Lecture 1

Missed

Lecture 2

Missed

Lecture 3 - 4/9/24

Question: How can we construct submanifolds?
If M, N are manifolds, $F \in C^\infty(M, N)$ and $c \in N$. When is $F^{-1}(c)$ a submanifold of M?

Definition 0.1. Let $S = F^{-1}(c)$. Say that c is a regular value of F if for all $p \in S$, the derivative $DF|_p : T_pM \to T_pN$ is surjective.

Proposition 1. *If c is a regular value, then S is an embedded submanifold of M of dimension* $\dim(M) - \dim(N)$.

Proof. ■

Example 0.1. S^n is a manifold. Consider $F : \mathbb{R}^m \to \mathbb{R}$ given by $(x_1, \ldots, x_n) \mapsto \sum_i x_i^2$. Observe that $F^{-1}(1) = S^n$, and that

$$DF|_p = 2(x_0, x_1, \ldots, x_n) \neq 0$$

when $p \in S^n$. Hence S^n is a manifold.

Example 0.2. The orthogonal group $O(n)$ is a manifold. We have $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$. Consider the set

$$N \stackrel{\text{def}}{=} \{A \in M_n(\mathbb{R}) \mid A^T = A\}$$

of symmetric $n \times n$ matrices. This is a manifold because it is a linear subspace of $M_n(\mathbb{R})$.
Define $F : M_n(\mathbb{R}) \to N$ by $A \mapsto AA^T$.
Then $F^{-1}(I_n) = O_n$. Show that F is regular.

Vector fields

The tangent bundle
Question: If we have a vector field $p \mapsto v_p \in T_pM$, what does it mean to say that this is smooth?
Direct approach:

Turn $TM \stackrel{\text{def}}{=} \coprod_p T_pM$ into a manifold. Then there is a natural map $\pi : TM \to M$ given by sending $v_p \to p$ and this map is smooth.

Definition 0.2. A vector field on some $U \subseteq M$ is a smooth map $X : U \to TM$ such that for all $p \in U$, we have $X(p) \in T_pX$, i.e. $\pi \circ X = \text{Id}$.
Write $\Gamma(TU)$ for the set of vector fields on U. This is an $\mathbb{R}$-vector space and a $C^\infty(M)$-module.
Note: Bisi uses the notation $\mathsf{Vect}(U)$, so the notation may vary here.
In fact this is a sheaf of $C^\infty(M)$-modules.
Let $(x_1, \dots, x_n)$ be local coordinates in a chart U of M. Then for any $p \in U$ and $v_p \in T_pM$, there are $\alpha_1, \dots, \alpha_n$ such that

$$v_p = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}|_p$$

This gives a bijection $\pi^{-1}(U) \to \phi(U) \times \mathbb{R}^n$ by $v_p \mapsto (x_1(p), \dots, x_n(p), \alpha_1, \dots, \alpha_n)$, yielding charts that make TM into a manifold of dimension $2n$ called the tangent bundle of M
Then:

(i) The projection map $\pi : TM \to M$ is smooth

(ii) Suppose that (U, ϕ) is a chart on M and $X \in \Gamma(TU)$. Write X_p for $X(p) \in T_pM$.

Then we may write:

$$X_p = \sum_{i=1}^n \alpha_i(p) \frac{\partial}{\partial x_i}|_p$$

Then X is smooth if and only if all α_i are smooth.

(iii) If $F \in C^\infty(M, N)$, then $DF : TM \to TN$, $v_p \mapsto DF_p(v_p)$ is smooth.

(iv) If $F \in C^\infty(M, N)$ and $X \in \Gamma(TU)$, then in general $DF(X)$ is not a vector field on N because e.g F need not be injective. This motivates the following definition:

Definition 0.3. Let M, N be manifolds and $X \in \Gamma(TM)$, $Y \in \Gamma(TN)$, $F \in C^\infty(M, N)$. Say that X, Y are F-related if the diagram commutes:

$$\begin{tikzcd} TM \arrow[r, "DF"] & TN \\ M \arrow[u, "X"] \arrow[r, "F"] & N \arrow[u, "Y"'] \end{tikzcd}$$

Question: What does $\Gamma(TU)$ look like?

Definition 0.4. Let $Der(C^\infty(M))$ be the set of all maps $\mathfrak{X} : C^\infty(N) \to C^\infty(M)$ that satisfy $\mathfrak{X}(fg) = f\mathfrak{X}(g) + g\mathfrak{X}(f)$. This is a $C^\infty(M)$-module. We identify it with $\Gamma(TM)$ as follows.
Given $X \in \Gamma(TM)$, define $\mathfrak{X}$ by $\mathfrak{X}(f)(p) = X_p(f)$. Given $\mathfrak{X} \in Der(C^\infty(M))$, define X by $X_p(f) = \mathfrak{X}(f)(p)$.
Product of the vector fields in one sense corresponds to composition of derivations in the other sense.

Definition 0.5. If $X, Y \in \Gamma(TM)$, the Lie Bracket$[X, Y] \in \Gamma(TM)$ is the vector field corresponding to the derivation $XY - YX$. Then $[-,-]$ is bilinear, antisymmetric, and satisfies the Jacobi identity. So $\Gamma(TM)$ is a Lie algebra.

Flows

Suppose that $X \in \Gamma(TM)$. An integral curve of X is a smooth curve $\gamma : I \to M$ such that $\dot{\gamma}(t) = X_{\gamma(t)}$. These will exist by ode stuff.

