Lecture 1

Let $(X, \mathcal{A}, \mu)$ be a measure space. Without any additional structure or information, we may define the Lebesgue integral $\int_X f\,d\mu$ for f an $\mathcal{A}-\mathcal{B}$ measurable function $f : X \to [-\infty, +\infty]$.
We only have a few examples without any work.

Example 0.1.
- For any set X, we can define the counting measure on $\mathcal{A} = 2^X$, which gives $\mu(A) = |A|$. If $X = \mathbb{N}$, then a measurable function is just a sequence (f_n), and $\int_X f\,d\mu = \sum f_n$
- We can also define the Dirac mass δ_p for a fixed $p \in X$ by

$$\delta_p(E) = \begin{cases} 1 & p \in E \\ 0 & p \notin E \end{cases}$$

  We have $\int_X f\,d\delta_p = f(p)$

To get another example of a measure we need to do some work.
Problem: We want a measure μ on $\mathbb{R}^n$ such that, for a rectangle,

$$\mu([a_1, b_1] \times \cdots \times [a_n, b_n]) = |a_1 - b_1| \cdots |a_n - b_n|$$

Once it is defined on all rectangles, it is defined on the minimal σ-algebra containing them, which is the Borel σ-algebra. In other words, this condition will completely specify a measure on the Borel σ-algebra $\mathcal{B}_{\mathbb{R}^n}$
If $X = \mathbb{R}^n$, or a general metric space, or even a general topological space, then $\mathcal{B}(X)$ denotes the σ-algebra generated by the open subsets of X.
Problem:
Suppose we have a distribution function $F : \mathbb{R} \to \mathbb{R}$, meaning F is monotone, positive, and $\lim_{x\to-\infty} f(x) = 0, \lim_{x\to\infty} f(x) = 1$, and continuous from the right. We want a Borel measure μ such that $F(t) = \mu((-\infty, t])$. Such a measure, denoted by λ_F, is called a Lebesgues-Stieltjes measure.
The corresponding integral is called a Lebesgue-Stieltjes integral.
If F is smooth, then $\int_{\mathbb{R}} \phi\,d\lambda_F = \int_{-\infty}^{\infty} \phi(x) d\,F(x)$.
The measure we want on $\mathbb{R}^n$ is denoted by λ^n.

The Carathéodory Construction

Suppose we have an outer measure $\gamma : 2^X \to [0, \infty]$. This means $\gamma(\varnothing) = 0$, $A \subset B \implies \gamma(A) \le \gamma(B)$ (monotone), and $\gamma(\cup_{i=1}^{\infty} E_i) \le \sum_{i=1}^{\infty} \gamma(E_i)$ (subadditive).
We can define a set S to be γ-measurable if for every testing set T, $\gamma(T) = \gamma(S \cap T) + \gamma(S^c \cap T)$.

Theorem 0.1. *(Carathéodory Extension Theorem)*

1. $\gamma(N) = 0 \implies N$ *is measurable.*

2. *The set of measurable sets forms a σ-algebra Γ.*

3. *γ restricted to Γ forms a measure.*

"Nothing in the above theorem can guarantee you that Γ is not trivial, i.e. $\Gamma = \{\varnothing, X\}$. Nevertheless, this is a very useful guy" - Dennis.

Definition 0.1. (Lebesgue outer measure on $\mathbb{R}^n$)
Let R be a rectangle in $\mathbb{R}^n$, that is $R = \prod_{i=1}^n [a_i, b_i]$. We have $\mathrm{Vol}(R) = |a_1 - b_1| \cdots |a_n - b_n|$. For any $E \subseteq \mathbb{R}^n$, we define

$$\mu^*(E) \overset{\text{def}}{=} \inf\{\sum_{j=1}^{\infty} \mathrm{Vol}(R_j) \mid E \subseteq \cup_{j=1}^{\infty} R_j\}$$

Proposition 1. *μ^* is an outer measure on $\mathbb{R}^n$ such that $\mu^*(R) = \mathrm{Vol}(R)$ for all rectangles R.*

