Lecture 1, 1/11/13

Section 1: Vocabulary and easy definitions

Homological algebra is the study of complexes of R-modules, where R is a ring with identity $1 \neq 0$. Notationally, R-Mod is the category of all left R-modules, and R-mod is the category of all finitely generated R-modules.

Definition 0.1. Let A_n " $\in$ " R-mod for $n \in \mathbb{Z}$ and $d_n \in \operatorname{Hom}_R(A_n, A_{n-1})$ such that $d_{n-1} \circ d_n = 0$ for all $n \in \mathbb{Z}$. Then the sequence

$$\cdots \xrightarrow{d_{n+2}} A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} \cdots$$

is called a complex of R-modules, assuming $\operatorname{im}(d_n) \subseteq \ker(d_{n-1})$. The sequence

$$0 \longrightarrow A_m \longrightarrow A_{m-1} \longrightarrow \cdots \longrightarrow A_{n+1} \longrightarrow A_n \longrightarrow 0$$

will occur more frequently. A complex $\mathbb{A}$ is an exact sequence if $\operatorname{im}(d_n) = \ker(d_{n-1})$ for all $n \in \mathbb{Z}$. This is called a short exact sequence if there are no more than 3 non-zero terms. Given a complex $\mathbb{A}$, the nth homology modules (or groups, in some cases) of $\mathbb{A}$ is

$$H_n(\mathbb{A}) = \frac{\ker(d_{n-1})}{\operatorname{im}(d_n)}$$

Remark. Given a short exact sequence (hereby abbv. as SES)

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

f is a mono and g is an epi, so $C \simeq B/\operatorname{im}(f)$. If A, B are known, but not f, then infinitely many C are available to complete the short exact sequence.

Example 0.1. Let $R = k$, a field, and take $A = B = k^{(\mathbb{N})} = \oplus_{i\in\mathbb{N}} k$.

(i) $0 \longrightarrow A \xrightarrow{\text{Id}} B \longrightarrow 0$ is a SES.

(ii) Define $f : A \to B$ by

$$f(b_i) = b_{2i} \text{ for } i \in \mathbb{N}$$
$$g(b_0) = \begin{cases} 0 & i \text{ even} \\ b_{\tau(i)} & i \text{ odd} \end{cases}$$

Where $\tau : (2\mathbb{N} - 1) \to \mathbb{N}$ is a bijection. If $A = B = C = \kappa^{(\mathbb{N})}$, then

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a SES.

(iii) Let $R = \mathbb{Z}$. Then

$$0 \longrightarrow 3\mathbb{Z} \xrightarrow{\iota} \mathbb{Z} \xrightarrow{\equiv} \mathbb{Z}/3\mathbb{Z} \longrightarrow 0$$

is a SES.

(iv) Let $R = \mathbb{Z}$. The sequence

$$0 \longrightarrow \overbrace{6\mathbb{Z}}^{A_1} \xrightarrow{\iota} \overbrace{\mathbb{Z}}^{A_0} \xrightarrow{\equiv} \overbrace{\mathbb{Z}/3\mathbb{Z}}^{A_{-1}} \longrightarrow 0$$

is a complex which is not exact. In fact, $H_0(\mathbb{A}) = \overbrace{3\mathbb{Z}}^{\ker(g)} / \underbrace{6\mathbb{Z}}_{\text{im}(f)} \cong \mathbb{Z}/2\mathbb{Z}$.

(v) Let $R = \kappa[x, y]$, κ a field. Let f be the inclusion $(x) \hookrightarrow R[x, y]$. The sequence

$$0 \longrightarrow (x) \xrightarrow{f} R \xrightarrow{g} \kappa[y] \longrightarrow 0$$

where

$$g\left(\sum_{i,j=0}^{\text{finite}} a_{ij}x^i y^j\right) = \sum_{j>0}^{\text{finite}} a_{\sigma_j} y^j$$

is exact.