Lecture 4 - 4/11/24

Lie Derivatives

Let X be a complete vector field (meaning any flow can be extended to all of $\mathbb{R}$), and let Θ be its flows. If $g \in^\infty (M)$, the

$$\mathscr{L}_X(g) = \frac{d}{dt}(\Theta_t^* g)|_{t=0}$$

Then $L_X(g)(p) = X(g(p))$. If $Y \in \Gamma(TM)$, then

$$\mathscr{L}_X(Y) = \frac{d}{dt}\Theta_t^*(Y)|_{t=0}$$

Lemma 1. $\mathscr{L}_X(Y) = [X, Y]$

Proof. Suppose that $g \in C^\infty(M)$, and first observe that

$$\Theta_t^*(Y)(g \circ \Theta_t) = Y(g) \circ \Theta_t$$

Now consider

$$\begin{aligned}\frac{\Theta_t^*(Y)(g) - Y(g)}{t} &= \frac{\Theta_t^*(Y)(g) - \Theta_t^*(Y)(g \circ \Theta_t)}{t} \\ &+ \frac{Y(g) \circ \Theta_t - Y(g)}{t} \\ &= \alpha_t + \beta_t,\end{aligned}$$

say.
We have

$$\lim_{t\to 0} \beta_t = \mathscr{L}_X(Y(g)) = X(Y(g))$$

Also

$$\begin{aligned}\lim_{t\to 0} \alpha_t &= \lim_{t\to 0} \Theta_t^*(Y)\left(\frac{g - g\circ\Theta_t}{t}\right)\\ &= Y(-\mathscr{L}_X(g))\\ &= -YX(g)\end{aligned}$$

Corollary 0.1. *Suppose that $X, Y \in \Gamma(TM)$ (with X, Y complete, and $f \in C^\infty(M)$. Then*

(i)

$$\begin{aligned}\mathscr{L}_X(fY) &= \mathscr{L}_X(f)Y + f\mathscr{L}_X(Y)\\ &= X(f)Y + f\mathscr{L}_X(Y)\end{aligned}$$

(ii)

$$\mathscr{L}_X(Y) = -\mathscr{L}_Y(X)$$

(iii)

$$\mathscr{L}_X[Y,Z] = [\mathscr{L}_X Y, Z] + [Y, \mathscr{L}_X Z]$$

Lie Groups:(At last...)

Definition 0.6. A Lie Group is a manifold G with a group sstructure such that the multiplication map $m : G\times G \to G$ and the inverse map $i : G \to G$ are smooth.
If $g \in G$, write $L_g : G \to G$ for the diffeomorphism $L_g(h) = gh$.

Definition 0.7. We say that $X \in \Gamma(TG)$ is left-invariant if

$$DL_g|_h(X_h) = X_{gh}$$

for all $g, h \in G$.
Notation: Denote by $\mathsf{Vect}^L(G)$ the set of left-invariant vector fields on G.
Using that fact that for all diffeomorphisms F, we have

$$F^*[X,Y] = [F^*X, F^*Y]$$

One checks that $\mathsf{Vect}^L(G)$ is a Lie subalgebra of $\Gamma(TG)$.

- A left-invariant vector field gives a tangent vector at the identity.
- Conversely, given a tangent vector at the identity, we obtain via translations a map $M \to TM$ which is a left-invariant vector field.

Lemma 2. *Given $\xi \in T_eG$, define*

$$X_\xi|_g = DL_g|_e(\xi) \in T_g(G)$$

$X_\xi \in \mathsf{Vect}^L(G)$, and the maps $\xi \to X_\xi$ is an isomorphism of vector spaces.

Proof. The inverse map is given by $X \mapsto X|_e$.
Left-invariance:

$$\begin{aligned} DL_h|_g(X_\xi|_h) &= DL_hL_g(DL_g|_e(\xi)) \\ &= X_\xi|_{hg} \end{aligned}$$

Smoothness:
Suppose that $f \in C^\infty(U)$ where U is open and contains e.
Let $\gamma : (-\varepsilon, \varepsilon) \to U$ be a smooth curve with $\dot\gamma(0) = \xi$. Then

$$\begin{aligned} X_\xi f|_g &= DL_g(\xi)(f) \\ &= \xi(f \circ L_g) \\ &= \frac{d}{dt}(f \circ L_g \circ \gamma)|_{t=0} \end{aligned}$$

Note that the map $(t, g) \mapsto f \circ L_g \circ \gamma(t)$ is smooth, it follows that X_ξ is smooth.

Definition 0.8. Let G be a Lie group. The Lie algebra $\mathfrak{g}$ of G is the Lie algebra T_eG whose Lie bracket is that induces by that of the isomorphism with $\mathsf{Vect}^L(G)$. So

$$[\xi, \eta] = [X_\xi, X_\eta]$$

Example 0.3. If G is abelian, the Lie bracket vanishes.

Example 0.4. For any vector space V and $v \in V$, we have $T_vV \cong V$. So the Lie algebra of V is V with trivial Lie bracket since the group is abelian.

