Lecture 1 - 7/1/25

Missed :(

Lecture 2 - 9/1/25

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Lecture 3 - 14/1/25

Character theory

Consider $\dim \operatorname{Hom}_G(\rho_i, \rho_j) = 1$ if $i = j$ and 0 if $i \neq j$ (meaning if $\rho_i \ncong \rho_j$)
Recall: Given a representation $\rho : G \to \mathrm{GL}_n(k)$, the character of ρ, χ_ρ, is given by $\chi_\rho : G \to k, g \mapsto \operatorname{tr}(\rho(g))$
For today, G will be finite, $k = \overline{k}$ will be algebraically closed, of characteristic 0.
Basic properties of characters:

1. Suppose $\rho : G \to \mathrm{GL}_n(k)$ is a representation: then $\chi_\rho(e) = n = \dim \rho$.

2. $\chi_\rho(g) = \chi_\rho(hgh^{-1})$ for all $g, h \in G$, i.e. χ_ρ is constant on each conjugacy class of G.

Definition 0.1. A function $f : G \to k$ which is constant on conjugacy classes is called a class function.

The ρ_i (isomorphism classes of reps) will form an ONB for the space of class functions.

Given $\rho_1 : G \to \mathrm{GL}_n(k), \rho_2 : G \to \mathrm{GL}_m(k)$, $\chi_{\rho_1 \oplus \rho_2} = \chi_{\rho_1} + \chi_{\rho_2}$

$\chi_{\rho_1 \otimes \rho_2} = \chi_{\rho_1}\chi_{\rho_2}$
To see this, let A, B be diagonalizable (which we have WLOG because the image of any finite group are all diagonalizable over an algebraically closed k of char 0, which follows from Jordan Normal form)

Then $\operatorname{tr}(A \otimes B) = \operatorname{tr}(A)\operatorname{tr}(B)$.

I can't see the board he's writing on very well, and also I am not sure how $A \otimes B$ was defined.

Claim. *$\chi_\rho : G \to k$ always factors through $\mathbb{Q}(\mu_\infty)$, the subfield of k containing $\mathbb{Q}$ (k has char 0) generated by all roots of unity ($k = \overline{k}$)*

Proof. Because G is finite, ρ_G has finite order, hence its eigenvalues are roots of unity, so the trace is the sum of roots of unity.

■

Definition 0.2. $\bar{\cdot} : \mathbb{Q}(\mu_\infty) \to \mathbb{Q}(\mu_\infty)$ is the unique field homomorphism with the property that $\overline{\zeta} = \zeta^{-1}$ for all roots of unity $\zeta \in \mathbb{Q}(\mu_\infty)$.

5 $\chi_{\rho^v} = \overline{\chi_\rho}$

Recall ρ^v is defined via the formula $g \cdot f = f(g^{-1} \cdot -)$ where f is a functional. We have

$$\begin{aligned}\chi_{\rho^v}(g) &= \operatorname{tr}(p(g^{-1})) \\ &= \sum_{\zeta \text{ is an eigenvalue of } \rho(G)} \zeta^{-1} \\ &= \sum \overline{\zeta} = \overline{\chi_\rho(g)}\end{aligned}$$

This also follows from the Hom-tensor adjunction because $\mathrm{Hom}_k(\rho_1, \rho_2) = \rho_1^v \otimes \rho_2$.

Definition 0.3. Let $\chi, \psi : G \to \mathbb{Q}(\mu_\infty)$ be class functions. We define their inner product by

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)}$$

This indeed is a positive definite non degenerate.
Let $\rho_1 : G \to \mathrm{GL}_n(k), \rho_2 : G \to \mathrm{GL}_m(k)$. What is $\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle$?

Theorem 0.1.

$$\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle = \dim_k \mathrm{Hom}_G(\rho_1, \rho_2) = \dim_k \mathrm{Hom}(\rho_1, \rho_2)^G$$

Corollary 0.2. *Suppose ρ_1, ρ_2 are irreducible. Then $\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle$ is 0 if ρ_1, ρ_2 are not isomorphic, and 1 if they are. So the ρ_i form an orthonormal basis for the space of class functions.*

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Proof. Let $R_G \in k[G]$ be the element given by

$$R_G = \frac{1}{|G|} \sum_{g \in G} eg$$

We want to show

1. for $v \in V^G, R_G \cdot v = v$.

2. For arbitrary $v \in V, R_G \cdot v \in V^G$

To check:

1. We have

$$\begin{aligned} R_G \cdot v &= \frac{1}{|G|} \sum_{g \in G} e_g \cdot v \\ &= \frac{1}{|G|} \sum_{g \in G} v \\ &= v \end{aligned}$$

2. Fix $g \in G$. Then

$$\begin{aligned} g \cdot R_G \cdot v &= g \cdot (\frac{1}{|G|} \sum_{h \in G} hv) \\ &= \frac{1}{|G|} \sum_{h \in G} gh \cdot v \\ &= \frac{1}{|G|} \sum_{h \in G} h \cdot v \\ &= R_G \cdot v \end{aligned}$$

Corollary 0.3. *Let V be a G-representation. Then* $\dim_k V^G = \operatorname{tr}(R_G|V)$

Proof.

Claim. $\operatorname{tr}(\textit{projection}) = \dim_k \operatorname{Im}$

Proof. Claim $\Longrightarrow$ Cor follows from $\operatorname{tr}(R_G) = \dim \operatorname{Im}(R_G|V) = \dim_k V^G$

∎

We can finally prove the theorem:

Proof.

$$\begin{aligned}
\dim_k \operatorname{Hom}_G(\rho_1, \rho_2) &= \dim_k \operatorname{Hom}_k(\rho_1, \rho_2)^G \\
&= \operatorname{tr}(R_G | \operatorname{Hom}_k(\rho_1, \rho_2)) \\
&= \operatorname{tr}(\frac{1}{|G|} \sum e_g | \operatorname{Hom}_k(\rho_1, \rho_2)) \\
&= \frac{1}{|G|} \operatorname{tr}(g | \hom_k(\rho_1, \rho_2)) \\
&= \frac{1}{|G|} \sum_{g \in G} \chi_{\hom_k(\rho_1, \rho_2)}(g) \\
&= \frac{1}{|G|} \sum_{g \in G} \overline{\chi_{\rho_1}} \chi_{\rho_2} \\
&= \langle \chi_{\rho_1}, \chi_{\rho_2} \rangle \\
&= \underbrace{\overline{\langle \chi_{\rho_2}, \chi_{\rho_1} \rangle}}_{\in \mathbb{Z}}
\end{aligned}$$

■

Lecture 4, 16/1/24

As always, G will be a finite group, $k = \overline{k}$ is an algebraically closed field of characteristic 0.
$\mathbb{Q}(\mu_\infty)$ is the algebraically closed subfield of $\mathbb{C}$ which contains all the roots of unity, and this comes with the complex conjugate $\bar{\cdot}, \zeta \mapsto \zeta^{-1}$.
Goal: Classify finite dimensional G-representations over k.
We have done:

1. Maschke's theorem, which states that any G-rep in V over k is semisimple.

2. Character theory: $V \sim \chi_V : G \to \mathbb{Q}(\mu_\infty) \subseteq k$, $g \mapsto \operatorname{tr}(g|V)$

Definition 0.4. $Cl(G)$ denotes the class functions $G \mapsto \mathbb{Q}(\mu_\infty)$, and it is equipped with an inner product,

$$\langle \psi, \varphi \rangle = \frac{1}{|G|} \sum_{g \in G} \psi(g) \overline{\varphi(g)}$$

Remark: There is an isomorphism $Cl(G) \simeq Z(\mathbb{Q}(\mu_\infty)[G])$, sending φ to $\sum_{g \in G} \phi(g) e_g$
Warning: They come with different ring structures which are not preserved by this isomorphism.

Last time we used the Reynolds operator to show $\langle \chi_V, \chi_W \rangle = \dim_k \mathrm{Hom}_G(V, W)$.
If ρ_1, ρ_2 are irreps of G, then $\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle$ is 1 if $\rho_1 \cong \rho_2$, and 0 otherwise.

Corollary 0.4. *# of conjugacy classes of irreducible representations of $G \leq \dim_{\mathbb{Q}(\mu_\infty)} Cl(G) =$ # of conjugacy classes of G*

Proof. If χ_{ρ_i} are orthonormal, then the number of conjugacy classes of irreps is equal to $\dim \mathrm{span}(\chi_{\rho_i}) \subseteq Cl(G)$, so this number is $\leq \dim Cl(G)$

■

Proposition 1. *Let V be a G-representation. Then*

$$\Phi_V : \oplus_{\rho_i \text{ irrep of } G} \rho_i \otimes_k \mathrm{Hom}_G(\rho_i, V) \to V$$

given by $v \otimes f \mapsto f(v)$ is an isomorphism.