(vi) Let $R = \kappa[x, y]$. Define $\mathbb{A}$ as

$$0 \longrightarrow \overbrace{(x)}^{A_1} \xrightarrow{f} \overbrace{R}^{A_0} \xrightarrow{g} \overbrace{\underbrace{\kappa}_{=R/(x,y)}}^{A_{-1}} \longrightarrow 0$$

where

$$g\left(\sum_{i,j=0}^{\text{finite}} a_{ij}x^i y^j\right) = a_\infty$$

then $\ker(g) = (x, y)$ and $\text{im}(f) = (x)$, so $\mathbb{A}$ is not exact. In fact,

$$\begin{aligned} H_0(\mathbb{A}) &= (x, y)/(x) \\ &\simeq (y) \\ &\simeq R \end{aligned}$$

Note: If R is an integral domain and $x \in R \setminus \{0\}$, then $(x) \simeq R$ (as R-modules, not as rings!), with isomorphism $r \mapsto rx$.
Typical questions addressed by homological algebra:

(i) Suppose

$$\mathbb{A} : \quad \cdots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \longrightarrow \cdots$$

is an exact sequence in R-mod and $F : R - \text{mod} \to S - \text{mod}$ is a functor. Is the sequence

$$\cdots \longrightarrow F(A_{n+1}) \xrightarrow{F(d_{n+1})} F(A_n) \xrightarrow{F(d_n)} F(A_{n-1}) \longrightarrow \cdots$$

exact? $F(\mathbb{A})$ is a complex when F is additive, but it may or may not be exact.

(ii) Given A, C " $\in$ " $R - \text{mod}$, characterize all modules B such that there exists an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

As an example, $R = \mathbb{Z}$, $A = C = \mathbb{A}/p\mathbb{Z}$, p prime, then

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \xrightarrow{f} \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \xrightarrow{g} \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

with $f : x \mapsto (x, 0)$ and $g : (x, y) \mapsto y$ is a SES. Alternatively, we could take $f : x + p\mathbb{Z} \mapsto px + p^2\mathbb{Z}$ and $g : y + p^2\mathbb{Z} \mapsto y + p\mathbb{Z}$ to make

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \xrightarrow{f} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{g} \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

a SES. These are the only possibilities in this case! In general, though, there are infinitely many possibilities for B. Why is this interesting? If R is an artinian ring and M " $\in$ " $R - \text{mod}$, then there are only finitely many simple $s_1, \ldots, s_n$ " $\in$ " $R - \text{mod}$ up to isomorphism. Moreover, for M " $\in$ " $R - \text{mod}$, there is a chain

$$M = M_0 \supset M_1 \supset \cdots \supset M_\ell = 0$$

such that M_i/M_{i+1} is simple for all $i < \ell$. If the answer to question (ii) is known, then all objects in $R - \text{mod}$ of fixed length ℓ are known up to isomorphism! Simply proved by induction.

Algebraic Topology

Definition 0.2. The standard n-simplex $\triangle_n$ in $\mathbb{R}^n$ is the convex hull of $v_0, v_1, \dots, v_n$, where $v_0 = 0$ and $v_i = (0, \dots, 0, \overbrace{1}^{i}, 0, \dots, 0)$ (so the standard basis).
An oriented simplex is $(\triangle_x, [\pi])$, where $[\pi]$ is an equivalence class of permutations of $\{0, \dots, n\}$, where $\pi \sim \pi' \iff \text{sgn}(\pi) = \text{sgn}(\pi')$. We write

$$(\triangle_x, \pi) = [v_{\pi(0)}, v_{\pi(1)}, \dots, v_{\pi(n)}$$

and identify $\triangle_n$ with $[0, 1, \dots, n]$. The ngative is $-[w_0, \dots, w_n]$.

