Lecture 1, 1/11/13

Section 1: Vocabulary and easy definitions

Homological algebra is the study of complexes of R-modules, where R is a ring with identity $1 \neq 0$. Notationally, R-Mod is the category of all left R-modules, and R-mod is the category of all finitely generated R-modules.

Definition 0.1. Let A_n" $\in$ "R-mod for $n \in \mathbb{Z}$ and $d_n \in \mathrm{Hom}_R(A_n, A_{n-1})$ such that $d_{n-1} \circ d_n = 0$ for all $n \in \mathbb{Z}$. Then the sequence

$$\cdots \xrightarrow{d_{n+2}} A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} \cdots$$

is called a complex of R-modules, assuming $\mathrm{im}(d_n) \subseteq \ker(d_{n-1})$. The sequence

$$0 \longrightarrow A_m \longrightarrow A_{m-1} \longrightarrow \cdots \longrightarrow A_{n+1} \longrightarrow A_n \longrightarrow 0$$

will occur more frequently. A complex $\mathbb{A}$ is an exact sequence if $\mathrm{im}(d_n) = \ker(d_{n-1})$ for all $n \in \mathbb{Z}$. This is called a short exact sequence if there are no more than 3 non-zero terms. Given a complex $\mathbb{A}$, the nth homology modules (or groups, in some cases) of $\mathbb{A}$ is

$$H_n(\mathbb{A}) = \frac{\ker(d_{n-1})}{\mathrm{im}(d_n)}$$

Remark. Given a short exact sequence (hereby abbv. as SES)

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

f is a mono and g is an epi, so $C \simeq B/\mathrm{im}(f)$. If A, B are known, but not f, then infinitely many C are available to complete the short exact sequence.

Example 0.1. Let $R = k$, a field, and take $A = B = k^{(\mathbb{N})} = \oplus_{i \in \mathbb{N}} k$.

(i) $0 \longrightarrow A \xrightarrow{\mathrm{Id}} B \longrightarrow 0$ is a SES.

(ii) Define $f : A \to B$ by

$$f(b_i) = b_{2i} \text{ for } i \in \mathbb{N}$$
$$g(b_0) = \begin{cases} 0 & i \text{ even} \\ b_{\tau(i)} & i \text{ odd} \end{cases}$$

Where $\tau : (2\mathbb{N}-1) \to \mathbb{N}$ is a bijection. If $A = B = C = \kappa^{(\mathbb{N})}$, then

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a SES.

(iii) Let $R = \mathbb{Z}$. Then

$$0 \longrightarrow 3\mathbb{Z} \xrightarrow{\iota} \mathbb{Z} \xrightarrow{\overline{\cdot}} \mathbb{Z}/3\mathbb{Z} \longrightarrow 0$$

is a SES.

(iv) Let $R = \mathbb{Z}$. The sequence

$$0 \longrightarrow \overbrace{6\mathbb{Z}}^{A_1} \xrightarrow{\iota} \overbrace{\mathbb{Z}}^{A_0} \xrightarrow{\overline{\cdot}} \overbrace{\mathbb{Z}/3\mathbb{Z}}^{A_{-1}} \longrightarrow 0$$

is a complex which is not exact. In fact, $H_0(\mathbb{A}) = \overbrace{3\mathbb{Z}}^{\ker(g)} / \underbrace{6\mathbb{Z}}_{\mathrm{im}(f)} \cong \mathbb{Z}/2\mathbb{Z}$.

(v) Let $R = \kappa[x,y]$, κ a field. Let f be the inclusion $(x) \hookrightarrow R[x,y]$. The sequence

$$0 \longrightarrow (x) \xrightarrow{f} R \xrightarrow{g} \kappa[y] \longrightarrow 0$$

where

$$g\left(\sum_{i,j=0}^{\text{finite}} a_{ij}x^iy^j\right) = \sum_{j>0}^{\text{finite}} a_{\sigma_j}y^j$$

is exact.

(vi) Let $R = \kappa[x,y]$. Define $\mathbb{A}$ as

$$0 \longrightarrow \overbrace{(x)}^{A_1} \xrightarrow{f} \overbrace{R}^{A_0} \xrightarrow{g} \overbrace{\underbrace{\kappa}_{=R/(x,y)}}^{A_{-1}} \longrightarrow 0$$

where

$$g\left(\sum_{i,j=0}^{\text{finite}} a_{ij}x^iy^j\right) = a_\infty$$

then $\ker(g) = (x, y)$ and $\operatorname{im}(f) = (x)$, so $\mathbb{A}$ is not exact. In fact,

$$\begin{aligned} H_0(\mathbb{A}) &= (x, y)/(x) \\ &\simeq (y) \\ &\simeq R \end{aligned}$$

Note: If R is an integral domain and $x \in R \setminus \{0\}$, then $(x) \simeq R$ (as R-modules, <u>not</u> as rings!), with isomorphism $r \mapsto rx$.
Typical questions addressed by homological algebra:

(i) Suppose

$$\mathbb{A}: \quad \cdots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \longrightarrow \cdots$$

is an exact sequence in R-mod and $F : R - \text{mod} \to S - \text{mod}$ is a functor. Is the sequence

$$\cdots \longrightarrow F(A_{n+1}) \xrightarrow{F(d_{n+1})} F(A_n) \xrightarrow{F(d_n)} F(A_{n-1}) \longrightarrow \cdots$$

exact? $F(\mathbb{A})$ is a complex when F is additive, but it may or may not be exact.

(ii) Given A, C "$\in$" $R - \text{mod}$, characterize all modules B such that there exists an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

As an example, $R = \mathbb{Z}$, $A = C = \mathbb{A}/p\mathbb{Z}$, p prime, then

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \xrightarrow{f} \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \xrightarrow{g} \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

with $f : x \mapsto (x, 0)$ and $g : (x, y) \mapsto y$ is a SES. Alternatively, we could take $f : x + p\mathbb{Z} \mapsto px + p^2\mathbb{Z}$ and $g : y + p^2\mathbb{Z} \mapsto y + p\mathbb{Z}$ to make

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \xrightarrow{f} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{g} \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

a SES. These are the only possibilities in this case! In general, though, there are infinitely many possibilities for B. Why is this interesting? If R is an artinian ring and M "$\in$" $R - \text{mod}$, then there are only finitely many simple $s_1, \ldots, s_n$ "$\in$" $R - \text{mod}$ up to isomorphism. Moreover, for M "$\in$" $R - \text{mod}$, there is a chain

$$M = M_0 \supset M_1 \supset \cdots \supset M_\ell = 0$$

such that M_i/M_{i+1} is simple for all $i < \ell$. If the answer to question (ii) is known, then all objects in $R - \text{mod}$ of fixed length ℓ are known up to isomorphism! Simply proved by induction.

Algebraic Topology

Definition 0.2. The standard n-simplex $\triangle_n$ in $\mathbb{R}^n$ is the convex hull of $v_0, v_1, \ldots, v_n$, where $v_0 = 0$ and $v_i = (0, \ldots, 0, \overbrace{1}^{i}, 0, \ldots, 0)$ (so the standard basis).
An oriented simplex is $(\triangle_x, [\pi])$, where $[\pi]$ is an equivalence class of permutations of $\{0, \ldots, n\}$, where $\pi \sim \pi' \iff \operatorname{sgn}(\pi) = \operatorname{sgn}(\pi')$. We write

$$(\triangle_x, \pi) = [v_{\pi(0)}, v_{\pi(1)}, \ldots, v_{\pi(n)}$$

and identify $\triangle_n$ with $[0, 1, \ldots, n]$. The ngative is $-[w_0, \ldots, w_n]$.

Definition 0.3. Let X be a topological space. An n-simplex in X is a continuous map

$$\sigma : \triangle_n \to X$$

The group of n-chains of X, $S_n(x)$, is the free abelian group having as basis the n-simplices in X. The singular chain complex of X is

$$\cdots \longrightarrow S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots \longrightarrow S_0(X) \longrightarrow 0$$

denoted $\mathbb{S}$, where $\partial_n : S_n(X) \to S_{n-1}(X)$ is the nth boundary map, which can be defined if we define $\partial_n(\sigma)$ for all n-simplices σ in X (i.e. in the basis of $S_n(X)$). Consider the map

$$\begin{aligned} \tau_i : \mathbb{R}^{n-1} &\to \mathbb{R}^n \\ (a_1, \ldots, a_{n-1}) &\mapsto (a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_n) \end{aligned}$$

For $i \in \{0, \ldots, n\}$. Then τ_i is continuous and $\tau_i(\triangle_{n-1}) = \triangle_n$. Define

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma(\tau_i)$$

Theorem 0.1. *$\partial_{n-1} \circ \partial_n = 0$ for all $n \in \mathbb{N}$, i.e. $\mathbb{S}$ is a complex in $\mathbb{Z}$-mod.*

Definition 0.4. The group of n-cycles is $Z_n(X) = \ker(\partial_{n-1})$, and the group of n-boundaries is $B_n = \operatorname{im}(\partial_n)$.
The nth homology group is $H_n(X) = Z_n(X)/B_n(X)$.

Lecture 2, 1/13/23

Chapter I: Categories and functors

There is a definition page on the Gaucho that has all the most basic definitions - objects, morphisms, compositions, etc.

If $f \in \mathrm{Hom}_C(A, B)$, we often write $A \xrightarrow{f} B$ even if f is not literally a map.

Example 0.2. **1.** The category of all sets, Set. The object class consists of all sets, and the morphisms are just set maps.

2. The category of all topological spaces, Top. The object class consists of all topological spaces, and the morphisms are continuous functions.

3. The category of all groups, Grp. The object class consists of all groups, and the morphisms are group homomorphisms.

4. Let $(P, \leq)$ be a partially ordered set with a relation $\leq$ which is reflexive, antisymmetric, and transitive. Then we can make P into a category, whose objects are the elements of p, and for $u, s \in P$, $\mathrm{Hom}_P(u, s) = \begin{cases} (u,s) & u \leq s \\ \varnothing & u \not\leq s \end{cases}$. We define the composition $(s, t)(u, s) \stackrel{\text{def}}{=} (u, t)$.

5. The opposite category of a category C, C^{op}.

6. Let R be a ring. R-Mod is the category of left R modules. R-mod is the finitely generated R-modules, and similarly for Mod-R and mod-R, which are the right R-modules.

7. R-comp. The object class consists of complexes of left R-modules.

Let $\mathbb{A}, \mathbb{A}'$ be objects of R-comp. Note: it is problematic to say “$\mathbb{A}, \mathbb{A}' \in R$-comp, as R-comp is not a set!

