

# Lecture 1

We will be using Hatcher's Algebraic Topology. The topology sequence is usually something like

$A$  Topological Spaces

$B$  Cell Complexes

$C$  Manifolds

## Theorem 0.1. (BIG Theorem)

*Given a “reasonably nice” space, there is a bijection between connected covers of a space and subgroups of the fundamental group.*

## Categories:

*Algebraic structures that are much flabbier than a group. They consist of*

- *A collection of arrows*
- *A partial binary operation on these arrows*
- *Objects, which arrows go between*

*We also want a composition law. That is, for objects and arrows*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

*there is an arrow  $A \xrightarrow{g \circ f} C$ . We want this composition to be associative, that is  $(f \circ g) \circ h = f \circ (g \circ h)$ , and we want objects to have identity arrows.*

*Not all functions have inverses. Using sets and functions as an example, we have described the category *Set*.*

*Here are some more examples of categories:*

**Example 0.1.**     • Groups and group homomorphisms (*Grp*)

- Topological spaces and continuous functions (*Top*)
- etc.

We can make the following new category.

**Definition 0.1.** We denote by  $\mathbf{Top}^*$  the category of based topological spaces, whose objects are pairs  $(X, x_0)$ , where  $X$  is a topological space and  $x_0 \in X$ , and whose morphisms are continuous functions  $f : (X, x_0) \rightarrow (Y, y_0)$  such that  $f(x_0) = y_0$ .

## Goal:

Our goal is to get a functor from  $\mathbf{Top}$  to  $\mathbf{Grp}$ . The fundamental group functor  $\pi_1$  will go from  $\mathbf{Top}^*$  to  $\mathbf{Grp}$ .

## Lecture 2

### Topology review:

**Definition 0.2.** A topological space is a set  $X$  along with a collection of subsets of  $X$  called “open sets,” such that  $X, \emptyset$  are open, and the arbitrary union and finite intersection of open sets are open.

Notice the following diagram commutes using the product topology

$$\begin{array}{ccccc}
 & & Z & & \\
 & f \swarrow & \vdots \exists! & \searrow g & \\
 X & \xleftarrow{P_X} & X \times Y & \xrightarrow{P_Y} & Y
 \end{array}$$

And in general

$$\begin{array}{ccc}
 & Z & \\
 & \vdots \exists! & \\
 \prod_{\alpha \in A} X_\alpha & \xrightarrow{P_\alpha} & X_\alpha
 \end{array}
 \begin{array}{c}
 \nearrow f_\alpha \\
 \\
 \\
 \end{array}$$

Maps are continuous; functions are not.

**Lemma 1.** (*Gluing lemma*)

Suppose  $f : A \rightarrow Y$ ,  $g : B \rightarrow Y$  are continuous, and  $f(x) = g(x)$  for all  $x \in A \cap B$ . Then  $f \cup g : A \cup B \rightarrow Y$  is continuous. This only holds as long as  $A, B \subseteq X$  are closed.

### Same Shape, Same Map

(maps up to wriggling things around a bit)

**Definition 0.3.** Two maps are homotopic if there exists a parametrized map  $f_t : X \rightarrow Y$  such that  $f_0 = f, f_1 = g$  for  $f, g : X \rightarrow Y$ . Equivalently, and more precisely, if there exists a map  $F : X \times [0, 1] \rightarrow Y$  such that  $F(x, 0) = f(x), F(x, 1) = g(x)$  for all  $x \in X$ .

$X, Y$  topological spaces are said to have the same shape if there exist maps  $f : X \rightarrow Y, g : Y \rightarrow X$  such that  $g \circ f \simeq \text{Id}_X$  and  $f \circ g \simeq \text{Id}_Y$ . We may say that  $X, Y$  have the same homotopy type

**Definition 0.4.** A deformation retraction from  $X \rightarrow A \subseteq X$  is a map from  $X \times I \rightarrow X$  such that, for all  $x \in A$ , and  $s, t \in I$ ,

$$\begin{aligned} f_0(x) &= x & \forall x \in X \\ f_1(x) &\in A & \forall x \in X \\ f_t(x) &= f_s(x) & \forall x \in A \end{aligned}$$

## Lecture 3, 1/13/23

**Definition 0.5.** Let  $X$  be a topological space. A retraction is a map  $r : X \rightarrow X$  such that  $r \circ r = r$ . That is,  $r(r(x)) = r(x)$  for any  $x \in X$ . Let  $A = r(X)$ . Then  $r|_A = \text{Id}_A$ .