Definition 0.3. Let X be a topological space. An n-simplex in X is a continuous map

$$\sigma : \triangle_n \to X$$

The group of n-chains of X, $S_n(x)$, is the free abelian group having as basis the n-simplices in X. The singular chain complex of X is

$$\cdots \longrightarrow S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots \longrightarrow S_0(X) \longrightarrow 0$$

denoted $\mathbb{S}$, where $\partial_n : S_n(X) \to S_{n-1}(X)$ is the nth boundary map, which can be defined if we define $\partial_n(\sigma)$ for all n-simplices σ in X (i.e. in the basis of $S_n(X)$). Consider the map

$$\begin{aligned} \tau_i : \mathbb{R}^{n-1} &\to \mathbb{R}^n \\ (a_1, \dots, a_{n-1}) &\mapsto (a_1, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_n) \end{aligned}$$

For $i \in \{0, \dots, n\}$. Then τ_i is continuous and $\tau_i(\triangle_{n-1}) = \triangle_n$. Define

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma(\tau_i)$$

Theorem 0.1. *$\partial_{n-1} \circ \partial_n = 0$ for all $n \in \mathbb{N}$, i.e. $\mathbb{S}$ is a complex in $\mathbb{Z}$-mod.*

Definition 0.4. The group of n-cycles is $Z_n(X) = \ker(\partial_{n-1})$, and the group of n-boundaries is $B_n = \text{im}(\partial_n)$.
The nth homology group is $H_n(X) = Z_n(X)/B_n(X)$.

Lecture 2, 1/13/23

Chapter I: Categories and functors

There is a definition page on the Gaucho that has all the most basic definitions - objects, morphisms, compositions, etc.

If $f \in \mathrm{Hom}_C(A, B)$, we often write $A \xrightarrow{f} B$ even if f is not literally a map.

Example 0.2. **1.** The category of all sets, Set. The object class consists of all sets, and the morphisms are just set maps.

2. The category of all topological spaces, Top. The object class consists of all topological spaces, and the morphisms are continuous functions.

3. The category of all groups, Grp. The object class consists of all groups, and the morphisms are group homomorphisms.

4. Let $(P, \leq)$ be a partially ordered set with a relation $\leq$ which is reflexive, antisymmetric, and transitive. Then we can make P into a category, whose objects are the elements of p, and for $u, s \in P$, $\mathrm{Hom}_P(u, s) = \begin{cases} (u, s) & u \leq s \\ \varnothing & u \not\leq s \end{cases}$. We define the composition $(s, t)(u, s) \stackrel{\text{def}}{=} (u, t)$.

5. The opposite category of a category C, C^{op}.

6. Let R be a ring. R-Mod is the category of left R modules. R-mod is the finitely generated R-modules, and similarly for Mod-R and mod-R, which are the right R-modules.

7. R-comp. The object class consists of complexes of left R-modules.

Let $\mathbb{A}, \mathbb{A}'$ be objects of R-comp. Note: it is problematic to say "$\mathbb{A}, \mathbb{A}' \in R$-comp, as R-comp is not a set!

Say $\mathbb{A} = \cdots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \longrightarrow \cdots$, and similarly for $\mathbb{A}'$. An element of $\mathrm{Hom}_{R-comp}(\mathbb{A}, \mathbb{A}')$ will be a sequence of R-module homomorphisms $f_n : A_n \to A'_n$ which make the following diagram commute:

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & A_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \arrow[d] \\
\cdots \arrow[r] & A'_n \arrow[r, "d'_n"'] & A'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

8. The category of rings Ring, whose obejcts are rings and whose morphisms are ring homomorphisms.

9. The category of $\mathbb{Z}$-modules is usually denoted Ab. This is also the category of Abelian groups, and is the prototypical example of an Abelian category.

Definition 0.5. A category $\mathcal{C}$ is called pre-additive if for all A, B objects of $\mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is an additive Abelian group (additive means we use the symbol "+") such that for all eligible morphisms f, g, h, k,

$$h(f+g) = hf + hg$$
$$(f+g)k = fk + gk$$

where "elibigle" means that these expressions make sense and are well-defined.