Say $\mathbb{A} = \cdots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \longrightarrow \cdots$, and similarly for $\mathbb{A}'$. An element of $\mathrm{Hom}_{R-comp}(\mathbb{A}, \mathbb{A}')$ will be a sequence of R-module homomorphisms $f_n : A_n \to A'_n$ which make the following diagram commute:

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & A_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \arrow[d] \\
\cdots \arrow[r] & A'_n \arrow[r, "d'_n"'] & A'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

8. The category of rings Ring, whose obejcts are rings and whose morphisms are ring homomorphisms.

9. The category of $\mathbb{Z}$-modules is usually denoted Ab. This is also the category of Abelian groups, and is the prototypical example of an Abelian category.

Definition 0.5. A category $\mathcal{C}$ is called pre-additive if for all A, B objects of $\mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is an additive Abelian group (additive means we use the symbol "+") such that for all eligible morphisms f, g, h, k,

$$h(f+g) = hf + hg$$
$$(f+g)k = fk + gk$$

where "elibigle" means that these expressions make sense and are well-defined.

Example 0.3. **1.** R-mod (in particular Ab)

2. R-comp

3. Ring fails to be pre-additive, because the identity morphisms add to be something which is not the identity morphism.

Definition 0.6. Let $\mathcal{C}, \mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ consists of an assignment $F_0 : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$, and for each pair of objects A, B "$\in$" $\mathrm{Obj}(\mathcal{C})$, a map (this actually is a map because we assume hom-sets are in fact sets). $F_{A,B} : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ such that, for all eligible morphisms f, g, and all A "$\in$" $\mathcal{C}$

(a) $F(\mathrm{Id}_A) = \mathrm{Id}_{F(A)}$

(b) $F(f \circ g) = F(f) \circ F(g)$

Example 0.4. **1.** Let $\mathcal{C}$ be a category. Then we have the identity functor $\mathrm{Id}_{\mathcal{C}}$, which assigns $\mathrm{Id}_{\mathcal{C}}(A) = A$, and $\mathrm{Id}_{\mathcal{A}}(f) = f$ for any eligible A "$\in$" $\mathrm{Obj}(\mathcal{D})$ and morphisms f.

2. Functors $\pi_n : \mathsf{Top} \to \mathsf{Grp}$ which sends $X \mapsto \pi_n(X)$

3. $\mathbb{S} : \mathsf{Top} \to \mathbb{Z}$-comp, which sends $X \mapsto \mathbb{S}(X)$, which is a complex

$$\cdots \longrightarrow S_{n+1}(X) \xrightarrow{\partial_{n+1}} S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots \longrightarrow S_0(x) \xrightarrow{\partial_0} 0$$

Let $\phi : X \to Y$ be continuous for X, Y "$\in$" Top. Then $\mathbb{S}(\phi)_n : S_n(X) \to S_n(Y)$ is given by $\sigma \mapsto \phi \circ \sigma$, and we can extend this for σ an n-simplex of X.

Lecture 4, 1/18/23

Functors:

Definition 0.7. Let $\mathcal{C}, \mathcal{D}$ be categories. A covariant functor from $\mathcal{C}$ to $\mathcal{D}$ consists of "maps" F_0 and $F|_{A,B}$ for any $A, B \in \mathrm{Obj}(\mathcal{C})$ such that

- $F_0 : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$
- $F_{A,B} : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_0 A, F_0 B)$ for any A, B" $\in$ "$\mathrm{Obj}(\mathcal{C})$

such that

(a) $F_{A,C}(fg) = F_{B,C}(f)F_{A,B}(g)$ for all eligible f, g

(b) $F_{A,A}(\mathrm{Id}_A) = \mathrm{Id}_{F(A)}$

from here on we don't care at all about indices. For simplicity, we will denote the action of a functor F as simply FA or Ff.

Definition 0.8. A contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ amounts to a covariant functor from $\mathcal{C}$ to $\mathcal{D}^{\mathrm{op}}$.
More examples of functors

Example 0.5. Homology functors $H_n : R - comp \to \mathbb{Z} - mod$ which sends $\mathbb{A}$ to $H_n(A)$.
That is, $\mathbb{A} \to F\mathbb{A} = \frac{\ker(d_n)}{\mathrm{Im}(d_{n+1})}$
Let $f \in \mathrm{Hom}_{R-comp}(\mathbb{A}, \mathbb{A}')$. That is, the following diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & A_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \arrow[d] \\
\cdots \arrow[r] & A'_n \arrow[r, "d'_n"'] & A'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Ff acts by $a_n + \mathrm{Im}(d_{n+1}) \to f_n(a_n) + \mathrm{Im}(d'_{n+1})$. Let's prove that this is actually well-defined.
Check
First, $a_n \in \ker(d_n)$ implies $f_n(a_n) \in \ker(d'_n)$. This can be seen by doing a diagram chase on the above diagram. Since $d_n(a_n) = 0$, we have $0 = f_{n-1}d_n(a_n) = d'_n f_n(a_n)$, i.e. $f_n(a_n) \in \ker(d'_n)$.
"Don't do much thinking. It's almost harmful" - Birge on doing diagram chasing. Also "follow your nose."
Now, $a_n \in \mathrm{Im}(d_{n+1})$ implies $f_n(a_n) \in \mathrm{Im}(d'_{n+1})$. So $a_n = d_{n+1}(x)$ with $x \in A_{n+1}$. hence $f_n(a_n) = f_n d_{n+1}(x) = a'_{n+1} f_{n+1}(x) \in \mathrm{Im}(d'_{n+1})$.

Example 0.6. Let $\mathcal{C}, \mathcal{D}$ be pre-additive categories (definition on the top of page 6). A functor F "from" $\mathcal{C}$ to $\mathcal{D}$ is called <u>additive</u> if, for all A, B" $\in$ "$\mathrm{Obj}(\mathcal{C})$, the map $F : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ is a homomorphism of abelian groups.

Remark. Note that $H_n : R - comp \to \mathbb{Z} - mod$ is an additive functor. The π_n functor is <u>not</u> additive, as Top is not preadditive.

Example 0.7. <u>Forgetful functors</u> e.g. $F : R - mod \to \mathbb{Z} - mod$ which sends $M \mapsto M$, where the M on the left hand side is an R-module, and M on the right is just an abelian group, which is a $\mathbb{Z}$-module. Or $F : R - mod \to \mathsf{Set}$ which sends an R-module M to the set of its elements, "forgetting" the module structure.
Moreover, if $\mathcal{C}, \mathcal{D}$ are pre-additive, and $F : \mathcal{C} \to \mathcal{D}$ is a forgetful functor of some sort, then F is additive.

Example 0.8. Let $F : R - mod \to S - mod$ be an additive functor. Then F induces an additive functor $\tilde{F} : R - comp \to S - comp$, sending $\mathbb{A}$ to $F(\mathbb{A})$.
If $\mathbb{A}$ is a complex

$$\cdots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \longrightarrow \cdots$$

then $F(\mathbb{A})$ is

$$\cdots \longrightarrow F(A_n) \xrightarrow{F(d_{n+1})} F(A_n) \xrightarrow{F(d_n)} F(A_{n-1}) \longrightarrow \cdots$$

An extremely important question: if $\mathbb{A}$ is exact, is $F(\mathbb{A})$ exact? If not, how far does it deviate from being an exact sequence?

Example 0.9. Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{E}$ be functors. Then $G \circ F : \mathcal{C} \to \mathcal{E}$ is a functor. <u>WARNING:</u> we use $\circ$ but this isn't actually a function composition. This is just notation!!!
$G \circ F$ acts how one might think: for A" $\in$ "$\mathrm{Obj}(\mathcal{C})$, $G \circ F(A) = G(F(A))$, and for $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $G \circ F(f) = G(F(f)) \in \mathrm{Hom}_{\mathcal{E}}(G(F(A)), G(F(B)))$.
Of interest to us: $H_n \circ \tilde{F}$, where $F : R - mod \to S - mod$ is additive. This functor sends a complex $\mathbb{A}$ to $H_n(F(\mathbb{A}))$. This is especially of interest if $\mathbb{A}$ is exact, but $F(\mathbb{A})$ is not.

Remark. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F sends isomorphisms in $\mathcal{C}$ to isomorphisms in $\mathcal{D}$. This is immediate from the definition of a functor.

Section 2: two types of functors that will follow us

(i) Hom-functors: Whenever $\mathcal{C}$ is a category, there is a bifunctor

$$\mathrm{Hom}_{\mathcal{C}}(-, -) : \mathcal{C} \times \mathcal{C} \to \mathsf{Set}$$

which sends a pair (A, B) to $\text{Hom}_{\mathcal{C}}(A, B)$, and on maps (note that this is covariant in the first factor and contravariant inh the second), they act as follows. Let $f : A \to A', g : B \to B'$ be morphisms in $\mathcal{C}$. Then

$$\text{Hom}(f, g) : \text{Hom}_{\mathcal{C}}(A', B) \to \text{Hom}_{\mathcal{C}}(A, B')$$

acts by $\phi \mapsto g \circ \phi \circ f$

Lecture 5, 1/20/23

Whenever $\mathcal{C}$ is a category, there is a bifunctor $\text{Hom}_{\mathcal{C}}(-, -) : \mathcal{C} \times \mathcal{C} \to \mathsf{Set}$, which sends (A, B) to $\text{Hom}_{\mathcal{C}}(A, B)$. On maps, when $f : A \to A'$ and $g : B \to B'$ are morphisms, then

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(f, g) : \text{Hom}_{\mathcal{C}}(A', B) &\to \text{Hom}_{\mathcal{C}}(A, B') \\ \varphi &\mapsto g \circ \varphi \circ f \end{aligned}$$

We will split this into two parts. Let $C“ \in ”\mathcal{C}$. Then we have a covariant functor

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(C, -) : \mathcal{C} &\to \mathcal{C} \\ C' &\mapsto \text{Hom}_{\mathcal{C}}(C, C') \\ g &\mapsto \text{Hom}_{\mathcal{C}}(C, A) \to \text{Hom}_{\mathcal{C}}(C, B) \\ \varphi &\mapsto g \circ \varphi \end{aligned}$$

We also have the contravariant functor $\text{Hom}_{\mathcal{C}}(-, D)$, which acts similarly.
As a special case, consider $\mathcal{C} = R - mod$. Then

$$\begin{aligned} \text{Hom}_R(M, -) : R - mod \to \mathbb{Z} - mod \\ \text{Hom}_R(-, N) : R - mod \to \mathbb{Z} - mod \end{aligned}$$

but we can have additional structure on $\text{Hom}_R(M, N)$. Suppose ${}_RM_S$ is a bimodule (S is a ring and $(rm)s = r(ms)$) and let ${}_RN_T$ be an $R-T$ module. Then $\text{Hom}_R(M, N)$ is a left S, right T bimodule. For $f \in \text{Hom}_R(M, n)$, $s \in S, t \in T$, define

$$\begin{aligned} (sf)(m) &= f(ms) \\ (ft)(m) &= f(m)t \end{aligned}$$

If R is commutative, then

$$\begin{aligned}\mathrm{Hom}_R(M,-) &: R-mod \to R-mod = Mod-R\\ \mathrm{Hom}_R(-,N) &: R-mod \to R-mod = Mod-R\end{aligned}$$

If ${}_RM_S$ is a bimodule, then

$$\mathrm{Hom}_R(M,-) : R-mod \to S-mod$$

If ${}_RN_T$ is a bimodule, then

$$\mathrm{Hom}(-,N) : R-mod \to Mod-T$$

Basic properties:

(i) $M \text{``}\in\text{''} R-mod \implies \underbrace{\mathrm{Hom}_R(R,M) \cong M}_{f\mapsto f(1)}$ in $\mathbb{Z}-mod$

(ii) $\mathrm{Hom}_R(\otimes_{i\in I} M_i, N) \cong \prod_{i\in I} \mathrm{Hom}_R(M_i,N)$. Prove this!

