

Lecture 1

Let (X, \mathcal{A}, μ) be a measure space. Without any additional structure or information, we may define the Lebesgue integral $\int_X f d\mu$ for f an $\mathcal{A} - \mathcal{B}$ measurable function $f : X \rightarrow [-\infty, +\infty]$.

We only have a few examples without any work.

Example 0.1. • For any set X , we can define the counting measure on $\mathcal{A} = 2^X$, which gives $\mu(A) = |A|$. If $X = \mathbb{N}$, then a measurable function is just a sequence (f_n) , and $\int_X f d\mu = \sum f_n$

- We can also define the Dirac mass δ_p for a fixed $p \in X$ by

$$\delta_p(E) = \begin{cases} 1 & p \in E \\ 0 & p \notin E \end{cases}$$

We have $\int_X f d\delta_p = f(p)$

To get another example of a measure we need to do some work.

Problem: We want a measure μ on \mathbb{R}^n such that, for a rectangle,

$$\mu([a_1, b_1] \times \cdots \times [a_n, b_n]) = |a_1 - b_1| \cdots |a_n - b_n|$$

Once it is defined on all rectangles, it is defined on the minimal σ -algebra containing them, which is the Borel σ -algebra. In other words, this condition will completely specify a measure on the Borel σ -algebra $\mathcal{B}_{\mathbb{R}^n}$

If $X = \mathbb{R}^n$, or a general metric space, or even a general topological space, then $\mathcal{B}(X)$ denotes the σ -algebra generated by the open subsets of X .

Problem:

Suppose we have a distribution function $F : \mathbb{R} \rightarrow \mathbb{R}$, meaning F is monotone, positive, and $\lim_{x \rightarrow -\infty} f(x) = 0, \lim_{x \rightarrow \infty} f(x) = 1$, and continuous from the right. We want a Borel measure μ such that $F(t) = \mu((-\infty, t])$. Such a measure, denoted by λ_F , is called a Lebesgue-Stieltjes measure.

The corresponding integral is called a Lebesgue-Stieltjes integral.

If F is smooth, then $\int_{\mathbb{R}} \phi d\lambda_F = \int_{-\infty}^{\infty} \phi(x) dF(x)$.

The measure we want on \mathbb{R}^n is denoted by λ^n .

The Carathéodory Construction

Suppose we have an outer measure $\gamma : 2^X \rightarrow [0, \infty]$. This means $\gamma(\emptyset) = 0$, $A \subset B \implies \gamma(A) \leq \gamma(B)$ (monotone), and $\gamma(\cup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \gamma(E_i)$ (subadditive).

We can define a set S to be γ -measurable if for every testing set T , $\gamma(T) = \gamma(S \cap T) + \gamma(S^c \cap T)$.

Theorem 0.1. (*Carathéodory Extension Theorem*)

1. $\gamma(N) = 0 \implies N$ is measurable.
2. The set of measurable sets forms a σ -algebra Γ .
3. γ restricted to Γ forms a measure.

“Nothing in the above theorem can guarantee you that Γ is not trivial, i.e. $\Gamma = \{\emptyset, X\}$. Nevertheless, this is a very useful guy” - Dennis.

Definition 0.1. (Lebesgue outer measure on \mathbb{R}^n)

Let R be a rectangle in \mathbb{R}^n , that is $R = \prod_{i=1}^n [a_i, b_i]$. We have $\text{Vol}(R) = |a_1 - b_1| \cdots |a_n - b_n|$. For any $E \subseteq \mathbb{R}^n$, we define

$$\mu^*(E) \stackrel{\text{def}}{=} \inf \left\{ \sum_{j=1}^{\infty} \text{Vol}(R_j) \mid E \subseteq \bigcup_{j=1}^{\infty} R_j \right\}$$

Proposition 1. μ^* is an outer measure on \mathbb{R}^n such that $\mu^*(R) = \text{Vol}(R)$ for all rectangles R .

Proof. The first and second axioms are trivial, so we will just prove the subadditivity. Let E be some set. By definition, for any ε , there is some cover R_j by rectangles such that

$$-\varepsilon + \sum_{j=1}^{\infty} \text{Vol}(R_j) \leq \mu^*(E) \leq \sum_{j=1}^{\infty} \text{Vol}(R_j)$$

meaning that $\sum_{j=1}^{\infty} \text{Vol}(R_j) \leq \mu^*(E) + \varepsilon$. So for each E_k , there is a sequence R_j^k which covers E_k , such that $\sum_{j=1}^{\infty} \text{Vol}(R_j^k) \leq \mu^*(E_k) + \frac{\varepsilon}{2^k}$.

So $\{R_j^k\}_{j,k \in \mathbb{N}}$ forms a cover of $\bigcup_{j=1}^{\infty} E_j$. Thus

$$\begin{aligned} \mu^*\left(\bigcup_{k=1}^{\infty} E_k\right) &\leq \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \text{Vol}(R_j^k) \\ &\leq \sum_{k=1}^{\infty} \left(\mu^*(E_k) + \frac{\varepsilon}{2^k} \right) \\ &= \sum_{k=1}^{\infty} \mu^*(E_k) + \varepsilon \end{aligned}$$

This is true for any positive ε . Taking the limit as $\varepsilon \rightarrow 0$ gives the result. ■

Now, fix a rectangle R . Note that R itself forms a cover of R , so by the definition, $\mu^*(R) \leq \text{Vol}(R)$. For $\varepsilon > 0$, we can take an almost-optimal cover (R_j) such that $\sum_{j=1}^{\infty} \text{Vol}(R_j) \leq \text{Vol}(R) + \varepsilon$. We can rig it such that $|\text{Vol}(R_j) - \text{Vol}(R)| \leq \frac{\varepsilon}{2^j}$. Because $R \subset \cup_{j=1}^{\infty} R_j$, and R_j is an open cover, by compactness of R there is a finite subcover, and the volume of R is less than or equal to the sum of the volumes of these finitely many R_j . So the volume of R is less than or equal to $\mu^*(R) + 2\varepsilon$. So $\text{Vol}(R) = \mu^*(R)$.

Proposition 2. *Every rectangle R in \mathbb{R}^n is Carathéodory measurable).*

Proof. I missed this lol. Apparently Dennis denotes \mathcal{M}_{λ^*} by \mathcal{L}^n . ■

Definition 0.2. A set is said to be G_{δ} if it is the countable intersection of open sets. A set is said to be F_{σ} if it is the countable union of closed sets.

Theorem 0.2. 1. *For all $E \in \mathcal{L}^n$, $\lambda^N(E) = \inf\{\lambda^n(O) \mid \text{open } O \supseteq E\}$.*

2. *$E \in \mathcal{L}^n$ if and only if $E = H \setminus Z$, where H is G_{δ} , and $\lambda^*(Z) = 0$.*

3. *$E \in \mathcal{L}^n$ if and only if $E = H \cup Z$, where H is F_{σ} and $\lambda^*(Z) = 0$.*

4. $\lambda^n(E) = \sup\{\lambda^n(C) \mid \text{closed } C \subseteq E\}$

Proof. It suffices to prove the first statement, as the others will follow by passing to a complement. ■

Definition 0.3. Suppose X is a metric space. A measure on X is a Radon measure if it is Borel (meaning defined on a σ -algebra containing Borel sets), and for any Borel E , $\mu(E) = \inf\{\mu(O) \mid \text{open } O \supseteq E\}$, and for any compact $C \subseteq X$, $\mu(C) < \infty$.

Theorem 0.3. (*Riesz*)

Let $X \subseteq \mathbb{R}^n$ be compact. Let $C(X)$ denote the vector space of all continuous functions on X . This admits a norm $\|f\|_{C(X)} = \sup_X |f|$, making it a Banach space.

Define $C^(X) = \{\phi : C(X) \rightarrow \mathbb{R}, \phi \text{ is linear and continuous}\}$.*

For all $\phi \in C^(X)$, there exists a Radon measure $\mu = \mu_+$, and a function $M : X \rightarrow \{\pm 1\}$ which is Borel, such that*

$$\phi(f) = \int_X f(x)M(x) d\mu(x)$$

for all $f \in C(X)$.

Proof. ■

Lecture 2, 1/17/23

Note: This is the first lecture with Davit. Davit will always use μ to refer to an outer measure, not a measure. The book will be “Measure theory and fine properties of functions.” According to Davit, this is the correct book to be using.

Definition 0.4. Let X be a nonempty set. A mapping $\mu : 2^X \rightarrow [0, +\infty]$ is called a measure if it satisfies the following 2 properties.

1. $\mu(\emptyset) = 0$.
2. (Countable subadditivity and monotonicity) If $A, A_1, A_2, \dots \subseteq X$ and $A \subseteq \bigcup_{i=1}^{\infty} A_i$ then $\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$

Remark. From the second definition, we can automatically get monotonicity, i.e. if $A \subseteq B$, then $\mu(A) \leq \mu(B)$. This is because, as written, the second definition is a statement not just about $\bigcup_{i=1}^{\infty} A_i$, but about any subset of it. Indeed, let $A = A$, $A_1 = B$, and $A_k = \emptyset$ for $k \geq 2$. Then we have $\mu(A) \leq \mu(\bigcup_{i=1}^{\infty} A_i) = \mu(A \cup B)$.

We will write “ μ is a measure on X ” to mean that μ satisfies the above definition (that is, μ is an outer measure).

Definition 0.5. Let X be a nonempty set and let μ be a measure on X . For a fixed set $C \subseteq X$, define the restriction measure $\nu = \mu|_C$ by $\nu(A) = \mu|_A(A) = \mu(A \cap C)$.

Remark. It is easy to prove that $\mu|_C$ is a measure on X .

Definition 0.6. (Carathéodory’s condition). Let X be a nonempty set and let μ be a measure on X . A subset $A \subseteq X$ is called μ -measurable if, for all subset $B \subseteq X$, we have

$$\mu(B) = \mu(B \cap A) + \mu(B \setminus A)$$

Remark. X and \emptyset are easily seen to be μ -measurable.

Theorem 0.4. (Carathéodory extension theorem)

The collection of μ -measurable sets on a set X is a σ -algebra.

Theorem 0.5. Let X be a nonempty set and let μ be a measure on X . Then the following holds:

1. \emptyset and X are μ -measurable.
2. $A \subseteq X$ is μ -measurable if and only if $X \setminus A$ is μ -measurable.
3. If $A \subseteq X$ is such that $\mu(A) = 0$, then A is μ -measurable.
4. Let $C \subseteq X$. Then anything which is μ -measurable is $\mu|_C$ -measurable.

Remark. A measure is also finitely subadditive, which says that if $A \subseteq A_1 \cup \dots \cup A_n$, then $\mu(A) \leq \sum_{i=1}^n \mu(A_i)$. So, to check that $\mu(B) = \mu(B \cap A) + \mu(B \setminus A)$, it will suffice to check

$$\mu(B) \geq \mu(B \cap A) + \mu(B \setminus A)$$

Proof. Part 1 is obvious.

Suppose that A is μ -measurable. Then $\mu(B \cap A) = \mu(B \setminus A^c)$ and $\mu(B \cap A^c) = \mu(B \setminus A)$ so $\mu(B \cap A) + \mu(B \setminus A) = \mu(B \cap A^c) + \mu(B \setminus A^c)$. So A is μ -measurable if and only if A^c is.

Suppose that $\mu(A) = 0$. Then $\mu(B \cap A) \leq \mu(A), \mu(B)$, so $\mu(B \cap A) = 0$ for any $B \subseteq X$. Now, $B \setminus A \subseteq B$, so by monotonicity $\mu(B \setminus A) \leq \mu(B)$. So $\mu(B \cap A) + \mu(B \setminus A) \leq \mu(B)$ for all $B \subseteq X$, so we are done.

Let A be μ -measurable. Then for any $B \subseteq X$ we have

$$\begin{aligned} \nu(B) &= \mu|_C(B) = \mu(B \cap C) \\ &= \mu((B \cap C) \cap A) + \mu((B \cap C) \setminus A) \\ &= \nu(B \cap A) + \mu((B \setminus A) \cap C) \\ &= \nu(B \cap A) + \nu(B \setminus A) \end{aligned}$$

■

Theorem 0.6. Let X be a nonempty set and let μ be a measure on X . Assume $A_1, A_2, \dots, A_n \subseteq X$ are μ -measurable. Then

1. $\bigcup_{k=1}^n A_k$ and $\bigcap_{k=1}^n A_k$ are also μ -measurable.
2. If the A_i are disjoint, then $\mu(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i)$

Proof. We prove part 2 first. Because each A_i is measurable,

$$\begin{aligned} \mu(\bigcup_{k=1}^n A_k) &= \mu((\bigcup_{k=1}^n A_k) \cap A_n) + (\mu(\bigcup_{k=1}^n A_k) \setminus A_n) \\ &= \mu(\bigcup_{i=1}^{n-1} A_k) + \mu(A_n) = \dots = \sum_{k=1}^n \mu(A_k) \end{aligned}$$

Now we prove part 1. Let $A, B \subseteq X$ be μ -measurable and disjoint. Then for any $C \subseteq X$, $\mu(C) = \mu(C \cap A) + \mu(C \setminus A)$, and similarly for B . This is equal to

$$\begin{aligned} \mu(C) &= \mu(C \cap A) + \mu((C \setminus A) \cap B) + \mu(C \setminus A \setminus B) \\ &= \mu(C \cap A) + \mu(C \cap B) + \mu(C \setminus (A \cup B)) + \mu(C \cap (A \cup B)) (?) \\ &= \mu(C \cap (A \cup B) \cap A) + \mu(C \cap (A \cup B) \setminus A) \\ &= \mu(C \cap A) + \mu(C \cap B) \\ &= \mu(C \cap (A \cup B)) + \mu(C \setminus (A \cup B)) \end{aligned}$$

So $A \cup B$ is μ -measurable. (I got a bit lost in the arithmetic, sorry)

Next, we show if $A, B \subseteq X$ are μ -measurable, then $A \cap B$ is μ -measurable. This is straightforward. We will continue next time.

Lecture 3, 1/19/23

We will continue our proof of the theorem. Assume $A, B \subseteq X$ are μ -measurable. We aim to show that $A \cap B$ is also μ -measurable. We need to show that, for any $C \subseteq X$, we have $\mu(C) = \mu(C \cap (A \cap B)) + \mu(C \setminus (A \cap B))$. Because A, B are μ -measurable, we have

$$\begin{aligned} \mu(C) &= \mu(C \cap A) + \mu(C \setminus A) \\ &= \mu((C \cap A) \cap B) + \mu((C \cap A) \setminus B) + \mu(C \setminus A) \\ &= \mu(C \cap (A \cap B)) + \mu((C \cap A) \setminus B) + \mu(C \setminus A) \\ &\geq \mu(C \cap (A \cap B)) + \mu(C \setminus (A \cap B)) \end{aligned}$$

The opposite inequality follows by subadditivity, so we have equality.

By induction, we get that also $\cap_{k=1}^n A_k$ is μ -measurable. For the union, we can get it using the fact that $\cup_{k=1}^n A_k = X \setminus \cap_{k=1}^n A_k^c$. ■

Remark. If A, B are μ -measurable, then $A \setminus B$ is μ -measurable. This follows from $A \setminus B = A \cap (X \setminus B)$

Theorem 0.7. Let X be a nonempty set, and μ a measure on X . Assume $\{A_k\}_{k=1}^\infty \subseteq X$ are μ -measurable. Then

1. If the A_k are disjoint, then we have countable additivity:

$$\mu\left(\bigcup_{k=1}^\infty A_k\right) = \sum_{k=1}^\infty \mu(A_k)$$

If $A_1 \subseteq A_2 \subseteq \dots$, then

$$\lim_{k \rightarrow \infty} \mu(A_k) = \mu\left(\bigcup_{k=1}^\infty A_k\right)$$

If $A_1 \supseteq A_2 \supseteq \dots$, and $\mu(A_1) < \infty$, then

$$\lim_{k \rightarrow \infty} \mu(A_k) = \mu\left(\bigcap_{k=1}^\infty A_k\right)$$

Proof. We have from before that if the A_k are pairwise disjoint, then $\mu(\cup_{k=1}^n A_k) = \sum_{k=1}^n \mu(A_k)$ for any $n \in \mathbb{N}$. Because $\cup_{k=1}^n A_k \subseteq \cup_{k=1}^\infty A_k$, we must have that $\mu(\cup_{k=1}^n A_k) \leq \mu(\cup_{k=1}^\infty A_k)$. Using the previous fact, and passing to a limit, we have

$$\sum_{k=1}^{\infty} \mu(A_k) \leq \mu(\cup_{k=1}^{\infty} A_k)$$

The opposite equality is automatically true by the countable subadditivity of μ , so we get equality. This completes the proof of 1. Now for part 2.