Example 0.3. **1.** R-mod (in particular Ab)

2. R-comp

3. Ring fails to be pre-additive, because the identity morphisms add to be something which is not the identity morphism.

Definition 0.6. Let $\mathcal{C}, \mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ consists of an assignment $F_0 : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$, and for each pair of objects A, B" $\in$ "$\mathrm{Obj}(\mathcal{C})$, a map (this actually is a map because we assume hom-sets are in fact sets). $F_{A,B} : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ such that, for all eligible morphisms f, g, and all A" $\in$ "$\mathcal{C}$

(a) $F(\mathrm{Id}_A) = \mathrm{Id}_{F(A)}$

(b) $F(f \circ g) = F(f) \circ F(g)$

Example 0.4. **1.** Let $\mathcal{C}$ be a category. Then we have the identity functor $\mathrm{Id}_{\mathcal{C}}$, which assigns $\mathrm{Id}_{\mathcal{C}}(A) = A$, and $\mathrm{Id}_{\mathcal{A}}(f) = f$ for any eligible A" $\in$ "$\mathrm{Obj}(\mathcal{D})$ and morphisms f.

2. Functors $\pi_n : \mathsf{Top} \to \mathsf{Grp}$ which sends $X \mapsto \pi_n(X)$

3. $\mathbb{S} : \mathsf{Top} \to \mathbb{Z}$-comp, which sends $X \mapsto \mathbb{S}(X)$, which is a complex

$$\cdots \longrightarrow S_{n+1}(X) \xrightarrow{\partial_{n+1}} S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots \longrightarrow S_0(x) \xrightarrow{\partial_0} 0$$

Let $\phi : X \to Y$ be continuous for X, Y" $\in$ "Top. Then $\mathbb{S}(\phi)_n : S_n(X) \to S_n(Y)$ is given by $\sigma \mapsto \phi \circ \sigma$, and we can extend this for σ an n-simplex of X.

Lecture 4, 1/18/23

Functors:

Definition 0.7. Let $\mathcal{C}, \mathcal{D}$ be categories. A covariant functor from $\mathcal{C}$ to $\mathcal{D}$ consists of "maps" F_0 and $F|_{A,B}$ for any $A, B \in \text{Obj}(\mathcal{C})$ such that

- $F_0 : \text{Obj}(\mathcal{C}) \to \text{Obj}(\mathcal{D})$
- $F_{A,B} : \text{Hom}_{\mathcal{C}}(A, B) \to \text{Hom}_{\mathcal{D}}(F_0A, F_0B)$ for any A, B" $\in$ "$\text{Obj}(\mathcal{C})$

such that

(a) $F_{A,C}(fg) = F_{B,C}(f)F_{A,B}(g)$ for all eligible f, g

(b) $F_{A,A}(\text{Id}_A) = \text{Id}_{F(A)}$

from here on we don't care at all about indices. For simplicity, we will denote the action of a functor F as simply FA or Ff.

Definition 0.8. A contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ amounts to a covariant functor from $\mathcal{C}$ to $\mathcal{D}^{\text{op}}$.
More examples of functors

Example 0.5. Homology functors $H_n : R - comp \to \mathbb{Z} - mod$ which sends $\mathbb{A}$ to $H_n(A)$.
That is, $\mathbb{A} \to F\mathbb{A} = \frac{\ker(d_n)}{\text{Im}(d_{n+1})}$
Let $f \in \text{Hom}_{R-comp}(\mathbb{A}, \mathbb{A}')$. That is, the following diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & A_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \arrow[d] \\
\cdots \arrow[r] & A'_n \arrow[r, "d'_n"'] & A'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Ff acts by $a_n + \text{Im}(d_{n+1}) \to f_n(a_n) + \text{Im}(d'_{n+1})$. Let's prove that this is actually well-defined.
Check
First, $a_n \in \ker(d_n)$ implies $f_n(a_n) \in \ker(d'_n)$. This can be seen by doing a diagram chase on the above diagram. Since $d_n(a_n) = 0$, we have $0 = f_{n-1}d_n(a_n) = d'_n f_n(a_n)$, i.e. $f_n(a_n) \in \ker(d'_n)$.
"Don't do much thinking. It's almost harmful" - Birge on doing diagram chasing.
Also "follow your nose."
Now, $a_n \in \text{Im}(d_{n+1})$ implies $f_n(a_n) \in \text{Im}(d'_{n+1})$. So $a_n = d_{n+1}(x)$ with $x \in A_{n+1}$. hence $f_n(a_n) = f_n d_{n+1}(x) = a'_{n+1} f_{n+1}(x) \in \text{Im}(d'_{n+1})$.

