Lecture 1

Let $(X, \mathcal{A}, \mu)$ be a measure space. Without any additional structure or information, we may define the Lebesgue integral $\int_X f \, d\mu$ for f an $\mathcal{A} - \mathcal{B}$ measurable function $f : X \to [-\infty, +\infty]$.
We only have a few examples without any work.

Example 0.1.
- For any set X, we can define the counting measure on $\mathcal{A} = 2^X$, which gives $\mu(A) = |A|$. If $X = \mathbb{N}$, then a measurable function is just a sequence (f_n), and $\int_X f \, d\mu = \sum f_n$
- We can also define the Dirac mass δ_p for a fixed $p \in X$ by

$$\delta_p(E) = \begin{cases} 1 & p \in E \\ 0 & p \notin E \end{cases}$$

  We have $\int_X f \, d\delta_p = f(p)$

To get another example of a measure we need to do some work.
Problem: We want a measure μ on $\mathbb{R}^n$ such that, for a rectangle,

$$\mu([a_1, b_1] \times \cdots \times [a_n, b_n]) = |a_1 - b_1| \cdots |a_n - b_n|$$

Once it is defined on all rectangles, it is defined on the minimal σ-algebra containing them, which is the Borel σ-algebra. In other words, this condition will completely specify a measure on the Borel σ-algebra $\mathcal{B}_{\mathbb{R}^n}$
If $X = \mathbb{R}^n$, or a general metric space, or even a general topological space, then $\mathcal{B}(X)$ denotes the σ-algebra generated by the open subsets of X.
Problem:
Suppose we have a distribution function $F : \mathbb{R} \to \mathbb{R}$, meaning F is monotone, positive, and $\lim_{x \to -\infty} f(x) = 0, \lim_{x \to \infty} f(x) = 1$, and continuous from the right. We want a Borel measure μ such that $F(t) = \mu((-\infty, t])$. Such a measure, denoted by λ_F, is called a Lebesgues-Stieltjes measure.
The corresponding integral is called a Lebesgue-Stieltjes integral.
If F is smooth, then $\int_{\mathbb{R}} \phi \, d\lambda_F = \int_{-\infty}^{\infty} \phi(x) d\, F(x)$.
The measure we want on $\mathbb{R}^n$ is denoted by λ^n.

The Carathéodory Construction

Suppose we have an outer measure $\gamma : 2^X \to [0, \infty]$. This means $\gamma(\varnothing) = 0$, $A \subset B \implies \gamma(A) \leq \gamma(B)$ (monotone), and $\gamma(\cup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \gamma(E_i)$ (subadditive).
We can define a set S to be γ-measurable if for every testing set T, $\gamma(T) = \gamma(S \cap T) + \gamma(S^c \cap T)$.

Theorem 0.1. *(Carathéodory Extension Theorem)*

1. $\gamma(N) = 0 \implies N$ *is measurable.*

2. *The set of measurable sets forms a σ-algebra Γ.*

3. γ *restricted to Γ forms a measure.*

"Nothing in the above theorem can guarantee you that Γ is not trivial, i.e. $\Gamma = \{\varnothing, X\}$. Nevertheless, this is a very useful guy" - Dennis.

Definition 0.1. (Lebesgue outer measure on $\mathbb{R}^n$)
Let R be a rectangle in $\mathbb{R}^n$, that is $R = \prod_{i=1}^n [a_i, b_i]$. We have $\mathrm{Vol}(R) = |a_1 - b_1| \cdots |a_n - b_n|$. For any $E \subseteq \mathbb{R}^n$, we define

$$\mu^*(E) \overset{\text{def}}{=} \inf\{\sum_{j=1}^{\infty} \mathrm{Vol}(R_j) \mid E \subseteq \cup_{j=1}^{\infty} R_j\}$$

Proposition 1. *μ^* is an outer measure on $\mathbb{R}^n$ such that $\mu^*(R) = \mathrm{Vol}(R)$ for all rectangles R.*

Proof. The first and second axioms are trivial, so we will just prove the subadditivity. Let E be some set. By definition, for any ε, there is some cover R_j by recrtangles such that

$$-\varepsilon + \sum_{j=1}^{\infty} \mathrm{Vol}(R_j) \leq \mu^*(E) \leq \sum_{j=1}^{\infty} \mathrm{Vol}(R_j)$$

meaning that $\sum_{j=1}^{\infty} \mathrm{Vol}(R_j) \leq \mu^*(E) + \varepsilon$. So for each E_k, there is a sequence R_j^k which covers E_k, such that $\sum_{j=1}^{\infty} \mathrm{Vol}(R_j^k) \leq \mu^*(E) + \frac{\varepsilon}{2^k}$.
So $\{R_j^k\}_{j,k \in \mathbb{N}}$ forms a cover of $\cup_{j=1}^{\infty} E_j$. Thus

$$\begin{aligned}
\mu^*(\cup_{k=1}^{\infty} E_k) &\leq \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \mathrm{Vol}(R_j^k) \\
&\leq \sum_{k=1}^{\infty} \left(\mu^*(E_k) + \frac{\varepsilon}{2^k} \right) \\
&= \sum_{k=1}^{\infty} \mu^*(E_k) + \varepsilon
\end{aligned}$$

