

Lecture 1

Let (X, \mathcal{A}, μ) be a measure space. Without any additional structure or information, we may define the Lebesgue integral $\int_X f d\mu$ for f an $\mathcal{A} - \mathcal{B}$ measurable function $f : X \rightarrow [-\infty, +\infty]$.

We only have a few examples without any work.

Example 0.1. • For any set X , we can define the counting measure on $\mathcal{A} = 2^X$, which gives $\mu(A) = |A|$. If $X = \mathbb{N}$, then a measurable function is just a sequence (f_n) , and $\int_X f d\mu = \sum f_n$

- We can also define the Dirac mass δ_p for a fixed $p \in X$ by

$$\delta_p(E) = \begin{cases} 1 & p \in E \\ 0 & p \notin E \end{cases}$$

We have $\int_X f d\delta_p = f(p)$

To get another example of a measure we need to do some work.

Problem: We want a measure μ on \mathbb{R}^n such that, for a rectangle,

$$\mu([a_1, b_1] \times \cdots \times [a_n, b_n]) = |a_1 - b_1| \cdots |a_n - b_n|$$

Once it is defined on all rectangles, it is defined on the minimal σ -algebra containing them, which is the Borel σ -algebra. In other words, this condition will completely specify a measure on the Borel σ -algebra $\mathcal{B}_{\mathbb{R}^n}$

If $X = \mathbb{R}^n$, or a general metric space, or even a general topological space, then $\mathcal{B}(X)$ denotes the σ -algebra generated by the open subsets of X .

Problem:

Suppose we have a distribution function $F : \mathbb{R} \rightarrow \mathbb{R}$, meaning F is monotone, positive, and $\lim_{x \rightarrow -\infty} f(x) = 0, \lim_{x \rightarrow \infty} f(x) = 1$, and continuous from the right. We want a Borel measure μ such that $F(t) = \mu((-\infty, t])$. Such a measure, denoted by λ_F , is called a Lebesgue-Stieltjes measure.

The corresponding integral is called a Lebesgue-Stieltjes integral.

If F is smooth, then $\int_{\mathbb{R}} \phi d\lambda_F = \int_{-\infty}^{\infty} \phi(x) dF(x)$.

The measure we want on \mathbb{R}^n is denoted by λ^n .

The Carathéodory Construction

Suppose we have an outer measure $\gamma : 2^X \rightarrow [0, \infty]$. This means $\gamma(\emptyset) = 0$, $A \subset B \implies \gamma(A) \leq \gamma(B)$ (monotone), and $\gamma(\cup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \gamma(E_i)$ (subadditive).

We can define a set S to be γ -measurable if for every testing set T , $\gamma(T) = \gamma(S \cap T) + \gamma(S^c \cap T)$.

Theorem 0.1. (*Carathéodory Extension Theorem*)

1. $\gamma(N) = 0 \implies N$ is measurable.
2. The set of measurable sets forms a σ -algebra Γ .
3. γ restricted to Γ forms a measure.

“Nothing in the above theorem can guarantee you that Γ is not trivial, i.e. $\Gamma = \{\emptyset, X\}$. Nevertheless, this is a very useful guy” - Dennis.

Definition 0.1. (Lebesgue outer measure on \mathbb{R}^n)

Let R be a rectangle in \mathbb{R}^n , that is $R = \prod_{i=1}^n [a_i, b_i]$. We have $\text{Vol}(R) = |a_1 - b_1| \cdots |a_n - b_n|$. For any $E \subseteq \mathbb{R}^n$, we define

$$\mu^*(E) \stackrel{\text{def}}{=} \inf \left\{ \sum_{j=1}^{\infty} \text{Vol}(R_j) \mid E \subseteq \bigcup_{j=1}^{\infty} R_j \right\}$$

Proposition 1. μ^* is an outer measure on \mathbb{R}^n such that $\mu^*(R) = \text{Vol}(R)$ for all rectangles R .

Proof. The first and second axioms are trivial, so we will just prove the subadditivity. Let E be some set. By definition, for any ε , there is some cover R_j by rectangles such that

$$-\varepsilon + \sum_{j=1}^{\infty} \text{Vol}(R_j) \leq \mu^*(E) \leq \sum_{j=1}^{\infty} \text{Vol}(R_j)$$

meaning that $\sum_{j=1}^{\infty} \text{Vol}(R_j) \leq \mu^*(E) + \varepsilon$. So for each E_k , there is a sequence R_j^k which covers E_k , such that $\sum_{j=1}^{\infty} \text{Vol}(R_j^k) \leq \mu^*(E_k) + \frac{\varepsilon}{2^k}$.

So $\{R_j^k\}_{j,k \in \mathbb{N}}$ forms a cover of $\bigcup_{j=1}^{\infty} E_j$. Thus

$$\begin{aligned} \mu^*\left(\bigcup_{k=1}^{\infty} E_k\right) &\leq \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \text{Vol}(R_j^k) \\ &\leq \sum_{k=1}^{\infty} \left(\mu^*(E_k) + \frac{\varepsilon}{2^k} \right) \\ &= \sum_{k=1}^{\infty} \mu^*(E_k) + \varepsilon \end{aligned}$$

This is true for any positive ε . Taking the limit as $\varepsilon \rightarrow 0$ gives the result. ■

Now, fix a rectangle R . Note that R itself forms a cover of R , so by the definition, $\mu^*(R) \leq \text{Vol}(R)$. For $\varepsilon > 0$, we can take an almost-optimal cover (R_j) such that $\sum_{j=1}^{\infty} \text{Vol}(R_j) \leq \text{Vol}(R) + \varepsilon$. We can rig it such that $|\text{Vol}(R_j) - \text{Vol}(R)| \leq \frac{\varepsilon}{2^j}$. Because $R \subset \cup_{j=1}^{\infty} R_j$, and R_j is an open cover, by compactness of R there is a finite subcover, and the volume of R is less than or equal to the sum of the volumes of these finitely many R_j . So the volume of R is less than or equal to $\mu^*(R) + 2\varepsilon$. So $\text{Vol}(R) = \mu^*(R)$.