Proof. The first and second axioms are trivial, so we will just prove the subadditivity. Let E be some set. By definition, for any ε, there is some cover R_j by recrtangles such that

$$-\varepsilon + \sum_{j=1}^{\infty} \mathrm{Vol}(R_j) \leq \mu^*(E) \leq \sum_{j=1}^{\infty} \mathrm{Vol}(R_j)$$

meaning that $\sum_{j=1}^{\infty} \mathrm{Vol}(R_j) \leq \mu^*(E) + \varepsilon$. So for each E_k, there is a sequence R_j^k which covers E_k, such that $\sum_{j=1}^{\infty} \mathrm{Vol}(R_j^k) \leq \mu^*(E) + \frac{\varepsilon}{2^k}$.
So $\{R_j^k\}_{j,k \in \mathbb{N}}$ forms a cover of $\cup_{j=1}^{\infty} E_j$. Thus

$$\begin{aligned}
\mu^*(\cup_{k=1}^{\infty} E_k) &\leq \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \mathrm{Vol}(R_j^k) \\
&\leq \sum_{k=1}^{\infty} \left(\mu^*(E_k) + \frac{\varepsilon}{2^k} \right) \\
&= \sum_{k=1}^{\infty} \mu^*(E_k) + \varepsilon
\end{aligned}$$

This is true for any positive ε. Taking the limit as $\varepsilon \to 0$ gives the result.

∎

Now, fix a rectangle R. Note that R itself forms a cover of R, so by the definition, $\mu^*(R) \leq \operatorname{Vol}(R)$. For $\varepsilon > 0$, we can take an almost-optimal cover (R_j) such that $\sum_{j=1}^{\infty} \operatorname{Vol}(R_j) \leq \operatorname{Vol}(R) + \varepsilon$. We can rig it such that $|\operatorname{Vol}(R_j) - \operatorname{Vol}(R)| \leq \frac{\varepsilon}{2^j}$. Because $R \subset \cup_{j=1}^{\infty} R_j$, and R_j is an open cover, by compactness of R there is a finite subcover, and the volume of R is less than or equal to the sum of the volumes of these finitely many R_j. So the volume of R is less than or equal to $\mu^*(R) + 2\varepsilon$. So $\operatorname{Vol}(R) = \mu^*(R)$.

Proposition 2. *Every rectangle R in $\mathbb{R}^n$ is Carathéodory measurable).*

Proof. I missed this lol. Apparently Dennis denotes $\mathcal{M}_{\lambda^*}$ by $\mathscr{L}^n$. ■

Definition 0.2. A set is said to be $\underline{G_\delta}$ if it is the countable intersection of open sets. A set is said to be $\underline{F_\sigma}$ if it is the countable union of closed sets.

Theorem 0.2. 1. *For all $E \in \mathscr{L}^n$, $\lambda^N(E) = \inf\{\lambda^n(O) \mid \text{open } O \supseteq E\}$.*

2. *$E \in \mathscr{L}^n$ if and only if $E = H \setminus Z$, where H is G_δ, and $\lambda^*(Z) = 0$.*

3. *$E \in \mathscr{L}^n$ if and only if $E = H \cup Z$, where H is F_σ and $\lambda^*(Z) = 0$.*

4. $\lambda^n(E) = \sup\{\lambda^n(C) \mid \text{closed } C \subseteq E\}$

Proof. It suffices to prove the first statement, as the others will follow by passing to a complement. ■

Definition 0.3. Suppose X is a metric space. A measure on X is a Radon measure if it is Borel (meaning defined on a σ-algebra containing Borel sets), and for any Borel E, $\mu(E) = \inf\{\mu(O) \mid \text{open } O \supseteq E\}$, and for any compact $C \subseteq X$, $\mu(C) < \infty$.

