Lecture 1

We will be using Hatcher's Algebraic Topology. The topology sequence is usually something like

$$A\text{Topological Spaces}$$
$$B\text{Cell Complexes}$$
$$C\text{Manifolds}$$

Theorem 0.1. *(BIG Theorem)*
Given a "reasonably nice" space, there is a bijection between connected covers of a space and subgroups of the fundamental group.

Categories:

Algebraic structures that are much flabbier than a group. They consist of

- *A collection of arrows*
- *A partial binary operation on these arrows*
- *Objects, which arrows go between*

We also want a composition law. That is, for objects and arrows

$$A \xrightarrow{f} B \xrightarrow{g} C$$

there is an arrow $A \xrightarrow{g\circ f} C$ *. We want this composition to be associative, that is* $(f \circ g) \circ h = f \circ (g \circ h)$*, and we want objects to have identity arrows.*
Not all functions have inverses. Using sets and functions as an example, we have described the category Set.
Here are some more examples of categories:

Example 0.1.
- Groups and group homomorphisms (Grp)
- Topological spaces and continuous functions (Top)
- etc.

We can make the following new category.

Definition 0.1. We denote by Top^* the category of based topological spaces, whose objects are pairs (X, x_0), where X is a topological space and $x_0 \in X$, and whose morphisms are continuous functions $f : (X, x_0) \to (Y, y_0)$ such that $f(x_0) = y_0$.

Goal:

Our goal is to get a functor from Top to Grp. The fundamental group functor π_1 will go from Top^* to Grp.

Lecture 2

Topology review:

Definition 0.2. A topological space is a set X along with a collection of subsets of X called "open sets," such that $X, \varnothing$ are open, and the arbitrary union and finite intersection of open sets are open.
Notice the following diagram commutes using the product topology

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[d, dotted, "\exists!"] \arrow[dr, "g"] & \\ X & X \times Y \arrow[l, "P_X"'] \arrow[r, "P_Y"] & Y \end{tikzcd}$$

And in general

$$\begin{tikzcd} Z \arrow[d, dotted, "\exists!"'] \arrow[dr, "f_\alpha"] & \\ \prod_{\alpha \in A} X_\alpha \arrow[r, "P_\alpha"] & X_\alpha \end{tikzcd}$$

Maps are continuous; functions are not.

Lemma 1. *(Gluing lemma)*
Suppose $f : A \to Y$, $g : B \to Y$ are continuous, and $f(x) = g(x)$ for all $x \in A \cap B$. Then $f \cup g : A \cup B \to Y$ is continuous. This only holds as long as $A, B \subseteq X$ are closed.

Same Shape, Same Map

(maps up to wriggling things around a bit)

Definition 0.3. Two maps are homotopic if there exists a parametrized map $f_t : X \to Y$ such that $f_0 = f, f_1 = g$ for $f, g : X \to Y$. Equivalently, and more precisely, if there exists a map $F : X \times [0,1] \to Y$ such that $F(x,0) = f(x), F(x,1) = g(x)$ for all $x \in X$.

X, Y topological spaces are said to have the same shape if there exist maps $f : X \to Y, g : Y \to X$ such that $g \circ f \simeq \mathrm{Id}_X$ and $f \circ g \simeq \mathrm{Id}_Y$. We may say that X, Y have the same homotopy type

Definition 0.4. A deformation retraction from $X \to A \subseteq X$ is a map from $X \times I \to X$ such that, for all $x \in A$, and $s, t \in I$,

$$\begin{aligned} f_0(x) &= x && \forall x \in X \\ f_1(x) &\in A && \forall x \in X \\ f_t(x) &= f_s(x) && \forall x \in A \end{aligned}$$

Lecture 3, 1/13/23

Definition 0.5. Let X be a topological space. A retraction is a map $r : X \to X$ such that $r \circ r = r$. That is, $r(r(x)) = r(x)$ for any $x \in X$.
Let $A = r(X)$. Then $r|_A = \mathrm{Id}_A$.

Definition 0.6. Let $F : X \times I \to Y$. We say $f_0 \simeq f_1 \,\mathrm{rel}\, A \subseteq X$ are homotopic relative to A if, for any $x \in A$, $f_t(x)$ is independent of t. That is, for any $s, t \in I$, $f_s(x) = f_t(x)$ for any $x \in A$.
For any map $f : X \to Y$, there exists a space $Z \simeq Y$ via $g : Y \to Z$ such that $g \circ f : X \to Z$ is injective. That is, in the following diagram, we have a bijection between homotopy classes of maps f and homotopy classes of maps $g \circ f$, and we can do this in a way that rigs $g \circ f$ to be injective.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g \circ f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

Definition 0.7. Given a map $f : X \to Y$ we can construct the Mapping Cylinder M_f by setting $M_f = X \times I \coprod Y / \sim$, where $(x, 0) \sim f(x)$.
The visual intuition should be taking the disjoint union of X and Y, and tieing a string between x and $f(x)$ for each point.

