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}^-$, we have $A = B \otimes C$. So

$$\begin{aligned} U &= A \cdot v \\ &= (C \otimes B) \cdot v \\ &= C \cdot v && \text{since } v \text{ is an eigenvector of } \mathfrak{b} \end{aligned}$$

So now the PBW theorem implies that the monomials $Y_{\beta_1}^{m_1} \cdots Y_{\beta_k}^{m_k}$ with $m_k \in \mathbb{N}$ form a basis of C, hence (1) follows.

2. An earlier proposition (from last time) shows that each element $Y_{\beta_1}^{m_1} \cdots Y_{\beta_k}^{m_k}$ has weight

$$\omega - \sum m_j \beta_j$$

and so each β_j is a linear combination of the α_i with nonnegative integer coefficients. This implies (2)

3. Observe that $\omega - \sum m_j \beta_j$ cannot equal ω unless $m_j = 0$ for all j.

4. Suppose that $U = U_1 \oplus U_2$. Then $U_\omega = U_{1,\omega} \oplus U_{2,\omega}$. Then something

■

Theorem 0.36. *Suppose that V is an irreducible $\mathfrak{g}$-module containing a primitive element v of weight ω.*
Then:

(a) The vector v is (up to scalar multiplication) the only primitive element in V. (we say that ω is the highest weight of V.

(b) The weights λ of V have the form

$$\lambda = \omega - \sum m_i \alpha_i, m_i \in \mathbb{N}$$

They are of finite multiplicity, and ω has multiplicity 1.

We have

$$V = \bigoplus_\lambda V_\lambda$$

(c) Two irreducible $\mathfrak{g}$-modules V_1, V_2, with highest weights ω_1, ω_2, are isomorphic if and only if $\omega_1 = \omega_2$.

Proof.

(b) Observe that $U = \mathfrak{g} \cdot v$ is non-zero, and so $U = V$, because V is irreducible. Now apply the previous proposition.

(a) Suppose that $z \in V$ is a primitive element of weight ζ. Then we may write

$$\zeta = \omega - \sum m_i \alpha_i, m_i \in \mathbb{N}$$

Similarly, interchanging the roles of ω and ζ gives

$$\omega = \zeta - \sum m_i \alpha_i, m_i \in \mathbb{N}$$

Hence $\omega = \zeta$.

(c) It is sufficient to prove that if $\omega_1 = \omega_2$, then $V_1 = V_2$. Let $v_i \in V_i$ $(i = 1, 2)$ be primitive of weight $\omega = \omega_1 = \omega_2$.

Consider $V = V_1 + V_2$. This has $v_1 + v_2$ as a primitive element of weight ω.

Set $U = \mathfrak{g} \cdot v$.

Then

$$pr_2 : V \to V_2$$

given by sending $U \subseteq V$ to V_2 via f_2. Exercise: show that f_2 is an isomorphism. So $U \simeq V_2$, similarly $U \simeq V_1$.

But do these things (irreducible highest-weight modules) exist?

Theorem 0.37. *For each weight ω, there is an irreducible $\mathfrak{g}$-module with highest weight equal to ω.*

Proof. Let L_ω be a one-dimensional $\mathfrak{b}$-module with basis vector v satisfying

$$H \cdot v = \omega(H)v$$

for all $H \in \mathfrak{h}$ and

$$X \cdot v = 0$$

for all $X \in \mathfrak{n}$.
Now view L_ω as a $U(\mathfrak{b})$-module, and consider the $U(\mathfrak{g})$-module

$$M(\omega) \stackrel{\text{def}}{=} U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} L_\omega$$

This is called the <u>Verma module of highest weight ω</u>.
Then

(i) The module $M(\omega)$ is generated by $1 \otimes v$.

(ii) The element $1 \otimes v \neq 0$ because the PBW theorem implies that $U(\mathfrak{g})$ is a free $U(\mathfrak{b})$-module having a basis containing the unit element 1.

(iii) $1 \otimes v$ is primitive of weight ω.

Now consider

$$M(\omega)^- \stackrel{\text{def}}{=} \sum_{\tau \neq \omega} M(\omega)_\tau$$

Claim. *If U is a proper $\mathfrak{g}$-submodule of $M(\omega)$, then $U \subseteq M(\omega)^-$.*
For U is an $\mathfrak{h}$-module, and so

$$U = \sum_\pi U_\pi$$

If $U_\omega \neq 0$, then $1 \otimes v \in U$, and so $U = M(\omega)$. This proves the claim.

■

So let $K(\omega)$ be the $\mathfrak{g}$-submodule of $M(\omega)$ generated by all proper submodules of $M(\omega)$. Then

$$L(\omega) \stackrel{\text{def}}{=} M(\omega)/K(\omega)$$

is irreducible, of highest weight ω.

■

Question
When is $L(\omega)$ finite-dimensional?

Theorem 0.38. *Suppose that ω is a weight. If $L(\omega)$ is finite-dimensional, then ω is a dominant weight, i.e. $\omega(H_\alpha) \geq 0$ for all Δ^+.*

Proof. If v is a primitive element of $L(\omega)$ for $\mathfrak{g}$, then v is also a primitive element for $\mathfrak{s}^\alpha \simeq \mathfrak{sl}_2$, and so $\omega(H_\alpha) \geq 0$.
This condition is in fact also sufficient!

■

The essential point is that the weights of $L(\omega)$ are invariant under the action of the Weyl group W.
$\Delta = \{\alpha_1, \cdot, \alpha_n\}$ is a base of R, and write H_i for H_{α_i}, etc.
Suppose that v is a primitive element of $L(\omega)$, and $1 \leq i \leq n$.

(I) Show that the space V_i spanned by

$$\{Y_i^p \cdot v\}_{1 \leq p \leq n}$$

is a finite-dimensional $\mathfrak{s}^i$-module.

(II) Show that $L(\omega)$ is a sum of finite dimensional $\mathfrak{s}^i$-modules.

(III) Let P_ω be the collection of weights of $L(\omega)$. Using (II), show that P_ω is invariant under the reflection corresponding to α_i, which in turn implies that P_ω is invariant under the action of the entire Weyl group W.

(IV) Show that P_ω has to be finite, and this is enough, because we have finitely many weights, each with finite multiplicity, so the whole thing has to be finite dimensional.

This is the highest weight theorem.