Example 0.6. Let $\mathcal{C}, \mathcal{D}$ be pre-additive categories (definition on the top of page 6). A functor F "from" $\mathcal{C}$ to $\mathcal{D}$ is called additive if, for all A, B" $\in$ "$\mathrm{Obj}(\mathcal{C})$, the map $F : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ is a homomorphism of abelian groups.

Remark. Note that $H_n : R - comp \to \mathbb{Z} - mod$ is an additive functor. The π_n functor is not additive, as Top is not preadditive.

Example 0.7. Forgetful functors e.g. $F : R - mod \to \mathbb{Z} - mod$ which sends $M \mapsto M$, where the M on the left hand side is an R-module, and M on the right is just an abelian group, which is a $\mathbb{Z}$-module. Or $F : R - mod \to \mathsf{Set}$ which sends an R-module M to the set of its elements, "forgetting" the module structure.
Moreover, if $\mathcal{C}, \mathcal{D}$ are pre-additive, and $F : \mathcal{C} \to \mathcal{D}$ is a forgetful functor of some sort, then F is additive.

Example 0.8. Let $F : R - mod \to S - mod$ be an additive functor. Then F induces an additive functor $\tilde{F} : R - comp \to S - comp$, sending $\mathbb{A}$ to $F(\mathbb{A})$.
If $\mathbb{A}$ is a complex

$$\cdots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \longrightarrow \cdots$$

then $F(\mathbb{A})$ is

$$\cdots \longrightarrow F(A_n) \xrightarrow{F(d_{n+1})} F(A_n) \xrightarrow{F(d_n)} F(A_{n-1}) \longrightarrow \cdots$$

An extremely important question: if $\mathbb{A}$ is exact, is $F(\mathbb{A})$ exact? If not, how far does it deviate from being an exact sequence?

Example 0.9. Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{E}$ be functors. Then $G \circ F : \mathcal{C} \to \mathcal{E}$ is a functor. WARNING: we use $\circ$ but this isn't actually a function composition. This is just notation!!!
$G \circ F$ acts how one might think: for A" $\in$ "$\mathrm{Obj}(\mathcal{C})$, $G \circ F(A) = G(F(A))$, and for $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $G \circ F(f) = G(F(f)) \in \mathrm{Hom}_{\mathcal{E}}(G(F(A)), G(F(B)))$.
Of interest to us: $H_n \circ \tilde{F}$, where $F : R - mod \to S - mod$ is additive. This functor sends a complex $\mathbb{A}$ to $H_n(F(\mathbb{A}))$. This is especially of interest if $\mathbb{A}$ is exact, but $F(\mathbb{A})$ is not.

Remark. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F sends isomorphisms in $\mathcal{C}$ to isomorphisms in $\mathcal{D}$. This is immediate from the definition of a functor.

Section 2: two types of functors that will follow us

(i) Hom-functors: Whenever $\mathcal{C}$ is a category, there is a bifunctor

$$\mathrm{Hom}_{\mathcal{C}}(-, -) : \mathcal{C} \times \mathcal{C} \to \mathsf{Set}$$

which sends a pair (A, B) to $\mathrm{Hom}_{\mathcal{C}}(A, B)$, and on maps (note that this is co-variant in the first factor and contravariant inh the second), they act as follows. Let $f : A \to A', g : B \to B'$ be morphisms in $\mathcal{C}$. Then

$$\mathrm{Hom}(f, g) : \mathrm{Hom}_{\mathcal{C}}(A', B) \to \mathrm{Hom}_{\mathcal{C}}(A, B')$$

acts by $\phi \mapsto g \circ \phi \circ f$

Lecture 5, 1/20/23

Whenever $\mathcal{C}$ is a category, there is a bifunctor $\mathrm{Hom}_{\mathcal{C}}(-, -) : \mathcal{C} \times \mathcal{C} \to \mathsf{Set}$, which sends (A, B) to $\mathrm{Hom}_{\mathcal{C}}(A, B)$. On maps, when $f : A \to A'$ and $g : B \to B'$ are morphisms, then