(iii) $\mathrm{Hom}_R(M, \prod_{i\in I} N_i) \cong \prod_{i\in I} \mathrm{Hom}_R(M,N_i)$. Prove this!

Definition 0.9. Let $M \text{``}\in\text{''} Mod-R$, $N \text{``}\in\text{''} R-mod$. Then an abelian group T is called a <u>tensor product</u> of M and N if there exists a map

$$\tau : M \times N \to T$$

Which is $\mathbb{Z}$-bilinear and <u>R-balanced</u>, i.e.

$$\tau(mr,n) = \tau(m,rn)$$

with the following universal property.
Whenever A is an abelian group and $\sigma : M \times N \to A$ is $\mathbb{Z}$-bilinear and R-balanced, there exists a unique $\mathbb{Z}$-linear map $\sigma' : T \to A$ such that this diagram commutes:

$$\begin{tikzcd} M \times N \arrow[r, "\tau"] \arrow[dr, "\sigma"'] & T \arrow[d, "\sigma'"] \\ & A \end{tikzcd}$$

We denote $T = M \otimes_R N$.

Theorem 0.2. *If $M \text{``}\in\text{''} Mod-R$ and $N \text{``}\in\text{''} R-mod$, then a tensor product $M \otimes_R N$ exists and is unique up to isomorphism.*

Proof. Let F be the free abelian group with basis $M \times N$, i.e.

$$F = \otimes_{m \in M, n \in N} \mathbb{Z}(m, n)$$

Define

$$M \otimes_R N = F/U$$

where U is the submodule generated by all elements of the form

$$\begin{aligned}(m_1 + m_2, n) - (m, n) - (m_2, n)\\ (m, n_1 + n_2) - (m, n_1) - (m, n_2)\\ (mr, n) - (m, rn)\end{aligned}$$

for all eligible $m_i, m \in M, n_i, n \in N, r \in R$.
Define

$$\begin{aligned}\tau :&M \times N \to M \otimes_R N\\ &(m, n) \mapsto m \otimes n\end{aligned}$$

Then τ is $\mathbb{Z}$-bilinear and R-balanced (check!). Moreover, $M \otimes_R N$ with τ satisfies the universal proeprty: let A be an abelian group and $\sigma : M \times N \to A$ be $\mathbb{Z}$-bilinear and R-balanced. Define

$$\begin{aligned}\tilde{\sigma} :&F \to A\\ &(m, n) \mapsto \sigma(m, n)\end{aligned}$$

and extend linearly. By construction, $\tilde{\sigma}(U) = 0$, i.e. $U \subseteq \ker(\tilde{\sigma})$. Hence there exists $\sigma' : F/U \to A$ with the property that

$$\sigma'(m, n) = \tilde{\sigma}((m, n) + n) = \tilde{\sigma}(m \otimes n)$$

Now show σ' is unique, and the proof is complete. ∎

Lecture 6, 1/23/23

Our two mainstay types of functors:

(i) Hom functors.

(ii) Tensor functors. For (M,N)"$\in$"$Mod-R\times R-mod$, we constructed an abelian group $M\otimes_R N = R^{(M\times N)}/u$, together with $\tau : M\times N \to M\otimes_R N$ given by $\tau(m,n) = m\otimes n = (m,n)+u$ such that $(M\otimes_R N, \tau)$ has the key universal property.

Note: The elements $m\otimes n \in M\times N$ form a generating set of $M\otimes_R N$, but not a basis.

The tensor functor

We have a bifunctor $-\otimes- : Mod-R\times R-mod \to \mathbb{Z}-mod$, $(M,N)\mapsto M\otimes_R N$. Let $(f,g), f\in \mathrm{Hom}_R(M,M'), g\in \mathrm{Hom}(N,N')$. Then

$$\begin{aligned} f\otimes g : &M\otimes_R N \to M'\otimes_R N' \\ &m\otimes n \mapsto f(m)\otimes g(n) \end{aligned}$$

To show this is well-defined, check that $\phi : M\times N \to M'\otimes N'$, $(m,n)\mapsto f(m)\otimes g(n)$ is $\mathbb{Z}$-bilinear and R-balanced.
Split $-\otimes_R-$ into two functors. So, we have a functor $M\otimes_R - : R-mod \to \mathbb{Z}-mod$ and a functor $-\otimes_R N : Mod-R \to \mathbb{Z}-mod$. The action on objects and morphisms is clear from the discussion up to now.

Additional structure on $M\otimes_R N$

Suppose ${}_S M_R$ and ${}_R N_T$ are bimodules. Then $M\otimes_R N$ is a $S-T$ bimodule, with

$$s(m\otimes n)t = (sm)\otimes(nt)$$

It is an exercise to check well-definedness.
Uses
Suppose ${}_{\mathbb{R}}V$ is a real vector space. We want to "complexify" V, making it a complex vector space. We could consider $\mathbb{C}\times V$, and define $c(d,v) = (cd,v)$. But this does not define a $\mathbb{C}$-vector space, because multiplication must be multilinear. But $\mathbb{C}\otimes_{\mathbb{R}} V$ will do it.
Basic properties
Consider $R\otimes_R M$. This is in fact isomorphic to M. Not just as Abelian groups, but as left R-modules. This is because R satisfies the associative law relative to multiplication. One isomorphism between them is $m\mapsto 1\otimes m$.
In general, unlike the hom-functor, the tensor functor will not commute with direct products/coproducts, unless "the sky is very benevolent."

The meaning of $m\otimes n$ depends on the meaning of M,N!

Example 0.10. Consider $2 \otimes \overline{1} \in \mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})$. This is the same as $1 \otimes \overline{2} = 1 \otimes 0 = 0$. By contrast, look at $2 \otimes \overline{1} \in 2\mathbb{Z} \otimes (\mathbb{Z}/2\mathbb{Z})$. This is nonzero! Let's show that. We know $2\mathbb{Z} \cong \mathbb{Z}$, with an isomorphism given by $x \mapsto \frac{x}{2}$. So

$$\begin{aligned} f \otimes \mathrm{Id}_{\mathbb{Z}/2\mathbb{Z}} :& 2\mathbb{Z} \otimes (\mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z} \otimes (\mathbb{Z}/2\mathbb{Z}) \\ & x \otimes y \mapsto f(x) \otimes y \end{aligned}$$

But functors take isomorphisms to isomorphisms, so $\underbrace{2 \otimes \overline{1}}_{\in 2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z}} \mapsto \overbrace{1 \otimes \overline{1}}^{\in \mathbb{Z}/2\mathbb{Z}} \neq 0$. Why is this last term nonzero? Because $\mathbb{Z} \otimes (\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}$, with the isomorphism sending $1 \otimes \overline{1}$ to $\overline{1}$, which is not zero.

Natural Transformations, Equivalences, and Dualities

Definition 0.10. **1.** Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A morphism of functors, or a natural transformation from F to G, is a family $(\phi(C))_{C \text{``}\in\text{''} \mathrm{Obj}(\mathcal{C})}$ of morphisms, $\phi(C) : F(C) \to G(C)$ such that for any $f \in \mathrm{Hom}_{\mathcal{C}}(C, C')$, the square

$$\begin{tikzcd} F(C) \arrow[r, "F(f)"] \arrow[d, "\phi(C)"'] & F(C') \arrow[d, "\phi(C)"] \\ G(C) \arrow[r, "G(f)"'] & G(C') \end{tikzcd}$$

commutes for all eligible morphisms f in the category $\mathcal{C}$. This is a covariant equivalence. A contravariant equivalence is an equivalence between contravariant functors, i.e. it makes the following square commute.

$$\begin{tikzcd} F(C) \arrow[d, "\phi(C)"'] & F(C') \arrow[l, "F(f)"'] \arrow[d, "\phi(C)"] \\ G(C) & G(C') \arrow[l, "G(f)"] \end{tikzcd}$$

2. Call $(\phi(C))_{C \text{``}\in\text{''} \mathrm{Obj}(\mathcal{C})}$ an isomorphism of functors, or a natural equivalence, if $\phi(C)$ is an isomorphism for each C "$\in$" $\mathrm{Obj}(\mathcal{C})$.

3. Two categories $\mathcal{C}, \mathcal{D}$ are equivalent categories if there are functors $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F \simeq \mathrm{Id}_{\mathcal{C}}$ and $F \circ G \simeq \mathrm{Id}_{\mathcal{D}}$, with " $\simeq$ " meaning "is naturally equivalent to." The F, G are called "mutually inverse equivalences."

4. A contravariant equivalence is called a duality.

5. Let R, S be rings. Call R, S Morita equivalent, denoted $R \sim S$, if $R{-}mod, S{-}mod$ are naturally equivalent. This is equivalent to saying $mod - R, mod - S$ are equivalent.

Lecture 7, 1/25/23

Definition 0.11. Let $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$ be functors. We say that (F, G) form an adjoint pair if the following two bifunctors $\mathcal{C} \times \mathcal{D} \to \mathsf{Set}$ are naturally isomorphic:

$$\mathrm{Hom}_{\mathcal{D}}(F(-), -) \cong \mathrm{Hom}_{\mathcal{C}}(-, G(-))$$

That is, for every (C, D)" $\in$ "$\mathcal{C} \times \mathcal{D}$, we have an isomorphism

$$\phi(C, D) : \mathrm{Hom}_{\mathcal{D}}(F(C), D) \to \mathrm{Hom}_{\mathcal{C}}(C, G(D))$$

and the collection of all $\phi(C, D)$ form a natural isomorphism.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(F(C), D) \arrow[r] \arrow[d, "\phi_{C,D}"] & \mathrm{Hom}_{\mathcal{D}}(F(C'), D') \arrow[d, "\phi_{C',D'}"] \\
\mathrm{Hom}_{\mathcal{C}}(C, G(D)) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(C', G(D'))
\end{tikzcd}$$

Example 0.11.

