Lecture 1, 1/11/13

Section 1: Vocabulary and easy definitions

Homological algebra is the study of complexes of R-modules, where R is a ring with identity $1 \neq 0$. Notationally, R-Mod is the category of all left R-modules, and R-mod is the category of all finitely generated R-modules.

Definition 0.1. Let A_n“ $\in$ ”R-mod for $n \in \mathbb{Z}$ and $d_n \in \mathrm{Hom}_R(A_n, A_{n-1})$ such that $d_{n-1} \circ d_n = 0$ for all $n \in \mathbb{Z}$. Then the sequence

$$\cdots \xrightarrow{d_{n+2}} A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} \cdots$$

is called a complex of R-modules, assuming $\mathrm{im}(d_n) \subseteq \ker(d_{n-1})$. The sequence

$$0 \longrightarrow A_m \longrightarrow A_{m-1} \longrightarrow \cdots \longrightarrow A_{n+1} \longrightarrow A_n \longrightarrow 0$$

will occur more frequently. A complex $\mathbb{A}$ is an exact sequence if $\mathrm{im}(d_n) = \ker(d_{n-1})$ for all $n \in \mathbb{Z}$. This is called a short exact sequence if there are no more than 3 non-zero terms. Given a complex $\mathbb{A}$, the nth homology modules (or groups, in some cases) of $\mathbb{A}$ is

$$H_n(\mathbb{A}) = \frac{\ker(d_{n-1})}{\mathrm{im}(d_n)}$$

Remark. Given a short exact sequence (hereby abbv. as SES)

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

f is a mono and g is an epi, so $C \simeq B/\mathrm{im}(f)$. If A, B are known, but not f, then infinitely many C are available to complete the short exact sequence.

Example 0.1. Let $R = k$, a field, and take $A = B = k^{(\mathbb{N})} = \oplus_{i\in\mathbb{N}} k$.

(i) $0 \longrightarrow A \xrightarrow{\mathrm{Id}} B \longrightarrow 0$ is a SES.

(ii) Define $f : A \to B$ by

$$f(b_i) = b_{2i} \text{ for } i \in \mathbb{N}$$
$$g(b_0) = \begin{cases} 0 & i \text{ even} \\ b_{\tau(i)} & i \text{ odd} \end{cases}$$

Where $\tau : (2\mathbb{N} - 1) \to \mathbb{N}$ is a bijection. If $A = B = C = \kappa^{(\mathbb{N})}$, then

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a SES.

(iii) Let $R = \mathbb{Z}$. Then

$$0 \longrightarrow 3\mathbb{Z} \xrightarrow{\iota} \mathbb{Z} \xrightarrow{\overline{\cdot}} \mathbb{Z}/3\mathbb{Z} \longrightarrow 0$$

is a SES.

(iv) Let $R = \mathbb{Z}$. The sequence

$$0 \longrightarrow \overbrace{6\mathbb{Z}}^{A_1} \xrightarrow{\iota} \overbrace{\mathbb{Z}}^{A_0} \xrightarrow{\overline{\cdot}} \overbrace{\mathbb{Z}/3\mathbb{Z}}^{A_{-1}} \longrightarrow 0$$

is a complex which is not exact. In fact, $H_0(\mathbb{A}) = \overbrace{3\mathbb{Z}}^{\ker(g)} / \underbrace{6\mathbb{Z}}_{\mathrm{im}(f)} \cong \mathbb{Z}/2\mathbb{Z}$.

(v) Let $R = \kappa[x, y]$, κ a field. Let f be the inclusion $(x) \hookrightarrow R[x, y]$. The sequence

$$0 \longrightarrow (x) \xrightarrow{f} R \xrightarrow{g} \kappa[y] \longrightarrow 0$$

where

$$g\left(\sum_{i,j=0}^{\text{finite}} a_{ij} x^i y^j\right) = \sum_{j>0}^{\text{finite}} a_{\sigma_j} y^j$$

is exact.

(vi) Let $R = \kappa[x, y]$. Define $\mathbb{A}$ as

$$0 \longrightarrow \overbrace{(x)}^{A_1} \xrightarrow{f} \overbrace{R}^{A_0} \xrightarrow{g} \overbrace{\underbrace{\kappa}_{=R/(x,y)}}^{A_{-1}} \longrightarrow 0$$

where

$$g\left(\sum_{i,j=0}^{\text{finite}} a_{ij} x^i y^j\right) = a_\infty$$

then $\ker(g) = (x, y)$ and $\mathrm{im}(f) = (x)$, so $\mathbb{A}$ is not exact. In fact,

$$\begin{aligned} H_0(\mathbb{A}) &= (x, y)/(x) \\ &\simeq (y) \\ &\simeq R \end{aligned}$$

Note: If R is an integral domain and $x \in R \setminus \{0\}$, then $(x) \simeq R$ (as R-modules, not as rings!), with isomorphism $r \mapsto rx$.

Typical questions addressed by homological algebra:

(i) Suppose

$$\mathbb{A}: \quad \cdots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \longrightarrow \cdots$$

is an exact sequence in R-mod and $F : R - \mathrm{mod} \to S - \mathrm{mod}$ is a functor. Is the sequence

$$\cdots \longrightarrow F(A_{n+1}) \xrightarrow{F(d_{n+1})} F(A_n) \xrightarrow{F(d_n)} F(A_{n-1}) \longrightarrow \cdots$$

exact? $F(\mathbb{A})$ is a complex when F is additive, but it may or may not be exact.

(ii) Given A, C "$\in$" $R - \mathrm{mod}$, characterize all modules B such that there exists an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

As an example, $R = \mathbb{Z}$, $A = C = \mathbb{A}/p\mathbb{Z}$, p prime, then

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \xrightarrow{f} \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \xrightarrow{g} \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

with $f : x \mapsto (x, 0)$ and $g : (x, y) \mapsto y$ is a SES. Alternatively, we could take $f : x + p\mathbb{Z} \mapsto px + p^2\mathbb{Z}$ and $g : y + p^2\mathbb{Z} \mapsto y + p\mathbb{Z}$ to make

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \xrightarrow{f} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{g} \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

a SES. These are the only possibilities in this case! In general, though, there are infinitely many possibilities for B. Why is this interesting? If R is an artinian ring and M "$\in$" $R - \mathrm{mod}$, then there are only finitely many simple $s_1, \dots, s_n$ "$\in$" $R-\mathrm{mod}$ up to isomorphism. Moreover, for M "$\in$" $R-\mathrm{mod}$, there is a chain

$$M = M_0 \supset M_1 \supset \cdots \supset M_\ell = 0$$

such that M_i/M_{i+1} is simple for all $i < \ell$. If the answer to question (ii) is known, then all objects in $R - \mathrm{mod}$ of fixed length ℓ are known up to isomorphism! Simply proved by induction.