Claim. *$X \hookrightarrow M_f, Y \hookrightarrow M_f$, and the latter is in fact a homotopy equivalence. Further, the injection $X \hookrightarrow M_f$ is homotopic to $f(X) \hookrightarrow M_f$.*

Proof. You can construct a homotopy which "squishes" the cylinder down to $f(X)$. ∎

Definition 0.8. A space X is contractible if it has the homotopy type of a point. A map is null-homotopic if it is homotopic to a constant map. So X is contractible if the identity is null-homotopic.
Now he's drawing an example. The example is Bing's House with 2 rooms, which I will not reproduce here. But the point is that it's contractible, but not obviously so.

Cell Complexes

Cell complexes are topological spaces which are built up inductively out of closed balls in Euclidean space. We write $\mathbb{D}^n := \{\vec{x} \in \mathbb{R}^n \mid \|\vec{x}\| \leq 1\}$, and $e^n := \{\vec{x} \in \mathbb{R}^n \mid \|\vec{x}\| < 1\}$. We can see that $e^n = \operatorname{int} \mathbb{D}^n$, and $\partial \mathbb{D}^n = \mathbb{S}^{n-1}$.

Base step

Start with some collection of points X^0, the 0-skeleton, with the discrete topology.

Inductive step

Let X^{n-1} be the $n-1$ skeleton, which has already been build and defined. Select some collection of n-dimensional balls $\{\mathbb{D}^n\}_{\alpha \in A}$, and some continuous "attaching map" $\varphi_\alpha : \partial \mathbb{D}^n_\alpha \to X^{n-1}$. Then

$$(X^n = X^{n-1} \coprod_{\alpha \in A} \mathbb{D}^n)/(x \sim \varphi_\alpha(x) \forall x \in \partial \mathbb{D}^n)$$

Lecture 4, 1/18/23

A space X is a cell complex if it has been constructed using the above inductive procedure. If $n = \infty$, we use the weak topology, in which the open sets are the sets which are open when intersected with each X^n.
For every $\mathbb{D}^n_\alpha$ and corresponding "attaching map $\varphi_\alpha : \partial \mathbb{D}^n_\alpha \to X^{n-1}$, there is a subset of X^n homeomorphic to $\operatorname{int}(\mathbb{D}^n_\alpha)$, via the composition

$$\operatorname{int}(\mathbb{D}^n_\alpha) \hookrightarrow \mathbb{D}^n_\alpha \hookrightarrow X^{n-1} \coprod_\alpha \mathbb{D}^n_\alpha \to X^n$$

which we call $\Phi_\alpha : \mathbb{D}^n_\alpha \to X^{n-1}$. So the attaching map $\phi_\alpha : \partial \mathbb{D}^n_\alpha \to X^{n-1}$ extends to a "characteristic map" Φ_α.
We will now see many examples of things.

Example 0.2. If you stop after constructing X^1, it's a graph.

Example 0.3. $\mathbb{S}^n$ has a cell structure with one e_0 and one e_n.

Example 0.4. Consider $\mathbb{RP}^2$. This can be expressed as $(\mathbb{R}^3 \setminus \{0\})/(\vec{x} \sim \lambda\vec{x}, \lambda \neq 0)$. We can replace 2 with any n and get $\mathbb{RP}^n$. Indeed, we can replace $\mathbb{R}$ with $\mathbb{C}$, $\mathbb{H}$, or indeed any field.
Homogenous coordinates

For $(x, y, z) \neq (0, 0, 0)$, we have $[x, y, z] \stackrel{\text{def}}{=} \{(\lambda x, \lambda y, \lambda z) \mid \lambda \neq 0\}$. For example, $[1, 2, 3] = [2, 4, 6]$.

Lecture 5, 1/20/23

Definition 0.9. A subcomplex of a complex X is a closed disjoint union of open cells $e^{n_i}_{\alpha_i}$ in X such that they form a cell complex on their own.
Remark. We keep talking about "CW Complexes." The C is for "closure finite," and the W is for "weak topology."
Recall: $\mathbb{RP}^n \stackrel{\text{def}}{=} \mathbb{R}^n/(x \sim \lambda x, \lambda \neq 0)$. $\mathbb{CP}^n$ can be defined similarly.
We can write $\mathbb{RP}^n = e^0 \cup e^1 \cup e^2 \cup \cdots \cup e^n$, and $\mathbb{CP}^n = e^0 \cup e^2 \cup e^3 \cup \cdots \cup e^{2n}$. We can do the same thing with the quaternions.
Next time, we will cover operations on complexes.

Lecture 6, 1/23/23

This lecture, we will cover operations on cell complexes, and two big theorems.