Define $B_k = A_k \setminus A_{k-1}$, where $A_0 \stackrel{\text{def}}{=} \emptyset$. We have $A_k = \cup_{i=1}^k B_i$. Note that the B_i are disjoint. So we have

$$\mu(A_k) = \sum_{i=1}^k \mu(B_i)$$

So, in the limit,

$$\lim_{k \rightarrow \infty} \mu(A_k) = \lim_{k \rightarrow \infty} \sum_{i=1}^k \mu(B_i) = \sum_{i=1}^{\infty} \mu(B_i)$$

So

$$\mu(\cup_{i=1}^{\infty} B_i) = \mu(\cup_{k=1}^{\infty} A_k)$$

Finally, let $A_1 \supseteq A_2 \supseteq \cdots$, $\mu(A_1) < \infty$. Define $B_k = A_1 \setminus A_k$. This is decreasing sequence of μ -measurable sets, so by the previous part,

$$\lim_{k \rightarrow \infty} \mu(B_k) = \mu(\cup_{k=1}^{\infty} B_k)$$

So

$$\begin{aligned} \mu(B_k) &= \mu(A_1 \setminus A_k) = \mu(A_1) - \mu(A_k) \implies \\ \lim_{k \rightarrow \infty} \mu(B_k) &= \lim_{k \rightarrow \infty} (\mu(A_1) - \mu(A_k)) = \mu(A_1) - \lim_{k \rightarrow \infty} \mu(A_k) \\ &= \mu(\cup_{k=1}^{\infty} B_k) = \mu(\cup_{k=1}^{\infty} (A_1 \setminus A_k)) = \mu(A_1 \setminus \cap_{k=1}^{\infty} A_k) \\ &\geq \mu(A_1) - \mu(\cap_{k=1}^{\infty} A_k) \end{aligned}$$

So $\lim_{k \rightarrow \infty} \mu(A_k) \leq \mu(\cap_{k=1}^{\infty} A_k)$. The opposite inequality follows easily by monotonicity. ■

We are ready to prove the Carathéodory extension theorem.

Proof. Let $A_1, A_2, \dots \subseteq X$ be μ -measurable. It will suffice to prove that $\cup_{k=1}^{\infty} A_k$ is μ -measurable. So we need to check that, for any B ,

$$\mu(B) = \mu(B \cap (\cup_{k=1}^{\infty} A_k)) + \mu(B \setminus \cup_{k=1}^{\infty} A_k)$$

Fix $B \subseteq X$, and consider $\nu = \mu|_B$. Recall this is defined as $\nu(C) = \mu(B \cap C)$. We would like

$$\nu(B) = \nu(\cup_{k=1}^{\infty} A_k) + \nu(B \setminus \cup_{k=1}^{\infty} A_k)$$

To this end,

$$\nu(\cup_{k=1}^{\infty} A_k) = \lim_{k \rightarrow \infty} \nu(\cup_{i=1}^k A_i)$$

Without loss of generality, $\nu(B) < \infty$. If $\nu(B) = \infty$, then we are done trivially. As before, define $B_k = B \setminus \cup_{i=1}^k A_i$. Then

$$\nu(B \setminus \cup_{k=1}^{\infty} A_k) = \lim_{k \rightarrow \infty} \nu(B \setminus \cup_{i=1}^k A_i)$$

So,

$$\begin{aligned} \nu(\cup_{k=1}^{\infty} A_k) + \nu(B \setminus \cup_{k=1}^{\infty} A_k) &= \lim_{k \rightarrow \infty} \left(\nu(\cup_{i=1}^k A_i) + \nu(B \setminus \cup_{i=1}^k A_i) \right) \\ &= \lim_{k \rightarrow \infty} \nu(B) = \nu(B) \end{aligned}$$

so we are done. ■

Definition 0.7. Let X be a nonempty set, and let μ be a measure on X . Then μ is said to be

1. A regular measure if, for any $A \subset X$, there exists a μ -measurable $B \subseteq X$ such that $A \subseteq B$, and $\mu(A) = \mu(B)$.
2. A Borel measure if all Borel sets (i.e. the elements of the Borel σ -algebra) are measurable. This only applies if X is also a topological space, of course.
3. A Borel-regular measure if μ is Borel, and for any $A \subseteq X$, there exists a Borel set $B \subseteq X$ such that $A \subseteq B$ and $\mu(A) = \mu(B)$.
4. A Radon measure if it is Borel-regular and $\mu(K) < \infty$ if K is compact.

Remark. Note that being Borel and regular is weaker than being Borel-regular.

Theorem 0.8. (Increasing sets for regular measures) Let X be a nonempty set, and let μ be a regular measure on X . Assume $A_1 \subseteq A_2 \subseteq \cdots \subseteq X$. Then

$$\lim_{k \rightarrow \infty} \mu(A_k) = \mu \left(\bigcup_{k=1}^{\infty} A_k \right)$$

Remark. The sets A_k need not be μ -measurable.

Proof. For all A_k , there is a $C_k \subseteq X$ which is μ -measurable, $A_k \subseteq C_k$, and $\mu(A_k) = \mu(C_k)$. Let $D_k = \bigcap_{i \geq k} C_i$. For $i \geq k$, we can see $A_k \subseteq A_i \subseteq C_i$. $A_k \subseteq \bigcup_{i \geq k} C_i = D_k$, then $\mu(A_k) \leq \mu(D_k)$. On the other hand, $D_k \subseteq C_k$, so $\mu(D_k) \leq \mu(C_k) = \mu(A_k)$. So

- $\mu(A_k) = \mu(D_k)$
- $A_k \subseteq D_k$
- D_k is μ -measurable and $D_1 \subseteq D_2 \subseteq \dots$

So

$$\begin{aligned} \lim_{k \rightarrow \infty} \mu(A_k) &= \lim_{k \rightarrow \infty} \mu(D_k) \\ &= \mu \left(\bigcup_{k=1}^{\infty} D_k \right) \\ &\geq \mu \left(\bigcup_{k=1}^{\infty} A_k \right) \end{aligned}$$

Because $\bigcup_{k=1}^{\infty} A_k \subseteq \bigcup_{k=1}^{\infty} D_k$,

$$\lim_{k \rightarrow \infty} \mu(A_k) \geq \mu \left(\bigcup_{k=1}^{\infty} A_k \right)$$

But $A_k \subseteq \bigcup_{k=1}^{\infty} A_k$, so the opposite inequality is also true, so we have equality. ■

Lecture 4, 1/24/23

Theorem 0.9. (Restriction and Radon measures)

Let X be a topological space and let μ be a Borel-regular measure on X . Let $A \subseteq X$ be μ -measurable with $\mu(A) < \infty$. Then the restriction measure $\nu = \mu|_A$ is Radon.

Proof. First, ν is a finite measure, as $\nu(X) = \mu(A \cap X) = \mu(A) < \infty$ for any $C \subseteq X$. It is clear that ν is Borel, as μ is Borel. Next, we show ν is Borel-Regular. Without loss of generality, we may assume that A is Borel, because μ is Borel-regular. Explicitly, we know there is a Borel set $B \subseteq X$ such that $A \subseteq B$ and $\mu(B) = \mu(A)$. We will show $\mu|_A = \mu|_B$.

We have $\mu(B) = \mu(B \cap A) + \mu(B \setminus A) = \mu(A) + \mu(B \setminus A)$. So $\mu(B \setminus A) = 0$.
So, for all $C \subseteq X$,

$$\begin{aligned}\mu|_B(C) &= \mu(B \cap C) \\ &= \mu((B \cap C) \cap A) + \mu((B \cap C) \setminus A) \\ &= \mu(C \cap A) + \mu((B \cap C) \setminus A) \\ &\leq \mu|_A(C) + \mu(B \setminus A) \\ &= \mu|_A(C)\end{aligned}$$

But $(A \cap C) \subseteq (B \cap C)$, so $\mu|_A(C) \leq \mu|_B(C)$, so we may conclude that $\mu|_A = \mu|_B$.
So assume A is Borel. Fix $C \subseteq X$. We need to prove that there exists a Borel $D \subseteq X$ such that $C \subseteq D$ and $\nu(C) = \nu(D)$. There exists a Borel $E \subseteq X$ such that $C \cap A \subseteq E$, and $\mu(C \cap A) = \mu(E)$. So $D = E \cup (X \setminus A)$ is Borel and $C \subseteq D$.
So

$$\begin{aligned}\nu(D) &= \mu((E \cup (X \setminus A)) \cap A) \\ &= \mu(E \cap A) \\ &\leq \mu(E) \\ &= \mu(C \cap A) \\ &= \nu(C)\end{aligned}$$

$C \subseteq D$ so $\nu(C) \leq \nu(D)$, so $\nu(C) = \nu(D)$. ■

Theorem 0.10. (Carathéodory Criterion for being Borel)

Let X be a metric space and let μ be a measure on X . Then μ is Borel if and only if, for all $A, B \subseteq X$ with $d(A, B) > 0$ (meaning $\inf\{d(a, b) \mid a \in A, b \in B\} > 0$),

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

Proof. \Rightarrow

Suppose μ is Borel. We will use \overline{B} to denote the closure of B . Then $d(A, \overline{B}) = d(A, B) > 0$. By measurability of \overline{B} ,

$$\mu(A \cup B) = \mu((A \cup B) \cap \overline{B}) + \mu((A \cup B) \setminus \overline{B}) = \mu(B) = \mu(A)$$

\Leftarrow

Suppose that, for A, B with $d(A, B) > 0$, $\mu(A \cup B) = \mu(A) + \mu(B)$. We will show that this implies μ is Borel. Let us show that every closed subset $C \subseteq X$ is μ -measurable.

So we have to prove that for every $A \subseteq X$,

$$\mu(A) = \mu(A \cap C) + \mu(A \setminus C)$$

We have \leq trivially. Assume $\mu(A) < \infty$; otherwise, this equality holds trivially.

Define for every $n \in \mathbb{N}$ the set $C_n = \{x \in X \mid d(x, C) \leq \frac{1}{n}\}$. We can see $d(A \setminus C_n, C) \geq \frac{1}{n} > 0$. So

$$\begin{aligned} \mu((A \setminus C_n) \cup (A \cap C)_{\subseteq A}) &= \mu(A \setminus C_n) + \mu(A \cap C) \\ &\leq \mu(A) \end{aligned}$$

So $\mu(A \setminus C_n) + \mu(A \cap C) \leq \mu(A)$ for all $n \in \mathbb{N}$. We will prove that $\lim_{n \rightarrow \infty} \mu(A \setminus C_n) = \mu(A \setminus C)$.

Consider the annuli $R_n = \{x \in A \mid \frac{1}{n+1} < d(x, C) \leq \frac{1}{n}\}$. We have

$$(A \setminus C_1) \bigcup_{n=1}^{\infty} R_n \subseteq A \setminus C$$

C is closed, so in fact we have equality above. Why? If a point belongs to $A \setminus C$, then it does not belong to C , so $d(x, C) > 0$. So there is an $n \in \mathbb{N}$ such that $x \in R_n$ or $x \in A \setminus C_1$. We have

$$\begin{aligned} \mu\left(\bigcup_{k=0}^n R_{2k+1}\right) &= \sum_{k=0}^n \mu(R_{2k+1}) \leq \mu(A) \\ \mu\left(\bigcup_{k=1}^n R_{2k}\right) &= \sum_{k=1}^n \mu(R_{2k}) \leq \mu(A) \end{aligned}$$

So $\sum_{n=1}^{\infty} \mu(R_n) \leq 2\mu(A) < \infty$, so $\lim_{n \rightarrow \infty} (\sum_{k=n}^{\infty} \mu(R_k)) = 0$. So $(A \setminus C_n) \bigcup_{k=n}^{\infty} R_k = A \setminus C$

So by subadditivity,

$$\mu(A \setminus C) \leq \mu(A \setminus C_n) + \sum_{k=n}^{\infty} \mu(R_k)$$

So as $n \rightarrow \infty$,

$$\mu(A \setminus C) \leq \liminf_{n \rightarrow \infty} \mu(A \setminus C_n) \leq \mu(A \setminus C)$$

This completes the proof.

It is time for our third section.

Approximation by open, closed, and compact sets

Theorem 0.11. Let μ be a Borel measure on \mathbb{R}^n , and let $B \subseteq \mathbb{R}^n$ be a Borel set.

1. *If $\mu(B) < \infty$, then for any $\varepsilon > 0$, there exists a closed $C \subseteq B$ such that $\mu(B \setminus C) < \varepsilon$.*
2. *If μ is a Radon measure, then for all $\varepsilon > 0$, there exists an open $U \supseteq B$ such that $\mu(U \setminus B) < \varepsilon$.*

Proof. 1. Let $\nu = \mu|_B$, a finite measure on \mathbb{R}^n .

Define the collection $\mathcal{F} = \{A \subseteq \mathbb{R}^n \mid A \text{ is } \mu\text{-measurable and for all } \varepsilon > 0, \text{ there exists a closed } C \subseteq A \text{ such that } \mu(A \setminus C) < \varepsilon\}$

Our goal is to show that $\mathcal{B}_{\mathbb{R}^n} \subseteq \mathcal{F}$. Davit uses “ σ_B ” to indicate the Borel σ -algebra.

By previous discussion, \mathcal{F} contains all closed sets.

Now, if $A_1, A_2, \dots \in \mathcal{F}$, then $\bigcup_{k=1}^{\infty} A_k \in \mathcal{F}$. For all A_k , there exists a closed $C_k \subseteq A_k$, such that $\nu(A_k \setminus C_k) < \frac{\varepsilon}{2^k}$. Then by subadditivity,

$$\nu\left(\bigcup_{k=1}^{\infty} A_k \setminus \bigcup_{k=1}^{\infty} C_k\right) \leq \nu\left(\bigcup_{k=1}^{\infty} (A_k \setminus C_k)\right) \leq \sum_{k=1}^{\infty} \nu(A_k \setminus C_k) < \varepsilon$$

and $C = \bigcup_{k=1}^{\infty} C_k$ is closed.

Lecture 5, 1/26/23

Theorem 0.12. Let μ be a Borel measure on \mathbb{R}^n and let $B \subseteq \mathbb{R}^n$ be a Borel set.

1. *If $\mu(B) < \infty$, then for all $\varepsilon > 0$, there exists a closed $C \subseteq B$ such that $\mu(B \setminus C) < \varepsilon$.*
2. *If μ is a Radon measure, then for all $\varepsilon > 0$, there exists an open $U \subseteq \mathbb{R}^n$ such that $B \subseteq U$ and $\mu(U \setminus B) < \varepsilon$.*

Proof.

1. Let $\nu = \mu|_B$ be a finite Borel measure on \mathbb{R}^n . Define the collection

$$\mathcal{F} = \{A \subseteq \mathbb{R}^n : A \mu\text{-measurable and for all } \varepsilon > 0, \exists C \subseteq A, C \text{ closed, } \nu(A \setminus C) < \varepsilon\}$$

We want to show $\mathcal{B} \in \mathcal{F}$, where \mathcal{B} is the Borel set.

Step 1:

\mathcal{F} contains all closed sets

Step 2:

If $A_1, A_1, \dots, A_k \in \mathcal{F}$, then for all A_k , there exists a closed C_k such that $\nu(A_k \setminus C_k) < \frac{\varepsilon}{2^k}$. Thus,

$$\begin{aligned} \nu \left(\bigcup_{k=1}^{\infty} A_k \setminus \bigcup_{k=1}^{\infty} C_k \right) &\leq \nu \left(\bigcup_{k=1}^{\infty} (A_k \setminus C_k) \right) \\ &\leq \sum_{k=1}^{\infty} \nu(A_k \setminus C_k) < \varepsilon \end{aligned}$$

Furthermore, $\cap_{k=1}^{\infty} C_k$ is closed. Thus \mathcal{F} is closed under countable intersections.

Step 3:

We want to show countable unions belong to \mathcal{F} . If $A_1, A_2, \dots, A_k, \dots \in \mathcal{F}$, then for all A_k , there is a closed C_k such that $\nu(A_k \setminus C_k) < \frac{\varepsilon}{2^k}$. However, we do not know if $\cup_{k=1}^{\infty} C_k$ is closed. Note that $\nu(\cup_{k=1}^{\infty} A_k \setminus \cup_{k=1}^{\infty} C_k) = \lim_{m \rightarrow \infty} \nu(\cup_{k=1}^m A_k \setminus \cup_{k=1}^m C_k) < \varepsilon$. So there is an $m \in \mathbb{N}$ such that $\nu(\cup_{k=1}^m A_k \setminus \cup_{k=1}^m C_k) < \varepsilon$. Furthermore $C = \cup_{k=1}^m C_k$ is closed.