Algebraic Topology

Definition 0.2. The standard n-simplex $\triangle_n$ in $\mathbb{R}^n$ is the convex hull of $v_0, v_1, \ldots, v_n$, where $v_0 = 0$ and $v_i = (0, \ldots, 0, \overbrace{1}^{i}, 0, \ldots, 0)$ (so the standard basis).
An oriented simplex is $(\triangle_x, [\pi])$, where $[\pi]$ is an equivalence class of permutations of $\{0, \ldots, n\}$, where $\pi \sim \pi' \iff \mathrm{sgn}(\pi) = \mathrm{sgn}(\pi')$. We write

$$(\triangle_x, \pi) = [v_{\pi(0)}, v_{\pi(1)}, \ldots, v_{\pi(n)}$$

and identify $\triangle_n$ with $[0, 1, \ldots, n]$. The ngative is $-[w_0, \ldots, w_n]$.

Definition 0.3. Let X be a topological space. An n-simplex in X is a continuous map

$$\sigma : \triangle_n \to X$$

The group of n-chains of X, $S_n(x)$, is the free abelian group having as basis the n-simplices in X. The singular chain complex of X is

$$\cdots \longrightarrow S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots \longrightarrow S_0(X) \longrightarrow 0$$

denoted $\mathbb{S}$, where $\partial_n : S_n(X) \to S_{n-1}(X)$ is the nth boundary map, which can be defined if we define $\partial_n(\sigma)$ for all n-simplices σ in X (i.e. in the basis of $S_n(X)$). Consider the map

$$\begin{aligned} \tau_i : \mathbb{R}^{n-1} &\to \mathbb{R}^n \\ (a_1, \ldots, a_{n-1}) &\mapsto (a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_n) \end{aligned}$$

For $i \in \{0, \ldots, n\}$. Then τ_i is continuous and $\tau_i(\triangle_{n-1}) = \triangle_n$. Define

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma(\tau_i)$$

Theorem 0.1. *$\partial_{n-1} \circ \partial_n = 0$ for all $n \in \mathbb{N}$, i.e. $\mathbb{S}$ is a complex in $\mathbb{Z}$-mod.*

Definition 0.4. The group of n-cycles is $Z_n(X) = \ker(\partial_{n-1})$, and the group of n-boundaries is $B_n = \mathrm{im}(\partial_n)$.
The nth homology group is $H_n(X) = Z_n(X)/B_n(X)$.

Lecture 2, 1/13/23

Chapter I: Categories and functors

There is a definition page on the Gaucho that has all the most basic definitions - objects, morphisms, compositions, etc.

If $f \in \text{Hom}_C(A, B)$, we often write $A \xrightarrow{f} B$ even if f is not literally a map.

Example 0.2. **1.** The category of all sets, Set. The object class consists of all sets, and the morphisms are just set maps.

2. The category of all topological spaces, Top. The object class consists of all topological spaces, and the morphisms are continuous functions.

3. The category of all groups, Grp. The object class consists of all groups, and the morphisms are group homomorphisms.

4. Let $(P, \leq)$ be a partially ordered set with a relation $\leq$ which is reflexive, antisymmetric, and transitive. Then we can make P into a category, whose objects are the elements of p, and for $u, s \in P$, $\text{Hom}_P(u, s) = \begin{cases} (u, s) & u \leq s \\ \varnothing & u \not\leq s \end{cases}$. We define the composition $(s, t)(u, s) \stackrel{\text{def}}{=} (u, t)$.

5. The opposite category of a category C, C^{op}.

6. Let R be a ring. R-Mod is the category of left R modules. R-mod is the finitely generated R-modules, and similarly for Mod-R and mod-R, which are the right R-modules.

7. R-comp. The object class consists of complexes of left R-modules.

Let $\mathbb{A}, \mathbb{A}'$ be objects of R-comp. Note: it is problematic to say "$\mathbb{A}, \mathbb{A}' \in R$-comp, as R-comp is not a set!

Say $\mathbb{A} = \cdots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \longrightarrow \cdots$, and similarly for $\mathbb{A}'$. An element of $\text{Hom}_{R-comp}(\mathbb{A}, \mathbb{A}')$ will be a sequence of R-module homomorphisms $f_n : A_n \to A'_n$ which make the following diagram commute:

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & A_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \arrow[d] \\
\cdots \arrow[r] & A'_n \arrow[r, "d'_n"'] & A'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

8. The category of rings Ring, whose obejcts are rings and whose morphisms are ring homomorphisms.

9. The category of $\mathbb{Z}$-modules is usually denoted Ab. This is also the category of Abelian groups, and is the prototypical example of an Abelian category.

Definition 0.5. A category $\mathcal{C}$ is called pre-additive if for all A, B objects of $\mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is an additive Abelian group (additive means we use the symbol "+") such that for all eligible morphisms f, g, h, k,

$$h(f+g) = hf + hg$$
$$(f+g)k = fk + gk$$

where "elibigle" means that these expressions make sense and are well-defined.

Example 0.3. **1.** R-mod (in particular Ab)

2. R-comp

3. Ring fails to be pre-additive, because the identity morphisms add to be something which is not the identity morphism.

Definition 0.6. Let $\mathcal{C}, \mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ consists of an assignment $F_0 : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$, and for each pair of objects A, B"$\in$"$\mathrm{Obj}(\mathcal{C})$, a map (this actually is a map because we assume hom-sets are in fact sets). $F_{A,B} : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ such that, for all eligible morphisms f, g, and all A"$\in$"$\mathcal{C}$

(a) $F(\mathrm{Id}_A) = \mathrm{Id}_{F(A)}$

(b) $F(f \circ g) = F(f) \circ F(g)$

Example 0.4. **1.** Let $\mathcal{C}$ be a category. Then we have the identity functor $\mathrm{Id}_{\mathcal{C}}$, which assigns $\mathrm{Id}_{\mathcal{C}}(A) = A$, and $\mathrm{Id}_{\mathcal{A}}(f) = f$ for any eligible A"$\in$"$\mathrm{Obj}(\mathcal{D})$ and morphisms f.

2. Functors $\pi_n : \mathsf{Top} \to \mathsf{Grp}$ which sends $X \mapsto \pi_n(X)$

3. $\mathbb{S} : \mathsf{Top} \to \mathbb{Z}$-comp, which sends $X \mapsto \mathbb{S}(X)$, which is a complex

$$\cdots \longrightarrow S_{n+1}(X) \xrightarrow{\partial_{n+1}} S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots \longrightarrow S_0(x) \xrightarrow{\partial_0} 0$$

Let $\phi : X \to Y$ be continuous for X, Y"$\in$"Top. Then $\mathbb{S}(\phi)_n : S_n(X) \to S_n(Y)$ is given by $\sigma \mapsto \phi \circ \sigma$, and we can extend this for σ an n-simplex of X.