Operations on Cell Complexes

1. If X, Y have cell structures, then $X \times Y$ has a natural cell structure.

2. If (X, A) is a CW-pair, then X/A has a natural cell structure (X/A denotes identifying all points in A together).

$$\begin{tikzcd}
\mathbb{D}'_\alpha \supseteq \partial\mathbb{D}'_\alpha \arrow[d, "\phi_\alpha"] \arrow[dr] & \\
X^0 \arrow[r, "q"'] & (X/A)^0
\end{tikzcd}$$

3. Cones and Suspensions. The cone on X, CX, is defined as

$$CX = (X \times I)/(X \times \{0\})$$

Note that CX is contractible for any X. The suspension on X, SX, is defined as

$$SX = CX/(X \times \{1\})$$

If $f : X \to Y$ is a map, there exists a natural map $Sf : SX \to SY$. Indeed, if $f : X \to Y$, then $f \times \text{Id} : X \times I \to Y \times I$, and so we can factor $f \times \text{Id}$ through the quotient map:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, hook] \arrow[dr, dotted] & Y \arrow[d, hook] \\
X \times I \arrow[r, "f\times \text{Id}"] \arrow[d, "q"] \arrow[dr, dotted] & Y \times I \arrow[d] \\
SX \arrow[r, "\exists ! Sf"'] & SY
\end{tikzcd}$$

Note $S(\mathbb{S}^n) = \mathbb{S}^{n+1}$.

4. Joins. If X, Y are cell structures, then we define their join $X \star Y$ as

$$X \star Y = \frac{X \times Y \times I}{(x, y_1, 0) \sim (x, y_2, 0), (x_1, y, 1) \sim (x_2, y, 1)}$$

This is a useful construction for simplices.

5. Wedge product. If X, Y are cell structures, with distinguished points $x_0 \in X, y_0 \in Y$, then we define their wedge product $X \wedge Y$ as

$$X \wedge Y = \frac{X \coprod Y}{x_0 \sim y_0}$$

This is just gluing X and Y together at a distinguished point. This raises an obvious question: does the wedge product depend on the points x_0, y_0? Yes, but not if they are (connected) cell complexes!

If x_0 is a 0-cell of X, and y_0 a 0-cell of Y, then $X \vee Y$ has a natural cell structure AND $X \vee Y$ is a subcomplex of $X \times Y$.

6. Smash product. If X, Y are spaces with distinguished points x_0, y_0, then the smash product is defined as

$$X \wedge Y = \frac{X \times Y}{X \vee Y}$$

For example, the smash product $S^1 \wedge S^1$ is a Torus quotiented out by the longitudinal and meridian circles. By arguing from some cell nonsense, we can say this is S^2.

Here are two big theorems.

Theorem 0.2. *If (X, A) is a CW-pair, and A is contractible, then X/A is homotopy equivalent to X, with the quotient mapping itself providing a homotopy equivalence.*

Theorem 0.3. *Suppose (X_1, A) is a CW-pair, and $f, g : A \to Y$ are maps. If $f \simeq g$, and everything in sight is a cell complex, then*

$$X_1 \coprod_f Y \simeq X_1 \coprod_g Y$$

That is, if f, g are used as attaching maps, then the resulting spaces will be homotopy equivalent.

Lecture 7, 1/25/23

Definition 0.10. Let X be a cell complex. If we let f_i be the number of i-dimensional cells in the cell structure, then we define

$$\chi(X) = f_0 - f_1 + f_2 - f_3 + \cdots$$

The more general definition is the alternating sum of the Betti numbers of X, where the ith Betti number is $\dim H^i(X)$.

Definition 0.11. Let X be a topological space, and let $A \subseteq X$ be a subspace. We say that (X, A) has the homotopy extension property (HEP) if for all topological spaces Y and for all maps $f : X \times \{0\} \cup A \times I \to Y$, there exists an extension of f, $\overline{f} : X \times I \to Y$, such that $\overline{f}|_{X \times \{0\} \cup A \times I} = f$.
Slogan: "A homotopy on the subspace can be extended to a homotopy on the entire space."

Lecture 8, 1/27/23

Proposition 1. *(X, A) has the homotopy extension property if and only if $X \times I$ retracts to $X \times \{0\} \cup A \times I$.*

Proof. ■

Example 0.5. Does $(\mathbb{D}^2, \partial\mathbb{D}^2)$ have the property? Does $\mathbb{D}^2 \times I$ retract onto $\mathbb{D}^2 \times \{0\} \cup \partial\mathbb{D}^2 \times I$? Yes. This is easy to see by drawing a picture.
Here is a non-example. Let $X = I$, and let $A = \{\frac{1}{n}\}_{n\in\mathbb{N}} \cup \{0\}$. $X \times I$ is the square, and $X \times \{0\}$ is the bottom of the square, so $X \times \{0\} \cup A \times I$ is the comb space. The square doesn't retract to this.