Proof. First, we show it is surjective. By Maschke, $V = \oplus_{\rho_i \text{ reps } G} \rho_i^{n_i}$.
Let $v \in \rho_i^{n_i} \subseteq V$, $v = (v_1, \ldots, v_{n_i})$. Let $f_j : \rho_j \to \rho_i^{n_i}$ be the inclusion of the jth coordinate.
Then $\Phi_v(\sum_j v_j \otimes f_j) = v$.
Now we show injectivity.
We have

$$\dim_k \oplus \rho_i \otimes_k \mathrm{Hom}_G(\rho_i, V) = \dim_k V$$

This follows from

$$\dim_k \mathrm{Hom}_G(\rho_i, V) = n_i$$

This follows from

$$\begin{aligned} \mathrm{Hom}_G(\rho_i, V) &= \mathrm{Hom}_G(\rho_i, \oplus \rho_i^{n_i}) \\ &= \oplus_j \mathrm{Hom}_G(\rho_i, \rho_j)^{n_i} \\ &= \mathrm{Hom}_G(\rho_i, \rho_j)^{n_i} \end{aligned}$$

Which is n_i-dimensional

$$\dim_k \oplus \rho_i \otimes \mathrm{Hom}_G(\rho_i, V) = \sum n_i \dim_k \rho_i = \dim V$$

■

Corollary 0.5.

$$V \simeq \oplus_{\rho \text{ irreps of } G} \rho_i^{\langle \chi_{\rho_i}, \chi_V \rangle}$$

Proof. Enough to show $\rho_i^{\langle \rho_i, V \rangle} \simeq \rho_i \otimes_k \mathrm{Hom}_G(\rho_i, V)$, i.e. $\dim_k \mathrm{Hom}(\rho_i, V) = \langle \chi_{\rho_i}, \chi_{\rho_j} \rangle$.
But that's the theorem.

■

Corollary 0.6.
$$V \simeq \oplus_{\rho_i\, irreps} \rho_i^{\oplus n_i}$$
, then $\langle \chi_V, \chi_V \rangle = \sum_i n_i^2$

Proof. $\chi_V = \sum n_i \chi_{\rho_i}$ ■

Corollary 0.7. $V \simeq W \iff \chi_V = \chi_W$

Corollary 0.8. *V is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$.*

Proof. Write $V = \oplus_i \rho_i^{n_i}$: so $\langle \chi_V, \chi_V \rangle = \sum_i n_i^2$ is equal to 1 iff exactly 1 n_i is nonzero, and equal to 1. ■

Example 0.1. (The regular representation)
Let $G \curvearrowright k(G)$ via left multiplication.
$\chi_{k[G]}(g) = \mathrm{tr}(g|k[G])$, which is $|G|$ if g is the identity, and 0 otherwise.
Because $g \cdot e_{g'} = e_{gg'}$, we have
$$\mathrm{tr}(g|k[G]) = \#\{h \in G \mid gh = g\}$$
Remark: if X is a G-set (i.e. a set with a G-action), then the permutation representation, k^X, has character
$$\chi_{k^X}(g) = \#\{x \in X \mid g \cdot x = x\}$$
Corollary 0.9. *As a G-representation,*
$$k[G] \simeq \oplus_{\rho_i\ irrep}\ \rho_i^{\oplus \dim \rho_i}$$

Proof.
$$\begin{aligned}\langle \chi_{\rho_i}, \chi_{k[G]} \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_{\rho_i}(g) \overline{\chi_{k[G]}(g)} \\ &= \frac{1}{|G|} \chi_{\rho_i}(e) \overline{\chi_{k[G]}(e)} \\ &= \frac{1}{|G|} \dim \rho_i |G| \\ &= \dim \rho_i\end{aligned}$$
Because this representation is 0 except at the identity. ■

Remark: In fact, $\mathrm{Hom}_G(k[G], \rho_i) \simeq \rho_i$, AS A VECTOR SPACE.

Proof. $\mathrm{Hom}_G(k[G], \rho_i) = \mathrm{Hom}_{k[G]}(k[G], \rho_i) \simeq \rho_i$ AS A VECTOR SPACE ∎

Corollary 0.10. *Let ρ_i be the (conjugacy classes of) irreps of G, n_i the dimension of ρ_i.*
Then $\sum_i n_i^2 = |G|$.

Proof. $|G| = \dim_k k[G] = \dim_k \oplus_i \rho_i^{\oplus \dim \rho_i} = \sum n_i^2$ ∎

Theorem 0.11. *Let G be a finite group, $k = \overline{k}$ an algebraically closed field of characteristic 0, $\rho_1, \dots, \rho_n$ the irreps of G. Then $\{\chi_{\rho_i}\}$ is an orthonormal basis of $Cl(G)$.*

Proof. We know it's orthonormal (so in particular linearly independent), so it is left to show that this indeed spans all of $Cl(G)$.
What remains to show is that χ_{ρ_i} span $Cl(G)$.
It is enough to show that if $\psi \in Cl(G)$ with $\langle \psi, \chi_{\rho_i} \rangle = 0$ for all i, then $\psi = 0$, i.e. the orthogonal complement of the span of the χ_{ρ_i} is trivial.

Definition 0.5. If $\psi : G \to \mathbb{Q}(\mu_\infty)$ is a class function,

$$\gamma_\psi \stackrel{\text{def}}{=} \sum_{g \in G} \psi(g) e_g \in Z(k[G])$$

Example 0.2. If $\psi : G \to k$, $g \mapsto \frac{1}{|G|}$, $\gamma_\psi = R_G$.
We will compute what γ_ψ does to a representation.

Proposition 2. *If ρ is an irreducible representation of G, then $\gamma_\psi : \rho \to \rho$ is multiplication by the scalar $\frac{|G|}{\dim \rho} \langle \psi, \chi_{\rho^v} \rangle$*

Proof.

1. First, $\gamma_\psi : \rho \to \rho$ is a homomorphism of G-representations, which follows from $\gamma_\psi \cdot g \cdot v = g \cdot \gamma_\psi \cdot v$ for all $g \in G, v \in \rho$, as $\gamma_\psi \in Z(k[G])$.

2. By Schur, $\gamma_\psi : \rho \to \rho$ is a scalar.

3. $\gamma_\psi = \frac{\mathrm{tr}(\gamma_\psi|\rho)}{\dim \rho} \cdot \mathrm{Id}_\rho$, so

$$\mathrm{tr}(\gamma_\psi|\rho) = \mathrm{tr}(\sum_{g \in G} \psi(g) e_g|\rho) = \sum_{g \in G} \psi(g)\chi_\rho(g) = |G|\langle \psi, \overline{\chi_\rho} \rangle = |G|\langle \psi, \chi_{\rho^v} \rangle$$

∎

Now, consider $\gamma_\psi : k[G] \to k[G]$. This is zero as γ_ψ acts as zero on every irrep (because it pairs to zero with all the irreps), and because it sends 1 to γ_ψ, γ_ψ has to be zero.

∎

Corollary 0.12. *(of earlier claim)*
$\frac{\dim \rho_i}{|G|}\gamma_{\chi_{\rho_i^v}}$ *acts as* 1 *on* ρ_i*, and 0 on* ρ_j*, for* $\rho_i \neq \rho_j$ *are irreps.*

Proof. ■

Corollary 0.13. *Given any* $V = \oplus \rho_i^{\oplus n_i}$*,*

$$\frac{\dim \rho_i}{|G|}\gamma_{\chi_{\rho_i^v}}$$

acts as a projection onto $\rho_i^{n_i} \subseteq V$*, which is called the* ρ_i *isotypic part of* V*.*

Corollary 0.14. *#irreps of* G = *#conjugacy classes of* G

Proof. Let $\{\rho_i\}$ be the irreps of G (up to conjugacy (i.e isomorhpism)).
Then $\{\chi_{\rho_i}\}$ is a basis for $Cl(G)$, so # of irreps $= \dim_k Cl(G) =$ #conjugacy classes of G.
Remark: These two numbers are equal, but there is no natural or canonical bijection between the two sets in general.

Classifying rep'ns

Theorem 0.15. *G is abelian iff all irreps of G are 1-dimensional.*

Proof. Let V be an irrep. If G is commutative, then $\cdot g : V \to V$ is a G-homomorphism for all $g \in G$.
By Schur, each $g \in G$ acts as a scalar. Now every subspace of V is a subrep, hence V is 1-dimensinoal.
Now suppose that all irreps are 1-dimensional. Let n_i be the dimensions of the irreps ρ_i, and let c be the number of conjugacy classes (or equivalently the number of irreps) of G. Then $|G| = \sum_i n_i^2$, but this is at least c, because we are taking the sum of c positive numbers, but each n_i is 1, so each element of G is its own conjugacy class.

■

Example 0.3. Take $G = \mathbb{Z}/n\mathbb{Z}$
For each element $\zeta \in \mu_n \overset{\text{def}}{=}$nth roots of unity, consider $\chi_\zeta : \mathbb{Z}/n\mathbb{Z} \to k^*, a \mapsto \zeta^a$
This gives n distinct reps, which is the number of conjugacy classes, hence we have a complete list.

Example 0.4. S_3 has conjugacy classes $[e], [(12)], [(123)]$, so there are 3 irreducible representations. We have a trivial representation, whose character sends all conjugacy classes to 1.
We also have $sgn : S_3 \to \{\pm 1\} \subseteq k^*$, so χ_{sgn} sends $[e]$ to 1, $[(12)]$ to -1, and $[(123)]$ to 1.

