Lecture 1

We will be using Hatcher's Algebraic Topology. The topology sequence is usually something like

ATopological Spaces
BCell Complexes
CManifolds

Theorem 0.1. *(BIG Theorem)*
Given a "reasonably nice" space, there is a bijection between connected covers of a space and subgroups of the fundamental group.

Categories:

Algebraic structures that are much flabbier than a group. They consist of

- *A collection of arrows*
- *A partial binary operation on these arrows*
- *Objects, which arrows go between*

We also want a composition law. That is, for objects and arrows

$$A \xrightarrow{f} B \xrightarrow{g} C$$

there is an arrow $A \xrightarrow{g\circ f} C$ *. We want this composition to be associative, that is* $(f \circ g) \circ h = f \circ (g \circ h)$*, and we want objects to have identity arrows.*
Not all functions have inverses. Using sets and functions as an example, we have described the category Set.
Here are some more examples of categories:

Example 0.1.
- Groups and group homomorphisms (Grp)
- Topological spaces and continuous functions (Top)
- etc.

We can make the following new category.

Definition 0.1. We denote by Top^* the category of based topological spaces, whose objects are pairs (X, x_0), where X is a topological space and $x_0 \in X$, and whose morphisms are continuous functions $f : (X, x_0) \to (Y, y_0)$ such that $f(x_0) = y_0$.

Goal:

Our goal is to get a functor from Top to Grp. The fundamental group functor π_1 will go from Top^* to Grp.

Lecture 2

Topology review:

Definition 0.2. A topological space is a set X along with a collection of subsets of X called "open sets," such that $X, \varnothing$ are open, and the arbitrary union and finite intersection of open sets are open.
Notice the following diagram commutes using the product topology

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[d, dotted, "\exists!"] \arrow[dr, "g"] & \\ X & X \times Y \arrow[l, "P_X"'] \arrow[r, "P_Y"] & Y \end{tikzcd}$$

And in general

$$\begin{tikzcd} Z \arrow[d, dotted, "\exists!"'] \arrow[dr, "f_\alpha"] & \\ \prod_{\alpha \in A} X_\alpha \arrow[r, "P_\alpha"] & X_\alpha \end{tikzcd}$$

Maps are continuous; functions are not.

Lemma 1. *(Gluing lemma)*
Suppose $f : A \to Y$, $g : B \to Y$ are continuous, and $f(x) = g(x)$ for all $x \in A \cap B$. Then $f \cup g : A \cup B \to Y$ is continuous. This only holds as long as $A, B \subseteq X$ are closed.

Same Shape, Same Map

(maps up to wriggling things around a bit)

Definition 0.3. Two maps are homotopic if there exists a parametrized map $f_t : X \to Y$ such that $f_0 = f, f_1 = g$ for $f, g : X \to Y$. Equivalently, and more precisely, if there exists a map $F : X \times [0,1] \to Y$ such that $F(x,0) = f(x), F(x,1) = g(x)$ for all $x \in X$.

X, Y topological spaces are said to have the same shape if there exist maps $f : X \to Y, g : Y \to X$ such that $g \circ f \simeq \mathrm{Id}_X$ and $f \circ g \simeq \mathrm{Id}_Y$. We may say that X, Y have the same homotopy type

Definition 0.4. A deformation retraction from $X \to A \subseteq X$ is a map from $X \times I \to X$ such that, for all $x \in A$, and $s, t \in I$,

$$\begin{aligned} f_0(x) &= x && \forall x \in X \\ f_1(x) &\in A && \forall x \in X \\ f_t(x) &= f_s(x) && \forall x \in A \end{aligned}$$

Lecture 3, 1/13/23

Definition 0.5. Let X be a topological space. A retraction is a map $r : X \to X$ such that $r \circ r = r$. That is, $r(r(x)) = r(x)$ for any $x \in X$.
Let $A = r(X)$. Then $r|_A = \mathrm{Id}_A$.

