Lecture 1

Let $(X, \mathcal{A}, \mu)$ be a measure space. Without any additional structure or information, we may define the Lebesgue integral $\int_X f\, d\mu$ for f an $\mathcal{A} - \mathcal{B}$ measurable function $f : X \to [-\infty, +\infty]$.
We only have a few examples without any work.

Example 0.1.
- For any set X, we can define the counting measure on $\mathcal{A} = 2^X$, which gives $\mu(A) = |A|$. If $X = \mathbb{N}$, then a measurable function is just a sequence (f_n), and $\int_X f\, d\mu = \sum f_n$
- We can also define the Dirac mass δ_p for a fixed $p \in X$ by

$$\delta_p(E) = \begin{cases} 1 & p \in E \\ 0 & p \notin E \end{cases}$$

  We have $\int_X f\, d\delta_p = f(p)$

To get another example of a measure we need to do some work.
Problem: We want a measure μ on $\mathbb{R}^n$ such that, for a rectangle,

$$\mu([a_1, b_1] \times \cdots \times [a_n, b_n]) = |a_1 - b_1| \cdots |a_n - b_n|$$

Once it is defined on all rectangles, it is defined on the minimal σ-algebra containing them, which is the Borel σ-algebra. In other words, this condition will completely specify a measure on the Borel σ-algebra $\mathcal{B}_{\mathbb{R}^n}$
If $X = \mathbb{R}^n$, or a general metric space, or even a general topological space, then $\mathcal{B}(X)$ denotes the σ-algebra generated by the open subsets of X.
Problem:
Suppose we have a distribution function $F : \mathbb{R} \to \mathbb{R}$, meaning F is monotone, positive, and $\lim_{x \to -\infty} f(x) = 0, \lim_{x \to \infty} f(x) = 1$, and continuous from the right. We want a Borel measure μ such that $F(t) = \mu((-\infty, t])$. Such a measure, denoted by λ_F, is called a Lebesgues-Stieltjes measure.
The corresponding integral is called a Lebesgue-Stieltjes integral.
If F is smooth, then $\int_{\mathbb{R}} \phi\, d\lambda_F = \int_{-\infty}^{\infty} \phi(x) d\, F(x)$.
The measure we want on $\mathbb{R}^n$ is denoted by λ^n.

The Carathéodory Construction

Suppose we have an outer measure $\gamma : 2^X \to [0, \infty]$. This means $\gamma(\varnothing) = 0$, $A \subset B \implies \gamma(A) \leq \gamma(B)$ (monotone), and $\gamma(\cup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \gamma(E_i)$ (subadditive).
We can define a set S to be γ-measurable if for every testing set T, $\gamma(T) = \gamma(S \cap T) + \gamma(S^c \cap T)$.

Theorem 0.1. *(Carathéodory Extension Theorem)*

1. $\gamma(N) = 0 \implies N$ *is measurable.*

2. *The set of measurable sets forms a σ-algebra Γ.*

3. *γ restricted to Γ forms a measure.*

"Nothing in the above theorem can guarantee you that Γ is not trivial, i.e. $\Gamma = \{\varnothing, X\}$. Nevertheless, this is a very useful guy" - Dennis.

Definition 0.1. (Lebesgue outer measure on $\mathbb{R}^n$)
Let R be a rectangle in $\mathbb{R}^n$, that is $R = \prod_{i=1}^n [a_i, b_i]$. We have $\text{Vol}(R) = |a_1 - b_1| \cdots |a_n - b_n|$. For any $E \subseteq \mathbb{R}^n$, we define

$$\mu^*(E) \stackrel{\text{def}}{=} \inf\{\sum_{j=1}^{\infty} \text{Vol}(R_j) \mid E \subseteq \cup_{j=1}^{\infty} R_j\}$$

Proposition 1. *μ^* is an outer measure on $\mathbb{R}^n$ such that $\mu^*(R) = \text{Vol}(R)$ for all rectangles R.*

Proof. The first and second axioms are trivial, so we will just prove the subadditivity. Let E be some set. By definition, for any ε, there is some cover R_j by recrtangles such that

$$-\varepsilon + \sum_{j=1}^{\infty} \text{Vol}(R_j) \le \mu^*(E) \le \sum_{j=1}^{\infty} \text{Vol}(R_j)$$

meaning that $\sum_{j=1}^{\infty} \text{Vol}(R_j) \le \mu^*(E) + \varepsilon$. So for each E_k, there is a sequence R_j^k which covers E_k, such that $\sum_{j=1}^{\infty} \text{Vol}(R_j^k) \le \mu^*(E) + \frac{\varepsilon}{2^k}$.
So $\{R_j^k\}_{j,k \in \mathbb{N}}$ forms a cover of $\cup_{j=1}^{\infty} E_j$. Thus

$$\begin{aligned}
\mu^*(\cup_{k=1}^{\infty} E_k) &\le \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \text{Vol}(R_j^k) \\
&\le \sum_{k=1}^{\infty} \left(\mu^*(E_k) + \frac{\varepsilon}{2^k} \right) \\
&= \sum_{k=1}^{\infty} \mu^*(E_k) + \varepsilon
\end{aligned}$$

This is true for any positive ε. Taking the limit as $\varepsilon \to 0$ gives the result. ■

Now, fix a rectangle R. Note that R itself forms a cover of R, so by the definition, $\mu^*(R) \leq \text{Vol}(R)$. For $\varepsilon > 0$, we can take an almost-optimal cover (R_j) such that $\sum_{j=1}^{\infty} \text{Vol}(R_j) \leq \text{Vol}(R) + \varepsilon$. We can rig it such that $|\text{Vol}(R_j) - \text{Vol}(R)| \leq \frac{\varepsilon}{2^j}$. Because $R \subset \cup_{j=1}^{\infty} R_j$, and R_j is an open cover, by compactness of R there is a finite subcover, and the volume of R is less than or equal to the sum of the volumes of these finitely many R_j. So the volume of R is less than or equal to $\mu^*(R) + 2\varepsilon$. So $\text{Vol}(R) = \mu^*(R)$.

Proposition 2. *Every rectangle R in $\mathbb{R}^n$ is Carathéodory measurable).*

Proof. I missed this lol. Apparently Dennis denotes $\mathcal{M}_{\lambda^*}$ by $\mathscr{L}^n$. ■

Definition 0.2. A set is said to be $\underline{G_\delta}$ if it is the countable intersection of open sets. A set is said to be $\underline{F_\sigma}$ if it is the countable union of closed sets.

Theorem 0.2. 1. *For all $E \in \mathscr{L}^n$, $\lambda^N(E) = \inf\{\lambda^n(O) \mid \text{open } O \supseteq E\}$.*

2. *$E \in \mathscr{L}^n$ if and only if $E = H \setminus Z$, where H is G_δ, and $\lambda^*(Z) = 0$.*

3. *$E \in \mathscr{L}^n$ if and only if $E = H \cup Z$, where H is F_σ and $\lambda^*(Z) = 0$.*

4. *$\lambda^n(E) = \sup\{\lambda^n(C) \mid \text{closed } C \subseteq E\}$*

Proof. It suffices to prove the first statement, as the others will follow by passing to a complement. ■

Definition 0.3. Suppose X is a metric space. A measure on X is a <u>Radon measure</u> if it is Borel (meaning defined on a σ-algebra containing Borel sets), and for any Borel E, $\mu(E) = \inf\{\mu(O) \mid \text{open } O \supseteq E\}$, and for any compact $C \subseteq X$, $\mu(C) < \infty$.

Theorem 0.3. *(Riesz)*
Let $X \subseteq \mathbb{R}^n$ be compact. Let $C(X)$ denote the vector space of all continuous functions on X. This admits a norm $\|f\|_{C(X)} = \sup_X |f|$, making it a Banach space. Define $C^*(X) = \{\phi : C(X) \to \mathbb{R}, \phi \text{ is linear and continuous }\}$.
For all $\phi \in C^*(X)$, there exists a Radon measure $\mu = \mu_+$, and a function $M : X \to \{\pm 1\}$ which is Borel, such that

$$\phi(f) = \int_X f(x) M(x) \, d\mu(x)$$

for all $f \in C(X)$.

Proof. ■

Lecture 2, 1/17/23

Note: This is the first lecture with Davit. Davit will always use μ to refer to an outer measure, not a measure. The book will be "Measure theory and fine properties of functions." According to Davit, this is the correct book to be using.

Definition 0.4. Let X be a nonempty set. A mapping $\mu : 2^X \to [0, +\infty]$ is called a measure if it satisfies the following 2 properties.

1. $\mu(\varnothing) = 0$.

2. (Countable subadditivity and monotonicity) If $A, A_1, A_2, \cdots \subseteq X$ and $A \subseteq \cup_{i=1}^{\infty} A_i$ then $\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$

Remark. From the second definition, we can automatically get monotonicity, i.e. if $A \subseteq B$, then $\mu(A) \leq \mu(B)$. This is because, as written, the second definition is a statement not just about $\cup_{i=1}^{\infty} A_i$, but about any subset of it. Indeed, let $A = A, A_1 = B$, and $A_k = \varnothing$ for $k \geq 2$. Then we have $\mu(A) \leq \mu(\cup_{i=1}^{\infty} A_i) = \mu(A \cup B)$.
We will write "μ is a measure on X" to mean that μ satisfies the above definition (that is, μ is an outer measure).

Definition 0.5. Let X be a nonempty set and let μ be a measure on X. For a fixed set $C \subseteq X$, define the restriction measure $\nu = \mu|_C$ by $\nu(A) = \mu|_A(A) = \mu(A \cap C)$.

Remark. It is easy to prove that $\mu|_C$ is a measure on X.

Definition 0.6. (Carathéodory's condition). Let X be a nonempty set and let μ be a measure on X. A subset $A \subseteq X$ is called μ-measurable if, for all subset $B \subseteq X$, we have

$$\mu(B) = \mu(B \cap A) + \mu(B \setminus A)$$

Remark. X and $\varnothing$ are easily seen to be μ-measurable.

Theorem 0.4. *(Carathéodory extension theorem)*
The collection of μ-measurable sets on a set X is a σ-algebra.

Theorem 0.5. *Let X be a nonempty set and let μ be a measure on X. Then the following holds:*

1. *$\varnothing$ and X are μ-measurable.*

2. *$A \subseteq X$ is μ-measurable if and only if $X \setminus A$ is μ-measurable.*

3. *If $A \subseteq X$ is such that $\mu(A) = 0$, then A is μ-measurable.*

4. *Let $C \subseteq X$. Then anything which is μ-measurable is $\mu|_C$-measurable.*

Remark. A measure is also finitely subadditive, which says that if $A \subseteq A_1 \cup \cdots \cup A_n$, then $\mu(A) \leq \sum_{i=1}^n \mu(A_i)$. So, to check that $\mu(B) = \mu(B \cap A) + \mu(B \setminus A)$, it will suffice to check

$$\mu(B) \geq \mu(B \cap A) + \mu(B \setminus A)$$

Proof. Part 1 is obvious.
Suppose that A is μ-measurable. Then $\mu(B \cap A) = \mu(B \setminus A^c)$ and $\mu(B \cap A^c) = \mu(B \setminus A)$ so $\mu(B \cap A) + \mu(B \setminus A) = \mu(B \cap A^c) + \mu(B \setminus A^c)$. So A is μ-measurable if and only if A^c is.
Suppose that $\mu(A) = 0$. Then $\mu(B \cap A) \leq \mu(A), \mu(B)$, so $\mu(B \cap A) = 0$ for any $B \subseteq X$. Now, $B \setminus A \subseteq B$, so by monotonicity $\mu(B \setminus A) \leq \mu(B)$. So $\mu(B \cap A) + \mu(B \setminus A) \leq \mu(B)$ for all $B \subseteq X$, so we are done.
Let A be μ-measurable. Then for any $B \subseteq X$ we have

$$\begin{aligned}
\nu(B) &= \mu|_C(B) = \mu(B \cap C) \\
&= \mu((B \cap C) \cap A) + \mu((B \cap C) \setminus A) \\
&= \nu(B \cap A) + \mu((B \setminus A) \cap C) \\
&= \nu(B \cap A) + \nu(B \setminus A)
\end{aligned}$$

■

Theorem 0.6. Let X be a nonempty set and let μ be a measure on X.
Assume $A_1, A_2, \ldots, A_n \subseteq X$ are μ-measurable. Then