**Definition 0.6.** Let  $F : X \times I \rightarrow Y$ . We say  $f_0 \simeq f_1 \text{ rel } A \subseteq X$  are homotopic relative to  $A$  if, for any  $x \in A$ ,  $f_t(x)$  is independent of  $t$ . That is, for any  $s, t \in I$ ,  $f_s(x) = f_t(x)$  for any  $x \in A$ .

For any map  $f : X \rightarrow Y$ , there exists a space  $Z \simeq Y$  via  $g : Y \rightarrow Z$  such that  $g \circ f : X \rightarrow Z$  is injective. That is, in the following diagram, we have a bijection between homotopy classes of maps  $f$  and homotopy classes of maps  $g \circ f$ , and we can do this in a way that rigs  $g \circ f$  to be injective.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow g \circ f & \downarrow g \\ & & Z \end{array}$$

**Definition 0.7.** Given a map  $f : X \rightarrow Y$  we can construct the Mapping Cylinder  $M_f$  by setting  $M_f = X \times I \amalg Y / \sim$ , where  $(x, 0) \sim f(x)$ .

The visual intuition should be taking the disjoint union of  $X$  and  $Y$ , and tying a string between  $x$  and  $f(x)$  for each point.

**Claim.**  $X \hookrightarrow M_f, Y \hookrightarrow M_f$ , and the latter is in fact a homotopy equivalence. Further, the injection  $X \hookrightarrow M_f$  is homotopic to  $f(X) \hookrightarrow M_f$ .

*Proof.* You can construct a homotopy which “squishes” the cylinder down to  $f(X)$ . ■

**Definition 0.8.** A space  $X$  is contractible if it has the homotopy type of a point. A map is null-homotopic if it is homotopic to a constant map. So  $X$  is contractible if the identity is null-homotopic.

Now he's drawing an example. The example is Bing's House with 2 rooms, which I will not reproduce here. But the point is that it's contractible, but not obviously so.

## Cell Complexes

Cell complexes are topological spaces which are built up inductively out of closed balls in Euclidean space. We write  $\mathbb{D}^n := \{\vec{x} \in \mathbb{R}^n \mid \|\vec{x}\| \leq 1\}$ , and  $e^n := \{\vec{x} \in \mathbb{R}^n \mid \|\vec{x}\| < 1\}$ . We can see that  $e^n = \text{int } \mathbb{D}^n$ , and  $\partial \mathbb{D}^n = \mathbb{S}^{n-1}$ .

### Base step

Start with some collection of points  $X^0$ , the 0-skeleton, with the discrete topology.

### Inductive step

Let  $X^{n-1}$  be the  $n-1$  skeleton, which has already been build and defined. Select some collection of  $n$ -dimensional balls  $\{\mathbb{D}^n\}_{\alpha \in A}$ , and some continuous “attaching map”  $\varphi_\alpha : \partial \mathbb{D}_\alpha^n \rightarrow X^{n-1}$ . Then

$$(X^n = X^{n-1} \coprod_{\alpha \in A} \mathbb{D}_\alpha^n) / (x \sim \varphi_\alpha(x) \forall x \in \partial \mathbb{D}_\alpha^n)$$

## Lecture 4, 1/18/23

A space  $X$  is a cell complex if it has been constructed using the above inductive procedure. If  $n = \infty$ , we use the weak topology, in which the open sets are the sets which are open when intersected with each  $X^n$ .

For every  $\mathbb{D}_\alpha^n$  and corresponding “attaching map”  $\varphi_\alpha : \partial \mathbb{D}_\alpha^n \rightarrow X^{n-1}$ , there is a subset of  $X^n$  homeomorphic to  $\text{int}(\mathbb{D}_\alpha^n)$ , via the composition

$$\text{int}(\mathbb{D}_\alpha^n) \hookrightarrow \mathbb{D}_\alpha^n \hookrightarrow X^{n-1} \coprod_{\alpha} \mathbb{D}_\alpha^n \rightarrow X^n$$

which we call  $\Phi_\alpha : \mathbb{D}_\alpha^n \rightarrow X^{n-1}$ . So the attaching map  $\phi_\alpha : \partial \mathbb{D}_\alpha^n \rightarrow X^{n-1}$  extends to a “characteristic map”  $\Phi_\alpha$ .