Proposition 2. *Every rectangle R in \mathbb{R}^n is Carathéodory measurable).*

Proof. I missed this lol. Apparently Dennis denotes \mathcal{M}_{λ^*} by \mathcal{L}^n . ■

Definition 0.2. A set is said to be G_{δ} if it is the countable intersection of open sets. A set is said to be F_{σ} if it is the countable union of closed sets.

Theorem 0.2. 1. *For all $E \in \mathcal{L}^n$, $\lambda^N(E) = \inf\{\lambda^n(O) \mid \text{open } O \supseteq E\}$.*

2. *$E \in \mathcal{L}^n$ if and only if $E = H \setminus Z$, where H is G_{δ} , and $\lambda^*(Z) = 0$.*

3. *$E \in \mathcal{L}^n$ if and only if $E = H \cup Z$, where H is F_{σ} and $\lambda^*(Z) = 0$.*

4. $\lambda^n(E) = \sup\{\lambda^n(C) \mid \text{closed } C \subseteq E\}$

Proof. It suffices to prove the first statement, as the others will follow by passing to a complement. ■

Definition 0.3. Suppose X is a metric space. A measure on X is a Radon measure if it is Borel (meaning defined on a σ -algebra containing Borel sets), and for any Borel E , $\mu(E) = \inf\{\mu(O) \mid \text{open } O \supseteq E\}$, and for any compact $C \subseteq X$, $\mu(C) < \infty$.

Theorem 0.3. (*Riesz*)

Let $X \subseteq \mathbb{R}^n$ be compact. Let $C(X)$ denote the vector space of all continuous functions on X . This admits a norm $\|f\|_{C(X)} = \sup_X |f|$, making it a Banach space.

Define $C^(X) = \{\phi : C(X) \rightarrow \mathbb{R}, \phi \text{ is linear and continuous}\}$.*

For all $\phi \in C^(X)$, there exists a Radon measure $\mu = \mu_+$, and a function $M : X \rightarrow \{\pm 1\}$ which is Borel, such that*

$$\phi(f) = \int_X f(x)M(x) d\mu(x)$$

for all $f \in C(X)$.

Proof. ■

Lecture 2, 1/17/23

Note: This is the first lecture with Davit. Davit will always use μ to refer to an outer measure, not a measure. The book will be “Measure theory and fine properties of functions.” According to Davit, this is the correct book to be using.

Definition 0.4. Let X be a nonempty set. A mapping $\mu : 2^X \rightarrow [0, +\infty]$ is called a measure if it satisfies the following 2 properties.

1. $\mu(\emptyset) = 0$.
2. (Countable subadditivity and monotonicity) If $A, A_1, A_2, \dots \subseteq X$ and $A \subseteq \bigcup_{i=1}^{\infty} A_i$ then $\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$

Remark. From the second definition, we can automatically get monotonicity, i.e. if $A \subseteq B$, then $\mu(A) \leq \mu(B)$. This is because, as written, the second definition is a statement not just about $\bigcup_{i=1}^{\infty} A_i$, but about any subset of it. Indeed, let $A = A$, $A_1 = B$, and $A_k = \emptyset$ for $k \geq 2$. Then we have $\mu(A) \leq \mu(\bigcup_{i=1}^{\infty} A_i) = \mu(A \cup B)$.

We will write “ μ is a measure on X ” to mean that μ satisfies the above definition (that is, μ is an outer measure).

Definition 0.5. Let X be a nonempty set and let μ be a measure on X . For a fixed set $C \subseteq X$, define the restriction measure $\nu = \mu|_C$ by $\nu(A) = \mu|_A(A) = \mu(A \cap C)$.

Remark. It is easy to prove that $\mu|_C$ is a measure on X .

Definition 0.6. (Carathéodory’s condition). Let X be a nonempty set and let μ be a measure on X . A subset $A \subseteq X$ is called μ -measurable if, for all subset $B \subseteq X$, we have

$$\mu(B) = \mu(B \cap A) + \mu(B \setminus A)$$

Remark. X and \emptyset are easily seen to be μ -measurable.

Theorem 0.4. (Carathéodory extension theorem)

The collection of μ -measurable sets on a set X is a σ -algebra.

Theorem 0.5. Let X be a nonempty set and let μ be a measure on X . Then the following holds:

1. \emptyset and X are μ -measurable.
2. $A \subseteq X$ is μ -measurable if and only if $X \setminus A$ is μ -measurable.
3. If $A \subseteq X$ is such that $\mu(A) = 0$, then A is μ -measurable.
4. Let $C \subseteq X$. Then anything which is μ -measurable is $\mu|_C$ -measurable.

Remark. A measure is also finitely subadditive, which says that if $A \subseteq A_1 \cup \dots \cup A_n$, then $\mu(A) \leq \sum_{i=1}^n \mu(A_i)$. So, to check that $\mu(B) = \mu(B \cap A) + \mu(B \setminus A)$, it will suffice to check

$$\mu(B) \geq \mu(B \cap A) + \mu(B \setminus A)$$

Proof. Part 1 is obvious.

Suppose that A is μ -measurable. Then $\mu(B \cap A) = \mu(B \setminus A^c)$ and $\mu(B \cap A^c) = \mu(B \setminus A)$ so $\mu(B \cap A) + \mu(B \setminus A) = \mu(B \cap A^c) + \mu(B \setminus A^c)$. So A is μ -measurable if and only if A^c is.

Suppose that $\mu(A) = 0$. Then $\mu(B \cap A) \leq \mu(A), \mu(B)$, so $\mu(B \cap A) = 0$ for any $B \subseteq X$. Now, $B \setminus A \subseteq B$, so by monotonicity $\mu(B \setminus A) \leq \mu(B)$. So $\mu(B \cap A) + \mu(B \setminus A) \leq \mu(B)$ for all $B \subseteq X$, so we are done.