1. $\bigcup_{k=1}^n A_k$ *and* $\bigcap_{i=1}^n A_k$ *are also μ-measurable.*

2. *If the A_i are disjoint, then* $\mu(\bigcup_{i=1}^n A_i = \sum_{i=1}^n \mu(A_i)$

Proof. We prove part 2 first. Because each A_i is measurable,

$$\begin{aligned}
\mu(\cup_{k=1}^n A_k) &= \mu((\cup_{k=1}^n A_k) \cap A_n) + (\mu(\cup_{k=1}^n A_k) \setminus A_n) \\
&= \mu(\cup_{i=1}^{n-1} A_k) + \mu(A_n) = \cdots = \sum_{k=1}^n \mu(A_k)
\end{aligned}$$

Now we prove part 1. Let $A, B \subseteq X$ be μ-measurable and disjoint. Then for any $C \subseteq X$, $\mu(C) = \mu(C \cap A) + \mu(C \setminus A)$, and similarly for B. This is equal to

$$\begin{aligned}
\mu(C) &= \mu(C \cap A) + \mu((C \setminus A) \cap B) + \mu(C \setminus A \setminus B) \\
&= \mu(C \cap A) + \mu(C \cap B) + \mu(C \setminus (A \cup B)) + \mu(C \cap (A \cup B))(?) \\
&= \mu(C \cap (A \cup B) \cap A) + \mu(C \cap (A \cup B) \setminus A) \\
&= \mu(C \cap A) + \mu(C \cap B) \\
&= \mu(C \cap (A \cup B)) + \mu(C \setminus (A \cup B))
\end{aligned}$$

So $A \cup B$ is μ-measurable. (I got a bit lost in the arithmetic, sorry)
Next, we show if $A, B \subseteq X$ are μ-measurable, then $A \cap B$ is μ-measurable. This is straightforward. We will continue next time.

Lecture 3, 1/19/23

We will continue our proof of the theorem. Assume $A, B \subseteq X$ are μ-measurable. We aim to show that $A \cap B$ is also μ-measurable. We need to show that, for any $C \subseteq X$, we have $\mu(C) = \mu(C \cap (A \cap B)) + \mu(C \setminus (A \cap B))$. Because A, B are μ-measurable, we have

$$\begin{aligned}
\mu(C) &= \mu(C \cap A) + \mu(C \setminus A) \\
&= \mu((C \cap A) \cap B) + \mu((C \cap A) \setminus B) + \mu(C \setminus A) \\
&= \mu(C \cap (A \cap B)) + \mu((C \cap A) \setminus B) + \mu(C \setminus A) \\
&\geq \mu(C \cap (A \cap B)) + \mu(C \setminus (A \cap B))
\end{aligned}$$

The opposite inequality follows by subadditivity, so we have equality.
By induction, we get that also $\cap_{k=1}^n A_k$ is μ-measurable. For the union, we can get it using the fact that $\cup_{k=1}^n A_k = X \setminus \cap_{k=1}^n A_k$.

■

Remark. If A, B are μ-measurable, then $A \setminus B$ is μ-measurable. This follows from $A \setminus B = A \cap (X \setminus B)$

Theorem 0.7. Let X be a nonempty set, and μ a measure on X. Assume $\{A_k\}_{k=1}^\infty \subseteq X$ are μ-measurable. Then

1. *If the A_k are disjoint, then we have countable additivity:*

$$\mu\left(\cup_{k=1}^\infty A_k\right) = \sum_{k=1}^\infty \mu(A_k)$$

If $A_1 \subseteq A_2 \subseteq \cdots$, then

$$\lim_{k\to\infty} \mu(A_k) = \mu\left(\bigcup_{k=1}^\infty A_k\right)$$

If $A_1 \supseteq A_2 \supseteq \cdots$, and $\mu(A_1) < \infty$, then

$$\lim_{k\to\infty} \mu(A_k) = \mu\left(\bigcap_{k=1}^\infty A_k\right)$$

Proof. We have from before that if the A_k are pairwise disjoint, then $\mu(\cup_{k=1}^n A_k) = \sum_{k=1}^n \mu(A_k)$ for any $n \in \mathbb{N}$. Because $\cup_{k=1}^n A_k \subseteq \cup_{k=1}^\infty A_k$, we must have that $\mu(\cup_{k=1}^n A_k) \leq \mu(\cup_{k=1}^\infty A_k)$. Using the previous fact, and passing to a limit, we have

$$\sum_{k=1}^{\infty} \mu(A_k) \leq \mu(\cup_{k=1}^\infty A_k)$$

The opposite equality is automatically true by the countable subadditivity of μ, so we get equality. This completes the proof of 1. Now for part 2.
Define $B_k = A_k \setminus A_{k-1}$, where $A_0 \overset{\text{def}}{=} \varnothing$. We have $A_k = \cup_{i=1}^k B_i$. Note that the B_i are disjoint. So we have

$$\mu(A_k) = \sum_{i=1}^{k} \mu(B_i)$$

So, in the limit,

$$\lim_{k\to\infty} \mu(A_k) = \lim_{k\to\infty} \sum_{i=1}^{k} \mu(B_i) = \sum_{i=1}^{\infty} \mu(B_i)$$

So

$$\mu(\cup_{i=1}^\infty B_i) = \mu(\cup_{k=1}^\infty A_k)$$

Finally, let $A_1 \supseteq A_2 \supseteq \cdots, \mu(A_1) < \infty$. Define $B_k = A_1 \setminus A_k$. This is decreasing sequence of μ-measurable sets, so by the previous part,

$$\lim_{k\to\infty} \mu(B_k) = \mu(\cup_{k=1}^\infty B_k)$$

So

$$\begin{aligned}\mu(B_k) = \mu(A_1 \setminus A_k) = \mu(A_1) - \mu(A_k) \implies \\ \lim_{k\to\infty} \mu(B_k) = \lim_{k\to\infty} (\mu(A_1) - \mu(A_k)) = \mu(A_1) - \lim_{k\to\infty} \mu(A_k) \\ = \mu(\cup_{k=1}^\infty B_k) = \mu(\cup_{k=1}^\infty (A_1 \setminus A_k)) = \mu(A_1 \setminus \cap_{k=1}^\infty A_k) \\ \geq \mu(A_1) - \mu(\cap_{k=1}^\infty A_k)\end{aligned}$$

So $\lim_{k\to\infty} \mu(A_k) \leq \mu(\cap_{k=1}^\infty A_k)$. The opposite inequality follows easily by monotonicity.

■

We are ready to prove the Carathéodory extension theorem.

Proof. Let $A_1, A_2, \cdots \subseteq X$ be μ-measurable. It will suffice to prove that $\cup_{k=1}^\infty A_k$ is μ-measurable. So we need to check that, for any B,

$$\mu(B) = \mu(B \cap (\cup_{k=1}^\infty A_k)) + \mu(B \setminus \cup_{k=1}^\infty A_k)$$

Fix $B \subseteq X$, and consider $\nu = \mu|_B$. Recall this is defined as $\nu(C) = \mu(B \cap C)$. We would like

$$\nu(B) = \nu(\cup_{k=1}^{\infty}) + \nu(B \setminus \cup_{k=1}^{\infty} A_k)$$

To this end,

$$\nu(\cup_{k=1}^{\infty} A_k) = \lim_{k \to \infty} \nu(\cup_{i=1}^{k} A_i)$$

Without loss of generality, $\nu(B) < \infty$. If $\nu(B) = \infty$, then we are done trivially. As before, define $B_k = B \setminus \cup_{i=1}^{k} A_i$. Then

$$\nu(B \setminus \cup_{k=1}^{\infty} A_k) = \lim_{k \to \infty} \nu(B \setminus \cup_{i=1}^{k} A_i)$$

So,

$$\begin{aligned} \nu(\cup_{k=1}^{\infty} A_k) + \nu(B \setminus \cup_{k=1}^{\infty} A_k) &= \lim_{k \to \infty} \left(\nu(\cup_{i=1}^{k} A_i) + \nu(B \setminus \cup_{i=1}^{k} A_i) \right) \\ &= \lim_{k \to \infty} \nu(B) = \nu(B) \end{aligned}$$

so we are done.

■

Definition 0.7. Let X be a nonempty set, and let μ be a measure on X. Then μ is said to be

1. A <u>regular measure</u> if, for any $A \subset X$, there exists a μ-measurable $B \subseteq X$ such that $A \subseteq B$, and $\mu(A) = \mu(B)$.

2. A <u>Borel measure</u> if all Borel sets (i.e. the elements of the Borel σ-algebra) are measurable. This only applies if X is also a topological space, of course.

3. A <u>Borel-regular measure</u> if μ is Borel, and for any $A \subseteq X$, there exists a Borel set $B \subseteq X$ such that $A \subseteq B$ and $\mu(A) = \mu(B)$.

4. A <u>Radon measure</u> if it is Borel-regular and $\mu(K) < \infty$ if K is compact.

Remark. Note that being Borel and regular is weaker than being Borel-regular.

Theorem 0.8. *(Increasing sets <u>for regular measures</u>) Let X be a nonempty set, and let μ be a regular measure on X. Assume $A_1 \subseteq A_2 \subseteq \cdots \subseteq X$. Then*

$$\lim_{k \to \infty} \mu(A_k) = \mu\left(\bigcup_{k=1}^{\infty} A_k \right)$$

Remark. The sets A_k need not be μ-measurable.

Proof. For all A_k, there is a $C_k \subseteq X$ which is μ-measurable, $A_k \subseteq C_k$, and $\mu(A_k) = \mu(C_k)$. Let $D_k = \cap_{i \geq k} C_i$. For $i \geq k$, we can see $A_k \subseteq A_i \subseteq C_i$. $A_k \subseteq \cup_{i \geq k} C_i = D_k$, then $\mu(A_k) \leq \mu(D_k)$. On the other hand, $D_k \subseteq C_k$, so $\mu(D_k) \leq \mu(C_k) = \mu(A_k)$. So

- $\mu(A_k) = \mu(D_k)$
- $A_k \subseteq D_k$
- D_k is μ-measurable and $D_1 \subseteq D_2 \subseteq \cdots$

So

$$\begin{aligned}
\lim_{k\to\infty} \mu(A_k) &= \lim_{k\to\infty} \mu(D_k) \\
&= \mu\left(\bigcup_{k=1}^{\infty} D_k\right) \\
&\geq \mu\left(\bigcup_{k=1}^{\infty} A_k\right)
\end{aligned}$$

Because $\cup_{k=1}^{\infty} A_k \subseteq \cup_{k=1}^{\infty} D_k$,

$$\lim_{k\to\infty} \mu(A_k) \geq \mu\left(\bigcup_{k=1}^{\infty} A_k\right)$$

But $A_k \subseteq \cup_{k=1}^{\infty} A_k$, so the opposite inequality is also true, so we have equality. ∎

Lecture 4, 1/24/23

Theorem 0.9. *(Restriction and Radon measures)*
Let X be a topological space and let μ be a Borel-regular measure on X. Let $A \subseteq X$ be μ-measurable with $\mu(A) < \infty$. Then the restriction measure $\nu = \mu|_A$ is Radon.

Proof. First, ν is a finite measure, as $\nu(X) = \mu(A \cap X) = \mu(A) < \infty$ for any $C \subseteq X$. It is clear that ν is Borel, as μ is Borel. Next, we show ν is Borel-Regular. Without loss of generality, we may assume that A is Borel, because μ is Borel-regular. Explicitly, we know there is a Borel set $B \subseteq X$ such that $A \subseteq B$ and $\mu(B) = \mu(A)$. We will show $\mu|_A = \mu|_B$.

