Lecture 1

We will be using Hatcher's Algebraic Topology. The topology sequence is usually something like

$$A\text{Topological Spaces}$$
$$B\text{Cell Complexes}$$
$$C\text{Manifolds}$$

Theorem 0.1. *(BIG Theorem)*
Given a "reasonably nice" space, there is a bijection between connected covers of a space and subgroups of the fundamental group.

Categories:

Algebraic structures that are much flabbier than a group. They consist of

- *A collection of arrows*
- *A partial binary operation on these arrows*
- *Objects, which arrows go between*

We also want a composition law. That is, for objects and arrows

$$A \xrightarrow{f} B \xrightarrow{g} C$$

there is an arrow $A \xrightarrow{g \circ f} C$ *. We want this composition to be associative, that is* $(f \circ g) \circ h = f \circ (g \circ h)$*, and we want objects to have identity arrows.*
Not all functions have inverses. Using sets and functions as an example, we have described the category Set.
Here are some more examples of categories:

Example 0.1.
- Groups and group homomorphisms (Grp)
- Topological spaces and continuous functions (Top)
- etc.

We can make the following new category.

Definition 0.1. We denote by Top^* the category of based topological spaces, whose objects are pairs (X, x_0), where X is a topological space and $x_0 \in X$, and whose morphisms are continuous functions $f : (X, x_0) \to (Y, y_0)$ such that $f(x_0) = y_0$.

Goal:

Our goal is to get a functor from Top to Grp. The fundamental group functor π_1 will go from Top^* to Grp.

Lecture 2

Topology review:

Definition 0.2. A topological space is a set X along with a collection of subsets of X called "open sets," such that $X, \varnothing$ are open, and the arbitrary union and finite intersection of open sets are open.
Notice the following diagram commutes using the product topology

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[d, dotted, "\exists!"] \arrow[dr, "g"] & \\ X & X \times Y \arrow[l, "P_X"'] \arrow[r, "P_Y"] & Y \end{tikzcd}$$

And in general

$$\begin{tikzcd} Z \arrow[d, dotted, "\exists!"'] \arrow[dr, "f_\alpha"] & \\ \prod_{\alpha \in A} X_\alpha \arrow[r, "P_\alpha"] & X_\alpha \end{tikzcd}$$

Maps are continuous; functions are not.

Lemma 1. *(Gluing lemma)*
Suppose $f : A \to Y$, $g : B \to Y$ are continuous, and $f(x) = g(x)$ for all $x \in A \cap B$. Then $f \cup g : A \cup B \to Y$ is continuous. This only holds as long as $A, B \subseteq X$ are closed.

Same Shape, Same Map

(maps up to wriggling things around a bit)

Definition 0.3. Two maps are homotopic if there exists a parametrized map $f_t : X \to Y$ such that $f_0 = f, f_1 = g$ for $f, g : X \to Y$. Equivalently, and more precisely, if there exists a map $F : X \times [0, 1] \to Y$ such that $F(x, 0) = f(x), F(x, 1) = g(x)$ for all $x \in X$.

X, Y topological spaces are said to have the same shape if there exist maps $f : X \to Y, g : Y \to X$ such that $g \circ f \simeq \mathrm{Id}_X$ and $f \circ g \simeq \mathrm{Id}_Y$. We may say that X, Y have the same homotopy type

Definition 0.4. A deformation retraction from $X \to A \subseteq X$ is a map from $X \times I \to X$ such that, for all $x \in A$, and $s, t \in I$,

$$\begin{aligned} f_0(x) &= x && \forall x \in X \\ f_1(x) &\in A && \forall x \in X \\ f_t(x) &= f_s(x) && \forall x \in A \end{aligned}$$

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Definition 0.5. Let X be a topological space. A retraction is a map $r : X \to X$ such that $r \circ r = r$. That is, $r(r(x)) = r(x)$ for any $x \in X$.
Let $A = r(X)$. Then $r|_A = \mathrm{Id}_A$.