Example 0.5. Note $G = \mathrm{GL}_n(\mathbb{R})$ is an open subgroup of $M_n(\mathbb{R})$ and so is a manifold. We have

$$\begin{aligned} \mathfrak{gl}_n(\mathbb{R}) &= T_e(\mathrm{GL}_n(\mathbb{R})) \\ &= T_e(M_n(\mathbb{R})) \\ &= M_n(\mathbb{R}) \end{aligned}$$

If $A, B \in \mathrm{GL}_n(\mathbb{R})$, then $L_A(B) = AB$, $L_A(B + H) = AB + BH$, and so $DL_A|_B(H) = AH$

By e.g. computing its terms in local coordinates, it follows if $\xi, \eta \in \mathfrak{gl}_n(\mathbb{R})$, then under the identification of $\mathfrak{gl}_n(\mathbb{R})$ with $M_n(\mathbb{R})$, we have

$$[\xi, \eta] = \xi\eta - \eta\xi$$

Proposition 2. *Let G be a Lie group and suppose that $\xi \in \mathfrak{g}$. Then the integral curve γ for X_ξ through e exists for all time and $\gamma : \mathbb{R} \to G$ is a Lie group homomorphism.*

Proof. Let $\gamma : I \to G$ be a maximal integral curve of X_ξ, and suppose that $(-\varepsilon, \varepsilon) subseteq I$. Fix t_0 with $|t_0| < \varepsilon$, and consider $g_0 = \gamma(t_0)$.
Now set

$$\tilde{\gamma}(t) = L_{g_0}(\gamma(t))$$

for $|t| < \varepsilon$.

Claim. *$\tilde{\gamma}$ is an integral curve of X_ξ with $\tilde{\gamma}(0) = g_0$.*

Proof. To see this, observe that

$$\begin{aligned}
\dot{\tilde{\gamma}}(0) &= \frac{d}{dt} L_{g_0}(\gamma(t)) \\
&= DL_{g_0}\dot{\gamma}(t) \\
&= DL_{g_0} X_\xi|_{\gamma(t)} \\
&= X_\xi|_{g_0\gamma(t)} \\
&= X_\xi|_{\gamma(t)}
\end{aligned}$$

By patching these together, we see that $(t_0 - \varepsilon, t_0 + \varepsilon) \subseteq I$. Since we have a fixed ε that works for all t_0, it follows that $I = \mathbb{R}$.

■

Example 0.6. Let $G = \mathrm{GL}_n(\mathbb{R})$. If $\xi \in \mathfrak{gl}_n$, then set

$$e^\xi = \sum_{k \geq 0} \frac{\zeta^k}{k!}$$

Set $F(t) = e^{\xi t}$. Then this is in $\mathrm{GL}_n(\mathbb{R})$, $F(0) = I$, and $F'(t) = L_{F(t)}\xi$.
So $F(t)$ is an integral curve through I.

Definition 0.9. The exponential map of a Lie group G is

$$\begin{aligned}
\exp : \mathfrak{g} &\to G \\
\xi &\mapsto \gamma_\xi(1)
\end{aligned}$$

Where γ_ξ is the integral curve of X_ξ through $e \in G$.

Proposition 3.

(i) exp *is a smooth map*

(ii) *If* $F(t) = \exp(t\xi)$*, then* $F : \mathbb{R} \to G$ *is a Lie group homomorphism, and*

$$DF|_0\left(\frac{d}{dt}\right) = \xi$$

(iii) *The derivative* $D\exp : T_e\mathfrak{g} \cong \mathfrak{g} \to T_eG = \mathfrak{g}$ *is the identity map.*

(iv) exp *is a local diffeomorphism around* $0 \in \mathfrak{g}$*, so there exists an open* $U \subseteq G$ *containing 0 such that* $\exp : U \to \exp(U)$ *is a diffeomorphism.*

(v) exp *is natural, i.e. if* $f : G \to H$ *is a Lie group homomorphism, then the following diagram commutes:*

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\exp"] \arrow[d, "DF|_e"'] & G \arrow[d, "f"] \\
\mathfrak{h} \arrow[r, "\exp"] & H
\end{tikzcd}$$

Proof. Exercise ■

Definition 0.10. A Lie Subgroup of G is a subgroup H with a smooth structure making H an immersed submanifold of G via the inclusion $i : H \to G$.
So if H is a subgroup then $\mathfrak{h} \subseteq \mathfrak{g}$ is a Lie subalgebra.
If H is a closed subgroup of G, then G/H (with the quotient topology) is a differentiable manifold, and the quotient map is a submersion.

Theorem 0.2. *If* $\mathfrak{h} \subseteq \mathfrak{g}$ *is a subalgebra, then there exists a unique connected Lie subgroup* H *of* G *such that* $T_eH = \mathfrak{h}$

Theorem 0.3. *If* $\mathfrak{g}$ *is a finite dimensional Lie algebra, then there exists a unique simply connected Lie group whose Lie algebra is* $\mathfrak{g}$*.*

Theorem 0.4. *Let* G, H *be Lie groups with* G *simply connected. Then every Lie algebra homomorphism* $\mathfrak{g} \to \mathfrak{h}$ *lifts to a Lie group homomorphism* $G \to H$*.*

Proof. ■

Tori

Lecture 5, 4/16/24

A torus T^n is just

$$\mathbb{R}^n/\mathbb{Z}^n \cong (\mathbb{R}/\mathbb{Z})^n \cong (S^1)^n$$

General Fact:
If G is any connected topological group, then G is generated by any open neighborhood of the identity.
WLOG: Let $U = U^{-1}$ be an open neighborhood of the identity (if $U \neq U^{-1}$, take $U \cap U^{-1}$).
Then

$$\bigcup_{m=1}^{\infty} U^m$$

is an open subgroup of G, and so has open (and disjoint) cosets. This must be all of G since G is connected.
Consequences

(a) A homomorphism between connected Lie groups is determined by its corresponding homomorphism on Lie algebras (For the exponential map is a local homeomorphism and is natural).

(b) The map $\exp : \mathfrak{g} \to G$ is a homomorphism if and only if G is abelian.

For the exponential is a bijection is a neighborhood of the identity, which contains a set of generators of G. So, since $\mathfrak{g}$ is commutative, if exp is a homomorphism, then G is abelian.