Step 4:

In the homework, we showed that every open set is the countable union of closed balls. Since \mathcal{F} contains all closed sets, and is closed under countable unions, \mathcal{F} contains all open sets.

Step 5:

Consider the subset $G \subseteq \mathcal{F}$ given by $G = \{A \in \mathcal{F} \mid A^c \in \mathcal{F}\}$. We claim that G is a σ -algebra. Going through the axioms,

- (i) Clearly, $\emptyset \in G$.
- (ii) If $A \in G$, $A^c \in G$.

- (iii) If $A_1, A_2, \dots, A_k, \dots \in G$, then $\cup_{k=1}^{\infty} A_k \in G$. Why? $\cup_{k=1}^{\infty} A_k \in \mathcal{F}$ and $\mathbb{R}^n \setminus \cap_{k=1}^{\infty} (\mathbb{R}^n \setminus A_k) \in \mathcal{F}$, since each $\mathbb{R}^n \setminus A_k \in \mathcal{F}$, and \mathcal{F} is closed under countable intersections.

Step 6:

Since the complement of an open sets is a closed set, and since \mathcal{F} contains all open and closed sets, all open sets are contained in the σ -algebra G . Thus the Borel sets are contained in G , implying that they are contained in \mathcal{F} .

Note: Part 1 requires that X be a seperable metric space.

2. For all $m \in \mathbb{N}$, denote $U_m = B_m(0) = \{x \in \mathbb{R}^n \mid \|x\| < m\}$.

Note that $\mu(U_m \setminus B) \leq \mu(U_m) < \infty$. Thus there exists a closed $C_m \subseteq U_m \setminus B$ such that $\mu((U_m \setminus B) \setminus C_m) < \frac{\varepsilon}{2^m}$.

Note that $B \cap U_m \subseteq (U_m \setminus C_m)$, which is an open set.

Thus, $\mu((U_m \setminus C_m) \setminus (B \cap U_m)) = \mu((U_m \setminus C_m) \setminus B) < \frac{\varepsilon}{2}$.

Define $U = \cup_{m=1}^{\infty} (U_m \setminus C_m)$, which is an open set.

Thus $B = \cup_{m=1}^{\infty} (B \cap U_m) \subseteq \cup_{m=1}^{\infty} (U_m \setminus C_m) = U$.

Furthermore,

$$\begin{aligned} \mu(U \setminus B) &= \mu(\cup_{m=1}^{\infty} (U_m \setminus C_m) \setminus \cup_{m=1}^{\infty} (B \cap U_m)) \\ &\leq \mu(\cup_{m=1}^{\infty} (U_m \setminus C_m \setminus B \cap U_m)) \\ &\leq \sum_{m=1}^{\infty} \mu(U_m \setminus C_m \setminus B \cap U_m) \\ &< \varepsilon \end{aligned}$$

Note: Part 2 requires that for all $r > 0$, for all $x \in X$, $\mu(B_r(x)) < \infty$.

■

Theorem 0.13. (Approximation by compact and open sets).

Let μ be a Radon measure on \mathbb{R}^n . Then

1. *For all $A \subseteq \mathbb{R}^n$, $\mu(A) = \inf\{\mu(U) \mid A \subseteq U, U \text{ open}\}$.*
2. *For all μ -measurable $A \subseteq \mathbb{R}^n$, we have $\mu(A) = \sup\{\mu(K) \mid K \subseteq A, K \text{ compact}\}$.*

Proof. Shortly.

Note: If μ is the Lebesgue measure, we define the outer measure as

$$\begin{aligned}\mu(A) &= \inf \left\{ \sum_{k=1}^{\infty} (b_k - a_k) \mid A \subseteq \bigcup_{k=1}^{\infty} (a_k, b_k) \right\} \\ &= \inf \left\{ \sum_{k=1}^{\infty} (b_k - a_k) \mid A \subseteq \bigcup_{k=1}^{\infty} [a_k, b_k), [a_i, b_i) \cap [a_j, b_j) = \emptyset \text{ if } i \neq j \right\}\end{aligned}$$

Remark. Let X be a topological space, and let μ be a measure on X . If $A \subseteq X$ is such that for all $\varepsilon > 0$, there exists a μ -measurable $A_\varepsilon \subseteq A$, such that $\mu(A \setminus A_\varepsilon) < \varepsilon$. Then A is μ -measurable.

Proof. of remark.

Take $\varepsilon = \frac{1}{k}$. By part 1, there exists $A_k \subseteq A$ such that $0 \leq \mu(A) - \mu(A_k) < \frac{1}{k}$.

Let $B = \bigcup_{k=1}^{\infty} A_k$, and note that $B \subseteq A$, B is μ -measurable, and $\mu(A) - \frac{1}{k} \leq \mu(A_k) \leq \mu(B)$ for all $k \in \mathbb{N}$.

This implies that $\mu(B) = \mu(A)$, $B \subseteq A$.

Thus, $\mu(A) = \mu(A \cap B) + \mu(A \setminus B) = \mu(B) + \mu(A \setminus B)$.

Since $\mu(A) = \mu(B)$, this implies that $\mu(A \setminus B) = 0$, i.e. $A \setminus B$ is μ -measurable.

So $A = B \cup (A \setminus B)$ is μ -measurable. ■

We are now prepared for a proof of the theorem.

Proof. **1.** If $\mu(A) = \infty$, then this is trivial. So assume $\mu(A) < \infty$. If A is Borel, we use part 2 of theorem 1.

We have an open $U_\varepsilon \subseteq \mathbb{R}^n$ such that $A \subseteq U_\varepsilon$, and $\mu(U_\varepsilon \setminus A) < \varepsilon$.

Thus $\mu(A) \leq \mu(U_\varepsilon) \leq \mu(A) + \mu(U_\varepsilon \setminus A) < \mu(A) + \varepsilon$.

So we are done for Borel sets. If A is not Borel, then there exists a Borel $B \subseteq \mathbb{R}^n$ such that $A \subseteq B$ and $\mu(A) = \mu(B)$.

Note: For the above proof, we used a previous theorem, and the fact that μ is Borel-regular.

2. Lets prove that $\mu(A) = \sup\{\mu(C) \mid C \subseteq A, C \text{ closed}\}$. We have two cases:

- (i) $\mu(A) < \infty$. In this case, consider $\nu = \mu|_A$. ν is a finite Radon measure. We apply part 1 to A^c . There exists an open $U_\varepsilon \subseteq \mathbb{R}^n$ such that $\mathbb{R}^n \setminus A \subseteq U_\varepsilon$ and $\nu(U_\varepsilon) < \nu(A^c) + \varepsilon = \varepsilon$.

Set $C_\varepsilon = U_\varepsilon^c$, which is closed. Note that $C_\varepsilon \subseteq A$ and $\mu(A \setminus C_\varepsilon) = \mu(A \cap U_\varepsilon) = \nu(U_\varepsilon) < \varepsilon$.

Since $C \subseteq A$, $\mu(C_\varepsilon) \leq \mu(A)$. By countable subadditivity, $\mu(A) \leq \mu(C_\varepsilon) + \mu(A \setminus C_\varepsilon) < \mu(C_\varepsilon) + \varepsilon$.

(ii) Next time!

■

Lecture 6, 1/31/23

Remark. Let μ be a Borel measure on X . If for every $A \subseteq X$ one has $\mu(A) = \inf\{\mu(B) \mid A \subseteq B, B \text{ Borel}\}$, then μ has to be Borel-regular.

Proof. of remark.

Take $B_k \supseteq A$ such that $\mu(B_k) < \mu(A) + \frac{1}{k}$. Let $B = \bigcap_{k=1}^{\infty} B_k$. Note B is Borel, $A \subseteq B$, and $\mu(A) \leq \mu(B) \leq \mu(B_k) < \mu(A) + \frac{1}{k}$. So $\mu(A) = \mu(B)$.

■

Proof. of theorem from last time.

We showed that if $\mu(A) < \infty$, then $\mu(A) = \sup\{\mu(C) \mid C \subseteq A, C \text{ closed}\}$.

If $\mu(A) = \infty$, write $\mathbb{R}^n = \bigcup_{k=1}^{\infty} R_k$, where $R_k = \{x \in \mathbb{R}^n \mid k \leq \|x\| < k+1\}$. Thus $A = \bigcup_{k=1}^{\infty} A \cap R_k$.

For all k , $\mu(A \cap R_k) \leq \mu(R_k) < \infty$. So there exists $C_k \subseteq A \cap R_k$ such that $\mu(C_k) > \mu(A \cap R_k) - \frac{1}{2^k}$.

Thus $\mu(\bigcup_{k=1}^{\infty} C_k) = \sup_{k=1}^{\infty} \mu(C_k) \geq \sum_{k=1}^{\infty} (\mu(A \cap R_k) - \frac{1}{2^k}) \geq \mu(A) - 2 = \infty$.

This implies that $\lim_{m \rightarrow \infty} \mu(\bigcup_{k=1}^m C_k) = \mu(\bigcup_{k=1}^{\infty} C_k) = \infty$.

This proves the theorem for closed sets.

Now we prove the theorem for compact sets.

Case 1: $\mu(A) < \infty$. For all $\varepsilon > 0$, there exists a closed set $C_\varepsilon \subseteq A$ such that $\mu(C_\varepsilon) > \mu(A) - \varepsilon$. Consider $K_m = C_\varepsilon \cap B_m$, where $B_m = \{x \in \mathbb{R}^n \mid \|x\| \leq m\}$.

Note K_m is compact.

Thus $\lim_{m \rightarrow \infty} \mu(K_m) = \mu(\bigcup_{k=1}^{\infty} K_m) = \mu(C_\varepsilon) > \mu(A) - \varepsilon$.

Case 2: $\mu(A) = \infty$. For all $m \in \mathbb{N}$, there exists a closed $C_m \subseteq A$ such that $\mu(C_m) \geq m$. Apply the same procedure.

■

Covering theorems (Vitali's and Besicovitch)

Notation: we will work in \mathbb{R}^n . Closed balls will be denoted by B . For a given closed ball $B = B_r(x) = \{y \in \mathbb{R}^n \mid \|x - y\| \leq r\}$, $\hat{B} = cB = \{y \in \mathbb{R}^n \mid \|x - y\| \leq cr\} = B_{cr}(x)$.

Definition 0.8. Let $A \subseteq \mathbb{R}^n$ and let $\mathcal{F} = \{B \subseteq \mathbb{R}^n\}$ be a family of balls.

1. \mathcal{F} is a cover of A if $A \subseteq \bigcup_{B \in \mathcal{F}} B$.

2. \mathcal{F} is a fine cover of A if for all $x \in A$ and $\varepsilon > 0$, there exists a $B \in \mathcal{F}$ such that $x \in B$ and $\text{diam}(B) < \varepsilon$. Alternatively, for all $x \in X$, $\inf\{\text{diam}(B) \mid x \in B\} = 0$.

Theorem 0.14. (Vitali's Covering Theorem)

Let \mathcal{F} be a collection of nondegenerate closed balls in \mathbb{R}^n with diameters uniformly bounded, i.e. $\sup\{\text{diam}(B) \mid B \in \mathcal{F}\} < \infty$. Then there exists a subcollection of countably many disjoint balls $\{\hat{B}_i\}_{i=1}^\infty$, such that $\cup_{B \in \mathcal{F}} B \subseteq \cup_{i=1}^\infty \hat{B}_i$

Proof. Denote $D = \sup\{\text{diam}(B) \mid B \in \mathcal{F}\}$ and consider $\mathcal{F}_k = \{B \in \mathcal{F} \mid \frac{D}{2^k} < \text{diam}(B) \leq \frac{D}{2^{k-1}}\}$. Let $G_1 \subseteq \mathcal{F}_1$ be a maximal disjoint subcollection of balls in \mathcal{F} (we can produce this easily with Zorn's Lemma). It will be maximal in the sense that if we add another element, it will not be a disjoint set.

Assume G_1, \dots, G_{k-1} have been chose.

Let G_k be a maximal disjoint subcollection in \mathcal{F} such that the balls at G_k do not intersect with the balls in $\cup_{i=1}^{k-1} G_i$.

Set $G = \cup_{k=1}^\infty G_k \subseteq \mathcal{F}$. Let $B \in \mathcal{F}_m$, i.e. $\frac{D}{2^m} < \text{diam}(B) \leq \frac{D}{2^{m-1}}$.

Because G_m is maximal, there exists $\bar{B} \in \cup_{i=1}^m G_i$ such that $B \cap \bar{B} \neq \emptyset$. Thus $\text{diam}(\bar{B}) \geq \frac{D}{2^m} \geq \frac{1}{2} \text{diam}(B)$. Thus $B \subseteq \hat{\bar{B}}$.

Lecture 7, 2/2/23

Corollary 0.15. Let $A \subseteq \mathbb{R}^n$, and let the collection \mathcal{F} of nondegenerate closed balls be a fine cover of A such that $\sup\{\text{diam}(B) \mid B \in \mathcal{F}\} < \infty$.

Then for any finite number of balls $B_1, B_2, \dots, B_m \in \mathcal{F}$, one has

$$A \setminus \cup_{i=1}^m B_i \subseteq \cup_{B \in G \setminus \{B_1, \dots, B_m\}} \hat{B}$$

where G is the disjoint collection of balls guaranteed by Vitali's theorem.

Proof. Assume $x \in A \setminus \cup_{i=1}^m B_i$. Then $x \notin \cup_{i=1}^m B_i$, so $d(x, \cup_{i=1}^m B_i) > 0$, as B_i is closed for all i , and the finite union of closed sets is closed.

Let $d = d(x, \cup_{i=1}^m B_i)$. Because $x \in A$, there exists a ball $B = B_r(y) \in \mathcal{F}$ such that $x \in B_r(y)$ and $2r < d$. This gives us that $B \cap \cup_{i=1}^m B_i = \emptyset$. By the construction of G , there exists $\bar{B} \in G$ such that $B \cap \bar{B} \neq \emptyset$ and $B \subseteq \hat{\bar{B}}$. Now, $\bar{B} \neq B_i, i = 1, 2, \dots, m$, and therefore $x \in B \subseteq \hat{\bar{B}} \subseteq \cup_{\bar{B} \in G \setminus \{B_1, \dots, B_m\}} \hat{\bar{B}}$.

Theorem 0.16. (Filling open sets with closed balls)

Let $U \subseteq \mathbb{R}^n$ be open, and let $\delta > 0$. Then there is a countable collection G of nondegenerate closed, disjoint balls, such that

$$\sup\{\text{diam}(B) \mid B \in G\} \leq \delta$$

and $\mathcal{L}^n(U \setminus \cup_{i=1}^\infty B_i) = 0$, where \mathcal{L}^n denotes n -dimensional Lebesgue measure. Here, $G = \{B_i\}_{i=1}^\infty$.

Proof.

Case 1: $\mathcal{L}^n(U) < \infty$

Consider the collection of nondegenerate closed balls $\mathcal{F} = \{B \subseteq U \mid \text{diam}(B) \leq \delta\}$. Because U is open, $\cup_{B \in \mathcal{F}} B = U$. By Vitali's covering theorem, there exists a countable family G of disjoint balls such that $U \subseteq \cup_{i=1}^{\infty} \hat{B}_i$. So

$$\mathcal{L}^n(U) \leq \mathcal{L}^n(\cup_{i=1}^{\infty} \hat{B}_i) \leq \sum_{i=1}^{\infty} \mathcal{L}^n(\hat{B}_i)$$

By countable subadditivity, $\mathcal{L}^n(U) \leq 5^n \sum_{i=1}^{\infty} \mathcal{L}^n(B_i) = 5^n \mathcal{L}^n(\cup_{i=1}^{\infty} B_i)$. So

$$\mathcal{L}^n(U \setminus \cup_{i=1}^{\infty} B_i) = \mathcal{L}^n(U) - \mathcal{L}^n(\cup_{i=1}^{\infty} B_i) \leq (1 - \frac{1}{5^n}) \mathcal{L}^n(U)$$

Now

$$\lim_{m \rightarrow \infty} \mathcal{L}^n(U \setminus \cup_{i=1}^m B_i) = \mathcal{L}^n(U \setminus \cup_{i=1}^{\infty} B_i) \leq (1 - \frac{1}{5^n}) \mathcal{L}^n(U)$$

So there exists an index $m_1 \in \mathbb{N}$ such that $\mathcal{L}^n(U \setminus \cup_{i=1}^{m_1} B_i) \leq (1 - \frac{1}{2 \cdot 5^n}) \mathcal{L}^n(U)$.