Let A be μ -measurable. Then for any $B \subseteq X$ we have

$$\begin{aligned} \nu(B) &= \mu|_C(B) = \mu(B \cap C) \\ &= \mu((B \cap C) \cap A) + \mu((B \cap C) \setminus A) \\ &= \nu(B \cap A) + \mu((B \setminus A) \cap C) \\ &= \nu(B \cap A) + \nu(B \setminus A) \end{aligned}$$

■

Theorem 0.6. Let X be a nonempty set and let μ be a measure on X . Assume $A_1, A_2, \dots, A_n \subseteq X$ are μ -measurable. Then

1. $\bigcup_{k=1}^n A_k$ and $\bigcap_{i=1}^n A_k$ are also μ -measurable.
2. If the A_i are disjoint, then $\mu(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i)$

Proof. We prove part 2 first. Because each A_i is measurable,

$$\begin{aligned} \mu(\bigcup_{k=1}^n A_k) &= \mu((\bigcup_{k=1}^n A_k) \cap A_n) + (\mu(\bigcup_{k=1}^n A_k) \setminus A_n) \\ &= \mu(\bigcup_{i=1}^{n-1} A_k) + \mu(A_n) = \dots = \sum_{k=1}^n \mu(A_k) \end{aligned}$$

Now we prove part 1. Let $A, B \subseteq X$ be μ -measurable and disjoint. Then for any $C \subseteq X$, $\mu(C) = \mu(C \cap A) + \mu(C \setminus A)$, and similarly for B . This is equal to

$$\begin{aligned} \mu(C) &= \mu(C \cap A) + \mu((C \setminus A) \cap B) + \mu(C \setminus A \setminus B) \\ &= \mu(C \cap A) + \mu(C \cap B) + \mu(C \setminus (A \cup B)) + \mu(C \cap (A \cup B)) (?) \\ &= \mu(C \cap (A \cup B) \cap A) + \mu(C \cap (A \cup B) \setminus A) \\ &= \mu(C \cap A) + \mu(C \cap B) \\ &= \mu(C \cap (A \cup B)) + \mu(C \setminus (A \cup B)) \end{aligned}$$

So $A \cup B$ is μ -measurable. (I got a bit lost in the arithmetic, sorry)

Next, we show if $A, B \subseteq X$ are μ -measurable, then $A \cap B$ is μ -measurable. This is straightforward. We will continue next time.

Lecture 3, 1/19/23

We will continue our proof of the theorem. Assume $A, B \subseteq X$ are μ -measurable. We aim to show that $A \cap B$ is also μ -measurable. We need to show that, for any $C \subseteq X$, we have $\mu(C) = \mu(C \cap (A \cap B)) + \mu(C \setminus (A \cap B))$. Because A, B are μ -measurable, we have

$$\begin{aligned} \mu(C) &= \mu(C \cap A) + \mu(C \setminus A) \\ &= \mu((C \cap A) \cap B) + \mu((C \cap A) \setminus B) + \mu(C \setminus A) \\ &= \mu(C \cap (A \cap B)) + \mu((C \cap A) \setminus B) + \mu(C \setminus A) \\ &\geq \mu(C \cap (A \cap B)) + \mu(C \setminus (A \cap B)) \end{aligned}$$

The opposite inequality follows by subadditivity, so we have equality.

By induction, we get that also $\cap_{k=1}^n A_k$ is μ -measurable. For the union, we can get it using the fact that $\cup_{k=1}^n A_k = X \setminus \cap_{k=1}^n A_k^c$. ■

Remark. If A, B are μ -measurable, then $A \setminus B$ is μ -measurable. This follows from $A \setminus B = A \cap (X \setminus B)$

Theorem 0.7. Let X be a nonempty set, and μ a measure on X . Assume $\{A_k\}_{k=1}^\infty \subseteq X$ are μ -measurable. Then

1. If the A_k are disjoint, then we have countable additivity:

$$\mu\left(\bigcup_{k=1}^\infty A_k\right) = \sum_{k=1}^\infty \mu(A_k)$$

If $A_1 \subseteq A_2 \subseteq \dots$, then

$$\lim_{k \rightarrow \infty} \mu(A_k) = \mu\left(\bigcup_{k=1}^\infty A_k\right)$$

If $A_1 \supseteq A_2 \supseteq \dots$, and $\mu(A_1) < \infty$, then

$$\lim_{k \rightarrow \infty} \mu(A_k) = \mu\left(\bigcap_{k=1}^\infty A_k\right)$$

Proof. We have from before that if the A_k are pairwise disjoint, then $\mu(\cup_{k=1}^n A_k) = \sum_{k=1}^n \mu(A_k)$ for any $n \in \mathbb{N}$. Because $\cup_{k=1}^n A_k \subseteq \cup_{k=1}^\infty A_k$, we must have that $\mu(\cup_{k=1}^n A_k) \leq \mu(\cup_{k=1}^\infty A_k)$. Using the previous fact, and passing to a limit, we have

$$\sum_{k=1}^{\infty} \mu(A_k) \leq \mu(\cup_{k=1}^{\infty} A_k)$$

The opposite equality is automatically true by the countable subadditivity of μ , so we get equality. This completes the proof of 1. Now for part 2.