We have $\mu(B) = \mu(B \cap A) + \mu(B \setminus A) = \mu(A) + \mu(B \setminus A)$. So $\mu(B \setminus A) = 0$.
So, for all $C \subseteq X$,

$$\begin{aligned}
\mu|_B(C) &= \mu(B \cap C) \\
&= \mu((B \cap C) \cap A) + \mu((B \cap C) \setminus A) \\
&= \mu(C \cap A) + \mu((B \cap C) \setminus A) \\
&\leq \mu|_A(C) + \mu(B \setminus A) \\
&= \mu|_A(C)
\end{aligned}$$

But $(A \cap C) \subseteq (B \cap C)$, so $\mu|_A(C) \leq \mu|_B(C)$, so we may conclude that $\mu|_A = \mu|_B$.
So assume A is Borel. Fix $C \subseteq X$. We need to prove that there exists a Borel $D \subseteq X$ such that $C \subseteq D$ and $\nu(C) = \nu(D)$. There exists a Borel $E \subseteq X$ such that $C \cap A \subseteq E$, and $\mu(C \cap A) = \mu(E)$. So $D = E \cup (X \setminus A)$ is Borel and $C \subseteq D$.
So

$$\begin{aligned}
\nu(D) &= \mu((E \cup (X \setminus A)) \cap A) \\
&= \mu(E \cap A) \\
&\leq \mu(E) \\
&= \mu(C \cap A) \\
&= \nu(C)
\end{aligned}$$

$C \subseteq D$ so $\nu(C) \leq \nu(D)$, so $\nu(C) = \nu(D)$.

■

Theorem 0.10. *(Carathéodory Criterion for being Borel)*
Let X be a metric space and let μ be a measure on X. Then μ is Borel if and only if, for all $A, B \subseteq X$ with $d(A, B) > 0$ (meaning $\inf\{d(a, b) \mid a \in A, b \in B\} > 0$),

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

Proof. =>

Suppose μ is Borel. We will use $\overline{B}$ to denote the closure of B. Then $d(A, \overline{B}) = d(A, B) > 0$. By measurability of $\overline{B}$,

$$\mu(A \cup B) = \mu((A \cup B) \cap \overline{B}) + \mu((A \cup B) \setminus \overline{B}) = \mu(B) = \mu(A)$$

<=

Suppose that, for A, B with $d(A, B) > 0$, $\mu(A \cup B) = \mu(A) + \mu(B)$. We will show that this implies μ is Borel. Let us show that every closed subset $C \subseteq X$ is μ-measurable.

So we have to prove that for every $A \subseteq X$,

$$\mu(A) = \mu(A \cap C) + \mu(A \setminus C)$$

We have $\leq$ trivially. Assume $\mu(A) < \infty$; otherwise, this equality holds trivially.
Define for every $n \in \mathbb{N}$ the set $C_n = \{x \in X \mid d(x, C) \leq \frac{1}{n}\}$. We can see $d(A \setminus C_n, C) \geq \frac{1}{n} > 0$. So

$$\begin{aligned} \mu(\underbrace{(A \setminus C_n) \cup (A \cap C)}_{\subseteq A}) &= \mu(A \setminus C_n) + \mu(A \cap C) \\ &\leq \mu(A) \end{aligned}$$

SO $\mu(A \setminus C_n) + \mu(A \cap C) \leq \mu(A)$ for all $n \in \mathbb{N}$. We will prove that $\lim_{n\to\infty} \mu(A \setminus C_n) = \mu(A \setminus C)$.
Consider the annuli $R_n = \{x \in A \mid \frac{1}{n+1} < d(x, C) \leq \frac{1}{n}$. We have

$$(A \setminus C_1) \bigcup_{n=1}^{\infty} R_n \subseteq A \setminus C$$

C is closed, so in fact we have equality above. Why? If a point belongs to $A \setminus C$, then it does not belong to C, so $d(x, C) > 0$. So there is an $n \in \mathbb{N}$ such that $x \in R_n$ or $x \in A \setminus C_1$. We have

$$\mu\left(\bigcup_{k=0}^{n} R_{2k+1}\right) = \sum_{k=0}^{n} \mu(R_{2k+1}) \leq \mu(A)$$
$$\mu\left(\bigcup_{k=1}^{n} R_{2k}\right) = \sum_{k=1}^{n} \mu(R_{2k}) \leq \mu(A)$$

So $\sum_{n=1}^{\infty} \mu(R_n) \leq 2\mu(A) < \infty$, so $\lim_{n\to\infty}(\sum_{k=n}^{\infty} \mu(R_k)) = 0$. So $(A \setminus C_n) \bigcup_{k=n}^{\infty} R_k = A \setminus C$
So by subadditivity,

$$\mu(A \setminus C) \leq \mu(A \setminus C_n) + \sum_{k=n}^{\infty} \mu(R_k)$$

So as $n \to \infty$,

$$\mu(A \setminus C) \leq \liminf_{n\to\infty} \mu(A \setminus C_n) \leq \mu(A \setminus C)$$

This completes the proof.
It is time for our third section.

Approximation by open, closed, and compact sets

Theorem 0.11. *Let μ be a Borel measure on $\mathbb{R}^n$, and let $B \subseteq \mathbb{R}^n$ be a Borel set.*

1. *If $\mu(B) < \infty$, then for any $\varepsilon > 0$, there exists a closed $C \subseteq B$ such that $\mu(B \setminus C) < \varepsilon$.*

2. *If μ is a Radon measure, then for all $\varepsilon > 0$, there exists an open $U \supseteq B$ such that $\mu(U \setminus B) < \varepsilon$.*

Proof. **1.** Let $\nu = \mu|_B$, a finite measure on $\mathbb{R}^n$.

Define the collection $\mathscr{F} = \{A \subseteq \mathbb{R}^n \mid A$ is μ-measurable and for all $\varepsilon > 0$, there exists a closed $C \subseteq A$ such that $\mu(A \setminus C) < \varepsilon\}$

Our goal is to show that $\mathscr{B}_{\mathbb{R}^n} \subseteq \mathscr{F}$. Davit uses "$\sigma_B$" to indicate the Borel σ-algebra.

By previous discussion, $\mathscr{F}$ contains all closed sets.

Now, if $A_1, A_2, \cdots \in \mathscr{F}$, then $\bigcup_{k=1}^{\infty} A_k \in \mathscr{F}$. For all A_k, there exists a closed $C_k \subseteq A_k$, such that $\nu(A_k \setminus C_k) < \frac{\varepsilon}{2^k}$. Then by subadditivity,

$$\nu(\bigcup_{k=1}^{\infty} A_k \setminus \bigcup_{k=1}^{\infty} C_k) \leq \nu(\bigcup_{k=1}^{\infty}(A_k \setminus C_k)) \leq \sum_{k=1}^{\infty} \nu(A_k \setminus C_k) < \varepsilon$$

and $C = \bigcup_{k=1}^{\infty} C_k$ is closed.

Lecture 5, 1/26/23

Theorem 0.12. *Let μ be a Borel measure on $\mathbb{R}^n$ and let $B \subseteq \mathbb{R}^n$ be a Borel set.*

1. *If $\mu(B) < \infty$, then for all $\varepsilon > 0$, there exists a closed $C \subseteq B$ such that $\mu(B \setminus C) < \varepsilon$.*

2. *If μ is a Radon measure, then for all $\varepsilon > 0$, there exists an open $U \subseteq \mathbb{R}^n$ such that $B \subseteq U$ and $\mu(U \setminus B) < \varepsilon$.*

Proof.

1. Let $\nu = \mu|_B$ be a finite Borel measure on $\mathbb{R}^n$. Define the collection

$$\mathscr{F} = \{A \subseteq \mathbb{R}^n : A\mu\text{−measurable and for all } \varepsilon > 0, \exists C \subseteq A, C \text{ closed}, \nu(A \backslash C) < \varepsilon\}$$

We want to show $\mathcal{B} \in \mathscr{F}$, where $\mathcal{B}$ is the Borel set.

Step 1:

$\mathscr{F}$ contains all closed sets

Step 2:

If $A_1, A_1, \ldots, A_k \in \mathscr{F}$, then for all A_k, there exists a closed C_k such that $\nu(A_k \setminus C_k) < \frac{\varepsilon}{2^k}$. Thus,

$$\begin{aligned}\nu\left(\bigcup_{k=1}^{\infty} A_k \setminus \bigcup_{k=1}^{\infty} C_k\right) &\leq \nu\left(\bigcup_{k=1}^{\infty}(A_k \setminus C_k)\right)\\ &\leq \sum_{k=1}^{\infty} \nu(A_k \setminus C_k) < \varepsilon\end{aligned}$$

Furthermore, $\cap_{k=1}^{\infty} C_k$ is closed. Thus $\mathscr{F}$ is closed under countable intersections.

Step 3:

We want to show countable unions belong to $\mathscr{F}$. If $A_1, A_2, \ldots, A_k, \cdots \in \mathscr{F}$, then for all A_k, there is a closed C_k such that $\nu(A_k \setminus C_k) < \frac{\varepsilon}{2^k}$. However, we do not know if $\cup_{k=1}^{\infty} C_k$ is closed. Note that $\nu(\cup_{k=1}^{\infty} A_k \setminus \cup_{k=1}^{\infty} C_k) = \lim_{m \to \infty} \nu(\cup_{k=1}^{m} A_k \setminus \cup_{k=1}^{m} C_k) < \varepsilon$. So there is an $m \in \mathbb{N}$ such that $\nu(\cup_{k=1}^{m} A_k \setminus \cup_{k=1}^{m} C_k) < \varepsilon$. Furthermore $C = \cup_{k=1}^{m} C_k$ is closed.

Step 4:

In the homework, we showed that every open set is the countable union of closed balls. Since $\mathscr{F}$ contains all closed sets, and is closed under countable unions, $\mathscr{F}$ contains all open sets.

Step 5:

Consider the subset $G \subseteq \mathscr{F}$ given by $G = \{A \in \mathscr{F} \mid A^c \in \mathscr{F}\}$. We claim that G is a σ-algebra. Going through the axioms,

(i) Clearly, $\varnothing \in G$.

(ii) If $A \in G, A^c \in G$.

(iii) If $A_1, A_2, \dots, A_k, \dots \in G$, then $\cup_{k=1}^{\infty} A_k \in G$. Why? $\cup_{k=1}^{\infty} A_k \in \mathscr{F}$ and $\mathbb{R}^n \setminus \cap_{k=1}^{\infty}(\mathbb{R}^n \setminus A_k) \in \mathscr{F}$, since each $\mathbb{R}^n \setminus A_k \in \mathscr{F}$, and $\mathscr{F}$ is closed under countable intersections.

Step 6:

Since the complement of an open sets is a closed set, and since $\mathscr{F}$ contains all open and closed sets, all open sets are contained in the σ-algebra G. Thus the Borel sets are contained in G, implying that they are contained in $\mathscr{F}$.

Note: Part 1 requires that X be a seperable metric space.

2. For all $m \in \mathbb{N}$, denote $U_m = B_m(0) = \{x \in \mathbb{R}^n \mid \|x\| < m\}$.

Note that $\mu(U_m \setminus B) \leq \mu(U_m) < \infty$. Thus there exists a closed $C_m \subseteq U_m \setminus B$ such that $\mu((U_m \setminus B) \setminus C_m) < \frac{\varepsilon}{2^m}$.

Note that $B \cap U_m \subseteq (U_m \setminus C_m)$, which is an open set.

Thus, $\mu((U_m \setminus C_m) \setminus (B \cap U_m)_= \mu((U_m \setminus C_m) \setminus B_< \frac{\varepsilon}{2}$.

Define $U = \cup_{m=1}^{\infty}(U_m \setminus C_m)$, which is an open set.

Thus $B = \cup_{m=1}^{\infty}(B \cap U_m) \subseteq \cup_{m=1}^{\infty}(U_m \setminus C_m) = U$.

Furtheremore,

$$\begin{aligned}
\mu(U \setminus B) &= \mu(\cup_{m=1}^{\infty}(U_m \setminus C_m) \setminus \cup_{m=1}^{\infty}(B \cap U_m)) \\
&\leq \mu(\cup_{m=1}^{\infty}(U_m \setminus C_m \setminus B \cap U_m) \\
&\leq \sum_{m=1}^{\infty} \mu(U_m \setminus C_m \setminus B \cap U_m) \\
&< \varepsilon
\end{aligned}$$

Note: Part 2 requires that for all $r > 0$, for all $x \in X$, $\mu(B_r(x)) < \infty$.