Lecture 4, 1/18/23

Functors:

Definition 0.7. Let $\mathcal{C}, \mathcal{D}$ be categories. A covariant functor from $\mathcal{C}$ to $\mathcal{D}$ consists of "maps" F_0 and $F|_{A,B}$ for any $A, B \in \text{Obj}(\mathcal{C})$ such that

- $F_0 : \text{Obj}(\mathcal{C}) \to \text{Obj}(\mathcal{D})$
- $F_{A,B} : \text{Hom}_{\mathcal{C}}(A, B) \to \text{Hom}_{\mathcal{D}}(F_0 A, F_0 B)$ for any A, B"$\in$"$\text{Obj}(\mathcal{C})$

such that

(a) $F_{A,C}(fg) = F_{B,C}(f)F_{A,B}(g)$ for all eligible f, g

(b) $F_{A,A}(\text{Id}_A) = \text{Id}_{F(A)}$

from here on we don't care at all about indices. For simplicity, we will denote the action of a functor F as simply FA or Ff.

Definition 0.8. A contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ amounts to a covariant functor from $\mathcal{C}$ to $\mathcal{D}^{\text{op}}$.
More examples of functors

Example 0.5. Homology functors $H_n : R - comp \to \mathbb{Z} - mod$ which sends $\mathbb{A}$ to $H_n(A)$.
That is, $\mathbb{A} \to F\mathbb{A} = \frac{\ker(d_n)}{\text{Im}(d_{n+1})}$
Let $f \in \text{Hom}_{R-comp}(\mathbb{A}, \mathbb{A}')$. That is, the following diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & A_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \arrow[d] \\
\cdots \arrow[r] & A'_n \arrow[r, "d'_n"'] & A'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Ff acts by $a_n + \text{Im}(d_{n+1}) \to f_n(a_n) + \text{Im}(d'_{n+1})$. Let's prove that this is actually well-defined.
Check
First, $a_n \in \ker(d_n)$ implies $f_n(a_n) \in \ker(d'_n)$. This can be seen by doing a diagram chase on the above diagram. Since $d_n(a_n) = 0$, we have $0 = f_{n-1}d_n(a_n) = d'_n f_n(a_n)$, i.e. $f_n(a_n) \in \ker(d'_n)$.
"Don't do much thinking. It's almost harmful" - Birge on doing diagram chasing.
Also "follow your nose."
Now, $a_n \in \text{Im}(d_{n+1})$ implies $f_n(a_n) \in \text{Im}(d'_{n+1})$. So $a_n = d_{n+1}(x)$ with $x \in A_{n+1}$. hence $f_n(a_n) = f_n d_{n+1}(x) = a'_{n+1} f_{n+1}(x) \in \text{Im}(d'_{n+1})$.

Example 0.6. Let $\mathcal{C}, \mathcal{D}$ be pre-additive categories (definition on the top of page 6). A functor F "from" $\mathcal{C}$ to $\mathcal{D}$ is called <u>additive</u> if, for all A, B" $\in$ "$\text{Obj}(\mathcal{C})$, the map $F : \text{Hom}_{\mathcal{C}}(A, B) \to \text{Hom}_{\mathcal{D}}(F(A), F(B))$ is a homomorphism of abelian groups.

Remark. Note that $H_n : R - comp \to \mathbb{Z} - mod$ is an additive functor. The π_n functor is <u>not</u> additive, as Top is not preadditive.

Example 0.7. <u>Forgetful functors</u> e.g. $F : R - mod \to \mathbb{Z} - mod$ which sends $M \mapsto M$, where the M on the left hand side is an R-module, and M on the right is just an abelian group, which is a $\mathbb{Z}$-module. Or $F : R - mod \to \mathsf{Set}$ which sends an R-module M to the set of its elements, "forgetting" the module structure.
Moreover, if $\mathcal{C}, \mathcal{D}$ are pre-additive, and $F : \mathcal{C} \to \mathcal{D}$ is a forgetful functor of some sort, then F is additive.

Example 0.8. Let $F : R - mod \to S - mod$ be an additive functor. Then F induces an additive functor $\tilde{F} : R - comp \to S - comp$, sending $\mathbb{A}$ to $F(\mathbb{A})$.
If $\mathbb{A}$ is a complex

$$\cdots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \longrightarrow \cdots$$

then $F(\mathbb{A})$ is

$$\cdots \longrightarrow F(A_n) \xrightarrow{F(d_{n+1})} F(A_n) \xrightarrow{F(d_n)} F(A_{n-1}) \longrightarrow \cdots$$

An extremely important question: if $\mathbb{A}$ is exact, is $F(\mathbb{A})$ exact? If not, how far does it deviate from being an exact sequence?

Example 0.9. Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{E}$ be functors. Then $G \circ F : \mathcal{C} \to \mathcal{E}$ is a functor. <u>WARNING:</u> we use $\circ$ but this isn't actually a function composition. This is just notation!!!
$G \circ F$ acts how one might think: for A" $\in$ "$\text{Obj}(\mathcal{C})$, $G \circ F(A) = G(F(A))$, and for $f \in \text{Hom}_{\mathcal{C}}(A, B)$, $G \circ F(f) = G(F(f)) \in \text{Hom}_{\mathcal{E}}(G(F(A)), G(F(B)))$.
Of interest to us: $H_n \circ \tilde{F}$, where $F : R - mod \to S - mod$ is additive. This functor sends a complex $\mathbb{A}$ to $H_n(F(\mathbb{A}))$. This is especially of interest if $\mathbb{A}$ is exact, but $F(\mathbb{A})$ is not.

Remark. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F sends isomorphisms in $\mathcal{C}$ to isomorphisms in $\mathcal{D}$. This is immediate from the definition of a functor.