1. (a) $R \otimes_R - \cong \mathrm{Id}_{R-mod}$. $R \otimes_R - : R - mod \to R - mod$ is well-defined since ${}_R R_R$ is a bimodule.

(b) $\mathrm{Hom}_R({}_R R_R, -) \cong \mathrm{Id}_{mod-R}$

2. For any ring R, $R \sim M_n(R)$, where $\sim$ indicates Morita equivalence, defined above. Why? Let ${}_R F = R^n$, $S = \mathrm{End}_R(F) \cong M_n(R)$. Let $F^* =_R (\mathrm{Hom}({}_S F, R))_S$

Claim. The functors $\mathrm{Hom}(F, -) : mod - R \to mod - S$, $\mathrm{Hom}(F^*, -) : mod - S \to mod - R$ *are mutually inverse functors.*

Proof. We want to show that, for M"$\in$"$Mod - R$,

$$\mathrm{Id}_{Mod-R} \cong \mathrm{Hom}_S(F^*, \mathrm{Hom}_R(F, -))$$

Consider

$$\Phi(M) : m \mapsto (F^* = \mathrm{Hom}(F, R) \ni f \mapsto (x \mapsto mf(x)))$$

Check that this is a R-module hiomomorphisms, and in fact an isomorphism of R-modules.

Let $R = k$ be a field. Then we have a duality

$$k - mod \to k - mod$$

Let v"$\in$"$k - mod$, and consider

$$\Phi(V) : V \to V^{**} = \mathrm{Hom}_k(\mathrm{Hom}_k(V, k), k)$$

, and

$$x \mapsto (\mathrm{Hom}_k(V, K) \ni f \mapsto f(x) \in k)$$

A duality from $k - mod$ to $k - mod$.

We may extend Φ to a functor $k - Mod \to k - Mod$, but this is not surjective if $\dim V = \infty$ (homework problem). So we have a natural equivalence $\mathrm{Id}_{k-mod} \cong (-)^{**}$

Here is an examples of an adjoint pair. Let ${}_SB_R$ be an $S - R$ bimodule. Then the functor

$$B \otimes_R - : R \to R - mod$$

is a left adjoint to

$$\mathrm{Hom}_S(R, -) : S - mod \to R - mod$$

Lecture 8, 1/27/23

Section 4: Additive and Abelian categories

Definition 0.12. A pre-additive category $\mathcal{C}$ is called an additive category if it has a zero object, and finite direct sums/products.

Definition 0.13. An additive category $\mathcal{C}$ is called an Abelian category if every map f has a kernel and cokernal, and every mono is a kernel, and every epi is a cokernel.

Example 0.12. In the category of rings (we assume these are unital rings, so this category is <u>not</u> preadditive, recall) there are categorical epis that fail to be surjective. For example, $f : \mathbb{Z} \hookrightarrow \mathbb{Q}$ is a categorical epimorphism.
Let $g, h \in \mathrm{Hom}_{\mathsf{Ring}}(\mathbb{Q}, R)$ be such that $gf = hf$. Then $g|_{\mathbb{Z}} = h|_{\mathbb{Z}}$, so it follows that $g = h$.
In $R - mod, R - comp$, categorical monos coincide with injective homomorphisms, and similarly, categorical epimorphisms coincide with surjections.

Example 0.13. of Abelian categories:
$R - Mod$, in particular $\mathbb{Z} - Mod = \mathsf{Ab}$, $R - comp$. Let $\mathscr{T} - \mathsf{Ab}$ be the full subcategory of Ab consisting of the torsion Abelian groups.
Is the full subcategory of Ab consisting of torsion-free groups Abelian? No! The map $f : \mathbb{Z} \to \mathbb{Z}$ given by multiplication by 2 doesn't have a cokernel.
$R - mod$ is not Abelian if R is not left Noetherian!!!! (A ring is left Noetherian if every left ideal is finitely generated). But $R - Mod$
For example, let k be a field, and consider $R = k^{\mathbb{N}}$. Let $I = k^{(\mathbb{N})}$ (which means the direct sum, as opposed to the direct product). This is not a finitely generated left ideal, and $I \hookrightarrow R$. But $\pi : R \to R/I$ does <u>not</u> have a kernel in R-mod even though $R, R/I$ are in $R - mod$, because we will get something not in $R - mod$, but in $R - Mod$.
She started talking about some stuff we won't see until later, and said it was "music of the future."

Chapter 2: On the road to derived functors

Section 1: Exactness properties of functors

<u>Note:</u> We'll develop the theory for the Abelian category $R - mod$, but it easily adapts to arbitrary categories.

Definition 0.14. Let R, S be rings, F an additive functor from $R - mod$ to $S - mod$.

1. F is called <u>exact</u> if for all exact sequences

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

the sequence

$$0 \longrightarrow FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \longrightarrow 0$$

is also exact. If F is contravariant, then instead we want the sequence

$$0 \longrightarrow FC \xrightarrow{Fg} FB \xrightarrow{Ff} FA \longrightarrow 0$$

to be exact.

2. F is called <u>left-exact</u> or <u>right-exact</u> if it sends left (or right) exact sequences to left (or right) exact sequences

Lecture 9, 2/1/23

Recall:
A functor $F : R - mod \to S - mod$ is left-exact if for all exact sequences

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

the sequence

$$0 \longrightarrow FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC$$

is exact. Similarly, it is right exact if the image of

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact.

Remark. **1.** A functor $F : R - Mod \to S - Mod$ induces a functor $F : R - comp \to S - comp$.

2. If $F \cong G : R - mod \to S - mod$ (meaning F, G are naturally isomorphic), then F, G have the same exactness properties.

3. If $F : R - mod \to S - mod$ is an equivalence (or a duality) then F is exact.

Theorem 0.3. *(our favorite functors)*

1. *Let* M"$\in$"$R - Mod$. *Then* $\mathrm{Hom}_R(M, -)$ *and* $\mathrm{Hom}_R(-, M)$ *are left-exact functors from* $R - Mod \to \mathbb{Z} - Mod$.

2. *Let* M"$\in$"$Mod - R$. *Then* $M \otimes_R - : R - mod \to \mathbb{Z} - mod$ *is right exact.*

Proof. Note: Starting from now, I will denote the image of f under the functor $\mathrm{Hom}_R(M, -)$ by $f*$. The reason is that for some reason tikzcd won't let me but Hom_R inside an arrow's name.
We'll just prove part 1 for the covariant Hom. The contravariant case is homework. So, we want to show that $\mathrm{Hom}_R(M, -)$ is left-exact. Let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

be exact in $R - Mod$. Then its image is a complex

$$0 \longrightarrow \mathrm{Hom}_R(M, A) \xrightarrow{f*} \mathrm{Hom}_R(M, B) \xrightarrow{g*} \mathrm{Hom}_R(M, C)$$

For $\phi : M \to A$, $f * (\phi) = f \circ \phi$, and similarly for $g*$.

We first show that $f*$ is a mono: Indeed, from $f \circ \phi = 0$, we obtain $\phi = 0$, since f is a mono. So the sequence is exact at $\mathrm{Hom}_R(M, A)$.
We know $\mathrm{Im}(\mathrm{Hom}_R(M, f)) \subseteq \ker(\mathrm{Hom}_R(M, g))$. To show the reverse direction, let $\psi \in \ker(\mathrm{Hom}_R(M, g))$, i.e. $g \circ \psi = 0$, i.e. $\mathrm{Im}(\psi) \subseteq \ker(\psi)$.
Consider

$$0 \longrightarrow A \xrightarrow{\tilde{f}} \mathrm{Im}(f) \hookrightarrow B \longrightarrow C \longrightarrow 0$$

Set $\phi = \tilde{f}^{-1} \circ \psi$ and check that $\mathrm{Hom}_R(M, f)(\phi) = \underbrace{f \circ \tilde{f}^{-1}}_{=\mathrm{Id}_A} \circ \psi = \psi$. So $\psi \in \mathrm{Im}(\mathrm{Hom}(M, f))$. This gives exactness at $\mathrm{Hom}_R(M, B)$. So, we have shown that the covariant Hom functor is left-exact.
Now for part 2.
Let

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be exact in $R - Mod$. The sequence

$$M \otimes_R A \xrightarrow{f*} M \otimes_R B \xrightarrow{g*} M \otimes_R C \longrightarrow 0$$

is a complex. Clearly, $M \otimes_R g, m \otimes b \mapsto m \otimes_R (f(g))$ is an epi because g is an epi.
We have $\underbrace{\mathrm{Im}(M \otimes_R f)}_{E} \subseteq \ker(M \otimes_R g)$ (Birge - "The image of $M \otimes_R f$, which I baptize E, ... "), and we want the reverse inclusion. Factor $M \otimes_R g$ in the form $M \otimes B \to M \otimes_R B/E \to M \otimes_R C$. (I missed a bit here because TeXwriter was being wonky, so this proof is nonsense I think. It's a standard proof that can be googled tho). Then $M \otimes_R g = G \circ ?$, so $\ker(G) = \ker(M \otimes g)/E$. Thus suffices to show that G is a mono.
Plan: Construct $H \in \mathrm{Hom}_{\mathbb{Z}}(M \otimes_R C, M \otimes_R B/E)$ such that $HG = \mathrm{Id}_{M \otimes_R B/E}$
Define $H' : M \times C \to M \otimes_R B/E$ by $(m, c) \mapsto (m \otimes b + E)$, where $b \in B$ is such that $g(b) = c$. We check well-definedness.
Suppose $g(b) = g(b'), b, b' \in B$. Then $b - b' \in \ker(g) = \mathrm{Im}(f)$ by hypothesis, hence $m \otimes_R b - m \otimes_R b' = m \otimes (b - b') \in E$, thus $m \otimes_R b + E = m \otimes_R b' + E$. Check H' is $\mathbb{Z}$-bilinear and R-balanced.
Hence the universal property of the tensor product yields $H \in \mathrm{Hom}_{\mathbb{Z}}(M \otimes C', M \otimes B/E)$ with $H(m \otimes c) = m \otimes b$, where $g(b) = c$.

■

Example 0.14. Here are some examples showing that in general, we shouldn't expect better than this previous theorem. That is, some witnesses to the non-left(right) exactness of $\mathrm{Hom}_R(M, -)$(resp. $M \otimes_R -$)

Definition 0.15. (i) Let $A" \in "\mathbb{Z} - Mod$. Then $a \in A$ is a torsion element iff there exists $n \in \mathbb{Z} \setminus \{0\}$ such that $n \cdot a = 0$. We use $T(A)$ to denote the torsion elements of A, which will always be a subgroup.

We say that A is torsion iff $A = T(A)$, and we say A is torsion-free iff $T(A) = 0$.