∎

Theorem 0.13. *(Approximation by compact and open sets).*
Let μ be a Radon measure on $\mathbb{R}^n$. Then

1. *For all $A \subseteq \mathbb{R}^n$, $\mu(A) = \inf\{\mu(U) \mid A \subseteq U, U \text{ open }\}$.*

2. *For all μ-measurable $A \subseteq \mathbb{R}^n$, we have $\mu(A) = \sup\{\mu(K) \mid K \subseteq A, K \text{ compact }\}$.*

Proof. Shortly.
Note: If μ is the Lebesgue measure, we define the outer measure as

$$\begin{aligned}\mu(A) &= \inf\{\sum_{k=1}^{\infty}(b_k - a_k) \mid A \subseteq \cup_{k=1}^{\infty}(a_k, b_k)\} \\ &= \inf\{\sum_{k=1}^{\infty}(b_k - a_k) \mid A \subseteq \cup_{k=1}^{\infty}[a_k, b_k), [a_i, b_i) \cap [a_j, b_j) = \varnothing \text{ if } i \neq j\}\end{aligned}$$

Remark. Let X be a topological space, and let μ be a measure on X. If $A \subseteq X$ is such that for all $\varepsilon > 0$, there exists a μ-measurable $A_\varepsilon \subseteq A$, such that $\mu(A \setminus A_\varepsilon) < \varepsilon$. Then A is μ-measurable.

Proof. of remark.
Take $\varepsilon = \frac{1}{k}$. By part 1, there exists $A_k \subseteq A$ such that $0 \leq \mu(A) - \mu(A_k) < \frac{1}{k}$.
Let $b = \cup_{k=1}^{\infty} A_k$, and note that $B \subseteq A$, B is μ-measurable, and $\mu(A) - \frac{1}{k} \leq \mu(A_k) \leq \mu(B)$ for all $k \in \mathbb{N}$.
This implies that $\mu(B) = \mu(A), B \subseteq A$.
Thus, $\mu(A) = \mu(A \cap B) + \mu(A \setminus B) = \mu(B) + \mu(A \setminus B)$.
Since $\mu(A) = \mu(B)$, this implies that $\mu(A \setminus B) = 0$, i.e. $A \setminus B$ is μ-measurable.
So $A = B \cup (A \setminus B)$ is μ-measurable. ∎

We are now prepared for a proof of the theorem.

Proof. **1.** If $\mu(A) = \infty$, then this is trivial. So assume $\mu(A) < \infty$. If A is Borel, we use part 2 of theorem 1.

We have an open $U_\varepsilon \subseteq \mathbb{R}^n$ such that $A \subseteq U_\varepsilon$, and $\mu(U_\varepsilon \setminus A) < \varepsilon$.

Thus $\mu(A) \leq \mu(U_\varepsilon) \leq \mu(A) + \mu(U_\varepsilon \setminus A) < \mu(A) + \varepsilon$.

So we are odne for Borel sets. If A is not Borel, then there exists a Borel $B \subseteq \mathbb{R}^n$ such that $A \subseteq B$ and $\mu(A) = \mu(B)$.

Note: For the above proof, we used a previous theorem, and the fact that μ is Borel-regular.

2. Lets prove that $\mu(A) = \sup\{\mu(C) \mid C \subseteq A, C \text{ closed }\}$. We have two cases:

(i) $\mu(A) < \infty$. In this case, consider $\nu = \mu|_A$. ν is a finite Radon measure. We apply part 1 to A^c. There exists an open $U_\varepsilon \subseteq \mathbb{R}^n$ such that $\mathbb{R}^n \setminus A \subseteq U_\varepsilon$ and $\nu(U_\varepsilon) < \nu(A^c) + \varepsilon = \varepsilon$.
Set $C_\varepsilon = U_\varepsilon^c$, which is closed. Note that $C_\varepsilon \subseteq A$ and $\mu(A \setminus C_\varepsilon) = \mu(A \cap U_\varepsilon) = \nu(U_\varepsilon) < \varepsilon$.

Since $C \subseteq A$, $\mu(C_\varepsilon) \le \mu(A)$. By countable subadditivity, $\mu(A) \le \mu(C_\varepsilon) + \mu(A \setminus C_\varepsilon) < \mu(C_\varepsilon) + \varepsilon$.

(ii) Next time!

■

Lecture 6, 1/31/23

Remark. Let μ be a Borel measure on X. If for every $A \subseteq X$ one has $\mu(A) = \inf\{\mu(B) \mid A \subseteq B, B \text{ Borel }\}$, then μ has to be Borel-regular.

Proof. of remark.
Take $B_k \supseteq A$ such that $\mu(B_k) < \mu(A) + \frac{1}{k}$. Let $B = \cap_{k=1}^{\infty} B_k$. Note B is Borel, $A \subseteq B$, and $\mu(A) \le \mu(B) \le \mu(B_k) < \mu(A) + \frac{1}{k}$. So $\mu(A) = \mu(B)$.

■

Proof. of theorem from last time.
We showed that if $\mu(A) < \infty$, then $\mu(A) = \sup\{\mu(C) \mid C \subseteq A, C \text{ closed }\}$.
If $\mu(A) = \infty$, write $\mathbb{R}^n = \cup_{k=1}^{\infty} R_k$, where $R_k = \{x \in \mathbb{R}^n \mid k \le \|x\| < k+1\}$. Thus $A = \cup_{k=1}^{\infty} A \cap R_k$.
For all $k, \mu(A \cap R_k) \le \mu(R_k) < \infty$. So there exists $C_k \subseteq A \cap R_k$ such that $\mu(C_k) > \mu(A \cap R_k) - \frac{1}{2^k}$.
Thus $\mu(\cup_{k=1}^{\infty} C_k) = \sup_{k=1}^{\infty} \mu(C_k) \ge \sum_{k=1}^{\infty} *\mu(A \cap R_k) - \frac{1}{2^k}) \ge \mu(A) - 2 = \infty$.
This implies that $\lim_{m\to\infty} \mu(\cup_{k=1}^{m} C_k = \mu(\cup_{k=1}^{\infty} C_k) = \infty$.
This proves the theorem for closed sets.
Now we prove the theorem for compact sets.
Case 1: $\mu(A) < \infty$. For all $\varepsilon > 0$, there exists a closed set $C_\varepsilon \subseteq A$ such that $\mu(C_\varepsilon) > \mu(A) - \varepsilon$. Consider $K_m = C_\varepsilon \cap B_m$, where $B_m = \{x \in \mathbb{R}^n \mid \|x\| \le m\}$.
Note K_m is compact.
Thus $\lim_{m\to\infty} \mu(K_m) = \mu(\cup_{k=1}^{\infty} K_m) = \mu(C_\varepsilon) > \mu(A) - \varepsilon$.
Case 2: $\mu(A) = \infty$. For all $m \in \mathbb{N}$, there exists a closed $C_m \subseteq A$ such that $\mu(C_m) \ge m$. Apply the same procedure.

■

Covering theorems (Vitali's and Besicovitch)

Notation: we will work in $\mathbb{R}^n$. Closed balls will be denoted by B. For a given closed ball $B = B_r(x) = \{y \in \mathbb{R}^n \mid \|x - y\| \le r\}$, $\hat{B} = cB = \{y \in \mathbb{R}^n \mid \|x - y\| \le cr\} = B_{cr}(x)$.

Definition 0.8. Let $A \subseteq \mathbb{R}^n$ and let $\mathscr{F} = \{B \subseteq \mathbb{R}^n\}$ be a family of balls.

1. $\mathscr{F}$ is a <u>cover of A</u> if $A \subseteq \cup_{B\in\mathscr{F}} B$.

2. $\mathscr{F}$ is a <u>fine cover of A</u> if for all $x \in A$ and $\varepsilon > 0$, there exists a $B \in \mathscr{F}$ such that $x \in B$ and $\text{diam}(B) < \varepsilon$. Alternatively, for all $x \in X$, $\inf\{\text{diam}(B) \mid x \in B\} = 0$.

Theorem 0.14. *(Vitali's Covering Theorem)*
Let $\mathscr{F}$ be a collection of nondegenerate closed balls in $\mathbb{R}^n$ with diameters uniformly bounded, i.e. $\sup\{\text{diam}(B) \mid B \in \mathscr{F}\} < \infty$. *Then there exists a subcollection of countably many disjoint balls $\{\hat{B}_i\}_{i=1}^{\infty}$, such that $\cup_{b\in B} B \subseteq \cup_{i=1}^{\infty} \hat{B}_i$*

Proof. Denote $D = \sup\{\text{diam}(B) \mid B \in \mathscr{F}\}$ and consider $\mathscr{F}_k = \{B \in \mathscr{F} \mid \frac{D}{2^k} < \text{diam}(B) \leq \frac{D}{2^{k-1}}\}$. Let $G_1 \subseteq \mathscr{F}_1$ be a maximal disjoint subcollection of balls in $\mathscr{F}$ (we can produce this easily with Zorn's Lemma). It will be maximal in the sense that if we add another element, it will not be a disjoint set.
Assume $G_1, \ldots, G_{k-1}$ have been chose.
Let G_k bew a maximal disjoint subcollection in $\mathscr{F}$ such taht the balls at G_k do not intersect with the balls in $\cup_{i=1}^{k-1} G_i$.
Set $G = \cup_{k=1}^{\infty} G_k \subseteq \mathscr{F}$. Let $B \in \mathscr{F}_m$, i.e. $\frac{D}{2^m} < \text{diam}(B) \leq \frac{D}{2^{m-1}}$.
Because G_m is maximal, there exists $\overline{B} \in \cup_{i=1}^{m} G_i$ such that $B \cap \overline{B} = \varnothing$. Thus $\text{diam}(\overline{B}) \geq \frac{D}{2^m} \geq \frac{1}{2}\text{diam}(B)$. Thus $B \subseteq \hat{\overline{B}}$.

Lecture 7, 2/2/23

Corollary 0.15. *Let $A \subseteq \mathbb{R}^n$, and let the collection $\mathscr{F}$ of nondegenerate closed balls be a fine cover of A such that* $\sup\{\text{diam}(B) \mid B \in \mathscr{F}\} < \infty$.
Then for any finite number of balls $B_1, B_2, \ldots, B_m \in \mathscr{F}$, one has

$$A \setminus \cup_{i=1}^{m} B_i \subseteq \cup_{B \in G\setminus\{B_1,\ldots,B_m\}} \hat{B}$$

where G is the disjoint collection of balls guaranteed by Vitali's theorem.

Proof. Assume $x \in A \setminus \cup_{i=1}^{m} B_i$. Then $x \notin \cup_{i=1}^{m} B_i$, so $d(x, \cup_{i=1}^{m} B_i) > 0$, as B_i is closed for all i, and the finite union of closed sets is closed.
Let $d = d(x, \cup_{i=1}^{m} B_i)$. Because $x \in A$, there exists a ball $B = B_r(y) \in \mathscr{F}$ such that $x \in B_r(y)$ and $2r < d$. This gives us that $B \cap \cup_{i=1}^{m} B_i = \varnothing$. By the construction of G, there exists $\overline{B} \in G$ such that $B \cap \overline{B} = \varnothing$ and $B \subseteq \hat{\overline{B}}$. Now, $\overline{B} \neq B_i, i = 1, 2, \ldots, m$, and therefore $x \in B \subseteq \hat{\overline{B}} \subseteq \cup_{\overline{\overline{B}} \in G\setminus\{B_1,\ldots,B_n\}} \hat{\overline{\overline{B}}}$.

Theorem 0.16. *(Filling open sets with closed balls)*
Let $U \subseteq \mathbb{R}^n$ be open, and let $\delta > 0$. Then there is a countable collection G of nondegenerate closed, disjoint balls, such that

$$\sup\{\text{diam}(B) \mid B \in G\} \leq \delta$$

and $\mathscr{L}^n(U \setminus \cup_{i=1}^{\infty} B_i) = 0$, where $\mathscr{L}^n$ denotes n-dimensional Lebesgue measure. Here, $G = \{B_i\}_{i=1}^{\infty}$.

Proof.