Proposition 2. *If (X, A) is a CW pair, then (X, A) has the homotopy extension property.*

Proof. Later ■

Theorem 0.4. *If (X, A) has the homotopy extension property, and A is contractible, then the quotient map $X \to X/A$ is a homotopy equivalence.*

Proof. Consider identity map $\mathrm{Id} : A \to A$. We have a homotopy $F : A \times I \to A$ which is a witness to A being contractible. That is, $f_0 = \mathrm{Id}_A$, $f_1 \equiv \{p\}$ for some point $p \in A$.
Then there is an extension to a homotopy $H : X \times I \to X$. We have the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "f_t"] \arrow[d, "q"] & X \arrow[d, "q"] \\ X/A \arrow[r, "\overline{f_t}"] & X/A \end{tikzcd}$$

Because all of A goes to a point for $t = 1$, then by the universal property of quotients, there is a map g making the diagram commute:

$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, "q"] & X \arrow[d, "q"] \\ X/A \arrow[r, "\overline{f_1}"'] \arrow[ur, "g"] & X/A \end{tikzcd}$$

So qg is homotopic to the identity map, and gq is homotopic to the identity map. This completes the proof? ■

Lecture 9, 1/30/23

Definition 0.12. We say that (X, A) and (Y, A) are homotopy equivalent relative to A if there exist maps $f : X \to Y, g : Y \to X$ such that $f|_A = \mathrm{Id}_A, g|_A = \mathrm{Id}_A$ and $g \circ f \simeq \mathrm{Id}_X$ relative to A, and $f \circ g \simeq \mathrm{Id}_Y$ relative to A.

Theorem 0.5. *If (X, A) is a CW Pair, and $f, g : A \to X_0$ are homotopic maps, then $X_0 \coprod_f X_1 \simeq X_0 \coprod_g X_1$ relative to X_0.*

Proof. Bunch of pictures I can't write down. ■

Proposition 3. *If $(X, A), (Y, A)$ both have the homotopy extension property, and $f : X \to Y$ is a homotopy equivalence such that $f|_A = \mathrm{Id}_A$, then f is a homotopy equivalence relative to A.*

Proof. ■

Corollary 0.6. *If (X, A) has the homotopy extension property and $A \hookrightarrow X$ is a homotopy equivalence, then X deformation retracts to A.*

Proof. ■

Corollary 0.7. *A map $f : X \to Y$ is a homotopy equivalence if and only if X is a deformation retraction of M_f.*

Proof. ■

Lecture 10, 2/3/23

Definition 0.13. Given any path $f : I \to X$, we write $[f]$ for the set of paths $g : I \to X$ such that $g \simeq f$ relative to ∂I. If we don't fix endpoints, each $[f]$ would be a path-component. Sometimes we use π^0 to denote the set of path components.
Let $f, g : I \to \mathbb{R}^n, f \simeq g$ by $h_t(u) = tg(u) + (1 - t)f(u)$. Then $h_0 = f, h)1 = g$. h is called the "straight line homotopy."
In fact, we could change I to <u>any</u> topological space, and $f \simeq g$ <u>still</u>
We could also change $\mathbb{R}^n$ to any $U \subseteq \mathbb{R}^n, U$ convex, and $f \simeq g$ <u>still</u>.
We could change X to U, any metric space iwth unique shortest path which vary continuosly as the endpoints vary.
Concatenation: If $f(1) = g(0)$, then define $f \star g : I \to X$ by

$$(f \star g)(t) = \begin{cases} f(2t) & t \in [0, \frac{1}{2}] \\ g(2t - 1) & t \in [\frac{1}{2}, 1] \end{cases}$$

Definition 0.14. Assume $f(1) = g(0)$. Then $[f] \star [g] = [f \star g]$ is well defined by handwaving.
Constants: The constant path c_x is the path which is constantly x. Note $[c_{f(0)}] \star [f] = [f], [f] \star [c_{f(1)}] = [f]$.
Inverses: Define $\overline{f}(u) = f(1 - u)$. Note $[f][\overline{f}] = [c_x]$.
Associativity: $(f \star g) \star h \simeq f \star (g \star h)$.

Definition 0.15. A category where every $f : A \to B$ has an inverse (i.e. an arrow $f^{-1} : B \to A$ such that $ff^{-1} = \mathrm{Id}_B = f^{-1}f)$ is called a <u>groupoid</u>.
π_0 is a functor, (objects→objects, arrow→arrows, compositions→compositions, identities→identities)
$(X, x_0) \mapsto \pi_1(X, x_0) = \{[f] \mid f : I \to X, f(0) = f(1) = x_0\}$.

$(X, x_0) \xrightarrow{\pi_1} \pi_1(X, x_0)$.

$$\begin{tikzcd}
(X, x_0) \arrow[r, "\pi_1"] \arrow[d, "h"] & \pi_1(X, x_0) & f \arrow[d] \\
(Y, y_0) \arrow[r, "\pi_1"] & \pi_1(Y, y_0) & h \circ f
\end{tikzcd}$$

Lecture 11, 2/6/23

just homework review

Lecture 12, 2/8/23

Review for the midterm
Types of questions:

1. State Definitions (particularly important and complicated definitions)

2. Carefully state important theorems

3. Give key counterexamples

4. Prove easy propositions

5. Do simple constructions/modifications (applying our theorems)

HEP: For any $f : X \times \{0\} \cup A \times I$, there exists an extesnsion $\overline{f} : X \times I \to Y$. This is equivalent to $X \times I$ def retracting to $X \times \{0\} \cup A \times I$

$$\begin{tikzcd}
X \times \{0\} \cup A \times I \arrow[r, "f"] \arrow[d, shift left, "\iota"] & Y \\
X \times I \arrow[u, shift left, "r"] &
\end{tikzcd}$$

Questions from previous exams;
1. Carefully define the notion of a cell complex.
2. Let $f : S^1 \to S^1$ be continuous. Define the mapping cylinder M_f and describe an explicit cell structure on it.