Define $B_k = A_k \setminus A_{k-1}$, where $A_0 \stackrel{\text{def}}{=} \emptyset$. We have $A_k = \cup_{i=1}^k B_i$. Note that the B_i are disjoint. So we have

$$\mu(A_k) = \sum_{i=1}^k \mu(B_i)$$

So, in the limit,

$$\lim_{k \rightarrow \infty} \mu(A_k) = \lim_{k \rightarrow \infty} \sum_{i=1}^k \mu(B_i) = \sum_{i=1}^{\infty} \mu(B_i)$$

So

$$\mu(\cup_{i=1}^{\infty} B_i) = \mu(\cup_{k=1}^{\infty} A_k)$$

Finally, let $A_1 \supseteq A_2 \supseteq \cdots$, $\mu(A_1) < \infty$. Define $B_k = A_1 \setminus A_k$. This is decreasing sequence of μ -measurable sets, so by the previous part,

$$\lim_{k \rightarrow \infty} \mu(B_k) = \mu(\cup_{k=1}^{\infty} B_k)$$

So

$$\begin{aligned} \mu(B_k) &= \mu(A_1 \setminus A_k) = \mu(A_1) - \mu(A_k) \implies \\ \lim_{k \rightarrow \infty} \mu(B_k) &= \lim_{k \rightarrow \infty} (\mu(A_1) - \mu(A_k)) = \mu(A_1) - \lim_{k \rightarrow \infty} \mu(A_k) \\ &= \mu(\cup_{k=1}^{\infty} B_k) = \mu(\cup_{k=1}^{\infty} (A_1 \setminus A_k)) = \mu(A_1 \setminus \cap_{k=1}^{\infty} A_k) \\ &\geq \mu(A_1) - \mu(\cap_{k=1}^{\infty} A_k) \end{aligned}$$

So $\lim_{k \rightarrow \infty} \mu(A_k) \leq \mu(\cap_{k=1}^{\infty} A_k)$. The opposite inequality follows easily by monotonicity. ■

We are ready to prove the Carathéodory extension theorem.

Proof. Let $A_1, A_2, \dots \subseteq X$ be μ -measurable. It will suffice to prove that $\cup_{k=1}^{\infty} A_k$ is μ -measurable. So we need to check that, for any B ,

$$\mu(B) = \mu(B \cap (\cup_{k=1}^{\infty} A_k)) + \mu(B \setminus \cup_{k=1}^{\infty} A_k)$$

Fix $B \subseteq X$, and consider $\nu = \mu|_B$. Recall this is defined as $\nu(C) = \mu(B \cap C)$. We would like

$$\nu(B) = \nu(\cup_{k=1}^{\infty} A_k) + \nu(B \setminus \cup_{k=1}^{\infty} A_k)$$

To this end,

$$\nu(\cup_{k=1}^{\infty} A_k) = \lim_{k \rightarrow \infty} \nu(\cup_{i=1}^k A_i)$$

Without loss of generality, $\nu(B) < \infty$. If $\nu(B) = \infty$, then we are done trivially. As before, define $B_k = B \setminus \cup_{i=1}^k A_i$. Then

$$\nu(B \setminus \cup_{k=1}^{\infty} A_k) = \lim_{k \rightarrow \infty} \nu(B \setminus \cup_{i=1}^k A_i)$$

So,

$$\begin{aligned} \nu(\cup_{k=1}^{\infty} A_k) + \nu(B \setminus \cup_{k=1}^{\infty} A_k) &= \lim_{k \rightarrow \infty} \left(\nu(\cup_{i=1}^k A_i) + \nu(B \setminus \cup_{i=1}^k A_i) \right) \\ &= \lim_{k \rightarrow \infty} \nu(B) = \nu(B) \end{aligned}$$

so we are done. ■

Definition 0.7. Let X be a nonempty set, and let μ be a measure on X . Then μ is said to be

1. A regular measure if, for any $A \subset X$, there exists a μ -measurable $B \subseteq X$ such that $A \subseteq B$, and $\mu(A) = \mu(B)$.
2. A Borel measure if all Borel sets (i.e. the elements of the Borel σ -algebra) are measurable. This only applies if X is also a topological space, of course.
3. A Borel-regular measure if μ is Borel, and for any $A \subseteq X$, there exists a Borel set $B \subseteq X$ such that $A \subseteq B$ and $\mu(A) = \mu(B)$.
4. A Radon measure if it is Borel-regular and $\mu(K) < \infty$ if K is compact.

Remark. Note that being Borel and regular is weaker than being Borel-regular.

Theorem 0.8. (Increasing sets for regular measures) Let X be a nonempty set, and let μ be a regular measure on X . Assume $A_1 \subseteq A_2 \subseteq \cdots \subseteq X$. Then

$$\lim_{k \rightarrow \infty} \mu(A_k) = \mu \left(\bigcup_{k=1}^{\infty} A_k \right)$$

Remark. The sets A_k need not be μ -measurable.

Proof. For all A_k , there is a $C_k \subseteq X$ which is μ -measurable, $A_k \subseteq C_k$, and $\mu(A_k) = \mu(C_k)$. Let $D_k = \bigcap_{i \geq k} C_i$. For $i \geq k$, we can see $A_k \subseteq A_i \subseteq C_i$. $A_k \subseteq \bigcup_{i \geq k} C_i = D_k$, then $\mu(A_k) \leq \mu(D_k)$. On the other hand, $D_k \subseteq C_k$, so $\mu(D_k) \leq \mu(C_k) = \mu(A_k)$. So

- $\mu(A_k) = \mu(D_k)$
- $A_k \subseteq D_k$
- D_k is μ -measurable and $D_1 \subseteq D_2 \subseteq \dots$

So

$$\begin{aligned} \lim_{k \rightarrow \infty} \mu(A_k) &= \lim_{k \rightarrow \infty} \mu(D_k) \\ &= \mu \left(\bigcup_{k=1}^{\infty} D_k \right) \\ &\geq \mu \left(\bigcup_{k=1}^{\infty} A_k \right) \end{aligned}$$

Because $\bigcup_{k=1}^{\infty} A_k \subseteq \bigcup_{k=1}^{\infty} D_k$,

$$\lim_{k \rightarrow \infty} \mu(A_k) \geq \mu \left(\bigcup_{k=1}^{\infty} A_k \right)$$

But $A_k \subseteq \bigcup_{k=1}^{\infty} A_k$, so the opposite inequality is also true, so we have equality. ■

Lecture 4, 1/24/23

Theorem 0.9. (Restriction and Radon measures)

Let X be a topological space and let μ be a Borel-regular measure on X . Let $A \subseteq X$ be μ -measurable with $\mu(A) < \infty$. Then the restriction measure $\nu = \mu|_A$ is Radon.

