Lecture 1

Let $(X, \mathcal{A}, \mu)$ be a measure space. Without any additional structure or information, we may define the Lebesgue integral $\int_X f\,d\mu$ for f an $\mathcal{A} - \mathcal{B}$ measurable function $f : X \to [-\infty, +\infty]$.
We only have a few examples without any work.

Example 0.1.
- For any set X, we can define the counting measure on $\mathcal{A} = 2^X$, which gives $\mu(A) = |A|$. If $X = \mathbb{N}$, then a measurable function is just a sequence (f_n), and $\int_X f\,d\mu = \sum f_n$
- We can also define the Dirac mass δ_p for a fixed $p \in X$ by

$$\delta_p(E) = \begin{cases} 1 & p \in E \\ 0 & p \notin E \end{cases}$$

  We have $\int_X f\,d\delta_p = f(p)$

To get another example of a measure we need to do some work.
Problem: We want a measure μ on $\mathbb{R}^n$ such that, for a rectangle,

$$\mu([a_1, b_1] \times \cdots \times [a_n, b_n]) = |a_1 - b_1| \cdots |a_n - b_n|$$

Once it is defined on all rectangles, it is defined on the minimal σ-algebra containing them, which is the Borel σ-algebra. In other words, this condition will completely specify a measure on the Borel σ-algebra $\mathcal{B}_{\mathbb{R}^n}$
If $X = \mathbb{R}^n$, or a general metric space, or even a general topological space, then $\mathcal{B}(X)$ denotes the σ-algebra generated by the open subsets of X.
Problem:
Suppose we have a distribution function $F : \mathbb{R} \to \mathbb{R}$, meaning F is monotone, positive, and $\lim_{x\to-\infty} f(x) = 0, \lim_{x\to\infty} f(x) = 1$, and continuous from the right. We want a Borel measure μ such that $F(t) = \mu((-\infty, t])$. Such a measure, denoted by λ_F, is called a Lebesgues-Stieltjes measure.
The corresponding integral is called a Lebesgue-Stieltjes integral.
If F is smooth, then $\int_{\mathbb{R}} \phi\,d\lambda_F = \int_{-\infty}^{\infty} \phi(x) d\,F(x)$.
The measure we want on $\mathbb{R}^n$ is denoted by λ^n.

The Carathéodory Construction

Suppose we have an outer measure $\gamma : 2^X \to [0, \infty]$. This means $\gamma(\varnothing) = 0$, $A \subset B \implies \gamma(A) \leq \gamma(B)$ (monotone), and $\gamma(\cup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \gamma(E_i)$ (subadditive).
We can define a set S to be γ-measurable if for every testing set T, $\gamma(T) = \gamma(S \cap T) + \gamma(S^c \cap T)$.

Theorem 0.1. *(Carathéodory Extension Theorem)*

1. $\gamma(N) = 0 \implies N$ *is measurable.*

2. *The set of measurable sets forms a σ-algebra Γ.*

3. γ *restricted to Γ forms a measure.*

"Nothing in the above theorem can guarantee you that Γ is not trivial, i.e. $\Gamma = \{\varnothing, X\}$. Nevertheless, this is a very useful guy" - Dennis.

Definition 0.1. (Lebesgue outer measure on $\mathbb{R}^n$)
Let R be a rectangle in $\mathbb{R}^n$, that is $R = \prod_{i=1}^n [a_i, b_i]$. We have $\mathrm{Vol}(R) = |a_1 - b_1| \cdots |a_n - b_n|$. For any $E \subseteq \mathbb{R}^n$, we define

$$\mu^*(E) \stackrel{\text{def}}{=} \inf\{\sum_{j=1}^{\infty} \mathrm{Vol}(R_j) \mid E \subseteq \cup_{j=1}^{\infty} R_j\}$$

Proposition 1. *μ^* is an outer measure on $\mathbb{R}^n$ such that $\mu^*(R) = \mathrm{Vol}(R)$ for all rectangles R.*