3. What does it mean to say that computing the fundamental gorup is a functor from $\mathsf{Top}*$ to Grp
4. In each case, find a simpler space with the same homotopy type, briefly explain reasons and use pictures. a. suspension of disjoint union of 3 circles as a wedge product.
b. View S^2 as a subspace of S^3 and describe the quotient S^3/S^2.
c. Remove both the Z axis and the unit circle in the xy plane and describe a 2-dimensional object with the same homotopy type
5. Let X be a topological space, prove that $f : S^1 \to X$ extends to a map $F : D^2 \to X$ if and only if f is nullhomotopic.
6. Define the homotopy extension property and then prove that a pair (X, A) has the property if and only if there is a retraction $X \times \{0\} \cup A \times I$.
7. Give examples of each of the following: retract of a cell complex onto a cell complex which does not extend to a def retraction
Give an example of a contractible space that does not deformation retract to a point.

Lecture 13, 2/13/23

Proposition 4. *Let $h : I \to X$ be a path from x_1 to x_0. The map $\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$ defined by $\beta_h([f]) = [hf\overline{h}]$ is an isomorphism.*

Proof. ∎

Remark. As long as X is path-connected, $\pi_1(X, x_0)$ is well-defined, up to isomorphism, independent of x_0, but this isomorhpism is not canonical.
If $x_1 = x_0$, then h is a loop, and $[h] \in \pi_1(X, x_0)$.
So $[f] \mapsto [hf\overline{h}] = [h][f][h]^{-1}$. So β_h is an inner automorphism of $G = \pi_1(X, x_0)$.

Definition 0.16. A map $p : \hat{X} \to X$ is called a <u>covering map</u> if there exists an open cover $\{U_\alpha\}$ of X such that for all α, $p^{-1}(U_\alpha)$ can be decomposed into a disjoint union of subsets which are each homeomorphic to U_α and sent homeomorphically to U_α by p.
Fact: If $p : \hat{X} \to X$ is a covering map and $p(\hat{x_0}) = x_0$, then the induced map $p_\star : \pi_1(\hat{X}, \hat{x_0}) \to \pi_1(X, x_0)$ is injective.

Lecture 14, 2/15/23

Definition 0.17. If X is a path connected space, then X is said to be <u>simply connected</u> if $\pi_1(X, x_0) = 0$.

Proposition 5. *X is simply connected if and only if the homotopy class of a loop is determined entirely by its endpoints.*
That is, if $\gamma, \sigma : [0, 1] \to X$ are paths, and $\gamma(0) = \sigma(0), \gamma(1) = \sigma(1)$, then $\gamma \simeq \sigma$.

Proof. ■

Let f, g be paths as in the statement of the problem, and let $h : [0,1] \to X$ be a path with $h(0) = f(1) = g(1), h(1) = f(0) = g(0)$. Because $\pi_1(X, x_0)$ is trivial, we know that $f \star h$ is contractible relative to x_0. The same is true of $h \star g$. So

$$\begin{aligned}
[f] &= [f \star e] \\
&= [f \star (h \star g)] \\
&= [(f \star h) \star g] \\
&= [g]
\end{aligned}$$

■

Definition 0.18. Given $p : Y \to X$ and $f : Z \to X$, a map $\tilde{f} : Z \to Y$ such that $p \circ \tilde{f} = f$, then $\tilde{f}$ is called a lift of f.

$$\begin{tikzcd}
Z \arrow[d, dotted, "\tilde{f}"'] \arrow[dr, "f"] & \\
Y \arrow[r, "p"] & X
\end{tikzcd}$$

Theorem 0.8. *Let $\Phi : \mathbb{Z} \to \pi_1(S^1, (1,0))$ be given by $n \mapsto [\omega_n]$, where $\omega_n : I \to \mathbb{S}^1$ is defined by $s \mapsto (\cos(2\pi ns), \sin(2\pi ns))$ (this is the path which wraps around n times) is an isomorphism.*

Proof. One can think of the projection map $\rho : \mathbb{R} \to \mathbb{S}^1$ given by $\rho(s) = e^{2\pi is}$ as a projection from a "spiral" above the circle. So we have a map

$$\begin{tikzcd}
\{0\} \arrow[r] \arrow[d] & \mathbb{R} \arrow[d, "\rho"] \\
{[0,1]} \arrow[r, "f"] \arrow[ur, dotted, "\exists! \tilde{f}"] & \mathbb{S}^1
\end{tikzcd}$$

We need two facts:

(a) For all $f : I \to \mathbb{S}^1$ starting at $x_0 \in \mathbb{S}^1$, and for each choice of $\tilde{x_0} \in p^{-1}(x_0)$, there exists a unique path $\tilde{f} : I \to \mathbb{R}$ starting at $\tilde{x_0}$ which lifts f.