Definition 0.6. Let $F : X \times I \to Y$. We say $f_0 \simeq f_1 \operatorname{rel} A \subseteq X$ are homotopic relative to A if, for any $x \in A$, $f_t(x)$ is independent of t. That is, for any $s, t \in I$, $f_s(x) = f_t(x)$ for any $x \in A$.
For any map $f : X \to Y$, there exists a space $Z \simeq Y$ via $g : Y \to Z$ such that $g \circ f : X \to Z$ is injective. That is, in the following diagram, we have a bijection between homotopy classes of maps f and homotopy classes of maps $g \circ f$, and we can do this in a way that rigs $g \circ f$ to be injective.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g \circ f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

Definition 0.7. Given a map $f : X \to Y$ we can construct the Mapping Cylinder M_f by setting $M_f = X \times I \coprod Y / \sim$, where $(x, 0) \sim f(x)$.
The visual intuition should be taking the disjoint union of X and Y, and tieing a string between x and $f(x)$ for each point.

Claim. *$X \hookrightarrow M_f, Y \hookrightarrow M_f$, and the latter is in fact a homotopy equivalence. Further, the injection $X \hookrightarrow M_f$ is homotopic to $f(X) \hookrightarrow M_f$.*

Proof. You can construct a homotopy which "squishes" the cylinder down to $f(X)$. ■

Definition 0.8. A space X is contractible if it has the homotopy type of a point. A map is null-homotopic if it is homotopic to a constant map. So X is contractible if the identity is null-homotopic.
Now he's drawing an example. The example is Bing's House with 2 rooms, which I will not reproduce here. But the point is that it's contractible, but not obviously so.

Cell Complexes

Cell complexes are topological spaces which are built up inductively out of closed balls in Euclidean space. We write $\mathbb{D}^n := \{\vec{x} \in \mathbb{R}^n \mid \|\vec{x}\| \leq 1\}$, and $e^n := \{\vec{x} \in \mathbb{R}^n \mid \|\vec{x}\| < 1\}$. We can see that $e^n = \operatorname{int} \mathbb{D}^n$, and $\partial \mathbb{D}^n = \mathbb{S}^{n-1}$.

Base step

Start with some collection of points X^0, the 0-skeleton, with the discrete topology.

Inductive step

Let X^{n-1} be the $n-1$ skeleton, which has already been build and defined. Select some collection of n-dimensional balls $\{\mathbb{D}^n\}_{\alpha \in A}$, and some continuous "attaching map" $\varphi_\alpha : \partial \mathbb{D}^n_\alpha \to X^{n-1}$. Then

$$(X^n = X^{n-1} \coprod_{\alpha \in A} \mathbb{D}^n)/(x \sim \varphi_\alpha(x) \forall x \in \partial \mathbb{D}^n)$$

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A space X is a cell complex if it has been constructed using the above inductive procedure. If $n = \infty$, we use the weak topology, in which the open sets are the sets which are open when intersected with each X^n.
For every $\mathbb{D}^n_\alpha$ and corresponding "attaching map $\varphi_\alpha : \partial \mathbb{D}^n_\alpha \to X^{n-1}$, there is a subset of X^n homeomorphic to $\operatorname{int}(\mathbb{D}^n_\alpha)$, via the composition

$$\operatorname{int}(\mathbb{D}^n_\alpha) \hookrightarrow \mathbb{D}^n_\alpha \hookrightarrow X^{n-1} \coprod_\alpha \mathbb{D}^n_\alpha \to X^n$$

which we call $\Phi_\alpha : \mathbb{D}^n_\alpha \to X^{n-1}$. So the attaching map $\phi_\alpha : \partial \mathbb{D}^n_\alpha \to X^{n-1}$ extends to a "characteristic map" Φ_α.
We will now see many examples of things.

Example 0.2. If you stop after constructing X^1, it's a graph.