We will now see many examples of things.

**Example 0.2.** If you stop after constructing  $X^1$ , it's a graph.

**Example 0.3.**  $\mathbb{S}^n$  has a cell structure with one  $e_0$  and one  $e_n$ .

**Example 0.4.** Consider  $\mathbb{RP}^2$ . This can be expressed as  $(\mathbb{R}^3 \setminus \{0\})/(\vec{x} \sim \lambda\vec{x}, \lambda \neq 0)$ . We can replace 2 with any  $n$  and get  $\mathbb{RP}^n$ . Indeed, we can replace  $\mathbb{R}$  with  $\mathbb{C}$ ,  $\mathbb{H}$ , or indeed any field.

Homogenous coordinates

For  $(x, y, z) \neq (0, 0, 0)$ , we have  $[x, y, z] \stackrel{\text{def}}{=} \{(\lambda x, \lambda y, \lambda z) \mid \lambda \neq 0\}$ . For example,  $[1, 2, 3] = [2, 4, 6]$ .

## Lecture 5, 1/20/23

**Definition 0.9.** A subcomplex of a complex  $X$  is a closed disjoint union of open cells  $e_{\alpha_i}^{n_i}$  in  $X$  such that they form a cell complex on their own.

*Remark.* We keep talking about “CW Complexes.” The C is for “closure finite,” and the W is for “weak topology.”

Recall:  $\mathbb{RP}^n \stackrel{\text{def}}{=} \mathbb{R}^n/(x \sim \lambda x, \lambda \neq 0)$ .  $\mathbb{CP}^n$  can be defined similarly.

We can write  $\mathbb{RP}^n = e^0 \cup e^1 \cup e^2 \cup \dots \cup e^n$ , and  $\mathbb{CP}^n = e^0 \cup e^2 \cup e^4 \cup \dots \cup e^{2n}$ . We can do the same thing with the quaternions.

Next time, we will cover operations on complexes.

## Lecture 6, 1/23/23

This lecture, we will cover operations on cell complexes, and two big theorems.

### Operations on Cell Complexes

1. If  $X, Y$  have cell structures, then  $X \times Y$  has a natural cell structure.
2. If  $(X, A)$  is a CW-pair, then  $X/A$  has a natural cell structure ( $X/A$  denotes identifying all points in  $A$  together).

$$\begin{array}{ccc}
 \mathbb{D}'_{\alpha} \supseteq \partial \mathbb{D}'_{\alpha} & & \\
 \downarrow \phi_{\alpha} & \searrow & \\
 X^0 & \xrightarrow{q} & (X/A)^0
 \end{array}$$

3. Cones and Suspensions. The cone on  $X$ ,  $CX$ , is defined as

$$CX = (X \times I)/(X \times \{0\})$$

Note that  $CX$  is contractible for any  $X$ . The suspension on  $X$ ,  $SX$ , is defined as

$$SX = CX/(X \times \{1\})$$

If  $f : X \rightarrow Y$  is a map, there exists a natural map  $Sf : SX \rightarrow SY$ . Indeed, if  $f : X \rightarrow Y$ , then  $f \times \text{Id} : X \times I \rightarrow Y \times I$ , and so we can factor  $f \times \text{Id}$  through the quotient map:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & \searrow & \downarrow \\ X \times I & \xrightarrow{f \times \text{Id}} & Y \times I \\ \downarrow q & \searrow & \downarrow \\ SX & \xrightarrow{\exists! Sf} & SY \end{array}$$

Note  $S(\mathbb{S}^n) = \mathbb{S}^{n+1}$ .

4. Joins. If  $X, Y$  are cell structures, then we define their join  $X \star Y$  as

$$X \star Y = \frac{X \times Y \times I}{(x, y_1, 0) \sim (x, y_2, 0), (x_1, y, 1) \sim (x_2, y, 1)}$$

This is a useful construction for simplices.

5. Wedge product. If  $X, Y$  are cell structures, with distinguished points  $x_0 \in X, y_0 \in Y$ , then we define their wedge product  $X \wedge Y$  as

$$X \wedge Y = \frac{X \amalg Y}{x_0 \sim y_0}$$

This is just gluing  $X$  and  $Y$  together at a distinguished point. This raises an obvious question: does the wedge product depend on the points  $x_0, y_0$ ? Yes, but not if they are (connected) cell complexes!