(b) For all $\underbrace{f_t : I \to \mathbb{S}^1}_{F : I \times I \to \mathbb{S}^1}$ starting at $x_0 \in \mathbb{S}^1$, and for each choice of $\tilde{x_0} \in p^{-1}(x_0)$, there exists a unique lifted homotopy of paths $\tilde{f}_t : I \to \mathbb{R}$ which is a lift of f_t.

(c) For every $F : Y \times I \to \mathbb{S}^1$ and every $\tilde{F}|_{Y\times\{0\}} : Y \times \{0\} \to \mathbb{R}$ which lifts $F|_{Y\times\{0\}} : Y \times \{0\} \to \mathbb{S}^1$, there exists a unique $\tilde{F} : Y \times I \to \mathbb{R}$ lifting F.

$$\begin{tikzcd}
Y \times \{0\} \arrow[r, "\tilde{F}|_{Y\times\{0\}}"] \arrow[d, hook] & \mathbb{R} \arrow[d, "p"] \\
Y \times I \arrow[r, "F"'] \arrow[ur, dotted, "\exists!\tilde{F}"] & \mathbb{S}^1
\end{tikzcd}$$

The first guarantees surjectivity of Φ; the second injectivity.
Indeed, let $[f]$ be a homotopy class of maps based at x_0. Then, once we choose an element of $p^{-1}(x_0)$, this will lift to a path $[\tilde{f}]$ in $\mathbb{R}$ which is homotopic to one of the ω_n's. The second will then allow us to use that homotopy to extend to a homotopy between f and $p(\omega_n)$.
Note: There exists a lift of ω_n starting at $(1,0,0)$ in the spiral, $\tilde{\omega_n} : I \to \mathbb{R} \subseteq \mathbb{R}^3, s \mapsto (\cos(2\pi ns), \sin(2\pi ns), ns)$.
We will prove (c), which will prove the first two.
There exists an open cover $\{U_\alpha\}$ of $\mathbb{S}^1$ such that for all $\alpha, p^{-1}(U_\alpha) = \coprod_\beta \tilde{U}_\alpha^{(\beta)}$ where for each $\tilde{U}_\alpha^{(\beta)}$,

$$p|_{\tilde{U}_\alpha^{(\beta)}} : \tilde{U}_\alpha^{(\beta)} \to U_\alpha$$

is a homeomorphism.
Remark. This means that

$$(p|_{\tilde{U}_\alpha^{(\beta)}})^{-1} : U_\alpha \to \tilde{U}_\alpha^{(\beta)}$$

exists and is continuous.
From here we argue by compactness.

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Lecture 15, 2/17/23

We know that π_1 is functor from the category of based topological spaces to the category of groups. This tells us that if $A \hookrightarrow X$ is a homotopy equivalence, i.e. if X deformation retracts to A, with retraction r, then because $\iota \circ r = \mathrm{Id}_A$, we get that ι_* must be injective.
So if $A \subseteq X$ admits a retraction, then $\pi_1(A, x_0) \subseteq \pi_1(X, x_0)$

Corollary 0.9. *If X deformation retracts to A, then* $\pi_1(A, a) \xrightarrow{\iota_*} \pi_1(X, a)$ *is not just injective but is an isomorphism.*

Proof. It is easy to see that any element of $\pi_1(X, a)$ homotopes to an element of $\pi_1(A, a)$. ι_* then clearly inverts this, so we have an isomorphism.

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We only know two facts:

1. $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$

2. $\pi_1(\text{a single point}) = 0$

Just this fact tells us that there isn't a retraction from the disk to its boundary, because then $\mathbb{Z}$ would inject into the trivial group, which is clearly not the case.
This machinery finally will give us ways to prove negative statements - statements to the effect of "there is no continuous $f : X \to Y$ with such and such property".

Theorem 0.10. *(Fundamental theorem of algebra)*
Let $p(z) = z^n + z_1 z^{n-1} + \cdots + a_n$, $p : \mathbb{C} \to \mathbb{C}$ has a root.

Proof. Suppose p has no roots. Fix a positive real r. Then define the map $\mathbb{S}^1 \to \mathbb{S}^1$ by sending $s \in [0, 1]$ to $p(re^{2\pi is})/p(r)$. This starts and ends at $1 \in \mathbb{C}$. We know this is not zero, so we can divide by the length to get

$$f_r(s) = \frac{p(re^{2\pi is})/p(r)}{\|p(re^{2\pi is})/p(r)\|}$$

As an element of $\pi_1(\mathbb{S}^1, 1) \cong \mathbb{Z}$, if r is small enough, then $[f_r(s)] = 0$. But if r is large enough, then $[f_r(s)] = [z^n]$. This is because we can consider $p(z) = z^n + b(a_1 z^{n-1} + \cdots + a_n)$ and letting $b \to 0$. For this to work r must be large enough.
But $F : [0, 1] \times \mathbb{S}^1 \to \mathbb{S}^1$ given by $F(r, s) = f_r(s)$ is a homotopy between 0 and $[z^n]$, which is impossible.
Thus p must have a root.
The punchline is essentially that as the radius grows large, it must run into a root eventually, or this bad behavior will happen.

