Lecture 1

Foundational problems

1. Interpolation: given a set of points a_j, can we get an analytic f such that $f(a_j) = c_j$ for some fixed c_j?

2. Approximation: Given an analytif f, can we describe $f = \lim_m h_m$ as the limit of a sequence of functions h_m?

3. Value distribution: studying sets of the form $\{\lambda \mid f(\lambda) = c\}$. This is discrete in general.

Let $\Omega \subseteq \mathbb{C}$ be open. We denote by $\mathcal{O}(\Omega)$ the set of analytic functions $f : \Omega \to \mathbb{C}$.
How can we topologize this?
Let Ω be the unit disk, and let $f \in \mathcal{O}(\Omega)$. Consider $f = \frac{1}{z-1}$. The sup of the modulus of this blows up on Ω. So, the sup norm of the modulus will not work.
We will do it as follows.

Definition 0.1. Let $K \subseteq \Omega$ be compact. We define

$$\|f\|_{\infty,K} = \sup_{z \in K} |f(z)| < \infty$$

Definition 0.2. We say that a sequence $f_n \in \mathscr{O}(\Omega)$, $f_n \to f$ (uniformly on compact sets) if for every compact $K \subseteq \Omega$, $\|f - f_n\|_{\infty,K} \to 0$.

Theorem 0.1. *(Weirstrass)*
This notion of convergence induces a topology which is metrizable, and this metric is complete.

Proof. ■
Suppose $K_j \subseteq K_{j+1} \subseteq K_{j+1} \cdots \subset\subset \Omega$, and the union of the K_j is Ω.

Lemma 1. *Let $f_n \in \mathcal{O}(\Omega)$. Then $f_n \to f$ uniformly on compact sets, i.e. $\|f - f_n\|_{\Omega,K_j} \to 0$, if and only if it converges with respect to the following metric.*

Definition 0.3. For $f, g \in \mathcal{O}(\Omega)$, define

$$\sigma(f,g) = \sum_{j=1}^{\infty} \frac{\|f-g\|_{\infty,K_j}}{1 + \|f-g\|_{\infty,K_j}}$$

Proof. We will not prove that this is a metric. Let $f_n \in \mathcal{O}(\Omega)$. Let $h : \Omega \to \mathbb{C}$, with $d(f_n, h) \to 0$. Then h is analytic by the Cauchy formula. To see this, let $a \in \Omega$, and consider $\overline{B_\delta(a)} \subseteq \Omega$ for some $\delta > 0$. Then

$$f_n(z) = \frac{1}{2\pi i} \int_{|\zeta - a| = \delta} \frac{f_n(\zeta)}{\zeta - z}\, d\zeta$$

We may pass the limit inside and get

$$h(z) = \frac{1}{2\pi i}\int_{|\zeta-a|=\delta} \frac{h(\zeta)}{\zeta - z}\, d\zeta$$

meaning that h is analytic.
Now, suppose that f_n is a Cauchy sequence with respect to σ. So for all j, f_n is Cauchy on $(C(K_j), \|\cdot\|_\infty)$.
We now review the Cauchy-Riemann equations. We write $z = x + iy \in \mathbb{C}$. $\overline{\partial} = \frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$
Recall for a 1-form $\phi\, dz$, we define

$$d(\phi dz) = d\phi \wedge dz = \partial\phi dz \wedge dz + \overline{\partial}\phi d\overline{z} \wedge dz$$

Also, we say $dA = dx \wedge dz = \frac{1}{2i} d\overline{z} \wedge dz$ (area measure)

Theorem 0.2. *Let $\phi \in C^{(\ell)}(\Omega)$. Then $\overline{\partial}(\phi) = 0 \iff \phi \in \mathcal{O}(\Omega) \iff d(\phi dz) = 0$ (that is it is a closed form).*