Proof. First, ν is a finite measure, as $\nu(X) = \mu(A \cap X) = \mu(A) < \infty$ for any $C \subseteq X$. It is clear that ν is Borel, as μ is Borel. Next, we show ν is Borel-Regular. Without loss of generality, we may assume that A is Borel, because μ is Borel-regular. Explicitly, we know there is a Borel set $B \subseteq X$ such that $A \subseteq B$ and $\mu(B) = \mu(A)$. We will show $\mu|_A = \mu|_B$.

We have $\mu(B) = \mu(B \cap A) + \mu(B \setminus A) = \mu(A) + \mu(B \setminus A)$. So $\mu(B \setminus A) = 0$.
So, for all $C \subseteq X$,

$$\begin{aligned}\mu|_B(C) &= \mu(B \cap C) \\ &= \mu((B \cap C) \cap A) + \mu((B \cap C) \setminus A) \\ &= \mu(C \cap A) + \mu((B \cap C) \setminus A) \\ &\leq \mu|_A(C) + \mu(B \setminus A) \\ &= \mu|_A(C)\end{aligned}$$

But $(A \cap C) \subseteq (B \cap C)$, so $\mu|_A(C) \leq \mu|_B(C)$, so we may conclude that $\mu|_A = \mu|_B$.
So assume A is Borel. Fix $C \subseteq X$. We need to prove that there exists a Borel $D \subseteq X$ such that $C \subseteq D$ and $\nu(C) = \nu(D)$. There exists a Borel $E \subseteq X$ such that $C \cap A \subseteq E$, and $\mu(C \cap A) = \mu(E)$. So $D = E \cup (X \setminus A)$ is Borel and $C \subseteq D$.
So

$$\begin{aligned}\nu(D) &= \mu((E \cup (X \setminus A)) \cap A) \\ &= \mu(E \cap A) \\ &\leq \mu(E) \\ &= \mu(C \cap A) \\ &= \nu(C)\end{aligned}$$

$C \subseteq D$ so $\nu(C) \leq \nu(D)$, so $\nu(C) = \nu(D)$. ■

Theorem 0.10. (Carathéodory Criterion for being Borel)

Let X be a metric space and let μ be a measure on X . Then μ is Borel if and only if, for all $A, B \subseteq X$ with $d(A, B) > 0$ (meaning $\inf\{d(a, b) \mid a \in A, b \in B\} > 0$),

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

Proof. \Rightarrow

Suppose μ is Borel. We will use \overline{B} to denote the closure of B . Then $d(A, \overline{B}) = d(A, B) > 0$. By measurability of \overline{B} ,

$$\mu(A \cup B) = \mu((A \cup B) \cap \overline{B}) + \mu((A \cup B) \setminus \overline{B}) = \mu(B) = \mu(A)$$

\Leftarrow

Suppose that, for A, B with $d(A, B) > 0$, $\mu(A \cup B) = \mu(A) + \mu(B)$. We will show that this implies μ is Borel. Let us show that every closed subset $C \subseteq X$ is μ -measurable.

So we have to prove that for every $A \subseteq X$,

$$\mu(A) = \mu(A \cap C) + \mu(A \setminus C)$$

We have \leq trivially. Assume $\mu(A) < \infty$; otherwise, this equality holds trivially.

Define for every $n \in \mathbb{N}$ the set $C_n = \{x \in X \mid d(x, C) \leq \frac{1}{n}\}$. We can see $d(A \setminus C_n, C) \geq \frac{1}{n} > 0$. So

$$\begin{aligned} \mu((A \setminus C_n) \cup (A \cap C)_{\subseteq A}) &= \mu(A \setminus C_n) + \mu(A \cap C) \\ &\leq \mu(A) \end{aligned}$$

So $\mu(A \setminus C_n) + \mu(A \cap C) \leq \mu(A)$ for all $n \in \mathbb{N}$. We will prove that $\lim_{n \rightarrow \infty} \mu(A \setminus C_n) = \mu(A \setminus C)$.

Consider the annuli $R_n = \{x \in A \mid \frac{1}{n+1} < d(x, C) \leq \frac{1}{n}\}$. We have

$$(A \setminus C_1) \bigcup_{n=1}^{\infty} R_n \subseteq A \setminus C$$

C is closed, so in fact we have equality above. Why? If a point belongs to $A \setminus C$, then it does not belong to C , so $d(x, C) > 0$. So there is an $n \in \mathbb{N}$ such that $x \in R_n$ or $x \in A \setminus C_1$. We have

$$\begin{aligned} \mu\left(\bigcup_{k=0}^n R_{2k+1}\right) &= \sum_{k=0}^n \mu(R_{2k+1}) \leq \mu(A) \\ \mu\left(\bigcup_{k=1}^n R_{2k}\right) &= \sum_{k=1}^n \mu(R_{2k}) \leq \mu(A) \end{aligned}$$

So $\sum_{n=1}^{\infty} \mu(R_n) \leq 2\mu(A) < \infty$, so $\lim_{n \rightarrow \infty} (\sum_{k=n}^{\infty} \mu(R_k)) = 0$. So $(A \setminus C_n) \bigcup_{k=n}^{\infty} R_k = A \setminus C$

So by subadditivity,

$$\mu(A \setminus C) \leq \mu(A \setminus C_n) + \sum_{k=n}^{\infty} \mu(R_k)$$

So as $n \rightarrow \infty$,

$$\mu(A \setminus C) \leq \liminf_{n \rightarrow \infty} \mu(A \setminus C_n) \leq \mu(A \setminus C)$$

This completes the proof.

It is time for our third section.

Approximation by open, closed, and compact sets

Theorem 0.11. Let μ be a Borel measure on \mathbb{R}^n , and let $B \subseteq \mathbb{R}^n$ be a Borel set.