(ii) $a \in A$ is divisible iff $a \in nA$ for all $n \in \mathbb{Z} \setminus \{0\}$ (i.e. there exists $b \in A$ such that $a = nb$). An abelian group is called a divisible group if every element is divisible, i.e. if $nA = A$ for every $n \in \mathbb{Z} \setminus \{0\}$.

Remark. If $f \in \mathrm{Hom}_{\mathbb{Z}}(A, B)$ and A is divisible, then $f(A)$ is a divisible subbgroup of B.

Lecture 10, 2/3/23

Consider the sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

This sequence is exact. However, $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0$, and $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \neq 0$. So the image of this sequence under the functor $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, -)$ is not exact (in particular, it is not exact on the right).
The above argument will work for any integer instead of 2, so this sequence is a witness to the inexactness of the functor $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, -)$ for $n \geq 2$.
Consider an epi $g : \mathbb{Z}^{(\mathbb{N})} \to \mathbb{Q}$, and apply F. The map

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}^{(\mathbb{N})}) \xrightarrow{F(g)} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q})$$

cannot be an epi, as $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}^{(\mathbb{N})}) = 0$ (as $\mathbb{Q}$ is divisible while $\mathbb{Z}^{(\mathbb{N})}$ is reduced), but $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}) \neq 0$.

Example 0.15. Here is an example to show that the tensor functor is not left-exact. Let $R = \mathbb{Z}$. We consider the functor $F(-) = \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} -$, $n \geq 2$. Consider the inclusion $\mathbb{Z} \xhookrightarrow{\iota} \mathbb{Q}$. This is a mono. However, $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = 0$, so $F(\iota) : \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \to 0$. Hoever, $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{Z}/n\mathbb{Z} \neq 0$, so $F(\iota)$ is not a mono.

Short-term program

1. Find exact functors

2. Characterize the exact sequences $\mathbb{A}$ in $R - comp$ such that $F(\mathbb{A})$ is exact in $S - comp$ for any additive functor $F : R - mod \to S - mod$.

Theorem 0.4. If $F : R-mod \to S-mod$ is an exact functor (i.e. F takes short exact sequences to short exact sequences) then $F(\mathbb{A})$ is exact in $S-mod$ whenever

$$\mathbb{A} : \cdot \longrightarrow A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f} A_{n-1} \xrightarrow{f_{n-1}} \cdots$$

is exact in $R-mod$.

Proof. Suppose $F : R-mod \to S-mod$ is exact, and $\mathbb{A}$ as in the claim is an exact sequence in $R-mod$. We can factor each f_n

$$A_n \xrightarrow{\tilde{f}_n} \mathrm{Im}(A) \xhookrightarrow{\iota_n} A_{n-1}$$

$$x \longrightarrow f_n(x_n) \longrightarrow f_n(x_n)$$

Consider the short exact sequence

$$0 \longrightarrow \mathrm{Im}(f_{n+1}) \xrightarrow{\iota_{n+1}} A_n \xrightarrow{\tilde{f}_n} \mathrm{Im}(f_n) \longrightarrow 0$$

Since F is exact, we obtain an exact sequence

$$0 \longrightarrow F(\mathrm{Im}(f_{n+1})) \xrightarrow{F(\iota_{n+1})} F(A_n) \xrightarrow{F(\tilde{f}_n)} F(\mathrm{Im}(\tilde{f}_n)) \longrightarrow 0$$

In particular $\mathrm{Im}(F(\iota_{n+1})) = \ker(F(\tilde{f}_n))$ for $n \in \mathbb{N}$.

Claim.

(a) $\ker(F(f_n)) = \ker(F(\tilde{f}_n))$

(b) $\ker(F(\tilde{f}_n)) = \mathrm{Im}(F(\iota_{n+1})) = \mathrm{Im}(F(f_{n+1}))$

Proof. (a) $f_n = \iota_n \circ \tilde{f}_n$, so $F(f_n) = F(\iota_n) \circ F(\tilde{f}_n)$ by functoriality. Because F is exact, $F(\iota_n)$ is a mono, so $\ker(F(f_n)) = \ker(F(\tilde{f}_n))$.

(b) $F(f_{n+1}) = F(\iota_{n+1}) \circ F(\tilde{f}_{n+1})$. Because F is exact, $F(\tilde{f}_{n+1})$ is an epi.
So $\mathrm{im}(F(f_{n+1})) = \mathrm{Im}(F(\iota_{n+1}))$, so we have shown $F(\mathbb{A})$ is exact.

■

First installment of finding exact functors.

Proposition 1. *Let I be a set (index set). Consider the functors:*

$$\begin{aligned}(R-Mod)^I &\to R-Mod\\ (m_i)_{i\in I} &\mapsto \prod_{i\in I} m_i\\ (f_i)_{i\in I} &\mapsto \prod_{i\in I} f_i, (\vec{m})_j = f_j(m_j)\end{aligned}$$

$\otimes_{i\in I}$ works the same as before.
Both of them are exact .

Proof. Obvious ∎

First installment re (2)

Remark. Warning: Birge uses nonstandard notation. She says "X is a direct summand of Y" to mean $X \subseteq^{\bigoplus} Y$

Definition 0.16. Let A, B"$\in$"$R-mod, f \in \mathrm{Hom}_R(A,B)$. f is called split if $\ker f$ is a direct summand of A and $\mathrm{Im}(f)$ is a direct summand of B, i.e. there exist A', B'"$\in$"$R-Mod$ with $A = \ker(f)^{\oplus}A'$ and $B = \mathrm{Im}(f)^{\oplus}B'$.

Lecture 11, 2/6/23

Convention: Let M, N"$\in$"$R-Mod$. We say that N is a direct summand of M, written $N \subseteq^{\oplus} M$, if there exists U"$\in$"$R-Mod$ such that $M = N^{\oplus}U$.

Definition 0.17. Let $\mathbb{A}$"$\in$"$R-comp$. Then $\mathbb{A}$ is split if f_n is split for all n.

Proposition 2. *Let A, B, C be left R-modules, $f \in \mathrm{Hom}_R(A,B)$ and $h \in \mathrm{Hom}_R(B,C)$. Then the following conditions are equivalent:*

1. *The sequence* $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ *is split exact.*

2. *There exists $f' \in \mathrm{Hom}_R(B,A)$ and $g' \in \mathrm{Hom}_R(C,B)$ such that $ff' + g'g = \mathrm{Id}_B$, and both triangles in the diagram below commute:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[dl, dotted, "f'"] & C \arrow[d, equal] \\ A & & C \arrow[ul, dotted, "g'"] \end{tikzcd}$$

Proof. ■

Theorem 0.5. *Let* $\mathbb{A}$: $\cdots \longrightarrow A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \longrightarrow \cdots$ *"$\in$" $R-$comp.*

Then the following are equivalent:

1. *For all additive functors $F : R-Mod \to S-Mod$ (S any ring), the sequence $F(\mathbb{A})$ is split exact.*

2. *$\mathbb{A}$ is split exact.*

Proof. For $1 \implies 2$, apply $F = \mathrm{Id}_{R-Mod}$.
For $2 \implies 1$, suppose 2. We will show 1 only in the case $\mathbb{A} = \; 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$ is split exact. Move to general $\mathbb{A}$ as in the proof of previous theorem.
By the proposition, there exist maps $f' \in \mathrm{Hom}_R(B, A)$, $g' \in \mathrm{Hom}_R(C, B)$ with $\mathrm{Id}_B = ff' + g'g$. Let F be an additive functor as in 1. Then

$$\begin{aligned} \mathrm{Id}_{F(B)} &= F(\mathrm{Id}_B) \\ &= F(f)F(f') + F(g')F(g) \end{aligned}$$

So by the proposition, the sequence

$$0 \longrightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(c) \longrightarrow 0$$

is split exact.

■

Example 0.16. There exist exact sequences

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

such that $B \cong A \oplus C$, but the sequence fails to be split.
Take $R = \mathbb{Z}$, let $n \geq 2, A = n\mathbb{Z}, B = \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z})^{(\mathbb{N})}$, and $C = (\mathbb{Z}/n\mathbb{Z})^{(\mathbb{N})}$. Then $A \oplus C \cong B$.
Find $f \in \mathrm{Hom}_{\mathbb{Z}}(A, B), g \in \mathrm{Hom}_{\mathbb{Z}}(B, C)$, such that

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact, but not split.

Section 2: Projective Modules

Our aim is to characterize those M"$\in$"$R-Mod$ for which $\mathrm{Hom}_R(M,-) : R-Mod \to \mathbb{Z}-Mod$ is exact.

Definition 0.18.

1. ${}_RF$ is <u>free</u> if ${}_RF \cong_R R^{(I)}$.

   It is known that ${}_RF$ is <u>free</u> iff F has an R-basis, that is, a linearly independent generating set.

2. ${}_RP$"$\in$"$R-Mod$ is called <u>projective</u> if P is isomorphic to a direct summand of a free module.

Example 0.17.

1. Vector spaces

2. Let k be a field and $R = k \oplus k$ (ring product). Then R admits projective modules that fail to be free. Let $e_1 = (1,0), e_2 = (0,1)$. Then $Re_1 = k \times \{0\}, Re_2 = \{0\} \times k$. We have $R = Re_1 \oplus Re_2$, so the Re_i are projective. However, they are not free, because $\dim_k(R) = 2, \dim_k(Re_i) = 1$

Remark.

1. Let $(R_i)_{i\in I}$ be a family of left R-modules. Then $\bigoplus_{i\in I} P_i$ is projective if and only if P_i is projective for all $i \in I$.

2. $\prod_{i\in I} P_i$ need not be projective for infinite I.

3. Let R be a PID. The projective R-modules are precisely the free ones. Such R include $R = \mathbb{Z}, R = k[x], k$ a field.

4. $\mathbb{Z}^{\mathbb{N}}$ is <u>NOT</u> free (proof to come), and hence not projective.

5. (Serre's Conjecture) If $R = k[x_1, \ldots, x_n]$, k a field, is every projective R-module free? It turns out yes, and there are independent proofs by Suslin and <u>Quillen</u> (Quillen got the fields medal).

Lecture 12, 2/8/23

<u>True or false:</u> If $P`` \in " R - Mod$ is finitely generated projective, then there exists $n \in \mathbb{N}$ such that P is isomorphic to a dirrect summand ${}_R R^n$?
This is true. It is isomorphic to a direct summand of a free module $R^{(I)}$. Pick finite $I' \subseteq I$ with $x_k \in \oplus_{i \in I'} R_i$. Then $P \subseteq \oplus_{i \in I'} R \cong R^n$, with $n = |I'|$ and hence P is isomorphic to a direct summand of R^n.
Let $R = K[x, y]$. Then $P = (x)$ is not projective, despite being a submodule of R.
In general: If ${}_R V \subseteq_R U \subseteq_R M$, and $V \subseteq^{\oplus} M$, then $V \subseteq^{\oplus} U$. Look up the "modular law."