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Theorem 0.11. *(Brouwer Fixed Point)*
If $f : D^2 \to D^2$ is continuous, then there exists $x \in D^2$ such that $f(x) = x$.

Proof. Suppose this was not the case. Let f be such that for all $x, f(x) \neq x$. Define the function $g : D^2 \to \mathbb{S}^1$ by letting $g(x)$ be the point on $\mathbb{S}^1$ where the line going from $f(x)$ to x intersects it. This is continuous and well-defined. This is also a retraction $r : D^2 \to \mathbb{S}^1$, which is impossible by previous discussion.

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Theorem 0.12. *(Borsuk-Ulam)*
For every function $f : \mathbb{S}^2 \to \mathbb{R}^2$, there is some $x \in \mathbb{S}^2$ such that $f(x) = f(-x)$. In other words, if we flatten a sphere, there are a pair of antipodal points which get sent to the same point.

Proof. Suppose that this is not the case. Let f be a function $\mathbb{S}^2 \to \mathbb{R}^2$ such that $f(x) \neq f(-x)$ for all x. Define $g : \mathbb{S}^2 \to \mathbb{S}^1$ by

$$x \mapsto \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$$

Note $g(-x) = -g(x)$. Consider the function $h : s \mapsto (\cos(2\pi s), \sin(2\pi s), 0)$, which is an equatorial loop around $\mathbb{S}^2$, based at $(1, 0, 0)$. Then $gh : I \to S^1$ is a loop based at $(1, 0)$, and $h(s) = -h(s + \frac{1}{2})$.
We now lift to the spiral S:

$$\begin{tikzcd} & S \arrow[d] \\ I \arrow[ur, "\tilde{h}"] \arrow[r, "h"] & \mathbb{S}^1 \end{tikzcd}$$

We see that $\tilde{h}(s), \tilde{h}(s + \frac{1}{2})$ must differ by a half integer, i.e. $\tilde{h}(s) = \tilde{h}(s + \frac{1}{2}) + \frac{q}{2}$ with q an odd integer (this can be seen by drawing the spiral and looking at lifts of antipodal points).
So $\tilde{h}(1) = \tilde{h}(\frac{1}{2}) + \frac{q}{2}$. Because the integers are discrete, q can't change as s varies. Further, $\tilde{h}(0) = \tilde{h}(\frac{1}{2}) + \frac{q}{2}$, so $\tilde{h}(0) = \tilde{h}\frac{1}{2}) + \frac{q}{2} = \tilde{h}(1) + q$. So $[h] = -q$ is odd. So $[h]$ is not zero, so is a loop which is not contractible.
Hitting everything with π_1, get a map from $\pi_1(I)$ to $\pi_1(\mathbb{S}^1)$ which has image consisting of at least two points. But this is impossible, as $\pi_1(I) = 0$. So we have a contradiction, so we have finished the proof. ∎

Proposition 6. *For all $n \geq 2$, $\pi_1(\mathbb{S}^n) = 0$.*

Proof. Let $\gamma : I \to \mathbb{S}^2$ be a loop. If this loop misses one point, then taking away that point, then this is a loop in $\mathbb{R}^2$, which is contractible, so the loop is contractible.
What if γ is very poorly behaved? Well, then there is an open set whose preimage is a union of open intervals in I. Then some stuff happens idk. ∎

Theorem 0.13. *(Invariance of domain)*
$\mathbb{R}^2 \not\cong \mathbb{R}^m$ *for $m > 2$.*

Proof. I missed it :(∎

Lecture 16, 2/22/23

Definition 0.19. Let G_1, G_2, H be groups. Consider maps $f_i : G_i \to H$. We want a group $G_1 \star G_2$ such that there is a unique f making the following diagram commute:

$$\begin{tikzcd} & G_1 \star G_2 \arrow[d, dotted, "\exists! f"] & \\ G_1 \arrow[ur, "\iota_1"] \arrow[r] & H & G_2 \arrow[ul, "\iota_2"'] \arrow[l] \end{tikzcd}$$

Indeed, this is the coproduct in Grp, and is the <u>free product of the groups G_1 and G_2</u>. This can be given explicitly by using group presentations of G_1, G_2. If G_1, G_2 have presentations $G_i \cong \langle K_i \mid R_i \rangle$, where K_i is the set of generators and R_i the relations, then $G_1 \star G_2$ has presentation $\langle K_1 \coprod K_2 \mid R_1 \coprod R_2 \rangle$.