Consider $U_2 = U \setminus \cup_{i=1}^{m_1} B_i$ and the new collection $\mathcal{F}_i = \{B \mid B \subseteq U_2, \text{diam}(B) \leq \delta\}$. Then

$$\mathcal{L}^n(U_2) \leq q \cdot \mathcal{L}^n(U) < \infty$$

So there exist disjoint closed $B_{m_1+1}, \dots, B_{m_2} \in \mathcal{F}_2$ such that

$$\mathcal{L}^n(U_2 \setminus \cup_{i=m_1+1}^{m_2} B_i) \leq q \mathcal{L}^n(U)$$

So $\mathcal{L}^n(U \setminus \cup_{i=1}^{m_2} B_i) \leq q^2 \mathcal{L}^n(U)$.

k -th step

There are disjoint balls $B_1, B_2, \dots, B_{m_k} \subseteq U$ such that

$$\mathcal{L}^n(U \setminus \cup_{i=1}^{m_k} B_i) \leq q^k \mathcal{L}^n(U)$$

So

$$\mathcal{L}^n(U \setminus \cup_{i=1}^{\infty} B_i) \leq \mathcal{L}^n(U \setminus \cup_{i=1}^{m_k} B_i) \leq q^k \mathcal{L}^n(U)$$

The above is true for every k , so it follows that $\mathcal{L}^n(U \setminus \cup_{i=1}^{\infty} B_i) = 0$. $G = \{B_i\}_{i=1}^{\infty}$. This completes the proof in the case of $\mathcal{L}^n(U) < \infty$.

Case 2, $\mathcal{L}^n(U) = \infty$

Consider $U_m = U \cap \{x \in \mathbb{R}^n : m-1 < |x| < m\}$, $m = 1, 2, \dots$. We know $\mathcal{L}^n(\partial B_r(x)) = 0$ for all $B_r(x) \subseteq \mathbb{R}^n$ (this will be a homework problem).

■

“You can go look at the proof of Besicovitch in the book , but to be honest I never read that proof.” - Davit

Theorem 0.17. (Besicovitch's covering theorem)

There exists a number N_n that depends only on the space dimension n , with the following property.

If \mathcal{F} is any collection of nondegenerate closed balls in \mathbb{R}^n , with

$$\sup\{\text{diam}(B) \mid B \in \mathcal{F}\} < \infty$$

and $A = \{x \mid \exists B_r(x) \in \mathcal{F} \text{ (the centers of the balls)}\}$.

Then there exists N_n countable collections G_1, G_2, \dots, G_{N_n} , each of which are disjoint (as in, the balls in each collection are disjoint. This does not mean $G_i \cap G_j = \emptyset$) in \mathcal{F} such that

$$A \subseteq \bigcup_{i=1}^{N_n} (\bigcup_{B \in G_i} B)$$

Proof. In the book ■

Theorem 0.18. (More on filling open sets with Balls)

Let μ be a Borel measure on \mathbb{R}^n , and let \mathcal{F} be any collection of nondegenerate closed balls. Let $A = \{x \mid \exists B_r(x) \in \mathcal{F} \text{ (again, the set of centers)}\}$.

Assume $\mu(A) < \infty$ (we do not assume A is μ -measurable) and $\inf\{r : B_r(a) \in \mathcal{F}\} = 0$ for any $a \in A$.

Then for every open set $U \subseteq \mathbb{R}^n$, there exists a countable collection G of disjoint balls in \mathcal{F} such that

$$1. \bigcup_{B \in G} B \subseteq U$$

$$2. \mu(A \cap U \setminus \bigcup_{B \in G} B) = 0.$$

Proof. Consider the collection $\mathcal{F}_1 = \{B \mid B \in \mathcal{F}, B \subseteq U, \text{diam}(B) \leq 1\}$.

$$A \cap U = \{x \mid \exists B_r(x) \in \mathcal{F}_1\}.$$

Apply the theorem to \mathcal{F}_1 . Then there exist G_1, G_2, \dots, G_{N_n} countable collections of disjoint balls (each) in \mathcal{F}_1 such that

$$A \cap U \subseteq \bigcup_{i=1}^{N_n} (\bigcup_{B \in G_i} B)$$

Then $\mu(A \cap U) \leq \sum_{i=1}^{N_n} \mu((A \cap U) \cap (\bigcup_{B \in G_i} B))$. So there exists an index $k \in \{1, 2, \dots, N_n\}$ such that

$$\mu((A \cap U) \cap (\bigcup_{B \in G_k} B)) \geq \frac{1}{N_n} \mu(A \cap U)$$

Write $G_k = \{B_i\}_{i=1}^\infty$. Then

$$\mu((A \cap U) \cap (\bigcup_{i=1}^\infty B_i)) \geq \frac{1}{N_n} \mu(A \cap U)$$

Let $q = 1 - \frac{1}{2N_n}$.

$$\mu(A \cap U) \leq \sup_{i=1}^{N_n} \mu((A \cap U) \cap (\cup_{B \in G_i} B)).$$

There exists an index $k \in \{1, 2, \dots, N_n\}$ such that

$$\mu((A \cap U) \cap (\cup_{B \in G_k} B)) \geq \frac{1}{N_n} \mu(A \cap U)$$

Lecture 8, 2/7/23

We continue with the proof. Some review.

μ is a Borel measure on \mathbb{R}^n . $\mathcal{F} = \{B \mid B \subseteq \mathbb{R}^n, B\text{-nondegenerate}\}$. $A = \{a \in \mathbb{R}^n \mid \exists B_r(a) \in \mathcal{F}\}$. Assume $\mu(A) < \infty$. Assume $\inf\{r \mid B_r(a) \in \mathcal{F}\} = 0$ for all $a \in A$.

Then for all open $U \subseteq \mathbb{R}^n$, there exists $G = \{B_i\}_{i=1}^\infty \subseteq \mathcal{F}$ collection of disjoint balls such that $\mu(A \cap U \setminus \cup_{i=1}^\infty B_i) = 0$, $\cup_{i=1}^\infty B_i \subseteq U$.

Let $\mathcal{F}_1 = \{B \in \mathcal{F} \mid B \subseteq U, \text{diam}(B) \leq 1\}$.

New set of centers = $A \cap U$. By Besicovitch, there exist N_n collections $G_1, G_2, \dots, G_{N_n} \subseteq \mathcal{F}_1$ such that $A \cap U \subseteq \cup_{i=1}^\infty \cup_{B \in G_i} B$.

Then $\mu(A \cap U) \leq \sum_{i=1}^{N_n} \mu(\cup_{B \in G_i} (B \cap A \cap U))$.

So there exists an index $k \in \{1, 2, \dots, N_n\}$ such that

$$\mu((A \cap U) \cap (\cup_{B \in G_k} B)) \geq \frac{1}{N_n} \mu(A \cap U)$$

Let $\nu = \mu|_A$ - Borel. Let $G_k = \{B_i\}_{i=1}^\infty$. Then $\mu((A \cap U) \cap (\cup_{i=1}^\infty B_i)) = \nu(U \cap \cup_{i=1}^\infty B_i) = \lim_{n \rightarrow \infty} \nu(U \cap \cup_{i=1}^n B_i)$. So there exists $m_1 \in \mathbb{N}$ such that

$$\nu(U \cap \cup_{i=1}^{m_1} B_i) = \mu((A \cap U) \cap \cup_{i=1}^{m_1} B_i) \geq \frac{1}{2N_n} \mu(A \cap U)$$

So

$$\begin{aligned} \mu(A \cap U \setminus \cup_{i=1}^{m_1} B_i) &= \mu(A \cap U) - \mu(A \cap U \cap \cup_{i=1}^{m_1} B_i) \\ &\leq \underbrace{\left(1 - \frac{1}{2N_n}\right)}_{0 < q < 1} \mu(A \cap U) \end{aligned}$$

Let $U_2 = U \setminus \cup_{i=1}^{m_1} B_i$, $\mathcal{F}_2 = \{B \in \mathcal{F}_1 \mid B \subseteq U_2, \text{diam}(B) \leq 1\}$.

kth step

We have $B_{m_{n-1}+1}, \dots, B_{m_k} \in \mathcal{F}_k$ such that

$$\mu(A \cap U \setminus \cup_{i=1}^{m_k} B_i) \leq q^k \mu(A \cap U)$$

Let $G = \{B_i\}$.

■

Differentiation of Radon Measures

Definition 0.9. Let μ and ν be Radon measures on \mathbb{R}^n . Define, for any $x \in \mathbb{R}^n$, the upper derivative of μ with respect to ν by

$$\overline{D}_\mu \nu(x) = \begin{cases} \limsup_{r \rightarrow 0^+} \frac{\nu(B_r(x))}{\mu(B_r(x))} & \text{if } \mu(B_r(x)) > 0, \forall r > 0 \\ +\infty & \mu(B_r(x)) = 0 \text{ for some } r \end{cases}$$

The lower derivative is defined similarly:

$$\underline{D}_\mu \nu(x) = \begin{cases} \liminf_{r \rightarrow 0^+} \frac{\nu(B_r(x))}{\mu(B_r(x))} & \text{if } \mu(B_r(x)) > 0, \forall r > 0 \\ +\infty & \mu(B_r(x)) = 0 \text{ for some } r \end{cases}$$

$\overline{D}_\mu \nu, \underline{D}_\mu \nu : \mathbb{R}^n \rightarrow [0, \infty]$. These are sometimes also called the upper/lower density.

We say that ν is differentiable with respect to μ at the point x if $\overline{D}_\mu \nu(x) = \underline{D}_\mu \nu(x) < \infty$

This leads to several questions/goals:

1. Study the set where $\underline{D}_\mu \nu(x) = \overline{D}_\mu \nu(x) < \infty$. Is it μ -a.e.?
2. Do we have $\nu(B) = \int_B \underline{D}_\mu \nu(x) d\mu$ for Borel sets $B \subseteq \mathbb{R}^n$?

For question 1, the answer is yes, μ -almost everywhere in \mathbb{R}^n . The answer to question 2 is also yes, subject to the additional condition $\nu \ll \mu$.

Lemma 1. Let μ and ν be Radon measures on \mathbb{R}^n . Let $0 < \alpha < +\infty$.

(i) If $A \subseteq \{x \in \mathbb{R}^n \mid \underline{D}_\mu \nu(x) \leq \alpha\}$, then $\nu(A) \leq \alpha \mu(A)$

(ii) If $A \subseteq \{x \in \mathbb{R}^n \mid \overline{D}_\mu \nu(x) \geq \alpha\}$, then $\nu(A) \geq \alpha \mu(A)$.

Proof. We can assume without loss of generality that $\mu(\mathbb{R}^n), \nu(\mathbb{R}^n) < \infty$. This is because $\nu(A \cap B_R(0)) \leq \alpha \mu(A \cap B_R(0))$, and as $R \rightarrow +\infty$, the left converges to $\nu(A)$, and the right converges to $\alpha \mu(A)$, $B_R(0) = \{|x| < R\}$.

Fix $\varepsilon > 0$.

- (i) Let U be any open set such that $A \subseteq U$. Consider the collection of closed balls $\mathcal{F} = \{B_r(x) \subseteq U \mid x \in A, \nu(B_r(x)) \leq (\alpha + \varepsilon)\mu(B_r(x)), \text{diam}(B) \leq 1\}$

For any $a \in A$, we have $\inf\{r \mid B_r(a) \in \mathcal{F}\} = 0$ because $\underline{D}_\mu \nu(x) \leq \alpha$. Set of centers = A .

By the theorem we just proved, there exists a countable collection $G = \{B_i\}_{i=1}^\infty$ of disjoint balls in \mathcal{F} such that

$$\nu(A \cap U \setminus \cup_{i=1}^\infty B_i) = 0$$

So

$$\nu(A) \leq \nu(\cup_{i=1}^{\infty} B_i) + \nu(A \setminus \cup_{i=1}^{\infty} B_i) = \nu(\cup_{i=1}^{\infty} B_i)$$

By disjointness, this is equal to

$$\begin{aligned} \sum_{i=1}^{\infty} \nu(B_i) &\leq \sum_{i=1}^{\infty} (\alpha + \varepsilon) \mu(B_i) \\ &= (\alpha + \varepsilon) \mu(\cup_{i=1}^{\infty} B_i) \\ &\leq (\alpha + \varepsilon) \mu(U) \end{aligned}$$

So $\nu(A) \leq (\alpha + \varepsilon) \mu(U)$ for all $\varepsilon > 0, U \supseteq A$.

In the limit as $\varepsilon \rightarrow 0^+$, we have $\nu(A) \leq \alpha \mu(U)$ for all $U \supseteq A$. So $\nu(A) \leq \alpha \inf\{\mu(U) \mid A \subseteq U, U \text{-open}\} = \alpha \mu(A)$.

This completes the proof of 1. Proof of 2 is similar? ■

Theorem 0.19. Let μ and ν be Radon measures on \mathbb{R}^n . Then

- (i) ν is differentiable with respect to μ almost everywhere in \mathbb{R}^n .*
- (ii) $\underline{D}_\mu \nu(x) = \overline{D}_\mu \nu(x) < \infty$ μ -almost everywhere in \mathbb{R}^n .*
- (iii) $D_\mu \nu$ is μ -measurable.*

Proof. Let $I = \{x \in \mathbb{R}^n \mid \overline{D}_\mu \nu(x) = \infty\}$.

With the lemma we have just proven, it is easy to see that $\mu(I) = 0$.

Assume $\mu(\mathbb{R}^n), \nu(\mathbb{R}^n) < \infty$. Fix any $\alpha > 0$. Then $I = \{x \in \mathbb{R}^n \mid \overline{D}_\mu \nu(x) \geq \alpha\}$. So $\nu(I) \geq \alpha \mu(I)$, so

$$\mu(I) \leq \frac{1}{\alpha} \nu(I) \leq \frac{\nu(\mathbb{R}^n)}{\alpha}$$

As let $\alpha \uparrow +\infty$, we get $\mu(I) < \infty$.

Lecture 9, 2/9/23

Theorem 0.20. Let μ and ν be Radon measures on \mathbb{R}^n . Then

- 1. $D_\mu \nu(x)$ exists and is finite μ -almost everywhere.*
- 2. $D_\mu \nu(x)$ is μ -measurable.*

Proof. Let $I = \{x \in \mathbb{R}^n \mid \overline{D}_\mu \nu(x) = \infty\}$. We know that $\mu(I) = 0$, so we just have to prove 2. Assume within the ball $B_R(0) = \{x \in \mathbb{R}^n \mid |x| < R\}$, $D_\mu \nu(x)$ exists μ -almost everywhere in $B_R(x)$.

Define $X_{INE}(R) = \{x \in B_R(0) \mid \text{either } D_\mu \nu(x) = \infty \text{ or } D_\mu \nu(x) \text{ DNE}\}$.

Note $X_{INE}(R) \subseteq B_R(x)$, $X_{INE}(1) \subseteq X_{INE}(2) \subseteq \cdots \subseteq X_{INE}(m) \subseteq \cdots$.

Now $\mu(X_{INE}(m)) = 0$ for all $m \in \mathbb{N}$, thus $\mu(X_{INE}(\infty)) \leq \sum_{m=1}^{\infty} \mu(X_{INE}(m)) = 0$.

Since X_{INE} is a null set, it is measurable.

Assume $\nu(\mathbb{R}^n), \mu(\mathbb{R}^n) < \infty$. Let $X_{NE} = \{x \in \mathbb{R}^n \mid D_\mu \nu(x) \text{ DNE}\} \implies \mu(X_{NE}) = 0$.

For $0 < a < b < \infty$, define $J(a, b) = \{x \in \mathbb{R}^n \mid \underline{D}_\mu \nu(x) \leq a \text{ or } \overline{D}_\mu \nu(x) \geq b\}$.

Then $X_{NE} \subseteq I \cup (\cup_{0 < r, q < \infty} J(r, q))$. $\mu(X_{NE}) \leq \mu(I) + \sum_{0 < r, q < \infty} \mu(J(r, q))$.

By a previous lemma, $\nu(J(a, b)) \leq a\mu(J(a, b))$, $\nu(J(a, b)) \geq b\mu(J(a, b))$.

Since $a\mu(J(a, b)) \geq b\mu(J(a, b))$ for all $a < b$, $\mu(J(a, b)) = 0$ for all $0 < a < b < \infty$. So $\mu(X_{NE}) = 0$.