Section 2: two types of functors that will follow us

(i) Hom-functors: Whenever $\mathcal{C}$ is a category, there is a bifunctor

$$\text{Hom}_{\mathcal{C}}(-, -) : \mathcal{C} \times \mathcal{C} \to \mathsf{Set}$$

which sends a pair (A, B) to $\text{Hom}_{\mathcal{C}}(A, B)$, and on maps (note that this is co-variant in the first factor and contravariant inh the second), they act as follows. Let $f : A \to A', g : B \to B'$ be morphisms in $\mathcal{C}$. Then

$$\text{Hom}(f, g) : \text{Hom}_{\mathcal{C}}(A', B) \to \text{Hom}_{\mathcal{C}}(A, B')$$

acts by $\phi \mapsto g \circ \phi \circ f$

Lecture 5, 1/20/23

Whenever $\mathcal{C}$ is a category, there is a bifunctor $\text{Hom}_{\mathcal{C}}(-,-) : \mathcal{C} \times \mathcal{C} \to \mathsf{Set}$, which sends (A, B) to $\text{Hom}_{\mathcal{C}}(A, B)$. On maps, when $f : A \to A'$ and $g : B \to B'$ are morphisms, then

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(f, g) : \text{Hom}_{\mathcal{C}}(A', B) &\to \text{Hom}_{\mathcal{C}}(A, B') \\ \varphi &\mapsto g \circ \varphi \circ f \end{aligned}$$

We will split this into two parts. Let $C“\in”\mathcal{C}$. Then we have a covariant functor

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(C, -) : \mathcal{C} &\to \mathcal{C} \\ C' &\mapsto \text{Hom}_{\mathcal{C}}(C, C') \\ g &\mapsto \text{Hom}_{\mathcal{C}}(C, A) \to \text{Hom}_{\mathcal{C}}(C, B) \\ \varphi &\mapsto g \circ \varphi \end{aligned}$$

We also have the contravariant functor $\text{Hom}_{\mathcal{C}}(-, D)$, which acts similarly.
As a special case, consider $\mathcal{C} = R - mod$. Then

$$\begin{aligned} \text{Hom}_R(M, -) : R - mod \to \mathbb{Z} - mod \\ \text{Hom}_R(-, N) : R - mod \to \mathbb{Z} - mod \end{aligned}$$

but we can have additional structure on $\text{Hom}_R(M, N)$. Suppose ${}_RM_S$ is a bimodule (S is a ring and $(rm)s = r(ms)$) and let ${}_RN_T$ be an $R-T$ module. Then $\text{Hom}_R(M, N)$ is a left S, right T bimodule. For $f \in \text{Hom}_R(M, n)$, $s \in S, t \in T$, define

$$\begin{aligned} (sf)(m) = f(ms) \\ (ft)(m) = f(m)t \end{aligned}$$

If R is commutative, then

$$\begin{aligned} \operatorname{Hom}_R(M,-) &: R-mod \to R-mod = Mod-R \\ \operatorname{Hom}_R(-,N) &: R-mod \to R-mod = Mod-R \end{aligned}$$

If ${}_RM_S$ is a bimodule, then

$$\operatorname{Hom}_R(M,-) : R-mod \to S-mod$$

If ${}_RN_T$ is a bimodule, then

$$\operatorname{Hom}(-,N) : R-mod \to Mod-T$$

Basic properties:

(i) $M\text{``}\in\text{''}R-mod \implies \underbrace{\operatorname{Hom}_R(R,M) \cong M}_{f\mapsto f(1)}$ in $\mathbb{Z}-mod$

(ii) $\operatorname{Hom}_R(\otimes_{i\in I}M_i, N) \cong \prod_{i\in I}\operatorname{Hom}_R(M_i,N)$. Prove this!

(iii) $\operatorname{Hom}_R(M, \prod_{i\in I}N_i) \cong \prod_{i\in I}\operatorname{Hom}_R(M,N_i)$. Prove this!

Definition 0.9. Let $M\text{``}\in\text{''}Mod-R$, $N\text{``}\in\text{''}R-mod$. Then an abelian group T is called a tensor product of M and N if there exists a map

$$\tau : M\times N \to T$$

Which is $\mathbb{Z}$-bilinear and R-balanced, i.e.

$$\tau(mr,n) = \tau(m,rn)$$

with the following universal property.
Whenever A is an abelian group and $\sigma : M\times N \to A$ is $\mathbb{Z}$-bilinear and R-balanced, there exists a unique $\mathbb{Z}$-linear map $\sigma' : T \to A$ such that this diagram commutes:

$$\begin{tikzcd} M\times N \arrow[r, "\tau"] \arrow[dr, "\sigma"'] & T \arrow[d, "\sigma'"] \\ & A \end{tikzcd}$$

We denote $T = M\otimes_R N$.

Theorem 0.2. *If $M\text{``}\in\text{''}Mod-R$ and $N\text{``}\in\text{''}R-mod$, then a tensor product $M\otimes_R N$ exists and is unique up to isomorphism.*

Proof. Let F be the free abelian group with basis $M \times N$, i.e.

$$F = \otimes_{m \in M, n \in N} \mathbb{Z}(m, n)$$

Define

$$M \otimes_R N = F/U$$

where U is the submodule generated by all elements of the form

$$\begin{aligned}(m_1 + m_2, n) - (m, n) - (m_2, n)\\ (m, n_1 + n_2) - (m, n_1) - (m, n_2)\\ (mr, n) - (m, rn)\end{aligned}$$

for all eligible $m_i, m \in M, n_i, n \in N, r \in R$.
Define

$$\begin{aligned}\tau :&M \times N \to M \otimes_R N\\ &(m, n) \mapsto m \otimes n\end{aligned}$$

Then τ is $\mathbb{Z}$-bilinear and R-balanced (check!). Moreover, $M \otimes_R N$ with τ satisfies the universal proeprty: let A be an abelian group and $\sigma : M \times N \to A$ be $\mathbb{Z}$-bilinear and R-balanced. Define

$$\begin{aligned}\tilde{\sigma} :&F \to A\\ &(m, n) \mapsto \sigma(m, n)\end{aligned}$$

and extend linearly. By construction, $\tilde{\sigma}(U) = 0$, i.e. $U \subseteq \ker(\tilde{\sigma})$. Hence there exists $\sigma' : F/U \to A$ with the property that

$$\sigma'(m, n) = \tilde{\sigma}((m, n) + n) = \tilde{\sigma}(m \otimes n)$$

Now show σ' is unique, and the proof is complete. ∎

Lecture 6, 1/23/23

Our two mainstay types of functors:

(i) Hom functors.

(ii) Tensor functors. For (M, N) "$\in$" $Mod-R \times R-mod$, we constructed an abelian group $M \otimes_R N = R^{(M \times N)}/u$, together with $\tau : M \times N \to M \otimes_R N$ given by $\tau(m, n) = m \otimes n = (m, n) + u$ such that $(M \otimes_R N, \tau)$ has the key universal property.

Note: The elements $m \otimes n \in M \times N$ form a generating set of $M \otimes_R N$, but not a basis.

The tensor functor

We have a bifunctor $- \otimes - : Mod - R \times R - mod \to \mathbb{Z} - mod$, $(M, N) \mapsto M \otimes_R N$. Let $(f, g), f \in \mathrm{Hom}_R(M, M'), g \in \mathrm{Hom}(N, N')$. Then

$$\begin{aligned} f \otimes g : & M \otimes_R N \to M' \otimes_R N' \\ & m \otimes n \mapsto f(m) \otimes g(n) \end{aligned}$$

To show this is well-defined, check that $\phi : M \times N \to M' \otimes N'$, $(m, n) \mapsto f(m) \otimes g(n)$ is $\mathbb{Z}$-bilinear and R-balanced.
Split $- \otimes_R -$ into two functors. So, we have a functor $M \otimes_R - : R - mod \to \mathbb{Z} - mod$ and a functor $- \otimes_R N : Mod - R \to \mathbb{Z} - mod$. The action on objects and morphisms is clear from the discussion up to now.