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(f, g) : \mathrm{Hom}_{\mathcal{C}}(A', B) &\to \mathrm{Hom}_{\mathcal{C}}(A, B') \\ \varphi &\mapsto g \circ \varphi \circ f \end{aligned}$$

We will split this into two parts. Let $C`` \in "\mathcal{C}$. Then we have a covariant functor

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(C, -) : \mathcal{C} \to \mathcal{C} \\ C' &\mapsto \mathrm{Hom}_{\mathcal{C}}(C, C') \\ g &\mapsto \mathrm{Hom}_{\mathcal{C}}(C, A) \to \mathrm{Hom}_{\mathcal{C}}(C, B) \\ \varphi &\mapsto g \circ \varphi \end{aligned}$$

We also have the contravariant functor $\mathrm{Hom}_{\mathcal{C}}(-, D)$, which acts similarly.
As a special case, consider $\mathcal{C} = R - mod$. Then

$$\begin{aligned} \mathrm{Hom}_R(M, -) : R - mod \to \mathbb{Z} - mod \\ \mathrm{Hom}_R(-, N) : R - mod \to \mathbb{Z} - mod \end{aligned}$$

but we can have additional structure on $\mathrm{Hom}_R(M, N)$. Suppose ${}_RM_S$ is a bimodule (S is a ring and $(rm)s = r(ms)$) and let ${}_RN_T$ be an $R-T$ module. Then $\mathrm{Hom}_R(M, N)$ is a left S, right T bimodule. For $f \in \mathrm{Hom}_R(M, n)$, $s \in S, t \in T$, define

$$\begin{aligned} (sf)(m) = f(ms) \\ (ft)(m) = f(m)t \end{aligned}$$

If R is commutative, then

$$\begin{aligned}\operatorname{Hom}_R(M,-) : R-mod \to R-mod = Mod-R \\ \operatorname{Hom}_R(-,N) : R-mod \to R-mod = Mod-R\end{aligned}$$

If ${}_RM_S$ is a bimodule, then

$$\operatorname{Hom}_R(M,-) : R-mod \to S-mod$$

If ${}_RN_T$ is a bimodule, then

$$\operatorname{Hom}(-,N) : R-mod \to Mod-T$$

Basic properties:

(i) $M``\in" R-mod \implies \underbrace{\operatorname{Hom}_R(R,M) \cong M}_{f\mapsto f(1)}$ in $\mathbb{Z}-mod$

(ii) $\operatorname{Hom}_R(\otimes_{i\in I} M_i, N) \cong \prod_{i\in I} \operatorname{Hom}_R(M_i,N)$. Prove this!

(iii) $\operatorname{Hom}_R(M, \prod_{i\in I} N_i) \cong \prod_{i\in I} \operatorname{Hom}_R(M,N_i)$. Prove this!

Definition 0.9. Let $M``\in" Mod-R$, $N``\in" R-mod$. Then an abelian group T is called a tensor product of M and N if there exists a map

$$\tau : M\times N \to T$$

Which is $\mathbb{Z}$-bilinear and R-balanced, i.e.

$$\tau(mr,n) = \tau(m,rn)$$

with the following universal property.
Whenever A is an abelian group and $\sigma : M\times N \to A$ is $\mathbb{Z}$-bilinear and R-balanced, there exists a unique $\mathbb{Z}$-linear map $\sigma' : T \to A$ such that this diagram commutes:

$$\begin{tikzcd} M\times N \arrow[r, "\tau"] \arrow[dr, "\sigma"'] & T \arrow[d, "\sigma'"] \\ & A \end{tikzcd}$$

We denote $T = M\otimes_R N$.