Theorem 0.3. *(Riesz)*
Let $X \subseteq \mathbb{R}^n$ be compact. Let $C(X)$ denote the vector space of all continuous functions on X. This admits a norm $\|f\|_{C(X)} = \sup_X |f|$, making it a Banach space. Define $C^*(X) = \{\phi : C(X) \to \mathbb{R}, \phi \text{ is linear and continuous }\}$.
For all $\phi \in C^*(X)$, there exists a Radon measure $\mu = \mu_+$, and a function $M : X \to \{\pm 1\}$ which is Borel, such that

$$\phi(f) = \int_X f(x) M(x)\, d\mu(x)$$

for all $f \in C(X)$.

Proof. ■

Lecture 2, 1/17/23

Note: This is the first lecture with Davit. Davit will always use μ to refer to an outer measure, not a measure. The book will be "Measure theory and fine properties of functions." According to Davit, this is the correct book to be using.

Definition 0.4. Let X be a nonempty set. A mapping $\mu : 2^X \to [0, +\infty]$ is called a measure if it satisfies the following 2 properties.

1. $\mu(\varnothing) = 0$.

2. (Countable subadditivity and monotonicity) If $A, A_1, A_2, \cdots \subseteq X$ and $A \subseteq \cup_{i=1}^{\infty} A_i$ then $\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$

Remark. From the second definition, we can automatically get monotonicity, i.e. if $A \subseteq B$, then $\mu(A) \leq \mu(B)$. This is because, as written, the second definition is a statement not just about $\cup_{i=1}^{\infty} A_i$, but about any subset of it. Indeed, let $A = A, A_1 = B$, and $A_k = \varnothing$ for $k \geq 2$. Then we have $\mu(A) \leq \mu(\cup_{i=1}^{\infty} A_i) = \mu(A \cup B)$.
We will write "μ is a measure on X" to mean that μ satisfies the above definition (that is, μ is an outer measure).

Definition 0.5. Let X be a nonempty set and let μ be a measure on X. For a fixed set $C \subseteq X$, define the restriction measure $\nu = \mu|_C$ by $\nu(A) = \mu|_A(A) = \mu(A \cap C)$.

Remark. It is easy to prove that $\mu|_C$ is a measure on X.

Definition 0.6. (Carathéodory's condition). Let X be a nonempty set and let μ be a measure on X. A subset $A \subseteq X$ is called μ-measurable if, for all subset $B \subseteq X$, we have

$$\mu(B) = \mu(B \cap A) + \mu(B \setminus A)$$

Remark. X and $\varnothing$ are easily seen to be μ-measurable.

Theorem 0.4. *(Carathéodory extension theorem)*
The collection of μ-measurable sets on a set X is a σ-algebra.

Theorem 0.5. *Let X be a nonempty set and let μ be a measure on X. Then the following holds:*

1. *$\varnothing$ and X are μ-measurable.*

2. *$A \subseteq X$ is μ-measurable if and only if $X \setminus A$ is μ-measurable.*

3. *If $A \subseteq X$ is such that $\mu(A) = 0$, then A is μ-measurable.*

4. *Let $C \subseteq X$. Then anything which is μ-measurable is $\mu|_C$-measurable.*

Remark. A measure is also finitely subadditive, which says that if $A \subseteq A_1 \cup \cdots \cup A_n$, then $\mu(A) \leq \sum_{i=1}^n \mu(A_i)$. So, to check that $\mu(B) = \mu(B \cap A) + \mu(B \setminus A)$, it will suffice to check

$$\mu(B) \geq \mu(B \cap A) + \mu(B \setminus A)$$

Proof. Part 1 is obvious.
Suppose that A is μ-measurable. Then $\mu(B \cap A) = \mu(B \setminus A^c)$ and $\mu(B \cap A^c) = \mu(B \setminus A)$ so $\mu(B \cap A) + \mu(B \setminus A) = \mu(B \cap A^c) + \mu(B \setminus A^c)$. So A is μ-measurable if and only if A^c is.
Suppose that $\mu(A) = 0$. Then $\mu(B \cap A) \leq \mu(A), \mu(B)$, so $\mu(B \cap A) = 0$ for any $B \subseteq X$. Now, $B \setminus A \subseteq B$, so by monotonicity $\mu(B \setminus A) \leq \mu(B)$. So $\mu(B \cap A) + \mu(B \setminus A) \leq \mu(B)$ for all $B \subseteq X$, so we are done.
Let A be μ-measurable. Then for any $B \subseteq X$ we have