At long last, SVK

What is $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)$? That is, what is the fundamental group of the figure 8?
Last time, we showed that if r is a retraction, then ι_* is injective and r_* is surjective. So, because there is a retraction from $\mathbb{S}^1 \vee \mathbb{S}^1$ to $\mathbb{S}^1$, we know $\pi_1(\mathbb{S}^1)] \cong \mathbb{Z}$ injects into $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)$ somehow.
Further, we get two different ways for $\mathbb{Z}$ to embed into it, because there are two different copies of $\mathbb{S}^1$ to retract onto.
We're trying to make covering spaces of the figure 8. We eventually came to the Cayley graph for $\mathbb{F}_2$. Because this is a tree, it has trivial fundamental group, so homotopy classes are determined by endpoints, and this clearly gives $\mathbb{F}_2$.
$\mathbb{F}_2$, the free group generated by a and b, is $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)$. So $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1) \cong \mathbb{Z} \star \mathbb{Z}$. Be careful: earlier we used $\star$ to refer to the join, which this is not.

Definition 0.20. A covering space $\begin{array}{c} Y \\ \big\downarrow{\scriptstyle p} \\ X \end{array}$ such that Y is simply connected is called the <u>universal cover of X</u>.

Theorem 0.14. *If $\{(X_\alpha, x_\alpha)\}$ are "nice," then*

$$\pi_1(\bigvee_\alpha X_\alpha) = \star_\alpha \pi_1(X_\alpha)$$

where again, $\star$ means free product.

Proof. This is a direct corollary of Van Kampen's theorem.

Lecture 17, 2/24/23

Theorem 0.15. *(Siefert-Van Kampen)*

1. *If $X = \cup_\alpha A_\alpha$, with each A_α path connected and open, and such that there is some x_0 in each A_α, and if, for all $\alpha, \beta, A_\alpha \cap A_\beta$ is path connected, then the natural map $\Phi : \star_\alpha \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$*

2. *If, in addition, for all $\alpha, \beta, \gamma, A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then the kernel of Φ is normally generated by elements of the form*

$$i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$$

where $i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta, x_0) \to \pi_1(A_\alpha, x_0)$.

In other words, let N be the subgroup normally generated by these elements (i.e. smallest normal subgroup containing these elements). Then $\pi_1(X, x_0) \cong \frac{\star_\alpha \pi_1(A_\alpha, x_0)}{N}$

Proof.

1. Let $[f] \in \pi_1(X, x_0)$. Of course, $f : I \to X$ is a map from the interval to X which is a loop based at x_0. We want to express this as the concatenation of finitely many loops, such that each loop is based at x_0, and stays entirely within A_α, where the α depends on the curve.

For every point in the image of f, take an open neighborhood. Without loss of generality this lies entirely in at least one of the A_α. So we can pull this open set back by f. We can do this for every point, and obtain an open cover of I, which, by compactness, admits an finite subcover. This can be used to form finite cover of I, such that in each partition element f does not leave one of the A_α, and such that each open cover representing A_α overlaps with one representing A_β, and in this overlap, f is in both.

The intersections are where f is in the intersection of the A_α, and there we take wherever our curve is, and send it to x_0 and then send it back, making it a loop based at x_0.

So we take a loop, decompose it into finitely segments, such that each of these segments lies entirely within an A_α, and in the overlap between each segment, in the intersection, we draw a path to x_0 and then back (which is possible by path-connectedness), to make each segment a loop based at x_0, so the entire loop is their concatenation.

Each concatenation is an element of the free product of the $\pi_1(A_\alpha, x_0)$, so the map Φ is onto.

2. Suppose $[f] \in \star_\alpha \pi_1(A_\alpha)$, and suppose $\Phi([f]) = [c]$, where $c \equiv x_0$ is a constant map. We know that $[f]$ is the contatenation of loops, each of which stays entirely within one of the A_α. Further, we know that f extends to a map $f : \mathbb{D}^2 \to X$ because $[f] = [c]$. For every point in $I \times I$, it has an open neighborhood which lise entirely within the preimage of one of the A_α, and so we can cover the disk/unit square in the same way we covered the unit interval before.

So we have a finite open cover of $I \times I$. By real analysis, there is a Lebesgue number λ such that any rectangle of side length λ lives entirely in one of our open sets. We can cover $I \times I$ with rectangles of this size, and without loss of generality we can do this with at most triple intersections.

We can cellulate this into a grid, such that each bottom edge of each grid cell is entirely within an overlap. We will pull a similar trick as before, in every vertex in $I \times I$, we run from that vertex to x_0, staying in the areas with triple intersection the entire time by staying in the edges.

So we can get a loop which is homotopic to f, by "pushing it around" the squares. To "push through" a square, we have to take a "commutator" of the form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$.

Remark. This is natural in the categorical sense, because the free product is the coproduct in group. Indeed, because the A_α include into X, we have a family of maps $(\iota_i)_*$, so by the universal property of the coproduct, there is a map Φ which makes the diagram below commute

$$\begin{tikzcd}
& \star_\alpha \pi_1(A_\alpha) \arrow[d, "\Phi"'] \\
\pi_1(A_\alpha) \arrow[ur] \arrow[r, "(\iota_i)_*"] & \pi_1(X)
\end{tikzcd}$$