Case 1: $\mathscr{L}^n(U) < \infty$

Consider the collection of nondegenerate closed balls $\mathscr{F} = \{B \subseteq U \mid \mathrm{diam}(B) \leq \delta\}$. Because U is open, $\cup_{B\in\mathscr{F}} B = U$. By Vitali's covering theorem, there exists a countable family G of disjoint balls such that $U \subseteq \cup_{i=1}^{\infty} \hat{B}_i$. So

$$\mathscr{L}^n(U) \leq \mathscr{L}^n(\cup_{i=1}^{\infty} \hat{B}_i) \leq \sum_{i=1}^{\infty} \mathscr{L}^n(\hat{B}_i)$$

By countable subadditivity, $\mathscr{L}^n(U) \leq 5^n \sum_{i=1}^{\infty} \mathscr{L}^n(B_i) = 5^n \mathscr{L}^n(\cup_{i=1}^{\infty} B_i)$. So

$$\mathscr{L}^n(U \setminus \cup_{i=1}^{\infty} B_i) = \mathscr{L}^n(U) - \mathscr{L}^n(\cup_{i=1}^{\infty} B_i) \leq (1 - \frac{1}{5^n})\mathscr{L}^n(U)$$

Now

$$\lim_{m\to\infty} \mathscr{L}^n(U \setminus \cup_{i=1}^{m} B_i) = \mathscr{L}^n(U \setminus \cup_{i=1}^{\infty} B_i) \leq (1 - \frac{1}{5^n})\mathscr{L}^n(U)$$

So there exists an index $m_1 \in \mathbb{N}$ such that $\mathscr{L}^n(U \setminus \cup_{i=1}^{m_1} B_i) \leq (1 - \frac{1}{2\cdot 5^n})\mathscr{L}^n(U)$. Consider $U_2 = U \setminus \cup_{i=1}^{m_1} B_i$ and the new collection $\mathscr{F}_i = \{B \mid B \subseteq U_2, \mathrm{diam}(B) \leq \delta\}$. Then

$$\mathscr{L}^n(U_2) \leq q \cdot \mathscr{L}^n(U) < \infty$$

So there exist disjoint closed $B_{m_{1+1}}, \cdots, B_{m_2} \in \mathscr{F}_2$ such that

$$\mathscr{L}^n(U_2 \setminus \cup_{i=m_1+1}^{m_2} B_i) \leq q\mathscr{L}^n(U)$$

So $\mathscr{L}^n(U \setminus \cup_{i=1}^{m_2}) \leq q^2\mathscr{L}^n(U)$.
<u>k-th step</u>
There are disjoint balls $B_1, B_2, \ldots, B_{m_k} \subseteq U$ such that

$$\mathscr{L}^n(U \setminus \cup_{i=1}^{m_k}) \leq q^k \mathscr{L}^k(U)$$

So

$$\mathscr{L}^n(U \setminus \cup_{i=1}^{\infty} B_i) \leq \mathscr{L}^n(U \setminus \cup_{i=1}^{m_k}) \leq q^k \mathscr{L}^n(U)$$

The above is true for every k, so it follows that $\mathscr{L}^n(U \setminus \cup_{i=1}^{\infty} B_i) = 0$. $G = \{B_i\}_{i=1}^{\infty}$. This completes the proof in the case of $\mathscr{L}^n(U) < \infty$.

Case 2, $\mathscr{L}^n(U) = \infty$

Consider $U_m = U \cap \{x \in \mathbb{R}^n : m - 1 < |x| < m\}$, $m = 1, 2, \ldots$. We know $\mathscr{L}^n(\partial B_r(x)) = 0$ for all $B_r(x) \subseteq \mathbb{R}^n$ (this will be a homework problem).

■

"You can go look at the proof of Besicovitch in the book , but to be honest I never read that proof." - Davit

Theorem 0.17. *(Besicovitch's covering theorem)*
There exists a number N_n that depends only on the space dimension n, with the following property.
If $\mathscr{F}$ is any collection of nondegenerate closed balls in $\mathbb{R}^n$, with

$$\sup\{\operatorname{diam}(B) \mid B \in \mathscr{F}\} < \infty$$

and $A = \{x \mid \exists B_r(x) \in \mathscr{F}$ (the centers of the balls).
Then there exists N_n countable collections $G_1, G_2, \dots, G_{N_n}$, each of which are disjoint (as in, the balls in each collection are disjoint. This does not mean $G_i \cap G_j = \varnothing$) in $\mathscr{F}$ such that

$$A \subseteq \cup_{i=1}^{N_n} (\cup_{B \in G_i} B)$$

Proof. In the book

■

Theorem 0.18. *(More on filling open sets with Balls)*
Let μ be a Borel measure on $\mathbb{R}^n$, and let $\mathscr{F}$ be any collection of nondegenerate closed balls. Let $A = \{x \mid \exists B_r(x) \in \mathscr{F}$ (again, the set of centers).
Assume $\mu(A) < \infty$ (we do not assume A is μ-measurable) and $\inf\{r : B_r(a) \in \} = 0$ *for any $a \in A$.*
Then for every open set $U \subseteq \mathbb{R}^n$, there exists a countable collection G of disjoint balls in $\mathscr{F}$ such that

1. $\cup_{B \in G} B \subseteq U$

2. $\mu(A \cap U \setminus \cup_{B \in G} B) = 0.$

Proof. Consider the collection $\mathscr{F}_1 = \{B \mid B \in \mathscr{F}, B \subseteq U, \operatorname{diam}(B) \leq 1\}$.
$A \cap U = \{x \mid \exists B_r(x) \in \mathscr{F}_1\}$.
Apply the theorem to $\mathscr{F}_1$. Then there exist $G_1, G_2, \dots, G_{N_n}$ countable collections of disjoint balls (each) in $\mathscr{F}_1$ such that

$$A \cap U \subseteq \cup_{i=1}^{N_n} (\cup_{B \in G_1} B)$$

Then $\mu(A \cap U) \leq \sum_{i=1}^{N_n} \mu\left((A \cap U) \cap (\cup_{B \in G_i} B)\right)$. So there exists an index $k \in \{1, 2, \dots, N_n\}$ such that

$$\mu\left((A \cap U) \cap (\cup_{B \in G_k} B)\right) \geq \frac{1}{N_n} \mu(A \cap U)$$

Write $G_k = \{B_i\}_{i=1}^{\infty}$. Then

$$\mu\left((A \cap U) \cap (\cup_{i=1}^{\infty} B_i)\right) \geq \frac{1}{N_n} \mu(A \cap U)$$

Let $q = 1 - \frac{1}{2N_n}$.
$\mu(A \cap U) \leq \sup_{i=1}^{N_n} \mu\left((A \cap U) \cap (\cup_{B \in G_i B}\right)$.
There exists an index $k \in \{1, 2, \dots, N_n\}$ such that

$$\mu\left((A \cap U) \cap (\cup_{B \in G_k} G)\right) \geq \frac{1}{N_n} \mu(A \cap U)$$

Lecture 8, 2/7/23

We continue with the proof. Some review.
μ is a Borel measure on $\mathbb{R}^n$. $\mathscr{F} = \{B \mid B \subseteq \mathbb{R}^n, B\text{-nondegenerate}\}$. $A = \{a \in \mathbb{R}^n \mid \exists B_r(a) \in \mathscr{F}\}$. Assume $\mu(A) < \infty$. Assume $\inf\{r \mid B_r(a) \in \mathscr{F}\} = 0$ for all $a \in A$.
Then for all open $U \subseteq \mathbb{R}^n$, there exists $G = \{B_i\}_{i=1}^{\infty} \subseteq \mathscr{F}$ collection of disjoint balls such that $\mu(A \cap U \setminus \cup_{i=1}^{\infty} B_i) = 0$, $\cup_{i=1}^{\infty} B_i \subseteq U$.
Let $\mathscr{F}_1 = \{B \in \mathscr{F} \mid B \subseteq U, \operatorname{diam}(B) \leq 1\}$.
New set of centers $= A \cap U$. By Besicovitch, there exist N_n collections $G_1, G_2, \dots, G_{N_N} \subseteq \mathscr{F}_1$ such that $A \cap U \subseteq \cup_{i=1}^{\infty} \cup_{B \in G_i} B$.
Then $\mu(A \cap U) \leq \sum_{i=1}^{N_n} \mu(\cup_{B \in G_i} (B \cap A \cap U))$.
So there exists an index $k \in \{1, 2, \dots, N_n\}$ such that

$$\mu((A \cap U) \cap (\cup_{B \in G_k} B)) \geq \frac{1}{N_n} \mu(A \cap U)$$

Let $\nu = \mu|_A$ - Borel. Let $G_k = \{B_i\}_{i=1}^{\infty}$. Then $\mu((A \cap U) \cap (\cup_{i=1}^{\infty} B_i)) = \nu(U \cap \cup_{i=1}^{\infty} B_i) = \lim_{n \to \infty} \nu(U \cap \cup_{i=1}^{m} B_i)$. So there exists $m_1 \in \mathbb{N}$ such that

$$\nu(U \cap \cup_{i=1}^{m_1} B_i) = \mu((A \cap U) \cup_{i=1}^{m_1} B_i) \geq \frac{1}{2N_n} \mu(A \cap U)$$

So

$$\begin{aligned} \mu(A \cap U \setminus \cup_{i=1}^{m_i} B_i &= \mu(A \cap U) - \mu(A \cap U \cap \cup_{i=1}^{m_1} B_i) \\ &\leq \underbrace{(1 - \frac{1}{2N_n})}_{0<q<1} \mu(A \cap U) \end{aligned}$$

Let $U_2 = U \setminus \cup_{i=1}^{m_1} B_i$, $\mathscr{F}_2 = \{B \in \mathscr{F}_1 \mid B \subseteq U_2, \operatorname{diam}(B) \leq 1\}$.
<u>kth step</u>
We have $B_{m_{n-1}+1}, \dots, B_{m_k} \in \mathscr{F}_k$ such that

$$\mu(A \cap U \setminus \cup_{i=1}^{m_k}) \leq q^k \mu(A \cup U)$$

Let $G = \{B_i\}$.

■

Differentiation of Radon Measures

Definition 0.9. Let μ and ν be Radon measures on $\mathbb{R}^n$. Define, for any $x \in \mathbb{R}^n$, the upper derivative of μ with respect to ν by

$$\overline{D}_\mu \nu(x) = \begin{cases} \limsup_{r\to 0^+} \frac{\nu(B_r(x))}{\mu(B_r(x))} & \text{if } \mu(B_r(x)) > 0, \forall r > 0 \\ +\infty & \mu(B_r(x)) = 0 \text{ for some } r \end{cases}$$

The lower derivative is defined similarly:

$$\underline{D}_\mu \nu(x) = \begin{cases} \liminf_{r\to 0^+} \frac{\nu(B_r(x))}{\mu(B_r(x))} & \text{if } \mu(B_r(x)) > 0, \forall r > 0 \\ +\infty & \mu(B_r(x)) = 0 \text{ for some } r \end{cases}$$

$\overline{D_\mu}\nu, \underline{D}_\mu \nu : \mathbb{R}^n \to [0,\infty]$. These are sometimes also called the upper/lower density. We say that ν is differentiable with respect to μ at the point x if $\overline{D}_\mu\nu(x) = \underline{D}_\mu\nu(x) < \infty$

This leads to several questions/goals:

1. Study the set where $\underline{D}_\mu\nu(x) = \overline{D}_\mu\nu(x) < \infty$. Is it μ-a.e.?

2. Do we have $\nu(B) = \int_B D_\mu\nu(x)\, d\mu$ for Borel sets $B \subseteq \mathbb{R}^n$?

For question 1, the answer is yes, μ-almost everywhere in $\mathbb{R}^n$. The answer to question 2 is also yes, subject to the additional condition $\nu << \mu$.

Lemma 1. *Let μ and ν be Radon measures on $\mathbb{R}^n$. Let $0 < \alpha < +\infty$.*

(i) If $A \subseteq \{x \in \mathbb{R}^n \mid \underline{D}_\mu\nu(x) \leq \alpha\}$, then $\nu(A) \leq \alpha\mu(A)$

(ii) If $A \subseteq \{x \in \mathbb{R}^n \mid \overline{D}_\mu\nu(x) \geq \alpha\}$, then $\nu(A) \geq \alpha\mu(A)$.

Proof. We can assume without loss of generality that $\mu(\mathbb{R}^n), \nu(\mathbb{R}^n) < \infty$. This is because $\nu(A \cap B_R(0)) \leq \alpha\mu(A \cap B_R(0))$, and as $R \to +\infty$, the left converges to $\nu(A)$, and the right converges to $\alpha\mu(A)$, $B_R(0) = \{|x| < R\}$.
Fix $\varepsilon > 0$.