At this point we know there must be a third representation, *std*, and we can fill in its row in the character table somehow. *std* is given by $S_3 \curvearrowright \mathbb{C}^{\{1,2,3\}} / \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, with $\chi_{std} = \chi_{\mathbb{C}^{\{1,2,3\}}} - \chi_{triv}$, so $\chi_{st}(e) = 2, \chi_{std}(12) = 0, \chi_{std}(123) = -1$.
We claim that χ_{std} is irreducible. To see this, we compute

$$\langle \chi_{std}, \chi_{std} \rangle = \frac{1}{6}(2^2 + 3 * 0^2 + 2(-1)^2) = 1.$$

Example 0.5. $Q_8 = \langle \pm 1, \pm i, \pm j, \pm k \rangle$, with multiplication given as in the quaternion group, $i^2 = j^2 = k^2 = ijk = -1$.
Conjugacy classes: (e),-1, $\{\pm i\}$,$\{\pm j\}$,$\{\pm k\}$.
χ_{triv} sends them all to 1, of course.

Lecture 5, 21/1/25

	1	-1	{i, -i}	{j,-j}	{k,-k}
triv	1	1	1	1	1
i-ker	1	1	1	-1	-1
j-ker	1	1	-1	1	-1
k-ker	1	1	-1	-1	1
?	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Let $\mathbb{H} = \mathbb{R}\langle 1, i, j, k \rangle$. Then $Q_8 \curvearrowright \mathbb{H}$ by left multiplication, $\mathbb{H} \curvearrowleft \mathbb{C}$ by multiplication by i on the right. This example might be useful to think about for the homework.
Now let's get the character table for S_4.

conj class	0	(12)	(123)	(12)(134)	(1234)
size	1	6	8	3	6
sgn	1	-1	1	1	-1
$std = \mathbb{C}^4 / \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$	3	1	0	-1	- 1
$std \otimes sgn$	3	-1	0	-1	1
$std \circ \pi_{4 \to 3}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

If S_4 is the symmetries of a tetrahedron, then $\pi_{4 \to 3}$ is the map from S_4 to S_3 furnished by S_4 acting on pairs of sides, of which there are 3.
How does the structure of G interact with its representation theory?

Proposition 3. *(Homework)*
Let G, H be groups, (NOT necessarily finite!), $k = \overline{k}$ algebraically closed.
Then any irrep of $G \times H$ has the form $V \boxtimes W$, where

- *V is an irrep of G,*
- *W is an irrep of H*
- $(g,h) \cdot v \boxtimes w = (g \cdot v) \boxtimes (h \cdot w)$
  This is the same as tensoring the two reps of $G \times H$ we get from

$$\begin{tikzcd} & G \arrow[r] & \mathrm{GL}(V) \\ G \times H \arrow[ur] \arrow[dr] & & \\ & H \arrow[r] & \mathrm{GL}(W) \end{tikzcd}$$

Proof. HW ■

We have now classified (modulo the homework) all representations of all finite abelian groups.
In some sense, (the sense of Artin's theorem) is that the representation theory of a group is controlled by the rep theory of its abelian subgroups.
<u>Restriction & induction</u>
Let $H \subseteq G$ be a subgroup of H, G again finite.
We have a restriction functor $Res_H^G : Rep_G \to Rep_H$,

$$(\rho : G \to \mathrm{GL}(W)) \mapsto \rho|_H$$

There is a functor going the other way called induction, $Ind_H^G : Rep_H \to Rep_G$.

Definition 0.6. Let V be an H-representation. Then

$$Ind_H^G V \overset{\text{def}}{=} k[G] \otimes_{k[H]} V$$

Equivalent descriptions:

$$Ind_H^G(V) \overset{\text{def}}{=} \{\phi : G \to V \mid \phi(gh^{-1}) = h\phi(g) \forall g \in G, h \in H\}$$

An element of the former looks like $\sum_g e_g \otimes v_g$. Take $e_g e_h \otimes v = e_g \otimes (h \cdot v)$, $g \cdot \phi = g\phi(g^{-1}-)$. Think about this and see how this makes the descriptions the same. One more description:

$$Ind_H^G(V) = \bigoplus_{g \in G/H} g_i \cdot V$$

where $g \cdot \sum g_i v_i = \sum g_{j(i)} k_i \cdot V$ where $g_j g_i = g_{j(i)}$ (???)
Exercise: check the above is equivalent to the other two things.

Example 0.6.

1. $Ind_H^G triv = k^{G/H}$ follows from second description. By definition, $Ind_H^G triv = \{f : G \to k \mid f(gh^{-1}) = h \cdot f(g) = f(g)\} = \{f : G/H \to k\}$
2. $Ind_{(1)}^G k = k[G] \otimes_k k = k[G]$
3. Suppose $\chi : H \to \mathbb{C}^\times$ is a representation. What is $Ind_H^G \chi$? To find $Ind_H^G \chi(g)$, pick coset representative g_i from G/H, and we get permutation matrix for $G \curvearrowright G/H$ times the diagonal matrix whose ith entry is $\chi(h_i)$, where $gh_i^{-1} = g_{j(i)}h_i^{-1}$

Lecture 6, 23/1/25

Corrections:
In the homework, problem 4 part a) should include the assumption that the action of G on H by conjugation is inner, i.e. for all $g \in G$, the map $(\cdot)^g : H \to H$ sending $h \mapsto ghg^{-1}$ is $(\cdot)^{h'}$ for some $h' \in H$.
Remark: An example is if we take $G = A \times B, H = A \times \{1\}$. Then $(\cdot)^{(a,b)} = (\cdot)^{(a,1)}$
Last time:

- We did character tables for Q_8, S_4
- We stated the classification of irreducible representations of a product $G \times H$
- Classification of irreps of finite abelian groups
- Restriction & induction

Here is more on induction:
$\mathrm{Ind}_H^G(V) \overset{\text{def}}{=} k[G] \otimes_{k[H]} V$, where $k[G]$ is a right module and V is a left one. Tensoring a right with a left yields an abelian group (indeed a k-vector space), and it all works out because $k[G]$ is a left $k[G]$ module.
It is also the set $\{\phi : G \to V \mid \phi(gh^{-1}) = h \cdot \phi(g) \text{ for all } g \in G, h \in H\}$, where

$$g \cdot \phi = \phi(g^{-1} \cdot)$$

<u>Explanation</u>
An element of $k[G] \otimes_{k[H]} V$ is a formal sum $\sum e_g \otimes v_g$ such that $e_g e_h \otimes v = e_g \otimes (h \cdot v)$
How to recognize induced representations:

- Suppose V is a G-rep, $W \subseteq V$ is H-stable. When is $V \simeq \mathrm{Ind}_H^G W$?
- Consider $gW \subseteq V$. Because W is H-stable, this only depends on $[g] \in G/H$

Proposition 4. *$V = \mathrm{Ind}_H^G W$ if and only if $V = \oplus_{g \in G/H} gW$*

Proof. Sketch
Recall the third version, $\operatorname{Ind}_H^G V = \oplus_{g_i \in G/H} g_i U$ ■

Proposition 5.

$$\begin{aligned}\chi_{\operatorname{Ind}_H^G \rho}(u) &= \frac{1}{|H|} \sum_{g \in G, g^{-1}ug \in H} \chi_\rho(g^{-1}ug) \\ &= \sum_{x \in G/H} \hat{\chi}_\rho(x^{-1}ux)\end{aligned}$$

where $\hat{\chi}_\rho(v) = \begin{cases} \chi_\rho(v) & v \in H \\ 0 & \textit{otherwise} \end{cases}$

Proof. ■

Proposition 6. *Let $H \subseteq G$ be a subgroup of finite index. Then*

$$\operatorname{Hom}_G(\operatorname{Ind}_H^G V, W) \simeq \operatorname{Hom}_H(V, \operatorname{Res}_G^H W)$$

Proof. This is a special case of the tensor-hom adjunction:

$$\begin{aligned}\operatorname{Hom}_G(\operatorname{Ind}_H^G V, W) &\simeq \operatorname{Hom}_G(k[G] \otimes_{k[H]} V, W) \\ &= \operatorname{Hom}_H(V, \operatorname{Hom}_G(k[G], W)) \\ &= \operatorname{Hom}_H(V, \underbrace{W}_{\text{as an H-rep}}) \\ &= \operatorname{Hom}_H(V, \operatorname{Res}_G^H W)\end{aligned}$$

■

Corollary 0.16. *Let V be a representation of H, W is a representation of G, both finite. Then*

$$\langle \chi_{\operatorname{Ind}_H^G V}, \chi_W \rangle = \langle \chi_V, \chi_{\operatorname{Res}_G^H W} \rangle$$

Proof. These numbers are the dimensions of the hom-spaces, which are the same by the above. ■

Theorem 0.17 (Artin)**.** *Let G be a finite group, $k = \overline{k}, \operatorname{char} k = 0$. Then the map*

$$\bigoplus_{H \subseteq G cyclic} Cl(H) \twoheadrightarrow Cl(G)$$

For each cyclic group H, it acts on characters linearly, so we can extend that to $Cl(H)$, and we can extend that to $\oplus Cl(H)$

Proof. Remark: Let G be a finite group, $R(G)$ be the "representation ring of G",

$$R(G) = \oplus_{\rho_i \text{ irreps of } G}\mathbb{Z}[\rho_i]$$

with $[\rho_i] \cdot [\rho_j] = [\rho_i \otimes \rho_j]$, by writing $\rho_i \otimes \rho_j = \bigoplus_{\rho_k \text{ irreps}} \rho_k^{n_k}$

Proposition 7. *There is a map $R(G) \to Cl(G)$ sending $[\rho_i] \to \chi_{\rho_i}$. This is a ring homomorphism (because character of tensor product is pointwise product of characters). There is an induced map $R(G) \otimes_{\mathbb{Z}} k \to Cl(G)$ which is an isomorphism.*

Proof.