Theorem 0.6. *For $M`` \in " R - Mod$, the following are equivalent.*

1. *M is projective*

2. $\mathrm{Hom}_R(M, -) : R - mod \to \mathbb{Z} - mod$ *is exact.*

3. *Whenever $f \in \mathrm{Hom}_R(B, C)$ is an epi and $g \in \mathrm{Hom}_R(M, C)$, then there exists a $\phi \in \mathrm{Hom}_R(M, B)$ with $f \circ \phi = g$. That is, there is a ϕ making the following diagram commutes.*

$$\begin{tikzcd} B \arrow[r, two heads, "f"] & C \arrow[r] & 0 \\ & M \arrow[u, "g"] \arrow[ul, dotted, "\phi"] & \end{tikzcd}$$

4. *Every epi onto M splits. That is, every surjection onto M admits a section.*

For the following proof, we will abbreviate $\mathrm{Hom}_R(M, -)$ *by* $[M, -]$.

Proof.

0.1 (1) $\implies$ (3)

We have $M \subseteq_R^{\oplus} F$, F free on basis $(x_i)_{i \in I}$, $F = M \oplus N$.
Let f and g be as in 3. That is, we have

$$\begin{tikzcd} B \arrow[r, two heads, "f"] & C \\ & M \arrow[u, "g"] \\ & F \arrow[u, "\pi"] \end{tikzcd}$$

where $\pi : F \to M$ is the projection along N, and $\iota : M \hookrightarrow F$ the embedding. There is a $\psi : F \to B$ which makes this commute:

$$\begin{tikzcd}
B \arrow[r, two heads, "f"] & C \\
& M \arrow[u, "g"'] \\
& F \arrow[u, "\pi"'] \arrow[uul, dotted, "\psi"]
\end{tikzcd}$$

defined by $\psi(x_i) = b_i$ if $f(b_i) = g\pi(x_i)$. This is well-defined because F is free on the x_i, and the b_i exist because f is an epi Then, we have $f \circ \psi = g \circ \pi$, so $f \circ \psi \circ \iota = g \circ \pi \circ \iota$. But $\pi \circ \iota = \mathrm{Id}_M$, so $f \circ (\psi \circ \iota) = g$. But that means $\phi = \psi \circ \iota : M \to B$ makes the diagram commute:

$$\begin{tikzcd}
B \arrow[r, two heads, "f"] & C \\
& M \arrow[u, "g"'] \arrow[ul, dotted, "\phi"'] \\
& F \arrow[u, "\pi"'] \arrow[uul, dotted, "\psi"]
\end{tikzcd}$$

(3) $\implies$ (2)

Assume 3. Since $[M, -]$ is a left-exact functor, it suffices to show that $[M, -]$ takes epis to epis.

Let $B \xrightarrow{f} C \longrightarrow 0$ be exact. To show $[M, f] : [M, B] \to [M, C]$ is an epi, let $g \in [M, C]$. By assumption, there exists $\phi \in [M, B]$ with $f^*(\phi) = g$. But this means $\phi^*(f) = g$.

(2) $\implies$ (4)

Assume 2. Let $f : N \twoheadrightarrow M$ be an epi. Then by assumption, $[M, f] : [M, B] \twoheadrightarrow [M, C]$ is an epi. So there exists $\phi \in [M, N]$ with $f \circ \phi = \mathrm{Id}$. So the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[d, equal] & N \arrow[dl, two heads, "f"'] \\ M & \end{tikzcd}$$

Hence $N = \mathrm{Im}(\phi) \oplus \ker(f)$, so $\ker(f) \subseteq^{\oplus} N$, and f splits.

(4) $\implies$ (1)

Assume 4. If $(m_i)_{i\in I}$ is a generating set for M, then $F = R^{(I)} \twoheadrightarrow M$, given by $(r_i) \mapsto \sum_{i\in I} r_i m_i$, (which will be a finite sum because all but finitely many r_i are zero) is an epi.
By assumption, f splits, meaning $F = \ker(f) \oplus N$. Then $N \cong F/\ker(f) \cong M$. So M is isomorphic to a direct summand of F.
This completes the proof.

■

Further examples of projective modules and structure results

Example 0.18.

1. Let R be a ring. All M" $\in$ "$R-Mod$ are projective if and only every submodule U of any left R-module N is a direct summand, i.e. $N = U \oplus V$.

(=>) Let ${}_R U \subseteq_R N$. Take $M = N/U$ and epi $\pi : N \to N/U$. Since N/U is projective, π splits, meaning $U \subseteq^{\oplus} A$.

(<=) Easy

Equivalently, if R is semi-simple when viewed as a right module over itself, R_R.

Lecture 13, 2/10/23

Proposition 3. *(Baer)* $\mathbb{Z}^{\mathbb{N}}$ *is not free.*

Proof. (not by Baer)
Assume $\mathbb{Z}^{\mathbb{N}}$ is free, i.e. there exists an isomorphism $\mathbb{Z}^{\mathbb{N}} \cong \mathbb{Z}^{(I)}$ for some I. Then $|I| > \aleph_0$. Pick a countable subset $I' \subset I$ such that $f(\mathbb{Z}^{(\mathbb{N})}) \subseteq \mathbb{Z}^{(I')}$ and consider the map $\overline{f}$ induced by f,

$$\overline{f} : \mathbb{Z}^{\mathbb{N}}/\mathbb{Z}^{(\mathbb{N})} \to \mathbb{Z}^{(I)}/\mathbb{Z}^{(I')} \cong \mathbb{Z}^{(I\setminus I')}$$
$$x + \mathbb{Z}^{(\mathbb{N})} \mapsto f(x) + \mathbb{Z}^{(I')}$$

In particular, $\mathbb{Z}^{(I\setminus I')}$ is nonzero and free, because I is uncountable and I' is countable.
Now, if F is a free abelian group, F contains no nonzero elements which are divisible by 3^n for all $n \in \mathbb{N}$.
Let $S = \{(z_n)_{n\in\mathbb{N}} \in \mathbb{Z}^{\mathbb{N}} \mid 3^n \mid z_n \text{ for } n \in \mathbb{N}\}$. Note S is uncountable.
Since $\mathbb{Z}^{(I')}$ is countable, there exists $s \in S$ with $f(s) \in \mathbb{Z}^{(I')}$ and thus $f(s) + \mathbb{Z}^{(I')} \neq 0$ in $\mathbb{Z}^{(I\setminus I')}$.
Hence $\overline{f}(s + \mathbb{Z}^{(\mathbb{N})}) \neq 0 \in \underbrace{\mathbb{Z}^{(I\setminus I')}}_{\text{free}}$.
Thus $\mathbb{Z}^{(I\setminus I')}$ contains nonzero elements divisible by 3^n for all n, a contradiction.
Recall that projective means isomorphic to a summand of a free module $R^{(I)}$ for some I.

In general, projectives in R-Mod are <u>not</u> direct sums of finitely generated modules.

Theorem 0.7. *(Kaplensky)*
Let P "$\in$" R-Mod be projective. Then P is the sum of countably generated modules.

Proof. Not enough time. ■

Corollary 0.8. *If R is local, then all projective R-modules are free.*

Proof. ■

Section 3:Projective resolutions and projective dimension

<u>Idea:</u>
Let M "$\in$" R-mod. Approximate $f_0 : P_0 \twoheadrightarrow M \to 0$
<u>Error:</u> $\ker(f_0)$. Next approximate $f_1 : P_1 \to \ker(f_0) \to 0$, with P_1 projective. This results in a sequence

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[rr, "f_2"] \arrow[dr, two heads] & & P_1 \arrow[rr, "f_1"] \arrow[dr, two heads] & & P_0 \arrow[r, two heads, "f_0"] & M \arrow[r] & 0 \\
& & \ker(f_1) \arrow[ur] & & \ker(f_0) \arrow[ur] & & &
\end{tikzcd}$$

Definition 0.19. Let M "$\in$" R-Mod. A <u>projective resolution of M</u> is an exact sequence

$$\cdots \longrightarrow P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$$

where all P_i are exact.
Call such a projective resolution finite of length n if $P_n \neq 0$, but $P_m = 0$ for $m > n$.

Definition 0.20. We define the projective dimension of a module M as follows. If there is no finite projective resolution, then it is ∞. Otherwise it is the smallest n suuch there exists a projective resolution of length n. By convention, the projective dimension of the zero module is $-\infty$.
We denote this by $\operatorname{pdim} M$.
The (left) global dimension of a ring R is the defined as $\operatorname{gldim} R = \sup\{\operatorname{pdim} M \mid M \text{“} \in \text{”} R - Mod\}$. There is also of course the right global dimension, where instead we consider $Mod - R$.

Example 0.19.

- $\operatorname{gldim} \mathbb{Z} = 1$.
- $\operatorname{lgldim} R = 0$ if and only if R is semisimple, which happens if and only if $\operatorname{rgldim} R = 0$.
- $$\begin{pmatrix} K & \cdots & K \\ \vdots & \vdots & \\ 0 & \cdots & K \end{pmatrix} \not\supseteq \begin{pmatrix} 0 & K & \cdots K & \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$
  not semisimple, so $\operatorname{lgldim} R \geq 1$. In fact, $=$ holds (argument later).
- This is Hilbert's Syzygy theorem. It says that if K is a field, then
  $$\operatorname{gldim} K[x_1, \ldots, x_n] = n$$
- $R = \mathbb{Z}/p^n\mathbb{Z}$, p prime, $n \geq 2$.

$$\begin{tikzcd}
\mathbb{Z}/p^n\mathbb{Z} \arrow[rr] \arrow[dr] & & \underbrace{\mathbb{Z}/p^n\mathbb{Z}}_{P_0 = {}_R R} \arrow[r, two heads, "f_0"] & \underbrace{\mathbb{Z}/p^{n-1}\mathbb{Z}}_{{}_R M} \arrow[r] & 0 \\
& p\mathbb{Z}/p^n\mathbb{Z} \arrow[ur, hook] & & &
\end{tikzcd}$$

Lecture 14, 2/13/23

Lecture 15, 2/17/23

Section 4, part A: Injective modules

Definition 0.21. A module is called injective if, for every such diagram of R-modules

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r, "f"] \arrow[d, "\phi"] & M \\ & Q & \end{tikzcd}$$

with f injective, then there is a $\psi : M \to Q$ making the following diagram commute:

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r, "f"] \arrow[d, "\phi"] & M \arrow[dl, dotted, "\exists\psi"] \\ & Q & \end{tikzcd}$$

Proposition 4. *(Baer's Criterion)*
A module M "$\in$" $R - Mod$ is injective if and only if whenever $I \subseteq R$ is an ideal and $y \in \mathrm{Hom}_R(I, M)$, there exists $\Phi \in \mathrm{Hom}_R(R, M)$ such that $\Phi|_I = y$.