Corollary 0.3. $\ker(\overline{\partial} : C^{(1)}(\Omega) \to C(\Omega)) = \mathcal{O}(\Omega)$

Theorem 0.4. *(Cauchy, Cauchy-Paupeir (sp?))*
Suppose $\Omega \subset\subset \mathbb{C}$ be precompact (meaning the closure is compact), with $\partial\Omega$ piecewise smooth, and let $\phi \in C^{(1)}(\overline{\Omega})$. Then for any $z \in \Omega$, we have

$$\phi(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{\phi(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{\pi}\int \frac{(\overline{\partial}\phi)(\zeta)}{\zeta - z}\, dA(\zeta)$$

Proof. First, $\frac{\overline{\partial}\phi(\zeta)}{\zeta - z}$ is dA-integrable because, if we write $\zeta - z = re^{i\theta}$, then $dA(\zeta) = rdr \wedge d\theta$. So we have

$$\frac{\overline{\partial}\phi(\zeta)}{\zeta - z} dA(\zeta) = \frac{\overline{\partial}\phi(z + re^{i\theta})}{re^{i\theta}} rdr \wedge d\theta$$

Let $\varepsilon > 0$. Denote by $\Omega_\varepsilon = \Omega \setminus B_\varepsilon(z)$. If we take the differential 1-form $\omega = \frac{\phi(\zeta)}{\zeta - z} d\zeta$. We have

$$d\omega = \frac{\overline{\partial}\phi(\zeta)}{\zeta - z} d\overline{\zeta} \wedge d\zeta$$

Now, we use Stoke's theorem, which tells us $\int_{\partial\Omega_\varepsilon} \omega = \int_{\Omega_\varepsilon} d\omega$. So we get

$$\int_{\partial\Omega} \frac{\phi(\zeta)}{\zeta - z}\, d\zeta - \int_{\partial B_\varepsilon(z)} \frac{\phi(\zeta)}{\zeta - z}\, d\zeta = \int_{\Omega_\varepsilon} \frac{(\overline{\partial}\phi)(\zeta)}{\zeta - z}\, d\overline{\zeta} \wedge d\zeta$$

As we let $\varepsilon \to 0$,

$$\int_{\partial\Omega} \frac{\phi(\zeta)}{\zeta - z}\, d\zeta - \underbrace{\int_{-\pi}^{\pi} \frac{\phi(z+\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}} \varepsilon i e^{i\theta}\, d\theta}_{2\pi i \phi(z)} = \int_{\Omega} \frac{(\overline{\partial}\phi)(\zeta)}{\zeta - z}\, d\overline{\zeta} \wedge d\zeta$$

Definition 0.4. Let μ be a measure on $\mathbb{C}$. We define the Cauchy transform $C\mu$ by

$$u(\zeta) = (C\mu)(\zeta) \stackrel{\text{def}}{=} \int \frac{d\mu(z)}{z - \zeta}$$

The support of μ is the smallest closed set L_1 with the property that $\mu(\psi) = 0$ if $\text{supp}(\psi) \cap L_1 = \varnothing$

Theorem 0.5. *(Hörmander)*
Let μ be a Radon measure, with compact support. Then $u(\zeta)$ is analytic on the complement of $\text{supp}(\mu)$*. Further, if $\mu = \frac{1}{2\pi i}\phi(z) dz \wedge d\overline{z}$, and $\phi \in C^{(k)}$ on an open set ω, then $u \in C^{(k)}$ on ω and $\overline{\partial} u = \phi$ on ω.*

Proof. ∎

Let L be compact. Then $C(L)^* = \mathcal{M}(L)$ is a collection of Radon measures μ, such that $\phi \mapsto \mu(\phi)$ is linear and continuous with respect to the sup norm.

Example 0.1. Let $L = [0,1]$, $\mu(\phi) = \sum_{n=1}^{\infty} \phi(\frac{1}{n}) 3^{-n}$. Then $\text{supp}(\mu) = \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$.