1. These are vector spaces of the same dimension

2. The map is surjective because (for example,) characters of irreps span.

■

Corollary 0.18 (to Artin's theorem)**.** *The map (linear extension of $\oplus \operatorname{Ind}_H^G$)*

$$\bigoplus_{H \leq G \text{ cyclic}} R(H)_k \to R(G)_k$$

is surjective.
I.e. every representation of G is a "k-linear combo" of irreps induced from cyclic subgroups.

Corollary 0.19.

1. $\oplus_{H \leq G \text{ cyclic}} R(H)_{\mathbb{Q}} \to R(G)_{\mathbb{Q}}$ *is surjective, i.e. every irreducible character of G is a $\mathbb{Q}$-linear combination of characters induced from cyclic subgroups.*

2. $\oplus_{H \leq Q \text{ cyclic}} R(H) \to R(G)$ *has finite cokernel.*

Proof.

- (1) $\implies$ (2) because the image of Ind spans $R(G)$ rationally by (1), i.e. given $x \in R(G)$, there is N such that $N \cdot x \in \operatorname{Im}(\operatorname{Ind})$, so the cokernel is torsion, and torsion finitely generated abelian groups are finite.
- We know (1) by Artin, because $\operatorname{Ind}_{\mathbb{Q}} \otimes_{\mathbb{Q}} k$ is surjective, as rank r invariant under extension of scalars?

■

We now prove Artin's theorem:

Proof. It is enough to show that the adjoint map of $\oplus \operatorname{Ind}_H^G$ is injective. But $\langle \operatorname{Ind}\chi, \psi\rangle = \langle \chi, \operatorname{Res}\psi\rangle$, so

$$\bigoplus \operatorname{Res}_G^H : Cl(G) \to \bigoplus_{H \leq G \text{ cyclic}} Cl(H)$$

is adjoint to Ind. Now let ψ be in the kernel; then $\operatorname{Res}_G^H \psi \equiv 0$ for all H, which implies $\psi \equiv 0$, so we win. ■

Loose ends:

- Structure of $k[G]$
- Integral theory
- Corollary of all this discussion: if G is a finite group, ρ an irrep, then $\dim\rho \mid |G|$

Structure of $k[G]$ (and more generally, semisimple algebras)

Definition 0.7. Let k be a field, R a k-algebra (possibly non-commutative). Then R is semisimple if

1. R is finite dimensional as a k-vector space

2. All left R-modules which are finite-dimensional k-vector spaces are semisimple.

Theorem 0.20. *Let R be semisimple k-algebra. Then*

$$R \simeq \prod \operatorname{Mat}_{n_i}(D_i)$$

where D_i are division k-algebras.

Proof. (Take $R = k[G]$)
Conside R as a left R-module;

$$R \simeq \oplus M_i^{\oplus n_i}$$

where M_i is simple, all M_js are mutually non-isomorphic left R-modules.
Note $\operatorname{Hom}_{R-\text{mod}}(M_i, M_i)$ is a division algebra (otherwise we would have a morphism with a kernel, but M_i is simple).
Because $R^{op} \simeq \operatorname{Hom}_{R-\text{mod}}(R, R)$, this means

$$R \simeq \operatorname{Hom}_{R-\text{mod}}(\oplus M_i^{\oplus n_i}, \oplus M_i^{\oplus n_i})$$

Now, $\text{Hom}_{R-\text{mod}}(M_i, M_j) = 0$ for $i \neq j$ (again by simplicity and mutual nonisomorphicness) so

$$\text{Hom}_{R-\text{mod}}(R, R) \simeq \oplus_i \text{Hom}_{R-\text{mod}}(M_i^{n_i}, M_i^{n_i})$$

So if we take $D_i^{op} = \text{Mat}_{n_i}(\text{Hom}(M_i, M_i))$, we win.

■

Corollary 0.21. *Let $k = \overline{k}$. Then $R \simeq \oplus \text{Mat}_{n_i}(k)$*

Proof.

1. Finite dimensional central division algebras over an algebraically closed field are the field itself.

2. Or, same proof as in Schur,

$$\text{Hom}_{R-\text{mod}}(M_i, M_i) = k$$

■

Let's specialize to $R = k[G]$.
As a $k[G]$-module, $k[G] \simeq \rho_i^{\oplus n_i}$, so we have a map

$$k[g] \to \bigoplus_{\rho_i \text{ irrep}} \underline{\text{Hom}}_k(\rho_i, \rho_i) \simeq \oplus_{\rho_i \text{ irrep}} \rho_i \boxtimes \rho_i^v \simeq \oplus_{\rho_i \text{ irrep}} \rho_i \otimes \text{Hom}(\rho_i, k[G])$$

$$x \mapsto \text{right multiplication by } x$$

Recall: If V is any G-rep, then $V = \oplus \rho_i \otimes \text{Hom}_G(\rho_i, V)$,
so we have $k[G] \to \oplus \text{End}(\text{Hom}(\rho_i, k[G]))$

Claim. *This isomorphism of rings is $G \times G$-equivariant if we give* $\text{End}(\rho_i^{\dim \rho_i})$ *the $G \times G$ structure $\rho_i \boxtimes \rho_i^v$*

Proof. We need to check $\text{End}(\rho_i^{\dim_i})$ as a right G-module it is $(\rho_i^v)^{\dim \rho_i}$.
If $G \hookrightarrow G \times G$ by $g \mapsto (g, g^{-1})$, then it has an invariant in $\text{Hom}_G(\rho_i^{\dim \rho_i}, \rho_i^{\dim \rho_i})$,
As G-reps, $\text{Hom}(\rho_i, \rho_i) \simeq \rho_i \otimes \rho_i^v$

Claim. *Given a rep $V \boxtimes W$ of $G \times G$, the structure of V and $V \boxtimes W|_{(g, g^{-1})}$ determines W.*

Proof. ■

Lecture 7, 28/1/25

Substitute for today: Dr Jacob Tsimerman
Let $k = \overline{k}$ be an algebraically closed field of characteristic 0, G a finite group.
Let $(\rho_1, V_1), \dots, (\rho_n, V_n)$ be the irreducible left representations of G.

Theorem 0.22.
$$k[G] \cong \bigoplus_{i=1}^{n} \rho_i \boxtimes \rho_i^v = \bigoplus_{i=1}^{n} V_i \otimes V_i^*$$
as $G \times G$-reps $((g, g') \cdot v \otimes v^* = (g \cdot v) \otimes v^* + v \otimes (g' \cdot v^*))$

Proof. Let $W_i \stackrel{\text{def}}{=} \mathrm{Hom}_G(V_i, k[G])$. Then
$$k[G] \cong \bigoplus_{i=1}^{n} V_i \otimes W_i$$
as $G \times G$-representations because we get the right G-action for free.

Claim. *As right G-representations, $W_i \cong V_i^*$*

Proof.
Convention: Given an element $x = \sum_{g \in G} a_g(x) g \in k[G]$, we use $a_g : k[G] \to k$ to denote the g-th coefficient.
This has the property that $a_g(x \cdot g') = a_{g' g^{-1}}(x)$
Define $\psi : W_i \to V_i^*$ by
$$\psi(\phi) \stackrel{\text{def}}{=} a_1 \circ \phi$$

Claim. *ψ is an isomorphism*

Proof. Suppose $\phi \in W_i$. For $g \in G$, $a_g(\phi(v)) = a_1(g^{-1}\phi(v))$. But ϕ is a map of left G-modules, so this is $a_1(\phi(g^{-1}(v))) = \psi(\phi)(g^{-1}v)$.
So, we can write
$$\phi(v) = \sum_{g \in G} \psi(\phi)(g^{-1}v) \cdot v$$
So ϕ is entirely determined by $\psi(\phi)$, or in other words, ψ is injective.
On the other hand, let $\ell \in V^*$.
Consider $\phi_\ell \in W_i$, $\phi_\ell(v) = \sum_{g \in G} \ell(g^{-1}v) \cdot g$

Claim. $\phi_\ell \in W_i$

Proof. Let $g_0 \in G$. Then
$$\phi_\ell(g_0 v) = \sum_{g \in G} \ell(g^{-1} g_0 v) = \sum_{g \in G} \ell(g^{-1} v) \cdot g_0 g = g_0 \cdot \phi_\ell(v)$$

This shows that ψ is surjective. ■

Claim. *ψ respects the right G-action.*

Proof.

$$\begin{aligned}\psi(\phi^{g_0})(v) &= \psi(\phi)(g_0 v)\\ &= a_1(\phi(g_0 v))\\ &= a_1(g_0\phi(v))\\ &= a_{g_0^{-1}}(\phi(v))\end{aligned}$$

On the other hand,

$$\begin{aligned}\psi(\phi^{g_0} v) &= a_1(\phi^{g_0}(v))\\ &= a_1(\phi(v)g_0)\\ &= a_{g_0^{-1}}(\phi(v))\end{aligned}$$

So $\psi(\phi^{g_0}) = \psi(\phi)^{g_0}$ ■
This proves the theorem. ■
Matrix Coefficients
Let $\{v_1, \dots, v_n\}$ be a basis for an irreducible representation V.
Let $\{v_1^*, \dots, v_n^*\}$ be the dual basis for V^*.