If  $x_0$  is a 0-cell of  $X$ , and  $y_0$  a 0-cell of  $Y$ , then  $X \vee Y$  has a natural cell structure AND  $X \vee Y$  is a subcomplex of  $X \times Y$ .

6. Smash product. If  $X, Y$  are spaces with distinguished points  $x_0, y_0$ , then the smash product is defined as

$$X \wedge Y = \frac{X \times Y}{X \vee Y}$$

For example, the smash product  $S^1 \wedge S^1$  is a Torus quotiented out by the longitudinal and meridian circles. By arguing from some cell nonsense, we can say this is  $S^2$ .

Here are two big theorems.

*Theorem 0.2. If  $(X, A)$  is a CW-pair, and  $A$  is contractible, then  $X/A$  is homotopy equivalent to  $X$ , with the quotient mapping itself providing a homotopy equivalence.*

*Theorem 0.3. Suppose  $(X_1, A)$  is a CW-pair, and  $f, g : A \rightarrow Y$  are maps. If  $f \simeq g$ , and everything in sight is a cell complex, then*

$$X_1 \coprod_f Y \simeq X_1 \coprod_g Y$$

*That is, if  $f, g$  are used as attaching maps, then the resulting spaces will be homotopy equivalent.*

## Lecture 7, 1/25/23

*Definition 0.10.* Let  $X$  be a cell complex. If we let  $f_i$  be the number of  $i$ -dimensional cells in the cell structure, then we define

$$\chi(X) = f_0 - f_1 + f_2 - f_3 + \cdots$$

The more general definition is the alternating sum of the Betti numbers of  $X$ , where the  $i$ th Betti number is  $\dim H^i(X)$ .

*Definition 0.11.* Let  $X$  be a topological space, and let  $A \subseteq X$  be a subspace. We say that  $(X, A)$  has the homotopy extension property (HEP) if for all topological spaces  $Y$  and for all maps  $f : X \times \{0\} \cup A \times I \rightarrow Y$ , there exists an extension of  $f$ ,  $\bar{f} : X \times I \rightarrow Y$ , such that  $\bar{f}|_{X \times \{0\} \cup A \times I} = f$ .

Slogan: “A homotopy on the subspace can be extended to a homotopy on the entire space.”

## Lecture 8, 1/27/23

*Proposition 1.*  $(X, A)$  has the homotopy extension property if and only if  $X \times I$  retracts to  $X \times \{0\} \cup A \times I$ .

*Proof.* ■

*Example 0.5.* Does  $(\mathbb{D}^2, \partial\mathbb{D}^2)$  have the property? Does  $\mathbb{D}^2 \times I$  retract onto  $\mathbb{D}^2 \times \{0\} \cup \partial\mathbb{D}^2 \times I$ ? Yes. This is easy to see by drawing a picture.

Here is a non-example. Let  $X = I$ , and let  $A = \{\frac{1}{n}\}_{n \in \mathbb{N}} \cup \{0\}$ .  $X \times I$  is the square, and  $X \times \{0\}$  is the bottom of the square, so  $X \times \{0\} \cup A \times I$  is the comb space. The square doesn't retract to this.

*Proposition 2. If  $(X, A)$  is a CW pair, then  $(X, A)$  has the homotopy extension property.*

*Proof.* Later ■

*Theorem 0.4. If  $(X, A)$  has the homotopy extension property, and  $A$  is contractible, then the quotient map  $X \rightarrow X/A$  is a homotopy equivalence.*

*Proof.* Consider identity map  $\text{Id} : A \rightarrow A$ . We have a homotopy  $F : A \times I \rightarrow A$  which is a witness to  $A$  being contractible. That is,  $f_0 = \text{Id}_A$ ,  $f_1 \equiv \{p\}$  for some point  $p \in A$ .

Then there is an extension to a homotopy  $H : X \times I \rightarrow X$ . We have the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f_t} & X \\ \downarrow q & & \downarrow q \\ X/A & \xrightarrow{\bar{f}_t} & X/A \end{array}$$

Because all of  $A$  goes to a point for  $t = 1$ , then by the universal property of quotients, there is a map  $g$  making the diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{f_1} & X \\ \downarrow q & \nearrow g & \downarrow q \\ X/A & \xrightarrow{\bar{f}_1} & X/A \end{array}$$