Theorem 0.2. *If $M``\in" Mod-R$ and $N``\in" R-mod$, then a tensor product $M\otimes_R N$ exists and is unique up to isomorphism.*

Proof. Let F be the free abelian group with basis $M \times N$, i.e.

$$F = \otimes_{m \in M, n \in N} \mathbb{Z}(m,n)$$

Define

$$M \otimes_R N = F/U$$

where U is the submodule generated by all elements of the form

$$\begin{aligned}(m_1 + m_2, n) - (m,n) - (m_2, n) \\ (m, n_1 + n_2) - (m, n_1) - (m, n_2) \\ (mr, n) - (m, rn)\end{aligned}$$

for all eligible $m_i, m \in M, n_i, n \in N, r \in R$.
Define

$$\begin{aligned}\tau : &M \times N \to M \otimes_R N \\ &(m,n) \mapsto m \otimes n\end{aligned}$$

Then τ is $\mathbb{Z}$-bilinear and R-balanced (check!). Moreover, $M \otimes_R N$ with τ satisfies the universal proeprty: let A be an abelian group and $\sigma : M \times N \to A$ be $\mathbb{Z}$-bilinear and R-balanced. Define

$$\begin{aligned}\tilde{\sigma} : &F \to A \\ &(m,n) \mapsto \sigma(m,n)\end{aligned}$$

and extend linearly. By construction, $\tilde{\sigma}(U) = 0$, i.e. $U \subseteq \ker(\tilde{\sigma})$. Hence there exists $\sigma' : F/U \to A$ with the property that

$$\sigma'(m,n) = \tilde{\sigma}((m,n) + n) = \tilde{\sigma}(m \otimes n)$$

Now show σ' is unique, and the proof is complete. ■

Lecture 6, 1/23/23

Our two mainstay types of functors:

(i) Hom functors.

(ii) Tensor functors. For (M,N)"$\in$"$Mod-R\times R-mod$, we constructed an abelian group $M\otimes_R N = R^{(M\times N)}/u$, together with $\tau : M\times N \to M\otimes_R N$ given by $\tau(m,n) = m\otimes n = (m,n)+u$ such that $(M\otimes_R N, \tau)$ has the key universal property.

Note: The elements $m\otimes n \in M\times N$ form a generating set of $M\otimes_R N$, but not a basis.

The tensor functor

We have a bifunctor $-\otimes - : Mod-R\times R-mod \to \mathbb{Z}-mod$, $(M,N)\mapsto M\otimes_R N$. Let $(f,g), f\in \mathrm{Hom}_R(M,M'), g\in \mathrm{Hom}(N,N')$. Then

$$\begin{aligned} f\otimes g : &M\otimes_R N \to M'\otimes_R N' \\ &m\otimes n \mapsto f(m)\otimes g(n) \end{aligned}$$

To show this is well-defined, check that $\phi : M\times N \to M'\otimes N'$, $(m,n)\mapsto f(m)\otimes g(n)$ is $\mathbb{Z}$-bilinear and R-balanced.
Split $-\otimes_R -$ into two functors. So, we have a functor $M\otimes_R - : R-mod \to \mathbb{Z}-mod$ and a functor $-\otimes_R N : Mod-R\to \mathbb{Z}-mod$. The action on objects and morphisms is clear from the discussion up to now.

Additional structure on $M\otimes_R N$

Suppose ${}_SM_R$ and ${}_RN_T$ are bimodules. Then $M\otimes_R N$ is a $S-T$ bimodule, with

$$s(m\otimes n)t = (sm)\otimes(nt)$$

It is an exercise to check well-definedness.
Uses
Suppose ${}_{\mathbb{R}}V$ is a real vector space. We want to "complexify" V, making it a complex vector space. We could consider $\mathbb{C}\times V$, and define $c(d,v) = (cd, v)$. But this does not define a $\mathbb{C}$-vector space, because multiplication must be multilinear. But $\mathbb{C}\otimes_{\mathbb{R}} V$ will do it.
Basic properties
Consider $R\otimes_R M$. This is in fact isomorphic to M. Not just as Abelian groups, but as left R-modules. This is because R satisfies the associative law relative to multiplication. One isomorphism between them is $m\mapsto 1\otimes m$.
In general, unlike the hom-functor, the tensor functor will not commute with direct products/coproducts, unless "the sky is very benevolent."