Definition 0.8. Given $1 \le i, j \le m$, the matrix coefficient $a_{i,j}$ is given by

$$a_{i,j}(g) = v_i^*(g \cdot v_j)$$

This is a function from G to k.
Define $A_{i,j} \in k[G]$ by

$$A_{i,j} \overset{\text{def}}{=} \sum_{g\in G} a_{i,j}(g) \cdot g$$

Theorem 0.23.

$$\langle A_{i,j} \rangle_{1\le i,j\le m} = \rho \boxtimes \rho^v$$

where (ρ, V) is the G-rep.

Proof. ■

Theorem 0.24. *Let G be a finite group, $k = \overline{k}$ an algebraically closed field of characteristic 0.*
Let (ρ, V) be an irreducible representation of G.
Then $\dim V \mid |G|$

Proof.

Corollary 0.25. *If $d_1, \ldots, d_n$ is the dimensions of the irreps of G, then*

1. *$m =$number of conjugacy classes of G (often called m)*

2. *$d_i \| |G|$ for all i*

3. $\sum_{i=1}^{m} d_i^2 = |G|$

Proof. ■

Example 0.7. If $G = S_3$, $m = 3$, with conjugacy classes $[\mathrm{Id}], [(12)], [(123)]$, then we have $d_1 = 1, 1 + d_2^2 + d_3^2 = 6, d_2, d_3 \mid 6$.
So we must have $d_2 = 1, d_3 = 2$.
Recollections of algebraic integers

Definition 0.9. Let R be a commutative ring.
Then $x \in R$ is integral, or an algebraic integer, if x satisfies a monic integer polynomial.

Example 0.8.

- 3
- $\sqrt{5}$
- $\frac{1+\sqrt{5}}{2}$

Non-examples include

- $\frac{3}{7}$
- $\frac{1}{\sqrt{2}}$

Proposition 8. *The following are equivalent:*

1. *x is integral*

2. *The subring generated by x is a finitely generated $\mathbb{Z}$-module*

3. *The subring generated by x is contained in a finitely generated $\mathbb{Z}$-module in R.*

Proof. Let's start with (1) $\implies$ (2).
Suppose $x^N + \sum_{i=1}^{N-1} a_i x^i = 0$, $a_i \in \mathbb{Z}$.
Then $x^N \in \langle 1, x, \ldots, x^{N-1} \rangle_{\mathbb{Z}}$. But then $x^{N+1} \in \langle 1, x, \ldots, x^N \rangle_{\mathbb{Z}}$, so $x^{N+1} \in \langle 1, x, \ldots, x^{N-1} \rangle_{\mathbb{Z}}$.
So the subring generated by x equals $\langle 1, x, \ldots, x^{N-1} \rangle_{\mathbb{Z}}$.
(2) $\implies$ (3) is clear
So let's see (3) $\implies$ (1).

Let $A_N = \langle 1, x, x^{N-1} \rangle_{\mathbb{Z}}$. By assumption, there exists a finitely generated $\mathbb{Z}$-module $B \subset R$ such that $A_1 \subseteq A_2 \subseteq \cdots \subseteq B$
By Noetheriality, the sequence stabilizes, so there exists some M such that $A_M = A_{M-1}$, and so x^M is a finite linear combination of lower powers of x, so there are a_i such that

$$x^M + \sum_{i=1}^{M-1} a_i x^i = 0$$

Corollary 0.26. *The things on the list of non algebraic integers actually belong on the list!*

Proof. ■

Lecture 8, 30/1/25

Sub Prof: Mathilde Gerbelli-Gauthier
End Goal: G finite, ρ irrep of G over $k = \overline{k}$ algebraically closed of characteristic 0.
We want to show that $\dim \rho \mid |G|$
Strategy: Prove that $\frac{|G|}{\dim \rho}$ is an algebraic integer
As a corollary of the proof of the last prop, we get

Corollary 0.27. *Integral elements of R form a subring.*

Proof. ■

Integrality of characters

As always, let G be a finite group, $k = \overline{k}$ algebraically closed of characteristic 0, and $\rho : G \to \mathrm{GL}_n(k)$ just any representation (not necessarily irreducible).

Proposition 9.

1. *The values of the character of $\rho, \chi_\rho(g)$, are algebraic integers*

2. *Let $u = \sum_{g \in G} u(g)g$ be an element of $Z(k[G])$. Suppose that $u(g) \in k$ are algebraic integers. Then u is integral.*

At some point in the classes I missed we show that the indicators of conjugacy classes span the center of $k[G]$.

Proof.

1. $\chi_\rho(g)$ is a sum of roots of unity, hence a sum of algebraic integers, hence an algebraic integer.

2. Using a previous result, let $u(g)$ be the indicator function of a conjugacy class. But the sub-$\mathbb{Z}$-module of $Z(k[G])$ generated by the indicator functions is a subring (because the product of $1_{C_1} \cdot 1_{C_2}$ is a linear combination of the indicators of conjugacy classes, and the coefficient in front of each g is an integer).

   Thus each indicator of a conjugacy class is contained in a finitely generated $\mathbb{Z}$-module, and is integral.

■

Corollary 0.28. *Let ρ be an irrep of G and let $u \in Z(k[G])$ be as before. Then*

$$u_\rho = \frac{1}{\dim \rho} \sum_{g \in G} u(g)\chi_\rho(g) \in k$$

is an algebraic integer.

Proof.

Claim. *Given ρ, $u \mapsto \frac{1}{\dim \rho} \sum u(g)\chi_\rho(g)$ is a ring homomorphism*

Proof.

$$u_1 * u_2 \mapsto \left(\frac{1}{\dim \rho} \sum u_1(g)\chi_\rho(g) \right) \left(\frac{1}{\dim \rho} \sum u_2(g)\chi_\rho(g) \right)$$

■

The goal will be to define a ring-hom from $Z(k[G])$ to k sending u to u_ρ. Since u is integral , it maps to to an integral element of k.

$$u \mapsto \frac{|G|}{\dim \rho} \langle u, \chi_{\rho^v} \rangle = u_\rho$$

$$\sum u'(g)\chi_\rho(g) = |G| \langle u, \rho^v \rangle$$

Recall that $Z(k[G]) \curvearrowright \rho$ by G-homomorphism, that action induces a natural map

$$Z(k[G]) \mapsto \mathrm{Hom}_G(\rho, \rho) = k$$

So

$$u \mapsto \frac{|G|}{\dim \rho} \langle u, \chi_{\rho^v} \rangle$$

The matrix is scalar, so it suffices to compute its trace. Its trace is

$$\sum_{g \in G} u(g)\chi_\rho(g) = |G| \langle u, \chi_{\rho^v} \rangle$$

Dividing by $\dim \rho$ gives the result.

■?

Theorem 0.29. *Let G be a finite group, $k = \overline{k}$ an algebraically closed field of characteristic 0, V_ρ an irrep of G. Then* $\dim V \mid \| G|$

Proof. Set $u = \sum_{g \in G} \chi_\rho(g^{-1})g$. By the above, we have

$$\begin{aligned}\frac{1}{\dim \rho} \sum u(g)\chi_\rho(g) &= \frac{|G|}{\dim \rho}\langle \chi_{\rho^v}, \chi_{\rho^v}\rangle \\ &= \frac{|G|}{\dim \rho} \underbrace{\dim \mathrm{Hom}_G(\rho^v, \rho^v)}_{=1} \\ &= \frac{|G|}{\dim \rho}\end{aligned}$$

But the left hand side is an integral element of $\mathbb{Q}$, so the right hand side is an integral element of $\mathbb{Q}$, hence an integer.

■

Rep theory of the symmetric group

As always, $|G| < \infty, Char(k = \overline{k}) = 0$
Here are some key facts about the symmetric groups:

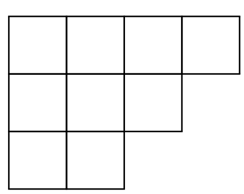
1. The number of irreps of S_n is equal to the number of conjugacy classes in S_n.

2. The conjugacy classes in S_n (aka cycle type) are in bijection with partitions of n.

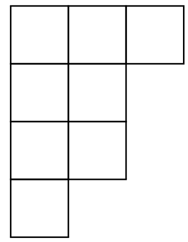
3. The irreps of S_n are also indexed by partitions of n.

Definition 0.10. A partition of n is a sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r)$ such that $\sum \lambda_i = n$.

Definition 0.11. The young diagram D_λ has λ_1 boxes in the first row, λ_2 in the second row, etc.
For example, the corresponding diagram for $\lambda = (4, 3, 1)$



The conjugate partition λ' is the one such that $D_{\lambda'}$ is obtained by D_λ by flipping along the diagonal.
If $\lambda = (4, 3, 1), \lambda' = (3, 2, 2, 1)$. Then $D_{\lambda'}$ is



Proejctions and young symmetrizers

An algorithm: start with λ

1. Number the booxes in your Young diagram D_λ from left to right, top to bottom: you now have a young tableaux.

1	2	3	4
5	6	7	
8			

2. Let $\cdot P \subseteq S_n$ be the subgroup of all permutations that preserve each row of our Young tableaux. E.g. $P \simeq S_4 \times S_3 \hookrightarrow S_8$.