This is true for any positive ε. Taking the limit as $\varepsilon \to 0$ gives the result. ■

Now, fix a rectangle R. Note that R itself forms a cover of R, so by the definition, $\mu^*(R) \leq \operatorname{Vol}(R)$. For $\varepsilon > 0$, we can take an almost-optimal cover (R_j) such that $\sum_{j=1}^{\infty} \operatorname{Vol}(R_j) \leq \operatorname{Vol}(R) + \varepsilon$. We can rig it such that $|\operatorname{Vol}(R_j) - \operatorname{Vol}(R)| \leq \frac{\varepsilon}{2^j}$. Because $R \subset \cup_{j=1}^{\infty} R_j$, and R_j is an open cover, by compactness of R there is a finite subcover, and the volume of R is less than or equal to the sum of the volumes of these finitely many R_j. So the volume of R is less than or equal to $\mu^*(R) + 2\varepsilon$. So $\operatorname{Vol}(R) = \mu^*(R)$.

Proposition 2. *Every rectangle R in $\mathbb{R}^n$ is Carathéodory measurable).*

Proof. I missed this lol. Apparently Dennis denotes $\mathcal{M}_{\lambda^*}$ by $\mathscr{L}^n$. ■

Definition 0.2. A set is said to be <u>G_δ</u> if it is the countable intersection of open sets. A set is said to be <u>F_σ</u> if it is the countable union of closed sets.

Theorem 0.2. **1.** *For all $E \in \mathscr{L}^n$, $\lambda^N(E) = \inf\{\lambda^n(O) \mid \text{open } O \supseteq E\}$.*

2. *$E \in \mathscr{L}^n$ if and only if $E = H \setminus Z$, where H is G_δ, and $\lambda^*(Z) = 0$.*

3. *$E \in \mathscr{L}^n$ if and only if $E = H \cup Z$, where H is F_σ and $\lambda^*(Z) = 0$.*

4. $\lambda^n(E) = \sup\{\lambda^n(C) \mid \text{closed } C \subseteq E\}$

Proof. It suffices to prove the first statement, as the others will follow by passing to a complement. ■

Definition 0.3. Suppose X is a metric space. A measure on X is a <u>Radon measure</u> if it is Borel (meaning defined on a σ-algebra containing Borel sets), and for any Borel E, $\mu(E) = \inf\{\mu(O) \mid \text{open } O \supseteq E\}$, and for any compact $C \subseteq X$, $\mu(C) < \infty$.

Theorem 0.3. *(Riesz)*
Let $X \subseteq \mathbb{R}^n$ be compact. Let $C(X)$ denote the vector space of all continuous functions on X. This admits a norm $\|f\|_{C(X)} = \sup_X |f|$, making it a Banach space. Define $C^*(X) = \{\phi : C(X) \to \mathbb{R}, \phi \text{ is linear and continuous }\}$.
For all $\phi \in C^*(X)$, there exists a Radon measure $\mu = \mu_+$, and a function $M : X \to \{\pm 1\}$ which is Borel, such that

$$\phi(f) = \int_X f(x) M(x) \, d\mu(x)$$

for all $f \in C(X)$.

Proof. ■

Lecture 2, 1/17/23

Note: This is the first lecture with Davit. Davit will always use μ to refer to an outer measure, not a measure. The book will be "Measure theory and fine properties of functions." According to Davit, this is the correct book to be using.

Definition 0.4. Let X be a nonempty set. A mapping $\mu : 2^X \to [0, +\infty]$ is called a measure if it satisfies the following 2 properties.

1. $\mu(\varnothing) = 0$.

2. (Countable subadditivity and monotonicity) If $A, A_1, A_2, \cdots \subseteq X$ and $A \subseteq \cup_{i=1}^{\infty} A_i$ then $\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$

Remark. From the second definition, we can automatically get monotonicity, i.e. if $A \subseteq B$, then $\mu(A) \leq \mu(B)$. This is because, as written, the second definition is a statement not just about $\cup_{i=1}^{\infty} A_i$, but about any subset of it. Indeed, let $A = A, A_1 = B$, and $A_k = \varnothing$ for $k \geq 2$. Then we have $\mu(A) \leq \mu(\cup_{i=1}^{\infty} A_i) = \mu(A \cup B)$.
We will write "μ is a measure on X" to mean that μ satisfies the above definition (that is, μ is an outer measure).