The meaning of $m\otimes n$ depends on the meaning of M,N!

Example 0.10. Consider $2 \otimes \overline{1} \in \mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})$. This is the same as $1 \otimes \overline{2} = 1 \otimes 0 = 0$. By contrast, look at $2 \otimes \overline{1} \in 2\mathbb{Z} \otimes (\mathbb{Z}/2\mathbb{Z})$. This is nonzero! Let's show that. We know $2\mathbb{Z} \cong \mathbb{Z}$, with an isomorphism given by $x \mapsto \frac{x}{2}$. So

$$\begin{aligned} f \otimes \mathrm{Id}_{\mathbb{Z}/2\mathbb{Z}} : & 2\mathbb{Z} \otimes (\mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z} \otimes (\mathbb{Z}/2\mathbb{Z}) \\ & x \otimes y \mapsto f(x) \otimes y \end{aligned}$$

But functors take isomorphisms to isomorphisms, so $\underbrace{2 \otimes \overline{1}}_{\in 2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z}} \mapsto \overbrace{1 \otimes \overline{1}}^{\in \mathbb{Z}/2\mathbb{Z}} \neq 0$. Why is this last term nonzero? Because $\mathbb{Z} \otimes (\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}$, with the isomorphism sending $1 \otimes \overline{1}$ to $\overline{1}$, which is not zero.

Natural Transformations, Equivalences, and Dualities

Definition 0.10. **1.** Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A <u>morphism of functors</u>, or <u>a natural transformation</u> from F to G, is a family $(\phi(C))_{C \text{``}\in\text{''} \mathrm{Obj}(\mathcal{C})}$ of morphisms, $\phi(C) : F(C) \to G(C)$ such that for any $f \in \mathrm{Hom}_{\mathcal{C}}(C, C')$, the square

$$\begin{tikzcd} F(C) \arrow[r, "F(f)"] \arrow[d, "\phi(C)"'] & F(C') \arrow[d, "\phi(C)"] \\ G(C) \arrow[r, "G(f)"'] & G(C') \end{tikzcd}$$

commutes for all eligible morphisms f in the category $\mathcal{C}$. This is a covariant equivalence. A contravariant equivalence is an equivalence between contravariant functors, i.e. it makes the following square commute.

$$\begin{tikzcd} F(C) \arrow[d, "\phi(C)"'] & F(C') \arrow[l, "F(f)"'] \arrow[d, "\phi(C)"] \\ G(C) & G(C') \arrow[l, "G(f)"] \end{tikzcd}$$

2. Call $(\phi(C))_{C \text{``}\in\text{''} \mathrm{Obj}(\mathcal{C})}$ an <u>isomorphism of functors</u>, or <u>a natural equivalence</u>, if $\phi(C)$ is an isomorphism for each $C \text{``} \in \text{''} \mathrm{Obj}(\mathcal{C})$.

3. Two categories $\mathcal{C}, \mathcal{D}$ are equivalent categories if there are functors $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F \simeq \text{Id}_{\mathcal{C}}$ and $F \circ G \simeq \text{Id}_{\mathcal{D}}$, with " $\simeq$ " meaning "is naturally equivalent to." The F, G are called "mutually inverse equivalences."

4. A contravariant equivalence is called a duality.

5. Let R, S be rings. Call R, S Morita equivalent, denoted $R \sim S$, if $R{-}mod, S{-}mod$ are naturally equivalent. This is equivalent to saying $Mod - R, mod - S$ are naturally equivalent.