Conversely, observe that multiplication $G \times G \to G$ induces $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ given by $(x, y) \mapsto x + y$. So if exp is a homeomorphism, then it follows via naturality of exp that G is abelian.

(c) If G is a connected, abelian Lie group, then $\exp : \mathfrak{g} \to G$ is surjective.

Lemma 3. *A discrete subgroup B of a finite dimensional vector space V is generated by linearly independent vectors $v_1, \ldots, v_k$.*

Proof. The proof is by induction on n, which is the dimension of B.
If $n = 1$, then V is generated by its smallest positive element, or $B = 0$.
If $n > 1$, choose a Euclidean metric on V and then choose an element v_1 of smallest nonzero norm. Let $V = \mathbb{R} \cdot v_1 \oplus W$ be an orthogonal splitting, and consider the projection $p : B \to W$.

Claim. *$p(B)$ contains no nonzero element of norm less than $\frac{|v_1|}{2}$*

Proof. For if $0 < p(v) < \frac{|v_1|}{2}$, where $v \in V$ is such an element, then for some $m \in \mathbb{Z}$, the projection of $v + mv_1$ has norm at most $\frac{|v_1|}{2}$. Then $v + mv_1 \in B$, but

$$0 < |v + mv_1| \frac{|v_1|}{2}$$

which contradicts the choice of v_1.
So $p(B)$ is discrete, and so by induction is generated in W by linearly independent vectors $v_2, \ldots, v_k$, $k \leq n$.

■

Thus we have an exact sequence

$$0 \longrightarrow \langle v_1 \rangle \longrightarrow B \xrightarrow{p} \langle v_2, \ldots, v_k \rangle \longrightarrow 0$$

■

Theorem 0.5. *Let G be a connected, abelian Lie group. Then $G \cong T^n \times \mathbb{R}^s$, where $T = \mathbb{R}/\mathbb{Z}$.*

Proof. The exponential map
$$\exp : \mathfrak{g} \to G$$
is a local bijection near the origin, so the kernel K is discrete. So K is generated by linearly independent vectors $X_1, \ldots, X_k \in \mathfrak{g}$. Complete this collection to a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$.
This determines an isomorphism from $\mathfrak{g} \cong \mathbb{R}^n$ with

$$K \cong \mathbb{Z}^\ell \times \{0\} \subseteq \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n$$

So

$$\mathfrak{g}/K \cong \mathbb{R}^n/(\mathbb{Z}^k \times \{0\}) = T^k \times \mathbb{R}^{n-k}$$

The homomorphism

$$T^k \times \mathbb{R}^{n-k} \cong \mathfrak{g}/K \to G$$

is a bijective local diffeomorphism, and so is an isomorphism of Lie groups.

■

Corollary 0.6. *A compact, abelian Lie group is isomorphic to a product of a Torus and a finite group.*

Proof. The connected component of the identity is a (possibly trivial) torus and so there is an exact sequence

$$0 \longrightarrow T \xrightarrow{\iota} G \xrightarrow{p} B \longrightarrow 0$$

As T is open in G, and G is compact, it follows that G is finite. The exact sequence splits because T is divisible, and this implies the result.

■

Now lets look at characters.

If G is a compact abelian Lie group, then

$$G \cong S^1 \times \cdots \times S^1 \times \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_k\mathbb{Z}$$

and we are interested in homomorphisms $G \to S^1$.

Theorem 0.7. *The irreducible complex representations of S^1 are given by the characters $z \to z^n$, $n \in \mathbb{Z}$.*
The irreducible complex characters of $T^n \cong \mathbb{R}^n/\mathbb{Z}^n$ are all of the form

$$\theta : [x] \mapsto \exp(2\pi i \alpha(x))$$

for $x \in \mathbb{R}^n$, and $\alpha(x) = \langle \alpha, x\rangle = \sum_i a_i x_i$, with $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$.

Proof. Suppose that θ is a character of T^n. Then we have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}^n \arrow[r] \arrow[d, "\alpha"] & Lie(T^n) = R^n \arrow[r] \arrow[d, "\alpha"] & T^n \arrow[r] \arrow[d, "\theta"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & Lie(S) = \mathbb{R} \arrow[r, "\exp"] & S^1 \arrow[r] & 1
\end{tikzcd}$$

The point is that α is a linear functional on $\mathbb{R}^n$ and takes integer values on $\mathbb{Z}^n$.

■

Corollary 0.8. *The real, non-trivial, irreducible representations of T^n are two dimensional, and are given by*

$$[x_1, \ldots, x_n] \mapsto \begin{pmatrix} \cos(2\pi)\langle a, x\rangle & \sin(2\pi)\langle a, x\rangle \\ -\sin(2\pi)\langle a, x\rangle & \cos(2\pi)\langle a, x\rangle \end{pmatrix}$$

Where the notation is as in the theorem. We have that a and $-a$ give equivalent representations.

Proof. If V is a real representation of T^n, then consider $V_{\mathbb{C}} \stackrel{\text{def}}{=} V \otimes_{\mathbb{R}} \mathbb{C}$.
If χ is a non-trivial irreducible complex representation occurring in $V_{\mathbb{C}}$, then $\overline{\chi}$ must be another.
Observe if $w, \overline{w}$ are corresponding eigenvectors, the vectors $v_+ = w + \overline{w}, v_- = i(w - \overline{w})$ are real and linearly independent, and form a basis for an invariant subspace of V of dimension 2.

■

Now suppose that T^n is a torus in a Lie group G, and consider the adjoint action of T^n on the Lie algebra $\mathfrak{g}$ of G.