Additional structure on $M \otimes_R N$

Suppose ${}_S M_R$ and ${}_R N_T$ are bimodules. Then $M \otimes_R N$ is a $S - T$ bimodule, with

$$s(m \otimes n)t = (sm) \otimes (nt)$$

It is an exercise to check well-definedness.
Uses
Suppose ${}_{\mathbb{R}}V$ is a real vector space. We want to "complexify" V, making it a complex vector space. We could consider $\mathbb{C} \times V$, and define $c(d, v) = (cd, v)$. But this does not define a $\mathbb{C}$-vector space, because multiplication must be multilinear. But $\mathbb{C} \otimes_{\mathbb{R}} V$ will do it.
Basic properties
Consider $R \otimes_R M$. This is in fact isomorphic to M. Not just as Abelian groups, but as left R-modules. This is because R satisfies the associative law relative to multiplication. One isomorphism between them is $m \mapsto 1 \otimes m$.
In general, unlike the hom-functor, the tensor functor will not commute with direct products/coproducts, unless "the sky is very benevolent."

The meaning of $m \otimes n$ depends on the meaning of M, N!

Example 0.10. Consider $2 \otimes \overline{1} \in \mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})$. This is the same as $1 \otimes \overline{2} = 1 \otimes 0 = 0$. By contrast, look at $2 \otimes \overline{1} \in 2\mathbb{Z} \otimes (\mathbb{Z}/2\mathbb{Z})$. This is nonzero! Let's show that. We know $2\mathbb{Z} \cong \mathbb{Z}$, with an isomorphism given by $x \mapsto \frac{x}{2}$. So

$$\begin{aligned} f \otimes \mathrm{Id}_{\mathbb{Z}/2\mathbb{Z}} :& 2\mathbb{Z} \otimes (\mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z} \otimes (\mathbb{Z}/2\mathbb{Z}) \\ & x \otimes y \mapsto f(x) \otimes y \end{aligned}$$

But functors take isomorphisms to isomorphisms, so $\underbrace{2 \otimes \overline{1}}_{\in 2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z}} \mapsto 1 \otimes \overbrace{\overline{1}}^{\in \mathbb{Z}/2\mathbb{Z}} \neq 0$. Why is this last term nonzero? Because $\mathbb{Z} \otimes (\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}$, with the isomorphism sending $1 \otimes \overline{1}$ to $\overline{1}$, which is not zero.

Natural Transformations, Equivalences, and Dualities

Definition 0.10. **1.** Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A morphism of functors, or a natural transformation from F to G, is a family $(\phi(C))_{C\text{“}\in\text{”}\mathrm{Obj}(\mathcal{C})}$ of morphisms, $\phi(C) : F(C) \to G(C)$ such that for any $f \in \mathrm{Hom}_{\mathcal{C}}(C, C')$, the square

$$\begin{tikzcd} F(C) \arrow[r, "F(f)"] \arrow[d, "\phi(C)"'] & F(C') \arrow[d, "\phi(C)"] \\ G(C) \arrow[r, "G(f)"'] & G(C') \end{tikzcd}$$

commutes for all eligible morphisms f in the category $\mathcal{C}$. This is a covariant equivalence. A contravariant equivalence is an equivalence between contravariant functors, i.e. it makes the following square commute.

$$\begin{tikzcd} F(C) \arrow[d, "\phi(C)"'] & F(C') \arrow[l, "F(f)"'] \arrow[d, "\phi(C)"] \\ G(C) & G(C') \arrow[l, "G(f)"] \end{tikzcd}$$

2. Call $(\phi(C))_{C\text{“}\in\text{”}\mathrm{Obj}(\mathcal{C})}$ an isomorphism of functors, or a natural equivalence, if $\phi(C)$ is an isomorphism for each C“ $\in$ ”$\mathrm{Obj}(\mathcal{C})$.

3. Two categories $\mathcal{C}, \mathcal{D}$ are equivalent categories if there are functors $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F \simeq \mathrm{Id}_{\mathcal{C}}$ and $F \circ G \simeq \mathrm{Id}_{\mathcal{D}}$, with " $\simeq$ " meaning "is naturally equivalent to." The F, G are called "mutually inverse equivalences."

4. A contravariant equivalence is called a duality.

5. Let R, S be rings. Call R, S Morita equivalent, denoted $R \sim S$, if $R{-}mod, S{-}mod$ are naturally equivalent. This is equivalent to saying $mod - R, mod - S$ are equivalent.

Lecture 7, 1/25/23

Definition 0.11. Let $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$ be functors. We say that (F, G) form an adjoint pair if the following two bifunctors $\mathcal{C} \times \mathcal{D} \to \mathsf{Set}$ are naturally isomorphic:

$$\mathrm{Hom}_{\mathcal{D}}(F(-), -) \cong \mathrm{Hom}_{\mathcal{C}}(-, G(-))$$

That is, for every (C, D)"$\in$"$\mathcal{C} \times \mathcal{D}$, we have an isomorphism

$$\phi(C, D) : \mathrm{Hom}_{\mathcal{D}}(F(C), D) \to \mathrm{Hom}_{\mathcal{C}}(C, G(D))$$

and the collection of all $\phi(C, D)$ form a natural isomorphism.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(F(C), D) \arrow[r] \arrow[d, "\phi_{C,D}"] & \mathrm{Hom}_{\mathcal{D}}(F(C'), D') \arrow[d, "\phi_{C',D'}"] \\
\mathrm{Hom}_{\mathcal{C}}(C, G(D)) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(C', G(D'))
\end{tikzcd}$$

Example 0.11.

1. (a) $R \otimes_R - \cong \mathrm{Id}_{R-mod}$. $R \otimes_R - : R - mod \to R - mod$ is well-defined since ${}_R R_R$ is a bimodule.

(b) $\mathrm{Hom}_R({}_R R_R, -) \cong \mathrm{Id}_{mod-R}$

2. For any ring R, $R \sim M_n(R)$, where $\sim$ indicates Morita equivalence, defined above. Why? Let ${}_R F = R^n$, $S = \mathrm{End}_R(F) \cong M_n(R)$. Let $F^* =_R (\mathrm{Hom}({}_S F, R))_S$

Claim. The functors $\mathrm{Hom}(F, -) : mod - R \to mod - S$, $\mathrm{Hom}(F^*, -) : mod - S \to mod - R$ *are mutually inverse functors.*

Proof. We want to show that, for M" $\in$ "$Mod - R$,

$$\mathrm{Id}_{Mod-R} \cong \mathrm{Hom}_S(F^*, \mathrm{Hom}_R(F, -))$$

Consider

$$\Phi(M) : m \mapsto (F^* = \mathrm{Hom}(F, R) \ni f \mapsto (x \mapsto mf(x)))$$

Check that this is a R-module hiomomorphisms, and in fact an isomorphism of R-modules.