(i) Let U be any open set such that $A \subseteq U$. Consider the collection of closed balls $\mathscr{F} = \{B_r(x) \subseteq U \mid x \in A, \nu(B_r(x)) \leq (\alpha+\varepsilon)\mu(B_r(x)), \mathrm{diam}(B) \leq 1\}$

For any $a \in A$, we have $\inf\{r \mid B_r(a) \in \mathscr{F}\} = 0$ because $\underline{D}_\mu\nu(x) \leq \alpha$. Set of centers $= A$.

By the theorem we just proved, there exists a countable collection $G = \{B_i\}_{i=1}^\infty$ of disjoint balls in $\mathscr{F}$ such that

$$\nu(A \cap U \setminus \cup_{i=1}^\infty B_i) = 0$$

So

$$\nu(A) \leq \nu(\cup_{i=1}^{\infty} B_i) + \nu(A \setminus \cup_{i=1}^{\infty} B_i) = \nu(\cup_{i=1}^{\infty} B_i)$$

By disjointness, this is equal to

$$\begin{aligned}\sum_{i=1}^{\infty} \nu(B_i) &\leq \sum_{i=1}^{\infty} (\alpha + \varepsilon)\mu(B_i) \\ &= (\alpha + \varepsilon)\mu(\cup_{i=1}^{\infty} B_i) \\ &\leq (\alpha + \varepsilon)\mu(U)\end{aligned}$$

So $\nu(A) \leq (\alpha + \varepsilon)\mu(U)$ for all $\varepsilon >), U \supseteq A$.

In the limit as $\varepsilon \to 0^+$, we have $\nu(A) \leq \alpha\mu(U)$ for all $U \supseteq A$. So $\nu(A) \leq \alpha \inf\{\mu(U) \mid A \subseteq U, U\text{-open}\} = \alpha\mu(A)$.

This completes the proof of 1. Proof of 2 is similar?

■

Theorem 0.19. *Let μ and ν be Radon measures on $\mathbb{R}^n$. Then*

(i) ν is differentiable with respect to μ almost everywhere in $\mathbb{R}^n$.

(ii) $\underline{D}_{\mu}\nu(x) = \overline{D}_{\mu}\nu(x) < \infty$ μ-almost everywhere in $\mathbb{R}^n$.

(iii) $D_{\mu}\nu$ is μ-measurable.

Proof. Let $I = \{x \in \mathbb{R}^n \mid \overline{D}_{\mu}\nu(x) = \infty\}$.
With the lemma we have just proven, it is easy to see that $\mu(I) = 0$.
Assume $\mu(\mathbb{R}^n), \nu(\mathbb{R}^n) < \infty$. Fix any $\alpha > 0$. Then $I = \{x \in \mathbb{R}^n \mid \overline{D}_{\mu}\nu(x) \geq \alpha\}$. So $\nu(I) \geq \alpha\mu(I)$, so

$$\mu(I) \leq \frac{1}{\alpha}\nu(I) \leq \frac{\nu(\mathbb{R}^n)}{\alpha}$$

As let let $\alpha \uparrow +\infty$, we get $\mu(I) < \infty$.

Lecture 9, 2/9/23

Theorem 0.20. *Let μ and ν be Radon measures on $\mathbb{R}^n$. Then*

1. *$D_{\mu}\nu(x)$ exists and is finite μ-almost everywhere.*

2. *$D_{\mu}\nu(x)$ is μ-measurable.*

Proof. Let $I = \{x \in \mathbb{R}^n \mid \overline{D}_\mu \nu(x) = \infty\}$. We know that $\mu(I) = 0$, so we just have to prove 2. Assume within the ball $B_R(0) = \{x \in \mathbb{R}^n \mid |x| < R\}$, $D_\mu\nu(x)$ exists μ-almost everywhere in $B_R(x)$.
Define $X_{INE}(R) = \{x \in B_R(0) \mid \text{either } D_\mu\nu(x) = \infty \text{ or } D_\mu\nu(x) \text{ DNE }\}$.
Note $X_{INE}(R) \subseteq B_R(x), X_{INE}(1) \subseteq X_{INE}(2) \subseteq \cdots \subseteq X_{INE}(m) \subseteq \cdots$.
Now $\mu(X_{INE}(m)) = 0$ for all $m \in \mathbb{N}$, thus $\mu(X_{INE}(\infty)) \leq \sum_{m=1}^{\infty} \mu(X_{INE}(m)) = 0$.
Since X_{INE} is a null set, it is measurable.
Assume $\nu(\mathbb{R}^n), \mu(\mathbb{R}^n) < \infty$. Let $X_{NE} = \{x \in \mathbb{R}^n \mid D_\mu\nu(x) \text{ DNE}\} \implies \mu(X_{NE}) = 0$.
For $0 < a < b < \infty$, define $J(a,b) = \{x \in \mathbb{R}^n \mid \underline{D}_\mu\nu(x) \leq a \text{ or } \overline{D}_\mu\nu(x) \geq b\}$.
Then $X_{NE} \subseteq I \cup (\cup_{0<r,q<\infty} J(r,q))$. $\mu(X_{NE}) \leq \mu(I) + \sum_{0<r,q<\infty} \mu(J(r,q))$.
By a previous lemma, $\nu(J(a,b)) \leq a\mu(J(a,b)), \nu(J(a,b)) \geq b\mu(J(a,b))$.
Since $a\mu(J(a,b)) \geq b\mu(J(a,b))$ for all $a < b$, $\mu(J(a,b)) = 0$ for all $0 < a < b < \infty$. So $\mu(X_{NE}) = 0$.

■?

Idea: Express $D_\mu\nu(x)$ as the limit of a sequence of μ-measurable functions.

Claim.

$$D_\mu\nu(x) = \begin{cases} \lim_{r\to 0^+} \frac{\nu(B_r(x))}{\mu(B_r(x))} & x \in \mathbb{R}^n \setminus N \\ \infty \textit{ or not defined in } N & \textit{where } \mu(N) = 0 \end{cases}$$

Proof. For fixed $r > 0$, define $f_r(x) : \mathbb{R}^n \to [0, \infty]$ by

$$f_r(x) = \begin{cases} \frac{\nu(B_r(x))}{\mu(B_r(x))} & \text{if } \mu(B_r(x)) > 0 \\ \infty & \text{otherwise.} \end{cases}$$

Claim. For any $r > 0$, the function $f_r(x)$ is μ-measurable.

Claim. For every $r > 0$, the function $g_r(x) : \mathbb{R}^n \to [0, \infty)$ defined by $g_r(x) = \mu(B_r(x))$ is upper semicontinuous, i.e. if $x_k \to x$, then $\limsup g_r(x_k) \leq g_r(x)$.

Proof. of second claim
Let $x_k \to x$ be a convergent sequence.
Define $\varphi(x) = \chi_{B_r(x)}$ and note $\limsup_{x_k \to x} \varphi(x_k) \leq \varphi(x)$.
$\limsup_{x_k \to x} \chi_{B_r(x_k)}(y) \leq \chi_{B_r(x)}(y)$.
If $y \in B_r(x)$ then $\chi_{B_r(x)}(y) = 1$, so we're done.
If $y \notin B_r(x)$ then $|x - y| = r + \delta$ where $\delta > 0$ so there exists $K \in \mathbb{N}$ such that for all $k \geq K$, $|x_k - x| < \delta \implies y \notin B_r(x_k)$.
So $\chi_{B_r(x_k)}(y) = 0 \implies \limsup_{x_k \to x} \chi_{B_r(x)}(y) = 0$.
Note $\liminf_{x_k \to x}(1 - \chi_{B_r(x_k)}(y)) \geq 1 - \chi_{B_r(x)}(y)$.
Thus $\int_{B_r(x)} \liminf_{x_k \to x}(1 - \chi_{B_r(x_k)}(y))\, d\mu(y) \geq \int_{B_r(x)} 1 - \chi_{B_r(x)}(y)$
By Fatou's lemma,

$$\liminf_{k\to\infty} \int_{B_r(x)} (1 - \chi_{B_r(x_k)}(y))\, d\mu(y) \geq \int_{B_r(x)} (1 - \chi_{B_r(x)}(y))$$

Thus $\liminf_{k\to\infty} f(\mu(B_{2r}(x))) - \mu(B_r(x_k)) \geq \mu(B_{2r}(x)) - \mu(B_r(x))$
So $\limsup_{k\to\infty} \mu(B_r(x_k)) \leq \mu(B_r(x))$

■

Proof. of first claim
Denote $I_r = \{x \in \mathbb{R}^n \mid \mu(B_r(x)) = 0\}$.
$I_r \subseteq I$, $\mu(I) = 0$, so $\mu(I_r) = 0$ for all $r > 0$.
Furthermore, if $0 < r_1 < r_2$, $I_{r_2} \subset I_{r_1} \subset I$.

■

Claim.

$$D_\mu \nu(x) = \begin{cases} \lim_{k\to\infty} \frac{\nu(B_{\frac{1}{k}}(x))}{\mu(B_{\frac{1}{k}}(x))} & \mu(B_{\frac{1}{k}}(x) > 0 \\ \infty & \mu(B_{\frac{1}{k}}(x)) = 0 \end{cases}$$

If $D_\mu \nu(x) < \infty$ μ-almost everywhere, then $D_\mu \nu(x)$ is μ-measurable.

Proof. ■

Definition 0.10. Let μ and ν be Borel measures on $\mathbb{R}^n$. Then

1. ν is <u>absolutely continuous with respect to μ</u> if $\mu(A) = 0 \implies \nu(A) = 0$ for all $A \subseteq \mathbb{R}^n$. We write $\nu << \mu$.

2. μ and ν are <u>mutually singular</u> if there exists a Borel set $B \subseteq \mathbb{R}^n$ such that $\nu(B) = \mu(\mathbb{R}^n \setminus B) = 0$. We write $\nu \perp \mu$.

Theorem 0.21. *(Radon-Nikodym)*
Let μ and ν be Radon measures on $\mathbb{R}^n$ such that $\nu << \mu$. Then for any μ-measurable set $A \subseteq \mathbb{R}^n$, one has

1. $\nu(A) = \int_A D_\mu \nu(x)\, d\mu$

2. *For any $f : \mathbb{R}^n \to \mathbb{R}$ μ-measurable, one has $\int_A f(x)\, d\nu = \int_A f(x) D_\mu \nu(x)\, d\mu$.*

Proof. For the first part, observe that if A is μ-measurable, then A is also ν-measurable. Why? If there exists a Borel set $B \subseteq \mathbb{R}^n$ such that $A \subseteq B, \mu(A) = \mu(B)$, then $\mu(B \setminus A) = 0$. Since $\nu << \mu$, $\nu(B \setminus A) = 0$, so $B \setminus A$ is ν-measurable. So $A = B \setminus (B \setminus A)$. so A is ν-measurable.

Corollary 0.22. *If f is μ-measurable, then it is ν-measurable.*

Lecture 10, 2/21/23

Claim. If $A \subseteq \mathbb{R}^n$ is μ-measurable, then A is also ν-measurable. If $f : \mathbb{R}^n \to \mathbb{R}$ is μ measurable, then f is also ν-measurable.