Proposition 4. *Under this adjoint action, the T^n-module $\mathfrak{g}$ decomposes as:*

$$\mathfrak{g} = V_0 \oplus \bigoplus_{i=1}^{r} V_{\theta_i}$$

where T^n acts trivially in V_0 and the V_{θ_i} are irreducible T^n-modules.
The torus T^n is maximal iff $Lie(T^n) = V_0$.

Proof. The first part is obvious. Observe that plainly $Lie(T^n) \subseteq V_0$. If T^n is not maximal, then $T^n \subsetneq T^m$ and

$$Lie(T^n) \subsetneq Lie(T^m) \subseteq W_0 \subseteq V_0$$

Whence the part of $\mathfrak{g}$ on which T^n acts trivially. Since $Lie(T^n) \neq Lie(T^m)$, it follows that $Lie(T^n) \neq V_0$.
Conversely, suppose that $Lie(T^n) \neq V_0$. Then there is some vector $x \in V_0 \setminus Lie(T^n)$. Then the one-parameter subgroup $H = \{\exp(tX) \mid t \in \mathbb{R}\}$ is invariant under conjugation by T^n.
Hence the closure of $H \cdot T^n$ is a compact connected abelian Lie group, and so is a Torus and contains T^n. ■

Corollary 0.9. $\dim(G) - \dim(T)$ *is even.*

Proof. ■

Definition 0.11. An element $t \in T^n$ is called a <u>generator</u> if $\{t^n \mid n \in \mathbb{Z}\}$ is dense in T^n.

Theorem 0.10 (Kronecker)**.** *A vector $v \in \mathbb{R}^n$ represents a generator of T^n if and only if 1, and the components $v_1, \ldots, v_n$ of T are linearly independent over $\mathbb{Q}$.*
(So the set of generators is dense in T^n).

Lecture 6 - 4/18/24

Tori (cont)

Proof. Observe that the exact sequence

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{R}^n \longrightarrow T^n \longrightarrow 0$$

identifies $\mathbb{R}^n = Lie(\mathbb{R}^n) = Lie(T^n)$ and identifies the projection $\mathbb{R}^n \to T^n$ with the exponential map. So a homomorphism $f : T^n \to S^1$ induces the following diagrams:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}^n \arrow[r] \arrow[d] & \mathbb{R}^n \arrow[r] \arrow[d, "Df|_n"] & T^n \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{R} \arrow[r] & S^1 \arrow[r] & 0
\end{tikzcd}$$

whence

$$(Df|_0)(v_1, \dots, v_n) = \sum_i \alpha_i v_i$$

with $\alpha_i \in \mathbb{Z}$.
Now observe that the following are equivalent:

(i) $1, v_1, \dots, v_n$ are linearly dependent over $\mathbb{Q}$.

(ii) $\sum_i q_i v_i \in \mathbb{Q}$ for some $(q_1, \dots, q_n) \neq 0 \in \mathbb{Q}^n$

(iii) $\sum_i \alpha_i v_i \in \mathbb{Z}$ for some $0 \neq (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$

(iv) $v \pmod{\mathbb{Z}}^n$ is in the kernel of some non-trivial homomorphism $f : T^n \to S^1$

(v) $v \pmod{\mathbb{Z}}^n$ is not a generator.

■

Definition 0.12. A subgroup T of G is a maximal torus if T is a torus and is not properly contained in any other subtorus of G.
So a maximal torus is the same thing as a maximal connected compact abelian subgroup of G.

Definition 0.13. Let T be a maximal torus in G and let $N = N(T)$ be the normalizer of T in G.
The groups $W = N(T)/T$ is called the Weyl group of G (we'll see later this is independent of the choice of T).
The normalizer $N(T)$ acts on T via conjugation:

$$N \times T \to T;\ (m, t) \mapsto mtm^{-1}$$

and this induces an action of W on T:

$$W \times T \to T;\ (nT, t) \mapsto ntn^{-1}$$

Theorem 0.11. *The group W is finite.*

Proof. Let N° be the connected component of the identity of N. We show that $N^\circ = T$. Then, as N is compact, the group $W = N/N^\circ$ is compact, and discrete, and so is finite.

■

View the action of N on T as a continuous map

$$\psi : N \to \mathrm{Aut}(T) \cong \mathrm{GL}_n(\mathbb{Z});\ n \mapsto \mathrm{Ad}_n|_{Lie(T)}$$

The image $\psi(N^\circ)$ in $\mathrm{GL}_n(\mathbb{Z})$ is connected and so $\psi(N^\circ) = I$, because $\mathrm{GL}_n(\mathbb{Z})$ is discrete. Hence N° acts trivially on T.

So if $\alpha : \mathbb{R} \to N^\circ$ is a one-parameter subgroup, then $\alpha(\mathbb{R}) \cdot T$ is a connected, abelian group containing T. So $\alpha(\mathbb{R}) \cdot T = T$, and therefore $\alpha(\mathbb{R}) \subseteq T$.
Since the groups $\alpha(\mathbb{R})$ cover an open neighborhood of the identity of N°, they generate N°, and so $N^\circ = T$.

Theorem 0.12. *Any two maximal torii in a compact, connected, Lie group are conjugate, and every element of G is contained in a maximal torus.*

Proof. Main Idea: Look for fixed points of the diffeomorphism

$$L_g : G/T \to G/T;\ xT \mapsto gxT, g \in G$$

Lemma 4. *A coset xT is a fixed point of L_g if and only if $g \in x^{-1}Tx$*

Proof. We have $gxT = xT$ iff $x^{-1}gxT = T$, and this happens if and only if $x^{-1}gx \in T$.