Let $R = k$ be a field. Then we have a duality

$$k - mod \to k - mod$$

Let v" $\in$ "$k - mod$, and consider

$$\Phi(V) : V \to V^{**} = \mathrm{Hom}_k(\mathrm{Hom}_k(V, k), k)$$

, and

$$x \mapsto (\mathrm{Hom}_k(V, K) \ni f \mapsto f(x) \in k)$$

A duality from $k - mod$ to $k - mod$.

We may extend Φ to a functor $k - Mod \to k - Mod$, but this is not surjective if $\dim V = \infty$ (homework problem). So we have a natural equivalence $\mathrm{Id}_{k-mod} \cong (-)^{**}$

Here is an examples of an adjoint pair. Let ${}_S B_R$ be an $S - R$ bimodule. Then the functor

$$B \otimes_R - : R \to R - mod$$

is a left adjoint to

$$\mathrm{Hom}_S(R, -) : S - mod \to R - mod$$

Lecture 8, 1/27/23

Section 4: Additive and Abelian categories

Definition 0.12. A pre-additive category $\mathcal{C}$ is called an additive category if it has a zero object, and finite direct sums/products.

Definition 0.13. An additive category $\mathcal{C}$ is called an Abelian category if every map f has a kernel and cokernal, and every mono is a kernel, and every epi is a cokernel.

Example 0.12. In the category of rings (we assume these are unital rings, so this category is not preadditive, recall) there are categorical epis that fail to be surjective. For example, $f : \mathbb{Z} \hookrightarrow \mathbb{Q}$ is a categorical epimorphism.
Let $g, h \in \mathrm{Hom}_{\mathsf{Ring}}(\mathbb{Q}, R)$ be such that $gf = hf$. Then $g|_{\mathbb{Z}} = h|_{\mathbb{Z}}$, so it follows that $g = h$.
In $R - mod, R - comp$, categorical monos coincide with injective homomorphisms, and similarly, categorical epimorphisms coincide with surjections.

Example 0.13. of Abelian categories:
$R - Mod$, in particular $\mathbb{Z} - Mod = \mathsf{Ab}$, $R - comp$. Let $\mathscr{T} - \mathsf{Ab}$ be the full subcategory of Ab consisting of the torsion Abelian groups.
Is the full subcategory of Ab consisting of torsion-free groups Abelian? No! The map $f : \mathbb{Z} \to \mathbb{Z}$ given by multiplication by 2 doesn't have a cokernel.
$R - mod$ is not Abelian if R is not left Noetherian!!!! (A ring is left Noetherian if every left ideal is finitely generated). But $R - Mod$
For example, let k be a field, and consider $R = k^{\mathbb{N}}$. Let $I = k^{(\mathbb{N})}$ (which means the direct sum, as opposed to the direct product). This is not a finitely generated left ideal, and $I \hookrightarrow R$. But $\pi : R \to R/I$ does not have a kernel in R-mod even though $R, R/I$ are in $R - mod$, because we will get something not in $R - mod$, but in $R - Mod$.
She started talking about some stuff we won't see until later, and said it was "music of the future."

Chapter 2: On the road to derived functors

Section 1: Exactness properties of functors

Note: We'll develop the theory for the Abelian category $R - mod$, but it easily adapts to arbitrary categories.

Definition 0.14. Let R, S be rings, F an additive functor from $R - mod$ to $S - mod$.

1. F is called exact if for all exact sequences

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

the sequence

$$0 \longrightarrow FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \longrightarrow 0$$

is also exact. If F is contravariant, then instead we want the sequence

$$0 \longrightarrow FC \xrightarrow{Fg} FB \xrightarrow{Ff} FA \longrightarrow 0$$

to be exact.

2. F is called left-exact or right-exact if it sends left (or right) exact sequences to left (or right) exact sequences

Lecture 9, 2/1/23

Recall:
A functor $F : R - mod \to S - mod$ is left-exact if for all exact sequences

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

the sequence

$$0 \longrightarrow FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC$$

is exact. Similarly, it is right exact if the image of

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact.

Remark. **1.** A functor $F : R - Mod \to S - Mod$ induces a functor $F : R - comp \to S - comp$.

2. If $F \cong G : R - mod \to S - mod$ (meaning F, G are naturally isomorphic), then F, G have the same exactness properties.

3. If $F : R - mod \to S - mod$ is an equivalence (or a duality) then F is exact.

Theorem 0.3. *(our favorite functors)*

1. *Let* M "$\in$" $R - Mod$. *Then* $\mathrm{Hom}_R(M, -)$ *and* $\mathrm{Hom}_R(-, M)$ *are left-exact functors from* $R - Mod \to \mathbb{Z} - Mod$.

2. *Let* M "$\in$" $Mod - R$. *Then* $M \otimes_R - : R - mod \to \mathbb{Z} - mod$ *is right exact.*

Proof. Note: Starting from now, I will denote the image of f under the functor $\mathrm{Hom}_R(M, -)$ by $f*$. The reason is that for some reason tikzcd won't let me but Hom_R inside an arrow's name.
We'll just prove part 1 for the covariant Hom. The contravariant case is homework. So, we want to show that $\mathrm{Hom}_R(M, -)$ is left-exact. Let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

be exact in $R - Mod$. Then its image is a complex

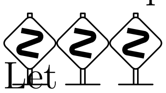
$$0 \longrightarrow \mathrm{Hom}_R(M, A) \xrightarrow{f*} \mathrm{Hom}_R(M, B) \xrightarrow{g*} \mathrm{Hom}_R(M, C)$$

For $\phi : M \to A$, $f * (\phi) = f \circ \phi$, and similarly for $g*$.

We first show that $f*$ is a mono: Indeed, from $f \circ \phi = 0$, we obtain $\phi = 0$, since f is a mono. So the sequence is exact at $\mathrm{Hom}_R(M, A)$.
We know $\mathrm{Im}(\mathrm{Hom}_R(M, f)) \subseteq \ker(\mathrm{Hom}_R(M, g))$. To show the reverse direction, let $\psi \in \ker(\mathrm{Hom}_R(M, g))$, i.e. $g \circ \psi = 0$, i.e. $\mathrm{Im}(\psi) \subseteq \ker(\psi)$.
Consider

$$0 \longrightarrow A \xrightarrow{\tilde{f}} \mathrm{Im}(f) \hookrightarrow B \longrightarrow C \longrightarrow 0$$

Set $\phi = \tilde{f}^{-1} \circ \psi$ and check that $\mathrm{Hom}_R(M, f)(\phi) = \underbrace{f \circ \tilde{f}^{-1}}_{=\mathrm{Id}_A} \circ \psi = \psi$. So $\psi \in \mathrm{Im}(\mathrm{Hom}(M, f))$. This gives exactness at $\mathrm{Hom}_R(M, B)$. So, we have shown that the covariant Hom functor is left-exact.
Now for part 2.



