Lecture 1

We will be using Hatcher's Algebraic Topology. The topology sequence is usually something like

ATopological Spaces
BCell Complexes
CManifolds

Theorem 0.1. *(BIG Theorem)*
Given a "reasonably nice" space, there is a bijection between connected covers of a space and subgroups of the fundamental group.

Categories:

Algebraic structures that are much flabbier than a group. They consist of

- *A collection of arrows*
- *A partial binary operation on these arrows*
- *Objects, which arrows go between*

We also want a composition law. That is, for objects and arrows

$$A \xrightarrow{f} B \xrightarrow{g} C$$

there is an arrow $A \xrightarrow{g\circ f} C$. *We want this composition to be associative, that is* $(f \circ g) \circ h = f \circ (g \circ h)$, *and we want objects to have identity arrows.*
Not all functions have inverses. Using sets and functions as an example, we have described the category Set.
Here are some more examples of categories:

Example 0.1.
- Groups and group homomorphisms (Grp)
- Topological spaces and continuous functions (Top)
- etc.

We can make the following new category.

Definition 0.1. We denote by Top^* the category of based topological spaces, whose objects are pairs (X, x_0), where X is a topological space and $x_0 \in X$, and whose morphisms are continuous functions $f : (X, x_0) \to (Y, y_0)$ such that $f(x_0) = y_0$.

Goal:

Our goal is to get a functor from Top to Grp. The fundamental group functor π_1 will go from Top^* to Grp.

Lecture 2

Topology review:

Definition 0.2. A topological space is a set X along with a collection of subsets of X called "open sets," such that $X, \varnothing$ are open, and the arbitrary union and finite intersection of open sets are open.
Notice the following diagram commutes using the product topology

$$\begin{tikzcd}
 & Z \arrow[dl, "f"'] \arrow[d, dotted, "\exists!"] \arrow[dr, "g"] & \\
X & X \times Y \arrow[l, "P_X"'] \arrow[r, "P_Y"] & Y
\end{tikzcd}$$

And in general

$$\begin{tikzcd}
Z \arrow[d, dotted, "\exists!"'] \arrow[dr, "f_\alpha"] & \\
\prod_{\alpha \in A} X_\alpha \arrow[r, "P_\alpha"] & X_\alpha
\end{tikzcd}$$

Maps are continuous; functions are not.

Lemma 1. *(Gluing lemma)*
Suppose $f : A \to Y$, $g : B \to Y$ are continuous, and $f(x) = g(x)$ for all $x \in A \cap B$. Then $f \cup g : A \cup B \to Y$ is continuous. This only holds as long as $A, B \subseteq X$ are closed.

Same Shape, Same Map

(maps up to wriggling things around a bit)

Definition 0.3. Two maps are homotopic if there exists a parametrized map $f_t : X \to Y$ such that $f_0 = f, f_1 = g$ for $f, g : X \to Y$. Equivalently, and more precisely, if there exists a map $F : X \times [0, 1] \to Y$ such that $F(x, 0) = f(x), F(x, 1) = g(x)$ for all $x \in X$.

X, Y topological spaces are said to have the same shape if there exist maps $f : X \to Y, g : Y \to X$ such that $g \circ f \simeq \mathrm{Id}_X$ and $f \circ g \simeq \mathrm{Id}_Y$. We may say that X, Y have the same homotopy type

Definition 0.4. A deformation retraction from $X \to A \subseteq X$ is a map from $X \times I \to X$ such that, for all $x \in A$, and $s, t \in I$,

$$\begin{aligned} f_0(x) &= x \\ f_1(x) &\in A \\ f_t(x) &= f_s(x) \end{aligned}$$