Definition 0.6. Let $F : X \times I \to Y$. We say $f_0 \simeq f_1 \,\mathrm{rel}\, A \subseteq X$ are homotopic relative to A if, for any $x \in A$, $f_t(x)$ is independent of t. That is, for any $s, t \in I$, $f_s(x) = f_t(x)$ for any $x \in A$.
For any map $f : X \to Y$, there exists a space $Z \simeq Y$ via $g : Y \to Z$ such that $g \circ f : X \to Z$ is injective. That is, in the following diagram, we have a bijection between homotopy classes of maps f and homotopy classes of maps $g \circ f$, and we can do this in a way that rigs $g \circ f$ to be injective.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g \circ f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

Definition 0.7. Given a map $f : X \to Y$ we can construct the Mapping Cylinder M_f by setting $M_f = X \times I \coprod Y / \sim$, where $(x, 0) \sim f(x)$.
The visual intuition should be taking the disjoint union of X and Y, and tieing a string between x and $f(x)$ for each point.

Claim. *$X \hookrightarrow M_f, Y \hookrightarrow M_f$, and the latter is in fact a homotopy equivalence. Further, the injection $X \hookrightarrow M_f$ is homotopic to $f(X) \hookrightarrow M_f$.*

Proof. You can construct a homotopy which "squishes" the cylinder down to $f(X)$. ■

Definition 0.8. A space X is contractible if it has the homotopy type of a point. A map is null-homotopic if it is homotopic to a constant map. So X is contractible if the identity is null-homotopic.
Now he's drawing an example. The example is Bing's House with 2 rooms, which I will not reproduce here. But the point is that it's contractible, but not obviously so.

Cell Complexes

Cell complexes are topological spaces which are built up inductively out of closed balls in Euclidean space. We write $\mathbb{D}^n := \{\vec{x} \in \mathbb{R}^n \mid \|\vec{x}\| \leq 1\}$, and $e^n := \{\vec{x} \in \mathbb{R}^n \mid \|\vec{x}\| < 1\}$. We can see that $e^n = \operatorname{int} \mathbb{D}^n$, and $\partial \mathbb{D}^n = \mathbb{S}^{n-1}$.

Base step

Start with some collection of points X^0, the 0-skeleton, with the discrete topology.

Inductive step

Let X^{n-1} be the $n-1$ skeleton, which has already been build and defined. Select some collection of n-dimensional balls $\{\mathbb{D}^n\}_{\alpha\in A}$, and some continuous "attaching map" $\varphi_\alpha : \partial\mathbb{D}^n_\alpha \to X^{n-1}$. Then

$$(X^n = X^{n-1} \coprod_{\alpha\in A} \mathbb{D}^n)/(x \sim \varphi_\alpha(x) \forall x \in \partial\mathbb{D}^n)$$

Lecture 4, 1/18/23

A space X is a cell complex if it has been constructed using the above inductive procedure. If $n = \infty$, we use the weak topology, in which the open sets are the sets which are open when intersected with each X^n.
For every $\mathbb{D}^n_\alpha$ and corresponding "attaching map $\varphi_\alpha : \partial\mathbb{D}^n_\alpha \to X^{n-1}$, there is a subset of X^n homeomorphic to $\operatorname{int}(\mathbb{D}^n_\alpha)$, via the composition

$$\operatorname{int}(\mathbb{D}^n_\alpha) \hookrightarrow \mathbb{D}^n_\alpha \hookrightarrow X^{n-1} \coprod_\alpha \mathbb{D}^n_\alpha \to X^n$$

which we call $\Phi_\alpha : \mathbb{D}^n_\alpha \to X^{n-1}$. So the attaching map $\phi_\alpha : \partial\mathbb{D}^n_\alpha \to X^{n-1}$ extends to a "characteristic map" Φ_α.
We will now see many examples of things.

Example 0.2. If you stop after constructing X^1, it's a graph.

Example 0.3. $\mathbb{S}^n$ has a cell structure with one e_0 and one e_n.

Example 0.4. Consider $\mathbb{RP}^2$. This can be expressed as $(\mathbb{R}^3 \setminus \{0\})/(\vec{x} \sim \lambda\vec{x}, \lambda \neq 0)$. We can replace 2 with any n and get $\mathbb{RP}^n$. Indeed, we can replace $\mathbb{R}$ with $\mathbb{C}$, $\mathbb{H}$, or indeed any field.

<u>Homogenous coordinates</u>

For $(x, y, z) \neq (0, 0, 0)$, we have $[x, y, z] \overset{\text{def}}{=} \{(\lambda x, \lambda y, \lambda z) \mid \lambda \neq 0\}$. For example, $[1, 2, 3] = [2, 4, 6]$.