Example 0.3. $\mathbb{S}^n$ has a cell structure with one e_0 and one e_n.

Example 0.4. Consider $\mathbb{RP}^2$. This can be expressed as $(\mathbb{R}^3 \setminus \{0\})/(\vec{x} \sim \lambda\vec{x}, \lambda \neq 0)$. We can replace 2 with any n and get $\mathbb{RP}^n$. Indeed, we can replace $\mathbb{R}$ with $\mathbb{C}$, $\mathbb{H}$, or indeed any field.
<u>Homogenous coordinates</u>
For $(x, y, z) \neq (0, 0, 0)$, we have $[x, y, z] \overset{\text{def}}{=} \{(\lambda x, \lambda y, \lambda z) \mid \lambda \neq 0\}$. For example, $[1, 2, 3] = [2, 4, 6]$.

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Definition 0.9. A <u>subcomplex</u> of a complex X is a closed disjoint union of open cells $e^{n_i}_{\alpha_i}$ in X such that they form a cell complex on their own.
Remark. We keep talking about "CW Complexes." The C is for "closure finite," and the W is for "weak topology."
Recall: $\mathbb{RP}^n \overset{\text{def}}{=} \mathbb{R}^n/(x \sim \lambda x, \lambda \neq 0)$. $\mathbb{CP}^n$ can be defined similarly.
We can write $\mathbb{RP}^n = e^0 \cup e^1 \cup e^2 \cup \cdots \cup e^n$, and $\mathbb{CP}^n = e^0 \cup e^2 \cup e^3 \cup \cdots \cup e^{2n}$. We can do the same thing with the quaternions.
Next time, we will cover operations on complexes.

Lecture 6, 1/23/23

This lecture, we will cover operations on cell complexes, and two big theorems.

Operations on Cell Complexes

1. If X, Y have cell structures, then $X \times Y$ has a natural cell structure.

2. If (X, A) is a CW-pair, then X/A has a natural cell structure (X/A denotes identifying all points in A together).

$$\begin{tikzcd} \mathbb{D}'_\alpha \supseteq \partial\mathbb{D}'_\alpha \arrow[d, "\phi_\alpha"] \arrow[dr] & \\ X^0 \arrow[r, "q"'] & (X/A)^0 \end{tikzcd}$$

3. Cones and Suspensions. The cone on X, CX, is defined as

$$CX = (X \times I)/(X \times \{0\})$$

Note that CX is contractible for any X. The suspension on X, SX, is defined as

$$SX = CX/(X \times \{1\})$$

If $f : X \to Y$ is a map, there exists a natural map $Sf : SX \to SY$. Indeed, if $f : X \to Y$, then $f \times \mathrm{Id} : X \times I \to Y \times I$, and so we can factor $f \times \mathrm{Id}$ through the quotient map:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, hook] \arrow[dr, dotted] & Y \arrow[d, hook] \\
X \times I \arrow[r, "f \times \mathrm{Id}"] \arrow[d, "q"] \arrow[dr, dotted] & Y \times I \arrow[d] \\
SX \arrow[r, "\exists ! Sf"'] & SY
\end{tikzcd}$$

Note $S(\mathbb{S}^n) = \mathbb{S}^{n+1}$.

4. Joins. If X, Y are cell structures, then we define their join $X \star Y$ as

$$X \star Y = \frac{X \times Y \times I}{(x, y_1, 0) \sim (x, y_2, 0), (x_1, y, 1) \sim (x_2, y, 1)}$$

This is a useful construction for simplices.

5. Wedge product. If X, Y are cell structures, with distinguished points $x_0 \in X, y_0 \in Y$, then we define their wedge product $X \wedge Y$ as

$$X \wedge Y = \frac{X \coprod Y}{x_0 \sim y_0}$$

This is just gluing X and Y together at a distinguished point. This raises an obvious question: does the wedge product depend on the points x_0, y_0? Yes, but not if they are (connected) cell complexes!