Proof. online, also proven in 220B.

■

Section 4, part B: Injective modules over a PID

Theorem 0.9. *Let R be a PID and R "$\in$" $R - Mod$. Then the following are equivalent:*

(i) M is injective.

(ii) M is divisible, meaning that $aM = M$ for all nonzero $a \in R$.

Proof. First, assume (1). And let a be a nonzero element of R. Fix $x \in M$ and consider

$$\begin{tikzcd} I = Ra \arrow[r, hook] \arrow[d, "g"'] & R \arrow[dl, dotted, "\exists\psi"] \\ M & \end{tikzcd}$$

So $g(a) = g(a \cdot 1) = \psi(a \cdot 1) = a \cdot \psi(1) \in aM$.
Assume (2). We'll apply Baer. So let ${}_R I \hookrightarrow_R R$ and $\varphi \in \mathrm{Hom}_R(I, M)$.
We know $I = Ra$, and without loss of generality $a \neq 0$. So by hypothesis there exists an $x \in M$ with $\varphi(a) = a \cdot x$. Define $\phi \in \mathrm{Hom}_R(R, M), \gamma \mapsto \gamma x$, check that $\phi|_I = \varphi$, and that does it.

■

So now we know the injective Abelian groups are precisely the divisible Abelian groups.

Example 0.20.
$\mathbb{Q}, \mathbb{Z}(p^\infty)$, Prüfer groups for p prime.
Above, $\mathbb{Z}(p^\infty)$ is defined as follows. Start with $\oplus\mathbb{Z}[x_i]/U(p)$, where $\mathbb{Z}[x_i]$ is a free group, and $U(p)$ is the subgroup of $\sum_{i\in\mathbb{N}} \mathbb{Z}[x_i]$ generated by $px_1 + \cdots + px_{i-1} - x_i$ for all $i \in \mathbb{N}$.
We find $\mathbb{Z}[\overline{x_i}] \cong \mathbb{Z}/p^i\mathbb{Z}$.

Remark. If T is a torsion Abelian group, then T is the direct sum $\oplus_{p \text{ prime}} T_p$, where $T_p = \{x \in T \mid p^n x = 0 \text{ for some } n\}$.

Theorem 0.10. *A"$\in$"$\mathbb{Z} - Mod$ is divisible iff* $A \cong \mathbb{Q}^{(I)} \oplus \bigoplus_{p\ prime}(\mathbb{Z}/(p^\infty))$

Proof. exercise

■

Part C: Injective resolutions

Theorem 0.11. *(Eckmann)*
Every left R-module is a submodule of an injective module.

Lemma 1. *If M"$\in$"$\mathbb{Z} - Mod$, then there exists a divisible D"$\in$"$\mathbb{Z} - Mod$ such that M is isomorphic to a submodule of D.*

Proof. We know $M \cong \mathbb{Z}^{(I)}/K$ for some subgroup $K \subseteq \mathbb{Z}^{(I)}$. But everything here is divisible.

■

Lecture 16, 2/22/23

Consider the bimodule ${}_\mathbb{Z}R_R$ and note $R \otimes_R -$ is exact, as $R \otimes_R -$ is isomorphic to the identity.

Lemma 2. *If D"$\in$"$\mathbb{Z} - mod$ is a divisible Abelian group, then the left R-module ${}_R\mathrm{Hom}_\mathbb{Z}(R_R, D)$ is injective.*

Proof. Set $E = \text{Hom}_{\mathbb{Z}}(R, D)$"$\in$"$\mathbb{R}$-Mod for a divisible D. To show injectivity of ${}_RE$, let

$$0 \longrightarrow {}_RU \xrightarrow{f} {}_RV \xrightarrow{g} {}_RW \longrightarrow 0$$

be an exact sequence in R-Mod.
Then

$$0 \longrightarrow R \otimes_R U \longrightarrow R \otimes_R V \longrightarrow R \otimes_R W \longrightarrow 0$$

is exact. Since ${}_{\mathbb{Z}}D$ is injective, the sequence

$$0 \longrightarrow \text{Hom}_{\mathbb{Z}}(R \otimes_R W, D) \xrightarrow{g^*} \text{Hom}_{\mathbb{Z}}(R \otimes_R V, D) \xrightarrow{f^*} \text{Hom}_{\mathbb{Z}}(R \otimes_R U, D) \longrightarrow 0$$

is exact, where f^*, g^* are the maps induced from f and g. But by the tensor-hom adjunction, we know $\text{Hom}_{\mathbb{Z}}(R \otimes_R -, D) \cong \text{Hom}_R(-, E)$ naturally. So the sequence

$$0 \longrightarrow \text{Hom}_R(W, E) \longrightarrow \text{Hom}_R(V, E) \longrightarrow \text{Hom}_R(U, E) \longrightarrow 0$$

is exact. Thus E is injective.

■

Recall that we are trying to prove that every left R-module is a submodule of an injective module.

Proof. We have ${}_RM \cong \text{Hom}_R(R_R, M) \subseteq \text{Hom}_{\mathbb{Z}}(R, M)$. So $M_{\mathbb{Z}} \subseteq_{\mathbb{Z}} D$ in $\mathbb{Z}-$Mod, where D is divisible and $\text{Hom}_{\mathbb{Z}}(R, -)$ is left-exact. But $\text{Hom}_{\mathbb{Z}}(R, M)$ is isomorphic to a submodule of ${}_R\text{Hom}_{\mathbb{Z}}(R_R, D)$.
Watch out! All iso's/embeddings above respect the left R-module structure.
But ${}_R\text{Hom}_{\mathbb{Z}}(R_R, D)$ is injective by previous lemma.

■

Corollary 0.12. *Every M"$\in$"$R - Mod$ has an injective resolution.*

Proof. Let M be an R-module. We know that there is an inclusion $f_0 : M \to E_0$, where E_0 is injective. The cokernel will then embed into E_1, and the induced map from E_0 to E_1 is called f_1, and we continue, as in the diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, hook, "f_0"] & E_0 \arrow[rr, "f_1"] \arrow[dr, two heads] & & E_1 \arrow[rr, "f_2"] \arrow[dr, two heads] & & E_2 \arrow[r] & \cdots \\
& & & \text{coker}(f_0) \arrow[ur, hook] & & \text{coker}(f_1) \arrow[ur, hook] & &
\end{tikzcd}$$

■

Definition 0.22. Let M"$\in$"R-Mod.

1. An <u>injective resolution of M</u> is any exact sequence

$$0 \longrightarrow M \xrightarrow{f_0} E_0 \xrightarrow{f_1} E_1 \longrightarrow \cdots$$

such that E_i is injective for all i. If $E_i = 0$ for $i >> 0$, define the length in analogy with projective case.

2. The <u>nth cozyzygy</u> of M is $\mathrm{Im}(f_n) \cong \mathrm{coker}(f_{n-1})$, which is unique up to injective direct summands.

3. The <u>injective resolution</u>, $\mathrm{idim}\, M$, is the minimal length of all finite injective resolutions, assuming any exist. Otherwise it is ∞.

Remark. We'll see that $\sup\{\mathrm{idim}\, M \mid M \text{“} \in \text{”} R - Mod\} = \mathrm{lgldim}\, R$.

Theorem 0.13. *For $M \text{“} \in \text{”}$R-Mod, and $n \in \mathbb{N} \cup \{0\}$, the following are equivalent:*

1. *(a)* $\mathrm{idim}\, M \leq n$

(b) There exists an injective resolution of M with an injective n-th cosyzygy

(c) In all injective resolutions of M, then n-th cosyzygy is injective.

2. $\mathrm{idim}\, M = \infty$ *iff all (one) injective resolutions has no injective cosyzygies.*

Proof. ■

If $\mathbb{Z}_D$ is divisible, and R is any ring, then ${}_R\mathrm{Hom}_{\mathbb{Z}}(R, D)$ is injective. The only bimodule structure on ${}_{\mathbb{Z}}R_R$ we use is that R is flat as a left R-module.

Lecture 17, 2/25/23

Lecture 18, 2/27/23

Last time, we stated and proved the so-called snake lemma. Pay special attention to the map ∂ and how it is defined, and indeed well-defined.

Theorem 0.14. *(Long exact homology sequence)*

Suppose $\mathscr{A} : \ 0 \longrightarrow \mathbb{A} \xrightarrow{f} \mathbb{A}' \xrightarrow{f'} \mathbb{A}'' \longrightarrow 0$ $(\star)$ *is an exact sequence in $R - comp$. Then for each $n \in \mathbb{Z}$, there exists $\partial_n \in \mathrm{Hom}_R(H_n(\mathbb{A}''), H_n(\mathbb{A}))$, such that the following long sequence is exact:*

$$\cdots \longrightarrow H_n(\mathbb{A}) \xrightarrow{H_n(f)} H_n(\mathbb{A}') \xrightarrow{H_n(f')} H_n(\mathbb{A}'') \xrightarrow{\partial_n} H_{n-1}(\mathbb{A}) \xrightarrow{H_{n-1}(f')} H_{n-1}(\mathbb{A}') \longrightarrow \cdots$$

Morover, this is natural in the sense that for any $\mathscr{A}$ as above, the family $(\partial_n)_{n\in\mathbb{Z}} = (\partial_n^{\mathscr{A}})$ satisfies the following condition:

Whenever

$$\begin{tikzcd}
\mathscr{A} : 0 \arrow[r] & \mathbb{A} \arrow[r, "f"] \arrow[d, "h"] & \mathbb{A}' \arrow[r, "f'"] \arrow[d, "h'"] & \mathbb{A}'' \arrow[r] \arrow[d, "h''"] & 0 \\
\mathscr{L} : 0 \arrow[r] & B \arrow[r, "g"'] & B' \arrow[r, "g'"'] & B'' \arrow[r] & 0
\end{tikzcd}$$

the following diagrams commute:

$$\begin{tikzcd}
H_n(\mathbb{A}'') \arrow[r, "\partial_n^{\mathscr{A}}"] \arrow[d, "H_n(h'')"] & H_{n-1}(\mathbb{A}) \arrow[d, "H_{n-1}(h)"] \\
H_n(B'') \arrow[r, "\partial_n^{\mathscr{L}}"'] & H_{n-1}(B)
\end{tikzcd}$$

Proof. The exact sequence $(\star)$ translates into an infinite commutative grid

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A_{n+1} \arrow[r] \arrow[d, "d_{n+1}"] & A'_{n+1} \arrow[r] \arrow[d, "d'_{n+1}"] & A''_{n+1} \arrow[r] \arrow[d, "d_{n+1}"] & 0 \\
0 \arrow[r] & A_n \arrow[r] \arrow[d] & A'_n \arrow[r] \arrow[d] & A''_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r] \arrow[d] & A'_{n-1} \arrow[r] \arrow[d] & A''_{n-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

We'll extract the commutative diagrams

$$\begin{tikzcd}
0 \arrow[r] & \frac{A_n}{\mathrm{Im}(d_{n+1})} \arrow[r, "\overline{f_n}"] \arrow[d, "\overline{d_n}"] & \frac{A'_n}{\mathrm{Im}(d'_{n+1})} \arrow[r] \arrow[d, "\overline{d'_n}"] & \frac{A''_n}{\mathrm{Im}(d''_{n+1})} \arrow[r] \arrow[d, "\overline{d''_n}"] & 0 \\
0 \arrow[r] & \ker(d_{n-1}) \arrow[r] & \ker(d'_{n-1}) \arrow[r] & \ker(d''_{n-1}) \arrow[r] & 0
\end{tikzcd}$$

Then we'll apply the snake lemma.