$$\begin{aligned}
\nu(B) &= \mu|_C(B) = \mu(B \cap C) \\
&= \mu((B \cap C) \cap A) + \mu((B \cap C) \setminus A) \\
&= \nu(B \cap A) + \mu((B \setminus A) \cap C) \\
&= \nu(B \cap A) + \nu(B \setminus A)
\end{aligned}$$

■

Theorem 0.6. *Let X be a nonempty set and let μ be a measure on X.*
Assume $A_1, A_2, \ldots, A_n \subseteq X$ are μ-measurable. Then

1. $\bigcup_{k=1}^n A_k$ *and* $\bigcap_{i=1}^n A_k$ *are also μ-measurable.*

2. *If the A_i are disjoint, then* $\mu(\bigcup_{i=1}^n A_i = \sum_{i=1}^n \mu(A_i)$

Proof. We prove part 2 first. Because each A_i is measurable,

$$\begin{aligned}
\mu(\cup_{k=1}^n A_k) &= \mu((\cup_{k=1}^n A_k) \cap A_n) + (\mu(\cup_{k=1}^n A_k) \setminus A_n) \\
&= \mu(\cup_{i=1}^{n-1} A_k) + \mu(A_n) = \cdots = \sum_{k=1}^n \mu(A_k)
\end{aligned}$$

Now we prove part 1. Let $A, B \subseteq X$ be μ-measurable and disjoint. Then for any $C \subseteq X$, $\mu(C) = \mu(C \cap A) + \mu(C \setminus A)$, and similarly for B. This is equal to

$$\begin{aligned}
\mu(C) &= \mu(C \cap A) + \mu((C \setminus A) \cap B) + \mu(C \setminus A \setminus B) \\
&= \mu(C \cap A) + \mu(C \cap B) + \mu(C \setminus (A \cup B)) + \mu(C \cap (A \cup B))(?) \\
&= \mu(C \cap (A \cup B) \cap A) + \mu(C \cap (A \cup B) \setminus A) \\
&= \mu(C \cap A) + \mu(C \cap B) \\
&= \mu(C \cap (A \cup B)) + \mu(C \setminus (A \cup B))
\end{aligned}$$

So $A \cup B$ is μ-measurable. (I got a bit lost in the arithmetic, sorry)
Next, we show if $A, B \subseteq X$ are μ-measurable, then $A \cap B$ is μ-measurable. This is straightforward. We will continue next time.

Lecture 3, 1/19/23

We will continue our proof of the theorem. Assume $A, B \subseteq X$ are μ-measurable. We aim to show that $A \cap B$ is also μ-measurable. We need to show that, for any $C \subseteq X$, we have $\mu(C) = \mu(C \cap (A \cap B)) + \mu(C \setminus (A \cap B))$. Because A, B are μ-measurable, we have

$$\begin{aligned}
\mu(C) &= \mu(C \cap A) + \mu(C \setminus A) \\
&= \mu((C \cap A) \cap B) + \mu((C \cap A) \setminus B) + \mu(C \setminus A) \\
&= \mu(C \cap (A \cap B)) + \mu((C \cap A) \setminus B) + \mu(C \setminus A) \\
&\geq \mu(C \cap (A \cap B)) + \mu(C \setminus (A \cap B))
\end{aligned}$$

The opposite inequality follows by subadditivity, so we have equality.
By induction, we get that also $\cap_{k=1}^{n} A_k$ is μ-measurable. For the union, we can get it using the fact that $\cup_{k=1}^{n} A_k = X \setminus \cap_{k=1}^{n} A_k$.