Definition 0.5. Let X be a nonempty set and let μ be a measure on X. For a fixed set $C \subseteq X$, define the restriction measure $\nu = \mu|_C$ by $\nu(A) = \mu|_A(A) = \mu(A \cap C)$.

Remark. It is easy to prove that $\mu|_C$ is a measure on X.

Definition 0.6. (Carathéodory's condition). Let X be a nonempty set and let μ be a measure on X. A subset $A \subseteq X$ is called μ-measurable if, for all subset $B \subseteq X$, we have

$$\mu(B) = \mu(B \cap A) + \mu(B \setminus A)$$

Remark. X and $\varnothing$ are easily seen to be μ-measurable.

Theorem 0.4. *(Carathéodory extension theorem)*
The collection of μ-measurable sets on a set X is a σ-algebra.

Theorem 0.5. *Let X be a nonempty set and let μ be a measure on X. Then the following holds:*

1. *$\varnothing$ and X are μ-measurable.*

2. *$A \subseteq X$ is μ-measurable if and only if $X \setminus A$ is μ-measurable.*

3. *If $A \subseteq X$ is such that $\mu(A) = 0$, then A is μ-measurable.*

4. *Let $C \subseteq X$. Then anything which is μ-measurable is $\mu|_C$-measurable.*

Remark. A measure is also finitely subadditive, which says that if $A \subseteq A_1 \cup \cdots \cup A_n$, then $\mu(A) \leq \sum_{i=1}^n \mu(A_i)$. So, to check that $\mu(B) = \mu(B \cap A) + \mu(B \setminus A)$, it will suffice to check

$$\mu(B) \geq \mu(B \cap A) + \mu(B \setminus A)$$

Proof. Part 1 is obvious.
Suppose that A is μ-measurable. Then $\mu(B \cap A) = \mu(B \setminus A^c)$ and $\mu(B \cap A^c) = \mu(B \setminus A)$ so $\mu(B \cap A) + \mu(B \setminus A) = \mu(B \cap A^c) + \mu(B \setminus A^c)$. So A is μ-measurable if and only if A^c is.
Suppose that $\mu(A) = 0$. Then $\mu(B \cap A) \leq \mu(A), \mu(B)$, so $\mu(B \cap A) = 0$ for any $B \subseteq X$. Now, $B \setminus A \subseteq B$, so by monotonicity $\mu(B \setminus A) \leq \mu(B)$. So $\mu(B \cap A) + \mu(B \setminus A) \leq \mu(B)$ for all $B \subseteq X$, so we are done.
Let A be μ-measurable. Then for any $B \subseteq X$ we have

$$\begin{aligned}
\nu(B) &= \mu|_C(B) = \mu(B \cap C) \\
&= \mu((B \cap C) \cap A) + \mu((B \cap C) \setminus A) \\
&= \nu(B \cap A) + \mu((B \setminus A) \cap C) \\
&= \nu(B \cap A) + \nu(B \setminus A)
\end{aligned}$$

■

Theorem 0.6. *Let X be a nonempty set and let μ be a measure on X.*
Assume $A_1, A_2, \ldots, A_n \subseteq X$ are μ-measurable. Then

1. $\bigcup_{k=1}^n A_k$ *and* $\bigcap_{i=1}^n A_k$ *are also μ-measurable.*

2. *If the A_i are disjoint, then* $\mu(\bigcup_{i=1}^n A_i = \sum_{i=1}^n \mu(A_i)$

Proof. We prove part 2 first. Because each A_i is measurable,

$$\begin{aligned}
\mu(\cup_{k=1}^n A_k) &= \mu((\cup_{k=1}^n A_k) \cap A_n) + (\mu(\cup_{k=1}^n A_k) \setminus A_n) \\
&= \mu(\cup_{i=1}^{n-1} A_k) + \mu(A_n) = \cdots = \sum_{k=1}^n \mu(A_k)
\end{aligned}$$

Now we prove part 1. Let $A, B \subseteq X$ be μ-measurable and disjoint. Then for any $C \subseteq X$, $\mu(C) = \mu(C \cap A) + \mu(C \setminus A)$, and similarly for B. This is equal to

$$\begin{aligned}
\mu(C) &= \mu(C \cap A) + \mu((C \setminus A) \cap B) + \mu(C \setminus A \setminus B) \\
&= \mu(C \cap A) + \mu(C \cap B) + \mu(C \setminus (A \cup B)) + \mu(C \cap (A \cup B))(?) \\
&= \mu(C \cap (A \cup B) \cap A) + \mu(C \cap (A \cup B) \setminus A) \\
&= \mu(C \cap A) + \mu(C \cap B) \\
&= \mu(C \cap (A \cup B)) + \mu(C \setminus (A \cup B))
\end{aligned}$$

So $A \cup B$ is μ-measurable. (I got a bit lost in the arithmetic, sorry)
Next, if $A, B \subseteq X$ are μ-measurable, then $A \cap B$ is μ-measurable.