Proof.
Fix $t > 1$ (t-truncation argument)
Consider the sets

$$A_m = \{x \in A \mid t^m \le D_\mu\nu(x) < t^{m+1}\},\ m \in \mathbb{Z}$$

Let $A = \cup_{m\in\mathbb{Z}} A_m \cup (A \cap \tilde{I}) \cup (A \cap \{x \mid D_\mu\nu(x) \text{ does not exist}\} \cup (A \cap Z)$, where

$$\tilde{I} \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \mid D_\mu\nu(x) = +\infty\}$$
$$Z \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \mid D_\mu\nu(x) = 0\}$$

Note $\nu((A \cap \tilde{I}) \cup (A \cap \{x \mid D_\mu\nu(x) DNE) \cup (A \cap Z)) = 0$, so $\nu(A) = \nu(\cup_{m\in\mathbb{Z}} A_m) = \sum_{m\in\mathbb{Z}} \nu(A_m)$.
Why is this set ν-null?
First, note $\mu(\{x \mid D_\mu\nu(x) DNE\}) = 0$ implies $\nu(\{x \mid D_\mu\nu(x) DNE\}) = 0$.
$\tilde{I} \subseteq I$, and $\mu(I) = 0$, so $\mu(\tilde{I}) = 0$, so $\mu(\tilde{I} \cap A) = 0$, so $\nu(\tilde{I} \cap A) = 0$.
Fix $\alpha > 0$. $\nu(C_\alpha) \le \alpha\mu(C_\alpha)$. So $\nu(Z) \le \nu(C_\alpha) \le \alpha\mu(C_\alpha) \le \alpha\mu(\mathbb{R}^n)$, which goes to 0.
So $\nu(Z) = 0$.
Thus $\nu(\tilde{I}) = \int_{\tilde{I}} D_\mu\nu(x)\,d\mu, \nu(Z) = \int_Z D_\mu\nu(x)\,d\mu = 0$.
Now $t^n\mu(A_m) \le \nu(A_m) \le t^{m+1}\mu(A_)$.
So $\sum_{m\in\mathbb{Z}} t^m\mu(A_m) \le \sum_{m\in\mathbb{Z}} \nu(A_m) \le \sum_{m\in\mathbb{Z}} t^{m+1}\mu(A_m) = t\sum_{m\in\mathbb{Z}} t^m\mu(A_m)$.
So $\int_{A_m} t^m\,d\mu \le \int_{A_m} D_\mu\nu(x)\,d\mu \le \int_{A_m} t^{m+1}\,d\mu$.
By monotone convergence theorem,$\sum_{m\in\mathbb{Z}} \int_{A_m} D_\mu\nu(x)\,d\mu = \int_{\cup A_m} D\mu\nu(x)\,d\mu$.
So $\sum_{m\in\mathbb{Z}} t^n\mu(A_m) \le \int_{\cup A_m} D_\mu\nu(x)\,d\mu \le t\sum_{m\in\mathbb{Z}} t^m\mu(A_m)$.
So $\frac{1}{t}\int_{\cup A_m} D_\mu\nu(A)\,d\mu \le \nu(A) \le t\int_{\cup A_m} D_\mu\nu(x)\,d\mu, \forall t > 1$.
So $\frac{1}{t}\int_A D_\mu\nu(x)\,d\mu \le \nu(A) \le t\int_A D_\mu\nu(x)\,d\mu$.
We let $t \to 1$ to get $\nu(A) = \cup_A D_\mu\nu(x)\,d\mu$.
We now prove that $\int_A f(x)\,d\nu = \int_A f(x)D_\mu\nu(x)\,d\mu$.
Let $f^+(x) = \max(f(x), 0)$ and $f^-(x) = -\min(0, f(x))$. Note $f = f^+ - f^-$.
Assume $f(x) \ge 0$ for all $x \in \mathbb{R}^n$.

Lemma 2. Let X be a nonempty set, and let μ be a measure on X, and let $f : X \to [0, \infty]$ be μ-measurable. Then there exist μ-measurable sets $\{A_k\}_{k=1}^\infty$ such that

$$f(x) = \sum_{k=1}^\infty \frac{1}{k}\chi_{A_k}(x)$$

Remark. The $\{A_k\}$ need not be disjoint.

Proof. Define $A_1 = \{x \mid f(x) \geq 1\}$.
Assuming $A_1, A_2, \dots, A_{k-1}$ are well-defined, define $A_k = \{x \in X \mid f(x) \geq \frac{1}{k} + \sum_{i=1}^{k-1} \frac{1}{i}\chi_{A_i}(x)\}$
By construction/induction, each A_k is μ-measurable.
We claim $f(x) = \sum_{k=1}^{\infty} \frac{1}{k}\chi_{A_k}(x)$.

1. Fix $n \in \mathbb{N}$. We claim $f(x) \geq \sum_{k=1}^{n} \frac{1}{k}\chi_{A_k}(x)$. Let $m \in \{1, \dots, n\}$ be the largest integer such that $x \in A_m$. Then

$$f(x) \geq \frac{1}{m} + \sum_{k=1}^{m-1} \frac{1}{k}\chi_{A_k}(x) = \sum_{k=1}^{m} \frac{1}{k}\chi_{A_k}(x) \leq \sum_{k=1}^{n} \frac{1}{k}\chi_{A_k}(x)$$

2. If $f(x) = \infty$, $x \in A_k$ for all $k \in \mathbb{N}$ so $\sum_{k=1}^{m} \frac{1}{k}\chi_{A_k}(x) = \sum_{k=1}^{\infty} \frac{1}{k} = \infty$. If $f(x) < \infty$, then there exists a sequence of naturals $\{n_\ell\}$ such that $x \notin A_{n_\ell}$. So

$$f(x) - \sum_{k=1}^{n_\ell - 1} \frac{1}{k}\chi_{A_k}(x) < \frac{1}{n_\ell},\ \ell = 1, 2, \dots.$$

We let $\ell \to \infty$ to get $f(x) = \sum_{k=1}^{\infty} \frac{1}{k}\chi_{A_k}(x)$

We are now ready to finish the proof.
Let $f_m = \sum_{k=1}^{m} \frac{1}{k}\chi_{A_k}(x)$, $A_k \subseteq A$.
Then $\int_A f_m(x)\, d\nu = \int_A f_m(x) D_\mu \nu(x)\, d\nu(x)$.
By MCT,

$$\int_A f(x)\, d\mu = \int_A f(x) D_\mu \nu(x)\, d\nu$$

∎

Theorem 0.23. (Lebesgue Decomposition Theorem)
Let μ and ν be Radon measures on $\mathbb{R}^n$. Then there exists Radon measures ν_{ac}, ν_s on $\mathbb{R}^n$ such that $\nu = \nu_{ac} + \nu_s$, and

1. $\nu_{ac} << \mu, \nu_s \perp \mu$

2. $D_\mu \nu_s(x) = 0$ *for μ-almost every $x \in \mathbb{R}^n$. Further, for all Borel $B \subseteq \mathbb{R}^n$, we have*

$$\nu(B) = \int_B D_\mu \nu_{ac}(x)\, d\mu + \nu_s(B)$$

Proof. Consider the collection $\mathscr{F} = \{A \subseteq \mathbb{R}^n \setminus A) = 0, A \text{ Borel }\}$. Let $m = \inf\{nu(A) \mid A \in \mathscr{F}\}$.

Lecture 11, 2/23/23

Theorem 0.24. (Lebesgue Decomposition Theorem).
Let μ and ν be Radon measures on $\mathbb{R}^n$. Then there exists ν_{ac}, ν_s Radon measures on $\mathbb{R}^n$ such that $\nu = \nu_{ac} + \nu_s$,

1. $\nu_{ac} << \nu_s, \nu_s \perp \mu$

2. $D_\mu \nu_s(x) = 0$ *for μ-almost every $x \in \mathbb{R}^n$, and for all Borel $B \subseteq \mathbb{R}^n$,*

$$\nu(B) = \int_B D_\mu \nu_{ac}(x)\, d\mu + \nu_s(B)$$

Proof. First, assume μ, ν are finite. Consider the collection $\mathscr{F} = \{A \subseteq \mathbb{R}^n \mid \mu(\mathbb{R}^n \setminus A) = 0, A \text{ Borel}\}$.
Let $m = \inf\{\nu(A) \mid A \in \mathscr{F}\}$.
Note $0 \leq m < \infty$, so we can choose $A_k \in \mathscr{F}$ so that $m \leq \nu(A_k) < m + \frac{1}{k}$.
Define $B = \cap_{k=1}^\infty A_k$. By continuity from above, $\nu(B) = m$. Furthermore, B is Borel.
Note $\mu(\mathbb{R}^n \setminus B) = \mu(\mathbb{R}^N \setminus \cap_{k=1}^\infty A_k) = \mu(\cup_{k=1}^\infty (\mathbb{R}^n \setminus A_k)) \leq \sum_{k=1}^\infty \mu(\mathbb{R}^n \setminus A_k) = 0$.
So $B \in \mathscr{F}$.
Note μ is supported on B. Define $\nu_{ac} = \nu|_B$, $\nu_s = \nu_{\mathbb{R}^n \setminus B}$. Obviously, $\nu_s \perp \nu_{ac}$.
We claim $\nu_{ac} << \mu$. Towards contradiction, suppose there is a $C \subseteq \mathbb{R}^n$ such that $\mu(C) = 0$ but $\nu_{ac} > 0$.
Without loss of generality, assume C is Borel. Further, assume $C \subseteq B$. Then $B \setminus C \in \mathscr{F}$, since $\mu(\mathbb{R}^n \setminus (B \setminus C)) \leq \mu(\mathbb{R}^n \setminus B) + \mu(C) = 0$. Furthermore, $\nu(B \setminus C) = \nu(B) - \nu_{ac}(C) < \nu(B)$, a contradiction.
Now we show $D_\mu \nu_s(x) = 0$ for μ-almost every $x \in \mathbb{R}^n$.
For every $\alpha > 0$, set $C_\alpha = \{x \in \mathbb{R}^n \mid \overline{D_\mu \nu_s}(x) \geq \alpha\}$.
By lemma, $\nu_s(C_\alpha) \geq \alpha \mu(C_\alpha)$. Clearly, $C_\alpha = (C_\alpha \cap B) \cup (C_\alpha \setminus B)$. Note $\mu(C_\alpha \setminus B) = 0$, $\mu(C_\alpha \cap B) = 0$. So $\mu(C_\alpha) = 0$.
Since $\{x \mid D_\mu \nu_s(x) > 0\} = \cup_{k=1}^\infty C_{\frac{1}{k}}$, $\mu(\{x \mid D_\mu \nu_s(x) > 0\}) = 0$.
Let $C_m = \{x \mid |x| < m\}$. There exists $B_m \subseteq C_m$ such that

- $\mu(C_m \setminus B_m) = 0$
- $\nu_{ac}^m = \nu_{C_m} << \mu$
- $\nu_s^m = \nu_{C_m \setminus B_m} \perp \mu$
- $D_\mu \nu_{ac}^m(x) = 0$ for μ-almost every $x \in C_m$

Set $B = \cup_{i=1}^\infty B_i$, $\nu_{ac} = \nu_B, \nu_s = \nu_{\mathbb{R}^n \setminus B}$

1. $\mu(\mathbb{R}^n \setminus B) = \mu(\cup_{k=1}^\infty C_k \setminus \cup_{k=1}^\infty B_k) \leq \mu(\cup_{k=1}^\infty (C_k \setminus B_k)) \leq \sum_{k=1}^\infty \mu(C_k \setminus B_k) = 0$

2. We claim $\nu_s \perp \mu$, $\nu_{ac} << \mu$. If $\mu(A) = 0$, $\mu(A \cap C_m) = 0$ for all $m \in \mathbb{N}$. So $\nu_{ac}(A) = \nu_{ac}(\cup_{k=1}^{\infty} C_k \cap A) \leq \sum_{k=1}^{\infty} \nu_{ac}(C_k \cap A) = \sum_{k=1}^{\infty} \nu(B \cap C_k \cap A)$.
Fix: New $\overline{B_m} = \cup_{k=1}^{m} B_k$. This will give us $= \sum_{k=1}^{\infty} \nu_{ac}^{k}(C_k \cap A) = 0$.

■?

Differentiation of Integrals, Lebesgue's Point

Definition 0.11.

$$L^1(\mathbb{R}^n, \mu) \stackrel{\text{def}}{=} \{f : \mathbb{R}^n \to [-\infty, \infty] \mid f \text{ is } \mu - \text{measurable }, \int_{\mathbb{R}^n} |f| \, d\mu < \infty\}$$

$$L^1_{loc}(\mathbb{R}^n, \mu) \stackrel{\text{def}}{=} \{f : \mathbb{R}^n \to [-\infty, \infty] \mid f \text{ is } \mu\text{−measurable }, \int_K |f| \, d\mu < \infty \text{ for all } K \text{ compact}\}$$

Theorem 0.25. *(Lebesgue-Besicovitch differentiation theorem)*
Let μ be a Radon measure on $\mathbb{R}^n$ and let $f \in L^1_{loc}(\mathbb{R}^n, \mu)$. Then

$$\lim_{r \to 0^+} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(y) \, dy = f(x)$$

for μ-almost every $x \in \mathbb{R}^n$.