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Remark. If A, B are μ-measurable, then $A \setminus B$ is μ-measurable. This follows from $A \setminus B = A \cap (X \setminus B)$

Theorem 0.7. Let X be a nonempty set, and μ a measure on X. Assume $\{A_k\}_{k=1}^{\infty} \subseteq X$ are μ-measurable. Then

1. *If the A_k are disjoint, then we have countable additivity:*

$$\mu\left(\cup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$$

If $A_1 \subseteq A_2 \subseteq \cdots$, then

$$\lim_{k \to \infty} \mu(A_k) = \mu\left(\bigcup_{k=1}^{\infty} A_k\right)$$

If $A_1 \supseteq A_2 \supseteq \cdots$, and $\mu(A_1) < \infty$, then

$$\lim_{k \to \infty} \mu(A_k) = \mu\left(\bigcap_{k=1}^{\infty} A_k\right)$$

Proof. We have from before that if the A_k are pairwise disjoint, then $\mu(\cup_{k=1}^n A_k) = \sum_{k=1}^n \mu(A_k)$ for any $n \in \mathbb{N}$. Because $\cup_{k=1}^n A_k \subseteq \cup_{k=1}^\infty A_k$, we must have that $\mu(\cup_{k=1}^n A_k) \leq \mu(\cup_{k=1}^\infty A_k)$. Using the previous fact, and passing to a limit, we have

$$\sum_{k=1}^\infty \mu(A_k) \leq \mu(\cup_{k=1}^\infty A_k)$$

The opposite equality is automatically true by the countable subadditivity of μ, so we get equality. This completes the proof of 1. Now for part 2.
Define $B_k = A_k \setminus A_{k-1}$, where $A_0 \stackrel{\text{def}}{=} \varnothing$. We have $A_k = \cup_{i=1}^k B_i$. Note that the B_i are disjoint. So we have

$$\mu(A_k) = \sum_{i=1}^k \mu(B_i)$$

So, in the limit,

$$\lim_{k\to\infty} \mu(A_k) = \lim_{k\to\infty} \sum_{i=1}^k \mu(B_i) = \sum_{i=1}^\infty \mu(B_i)$$

So

$$\mu(\cup_{i=1}^\infty B_i) = \mu(\cup_{k=1}^\infty A_k)$$

Finally, let $A_1 \supseteq A_2 \supseteq \cdots, \mu(A_1) < \infty$. Define $B_k = A_1 \setminus A_k$. This is decreasing sequence of μ-measurable sets, so by the previous part,

$$\lim_{k\to\infty} \mu(B_k) = \mu(\cup_{k=1}^\infty B_k)$$

So

$$\begin{aligned}
\mu(B_k) = \mu(A_1 \setminus A_k) = \mu(A_1) - \mu(A_k) \implies& \\
\lim_{k\to\infty} \mu(B_k) = \lim_{k\to\infty} (\mu(A_1) - \mu(A_k)) &= \mu(A_1) - \lim_{k\to\infty} \mu(A_k) \\
= \mu(\cup_{k=1}^\infty B_k) = \mu(\cup_{k=1}^\infty (A_1 \setminus A_k)) &= \mu(A_1 \setminus \cap_{k=1}^\infty A_k) \\
&\geq \mu(A_1) - \mu(\cap_{k=1}^\infty A_k)
\end{aligned}$$

So $\lim_{k\to\infty} \mu(A_k) \leq \mu(\cap_{k=1}^\infty A_k)$. The opposite inequality follows easily by monotonicity.

■

We are ready to prove the Carathéodory extension theorem.