If x_0 is a 0-cell of X, and y_0 a 0-cell of Y, then $X \vee Y$ has a natural cell structure AND $X \vee Y$ is a subcomplex of $X \times Y$.

6. Smash product. If X, Y are spaces with distinguished points x_0, y_0, then the smash product is defined as

$$X \wedge Y = \frac{X \times Y}{X \vee Y}$$

For example, the smash product $S^1 \wedge S^1$ is a Torus quotiented out by the longitudinal and meridian circles. By arguing from some cell nonsense, we can say this is S^2.

Here are two big theorems.

Theorem 0.2. *If (X, A) is a CW-pair, and A is contractible, then X/A is homotopy equivalent to X, with the quotient mapping itself providing a homotopy equivalence.*

Theorem 0.3. *Suppose (X_1, A) is a CW-pair, and $f, g : A \to Y$ are maps. If $f \simeq g$, and everything in sight is a cell complex, then*

$$X_1 \coprod_f Y \simeq X_1 \coprod_g Y$$

That is, if f, g are used as attaching maps, then the resulting spaces will be homotopy equivalent.

Lecture 7, 1/25/23

Definition 0.10. Let X be a cell complex. If we let f_i be the number of i-dimensional cells in the cell structure, then we define

$$\chi(X) = f_0 - f_1 + f_2 - f_3 + \cdots$$

The more general definition is the alternating sum of the Betti numbers of X, where the ith Betti number is $\dim H^i(X)$.

Definition 0.11. Let X be a topological space, and let $A \subseteq X$ be a subspace. We say that (X, A) has the homotopy extension property (HEP) if for all topological spaces Y and for all maps $f : X \times \{0\} \cup A \times I \to Y$, there exists an extension of f, $\overline{f} : X \times I \to Y$, such that $\overline{f}|_{X \times \{0\} \cup A \times I} = f$.
Slogan: "A homotopy on the subspace can be extended to a homotopy on the entire space."

Lecture 8, 1/27/23

Proposition 1. *(X, A) has the homotopy extension property if and only if $X \times I$ retracts to $X \times \{0\} \cup A \times I$.*

Proof. ■

Example 0.5. Does $(\mathbb{D}^2, \partial\mathbb{D}^2)$ have the property? Does $\mathbb{D}^2 \times I$ retract onto $\mathbb{D}^2 \times \{0\} \cup \partial\mathbb{D}^2 \times I$? Yes. This is easy to see by drawing a picture.
Here is a non-example. Let $X = I$, and let $A = \{\frac{1}{n}\}_{n \in \mathbb{N}} \cup \{0\}$. $X \times I$ is the square, and $X \times \{0\}$ is the bottom of the square, so $X \times \{0\} \cup A \times I$ is the comb space. The square doesn't retract to this.

Proposition 2. *If (X, A) is a CW pair, then (X, A) has the homotopy extension property.*

Proof. Later ■

Theorem 0.4. *If (X, A) has the homotopy extension property, and A is contractible, then the quotient map $X \to X/A$ is a homotopy equivalence.*

Proof. Consider identity map $\mathrm{Id} : A \to A$. We have a homotopy $F : A \times I \to A$ which is a witness to A being contractible. That is, $f_0 = \mathrm{Id}_A$, $f_1 \equiv \{p\}$ for some point $p \in A$.
Then there is an extension to a homotopy $H : X \times I \to X$. We have the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "f_t"] \arrow[d, "q"] & X \arrow[d, "q"] \\ X/A \arrow[r, "\overline{f_t}"] & X/A \end{tikzcd}$$

Because all of A goes to a point for $t = 1$, then by the universal property of quotients, there is a map g making the diagram commute:

$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, "q"] & X \arrow[d, "q"] \\ X/A \arrow[r, "\overline{f_1}"'] \arrow[ur, "g"] & X/A \end{tikzcd}$$

So qg is homotopic to the identity map, and gq is homotopic to the identity map. This completes the proof? ■

Lecture 9, 1/30/23

Definition 0.12. We say that (X, A) and (Y, A) are homotopy equivalent relative to A if there exist maps $f : X \to Y, g : Y \to X$ such that $f|_A = \mathrm{Id}_A, g|_A = \mathrm{Id}_A$ and $g \circ f \simeq \mathrm{Id}_X$ relative to A, and $f \circ g \simeq \mathrm{Id}_Y$ relative to A.