3. $Q \subseteq S_n$ the subgroup that preserves each column of the same Young tableau e.g. $Q \simeq S_3 \times S_2 \times S_2 \hookrightarrow S_8$.

In $\mathbb{C}[S_n]$, define $a = \sum_{p \in P} e_p, b = \sum_{q \in Q} sgn(q) e_q$

4. Suppose that V is a vector space, and $S_n \curvearrowright V^{\otimes n}$ by permuting factors.

The element a symmetrizes along the rows, and projects onto

$$Sym^{\lambda_1}(V) \otimes \cdots \otimes Sym^{\lambda_n}(V)$$

up to an isomorphism.

5. The element b alternates along the columns and projects onto a tensor product of exterior powers indexed by λ':

$$\bigwedge^{\lambda_1'}(V) \otimes \cdots \otimes \bigwedge^{\lambda_n'}(V)$$

6. Set $c = ab$. This is called the Young Symmetrizer

Here are some examples of Young symmetrizers: If $\lambda = (1, \ldots, 1)$, then c gives the sign representation. $\lambda = (n)$ gives the trivial rep.

Irreducibility and idempotency

Theorem 0.30. *A suitable nonzero scalar of $c = ab$ is an idempotent in $\mathbb{C}[S_n]$. Its image, when acting on the regular representation, is irreducible, and denoted V_λ. Distinct partitions give rise to distinct (meaning nonisomorphic) representations and every irep arises from this process for a unique partition.*

Corollary 0.31. *Every representation of S_n is defined over $\mathbb{Q}$.*

Proof. ∎

Example 0.9.

- For S_3, $triv = (4), sgn = (1, 1, 1), std = (2, 1)$
- For S_4, $triv = (4), sgn = (1, 1, 1, 1), std = (3, 1), std \otimes sgn = (2, 1, 1), S_4 \to S_3 = (2, 2)$
- In general, $(d, 1, \cdots, 1)$ corresponds to various exterior powers of the standard representation.

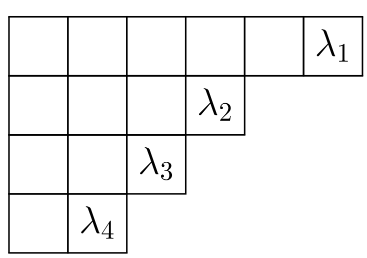
Theorem 0.32. *(Hook-length formula)*
Label each box b in a young diagram (boxes to the right of b) + (boxes below). These are called hook lengths. Then $\dim V_\lambda = \frac{n!}{\prod(\textit{hook lengths of } b)}$

Proof. ∎

Lecture 9, 4/2/25

Let $n \in \mathbb{Z}_{>0}$. Our goal is to classify irreps of S_n. Recall:

Theorem 0.33. *For each partition λ of n, there exists a unique isomorphism class of irrep V_λ of S_n, constructed as follows:*



where $\sum \lambda_i = n$ We let R be the subgroup of S_n which preserves the rows, Q the subgroup preserving the columns. We set

$$a \stackrel{\text{def}}{=} \sum_{g \in P} e_g \in \mathbb{C}[S_n]$$

$$b \overset{\text{def}}{=} \sum_{g \in Q} sgn(g) e_g \in \mathbb{C}[S_n]$$

$$c = ab$$

Then $V_\lambda \overset{\text{def}}{=} \mathbb{C}[S_n]c$ is an irrep of S_n.
Further, every irrep arises in this way.

Proof. Summary: WTS

1. $\dim \operatorname{Hom}_G(V_\lambda, V_\mu) = \delta_{\mu\lambda}$

2. Any irrep is some V_λ.

Remark:

1. There is an explicit dimension formula, the hook-length formula

2. There is an explicit formula for the character of V_λ due to Frobenius.

For more, look for Etingof's "Representation theory" notes for a course given at MIT.
We will begin the proof by writing down c_λ.

Lemma 1.

$$c_\lambda = \sum_{g = \underbrace{p}_{\in P_\lambda} \underbrace{q}_{\in Q_\lambda}} sgn(q) e_{pq}$$

Proof.

$$\begin{aligned} a_\lambda b_\lambda &= \left(\sum_{g \in P_\lambda} e_g \right) \cdot \left(\sum_{h \in Q_\lambda} sgn(h) e_h \right) \\ &= \sum_{g \in P_\lambda, h \in Q_\lambda} sgn(h) \underbrace{e_g e_h}_{e_{gh}} \end{aligned}$$

∎

Goal: Compute $c_\lambda^2 = a_\lambda b_\lambda a_\lambda b_\lambda$

Lemma 2. *For all $x \in \mathbb{C}[S_n]$, $a_\lambda x b_\lambda = \ell_\lambda(x) c_\lambda$, where $\ell_\lambda : \mathbb{C}[S_n] \to \mathbb{C}$ is some linear map.*

Corollary 0.34. $c_\lambda^2 = \ell_\lambda(b_\lambda a_\lambda) c_\lambda$

Proof. Check this on each $e_g \in \mathbb{C}[S_n]$, $g \in S_n$.

Case 1 $g \in P_\lambda Q_\lambda$

We have $g = pq, e_g = e_p e_q$.

$$\begin{aligned} a_\lambda e_g b_\lambda &= \left(\sum_{h\in P_\lambda} e_h\right) e_g \left(\sum_{u \in Q_\lambda} sgn(u) e_u\right) \\ &= \underbrace{\left(\sum_{h\in P_\lambda} e_h e_p\right)}_{a_\lambda} \underbrace{\sum_{u\in Q_\lambda} sgn(u) e_q e_u}_{sgn(q) b_\lambda} \\ &= sgn(q) c_\lambda b_\lambda \\ &= sgn(q) c_\lambda \end{aligned}$$

Case 2 $g \notin P_\lambda Q_\lambda$

In this case, $a_\lambda e_g b_\lambda = 0$. To see this, it is enough to show that there exists a transposition $t \in P_\lambda$ such that $g^{-1}tg \in Q_\lambda$, i.e. g sends two elements of $\{1, \dots, n\}$ in the same row of the Young diagram for λ, to two elements of the same column.

It is enough to show this because

$$\begin{aligned} a_\lambda g b_\lambda &= a_\lambda t g b_\lambda \\ &= a_\lambda g \overbrace{\left(g^{-1} t g\right)}^{sgn=-1} b_\lambda \\ &= -a_\lambda g b_\lambda \end{aligned}$$

This implies $a_\lambda g b_\lambda = 0$.

Now, suppose there do not exist 2 elements in the same row of λ sent to the same column of λ by g.

Then $g \in P_\lambda Q_\lambda$.

To see this, let T be the <u>standard</u> Young Tableau for $\lambda, T' = gT$, P' the stabilizer of rows of T', Q' the stabilizers of columns.

(i) By assumption, any two numbers in the first row of T lie in different columns of T'.

(ii) Then there exists $q_1' \in Q'$ such that $q'T'$ has the same elements in first row (perhaps in a different order).

(iii) Choose $p_1' \in P_\lambda$ such that $p_1' q_1' T'$ has the first row as T.

(iv) Likewise with the 2nd row and so on.

■

Corollary 0.35.

$$\ell_\lambda(b_\lambda a_\lambda) = \frac{n!}{\dim V_\lambda}$$

Proof. later

■

Lecture 10, 6/2/25

Note: For the finite group stuff we are using "Linear reps of finite groups" by Serre (first 3rd is for chemists apparently which is amusing). Specifically chapters 1-3, 6, 9 Other stuff is also on the quercus.
To finish the proof of the theorem, we have to show that the V_λ are irreducible and mutually non-isomorphic. Then, from a bijection between conjugacy classes and partitions, we will be done.
Last time we showed that $a_\lambda x b_\lambda = \ell_\lambda(x)c_\lambda$, and its corrolary, that $c_\lambda^2 = \ell_\lambda(b_\lambda a_\lambda)c_\lambda$

Corollary 0.36.

$$\ell_\lambda(b_\lambda a_\lambda) = \frac{n!}{\dim V_\lambda}$$

Proof. We know that $c_\lambda = \alpha \cdot p_\lambda$, where p_λ is an idempotent.

$$\begin{aligned} c_\lambda^2 &= \alpha^2 p_\lambda^2 \\ &= \alpha^2 p_\lambda \\ &= \alpha c_\lambda \end{aligned}$$

So $\alpha = \ell_\lambda(b_\lambda a_\lambda)$ so we calculate the trace of c_λ : Trace of an idempotent is dim of its image, and c_λ has the same image as p_λ

$$\begin{aligned} \operatorname{tr}(c_\lambda) &= \alpha \cdot \dim \operatorname{Im}(c_\lambda) \\ &= \ell(b_\lambda a_\lambda) \cdot \dim \operatorname{Im}(c_\lambda) \\ &= \ell(b_\lambda a_\lambda) \cdot \dim V_\lambda \end{aligned}$$

Now, if this number is not zero, then we get an idempotent by dividing c_λ by this number. We calculate

$$\begin{aligned} \operatorname{tr}(c_\lambda) &= \sum_{pq \in P_\lambda Q_\lambda} \operatorname{tr}(\cdot e_{pg}) sgn(q) \\ &= \operatorname{tr}(\cdot \operatorname{Id}) \\ &= n! \end{aligned}$$

■

Goal: Compute $\dim_{\mathbb{C}} \operatorname{Hom}_{S_n}(V_\lambda, V_\mu) = \begin{cases} 1 & \lambda = \mu \\ 0 & \text{otherwise} \end{cases}$

We know $\operatorname{Hom}_{S_n}(V_\lambda, V_\mu) = \operatorname{Hom}_{S_n}(\mathbb{C}[S_n]c_\lambda, \mathbb{C}[S_n]c_\mu)$

Proposition 10. *Let A be a $\mathbb{C}$-algebra, $e \in A$ an idempotent, M an A-module. Then $\operatorname{Hom}_A(Ae, M) \simeq eM$ naturally.*

Proof. For $x \in eM$, we have a morphism $x \mapsto (a \mapsto ax)$, and $f \mapsto f(e)$.
e is an idempotent, so $1 - e$ is also an idempotent, so $1 = e + (1 - e)$, so $A \simeq Ae \oplus A(1 - e)$, so $\operatorname{Hom}(Ae, M) \simeq \operatorname{Hom}(A/A(1 - e), M) = \{f : A \to M \mid f(e) = f(1)\} = \{x \in M \mid x \in eM\} = eM$

■

Now let's prove the main theorem.