■?

Idea: Express $D_\mu \nu(x)$ as the limit of a sequence of μ -measurable functions.

Claim.

$$D_\mu \nu(x) = \begin{cases} \lim_{r \rightarrow 0^+} \frac{\nu(B_r(x))}{\mu(B_r(x))} & x \in \mathbb{R}^n \setminus N \\ \infty \text{ or not defined in } N & \text{where } \mu(N) = 0 \end{cases}$$

Proof. For fixed $r > 0$, define $f_r(x) : \mathbb{R}^n \rightarrow [0, \infty]$ by

$$f_r(x) = \begin{cases} \frac{\nu(B_r(x))}{\mu(B_r(x))} & \text{if } \mu(B_r(x)) > 0 \\ \infty & \text{otherwise.} \end{cases}$$

Claim. For any $r > 0$, the function $f_r(x)$ is μ -measurable.

Claim. For every $r > 0$, the function $g_r(x) : \mathbb{R}^n \rightarrow [0, \infty)$ defined by $g_r(x) = \mu(B_r(x))$ is upper semicontinuous, i.e. if $x_k \rightarrow x$, then $\limsup g_r(x_k) \leq g_r(x)$.

Proof. of second claim

Let $x_k \rightarrow x$ be a convergent sequence.

Define $\varphi(x) = \chi_{B_r(x)}$ and note $\limsup_{x_k \rightarrow x} \varphi(x_k) \leq \varphi(x)$.

$\limsup_{x_k \rightarrow x} \chi_{B_r(x_k)}(y) \leq \chi_{B_r(x)}(y)$.

If $y \in B_r(x)$ then $\chi_{B_r(x)}(y) = 1$, so we're done.

If $y \notin B_r(x)$ then $|x - y| = r + \delta$ where $\delta > 0$ so there exists $K \in \mathbb{N}$ such that for all $k \geq K$, $|x_k - x| < \delta \implies y \notin B_r(x_k)$.

So $\chi_{B_r(x_k)}(y) = 0 \implies \limsup_{x_k \rightarrow x} \chi_{B_r(x)}(y) = 0$.

Note $\liminf_{x_k \rightarrow x} (1 - \chi_{B_r(x_k)}(y)) \geq 1 - \chi_{B_r(x)}(y)$.

Thus $\int_{B_r(x)} \liminf_{x_k \rightarrow x} (1 - \chi_{B_r(x_k)}(y)) d\mu(y) \geq \int_{B_r(x)} 1 - \chi_{B_r(x)}(y)$

By Fatou's lemma,

$$\liminf_{k \rightarrow \infty} \int_{B_r(x)} (1 - \chi_{B_r(x_k)}(y)) d\mu(y) \geq \int_{B_r(x)} (1 - \chi_{B_r(x)}(y))$$

Thus $\liminf_{k \rightarrow \infty} f(\mu(B_{2r}(x))) - \mu(B_r(x_k)) \geq \mu(B_{2r}(x)) - \mu(B_r(x))$

So $\limsup_{k \rightarrow \infty} \mu(B_r(x_k)) \leq \mu(B_r(x))$

■

Proof. of first claim

Denote $I_r = \{x \in \mathbb{R}^n \mid \mu(B_r(x)) = 0\}$.

$I_r \subseteq I$, $\mu(I) = 0$, so $\mu(I_r) = 0$ for all $r > 0$.

Furthermore, if $0 < r_1 < r_2$, $I_{r_2} \subset I_{r_1} \subset I$.

■

Claim.

$$D_\mu \nu(x) = \begin{cases} \lim_{k \rightarrow \infty} \frac{\nu(B_{\frac{1}{k}}(x))}{\mu(B_{\frac{1}{k}}(x))} & \mu(B_{\frac{1}{k}}(x)) > 0 \\ \infty & \mu(B_{\frac{1}{k}}(x)) = 0 \end{cases}$$

If $D_\mu \nu(x) < \infty$ μ -almost everywhere, then $D_\mu \nu(x)$ is μ -measurable.

Proof.

■

Definition 0.10. Let μ and ν be Borel measures on \mathbb{R}^n . Then

1. ν is absolutely continuous with respect to μ if $\mu(A) = 0 \implies \nu(A) = 0$ for all $A \subseteq \mathbb{R}^n$. We write $\nu \ll \mu$.
2. μ and ν are mutually singular if there exists a Borel set $B \subseteq \mathbb{R}^n$ such that $\nu(B) = \mu(\mathbb{R}^n \setminus B) = 0$. We write $\nu \perp \mu$.

Theorem 0.21. (Radon-Nikodym)

Let μ and ν be Radon measures on \mathbb{R}^n such that $\nu \ll \mu$. Then for any μ -measurable set $A \subseteq \mathbb{R}^n$, one has

1. $\nu(A) = \int_A D_\mu \nu(x) d\mu$
2. For any $f : \mathbb{R}^n \rightarrow \mathbb{R}$ μ -measurable, one has $\int_A f(x) d\nu = \int_A f(x) D_\mu \nu(x) d\mu$.

Proof. For the first part, observe that if A is μ -measurable, then A is also ν -measurable. Why? If there exists a Borel set $B \subseteq \mathbb{R}^n$ such that $A \subseteq B$, $\mu(A) = \mu(B)$, then $\mu(B \setminus A) = 0$. Since $\nu \ll \mu$, $\nu(B \setminus A) = 0$, so $B \setminus A$ is ν -measurable. So $A = B \setminus (B \setminus A)$, so A is ν -measurable.

Corollary 0.22. If f is μ -measurable, then it is ν -measurable.

Lecture 10, 2/21/23

Claim. If $A \subseteq \mathbb{R}^n$ is μ -measurable, then A is also ν -measurable. If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is μ -measurable, then f is also ν -measurable.

Proof.

Fix $t > 1$ (t -truncation argument)

Consider the sets

$$A_m = \{x \in A \mid t^m \leq D_\mu \nu(x) < t^{m+1}\}, \quad m \in \mathbb{Z}$$

Let $A = \cup_{m \in \mathbb{Z}} A_m \cup (A \cap \tilde{I}) \cup (A \cap \{x \mid D_\mu \nu(x) \text{ does not exist}\}) \cup (A \cap Z)$, where

$$\tilde{I} \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \mid D_\mu \nu(x) = +\infty\}$$

$$Z \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \mid D_\mu \nu(x) = 0\}$$

Note $\nu((A \cap \tilde{I}) \cup (A \cap \{x \mid D_\mu \nu(x) \text{ DNE}\}) \cup (A \cap Z)) = 0$, so $\nu(A) = \nu(\cup_{m \in \mathbb{Z}} A_m) = \sum_{m \in \mathbb{Z}} \nu(A_m)$.

Why is this set ν -null?

First, note $\mu(\{x \mid D_\mu \nu(x) \text{ DNE}\}) = 0$ implies $\nu(\{x \mid D_\mu \nu(x) \text{ DNE}\}) = 0$.

$\tilde{I} \subseteq I$, and $\mu(I) = 0$, so $\mu(\tilde{I}) = 0$, so $\mu(\tilde{I} \cap A) = 0$, so $\nu(\tilde{I} \cap A) = 0$.

Fix $\alpha > 0$. $\nu(C_\alpha) \leq \alpha \mu(C_\alpha)$. So $\nu(Z) \leq \nu(C_\alpha) \leq \alpha \mu(C_\alpha) \leq \alpha \mu(\mathbb{R}^n)$, which goes to 0.

So $\nu(Z) = 0$.

Thus $\nu(\tilde{I}) = \int_{\tilde{I}} D_\mu \nu(x) d\mu$, $\nu(Z) = \int_Z D_\mu \nu(x) d\mu = 0$.

Now $t^n \mu(A_m) \leq \nu(A_m) \leq t^{m+1} \mu(A_m)$.

So $\sum_{m \in \mathbb{Z}} t^m \mu(A_m) \leq \sum_{m \in \mathbb{Z}} \nu(A_m) \leq \sum_{m \in \mathbb{Z}} t^{m+1} \mu(A_m) = t \sum_{m \in \mathbb{Z}} t^m \mu(A_m)$.

So $\int_{A_m} t^m d\mu \leq \int_{A_m} D_\mu \nu(x) d\mu \leq \int_{A_m} t^{m+1} d\mu$.

By monotone convergence theorem, $\sum_{m \in \mathbb{Z}} \int_{A_m} D_\mu \nu(x) d\mu = \int_{\cup A_m} D_\mu \nu(x) d\mu$.

So $\sum_{m \in \mathbb{Z}} t^n \mu(A_m) \leq \int_{\cup A_m} D_\mu \nu(x) d\mu \leq t \sum_{m \in \mathbb{Z}} t^m \mu(A_m)$.

So $\frac{1}{t} \int_{\cup A_m} D_\mu \nu(x) d\mu \leq \nu(A) \leq t \int_{\cup A_m} D_\mu \nu(x) d\mu, \forall t > 1$.

So $\frac{1}{t} \int_A D_\mu \nu(x) d\mu \leq \nu(A) \leq t \int_A D_\mu \nu(x) d\mu$.

We let $t \rightarrow 1$ to get $\nu(A) = \int_A D_\mu \nu(x) d\mu$.

We now prove that $\int_A f(x) d\nu = \int_A f(x) D_\mu \nu(x) d\mu$.

Let $f^+(x) = \max(f(x), 0)$ and $f^-(x) = -\min(0, f(x))$. Note $f = f^+ - f^-$.

Assume $f(x) \geq 0$ for all $x \in \mathbb{R}^n$.

Lemma 2. Let X be a nonempty set, and let μ be a measure on X , and let $f : X \rightarrow [0, \infty]$ be μ -measurable. Then there exist μ -measurable sets $\{A_k\}_{k=1}^\infty$ such that

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k} \chi_{A_k}(x)$$

Remark. The $\{A_k\}$ need not be disjoint.

Proof. Define $A_1 = \{x \mid f(x) \geq 1\}$.

Assuming A_1, A_2, \dots, A_{k-1} are well-defined, define $A_k = \{x \in X \mid f(x) \geq \frac{1}{k} + \sum_{i=1}^{k-1} \frac{1}{i} \chi_{A_i}(x)\}$

By construction/induction, each A_k is μ -measurable.

We claim $f(x) = \sum_{k=1}^{\infty} \frac{1}{k} \chi_{A_k}(x)$.

1. Fix $n \in \mathbb{N}$. We claim $f(x) \geq \sum_{k=1}^n \frac{1}{k} \chi_{A_k}(x)$. Let $m \in \{1, \dots, n\}$ be the largest integer such that $x \in A_m$. Then

$$f(x) \geq \frac{1}{m} + \sum_{k=1}^{m-1} \frac{1}{k} \chi_{A_k}(x) = \sum_{k=1}^m \frac{1}{k} \chi_{A_k}(x) \leq \sum_{k=1}^n \frac{1}{k} \chi_{A_k}(x)$$

2. If $f(x) = \infty$, $x \in A_k$ for all $k \in \mathbb{N}$ so $\sum_{k=1}^m \frac{1}{k} \chi_{A_k}(x) = \sum_{k=1}^{\infty} \frac{1}{k} = \infty$. If $f(x) < \infty$, then there exists a sequence of naturals $\{n_\ell\}$ such that $x \notin A_{n_\ell}$. So

$$f(x) - \sum_{k=1}^{n_\ell-1} \frac{1}{k} \chi_{A_k}(x) < \frac{1}{n_\ell}, \quad \ell = 1, 2, \dots$$

We let $\ell \rightarrow \infty$ to get $f(x) = \sum_{k=1}^{\infty} \frac{1}{k} \chi_{A_k}(x)$

We are now ready to finish the proof.

Let $f_m = \sum_{k=1}^m \frac{1}{k} \chi_{A_k}(x)$, $A_k \subseteq A$.

Then $\int_A f_m(x) d\nu = \int_A f_m(x) D_\mu \nu(x) d\nu(x)$.

By MCT,

$$\int_A f(x) d\mu = \int_A f(x) D_\mu \nu(x) d\nu$$

■

Theorem 0.23. (Lebesgue Decomposition Theorem)

Let μ and ν be Radon measures on \mathbb{R}^n . Then there exists Radon measures ν_{ac}, ν_s on \mathbb{R}^n such that $\nu = \nu_{ac} + \nu_s$, and

1. $\nu_{ac} \ll \mu, \nu_s \perp \mu$
2. $D_\mu \nu_s(x) = 0$ for μ -almost every $x \in \mathbb{R}^n$. Further, for all Borel $B \subseteq \mathbb{R}^n$, we have

$$\nu(B) = \int_B D_\mu \nu_{ac}(x) d\mu + \nu_s(B)$$

Proof. Consider the collection $\mathcal{F} = \{A \subseteq \mathbb{R}^n \mid \nu(A) = 0, A \text{ Borel}\}$. Let $m = \inf\{\nu(A) \mid A \in \mathcal{F}\}$.

Lecture 11, 2/23/23

Theorem 0.24. (Lebesgue Decomposition Theorem).

Let μ and ν be Radon measures on \mathbb{R}^n . Then there exists ν_{ac}, ν_s Radon measures on \mathbb{R}^n such that $\nu = \nu_{ac} + \nu_s$,

1. $\nu_{ac} \ll \nu_s, \nu_s \perp \mu$
2. $D_\mu \nu_s(x) = 0$ for μ -almost every $x \in \mathbb{R}^n$, and for all Borel $B \subseteq \mathbb{R}^n$,

$$\nu(B) = \int_B D_\mu \nu_{ac}(x) d\mu + \nu_s(B)$$

Proof. First, assume μ, ν are finite. Consider the collection $\mathcal{F} = \{A \subseteq \mathbb{R}^n \mid \mu(\mathbb{R}^n \setminus A) = 0, A \text{ Borel}\}$.

Let $m = \inf\{\nu(A) \mid A \in \mathcal{F}\}$.

Note $0 \leq m < \infty$, so we can choose $A_k \in \mathcal{F}$ so that $m \leq \nu(A_k) < m + \frac{1}{k}$.

Define $B = \bigcap_{k=1}^\infty A_k$. By continuity from above, $\nu(B) = m$. Furthermore, B is Borel.

Note $\mu(\mathbb{R}^n \setminus B) = \mu(\mathbb{R}^n \setminus \bigcap_{k=1}^\infty A_k) = \mu(\bigcup_{k=1}^\infty (\mathbb{R}^n \setminus A_k)) \leq \sum_{k=1}^\infty \mu(\mathbb{R}^n \setminus A_k) = 0$.

So $B \in \mathcal{F}$.

Note μ is supported on B . Define $\nu_{ac} = \nu|_B$, $\nu_s = \nu_{\mathbb{R}^n \setminus B}$. Obviously, $\nu_s \perp \nu_{ac}$.

We claim $\nu_{ac} \ll \mu$. Towards contradiction, suppose there is a $C \subseteq \mathbb{R}^n$ such that $\mu(C) = 0$ but $\nu_{ac} > 0$.

Without loss of generality, assume C is Borel. Further, assume $C \subseteq B$. Then $B \setminus C \in \mathcal{F}$, since $\mu(\mathbb{R}^n \setminus (B \setminus C)) \leq \mu(\mathbb{R}^n \setminus B) + \mu(C) = 0$. Furthermore, $\nu(B \setminus C) = \nu(B) - \nu_{ac}(C) < \nu(B)$, a contradiction.

Now we show $D_\mu \nu_s(x) = 0$ for μ -almost every $x \in \mathbb{R}^n$.

For every $\alpha > 0$, set $C_\alpha = \{x \in \mathbb{R}^n \mid \overline{D_\mu \nu_s}(x) \geq \alpha\}$.

By lemma, $\nu_s(C_\alpha) \geq \alpha \mu(C_\alpha)$. Clearly, $C_\alpha = (C_\alpha \cap B) \cup (C_\alpha \setminus B)$. Note $\mu(C_\alpha \setminus B) = 0$, $\mu(C_\alpha \cap B) = 0$. So $\mu(C_\alpha) = 0$.

Since $\{x \mid D_\mu \nu_s(x) > 0\} = \bigcup_{k=1}^\infty C_{\frac{1}{k}}$, $\mu(\{x \mid D_\mu \nu_s(x) > 0\}) = 0$.