1. *If $\mu(B) < \infty$, then for any $\varepsilon > 0$, there exists a closed $C \subseteq B$ such that $\mu(B \setminus C) < \varepsilon$.*
2. *If μ is a Radon measure, then for all $\varepsilon > 0$, there exists an open $U \supseteq B$ such that $\mu(U \setminus B) < \varepsilon$.*

Proof. 1. Let $\nu = \mu|_B$, a finite measure on \mathbb{R}^n .

Define the collection $\mathcal{F} = \{A \subseteq \mathbb{R}^n \mid A \text{ is } \mu\text{-measurable and for all } \varepsilon > 0, \text{ there exists a closed } C \subseteq A \text{ such that } \mu(A \setminus C) < \varepsilon\}$

Our goal is to show that $\mathcal{B}_{\mathbb{R}^n} \subseteq \mathcal{F}$. Davit uses “ σ_B ” to indicate the Borel σ -algebra.

By previous discussion, \mathcal{F} contains all closed sets.

Now, if $A_1, A_2, \dots \in \mathcal{F}$, then $\bigcup_{k=1}^{\infty} A_k \in \mathcal{F}$. For all A_k , there exists a closed $C_k \subseteq A_k$, such that $\nu(A_k \setminus C_k) < \frac{\varepsilon}{2^k}$. Then by subadditivity,

$$\nu\left(\bigcup_{k=1}^{\infty} A_k \setminus \bigcup_{k=1}^{\infty} C_k\right) \leq \nu\left(\bigcup_{k=1}^{\infty} (A_k \setminus C_k)\right) \leq \sum_{k=1}^{\infty} \nu(A_k \setminus C_k) < \varepsilon$$

and $C = \bigcup_{k=1}^{\infty} C_k$ is closed.

Lecture 5, 1/26/23

Theorem 0.12. Let μ be a Borel measure on \mathbb{R}^n and let $B \subseteq \mathbb{R}^n$ be a Borel set.

1. *If $\mu(B) < \infty$, then for all $\varepsilon > 0$, there exists a closed $C \subseteq B$ such that $\mu(B \setminus C) < \varepsilon$.*
2. *If μ is a Radon measure, then for all $\varepsilon > 0$, there exists an open $U \subseteq \mathbb{R}^n$ such that $B \subseteq U$ and $\mu(U \setminus B) < \varepsilon$.*

Proof.

1. Let $\nu = \mu|_B$ be a finite Borel measure on \mathbb{R}^n . Define the collection

$$\mathcal{F} = \{A \subseteq \mathbb{R}^n : A \mu\text{-measurable and for all } \varepsilon > 0, \exists C \subseteq A, C \text{ closed, } \nu(A \setminus C) < \varepsilon\}$$

We want to show $\mathcal{B} \in \mathcal{F}$, where \mathcal{B} is the Borel set.

Step 1:

\mathcal{F} contains all closed sets

Step 2:

If $A_1, A_1, \dots, A_k \in \mathcal{F}$, then for all A_k , there exists a closed C_k such that $\nu(A_k \setminus C_k) < \frac{\varepsilon}{2^k}$. Thus,

$$\begin{aligned} \nu \left(\bigcup_{k=1}^{\infty} A_k \setminus \bigcup_{k=1}^{\infty} C_k \right) &\leq \nu \left(\bigcup_{k=1}^{\infty} (A_k \setminus C_k) \right) \\ &\leq \sum_{k=1}^{\infty} \nu(A_k \setminus C_k) < \varepsilon \end{aligned}$$

Furthermore, $\cap_{k=1}^{\infty} C_k$ is closed. Thus \mathcal{F} is closed under countable intersections.

Step 3:

We want to show countable unions belong to \mathcal{F} . If $A_1, A_2, \dots, A_k, \dots \in \mathcal{F}$, then for all A_k , there is a closed C_k such that $\nu(A_k \setminus C_k) < \frac{\varepsilon}{2^k}$. However, we do not know if $\cup_{k=1}^{\infty} C_k$ is closed. Note that $\nu(\cup_{k=1}^{\infty} A_k \setminus \cup_{k=1}^{\infty} C_k) = \lim_{m \rightarrow \infty} \nu(\cup_{k=1}^m A_k \setminus \cup_{k=1}^m C_k) < \varepsilon$. So there is an $m \in \mathbb{N}$ such that $\nu(\cup_{k=1}^m A_k \setminus \cup_{k=1}^m C_k) < \varepsilon$. Furthermore $C = \cup_{k=1}^m C_k$ is closed.

Step 4:

In the homework, we showed that every open set is the countable union of closed balls. Since \mathcal{F} contains all closed sets, and is closed under countable unions, \mathcal{F} contains all open sets.

Step 5:

Consider the subset $G \subseteq \mathcal{F}$ given by $G = \{A \in \mathcal{F} \mid A^c \in \mathcal{F}\}$. We claim that G is a σ -algebra. Going through the axioms,

- (i) Clearly, $\emptyset \in G$.
- (ii) If $A \in G$, $A^c \in G$.

- (iii) If $A_1, A_2, \dots, A_k, \dots \in G$, then $\cup_{k=1}^{\infty} A_k \in G$. Why? $\cup_{k=1}^{\infty} A_k \in \mathcal{F}$ and $\mathbb{R}^n \setminus \cap_{k=1}^{\infty} (\mathbb{R}^n \setminus A_k) \in \mathcal{F}$, since each $\mathbb{R}^n \setminus A_k \in \mathcal{F}$, and \mathcal{F} is closed under countable intersections.

Step 6:

Since the complement of an open sets is a closed set, and since \mathcal{F} contains all open and closed sets, all open sets are contained in the σ -algebra G . Thus the Borel sets are contained in G , implying that they are contained in \mathcal{F} .

Note: Part 1 requires that X be a seperable metric space.