Let

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be exact in $R - Mod$. The sequence

$$M \otimes_R A \xrightarrow{f*} M \otimes_R B \xrightarrow{g*} M \otimes_R C \longrightarrow 0$$

is a complex. Clearly, $M \otimes_R g, m \otimes b \mapsto m \otimes_R (f(g))$ is an epi because g is an epi.
We have $\underbrace{\mathrm{Im}(M \otimes_R f)}_{E} \subseteq \ker(M \otimes_R g)$ (Birge - "The image of $M \otimes_R f$, which I baptize E, ... "), and we want the reverse inclusion. Factor $M \otimes_R g$ in the form $M \otimes B \to M \otimes_R B/E \to M \otimes_R C$. (I missed a bit here because TeXwriter was being wonky, so this proof is nonsense I think. It's a standard proof that can be googled tho). Then $M \otimes_R g = G \circ ?$, so $\ker(G) = \ker(M \otimes g)/E$. Thus suffices to show that G is a mono.
Plan: Construct $H \in \mathrm{Hom}_{\mathbb{Z}}(M \otimes_R C, M \otimes_R B/E)$ such that $HG = \mathrm{Id}_{M \otimes_R B/E}$
Define $H' : M \times C \to M \otimes_R B/E$ by $(m, c) \mapsto (m \otimes b + E)$, where $b \in B$ is such that $g(b) = c$. We check well-definedness.
Suppose $g(b) = g(b'), b, b' \in B$. Then $b - b' \in \ker(g) = \mathrm{Im}(f)$ by hypothesis, hence $m \otimes_R b - m \otimes_R b' = m \otimes (b - b') \in E$, thus $m \otimes_R b + E = m \otimes_R b' + E$. Check H' is $\mathbb{Z}$-bilinear and R-balanced.
Hence the universal property of the tensor product yields $H \in \mathrm{Hom}_{\mathbb{Z}}(M \otimes C', M \otimes B/E)$ with $H(m \otimes c) = m \otimes b$, where $g(b) = c$.

∎

Example 0.14. Here are some examples showing that in general, we shouldn't expect better than this previous theorem. That is, some witnesses to the non-left(right) exactness of $\mathrm{Hom}_R(M, -)$(resp. $M \otimes_R -$)

Definition 0.15. (i) Let $A" \in "\mathbb{Z}-Mod$. Then $a \in A$ is a <u>torsion element</u> iff there exists $n \in \mathbb{Z} \setminus \{0\}$ such that $n \cdot a = 0$. We use $T(A)$ to denote the torsion elements of A, which will always be a subgroup.

We say that A is torsion iff $A = T(A)$, and we say A is <u>torsion-free</u> iff $T(A) = 0$.

(ii) $a \in A$ is <u>divisible</u> iff $a \in nA$ for all $n \in \mathbb{Z} \setminus \{0\}$ (i.e. there exists $b \in A$ such that $a = nb$). An abelian group is called a <u>divisible group</u> if every element is divisible, i.e. if $nA = A$ for every $n \in \mathbb{Z} \setminus \{0\}$.

Remark. If $f \in \operatorname{Hom}_{\mathbb{Z}}(A, B)$ and A is divisible, then $f(A)$ is a divisible subbgroup of B.

Lecture 10, 2/3/23

Consider the sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

This sequence is exact. However, $\operatorname{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0$, and $\operatorname{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \neq 0$. So the image of this sequence under the functor $\operatorname{Hom}(\mathbb{Z}/2\mathbb{Z}, -)$ is not exact (in particular, it is not exact on the right).
The above argument will work for any integer instead of 2, so this sequence is a witness to the inexactness of the functor $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, -)$ for $n \geq 2$.
Consider an epi $g : \mathbb{Z}^{(\mathbb{N})} \to \mathbb{Q}$, and apply F. The map

$$\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}^{(\mathbb{N})}) \xrightarrow{F(g)} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q})$$

cannot be an epi, as $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}^{(\mathbb{N})}) = 0$ (as $\mathbb{Q}$ is divisible while $\mathbb{Z}^{(\mathbb{N})}$ is reduced), but $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}) \neq 0$.

Example 0.15. Here is an example to show that the tensor functor is not left-exact. Let $R = \mathbb{Z}$. We consider the functor $F(-) = \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} -$, $n \geq 2$. Consider the inclusion $\mathbb{Z} \xhookrightarrow{\iota} \mathbb{Q}$. This is a mono. However, $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = 0$, so $F(\iota) : \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \to 0$. Hoever, $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{Z}/n\mathbb{Z} \neq 0$, so $F(\iota)$ is not a mono.

Short-term program

1. Find exact functors

2. Characterize the exact sequences $\mathbb{A}$ in $R-comp$ such that $F(\mathbb{A})$ is exact in $S-comp$ for any additive functor $F : R-mod \to S-mod$.

Theorem 0.4. If $F : R - mod \to S - mod$ is an exact functor (i.e. F takes short exact sequences to short exact sequences) then $F(\mathbb{A})$ is exact in $S - mod$ whenever

$$\mathbb{A} : \quad \cdot \longrightarrow A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f} A_{n-1} \xrightarrow{f_{n-1}} \cdots$$

is exact in $R - mod$.

Proof. Suppose $F : R - mod \to S - mod$ is exact, and $\mathbb{A}$ as in the claim is an exact sequence in $R - mod$. We can factor each f_n

$$A_n \xrightarrow{\tilde{f}_n} \mathrm{Im}(A) \xhookrightarrow{\iota_n} A_{n-1}$$

$$x \longrightarrow f_n(x_n) \longrightarrow f_n(x_n)$$

Consider the short exact sequence

$$0 \longrightarrow \mathrm{Im}(f_{n+1}) \xrightarrow{\iota_{n+1}} A_n \xrightarrow{\tilde{f}_n} \mathrm{Im}(f_n) \longrightarrow 0$$

Since F is exact, we obtain an exact sequence

$$0 \longrightarrow F(\mathrm{Im}(f_{n+1})) \xrightarrow{F(\iota_{n+1})} F(A_n) \xrightarrow{F(\tilde{f}_n)} F(\mathrm{Im}(\tilde{f}_n)) \longrightarrow 0$$

In particular $\mathrm{Im}(F(\iota_{n+1})) = \ker(F(\tilde{f}_n))$ for $n \in \mathbb{N}$.