Proof. Let $A_1, A_2, \cdots \subseteq X$ be μ-measurable. It will suffice to prove that $\cup_{k=1}^\infty A_k$ is μ-measurable. So we need to check that, for any B,

$$\mu(B) = \mu(B \cap (\cup_{k=1}^\infty A_k)) + \mu(B \setminus \cup_{k=1}^\infty A_k)$$

Fix $B \subseteq X$, and consider $\nu = \mu|_B$. Recall this is defined as $\nu(C) = \mu(B \cap C)$. We would like

$$\nu(B) = \nu(\cup_{k=1}^{\infty}) + \nu(B \setminus \cup_{k=1}^{\infty} A_k)$$

To this end,

$$\nu(\cup_{k=1}^{\infty} A_k) = \lim_{k\to\infty} \nu(\cup_{i=1}^{k} A_i)$$

Without loss of generality, $\nu(B) < \infty$. If $\nu(B) = \infty$, then we are done trivially. As before, define $B_k = B \setminus \cup_{i=1}^{k} A_i$. Then

$$\nu(B \setminus \cup_{k=1}^{\infty} A_k) = \lim_{k\to\infty} \nu(B \setminus \cup_{i=1}^{k} A_i)$$

So,

$$\begin{aligned} \nu(\cup_{k=1}^{\infty} A_k) + \nu(B \setminus \cup_{k=1}^{\infty} A_k) &= \lim_{k\to\infty} \Big(\nu(\cup_{i=1}^{k} A_i) + \nu(B \setminus \cup_{i=1}^{k} A_i)\Big) \\ &= \lim_{k\to\infty} \nu(B) = \nu(B) \end{aligned}$$

so we are done.

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Definition 0.7. Let X be a nonempty set, and let μ be a measure on X. Then μ is said to be

1. A <u>regular measure</u> if, for any $A \subset X$, there exists a μ-measurable $B \subseteq X$ such that $A \subseteq B$, and $\mu(A) = \mu(B)$.

2. A <u>Borel measure</u> if all Borel sets (i.e. the elements of the Borel σ-algebra) are measurable. This only applies if X is also a topological space, of course.

3. A <u>Borel-regular measure</u> if μ is Borel, and for any $A \subseteq X$, there exists a Borel set $B \subseteq X$ such that $A \subseteq B$ and $\mu(A) = \mu(B)$.

4. A <u>Radon measure</u> if it is Borel-regular and $\mu(K) < \infty$ if K is compact.

Remark. Note that being Borel and regular is weaker than being Borel-regular.

Theorem 0.8. *(Increasing sets <u>for regular measures</u>) Let X be a nonempty set, and let μ be a regular measure on X. Assume $A_1 \subseteq A_2 \subseteq \cdots \subseteq X$. Then*

$$\lim_{k\to\infty} \mu(A_k) = \mu\left(\bigcup_{k=1}^{\infty} A_k\right)$$

Remark. The sets A_k need not be μ-measurable.

Proof. For all A_k, there is a $C_k \subseteq X$ which is μ-measurable, $A_k \subseteq C_k$, and $\mu(A_k) = \mu(C_k)$. Let $D_k = \cap_{i \geq k} C_i$. For $i \geq k$, we can see $A_k \subseteq A_i \subseteq C_i$. $A_k \subseteq \cup_{i \geq k} C_i = D_k$, then $\mu(A_k) \leq \mu(D_k)$. On the other hand, $D_k \subseteq C_k$, so $\mu(D_k) \leq \mu(C_k) = \mu(A_k)$. So

- $\mu(A_k) = \mu(D_k)$
- $A_k \subseteq D_k$
- D_k is μ-measurable and $D_1 \subseteq D_2 \subseteq \cdots$

So

$$\begin{aligned}
\lim_{k\to\infty} \mu(A_k) &= \lim_{k\to\infty} \mu(D_k) \\
&= \mu\left(\bigcup_{k=1}^{\infty} D_k\right) \\
&\geq \mu\left(\bigcup_{k=1}^{\infty} A_k\right)
\end{aligned}$$

Because $\cup_{k=1}^{\infty} A_k \subseteq \cup_{k=1}^{\infty} D_k$,

$$\lim_{k\to\infty} \mu(A_k) \geq \mu\left(\bigcup_{k=1}^{\infty} A_k\right)$$

But $A_k \subseteq \cup_{k=1}^{\infty} A_k$, so the opposite inequality is also true, so we have equality. ∎

Lecture 4, 1/24/23

Theorem 0.9. *(Restriction and Radon measures)*
Let X be a topological space and let μ be a Borel-regular measure on X. Let $A \subseteq X$ be μ-measurable with $\mu(A) < \infty$. Then the restriction measure $\nu = \mu|_A$ is Radon.