Proof. Consider $f^+ = \max(f, 0)$ and $f^- = -\min(0, f)$. Note $f = f^+ - f^-$, f^+ and f^- are μ-measurable, $f^+, f^- \in L^1_{loc}(\mathbb{R}^n, \mu)$.
Define $\nu^+, \nu^- : 2^{\mathbb{R}^n} \to [0, \infty]$ as follows:

$$\nu^{\pm}(B) \stackrel{\text{def}}{=} \int_B f^{\pm}(x) \, d\mu, \text{ where } B \text{ is Borel}\}$$
$$\nu^{\pm}(A) \stackrel{\text{def}}{=} \inf\{\nu^{\pm}(B) \mid A \subseteq B, B \text{ Borel}\}$$

Lemma 3. *ν^{pm} are Radon measures with*

- *$\nu^{\pm} << \mu$*
- *$D_\mu \nu^{\pm}(x) = f^{\pm}(x)$ for μ-almost every $x \in \mathbb{R}^n$.*

Lemma 4. *Let $g \in L^1_{loc}(\mathbb{R}^n, \mu)$, where μ is a Borel measure on $\mathbb{R}^n$. Then*

$$\nu(B) = \int_B g(x) \, d\mu, B \text{ Borel}$$
$$\nu(A) = \inf\{\nu(B) \mid A \subseteq B, B \text{ Borel}\}$$

is a Radon measure, $\nu << \mu, D_\mu \nu(x) = g(x)$ for μ-almost every $x \in \mathbb{R}^n$.

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Lemma 5. *Let μ be a Borel measure on $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to [0, \infty]$ be such that $f \in L^1_{loc}(\mathbb{R}^n, \mu)$. Define $\nu(B) = \int_B f\, d\mu$ for all B Borel, $\nu(A) = \inf\{\nu(B) \mid A \subseteq B, B \text{ Borel}\}$. Then:*

1. *ν is a Radon measure on $\mathbb{R}^n$ (even if μ is just Borel)*

2. *If μ is Borel-regular, then $\nu << \mu$ and $\nu(A) = \int_A f\, d\mu$ for all $A \subseteq \mathbb{R}^n$ μ-measurable.*

3. *If μ is Radon, then $D_\mu \nu(x) = f(x)$ for μ-almost every $x \in \mathbb{R}^n$.*

Proof.

1. ν is a Borel-regular measure on $\mathbb{R}^n$, by homework 2, problem 2.

2. Assume A is μ-measurable.

Then there exists a Borel set B such that $A \subseteq B, \mu(A) = \mu(B)$. So $\mu(B \setminus A) = \mu(B) - \mu(A) = 0$.

$\nu(B) = \int_B f\, d\mu = \int_A f\, d\mu + \int_{B\setminus A} f\, d\mu = \int_A f\, d\mu$

$\nu(A) = \inf\{\nu(\tilde{B}) \mid A \subseteq \tilde{B}, \tilde{B} \text{ Borel}\}$.

So $\int_{\tilde{B}} f\, d\mu = \int_{\tilde{B}\setminus A} f\, d\mu + \int_A f\, d\mu \geq \int_A f\, d\mu = \nu(B)$. So $\nu(B) = \inf\{\text{“”}\} = \nu(A)$.

Assume $\mu(A) = 0$. Then $\nu(A) = \int_A f\, d\mu = 0$, so $\nu << \mu$.

3. We need a lemma

Lemma 6. *Let X be a nonempty set, and let μ be a measure on X. Assume $f \in L^1_{loc}(X, \mu)$. If $\int_A f(x)\, d\mu = 0$ for all A μ-measurable, then $f \equiv 0$ μ-almost everywhere.*

Proof. Assume f is nonnegative μ-almost everywhere. Let $A_n = \{x \in X \mid f(x) \geq \frac{1}{n}\}$. For all n, A_n is μ-measurable. Note $\{x \in X \mid f(x) > 0\} = \cup_{n=1}^\infty A_n$. We have

$$0 = \int_{A_n} f\, d\mu \geq \int_{A_n} \frac{1}{n}\, d\mu = \frac{1}{n}\mu(A_n) \geq 0$$

So $\mu(A_n) = 0$. Thus $\mu(\{x \in X \mid f(x) > 0\}) = 0$.
Now consider any $f \in L^1_{loc}(X, \mu)$. Let $f = f^+ - f^-$. Both $f^+, f^- = 0$ μ-almost everywhere.

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We now proceed with the proof. For all $A \subseteq \mathbb{R}^n$ μ-measurable and K compact, we have $\int_{A\cap K} f\, d\mu = \nu(A \cap K)$, and by the Radon-Nikodym theorem this equals $\int_{A\cap K} D_\mu\nu(x)\, d\mu$. Thus $\int_{A\cap K} D_\mu\nu(x) - f(x)\, d\mu = 0$. So by lemma 2, $D_\mu\nu(x) = f(x)$ μ-almost everywhere. Take $K = B_1, B_2, \ldots$.

Theorem 0.26. *(Lebesgue-Besicovitch Theorem):*
Let μ be a Radon measure on $\mathbb{R}^n$ and let $f \in L^1_{loc}(\mathbb{R}^n, \mu)$. Then $\lim_{r \to 0^+} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f \, d\mu = f(x)$ μ-almost everywhere.

Proof. Write $f = f^+ - f^-$ and note $f^\pm \in L^1_{loc}(\mathbb{R}^n, \mu)$. Define $\nu^\pm(B) = \int_B f^\pm \, d\mu$ for all B Borel, and as in lemma 1, $\nu^\pm(A) = \inf\{\nu^\pm(B) \mid A \subseteq B, B \text{ Borel }\}$.
Then $\nu^\pm << \mu, \nu^\pm$ are Radon. Note that $D_\mu \nu^\pm(x) = f^\pm(x)$ μ-almost everywhere. So

$$\begin{aligned}
\lim_{r \to 0^+} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f \, d\mu &= \lim_{r \to 0^+} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} (f^+ - f^-) \, d\mu \\
&= \lim_{r \to 0^+} \left(\nu^+(B(x,r)) - \nu^-(B(x,r))\right) \\
&= \lim_{r \to 0^+} \frac{\nu^+(B(x,r))}{\mu(B(x,r))} - \lim_{r \to 0^+} \frac{\nu^-(B(x,r))}{\mu(B(x,r))} \\
&= D_\mu \nu^+(x) - D_\mu \nu^-(x) \\
&= f^+(x) - f^-(x) \\
&= f(x)\mu - a.e.
\end{aligned}$$

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Theorem 0.27. *Under the conditions of the previous theorem, we have*

$$\lim_{r \to 0^+} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, d\mu = 0$$

μ-almost everywhere.

Proof. We have

$$\begin{aligned}
|\frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(y) \, d\mu - f(x)| &= |\frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(y) - f(x) \, d\mu| \\
&\leq \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, d\mu \\
&\to 0
\end{aligned}$$

And then the previous theorem finishes the proof.

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Theorem 0.28. *Let μ be a Radon measure on $\mathbb{R}^n$ and let $f \in L^1_{loc}(\mathbb{R}^n, \mu)$. Then*

$$\underbrace{\lim_{r \to 0^+} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, d\mu = 0}_{*}$$

for μ-almost every $x \in \mathbb{R}^n$.

Definition 0.12. Points $x \in \mathbb{R}^n$ that satisfy the above equation are called Lebesgue points of f (for the measure μ)

Proof. Let $\mathbb{Q} = \{r_i\}_{i=1}^{\infty} \subseteq \mathbb{R}$ be an enumeration of the rationals. We know $\mathbb{Q}$ is dense in $\mathbb{R}$.
For every $i \in \mathbb{N}$, $\lim_{r \to 0^+} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - f(r_i)| \, d\mu = |f(x) - r_i|$ is satisfied μ-almost everywhere.
Let $A_i \subseteq \mathbb{R}^n$ be points where the equation is satisfied. Then $\mu(\mathbb{R}^n \setminus A_i) = 0$.
We have

$$\phi_i(x) = |f(x) - r_i| \in L^1_{loc}(\mathbb{R}^n, \mu)$$

because

$$|\phi_i(x)| \leq |f(x) + |r_i| \in L^1_{loc}(\mathbb{R}^n, \mu)$$

Set $A = \cap_{i=1}^{\infty} A_i$. We claim that $*$ is satisfied for all $x \in A$. Note

$$\mu(\mathbb{R}^n \setminus A) = \mu(\mathbb{R}^n \setminus \cap_{i=1}^{\infty} A_i) = \mu(\cup_{i=1}^{\infty} (R^n \setminus A_i)) \leq \sum_{i=1}^{\infty} \mu(\mathbb{R}^n \setminus A_i) = 0$$

Fix $\varepsilon > 0$. Choose r_m such that $|f(x) - r_m| < \varepsilon$.
Then

$$|f(y) - f(x)| \leq |f(y) - r_m| + |f(x) - r_m|$$

so

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, d\mu \leq 2|f(x) - r_m| < 2\varepsilon$$

We take the limit as $\varepsilon \to 0$ to get

$$\limsup_{r \to 0^+} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, d\mu = 0$$

thus

$$\lim_{r \to 0^+} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, d\mu = 0$$

∎

Definition 0.13. A measure μ has the doubling property if there exists a constant $C > 0$, such that

$$0 < \mu(B(x, 2r)) \leq C\mu(B(x, r)) < \infty$$

Remark. Note that $\mu = \lambda^n$ has the doubling property: there exists c_1, c_2 such that $\mu(B(x, c_1 r)) \leq c_2\mu(B(x, r))$ for all $r > 0, x \in \mathbb{R}^n$. Simply choose $c_1 = 2, c_2 = 2^n$. Because $c_1 > 1$, there exists a $k \in \mathbb{N}$ such taht $c_1^k \geq 2$, $\mu(B(x, 2r)) \leq \mu(B(x, c_1^k r)) \leq \cdots \leq c_1^k \mu(B(x, r))$

Theorem 0.29. *Let μ be a Radon measure on $\mathbb{R}^n$ with the doubling property. Choose $f \in L^1_{loc}(\mathbb{R}^n, \mu)$. Then $\lim_{r\to 0^+} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|\, d\mu = 0$ for μ-almost every $x \in \mathbb{R}^n$.*

Proof.

$$\begin{aligned}\frac{1}{\mu(B(x,r))} \int_{B(z(r),r)} |f(y) - f(x)|\, d\mu &\leq \frac{1}{\mu(B(x,r))} \int_{B(x,2r)} |f(y) - f(x)|\, d\mu \\ &\leq \frac{c_1^k}{\mu(B(z(r), 2r))} |f(y) - f(x)|\, d\mu\end{aligned}$$

This goes to 0 as $r \to 0$.

Theorem 0.30. *(Density)*
Let μ be a Radon measure on $\mathbb{R}^n$ and let $E \subseteq \mathbb{R}^n$ be μ-measurable. Then

1. $\lim_{r\to 0^+} \frac{\mu(E\cap B(x,r))}{\mu(B(x,r))} = 1$ *for μ-almost every $x \in E$*

2. $\lim_{r\to 0^+} \frac{\mu(E\cap B(x,r))}{\mu(B(x,r))} = 0$ *for μ-almost every $x \in E^c$.*

Proof. Apply previous theorem to $f(x) = \chi_E(x)$.
Choose $R_1 < R_2 < \cdots < R_k < \cdots$ such that $\lim_{k\to\infty} R_k = \infty$, $\mu 9\{x \mid ||x| = R_i\}) = 0$.
Then do the analysis for the sets $a - k = \{x \mid R_k < |x| < R_{k+1}\}$, $A_0 = \{x \mid |x| < R\}$.
Let $\nu_s = \sum \nu_s^i + \nu|_{\cup_{k=1}^\infty \{x||x|=R_k\}}, \nu_{ac} = \sum \nu_{ac}^i$.

∎

Lemma 7. *Let μ be a Radon measure on $\mathbb{R}^n$. Fix $x \in \mathbb{R}^n$. Then $\mu(\partial(B(x, r)) = 0$ for $r \in \mathbb{R}^{\setminus}\{r_1, r_2, \ldots\}$.*

Proof. It is sufficient to prove the statement within any fixed ball $B(x, r), R > 0$.
Then take $R = 1, 2, \ldots, \mathbb{R}^n = \cup_{r=1}^\infty B(x, r)$.
Define $A_k = \{r \in [0, R] \mid \mu(\partial B(x, r)) \geq \frac{1}{k}\}, k = 1, 2, \ldots$. Note $\{r \in [0, R] \mid \mu(\partial B(x, r)) >> 0\} = \cup_{k=1}^\infty A_k$.
$\infty > \mu(B(x, r)) \geq |A_k|\frac{1}{k}, |A_k| \leq k\mu(B(x, r))$