Proof. The first and second axioms are trivial, so we will just prove the subadditivity. Let E be some set. By definition, for any ε, there is some cover R_j by recrtangles such that

$$-\varepsilon + \sum_{j=1}^{\infty} \mathrm{Vol}(R_j) \leq \mu^*(E) \leq \sum_{j=1}^{\infty} \mathrm{Vol}(R_j)$$

meaning that $\sum_{j=1}^{\infty} \mathrm{Vol}(R_j) \leq \mu^*(E) + \varepsilon$. So for each E_k, there is a sequence R_j^k which covers E_k, such that $\sum_{j=1}^{\infty} \mathrm{Vol}(R_j^k) \leq \mu^*(E) + \frac{\varepsilon}{2^k}$.
So $\{R_j^k\}_{j,k \in \mathbb{N}}$ forms a cover of $\cup_{j=1}^{\infty} E_j$. Thus

$$\begin{aligned} \mu^*(\cup_{k=1}^{\infty} E_k) &\leq \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \mathrm{Vol}(R_j^k) \\ &\leq \sum_{k=1}^{\infty} \left(\mu^*(E_k) + \frac{\varepsilon}{2^k} \right) \\ &= \sum_{k=1}^{\infty} \mu^*(E_k) + \varepsilon \end{aligned}$$

This is true for any positive ε. Taking the limit as $\varepsilon \to 0$ gives the result. ■

Now, fix a rectangle R. Note that R itself forms a cover of R, so by the definition, $\mu^*(R) \leq \mathrm{Vol}(R)$. For $\varepsilon > 0$, we can take an almost-optimal cover (R_j) such that $\sum_{j=1}^{\infty} \mathrm{Vol}(R_j) \leq \mathrm{Vol}(R) + \varepsilon$. We can rig it such that $|\mathrm{Vol}(R_j) - \mathrm{Vol}(R)| \leq \frac{\varepsilon}{2^j}$. Because $R \subset \cup_{j=1}^{\infty} R_j$, and R_j is an open cover, by compactness of R there is a finite subcover, and the volume of R is less than or equal to the sum of the volumes of these finitely many R_j. So the volume of R is less than or equal to $\mu^*(R) + 2\varepsilon$. So $\mathrm{Vol}(R) = \mu^*(R)$.

Proposition 2. *Every rectangle R in $\mathbb{R}^n$ is Carathéodory measurable).*

Proof. I missed this lol. Apparently Dennis denotes $\mathcal{M}_{\lambda^*}$ by $\mathscr{L}^n$.

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Definition 0.2. A set is said to be <u>G_δ</u> if it is the countable intersection of open sets. A set is said to be <u>F_σ</u> if it is the countable union of closed sets.

Theorem 0.2. 1. *For all $E \in \mathscr{L}^n$, $\lambda^N(E) = \inf\{\lambda^n(O) \mid \text{open } O \supseteq E\}$.*

2. *$E \in \mathscr{L}^n$ if and only if $E = H \setminus Z$, where H is G_δ, and $\lambda^*(Z) = 0$.*

3. *$E \in \mathscr{L}^n$ if and only if $E = H \cup Z$, where H is F_σ and $\lambda^*(Z) = 0$.*

4. *$\lambda^n(E) = \sup\{\lambda^n(C) \mid \text{closed } C \subseteq E\}$*

Proof. It suffices to prove the first statement, as the others will follow by passing to a complement.

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Definition 0.3. Suppose X is a metric space. A measure on X is a <u>Radon measure</u> if it is Borel (meaning defined on a σ-algebra containing Borel sets), and for any Borel E, $\mu(E) = \inf\{\mu(O) \mid \text{open } O \supseteq E\}$, and for any compact $C \subseteq X$, $\mu(C) < \infty$.

Theorem 0.3. *(Riesz)*
Let $X \subseteq \mathbb{R}^n$ be compact. Let $C(X)$ denote the vector space of all continuous functions on X. This admits a norm $\|f\|_{C(X)} = \sup_X |f|$, making it a Banach space. Define $C^*(X) = \{\phi : C(X) \to \mathbb{R}, \phi \text{ is linear and continuous }\}$.
For all $\phi \in C^*(X)$, there exists a Radon measure $\mu = \mu_\phi$, and a function $M : X \to \{\pm 1\}$ which is Borel, such that

$$\phi(f) = \int_X f(x) M(x)\, d\mu_\phi(x)$$

for all $f \in C(X)$.