Proof. First, ν is a finite measure, as $\nu(X) = \mu(A \cap X) = \mu(A) < \infty$ for any $C \subseteq X$. It is clear that ν is Borel, as μ is Borel. Next, we show ν is Borel-Regular. Without loss of generality, we may assume that A is Borel, because μ is Borel-regular. Explicitly, we know there is a Borel set $B \subseteq X$ such that $A \subseteq B$ and $\mu(B) = \mu(A)$. We will show $\mu|_A = \mu|_B$.

We have $\mu(B) = \mu(B \cap A) + \mu(B \setminus A) = \mu(A) + \mu(B \setminus A)$. So $\mu(B \setminus A) = 0$.
So, for all $C \subseteq X$,

$$\begin{aligned}
\mu|_B(C) &= \mu(B \cap C) \\
&= \mu((B \cap C) \cap A) + \mu((B \cap C) \setminus A) \\
&= \mu(C \cap A) + \mu((B \cap C) \setminus A) \\
&\leq \mu|_A(C) + \mu(B \setminus A) \\
&= \mu|_A(C)
\end{aligned}$$

But $(A \cap C) \subseteq (B \cap C)$, so $\mu|_A(C) \leq \mu|_B(C)$, so we may conclude that $\mu|_A = \mu|_B$.
So assume A is Borel. Fix $C \subseteq X$. We need to prove that there exists a Borel $D \subseteq X$ such that $C \subseteq D$ and $\nu(C) = \nu(D)$. There exists a Borel $E \subseteq X$ such that $C \cap A \subseteq E$, and $\mu(C \cap A) = \mu(E)$. So $D = E \cup (X \setminus A)$ is Borel and $C \subseteq D$.
So

$$\begin{aligned}
\nu(D) &= \mu((E \cup (X \setminus A)) \cap A) \\
&= \mu(E \cap A) \\
&\leq \mu(E) \\
&= \mu(C \cap A) \\
&= \nu(C)
\end{aligned}$$

$C \subseteq D$ so $\nu(C) \leq \nu(D)$, so $\nu(C) = \nu(D)$.

■

Theorem 0.10. *(Carathéodory Criterion for being Borel)*
Let X be a metric space and let μ be a measure on X. Then μ is Borel if and only if, for all $A, B \subseteq X$ with $d(A, B) > 0$ (meaning $\inf\{d(a, b) \mid a \in A, b \in B\} > 0$),

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

Proof. =>

Suppose μ is Borel. We will use $\overline{B}$ to denote the closure of B. Then $d(A, \overline{B}) = d(A, B) > 0$. By measurability of $\overline{B}$,

$$\mu(A \cup B) = \mu((A \cup B) \cap \overline{B}) + \mu((A \cup B) \setminus \overline{B}) = \mu(B) = \mu(A)$$

<=

Suppose that, for A, B with $d(A, B) > 0$, $\mu(A \cup B) = \mu(A) + \mu(B)$. We will show that this implies μ is Borel. Let us show that every closed subset $C \subseteq X$ is μ-measurable.