■

Lemma 5. *Suppose that $t \in T$ is a generator. Then xT is a fixed point of L_t if and only if $x \in N(T)$. The number of fixed points of L_t is finite.*

Proof. The coset xT is a fixed point of L_t if and only iff $x^{-1}tx \in T$ if and only if $x^{-1}Tx = T$ because t is a topological generator of T.
So the fixed points of L_t are the elements of the Weyl group W and are finite in number.
Here are some facts from algebraic topology about fixed points:
Suppose that M is a compact manifold of dimension n, and let $f : M \to M$ be a diffeomorphism. There is a Lefschetz number $L(f)$ attached to f with the following properties:

(a) If $f : M \to M$ is homotopic to f, then $L(f_1) = L(f)$

(b) If $L(f) \neq 0$, then f has at least one fixed point.

(c) If f has only isolated fixed points, p_i, say, then one can attach an index k_i to p_i, such that $L(f) = (-1)^n \sum_i k_i$.

(d) If p is an isolated fixed point of f, and if 1 is not an eigenvalue of

$$Df|_p : T_pM \to T_pM$$

then the index of p is equal to the sign of $\det(I - Df|_p)$

Remark: Note that since G is connected, any two left -translation maps L_g and L_h are homotopic and so have the same Lefschetz number.

Lemma 6. *Suppose that $n \in N(T)$ and $t \in G$ is a generator. Then*

$$\det(I - DL_t|_0) = \det(I - DL_t|_n)$$

Proof. Since $n \in N(T)$, we have $Tn = nT$, and so for any $x \in G$, we have $xTn = xnT$.
So the right-translation action

$$R_n : G/T \to G/T; xT \mapsto xTn = xnT$$

is well-defined, and R_n commutes with any left translation L_g. So the following diagram commutes:

$$\begin{array}{ccc} T_e(G/T) & \xrightarrow{DR_n|_t} & T_n(G/T) \\ \Big\downarrow{\scriptstyle DL_t|_e} & & \Big\downarrow{\scriptstyle DL_t|_n} \\ T_e(G/T) & \xrightarrow{DR_n|_e} & T_n(G/T) \end{array}$$

So $DL_t|_e$ and $DL_t|_n$ are conjugate maps, whence

$$\det(I - DL_t|_e) = \det(I - DL_t|_n)$$

Lecture 7, 4/23/24

We continue the proof. We want to show that $L_g : G/T \to G/T$ has a fixed point. This will be done if we show that L_g has nonzero Lefschetz number.
We look at L_t for t a generator of T. Then L_t has fixed points corresponding to elements of $N(T)/T = W$. We must show that L_t has nonzero Lefschetz number.
We must compute the index:

$$\det(I - DL_t|_n) = \det(I - DL_t|_e)$$

Now recall that we have the following decomposition of $Lie(G)$ as a direct sum of T-modules (with respect to adjoint action). We have

$$Lie(G) = Lie(T) \oplus \left(\bigoplus V_{\theta_i}\right)$$

where the action of any $t \in T$ on V_{θ_i} is given by the matrix

$$\begin{pmatrix} \cos 2\pi\theta_i(t) & \sin 2\pi\theta_i(t) \\ -\sin 2\pi\theta_i(t) & \cos 2\pi\theta_i(t) \end{pmatrix}$$

Where θ_i are non-trivial linear functionals on $Lie(T)$ taking integer values on the integral lattice of T.

Lemma 7. *For any element $t \in T$, we have*

$$\det(I - DL_t|_e) = \prod_i (2 - 2\cos 2\pi\theta_i(t))$$

So if t is a generator of T, then

$$\det(I - DL_t|_e) > 0$$

Proof. As $txT = txt^{-1}T$, we see that the left translation map

$$L_t : G/T \to G/T$$

lifts to the conjugation map

$$\mathrm{ad}_t : G \to G$$

Since $D(\mathrm{ad}_t)|_e = \mathrm{Ad}_t$, we have the following diagram of T-equivariant maps.

$$\begin{tikzcd}
0 \arrow[r] & Lie(T) \arrow[r] \arrow[d, "\mathrm{Ad}_t"] & Lie(G) \arrow[r] \arrow[d, "\mathrm{Ad}_t"] & T_e(G/T) \arrow[r] \arrow[d, "DL_t|_e"] & 0 \\
0 \arrow[r] & Lie(T) \arrow[r] & Lie(G) \arrow[r] & T_e(G/T) \arrow[r] & 0
\end{tikzcd}$$

We deduce from this that $T_e(G/T) = \bigoplus_i V_{\theta_i}$
On each component V_{θ_i}, the action of $I - \mathrm{Ad}_t$ is given by the matrix

$$\begin{pmatrix} 1 - \cos 2\pi\theta_i(t) & \sin \pi\theta_i(t) \\ -\sin \pi\theta_i(t) & 1 - \cos \pi\theta_i(t) \end{pmatrix}$$

whose determinant is $2 - 2\cos \pi\theta_i(t) \geq 0$. If this is an equality, then $\theta_i(t)$ is an integer, whence the same is true of $\theta_i(t^n) = n\theta_i(t)$ for all $n \in \mathbb{Z}$. This is impossible if t is a generator because θ_i is nontrivial.
Hence the Leftschetz number of L_t is non-zero.
We have shown the following:

Theorem 0.13. *If G is a compact, connected Lie group, and suppose that T is a maximal torus in G, and that $g \in G$. Then g is contained in a conjugate of T.*

Proof. The map $L_g : G/T \to G/T$ has a fixed point. ∎

Corollary 0.14. *Any two maximal tori in a compact, connected Lie group are conjugate.*

Proof. Suppose that T, T' are maximal tori and $t \in T$ is a generator. Then $x^{-1}tx \in T'$ for some $x \in G$, and conjugation by x maps the closure of $\langle t \rangle$ (i.e. T) into the closure of $\langle x^{-1}Tx \rangle$, i.e. T'. ∎

The exponential map of a compact, connected Lie group is surjective.