Step 1

Define $\overline{d_n} : \frac{A_n}{\mathrm{Im}(d_{n+1})} \to \ker(d_{n-1})$ by $\overline{a_n} \mapsto d_n(a_n)$. This is well-defined because $d_n d_{n+1} = 0 = d_{n-1}d_n$.
Note $\ker(\overline{d_n}) = \frac{\ker(d_n)}{\mathrm{Im}(d_{n+1})} = H_n(\mathbb{A})$, and $\mathrm{coker}(\overline{d_n}) = \frac{\ker(d_{n-1})}{\mathrm{Im}(d_n)} = H_{n-1}(\mathbb{A})$. So we obtain exact sequences of the following ilk:

$$0 \longrightarrow H_n(\mathbb{A}) \xhookrightarrow{\iota_n} \frac{A_n}{\mathrm{Im}(d_{n+1})} \xrightarrow{\overline{d_n}} \ker(d_{n-1}) \xrightarrow{j_n} H_{n-1}(\mathbb{A}) \longrightarrow 0$$

Step 2

The snake lemma now yields maps $\partial_n \in \mathrm{Hom}_R(H_n(\mathbb{A}''), H_{n-1}(\mathbb{A}))$ such that the following diagram (refer to diagram on Gauchospace) commutes, and the sequence

$$\cdots \longrightarrow H_n(\mathbb{A}') \longrightarrow H_n(\mathbb{A}'') \xrightarrow{\partial_n} H_{n-1}(\mathbb{A}) \longrightarrow H_{n-1}(\mathbb{A}') \longrightarrow \cdots$$

is exact.

■

Section 2: Homotopy of complexes

Definition 0.23.

1. Let $u : \mathbb{A} \to \mathbb{A}'$ be a morphism in $R-comp$, say $u = (u_n)_{n\in\mathbb{Z}}$. Call u null-homotopic if there exist $s_n \in \mathrm{Hom}_R(A_n, A_{n-1})$ such that $u_n = d'_{n+1} \circ s_n + s_{n-1} \circ d_n$. So

$$\begin{tikzcd}
\cdots \arrow[r] & A_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "u_{n+1}"'] & A_n \arrow[r, "d_n"] \arrow[d, "u_n"] \arrow[dl, "s_n"'] & A_{n-1} \arrow[r] \arrow[d, "u_{n-1}"] \arrow[dl, "s_{n-1}"] & \cdots \\
\cdots \arrow[r] & A'_{n+1} \arrow[r, "d'_{n+1}"] & A'_n \arrow[r, "d'_n"] & A'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

2. Two chain maps f, g are called homotopy equivalent if $f - g$ is null-homotopic. We write $f \simeq g$.

3. Two chain complexes $\mathbb{A}, \mathbb{B}$ are homotopy equivalent if there are chain maps

Proposition 5. *Suppose* $u = (u_n) \in \mathrm{Hom}_{R-comp}(\mathbb{A}, \mathbb{A}')$ *is null-homotopic. Then* $H_n(u) = 0$ *for all* $n \in \mathbb{Z}$.

Proof. Let $s_n \in \mathrm{Hom}_R(A_n, A'_{n+1})$ be as in the definition, i.e. $u_n = d'_{n+1} \circ s_n + s_{n-1} \circ d_n$. Then

$$H_n(u) : \frac{\ker(d_n)}{\mathrm{Im}(d_{n+1})} \to \frac{\ker(d'_n)}{\mathrm{Im}(d'_{n+1})}$$

acts by $H_n(u)(\overline{a_n}) = \overline{u_n(a_n)}$.
Suppose $a_n \in \ker(d_n)$. Then

$$\begin{aligned} u_n(a_n) &= d'_{n+1}(s_n(a_n)) + s_{n-1}(\overbrace{d_n(a_n)}^{=0}) \\ &= d'_{n+1}(s_n(a_n)) \in \mathrm{Im}(d'_{n+1}) \end{aligned}$$

Lecture 19, 3/1/23

Definition 0.24.

1. Let M "$\in$" $R-$ Mod, and

$$\cdots \longrightarrow P_n \xrightarrow{d_n} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0$$

a projective resolution of M.

Then the deleted projective resolution is the complex obtained by deleting M and replacing it with 0, i.e.

$$\mathbb{P} : \cdots \longrightarrow P_n \xrightarrow{d_n} \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} 0$$

Note that this "just" a complex, not necessarily exact.

2. If

$$0 \longrightarrow M \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \cdots$$

is an injective resolution of M, then the deleted injective resolution is defined the same way, i.e. as

$$0 \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \cdots$$

Remark. A deleted projective resolution of M determines M up to isomorphism. As we'll see, conversely, any two deleted projective resolutions of a given M are homotopy equivalent.
Model problem: Look at the functor $X \otimes_R -$. We know this functor is right exact, and in general not left exact.
For

$$0 \longrightarrow M \xrightarrow{f} M' \xrightarrow{f'} M'' \longrightarrow 0$$

we want to get some information on the kernel of $X \otimes_R f : X \otimes_R M \to X \otimes_R M'$.
Step 1: One can construct projective resolutions of M, M', M'', and one obtains a short exact sequence

$$0 \longrightarrow \mathbb{P} \xrightarrow{\overline{f}} \mathbb{P}' \xrightarrow{\overline{f'}} \mathbb{P}'' \longrightarrow 0$$

where $\mathbb{P}, \mathbb{P}', \mathbb{P}''$ are deleted resolutions of M, M', M''.
In particular, if $\overline{f} = (f_n)_{n\in\mathbb{Z}}$, and $\overline{f'} = (\overline{f}_n)_{n\in\mathbb{Z}}$, we obtain split exact sequences

$$0 \longrightarrow P_n \xrightarrow{f_n} P'_n \xrightarrow{f'_n} P'' \longrightarrow 0$$

Due to projectivity of P''_n, this sequence is split, and hence

$$0 \longrightarrow F(P_n) \xrightarrow{F(f_n)} F(P'_n) \xrightarrow{F(\overline{f_n})} F(P''_n) \longrightarrow 0$$

is exact.
In other words,

$$\otimes: \ 0 \longrightarrow F(\mathbb{P}) \longrightarrow F(\mathbb{P}') \longrightarrow F(\mathbb{P}'') \longrightarrow 0$$

is exact.
Step 2:
The sequence $\otimes$ give rise to a long exact homology sequence

$$\begin{tikzcd} H_n(F(\mathbb{P})) \arrow[r] & H_{n-1}(F(\mathbb{P})) \arrow[r] & H_{n-1}(F(\mathbb{P}')) \arrow[r] & H_{n-1}(F(\mathbb{P}'')) \arrow[r] & \ddots \arrow[d] \\ & & & & H_0(F(\mathbb{P}'')) \end{tikzcd}$$

Step 3:
Since F is right-exact, $H_0(F(\mathbb{P})) \cong F(M)$, and similarly $H_0(F(\mathbb{P}')) \cong F(M')$, and $H_0(F(\mathbb{P}'')) \cong F(M'')$.

We obtain an exact sequence

$$
\begin{tikzcd}
\cdots \arrow[r] & H_1(F(\mathbb{P})) \arrow[r] & H_1(F(\mathbb{P}')) \arrow[r] & H_1(F(\mathbb{P}'')) \arrow[d, "\partial_1"] \\
 & & & F(M) \arrow[d, "F(f)"] \\
 & & & F(M') \arrow[d, "F(f')"] \\
 & & & F(M'') \arrow[d] \\
 & & & 0
\end{tikzcd}
$$

Lemma 3. (Comparison Lemma)
Let M, M' " $\in$ " $R-Mod$, and $f \in \mathrm{Hom}_R(M, M')$. Moreover, let $\mathbb{P}, \mathbb{P}'$ be deleted projective resolutions of M, M' respectively. Then there exists a map $\overline{f} = (f_n)_{n\geq 0} \in \mathrm{Hom}_{R-comp}(\mathbb{P}, \mathbb{P}')$, such that the following diagram commutes:

$$
\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r, "d_1"] \arrow[d, "f_1"] & P_0 \arrow[r, "d_0"] \arrow[d, "f_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & P_1' \arrow[r, "d_1'"'] & P_0' \arrow[r, "d_0'"'] & M' \arrow[r] & 0
\end{tikzcd}
$$

Moreover, given another map $\overline{g} \in \mathrm{Hom}_{R-comp}(\mathbb{P}, \mathbb{P}')$, such that the above diagram, with f_i replaced by g_i (the final $f : M \to M'$ stays), still commutes, then $\overline{f} \simeq \overline{g}$.

Remark. Here $\overline{f}$ is called "the" chain map lying over f.

Proof. Existence of $\overline{f}$
We find f_n by induction on $n \geq 0$.

- For $n = 0$, consider the diagram

$$\begin{tikzcd} & P_0 \arrow[d, "d_0"] \\ & M \arrow[d, "f"] \\ P_0' \arrow[r, "d_0'"'] \arrow[ruu, dotted, "\exists f_0"] & M' \end{tikzcd}$$

f_0 exists because P_0 is projective and d_0' is an epi.

- Let $n \geq 1$, and suppose $f_0, \ldots, f_{n-1}$ are given as required. Consider the diagram

$$\begin{tikzcd} & P_n \arrow[d, "d_n"] \\ & P_{n-1} \arrow[d, "f_{n-1}"] \\ P_{n-1}' \arrow[r, "d_n'"'] & P_{n-1}' \end{tikzcd}$$

The problem is d_n' is not necessarily an epi. So we replace P_{n-1}' by $\mathrm{Im}(d_n')$. We wish to show $\mathrm{Im}(f_{n-1}d_n) \subseteq \mathrm{Im}(d_n')$. But $\mathrm{Im}(d_n') = \ker(d_{n+1}')$. Compute $d_{n-1}' f_n d_n = d_{n-1}' d_n f_n$. We will continue next time.