Claim.

(a) $\ker(F(f_n)) = \ker(F(\tilde{f}_n))$

(b) $\ker(F(\tilde{f}_n)) = \mathrm{Im}(F(\iota_{n+1})) = \mathrm{Im}(F(f_{n+1}))$

Proof. (a) $f_n = \iota_n \circ \tilde{f}_n$, so $F(f_n) = F(\iota_n) \circ F(\tilde{f}_n)$ by functoriality. Because F is exact, $F(\iota_n)$ is a mono, so $\ker(F(f_n)) = \ker(F(\tilde{f}_n))$.

(b) $F(f_{n+1}) = F(\iota_{n+1}) \circ F(\tilde{f}_{n+1})$. Because F is exact, $F(\tilde{f}_{n+1})$ is an epi.
So $\mathrm{im}(F(f_{n+1})) = \mathrm{Im}(F(\iota_{n+1}))$, so we have shown $F(\mathbb{A})$ is exact.

■

First installment of finding exact functors.

Proposition 1. *Let I be a set (index set). Consider the functors:*

$$(R-Mod)^I \to R-Mod$$
$$(m_i)_{i\in I} \mapsto \prod_{i\in I} m_i$$
$$(f_i)_{i\in I} \mapsto \prod_{i\in I} f_i, (\vec{m})_j = f_j(m_j)$$

$\otimes_{i\in I}$ works the same as before.
Both of them are exact .

Proof. Obvious

■

First installment re (2)

Remark. Warning: Birge uses nonstandard notation. She says "X is a direct summand of Y" to mean $X \subseteq^{\bigoplus} Y$

Definition 0.16. Let A, B" $\in$ "$R-mod$, $f \in \text{Hom}_R(A,B)$. f is called split if $\ker f$ is a direct summand of A and $\text{Im}(f)$ is a direct summand of B, i.e. there exist A', B'" $\in$ "$R-Mod$ with $A = \ker(f)^{\oplus}A'$ and $B = \text{Im}(f)^{\oplus}B'$.

Lecture 11, 2/6/23

Convention: Let M, N" $\in$ "$R-Mod$. We say that N is a direct summand of M, written $N \subseteq^{\oplus} M$, if there exists U" $\in$ "$R-Mod$ such that $M = N^{\oplus}U$.

Definition 0.17. Let $\mathbb{A}$" $\in$ "$R-comp$. Then $\mathbb{A}$ is split if f_n is split for all n.

Proposition 2. *Let A, B, C be left R-modules, $f \in \text{Hom}_R(A,B)$ and $h \in \text{Hom}_R(B,C)$. Then the following conditions are equivalent:*

1. *The sequence* $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ *is split exact.*

2. *There exists $f' \in \text{Hom}_R(B,A)$ and $g' \in \text{Hom}_R(C,B)$ such that $ff' + g'g = \text{Id}_B$, and both triangles in the diagram below commute:*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[dl, dotted, "f'"] & C \arrow[d, equal] \arrow[l, dotted, "g'"'] \\
A & & C
\end{tikzcd}$$

Proof. ■

Theorem 0.5. *Let* $\mathbb{A}: \quad \cdots \longrightarrow A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \longrightarrow \cdots$ *"$\in$" $R-comp$.*

Then the following are equivalent:

1. *For all additive functors $F : R - Mod \to S - Mod$ (S any ring), the sequence $F(\mathbb{A})$ is split exact.*

2. *$\mathbb{A}$ is split exact.*

Proof. For $1 \implies 2$, apply $F = \mathrm{Id}_{R-Mod}$.
For $2 \implies 1$, suppose 2. We will show 1 only in the case $\mathbb{A} = \; 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$ is split exact. Move to general $\mathbb{A}$ as in the proof of previous theorem.
By the proposition, there exist maps $f' \in \mathrm{Hom}_R(B, A)$, $g' \in \mathrm{Hom}_R(C, B)$ with $\mathrm{Id}_B = ff' + g'g$. Let F be an additive functor as in 1. Then

$$\begin{aligned}\mathrm{Id}_{F(B)} &= F(\mathrm{Id}_B) \\ &= F(f)F(f') + F(g')F(g)\end{aligned}$$

So by the proposition, the sequence

$$0 \longrightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(c) \longrightarrow 0$$

is split exact.

■

Example 0.16. There exist exact sequences

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

such that $B \cong A \oplus C$, but the sequence fails to be split.
Take $R = \mathbb{Z}$, let $n \geq 2, A = n\mathbb{Z}, B = \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z})^{(\mathbb{N})}$, and $C = (\mathbb{Z}/n\mathbb{Z})^{(\mathbb{N})}$. Then $A \oplus C \cong B$.
Find $f \in \mathrm{Hom}_{\mathbb{Z}}(A, B), g \in \mathrm{Hom}_{\mathbb{Z}}(B, C)$, such that

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact, but not split.

Section 2: Projective Modules

Our aim is to characterize those M"$\in$"$R-Mod$ for which $\mathrm{Hom}_R(M,-) : R-Mod \to \mathbb{Z}-Mod$ is exact.

Definition 0.18.

1. ${}_RF$ is <u>free</u> if ${}_RF \cong_R R^{(I)}$.

It is known that ${}_RF$ is <u>free</u> iff F has an R-basis, that is, a linearly independent generating set.

2. ${}_RP$"$\in$"$R-Mod$ is called <u>projective</u> if P is isomorphic to a direct summand of a free module.

Example 0.17.

1. Vector spaces

2. Let k be a field and $R = k \oplus k$ (ring product). Then R admits projective modules that fail to be free. Let $e_1 = (1,0), e_2 = (0,1)$. Then $Re_1 = k \times \{0\}, Re_2 = \{0\} \times k$. We have $R = Re_1 \oplus Re_2$, so the Re_i are projective. However, they are not free, because $\dim_k(R) = 2, \dim_k(Re_i) = 1$

Remark.

1. Let $(R_i)_{i\in I}$ be a family of left R-modules. Then $\bigoplus_{i\in I} P_i$ is projective if and only if P_i is projective for all $i \in I$.

2. $\prod_{i\in I} P_i$ need not be projective for infinite I.

3. Let R be a PID. The projective R-modules are precisely the free ones. Such R include $R = \mathbb{Z}, R = k[x], k$ a field.

4. $\mathbb{Z}^{\mathbb{N}}$ is <u>NOT</u> free (proof to come), and hence not projective.

5. (Serre's Conjecture) If $R = k[x_1, \ldots, x_n]$, k a field, is every projective R-module free? It turns out yes, and there are independent proofs by Suslin and <u>Quillen</u> (Quillen got the fields medal).