2. For all $m \in \mathbb{N}$, denote $U_m = B_m(0) = \{x \in \mathbb{R}^n \mid \|x\| < m\}$.

Note that $\mu(U_m \setminus B) \leq \mu(U_m) < \infty$. Thus there exists a closed $C_m \subseteq U_m \setminus B$ such that $\mu((U_m \setminus B) \setminus C_m) < \frac{\varepsilon}{2^m}$.

Note that $B \cap U_m \subseteq (U_m \setminus C_m)$, which is an open set.

Thus, $\mu((U_m \setminus C_m) \setminus (B \cap U_m)) = \mu((U_m \setminus C_m) \setminus B) < \frac{\varepsilon}{2}$.

Define $U = \cup_{m=1}^{\infty} (U_m \setminus C_m)$, which is an open set.

Thus $B = \cup_{m=1}^{\infty} (B \cap U_m) \subseteq \cup_{m=1}^{\infty} (U_m \setminus C_m) = U$.

Furthermore,

$$\begin{aligned} \mu(U \setminus B) &= \mu(\cup_{m=1}^{\infty} (U_m \setminus C_m) \setminus \cup_{m=1}^{\infty} (B \cap U_m)) \\ &\leq \mu(\cup_{m=1}^{\infty} (U_m \setminus C_m \setminus B \cap U_m)) \\ &\leq \sum_{m=1}^{\infty} \mu(U_m \setminus C_m \setminus B \cap U_m) \\ &< \varepsilon \end{aligned}$$

Note: Part 2 requires that for all $r > 0$, for all $x \in X$, $\mu(B_r(x)) < \infty$.

■

Theorem 0.13. (Approximation by compact and open sets).

Let μ be a Radon measure on \mathbb{R}^n . Then

1. *For all $A \subseteq \mathbb{R}^n$, $\mu(A) = \inf\{\mu(U) \mid A \subseteq U, U \text{ open}\}$.*
2. *For all μ -measurable $A \subseteq \mathbb{R}^n$, we have $\mu(A) = \sup\{\mu(K) \mid K \subseteq A, K \text{ compact}\}$.*

Proof. Shortly.

Note: If μ is the Lebesgue measure, we define the outer measure as

$$\begin{aligned}\mu(A) &= \inf \left\{ \sum_{k=1}^{\infty} (b_k - a_k) \mid A \subseteq \bigcup_{k=1}^{\infty} (a_k, b_k) \right\} \\ &= \inf \left\{ \sum_{k=1}^{\infty} (b_k - a_k) \mid A \subseteq \bigcup_{k=1}^{\infty} [a_k, b_k), [a_i, b_i) \cap [a_j, b_j) = \emptyset \text{ if } i \neq j \right\}\end{aligned}$$

Remark. Let X be a topological space, and let μ be a measure on X . If $A \subseteq X$ is such that for all $\varepsilon > 0$, there exists a μ -measurable $A_\varepsilon \subseteq A$, such that $\mu(A \setminus A_\varepsilon) < \varepsilon$. Then A is μ -measurable.

Proof. of remark.

Take $\varepsilon = \frac{1}{k}$. By part 1, there exists $A_k \subseteq A$ such that $0 \leq \mu(A) - \mu(A_k) < \frac{1}{k}$.

Let $B = \bigcup_{k=1}^{\infty} A_k$, and note that $B \subseteq A$, B is μ -measurable, and $\mu(A) - \frac{1}{k} \leq \mu(A_k) \leq \mu(B)$ for all $k \in \mathbb{N}$.

This implies that $\mu(B) = \mu(A)$, $B \subseteq A$.

Thus, $\mu(A) = \mu(A \cap B) + \mu(A \setminus B) = \mu(B) + \mu(A \setminus B)$.

Since $\mu(A) = \mu(B)$, this implies that $\mu(A \setminus B) = 0$, i.e. $A \setminus B$ is μ -measurable.

So $A = B \cup (A \setminus B)$ is μ -measurable. ■

We are now prepared for a proof of the theorem.

Proof. **1.** If $\mu(A) = \infty$, then this is trivial. So assume $\mu(A) < \infty$. If A is Borel, we use part 2 of theorem 1.

We have an open $U_\varepsilon \subseteq \mathbb{R}^n$ such that $A \subseteq U_\varepsilon$, and $\mu(U_\varepsilon \setminus A) < \varepsilon$.

Thus $\mu(A) \leq \mu(U_\varepsilon) \leq \mu(A) + \mu(U_\varepsilon \setminus A) < \mu(A) + \varepsilon$.

So we are done for Borel sets. If A is not Borel, then there exists a Borel $B \subseteq \mathbb{R}^n$ such that $A \subseteq B$ and $\mu(A) = \mu(B)$.

Note: For the above proof, we used a previous theorem, and the fact that μ is Borel-regular.

2. Lets prove that $\mu(A) = \sup\{\mu(C) \mid C \subseteq A, C \text{ closed}\}$. We have two cases:

- (i) $\mu(A) < \infty$. In this case, consider $\nu = \mu|_A$. ν is a finite Radon measure. We apply part 1 to A^c . There exists an open $U_\varepsilon \subseteq \mathbb{R}^n$ such that $\mathbb{R}^n \setminus A \subseteq U_\varepsilon$ and $\nu(U_\varepsilon) < \nu(A^c) + \varepsilon = \varepsilon$.

Set $C_\varepsilon = U_\varepsilon^c$, which is closed. Note that $C_\varepsilon \subseteq A$ and $\mu(A \setminus C_\varepsilon) = \mu(A \cap U_\varepsilon) = \nu(U_\varepsilon) < \varepsilon$.

Since $C \subseteq A$, $\mu(C_\varepsilon) \leq \mu(A)$. By countable subadditivity, $\mu(A) \leq \mu(C_\varepsilon) + \mu(A \setminus C_\varepsilon) < \mu(C_\varepsilon) + \varepsilon$.

(ii) Next time!

■

Lecture 6, 1/31/23

Remark. Let μ be a Borel measure on X . If for every $A \subseteq X$ one has $\mu(A) = \inf\{\mu(B) \mid A \subseteq B, B \text{ Borel}\}$, then μ has to be Borel-regular.