Let $C_m = \{x \mid |x| < m\}$. There exists $B_m \subseteq C_m$ such that

- $\mu(C_m \setminus B_m) = 0$
- $\nu_{ac}^m = \nu_{C_m} \ll \mu$
- $\nu_s^m = \nu_{C_m \setminus B_m} \perp \mu$
- $D_\mu \nu_{ac}^m(x) = 0$ for μ -almost every $x \in C_m$

Set $B = \bigcup_{i=1}^\infty B_i$, $\nu_{ac} = \nu_B$, $\nu_s = \nu_{\mathbb{R}^n \setminus B}$

1. $\mu(\mathbb{R}^n \setminus B) = \mu(\bigcup_{k=1}^\infty C_k \setminus \bigcup_{k=1}^\infty B_k) \leq \mu(\bigcup_{k=1}^\infty (C_k \setminus B_k)) \leq \sum_{k=1}^\infty \mu(C_k \setminus B_k) = 0$

2. We claim $\nu_s \perp \mu$, $\nu_{ac} \ll \mu$. If $\mu(A) = 0$, $\mu(A \cap C_m) = 0$ for all $m \in \mathbb{N}$. So $\nu_{ac}(A) = \nu_{ac}(\cup_{k=1}^{\infty} C_k \cap A) \leq \sum_{k=1}^{\infty} \nu_{ac}(C_k \cap A) = \sum_{k=1}^{\infty} \nu(B \cap C_k \cap A)$.
 Fix: New $\overline{B_m} = \cup_{k=1}^m B_k$. This will give us $\sum_{k=1}^{\infty} \nu_{ac}^k(C_k \cap A) = 0$.

■?

Differentiation of Integrals, Lebesgue's Point

Definition 0.11.

$$L^1(\mathbb{R}^n, \mu) \stackrel{\text{def}}{=} \{f : \mathbb{R}^n \rightarrow [-\infty, \infty] \mid f \text{ is } \mu\text{-measurable, } \int_{\mathbb{R}^n} |f| d\mu < \infty\}$$

$$L^1_{loc}(\mathbb{R}^n, \mu) \stackrel{\text{def}}{=} \{f : \mathbb{R}^n \rightarrow [-\infty, \infty] \mid f \text{ is } \mu\text{-measurable, } \int_K |f| d\mu < \infty \text{ for all } K \text{ compact}\}$$

Theorem 0.25. (Lebesgue-Besicovitch differentiation theorem)

Let μ be a Radon measure on \mathbb{R}^n and let $f \in L^1_{loc}(\mathbb{R}^n, \mu)$. Then

$$\lim_{r \rightarrow 0^+} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} f(y) dy = f(x)$$

for μ -almost every $x \in \mathbb{R}^n$.

Proof. Consider $f^+ = \max(f, 0)$ and $f^- = -\min(0, f)$. Note $f = f^+ - f^-$, f^+ and f^- are μ -measurable, $f^+, f^- \in L^1_{loc}(\mathbb{R}^n, \mu)$.

Define $\nu^+, \nu^- : 2^{\mathbb{R}^n} \rightarrow [0, \infty]$ as follows:

$$\begin{aligned} \nu^{\pm}(B) &\stackrel{\text{def}}{=} \int_B f^{\pm}(x) d\mu, \text{ where } B \text{ is Borel} \\ \nu^{\pm}(A) &\stackrel{\text{def}}{=} \inf\{\nu^{\pm}(B) \mid A \subseteq B, B \text{ Borel}\} \end{aligned}$$

Lemma 3. ν^{\pm} are Radon measures with

- $\nu^{\pm} \ll \mu$
- $D_{\mu}\nu^{\pm}(x) = f^{\pm}(x)$ for μ -almost every $x \in \mathbb{R}^n$.

Lemma 4. Let $g \in L^1_{loc}(\mathbb{R}^n, \mu)$, where μ is a Borel measure on \mathbb{R}^n . Then

$$\begin{aligned} \nu(B) &= \int_B g(x) d\mu, B \text{ Borel} \\ \nu(A) &= \inf\{\nu(B) \mid A \subseteq B, B \text{ Borel}\} \end{aligned}$$

is a Radon measure, $\nu \ll \mu$, $D_{\mu}\nu(x) = g(x)$ for μ -almost every $x \in \mathbb{R}^n$.

Lecture 12, 2/28/23

Lemma 5. Let μ be a Borel measure on \mathbb{R}^n and let $f : \mathbb{R}^n \rightarrow [0, \infty]$ be such that $f \in L^1_{loc}(\mathbb{R}^n, \mu)$. Define $\nu(B) = \int_B f d\mu$ for all B Borel, $\nu(A) = \inf\{\nu(B) \mid A \subseteq B, B \text{ Borel}\}$. Then:

1. ν is a Radon measure on \mathbb{R}^n (even if μ is just Borel)
2. If μ is Borel-regular, then $\nu \ll \mu$ and $\nu(A) = \int_A f d\mu$ for all $A \subseteq \mathbb{R}^n$ μ -measurable.
3. If μ is Radon, then $D_\mu \nu(x) = f(x)$ for μ -almost every $x \in \mathbb{R}^n$.

Proof.

1. ν is a Borel-regular measure on \mathbb{R}^n , by homework 2, problem 2.
2. Assume A is μ -measurable.

Then there exists a Borel set B such that $A \subseteq B$, $\mu(A) = \mu(B)$. So $\mu(B \setminus A) = \mu(B) - \mu(A) = 0$.

$$\nu(B) = \int_B f d\mu = \int_A f d\mu + \int_{B \setminus A} f d\mu = \int_A f d\mu$$

$$\nu(A) = \inf\{\nu(\tilde{B}) \mid A \subseteq \tilde{B}, \tilde{B} \text{ Borel}\}.$$

So $\int_{\tilde{B}} f d\mu = \int_{\tilde{B} \setminus A} f d\mu + \int_A f d\mu \geq \int_A f d\mu = \nu(B)$. So $\nu(B) = \inf\{\nu(\tilde{B})\} = \nu(A)$.

Assume $\mu(A) = 0$. Then $\nu(A) = \int_A f d\mu = 0$, so $\nu \ll \mu$.

3. We need a lemma

Lemma 6. Let X be a nonempty set, and let μ be a measure on X . Assume $f \in L^1_{loc}(X, \mu)$. If $\int_A f(x) d\mu = 0$ for all A μ -measurable, then $f \equiv 0$ μ -almost everywhere.

Proof. Assume f is nonnegative μ -almost everywhere. Let $A_n = \{x \in X \mid f(x) \geq \frac{1}{n}\}$. For all n , A_n is μ -measurable. Note $\{x \in X \mid f(x) > 0\} = \cup_{n=1}^{\infty} A_n$. We have

$$0 = \int_{A_n} f d\mu \geq \int_{A_n} \frac{1}{n} d\mu = \frac{1}{n} \mu(A_n) \geq 0$$

So $\mu(A_n) = 0$. Thus $\mu(\{x \in X \mid f(x) > 0\}) = 0$.

Now consider any $f \in L^1_{loc}(X, \mu)$. Let $f = f^+ - f^-$. Both $f^+, f^- = 0$ μ -almost everywhere. ■

We now proceed with the proof. For all $A \subseteq \mathbb{R}^n$ μ -measurable and K compact, we have $\int_{A \cap K} f d\mu = \nu(A \cap K)$, and by the Radon-Nikodym theorem this equals $\int_{A \cap K} D_\mu \nu(x) d\mu$. Thus $\int_{A \cap K} D_\mu \nu(x) - f(x) d\mu = 0$. So by lemma 2, $D_\mu \nu(x) = f(x)$ μ -almost everywhere. Take $K = B_1, B_2, \dots$

Theorem 0.26. (Lebesgue-Besicovitch Theorem):

Let μ be a Radon measure on \mathbb{R}^n and let $f \in L^1_{loc}(\mathbb{R}^n, \mu)$. Then $\lim_{r \rightarrow 0^+} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} f d\mu = f(x)$ μ -almost everywhere.

Proof. Write $f = f^+ - f^-$ and note $f^\pm \in L^1_{loc}(\mathbb{R}^n, \mu)$. Define $\nu^\pm(B) = \int_B f^\pm d\mu$ for all B Borel, and as in lemma 1, $\nu^\pm(A) = \inf\{\nu^\pm(B) \mid A \subseteq B, B \text{ Borel}\}$.

Then $\nu^\pm \ll \mu$, ν^\pm are Radon. Note that $D_\mu \nu^\pm(x) = f^\pm(x)$ μ -almost everywhere. So

$$\begin{aligned} \lim_{r \rightarrow 0^+} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} f d\mu &= \lim_{r \rightarrow 0^+} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} (f^+ - f^-) d\mu \\ &= \lim_{r \rightarrow 0^+} (\nu^+(B(x, r)) - \nu^-(B(x, r))) \\ &= \lim_{r \rightarrow 0^+} \frac{\nu^+(B(x, r))}{\mu(B(x, r))} - \lim_{r \rightarrow 0^+} \frac{\nu^-(B(x, r))}{\mu(B(x, r))} \\ &= D_\mu \nu^+(x) - D_\mu \nu^-(x) \\ &= f^+(x) - f^-(x) \\ &= f(x) \mu - a.e. \end{aligned}$$

■

Theorem 0.27. Under the conditions of the previous theorem, we have

$$\lim_{r \rightarrow 0^+} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| d\mu = 0$$

μ -almost everywhere.

Proof. We have

$$\begin{aligned} \left| \frac{1}{\mu(B(x, r))} \int_{B(x, r)} f(y) d\mu - f(x) \right| &= \left| \frac{1}{\mu(B(x, r))} \int_{B(x, r)} f(y) - f(x) d\mu \right| \\ &\leq \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| d\mu \\ &\rightarrow 0 \end{aligned}$$

And then the previous theorem finishes the proof.

Lecture 13, 3/2/23

Theorem 0.28. Let μ be a Radon measure on \mathbb{R}^n and let $f \in L^1_{loc}(\mathbb{R}^n, \mu)$. Then

$$\underbrace{\lim_{r \rightarrow 0^+} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| d\mu}_{*} = 0$$

for μ -almost every $x \in \mathbb{R}^n$.

Definition 0.12. Points $x \in \mathbb{R}^n$ that satisfy the above equation are called Lebesgue points of f (for the measure μ)

Proof. Let $\mathbb{Q} = \{r_i\}_{i=1}^{\infty} \subseteq \mathbb{R}$ be an enumeration of the rationals. We know \mathbb{Q} is dense in \mathbb{R} .

For every $i \in \mathbb{N}$, $\lim_{r \rightarrow 0^+} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f(y) - f(r_i)| d\mu = |f(x) - r_i|$ is satisfied μ -almost everywhere.

Let $A_i \subseteq \mathbb{R}^n$ be points where the equation is satisfied. Then $\mu(\mathbb{R}^n \setminus A_i) = 0$.

We have

$$\phi_i(x) = |f(x) - r_i| \in L^1_{loc}(\mathbb{R}^n, \mu)$$

because

$$|\phi_i(x)| \leq |f(x)| + |r_i| \in L^1_{loc}(\mathbb{R}^n, \mu)$$

Set $A = \cap_{i=1}^{\infty} A_i$. We claim that $*$ is satisfied for all $x \in A$. Note

$$\mu(\mathbb{R}^n \setminus A) = \mu(\mathbb{R}^n \setminus \cap_{i=1}^{\infty} A_i) = \mu(\cup_{i=1}^{\infty} (\mathbb{R}^n \setminus A_i)) \leq \sum_{i=1}^{\infty} \mu(\mathbb{R}^n \setminus A_i) = 0$$

Fix $\varepsilon > 0$. Choose r_m such that $|f(x) - r_m| < \varepsilon$.

Then

$$|f(y) - f(x)| \leq |f(y) - r_m| + |f(x) - r_m|$$

so

$$\frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| d\mu \leq 2|f(x) - r_m| < 2\varepsilon$$

We take the limit as $\varepsilon \rightarrow 0$ to get

$$\limsup_{r \rightarrow 0^+} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| d\mu = 0$$

thus

$$\lim_{r \rightarrow 0^+} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| d\mu = 0$$



Definition 0.13. A measure μ has the doubling property if there exists a constant $C > 0$, such that

$$0 < \mu(B(x, 2r)) \leq C\mu(B(x, r)) < \infty$$

Remark. Note that $\mu = \lambda^n$ has the doubling property: there exists c_1, c_2 such that $\mu(B(x, c_1 r)) \leq c_2 \mu(B(x, r))$ for all $r > 0, x \in \mathbb{R}^n$. Simply choose $c_1 = 2, c_2 = 2^n$. Because $c_1 > 1$, there exists a $k \in \mathbb{N}$ such that $c_1^k \geq 2$, $\mu(B(x, 2r)) \leq \mu(B(x, c_1^k r)) \leq \cdots \leq c_1^k \mu(B(x, r))$

Theorem 0.29. Let μ be a Radon measure on \mathbb{R}^n with the doubling property. Choose $f \in L^1_{loc}(\mathbb{R}^n, \mu)$. Then $\lim_{r \rightarrow 0^+} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| d\mu = 0$ for μ -almost every $x \in \mathbb{R}^n$.

Proof.

$$\begin{aligned} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| d\mu &\leq \frac{1}{\mu(B(x, r))} \int_{B(x, 2r)} |f(y) - f(x)| d\mu \\ &\leq \frac{c_1^k}{\mu(B(x, 2r))} \int_{B(x, 2r)} |f(y) - f(x)| d\mu \end{aligned}$$

This goes to 0 as $r \rightarrow 0$.

Theorem 0.30. (Density)

Let μ be a Radon measure on \mathbb{R}^n and let $E \subseteq \mathbb{R}^n$ be μ -measurable. Then

1. $\lim_{r \rightarrow 0^+} \frac{\mu(E \cap B(x, r))}{\mu(B(x, r))} = 1$ for μ -almost every $x \in E$
2. $\lim_{r \rightarrow 0^+} \frac{\mu(E \cap B(x, r))}{\mu(B(x, r))} = 0$ for μ -almost every $x \in E^c$.

Proof. Apply previous theorem to $f(x) = \chi_E(x)$.

Choose $R_1 < R_2 < \cdots < R_k < \cdots$ such that $\lim_{k \rightarrow \infty} R_k = \infty$, $\mu(\{x \mid |x| = R_i\}) = 0$. Then do the analysis for the sets $A_k = \{x \mid R_k < |x| < R_{k+1}\}$, $A_0 = \{x \mid |x| < R_1\}$. Let $\nu_s = \sum \nu_s^i + \nu|_{\cup_{k=1}^{\infty} \{x \mid |x| = R_k\}}$, $\nu_{ac} = \sum \nu_{ac}^i$. ■

Lemma 7. Let μ be a Radon measure on \mathbb{R}^n . Fix $x \in \mathbb{R}^n$. Then $\mu(\partial(B(x, r))) = 0$ for $r \in \mathbb{R}^n \setminus \{r_1, r_2, \dots\}$.

Proof. It is sufficient to prove the statement within any fixed ball $B(x, R)$, $R > 0$.

Then take $R = 1, 2, \dots$, $\mathbb{R}^n = \cup_{r=1}^{\infty} B(x, r)$.

Define $A_k = \{r \in [0, R] \mid \mu(\partial B(x, r)) \geq \frac{1}{k}\}$, $k = 1, 2, \dots$. Note $\{r \in [0, R] \mid \mu(\partial B(x, r)) > 0\} = \cup_{k=1}^{\infty} A_k$.

$\infty > \mu(B(x, R)) \geq |A_k| \frac{1}{k}, |A_k| \leq k\mu(B(x, R))$

Lecture 14, 3/7/23

Increasing, BV, AV, and Lipschitz Functions

- BV: Bounded Variation
- AC: Absolutely Continuous

Almost everywhere differentiability of nondecreasing functions.

Theorem 0.31. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be increasing. Then $f'(x)$ exists and is finite λ -almost everywhere. Moreover, $f' : \mathbb{R} \rightarrow \mathbb{R}$ is Lebesgue-measurable ($f'(x) \geq 0$) and

$$\int_a^b f'(x) dx \leq f(b) - f(a)$$

for all $-\infty < b < a < \infty$.

Lemma 8. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be increasing. Then the set of discontinuous points of f is at most countable.

Proof. Since f is increasing, for all $x \in \mathbb{R}$, there exist $x+, x-$ such that $f(x+), f(x-)$ are finite, and

- $f(x-) = \lim_{\varepsilon \rightarrow 0^+} f(x - \varepsilon) \leq f(x)$
- $f(x+) = \lim_{\varepsilon \rightarrow 0^+} f(x + \varepsilon) \leq f(y)$ for all $y > x$.
- $f(x-) \leq f(x) \leq f(x+)$

$\lim_{y \rightarrow x} f(y)$ exists if $f(x-) = f(x+) = f(x)$. If $f(x+) > f(x-)$, then f is discontinuous at x . The intervals $(f(x-), f(x+))$ are disjoint for all discontinuities x . Each contains a unique rational, giving an injection to a countable set. So there are at most countably many discontinuities. ■

Given $f : \mathbb{R} \rightarrow \mathbb{R}$ increasing, define $g : \mathbb{R} \rightarrow \mathbb{R}$ as follows:

$$g(x) = \begin{cases} f(x) & f \text{ continuous at } x \\ f(x+) & \text{otherwise} \end{cases}$$

In other words, $g(x) = f(x+)$.