Proposition 11.

$$\dim_C \operatorname{Hom}_{S_n}(V_\lambda, V_\lambda) = 1$$

Thus, V_λ is irreducible

Proof.

$$\begin{aligned} \operatorname{Hom}_{S_n}(V_\lambda, V_\lambda) &= c_\lambda \mathbb{C}[S_n] c_\lambda \\ &\subseteq a_\lambda \mathbb{C}[S_n] b_\lambda \\ &\subseteq \operatorname{span}_{\mathbb{C}}(c_\lambda) \end{aligned}$$

So the dimension is at most 1. To see it is exactly 1, this space has $c_\lambda \cdot 1 \cdot c_\lambda \neq 0$
So $\dim = 1$, so V_λ is irreducible.

■

Now let λ, μ be two partitions of n. Sat $\lambda > \mu$ if the first $\lambda_i \neq \mu_i$ has $\lambda_i > \mu_i$, i.e. the lexicographical ordering. This is a total ordering, i.e. for any pair (λ, μ), exactly one of $\lambda = \mu, \lambda > \mu, \lambda < \mu$ is true.

Proposition 12. *If $\lambda > \mu$, then $a_\lambda \mathbb{C}[S_n] b_\mu = 0$.*

Proof. In a bit

■

Assuming this, then, if $\lambda \neq \mu$, we want to show that $\dim \operatorname{Hom}_{S_n}(V_\lambda, V_\mu) = 0$.

Proof. We have

$$\begin{aligned} \operatorname{Hom}_{S_n}(V_\lambda, V_\lambda) &= c_\lambda \mathbb{C}[S_n] c_\mu \\ &= a_\lambda b_\lambda \mathbb{C}[S_n] a_\mu b_\mu \\ &\subseteq a_\lambda \mathbb{C}[S_n] b_\mu \\ &= 0 \end{aligned}$$

if $\lambda > \mu$. But $\dim \mathrm{Hom}_{S_n}(V_\lambda, V_\mu) = \dim \mathrm{Hom}_{S_n}(V_\mu, V_\lambda)$, so one, hence both, are 0. ■

Now we prove the proposition

Proof. We will verify it on $e_g \in \mathbb{C}[S_n]$.

Claim. *There exist two numbers on the same row of the standard Young tableaux for λ, same column for $g \cdot$ (standard Young tableaux of μ)*

Proof. Homework ■

Example 0.10. If $g = \mathrm{Id}, \lambda_1 > \mu_1$,

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline & & & \\ \cline{1-3} & & \\ \cline{1-2}\end{array}\;,\quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} & \\ \cline{1-2}\end{array}$$

Let t be the transpotion for these two numbers. Then

$$\begin{aligned} a_\lambda g b_\lambda &= c_\lambda t g b_\mu \\ &= a_\lambda g g^{-1} t g b_\lambda \\ &= -a_\lambda g b_\mu \end{aligned}$$

The rep theory of $\mathrm{GL}_2(\mathbb{F}_p)$

Goal: Understand the irreps of $\mathrm{GL}_n(\mathbb{F}_q)$
What is the size of this group?
$|\mathrm{GL}_2(\mathbb{F}_q)| = (q^2-1)(q^2-q) = q(q^2-1)(q-1)$

Proof.

$$\mathrm{GL}_2(\mathbb{F}_q) = \{(v, w) \mid v, w \in (\mathbb{F}_q)^2 \text{ linearly independent }\}$$

So we can pick any v a nonzero vector, and any w not in the span of v. The number of such possible choices is $(q^2-1)(q^2-q)$ ■

Conjugacy classes:
What are the conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_q)$?
What are the reps of $\mathrm{GL}_2(\mathbb{F}_q)$ over $\mathbb{C}$? Besides the trivial one, we also have $P^1(\mathbb{F}_1) = \{1 - \mathit{dim} \text{ subspaces of } \mathbb{F}_q^2\}$.
This gives a permutation representation $\mathbb{C}^{P^1(\mathbb{F}_q)}$.
We have $\mathit{std} = \mathbb{C}^{P^1(\mathbb{F}_q)}/\mathbb{C}$ has dimension q. Let's compute the character of this representation. Let's call the first set of conjugacy classes above z_x, the second $d_{x,y}$, u_x, $t_{x,y}$

	z_x	$d_{x,y}$	u_x	$t_{x,y}$
triv	1	1	1	1
std	q	1	0	-1

We have

$$\begin{aligned}\langle std, std\rangle &= \frac{1}{q(q-1)^2(q+1)}\left((q-1)q^2 + \frac{q(q-1)(q-2)(q+1)}{2} + 0 + \frac{q^2(q-1)^2}{2}\right)\\ &= 1\end{aligned}$$

What other representation are there?
Choose $\chi : \mathbb{F}_q^\times \to \mathbb{C}$, and then $(\chi \circ \det)^n$, for $n = 1, \ldots, q-2$.
To construct more reps, we will examine some induces reps.

Definition 0.12. Let $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subseteq \mathrm{GL}_2(\mathbb{F}_q)$ (B is for Borel)

$|B| = q(q-1)^2$. Let U be all the matrices of the form $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$.
What is B/U? It is $\mathbb{F}_q^\times \times \mathbb{F}_q^\times$. We will take reps of this and view them as reps of B via the quotient map and induced reps.
For each $\psi : \mathbb{F}_q^\times \times \mathbb{F}_q^\times \to \mathbb{C}^\times$, we can consider the induction $\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{F}_1)}(\psi|_B)$. These are indexed by $\psi(\epsilon, 1)$ and $\psi(1, \epsilon)$.
Then $\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{F}_q)}(\psi_{a,b}|_B)$ has dimension $q+1$ and has character $(q+1)\psi(x)^2$
For $d_{x,y}$ we have $\psi(x,1) + \psi(1,x) + \psi(y,1) \cdot \psi(1,y)$
I have kind of lost the plot at this point I'm sorry.

Proposition 13. *Let $\chi = \sum n_i\rho_i \in R(G)$ be a virtual character of a finite group G. Then χ is the character of an honest irrep iff $\langle \chi, \chi\rangle = 1$, and $\chi(1) > 0$.*

Proof. If we write $\chi = \sum n_i'\rho_i'$, where ρ_i' are irreps, then $\langle \chi, \chi\rangle = \sum_i (n_i)^2 = 1$ by assumption. So at most one of the n_i are nonzero, and it must be ± 1. So $\chi = \pm\rho$ for some irrep ρ. If $\chi(1) > 0$, then $\chi(1) = \pm \dim \rho > 0$

Lecture 11, 11/2/25

For simplicity, we will assume q is odd (even is similar, but annoying to do uniformly)
Recall that $|\mathrm{GL}_2(\mathbb{F}_q)| = (q^2-1)(q^2-q)$

Conjugacy class	number of such conjugacy classes	size of each
$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, x \in \mathbb{F}_q^\times$	$q-1$	1
$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, x \neq y \in \mathbb{F}_q^\times$	$\frac{(q-1)(q-2)}{2}$	$q(q+1)$
$\begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}, x \in \mathbb{F}_q^\times$	$q-1$	q^2-1
$\begin{pmatrix} x & \epsilon y \\ y & x \end{pmatrix}$, ϵ a generator of $\mathbb{F}_q^\times$	$\frac{q(q-1)}{2}$	q^2-q

For the last one, $char \neq 2$

	z_x	$d_{x,y}$	u_x	$t_{x,y}$
triv	1	1	1	1
std	q	1	0	-1

We denote by B all matrices of the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, and by T the span of all matrices of the form $\begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}$

Let U denote the matrices of the form $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$. Then $B/U \cong (\mathbb{F}_q^\times)^2$ (picks out two diagonal entries of B).