Proof. of remark.

Take $B_k \supseteq A$ such that $\mu(B_k) < \mu(A) + \frac{1}{k}$. Let $B = \bigcap_{k=1}^{\infty} B_k$. Note B is Borel, $A \subseteq B$, and $\mu(A) \leq \mu(B) \leq \mu(B_k) < \mu(A) + \frac{1}{k}$. So $\mu(A) = \mu(B)$.

■

Proof. of theorem from last time.

We showed that if $\mu(A) < \infty$, then $\mu(A) = \sup\{\mu(C) \mid C \subseteq A, C \text{ closed}\}$.

If $\mu(A) = \infty$, write $\mathbb{R}^n = \bigcup_{k=1}^{\infty} R_k$, where $R_k = \{x \in \mathbb{R}^n \mid k \leq \|x\| < k+1\}$. Thus $A = \bigcup_{k=1}^{\infty} A \cap R_k$.

For all k , $\mu(A \cap R_k) \leq \mu(R_k) < \infty$. So there exists $C_k \subseteq A \cap R_k$ such that $\mu(C_k) > \mu(A \cap R_k) - \frac{1}{2^k}$.

Thus $\mu(\bigcup_{k=1}^{\infty} C_k) = \sup_{k=1}^{\infty} \mu(C_k) \geq \sum_{k=1}^{\infty} (\mu(A \cap R_k) - \frac{1}{2^k}) \geq \mu(A) - 2 = \infty$.

This implies that $\lim_{m \rightarrow \infty} \mu(\bigcup_{k=1}^m C_k) = \mu(\bigcup_{k=1}^{\infty} C_k) = \infty$.

This proves the theorem for closed sets.

Now we prove the theorem for compact sets.

Case 1: $\mu(A) < \infty$. For all $\varepsilon > 0$, there exists a closed set $C_\varepsilon \subseteq A$ such that $\mu(C_\varepsilon) > \mu(A) - \varepsilon$. Consider $K_m = C_\varepsilon \cap B_m$, where $B_m = \{x \in \mathbb{R}^n \mid \|x\| \leq m\}$.

Note K_m is compact.

Thus $\lim_{m \rightarrow \infty} \mu(K_m) = \mu(\bigcup_{k=1}^{\infty} K_m) = \mu(C_\varepsilon) > \mu(A) - \varepsilon$.

Case 2: $\mu(A) = \infty$. For all $m \in \mathbb{N}$, there exists a closed $C_m \subseteq A$ such that $\mu(C_m) \geq m$. Apply the same procedure.

■

Covering theorems (Vitali's and Besicovitch)

Notation: we will work in \mathbb{R}^n . Closed balls will be denoted by B . For a given closed ball $B = B_r(x) = \{y \in \mathbb{R}^n \mid \|x - y\| \leq r\}$, $\hat{B} = cB = \{y \in \mathbb{R}^n \mid \|x - y\| \leq cr\} = B_{cr}(x)$.

Definition 0.8. Let $A \subseteq \mathbb{R}^n$ and let $\mathcal{F} = \{B \subseteq \mathbb{R}^n\}$ be a family of balls.

1. \mathcal{F} is a cover of A if $A \subseteq \bigcup_{B \in \mathcal{F}} B$.

2. \mathcal{F} is a fine cover of A if for all $x \in A$ and $\varepsilon > 0$, there exists a $B \in \mathcal{F}$ such that $x \in B$ and $\text{diam}(B) < \varepsilon$. Alternatively, for all $x \in X$, $\inf\{\text{diam}(B) \mid x \in B\} = 0$.

Theorem 0.14. (Vitali's Covering Theorem)

Let \mathcal{F} be a collection of nondegenerate closed balls in \mathbb{R}^n with diameters uniformly bounded, i.e. $\sup\{\text{diam}(B) \mid B \in \mathcal{F}\} < \infty$. Then there exists a subcollection of countably many disjoint balls $\{\hat{B}_i\}_{i=1}^{\infty}$, such that $\cup_{B \in \mathcal{F}} B \subseteq \cup_{i=1}^{\infty} \hat{B}_i$

Proof. Denote $D = \sup\{\text{diam}(B) \mid B \in \mathcal{F}\}$ and consider $\mathcal{F}_k = \{B \in \mathcal{F} \mid \frac{D}{2^k} < \text{diam}(B) \leq \frac{D}{2^{k-1}}\}$. Let $G_1 \subseteq \mathcal{F}_1$ be a maximal disjoint subcollection of balls in \mathcal{F} (we can produce this easily with Zorn's Lemma). It will be maximal in the sense that if we add another element, it will not be a disjoint set.

Assume G_1, \dots, G_{k-1} have been chose.

Let G_k be a maximal disjoint subcollection in \mathcal{F} such that the balls in G_k do not intersect with the balls in $\cup_{i=1}^{k-1} G_i$.

Set $G = \cup_{k=1}^{\infty} G_k \subseteq \mathcal{F}$. Let $B \in \mathcal{F}_m$, i.e. $\frac{D}{2^m} < \text{diam}(B) \leq \frac{D}{2^{m-1}}$.

Because G_m is maximal, there exists $\bar{B} \in \cup_{i=1}^m G_i$ such that $B \cap \bar{B} \neq \emptyset$. Thus $\text{diam}(\bar{B}) \geq \frac{D}{2^m} \geq \frac{1}{2} \text{diam}(B)$. Thus $B \subseteq \hat{\bar{B}}$.

Corollary 0.15. Assume \mathcal{F} is a fine cover of A by closed balls such that $\sup\{\text{diam}(B) \mid B \in \mathcal{F}\} < \infty$. Then for the collection G in theorem 1, one has $A \setminus \cup_{k=1}^m B_k \subseteq \cup_{B \in G \setminus \{B_1, \dots, B_m\}} \hat{B}$ for all $B_1, \dots, B_m \in \mathcal{F}$