Lemma 9. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be increasing. Let $g(x) = f(x+)$ for all $x \in \mathbb{R}$. Then, if $g'(x)$ exists, then $f'(x)$ exists, and $f'(x) = g'(x)$.

Proof. :

Homework assignment ■

Theorem 0.32. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be increasing and right-continuous. Then $f'(x)$ exists and is finite λ -almost everywhere. Moreover, $f : \mathbb{R} \rightarrow \mathbb{R}$ is Lebesgue-measurable ($f'(x) \geq 0$) and

$$\int_a^b f'(x) dx \leq f(b) - f(a)$$

for all $-\infty < b < a < \infty$.

Proof. $\nu(A) = \inf\{\sum_{i=1}^{\infty} (f(b_i) - f(a_i)) : A \subseteq \cup_{i=1}^{\infty} (a_i, b_i]\}$ is a Radon measure on \mathbb{R} . Furthermore, $\nu(a, b] = f(b) - f(a)$ for all $a < b$, so $\nu(\{a\}) = f(a) - f(a')$ for all $a \in \mathbb{R}$. By the Lebesgue Decomposition Theorem, $\nu = \nu_{ac} + \nu_s$, where

- $\nu_{ac} \ll \lambda, \nu_s \perp \lambda$
- $D_\lambda \nu_s(x) = 0$ λ -almost everywhere in \mathbb{R} .
- $D_\lambda \nu_{ac}(x)$ and $D_\lambda \nu_s(x)$ are nonnegative, λ -measurable.
- $\nu_{ac} = \int_A D_\lambda \nu_{ac}(x) d\lambda = \int_A D_\lambda \nu_{ac}(x) dx$ for all λ -measurable $A \subseteq \mathbb{R}$.

I.e. there exists $B \subseteq \mathbb{R}$ Borel such that $\nu_{ac} = \nu|_B, \lambda(\mathbb{R} \setminus B) = 0, \nu_s = \nu|_{(\mathbb{R} \setminus B)}$
Now we need a proposition (PROOF NOT COMPLETE).

Proposition 3. Let $A \subseteq \mathbb{R}$ be the set of all points $x \in \mathbb{R}$, where

- $f(x+) = f(x-)$
- There exists $D_\lambda \nu_{ac}(x) < \infty, D_\lambda \nu_s(x) = 0$

Then $f'(x)$ exists and equals $D_\lambda \nu_{ac}(x)$.

Proof. By definition, we have

$$\begin{aligned} f'(x+) &= \lim_{h \rightarrow 0^+} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{\nu((x, x+h])}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{\nu_{ac}((x, x+h]) + \nu_s((x, x+h])}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{\nu_{ac}((x, x+h])}{h} + \lim_{h \rightarrow 0^+} \frac{\nu_s((x, x+h])}{h} \end{aligned}$$

Observe

$$0 \leq \lim_{h \rightarrow 0^+} \frac{\nu_s((x, x+h])}{h} \leq 2 \lim_{h \rightarrow 0^+} \frac{\nu_s((x-h, x+h])}{h} = 2D_\lambda \nu_s(x) = 0$$

So $\lim_{h \rightarrow 0^+} \frac{\nu_s((x, x+h])}{h} = 0$. Further,

$$\lim_{h \rightarrow 0^+} \frac{\nu_{ac}((x, x+h])}{h} = \lim_{h \rightarrow 0^+} \frac{\int_{(x, x+h]} D_\lambda \nu_{ac}(t) dt}{h} = \lim_{h \rightarrow 0^+} \frac{\int_{[x, x+h]} D_\lambda \nu_{ac}(t) dt}{h} = D_\lambda \nu_{ac}(x)$$

So $f'(x+) = D_\lambda \nu_{ac}(x) = 0$. A similar computation shows that $f(x-) = D_\lambda \nu_{ac}(x)$. ■

We proceed to the proof of the theorem.

Proof. We now need to show $\int_a^b f'(x) dx \leq f(b) - f(a)$. Indeed,

$$\begin{aligned} \int_a^b f'(x) dx &= \int_a^b D_\lambda \nu_{ac}(x) dx \\ &= \nu_{ac}((a, b]) \\ &\leq \nu((a, b]) \\ &= f(b) - f(a) \end{aligned}$$
■

We now show the theorem from the beginning.

Proof. Let $g(x) = f(x+)$. Note that $\int_a^b f'(x) dx = \int_a^b g'(x)$ by lemma, and by previous theorem this is less than or equal to $g(b) - g(a)$ for all $a < b$.

Furthermore, $g(a) = f(a+) \geq f(a)$. Choose (b_n) such that $g(b_n) = f(b_n)$, and $b_n \uparrow b$.
 $\int_a^{b_n} f'(x) dx = \int_a^{b_n} g'(x) dx \leq g(b_n) - g(a) \leq f(b_n) - f(a) \leq f(b) - f(a)$ ■

Example 0.2. Here are some examples of strict inequality: $f(x) = \begin{cases} 0 & x < 1 \\ 1 & x \geq 1 \end{cases} = \chi_{[1, \infty)}$

$f'(x) = 0$ for all $x \in (0, 1)$, and $f(1) - f(0) = 1$.

We don't even need a jump point, cf the Devil's staircase.

Lecture 15, 3/9/23

BV (Bounded Variation) Functions

Definition 0.14. Let $a, b \in \mathbb{R}, a < b$. A partition of $[a, b]$ is a finite sequence

$$P : a = x_0 < x_1 < \cdots < x_n = b$$

For $f : [a, b] \rightarrow \mathbb{R}$, define

$$V(f, P) = \sum_{i=1}^{n-1} |f(x_{i+1}) - f(x_i)|$$

and call it the variation of f at P . Define

$$\mathcal{V}_a^b(f) = \sup_P \mathcal{V}(f, P)$$

and call it the total variation of f over $[a, b]$. The function f is said to be of bounded variation in $[a, b]$ if $\mathcal{V}_a^b(f) < \infty$. The family of all such functions is denoted by $BV[a, b]$.

Theorem 0.33. Assume $f, g \in BV[a, b]$. Then

1. $f \pm g \in BV[a, b]$ and $\mathcal{V}_a^b(f \pm g) = \mathcal{V}_a^b f \pm \mathcal{V}_a^b g$
2. For all $c \in \mathbb{R}$, $cf \in BV[a, b]$ and $\mathcal{V}_a^b(cf) = |c| \mathcal{V}_a^b(f)$
3. f is bounded in $[a, b]$
4. If $|f(x)|, |g(x)| < M$ for all $x \in [a, b]$, then $\mathcal{V}_a^b(fg) \leq M(\mathcal{V}_a^b(f) + \mathcal{V}_a^b(g))$
5. $\mathcal{V}_a^b(f) \geq |f(b) - f(a)|$
6. $f \in BV[a, b] \iff f \in BV[a, c], f \in BV[c, b]$ with $\mathcal{V}_a^c(f) + \mathcal{V}_c^b(f) = \mathcal{V}_a^b(f)$ for all $c \in [a, b]$.
7. The function $v(x) = \mathcal{V}_a^x(f) : [a, b] \rightarrow \mathbb{R}$ is increasing and $v(x) \geq |f(x) - f(a)|$

Proof. 1. Triangle inequality

2. By definition

3. For all $x \in [a, b]$, $|f(x) - f(a)| + |f(x) - f(b)| \leq \mathcal{V}_a^b(f)$ so $|f(x)| \leq |f(a)| + \mathcal{V}_a^b(f) < \infty$

4. For all $P : a = x_0 < x_1 < \dots < x_n = b$,

$$\begin{aligned} \mathcal{V}_a^b(fg) &= \sum_{i=0}^{n-1} |f(x_{i+1})g(x_{i+1}) - f(x_i)g(x_i)| \\ &= \sum_{i=0}^{n-1} |g(x_{i+1})(f(x_{i+1}) - f(x_i)) + f(x_i)(g(x_{i+1}) - g(x_i))| \\ &\leq \sum_{i=0}^{n-1} |g(x_{i+1})(f(x_{i+1}) - f(x_i))| + \sum_{i=0}^{n-1} |f(x_i)(g(x_{i+1}) - g(x_i))| \\ &\leq M \sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)| + M \sum_{i=0}^{n-1} |g(x_{i+1}) - g(x_i)| \\ &= M \mathcal{V}_a^b(f) + M \mathcal{V}_a^b(g) \end{aligned}$$

5. Triangle Inequality

6. Given a partition P of $[a, b]$, by adding one (thus finitely many) points to P , the variational sum $\mathcal{V}(f, P)$ may only increase.

So, assume $f \in BV[a, b]$. We prove that $\mathcal{V}_a^c(f) + \mathcal{V}_c^b(f) = \mathcal{V}_a^b(f)$ for all $c \in (a, b)$.

First, observe that $f \in BV[a, c]$ and $f \in BV[c, b]$. $\mathcal{V}_a^c(f) + \mathcal{V}_c^b(f) \leq \mathcal{V}_a^b(f)$.

Fix $\varepsilon > 0$. There exist $P : a = x_0 < x_1 < \cdots < x_n = b$ such that $\mathcal{V}(f, P) > \mathcal{V}_a^b(f) - \varepsilon$.

$$\mathcal{V}_a^c(f) + \mathcal{V}_c^b(f) \geq \mathcal{V}(f, P_1) + \mathcal{V}(f, P_2) \geq \mathcal{V}_a^b(f, P) > \mathcal{V}_a^b(f) - \varepsilon$$

We take the limit as $\varepsilon \rightarrow 0$ to get the desired result.

For the opposite inequality, choose P_1 for $[a, c]$ and P_2 for $[c, b]$ such that

$$\mathcal{V}(f, P_1) > \mathcal{V}_a^c(f) - \varepsilon, \mathcal{V}(f, P_2) > \mathcal{V}_c^b(f) - \varepsilon$$

Then $\mathcal{V}_a^b(f) \geq \mathcal{V}(f, P_1 \cup P_2) = \mathcal{V}(f, P_1) + \mathcal{V}(f, P_2) > \mathcal{V}_a^c(f) + \mathcal{V}_c^b(f) - 2\varepsilon$.

7. If $x < y$, $x, y \in [a, b]$, $v(y) = \mathcal{V}_a^y(f) = \mathcal{V}_a^x(f) + \mathcal{V}_x^y(f) \geq \mathcal{V}_a^x(f) = v(x)$. $v(y) - f(x) = \mathcal{V}_x^y(f) \geq |f(y) - f(x)|$.

Theorem 0.34. For any $f \in BV[a, b]$, there exists $f_1, f_2 : [a, b] \rightarrow \mathbb{R}$ increasing such that $f = f_1 - f_2$

Remark. Denote $\text{Mon}^+[a, b] = \{f : [a, b] \rightarrow \mathbb{R} \mid f \text{ increasing in } [a, b]\}$.

$BV[a, b] \subseteq \text{Mon}^+[a, b] - \text{Mon}^+[a, b]$ If $f \in \text{Mon}^+[a, b]$, then $\mathcal{V}_a^b(f) = f(b) - f(a)$.

So $\text{Mon}^+[a, b] \subseteq BV[a, b]$.

On the other hand, we have $\text{Mon}^+[a, b] - \text{Mon}^+[a, b] \subseteq BV[a, b]$. In sum, we have $BV[a, b] = \text{Mon}^+[a, b] - \text{Mon}^+[a, b]$

Proof. $f(x) = v(x) - (v(x) - f(x))$ for all $x \in [a, b]$.

Define $f_1 = v, f_2 = v - f$. If $x < y$, $v(y) - f(y) \geq v(x) - f(x), v(y) - v(x) \geq f(y) - f(x)$. ■

Theorem 0.35. For any $f \in BV[a, b]$, $f'(x)$ exists and is finite almost everywhere in $[a, b]$. Moreover, $f'(x)$ is Lebesgue-measurable.

Proof. $f = f_1 - f_2$, both increasing on $[a, b]$. We have $f_1'(x)$ and $f_2'(x)$ exists and are finite almost everywhere in $[a, b]$ and are Lebesgue-measurable. Thus $f'(x) = f_1'(x) - f_2'(x)$ almost everywhere on $[a, b]$. ■

Absolutely Continuous Functions: AC[a,b]

Definition 0.15. Let $f : [a, b] \rightarrow \mathbb{R}$. Then f is absolutely continuous on $[a, b]$ if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\{(a_i, b_i)\}_{i=1}^{\infty} \subseteq [a, b]$ disjoint with $\lambda(\cup_{i=1}^{\infty} (a_i, b_i)) < \delta$, $\sum_{i=1}^{\infty} |f(b_i) - f(a_i)| < \varepsilon$.

We write $f \in \text{AC}[a, b]$

Example 0.3. 1. $f(x) = cx$

2. $f(x) \in C^1[a, b]$

3. $\text{Lip}[a, b] = \{f : [a, b] \rightarrow \mathbb{R} \mid \exists M \text{ such that } |f(x) - f(y)| \leq M|x - y| \text{ for all } x, y \in [a, b]\}$.

If $f \in C^1[a, b]$ then $|f'(x)| \leq M$ for all $x \in [a, b]$.

By MVT, $|\frac{f(y)-f(x)}{y-x}| = |f'(z)| \leq M$

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Remark. If we take just one interval in the definition of absolute continuity, then we reproduce uniform continuity.

Class 1: Lipschitz functions on $[a, b]$ are absolutely continuous on $[a, b]$.

Class 2: Continuously differentiable functions on $[a, b]$, called $C^1[a, b]$.

Proposition 4. $C^1[a, b] \subseteq \text{Lip}[a, b] \subseteq \text{AC}[a, b]$

Proof.

$$C^1[a, b] \subseteq \text{Lip}[a, b]$$

Choose $f \in C^1[a, b]$. Since $f'(x) \in C[a, b]$, $f'(x)$ is bounded by Weierstrass Theorem.

By MVT, for all $x, y \in [a, b]$, $\frac{f(x)-f(y)}{x-y} = f'(z)$, for some $z \in [x, y]$.

Because $|f'(z)| \leq M$, $|f(x) - f(y)| \leq M|y - x|$

So f is Lipschitz on $[a, b]$.

$$\text{Lip}[a, b] \subseteq \text{AC}[a, b]$$

Choose $f \in \text{Lip}[a, b]$. Then there exists M such that for all $x, y \in [a, b]$, $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in [a, b]$.

Let $\{(a_i, b_i)\}_{i=1}^{\infty} \subseteq [a, b]$ be disjoint intervals.

Then $\sum_{i=1}^{\infty} |f(b_i) - f(a_i)| \leq \sum_{i=1}^{\infty} M|b_i - a_i| = M \sum_{i=1}^{\infty} |b_i - a_i|$

So we can choose in the definition of absolutely continuous $\delta = \frac{\varepsilon}{M}$.

So $\sum_{i=1}^{\infty} (b_i - a_i) < \delta = \frac{\varepsilon}{M} \implies \sum_{i=1}^{\infty} |f(b_i) - f(a_i)| < \varepsilon$.

■

Our goal is to prove the following theorem:

Theorem 0.36. Absolutely continuous functions are almost everywhere differentiable. Namely, if $f \in AC[a, b]$, then $f'(x)$ exists and is finite almost everywhere in $[a, b]$. Moreover,

$$\int_a^b f'(x) dx = f(b) - f(a)$$

where $f'(x)$ is Lebesgue-integrable in $[a, b]$.

Remark. The fundamental theorem of calculus therefore holds for absolutely continuous functions.

Proof.

Proposition 5. $AC[a, b] \subseteq BV[a, b]$

Proof. Chose $f \in AC[a, b]$.

Then, choosing $\varepsilon = 1$ in the definition, there exists $\delta > 0$ such that for all $\{(a_i, b_i)\}_{i=1}^\infty$ disjoint, $\sum_{i=1}^\infty (b_i - a_i) < \delta$, then $\sum_{i=1}^\infty |f(b_i) - f(a_i)| < 1$.

We divide $[a, b]$ into N equal parts, where N is an integer large enough that $\frac{b-a}{N} < \delta$.

Let the dividing points be y_1, \dots, y_n .