So we have to prove that for every $A \subseteq X$,

$$\mu(A) = \mu(A \cap C) + \mu(A \setminus C)$$

We have $\leq$ trivially. Assume $\mu(A) < \infty$; otherwise, this equality holds trivially.
Define for every $n \in \mathbb{N}$ the set $C_n = \{x \in X \mid d(x, C) \leq \frac{1}{n}\}$. We can see $d(A \setminus C_n, C) \geq \frac{1}{n} > 0$. So

$$\begin{aligned} \mu((A \setminus C_n) \cup (A \cap C)_{\subseteq A}) &= \mu(A \setminus C_n) + \mu(A \cap C) \\ &\leq \mu(A) \end{aligned}$$

SO $\mu(A \setminus C_n) + \mu(A \cap C) \leq \mu(A)$ for all $n \in \mathbb{N}$. We will prove that $\lim_{n\to\infty} \mu(A \setminus C_n) = \mu(A \setminus C)$.
Consider the annuli $R_n = \{x \in A \mid \frac{1}{n+1} < d(x, C) \leq \frac{1}{n}$. We have

$$(A \setminus C_1) \bigcup_{n=1}^{\infty} R_n \subseteq A \setminus C$$

C is closed, so in fact we have equality above. Why? If a point belongs to $A \setminus C$, then it does not belong to C, so $d(x, C) > 0$. So there is an $n \in \mathbb{N}$ such that $x \in R_n$ or $x \in A \setminus C_1$. We have

$$\mu\left(\bigcup_{k=0}^{n} R_{2k+1}\right) = \sum_{k=0}^{n} \mu(R_{2k+1}) \leq \mu(A)$$
$$\mu\left(\bigcup_{k=1}^{n} R_{2k}\right) = \sum_{k=1}^{n} \mu(R_{2k}) \leq \mu(A)$$

So $\sum_{n=1}^{\infty} \mu(R_n) \leq 2\mu(A) < \infty$, so $\lim_{n\to\infty}(\sum_{k=n}^{\infty} \mu(R_k)) = 0$. So $(A \setminus C_n) \bigcup_{k=n}^{\infty} R_k = A \setminus C$
So by subadditivity,

$$\mu(A \setminus C) \leq \mu(A \setminus C_n) + \sum_{k=n}^{\infty} \mu(R_k)$$

So as $n \to \infty$,

$$\mu(A \setminus C) \leq \liminf_{n\to\infty} \mu(A \setminus C_n) \leq \mu(A \setminus C)$$

This completes the proof.
It is time for our third section.

Approximation by open, closed, and compact sets

Theorem 0.11. *Let μ be a Borel measure on $\mathbb{R}^n$, and let $B \subseteq \mathbb{R}^n$ be a Borel set.*

1. *If $\mu(B) < \infty$, then for any $\varepsilon > 0$, there exists a closed $C \subseteq B$ such that $\mu(B \setminus C) < \varepsilon$.*

2. *If μ is a Radon measure, then for all $\varepsilon > 0$, there exists an open $U \supseteq B$ such that $\mu(U \setminus B) < \varepsilon$.*

Proof. **1.** Let $\nu = \mu|_B$, a finite measure on $\mathbb{R}^n$.

Define the collection $\mathscr{F} = \{A \subseteq \mathbb{R}^n \mid A$ is μ-measurable and for all $\varepsilon > 0$, there exists a closed $C \subseteq A$ such that $\mu(A \setminus C) < \varepsilon\}$

Our goal is to show that $\mathscr{B}_{\mathbb{R}^n} \subseteq \mathscr{F}$. Davit uses "$\sigma_B$" to indicate the Borel σ-algebra.

By previous discussion, $\mathscr{F}$ contains all closed sets.

Now, if $A_1, A_2, \cdots \in \mathscr{F}$, then $\bigcap_{k=1}^{\infty} A_k \in \mathscr{F}$. For all A_k, there exists a closed $C_k \subseteq A_k$, such that $\nu(A_k \setminus C_k) < \frac{\varepsilon}{2^k}$. Then by subadditivity,

$$\nu(\bigcap_{k=1}^{\infty} A_k \setminus \bigcap_{k=1}^{\infty} C_k) \leq \nu(\bigcup_{k=1}^{\infty}(A_k \setminus C_k)) \leq \sum_{k=1}^{\infty} \nu(A_k \setminus C_k) < \varepsilon$$

and $C = \bigcap_{k=1}^{\infty} C_k$ is closed.