Proof. Every element of G is contained in a maximal torus, and the exponential map of a torus is surjective.

■

Definition 0.14. The dimension of a maximal torus of G is called the rank of G. Recall that the centralizer $Z(H)$ of a subgroup H of G is the subgroup

$$Z(H) = \{g \in G \mid gh = hg \forall h \in H\}$$

Theorem 0.15. *Let G be a compact, connected Lie group, and T a maximal torus of G. Then*

(i) $Z(T) = T$, so T is a maximal abelian subgroup of G

(ii) If S is a connected abelian subgroup of G, then $Z(S)$ is equal to the union of the maximal tori of G that contain S

(iii) $Z(G)$ is the intersection of all the maximal tori.

Proof.

(ii) The closure $\overline{S}$ of S is compact, abelian and connected and so is a torus; furthermore, $Z(S) = Z(\overline{S})$. We may assume that S is a torus.

Now suppose that $x \in Z(S)$, and let $B = \overline{\langle x, S \rangle}$. Then B is compact and abelian, so the connected component containing the identity B°, is equal to another torus. Therefore B/B° is a finite cyclic group, generated by xB°. Hence

$$B \simeq B^\circ \times (\text{finite cyclic thing})$$

and so B is the closure of a cyclic subgroup $\langle g \rangle$ of G. But g lies in some maximal torus T, therefore so does $S \cup \{x\}$, as required.

(i) Take $S = T$ as in (ii).

(iii) If $x \in Z(G)$, then plainly x lies in every maximal torus of G. Conversely, if x lies in every maximal torus of G, then x commutes with every element of G, because every element of G lies in a maximal torus.

■

Corollary 0.16. *The Weyl group W acts faithfully on the maximal torus T, i.e. the map*

$$W \to \text{Aut}(T)$$

is injective.

Lemma 8. *Two elements of the maximal torus T are conjugate in G if and only if they lie in the same orbit under the action of the Weyl group W.*

Proof. Suppose that $x, y \in T$, $g \in G$ with $gxg^{-1} = y$. Then conjugation by g induces a map:

$$c(g) : Z(x) \to Z(g);\ z \mapsto g^{-1}zg$$

and since $T \subseteq Z(x)$, we have $g^{-1}Tg \subseteq Z(G)$.
Hence T, $g^{-1}Tg$ are both maximal tori, contained in the connected component of the identity $Z(y)^\circ$ of $Z(y)$.
Thus for some $h \in Z(y)^\circ$, we have $(hg)^{-1}T(hg) = T$, and so $(hg)T$ is an element of W such that $wx = y$.
The upshot of this is that the inclusion $T \to G$ induces a bijection between the orbits of W on T and the conjugacy classes of G.

■

Proposition 5. *There is a dense open subset U of T such that if $t \in U$, then $|W|$ elements wtw^{-1} $(w \in W)$ are all distinct.*

Proof. For any $w \in W$, set

$$U_w = \{t \in T; t \neq wtw^{-1}\}$$

Then U_w is open (because it has closed complement).
If t is a generator of T, and $w \neq 1$, then $t \in U_w$ (otherwise if $n \in N(T)$ is a representative of w, then $n \in Z(t) = Z(T)$, and so $n \in T$, which contradicts $w \neq 1$).
Now Kronecker's Theorem implies that

$$U := \bigcap_{w \in U_w} U_w$$

is dense in T.
It follows from this that the map $\phi : G/T \times T \to G; (xT, t) \mapsto xtx^{-1}$ is a $|W|$-fold cover over a dense open set.

■

If f is any function in G, and dg, dt are normalized Haar measures on G and T, respectively, we have

$$\int_G f(g)\, dg = \frac{1}{|W|} \int_{G/T \times T} f(\phi(xT, t)) \cdot J(\phi(xT, t))\, dx \times dt$$

where $J(\phi(xT, t))$ is the Jacobian of ϕ.

Lecture 8, 4/30/24

Proposition 6. *If f is a class function on G and dg, dt are normalized Haar measures on G and T so that both have volume 1, then*

$$\int_G f(g)\, dg = \frac{1}{|W|} \int_T f(t) \det([\mathrm{Ad}_{t^{-1}} - I]|_{Lie(G/T)})\, dt$$

This is the Weyl integration formula

Proof. We have to compute the Jacobian $J\phi$ of ϕ. We do this by using a clever choice of charts on G/T and T.
Recall that we have a decomposition

$$Lie(G) = Lie(T) \oplus Lie(G/T)$$

(with respect to the adjoint action of T on $Lie(G)$).
Choose volume elements on $Lie(G)$ and $Lie(T)$ so that the Jacobians of the exponential maps are equal to 1.
Now parametrize a neighborhood of each $xT \in G/T$ by a chart based in a neighborhood of the origin in $Lie(G/T)$. This is given by

$$Lie(G/T) \ni x \mapsto xe^{x}T$$

Parametrise a neighborhood of each $t \in T$ by a chart based on a neighborhood of the origin in $Lie(T)$. This is given by:

$$Lie(T) \ni y \mapsto te^{y}$$

So a corresponding chart near $(xT, t) \in (G/T) \times T$ is given by

$$Lie(G/T) \times T \ni (x, y) \mapsto (xe^{x}T, te^{y}) \in G/T \times T$$

With respect to these coordinates, the map ϕ is given by

$$(x, y) \mapsto xe^{x}te^{y}e^{-x}x^{-1}$$

Now translation on the left is given by $t^{-1}x^{-1}$ and on the right by t. So we are reduced to computing the determinant at (eT, e) of a map which sends (eT, e) to e.
We are reduced to computing the Jacobian of the map

$$\begin{aligned}(x, y) \mapsto & t^{-1}e^{x}te^{y}e^{-x} \\ &= e^{\mathrm{Ad}_{t^{-1}}(x)}e^{y}e^{-x}\end{aligned}$$