Let $P : a = x_0 < x_1 < \dots < x_n = b$ be any partitions. Then

$$\begin{aligned} V(f, P) &\leq V(f, P \cup \{y_1, \dots, y_n\}) \\ &= V_a^{y_1}(f, P_1) + V_{y_1}^{y_2}(f, P_1) + \dots + V_{y_{N-1}}^{y_N}(f, P_N) \\ &< 1 + 1 + \dots + 1 \\ &= N \end{aligned}$$

So $V_a^b(f) = \sup_P V(f, P) \leq N < \infty$. Thus $f \in BV[a, b]$.

This completes the proof of the claim.

Corollary 0.37. $f \in AC[a, b]$ implies $f'(x)$ exists and is finite almost everywhere in $[a, b]$. ■

Question: Why do we have $\int_a^b f'(x) dx = f(b) - f(a)$?

Proposition 6. If $f \in AC[a, b]$, then the function $v(x) = V_a^x(f)$ is continuous in $[a, b]$.

Proof. We evaluate $|v(x) - v(y)|$.

Assume $a \leq x \leq y \leq b$.

Then $|v(x) - v(y)| = v(y) - v(x) = V_a^y(f) - V_a^x(f) = V_x^y(f) = \sup_P V_x^y(f, P)$.

Because $f \in AC[a, b]$, let $\varepsilon > 0$ be fixed and $\delta > 0$ the corresponding δ . If $|x - y| = y - x < \delta$, then $V_x^y(f, P) = \sum_{i=1}^n |f(x_i) - f(x_{i+1})| < \varepsilon$

Thus $V_x^y(f) = \sup_P V_x^y(f, P) < \varepsilon$

Thus $v(x) = \bigvee_a^x(f)$ is continuous.

Now suppose $\{(a_i, b_i)\}_{i=1}^\infty \subseteq [a, b]$ such that $\sum_{i=1}^\infty (b_i - a_i) < \delta$, then

$$\sum_{i=1}^\infty |v(b_i) - v(a_i)| = \sum_{i=1}^\infty \bigvee_{a_i}^{b_i}(f) < \varepsilon$$

(follow the same proof)

Proposition 7.

1. $\text{AC}[a, b] \pm \text{AC}[a, b] \subseteq \text{AC}[a, b]$
2. $c \text{AC}[a, b] = \text{AC}[a, b], c \neq 0$
3. $\text{AC}[a, b] \cdot \text{AC}[a, b] \subseteq \text{AC}[a, b]$

Proof. David skipped it

Remark. If $f \in \text{AC}[a, b]$, then $f \in C[a, b]$.

Assume now that f is absolutely continuous. Then $f \in \text{BV}[a, b]$, $f(x) = f(x) - (v(x) - f(x))$, $x \in [a, b]$ where $v(x) = \bigvee_a^x(f)$ for all $x \in [a, b]$.

We know that $v(x)$ and $v(x) - f(x)$ are monotone increasing on $[a, b]$.

By previous proposition, we know that $v \in \text{AC}[a, b]$.

Because $f \in \text{AC}[a, b]$, $v - f \in \text{AC}[a, b]$.

Let $f = f_1 - f_2$, where f_1 and f_2 are absolutely continuous and increasing. So it remains to prove the FTC for increasing AC functions.

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Theorem 0.38. Increasing absolutely continuous functions are almost everywhere differentiable. Namely, if $f \in \text{AC}[a, b]$, then $f'(x)$ exists and is finite almost everywhere in $[a, b]$. Moreover,

$$\int_a^b f'(x) dx = f(b) - f(a)$$

where $f'(x)$ is Lebesgue-integrable in $[a, b]$.

Proof. Assume $f : [a, b] \rightarrow \mathbb{R}$ is increasing and absolutely continuous. So $f \in \text{BV}[a, b]$ and thus $f'(x)$ exists and is finite μ -almost everywhere. We need to show $\int_a^b f'(x) dx = f(b) - f(a)$.

Claim: The Radon measure ν generated by f is absolutely continuous with respect to the Lebesgue Measure.

Recall:

We extend f to \mathbb{R} as

$$f(x) = \begin{cases} f(x) & x \in [a, b] \\ f(a) & x < a \\ f(b) & x > b \end{cases}$$

Of course, since f is absolutely continuous, it is right continuous. Thus ν is a Radon measure, where

$$\nu(A) = \inf \left\{ \sum_{i=1}^{\infty} (f(b_i) - f(a_i)) \mid A \subseteq \bigcup_{i=1}^{\infty} [a_i, b_i), [a_i, b_i) \text{ disjoint} \right\}$$

For all $[c, d] \subseteq \mathbb{R}$, $\nu([c, d]) = f(d) - f(c)$.

Assume $A \subseteq \mathbb{R}$ has zero Lebesgue measure.

We want to show $\nu(A) = 0$.

We have $\lambda(A) = \inf \left\{ \sum_{i=1}^{\infty} (b_i - a_i) \mid A \subseteq \bigcup_{i=1}^{\infty} [a_i, b_i), [a_i, b_i) \text{ disjoint} \right\}$

Because f is absolutely continuous in $[a, b]$, and we extended it as constant to the left and right of $[a, b]$, then f is also in AC in \mathbb{R} .

For all $\varepsilon > 0$, there exists $\delta > 0$ such that if $\sum_{i=1}^{\infty} (b_i - a_i) < \delta$, then $\sum_{i=1}^{\infty} |f(b_i) - f(a_i)| < \varepsilon$, where $[a_i, b_i] \subseteq \mathbb{R}$ are disjoint.

Fix $\varepsilon > 0$, and choose δ as above.

Since $\lambda(A) = 0$, there exists a cover of A , $\{[a_i, b_i]\}$ such that $\sum_{i=1}^{\infty} (b_i - a_i) < \delta$. Then by choice of δ , we have $\nu(A) \leq \sum_{i=1}^{\infty} (f(b_i) - f(a_i)) < \varepsilon$.

We take the limit as $\varepsilon \rightarrow 0$ to conclude that $\nu(A) = 0$. So $\nu \ll \lambda$

By the Radon-Nikodym theorem, $\nu[a, b] = \int_a^b D_\lambda \nu(x) dx = \int_a^b f'(x) dx$, since $f'(x) = D_\lambda \nu_{ac}(x) = D_\lambda \nu(x)$ for almost every $x \in [a, b]$.

However, $\nu[a, b] = \nu[a, b) + \nu\{b\} = f(b) - f(a) + 0 \implies f(b) - f(a) = \int_a^b f'(x) dx$

Summary:

Let $f \in \text{AC}[a, b]$. Then $f \in \text{BV}[a, b]$. So $f = f_1 - f_2$, where $f_1, f_2 \in \text{BV}[a, b]$, and are increasing in $[a, b]$. Moreover, $f_1 - f_2 \in \text{AC}[a, b]$. We proved that

$$\int_a^b f'_1(x) dx = f_1(b) - f_1(a), \int_a^b f'_2(x) dx = f_2(b) - f_2(a)$$

where f'_1, f'_2 exist and are finite for almost every $x \in [a, b]$.

Thus

$$\int_a^b f'(x) dx = \int_a^b f'_1(x) dx - \int_a^b f'_2(x) dx = f_1(b) - f_1(a) - (f_2(b) - f_2(a)) = f(b) - f(a)$$

■

Theorem 0.39. Let $f \in \text{AC}[a, b]$. Then $f'(x)$ exists and is finite almost everywhere in $[a, b]$. Moreover, $f'(x)$ is summable in $[a, b]$ (not only integrable) and the FTC holds, i.e. $\int_a^b f'(x) dx = f(b) - f(a)$.

Theorem 0.40. (Characterization/integral representation of AC functions): A function $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous in $[a, b]$ iff there exists $g(x) \in L^1[a, b]$ such that $f(x) = f(a) + \int_a^x g(t) dt$ for all $x \in [a, b]$

Remark. In that case, we have $g(t) = f'(t)$ for almost every $t \in [a, b]$.

Proof. \Rightarrow

Assume $f \in \text{AC}[a, b]$. Then we know $f'(x) \in L^1[a, b]$.

As $f \in \text{AC}[a, x]$ for all $x \in [a, b]$, by Newton-Leibniz we have $\int_a^x f'(t) dt = f(x) - f(a)$ for all $x \in [a, b]$.

Chose $g(t) = f'(t) \in L^1[a, b]$.

\Leftarrow

Now assume there exists $g(t) \in L^1[a, b]$ and that $f(x) = f(a) + \int_a^x g(t) dt$.

We need to show that $f \in \text{AC}[a, b]$, and $f'(t) = g(t)$ for almost every $x \in [a, b]$.

$f \in [a, b]$: Follows from absolute continuity of the integral.

Choose $\varepsilon > 0$. There exists $\delta > 0$ such that if $A \subseteq [a, b]$ is Lebesgue-measurable, $\lambda(A) < \delta$, then $\int_A |g(t)| dt < \varepsilon$.

If now $\{[a_i, b_i]\} \subseteq [a, b]$ are disjoint with $\sum_{i=1}^{\infty} (b_i - a_i) < \delta$, then

$$\begin{aligned} \sum_{i=1}^{\infty} |f(b_i) - f(a_i)| &= \sum_{i=1}^{\infty} \left| \int_{a_i}^{b_i} g(t) dt - \int_{a_i}^{a_i} g(t) dt \right| \\ &= \sum_{i=1}^{\infty} \left| \int_{a_i}^{b_i} g(t) dt \right| \\ &\leq \sum_{i=1}^{\infty} \int_{a_i}^{b_i} |g(t)| dt \\ &= \int_{\cup_{i=1}^{\infty} [a_i, b_i]} |g(t)| dt < \varepsilon \end{aligned}$$

Thus $f \in \text{AC}[a, b]$.

$f'(x) = g(x)$ for almost every $x \in [a, b]$: Because $g \in L^1[a, b]$, then x is a Lebesgue point of g for almost every $x \in [a, b]$.

Definition 0.16. A Lebesgue point is a point x such that

$$\lim_{r \rightarrow 0^+} \frac{1}{r} \int_x^{x+r} g(t) dt = g(x), \quad \lim_{r \rightarrow 0^+} \frac{1}{r} \int_{x-r}^x g(t) dt = g(x)$$

Thus $\lim_{r \rightarrow 0^+} \frac{1}{r} \int_x^{x+r} g(t) dt = \lim_{r \rightarrow 0^+} \frac{1}{r} ((f(x+r)) - f(x)) = f'(x+) = g(x)$

Similarly, $\lim_{r \rightarrow 0^+} \frac{1}{r} \int_{x-r}^x g(t) dt = f'(x-) = g(x)$.

Thus $f'(x+) = f'(x-) = g(x)$ for almost every $x \in [a, b]$. So $f'(x) = g(x)$ for almost every $x \in [a, b]$. ■

Example 0.4. A function that is increasing, BV, but not AC
Cantor's function.

The classical "middle thirds" Cantor set is constructed as follows:

1. Divide $[0, 1]$ into 3 equal parts, remove middle open interval.
2. Divide remaining intervals into 3 equal parts, remove middle open intervals,
3. Continue infinitely many times.

The remaining part of $[0, 1]$ is the Cantor set. This set has the following properties:

1. $\lambda(C) = 0$
2. $C \subseteq [0, 1]$ is compact, because $C = [0, 1] \setminus \bigcup_{i=1}^{\infty} (a_i, b_i)$, where $(a_i, b_i) \subseteq [0, 1]$ are disjoint open intervals.
3. C is nonempty. In particular, $\frac{1}{3} \in C$.

Cantor function: From the construction of the cantor set for all $x \in C$, x has the form $x = \sum_{n=1}^{\infty} \frac{2a_n}{3^n}$, where $a_n \in \{0, 1\}$. Every number $x \in [0, 1]$ can be written in base 3 as $x = \sum_{k=1}^{\infty} \frac{b_k}{3^k}$, $b_k \in \{0, 1, 2\}$.

This representation in base 3 is unique.

The points x that belong to the Cantor set have representation in base 3 coefficients 1 or 2.

$$C(x) = \begin{cases} \sum_{n=1}^{\infty} \frac{a_n}{2^n}, & x = \sum_{n=1}^{\infty} \frac{2a_n}{3^n} \in C \\ \sup_{y \leq x, y \in C} C(y) & x \notin C \end{cases}$$

Theorem 0.41. (Properties of the Cantor function): The Cantor function has the following properties:

1. Continuous on $[0, 1]$
2. Increasing on $[0, 1]$
3. $C'(x) = 0$ for almost every $x \in [0, 1]$
4. $C(x)$ is not absolutely continuous in $[0, 1]$.

Proof.

1. By definition, if we have a sequence $y_k = \sum_{n=1}^{\infty} \frac{2a_n^{(k)}}{3^n}$ converges to $x = \sum_{n=1}^{\infty} \frac{2b_n}{3^n}$, then we should have for all $N \in \mathbb{N}$, there exists $K_0 \in \mathbb{N}$ such that $a_n^{(k)} = b_n$ if $k \geq K_0$, $n = 1, \dots, N$.

Thus

$$|C(y_k) - C(x)| = \left| \sum_{n=1}^{\infty} \frac{2a_n^{(k)}}{3^n} - \sum_{n=1}^{\infty} \frac{2b_n}{3^n} \right| = \left| \sum_{n=1}^{\infty} \frac{2(a_n^{(k)} - b_n)}{3^n} \right| \leq \sum_{n=N+1}^{\infty} \frac{4}{3^n} < \frac{4}{3^{N+1}}$$

We take the limit as $N \rightarrow \infty$ to get $|C(y_k) - C(x)| \rightarrow 0$

So the Cantor function is continuous on the Cantor set.

2. By definition, $C(x)$ increases on the Cantor set. Now we show $C(x)$ increases on all of $[0, 1]$.

Choose $x < y$. If $x, y \in C$, we are done.

If $x \in C$, $C(y) = \sup_{z \leq y, z \in C} C(z) \geq C(x)$, because $x \in C$, $x \leq y$.

If $x \notin C$ and $y \in C$, $C(x) = \sup_{z \leq x, z \in C} C(z) \leq \sup_{z \leq y, z \in C} C(z) \leq C(y)$, because $C(x)$ increases on C

If $x \notin C$, $y \notin C$, $C(x) = \sup_{z \leq x, z \in C} C(z) \leq \sup_{z \leq y, z \in C} C(z) = C(y)$

1 Continued.

We want to show $C(x)$ is continuous on $[0, 1]$. Assume $x_0 \in [0, 1]$, need to prove $\lim_{x \rightarrow x_0} C(x) = C(x_0)$

Case 1: Suppose $x_0 \notin C$. Thus $x_0 \in (a_k, b_k)$, where (a_k, b_k) is an interval removed at some step in the construction of C .

$C(x)$ is constant on any removed interval, so it is continuous on these (open) removed intervals.

Case 2: Suppose $x_0 \in C$. Assume $x_n \rightarrow x_0$, $x_n \notin C$. Because $C(x)$ is increasing, without loss of generality we can assume x_n is increasing. If there exists $y_n \in C$ such that $x_n < y_n < x_0$, then we have $C(x_0) = \lim_{n \rightarrow \infty} C(y_n)$.

For all $n \in \mathbb{N}$, there exists $N_n \in \mathbb{N}$ such that $x_0 > x_{N_n} > y_n$.

Thus $C(x_0) > C(x_{N_n}) \geq C(y_n) \rightarrow C(x_0)$

Thus $C(x_n) \rightarrow C(x_0)$ as $n \rightarrow \infty$.

If no such sequence y_n exists, then there exists $N \in \mathbb{N}$ such that $(x_N, x_0) \subseteq [0, 1] \setminus C$. Since $x_0 \in C$, x_0 is the right endpoint of a removed interval.

Thus $C(x)$ is left-continuous at x_0 .

In the same way, we prove that $C(x)$ is right-continuous.

Therefore $C(x)$ is continuous on $[0, 1]$.

3. $C'(x) = 0$ for almost every $x \in [0, 1]$.

For every removed interval $(a_k, b_k) \subset [0, 1]$, $C(x)$ is constant on (a_k, b_k) .

Thus $C'(x) = 0$ for all $x \in [0, 1] \setminus C$.

Since $\lambda([0, 1] \setminus C) = 1$, $C'(x) = 0$ for almost every $x \in [0, 1]$.

4. $C(x) \notin \text{AC}[0, 1]$.

This follows from the fact that $0, 1 \in C$.

It is easy to show $C(0) = 0, C(1) = 1$.

Thus $\int_0^1 C'(x) dx = 0 < C(1) - C(0)$, therefore $C \notin \text{AC}[0, 1]$

■