Theorem 0.5. *If (X, A) is a CW Pair, and $f, g : A \to X_0$ are homotopic maps, then $X_0 \coprod_f X_1 \simeq X_0 \coprod_g X_1$ relative to X_0.*

Proof. Bunch of pictures I can't write down. ■

Proposition 3. *If $(X, A), (Y, A)$ both have the homotopy extension property, and $f : X \to Y$ is a homotopy equivalence such that $f|_A = \mathrm{Id}_A$, then f is a homotopy equivalence relative to A.*

Proof. ∎

Corollary 0.6. *If (X, A) has the homotopy extension property and $A \hookrightarrow X$ is a homotopy equivalence, then X deformation retracts to A.*

Proof. ∎

Corollary 0.7. *A map $f : X \to Y$ is a homotopy equivalence if and only if X is a deformation retraction of M_f.*

Proof. ∎

Lecture 10, 2/3/23

Definition 0.13. Given any path $f : I \to X$, we write $[f]$ for the set of paths $g : I \to X$ such that $g \simeq f$ relative to ∂I. If we don't fix endpoints, each $[f]$ would be a path-component. Sometimes we use π^0 to denote the set of path components.
Let $f, g : I \to \mathbb{R}^n, f \simeq g$ by $h_t(u) = tg(u) + (1-t)f(u)$. Then $h_0 = f, h)1 = g$. h is called the "straight line homotopy."
In fact, we could change I to <u>any</u> topological space, and $f \simeq g$ <u>still</u>
We could also change $\mathbb{R}^n$ to any $U \subseteq \mathbb{R}^n, U$ convex, and $f \simeq g$ <u>still</u>.
We could change X to U, any metric space iwth unique shortest path which vary continuosly as the endpoints vary.
Concatenation: If $f(1) = g(0)$, then define $f \star g : I \to X$ by

$$(f \star g)(t) = \begin{cases} f(2t) & t \in [0, \frac{1}{2}] \\ g(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

Definition 0.14. Assume $f(1) = g(0)$. Then $[f] \star [g] = [f \star g]$ is well defined by handwaving.
Constants: The constant path c_x is the path which is constantly x. Note $[c_{f(0)}] \star [f] = [f], [f] \star [c_{f(1)}] = [f]$.
Inverses: Define $\overline{f}(u) = f(1-u)$. Note $[f][\overline{f}] = [c_x]$.
Associativity: $(f \star g) \star h \simeq f \star (g \star h)$.

Definition 0.15. A category where every $f : A \to B$ has an inverse (i.e. an arrow $f^{-1} : B \to A$ such that $ff^{-1} = \mathrm{Id}_B = f^{-1}f$) is called a <u>groupoid</u>.
π_0 is a functor, (objects→objects, arrow→arrows, compositions→compositions, identities→identities)
$(X, x_0) \mapsto \pi_1(X, x_0) = \{[f] \mid f : I \to X, f(0) = f(1) = x_0\}$.

$(X, x_0) \xrightarrow{\pi_1} \pi_1(X, x_0)$.

$$\begin{tikzcd}
(X, x_0) \arrow[r, "\pi_1"] \arrow[d, "h"] & \pi_1(X, x_0) & f \arrow[d] \\
(Y, y_0) \arrow[r, "\pi_1"] & \pi_1(Y, y_0) & h \circ f
\end{tikzcd}$$

Lecture 11, 2/6/23

just homework review