Identify the tangent space at the origin of the real vector space $Lie(G/T) \times Lie(T)$ with itself.
Then the differential of the above map becomes

$$X + Y \mapsto (\mathrm{Ad}_{t^{-1}} - I)X + Y$$

The Jacobian is the determinant of the differential and so

$$J\phi = \det([\mathrm{Ad}_{t^{-1}} - I]|_{Lie(G/T)})$$

Now recall that

$$\int_G f(g)\,dg = \frac{1}{|W|}\int_{G/T\times T} f(\phi(xT,t))J(\phi(xT,t))\,dx\times dt$$

Since f is a class function, we have

$$f(\phi(xT,t))J(\phi(xT,t)) = f(t)\det([\mathrm{Ad}_{t^{-1}} - I]|_{Lie(G/T)})$$

is independent of x.
The result follows.

■

Example 0.7. Suppose that $G = U(n)$ and T is the diagonal torus. Set

$$t = \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix} \in T$$

Then if f is a class function in G, then

$$\int_G f(g)\,dg = \frac{1}{n!}\int_T f(diag(t_i))\prod_{i<j}|t_i - t_j|^2\,dt$$

To show this (given that the Weyl group of $U(n)$ is S_n), we need to check that

$$\det([\mathrm{Ad}_{t^{-1}} - I]|_{Lie(G/T)}) = \prod_{i<j}|t_i - t_j|^2$$

Recall that $\mathfrak{u}(n)_{\mathbb{C}} = \mathfrak{gl}(n,\mathbb{C}) = M_n(\mathbb{C})$
$Lie(G/T)_{\mathbb{C}}$ is spanned by the T-eigenspaces in $\mathfrak{u}(n)_{\mathbb{C}}$ corresponding to the non-trivial characters of T, and those are in turn spanned by the elementary matrices E_{ij} with $1 \leq i,j \leq n$ and $i < j$.
Compute that the eigenvalues of t on E_{ij} is $t_i t_j^{-1}$. This (after some computation) gives what we want.

Root Systems (redux)

Suppose that V is a real Euclidean space with inner product $\langle,\rangle$. Recall that for $\alpha \in V, \alpha \neq 0$, we have the reflection S_α in the hyperplane orthogonal to α given by

$$S_\alpha(x) = x - 2\frac{\langle x,\alpha\rangle}{\langle \alpha,\alpha\rangle}\alpha$$

So $S_\alpha(\alpha) = -\alpha$, and $S_\alpha(\beta) = \beta$ for all β with $\langle \beta,\alpha\rangle = 0$.
A finite set $R \subseteq V$ of nonzero vectors is a root system if:

1. For $\alpha \in R$, $S_\alpha(R) = R$

2. If $\alpha, \beta \in R$, then $2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$

3. If $\alpha, \lambda\alpha \in R$, $\lambda \in \mathbb{R}$, then $\lambda = \pm 1$

In practice, one is given a lattice Λ that spans V, and then $R \subseteq V$ sit inside Λ.
Suppose that G is a compact, connected Lie group, and that T is a maximal torus of G, of dimension r, say (so r is the rank of G).
We'll have

$$\Lambda = X^*(T) \simeq \mathbb{Z}^r$$

(the weight lattice and $V = R \otimes \Lambda$. How does this work?
If we identify $Lie(\mathbb{C}^\times)$ with $\mathbb{C}$, then $Lie(S^1)$ is identified with $i\mathbb{R}$.
Set $\mathfrak{g} = Lie(G), \mathfrak{t} = Lie(T)$
So a character $\lambda : T \to S^1$ of the torus gives

$$D\lambda|_{e^{i\pi}} \overset{\text{def}}{=} d\lambda : t \to i\mathbb{R}$$

$d\lambda(H) = \frac{d}{dt}\lambda(e^{tH})|_{t=0}$, $H \in \mathfrak{t}$
This induces

$$X^*(T) \to Hom(\mathfrak{t}, i\mathbb{R}) \simeq \mathrm{Hom}(i\mathfrak{t}, \mathbb{R})$$

If $\pi : G \to GL(V_1)$ is a complex representation, then $\pi|_T$ decomposes into a sum of one-dimensional characters of T. These are called the weights of π.
A root of G with respect to T is a non-zero weight of the adjoint representation.

Proposition 7. *Any maximal abelian subgroup $\mathfrak{h}$ of $\mathfrak{g}$ is the Lie algebra of a conjugate of T.*

Proof. Since $\mathfrak{h}$ is abelian, the exponential map on $\mathfrak{h}$ is a homomorphism, and so $\exp(\mathfrak{h})$ is a connected commutative group. It's closure H is a Lie subgroup of G, which is closed, connected, and abelian, thus is a torus. So H is contained in a maximal torus H_1, say, of G. We have $\mathfrak{h} \subseteq Lie(H_1)$, and by maximality these must be equal, and $H = H_1$. So by Cartan's theorem, H is conjugate to T. ∎