Given $\alpha, \beta : \mathbb{F}_q^\times \to \mathbb{C}^\times$ reps of $\mathbb{F}_q^\times$,, then we have a rep $\psi_{\alpha,\beta} : B \to B/U \simeq (\mathbb{F}_q^\times)^2 \to \mathbb{C}^\times$, where this second morphism is by $\alpha \boxtimes \beta$.

	z_x	$d_{x,y}$	u_x	$t_{x,y}$
triv	1	1	1	1
std	q	1	0	-1
$\operatorname{Ind}_B^{\mathrm{GL}_2(\mathbb{F}_q)} \psi_{\alpha,\beta}$	$(q+1)\alpha(x)\beta(x)$	$\alpha(x)\beta(y) + \alpha(y)\beta(x)$	$\alpha(x)\beta(x)$	0

This third line is irreducible if $\alpha \neq \beta$. Note $\operatorname{Ind}_B^{\mathrm{GL}_2} \psi_{\alpha,\beta} = \operatorname{Ind}_B^{\mathrm{GL}_2} \psi_{\beta,\alpha} \operatorname{Ind}_B^{\mathrm{GL}_2} \psi_{\alpha,\alpha} = \alpha \circ \det \oplus (\alpha \circ \det) \otimes (\alpha)(?)$

Recall we proved last time that if G is a finite group, $R(G)$ the representation ring. As a group, $R(G)$ is generated by the irreps of G. Multiplication is given by expressing the tensor of two irreps as a sum of two irreps. Given $V \in R(G)$, V is the class of an irrep if and only if $\langle \chi_V, \chi_V \rangle = 1$, and $\chi_V(\mathrm{Id}_G) > 0$.

Now, we can think of T as being isomorphic to $\{x + \sqrt{\varepsilon} y \in \mathbb{F}_{q^2}^\times\}$. Given a $\varphi : \mathbb{F}_{q^2}^\times \to \mathbb{C}^\times$, we have

	z_x	$d_{x,y}$	u_x	$t_{x,y}$
triv	1	1	1	1
std	q	1	0	-1
$\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{F}_q)} \psi_{\alpha,\beta}$	$(q+1)\alpha(x)\beta(x)$	$\alpha(x)\beta(y) + \alpha(y)\beta(x)$	$\alpha(x)\beta(x)$	0
$\mathrm{Ind}_T^{\mathrm{GL}_2(\mathbb{F}_q)} \varphi$	$q(q-1)\varphi(x)$	0	0	$\varphi(\zeta) + \varphi(\zeta)^q(?)$

where $\zeta = x + \sqrt{2}y(?)$. There's another very convoluted row I didn't quite catch you get by putting the others together in various ways, but that's all of them.
For more: wikipedia Deliegne-Lustztig(sp?) theory.
The story of $\mathrm{SL}_2(\mathbb{F}_q)$ is similar: restrict reps fo $\mathrm{SL}_2(\mathbb{F}_q)$, some of the α, β break up (into at most two pieces), there's some redundancies, and every rep of $\mathrm{SL}_2(\mathbb{F}_q)$ is a restriction.
Fun exercise: Show $\mathrm{PSL}_2(\mathbb{F}_q)$ is simple for $q > 3$ odd.

Lie Algebras

Let k be an arbitrary field, $\mathfrak{g}$ a finite dimensional k-vector space.

Definition 0.13. A Lie bracket $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$

- is Bilinear, meaning $[-,*] : \mathfrak{g} \to \mathfrak{g}$ is linear, as is $[*,-]$ for any $* \in \mathfrak{g}$
- is Alternating, meaning $[x,x] = 0$ for all $x \in \mathfrak{g}$. This implies $[x,y] = -[y,x]$. In characteristic 2, this is stronger!
- satisfies the Jacobi Identity:

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$$

A Lie Algebra is a finite dimensional k-vector space $\mathfrak{g}$ with a Lie bracket.

Corollary 0.37. $[x,y] = -[y,x]$

Proof. $[x+y, x+y] = [x,x] + [y,y] + [x,y] + [y,x] = 0$ ∎

Example 0.11.

- Let R be an associative k-algebra. Then $[x,y] = xy - yx$ will be a Lie bracket.
- $\mathrm{Mat}_{n\times n}(k)$ is a Lie algebra with

$$[A,B] = AB - BA$$

- We have $\mathfrak{sl}_n(k) \subseteq \mathrm{Mat}_{n\times n}(k)$ the set of all matrices with trace 0. Then $[A,B] = AB - BA$ is again a Lie bracket.

- A Lie algebra is abelian if $[-,-] = 0$ for all vectors.

Suppose R is a commutative k-algebra. A k-derivation $\delta : R \to R$ is a k-linear map such that $\delta(a) = 0$ for $a \in k$, and $\delta(xy) = \delta(x)y + x\delta(y)$.

Example 0.12.
We can take $R = k[t], \delta = \frac{\partial}{\partial t}$.
Fact:
$Der_k(R)$, the set of all k-derivations on R, is a Lie algebra, with $[\delta, \gamma] = \delta \circ \gamma - \gamma \circ \delta$.

Lecture 12, 13/2/25

Let's prove the above fact.

Claim. *$Der_k(R)$ is a Lie algebra, with bracket $[\delta, \gamma] = \delta \circ \gamma - \gamma \circ \delta$.*

Proof. We have

$$\begin{aligned}
\delta \circ \gamma(xy) &= \delta(\gamma(x)y + x\gamma(y)) \\
&= \delta\gamma(x)y + \gamma(x)\delta(y) + \delta(x)\gamma(y) + x\delta\gamma(y) \\
\gamma \circ \delta(xy) &= \cdots \\
(\delta \circ \gamma - \gamma \circ \delta)(xy) &= \delta\gamma(x)y + x\delta\gamma(y) - \gamma\delta(x)y - x\gamma\delta(y)
\end{aligned}$$

■

Example 0.13.

- If M is a smooth manifold, $Der_{\mathbb{R}}(C^\infty(M)) = C^\infty$ vector fields on M.
- If G is a Lie group (i.e. a C^∞ manifold equipped with a C^∞ group structure). Define $Lie(G)$ to be the space of left G-invariant vector fields. This has another description as the tangent space of the identity, T_eG.

Definition 0.14. Let $\mathfrak{g}$ be a Lie algebra over k. A representation of $\mathfrak{g}$ is a k-linear map $\rho : \mathfrak{g} \to \mathrm{Mat}_{n\times n}(k)$ which respects the Lie bracket, i.e. $\rho([x, y]) = [\rho(x), \rho(y)]$, where on the right hand side it's just the commutator of matrices.

Definition 0.15. Given Lie algebras $\mathfrak{g}, \mathfrak{h}$ over a field k, a homomorphism $f : \mathfrak{g} \to \mathfrak{h}$ is a k-linear map such that $f([x, y]) = [f(x), f(y)]$ for all $x, y \in \mathfrak{g}$.
Equivalently, a representation is a Lie algebra morphism $\rho : \mathfrak{g} \to \mathfrak{gl}_n$. Equivalently, $\mathfrak{gl}(V) = \mathrm{End}(V)$, so a rep is a morphism $\mathfrak{g} \to \mathfrak{gl}(V)$

Definition 0.16. The Universal Enveloping Algebra $U\mathfrak{g}$ is defined by

$$U\mathfrak{g} \stackrel{\text{def}}{=} \frac{\bigoplus_{n\geq 0} \mathfrak{g}^{\otimes n}}{\langle x \otimes y - y \otimes x - [x, y] \forall x, y \in \mathfrak{g}\rangle}$$

Main property:
$$\mathrm{Hom}_{\text{Assoc } k-\text{ alg}}(U\mathfrak{g}, R) = \mathrm{Hom}_{Lie}(\mathfrak{g}, R)$$

Definition 0.17. A (finite-dimensional) $\mathfrak{g}$-representation is the same as a left $U\mathfrak{g}$-module which is finite dimensional as a k-vector space.

Example 0.14. If $\mathfrak{g}$ is the Lie algebra of a Lie group, then $U\mathfrak{g}$ is the left-invariant differential operators.

Example 0.15. of representations

- $\mathrm{Id} : \mathfrak{g} = \mathfrak{gl}_n \to \mathfrak{gl}_n$. Inside of $\mathfrak{gl}_n$ is $\mathfrak{s}l_n$, and this gives a rep of it.
- $k, [-,-] = 0$ is repped by sending $*$ to $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$. The span of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a subrepresentation.
- $\rho_A : k \to \mathfrak{gl}_n, * \mapsto *A$

Definition 0.18. Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation. $W \subseteq V$ is a <u>subrepresentation</u> if for all $x \in \mathfrak{g}, w \in W, \rho(x)w \in W$. This is the same as saying W is a $U\mathfrak{g}$-submodule of V.

Example 0.16. Let $\mathfrak{b}_n$ be the Lie algebra of upper triangular matrices. It contains $\mathfrak{N}_n$, the strictly upper triangular matrices. The former is solvable, the latter nilpotent.
Given some representations, how can we make new ones?

Operations on representations

Let $\mathfrak{g}$ be a Lie algebra over k, V, W representations of $\mathfrak{g}$, i.e. are equipped with $\rho_V : \mathfrak{g} \to \mathfrak{gl}(V), \rho_W : \mathfrak{g} \to \mathfrak{gl}(W)$.

- $\rho_V \oplus \rho_W : \mathfrak{g} \to \mathfrak{gl}(V) \oplus \mathfrak{gl}(W) \to \mathfrak{gl}(V \oplus W)$ via the block matrix $\begin{pmatrix} \rho_V(x) & 0 \\ 0 & \rho_W(x) \end{pmatrix}$
- V^* is a rep via $\rho_{V^*} : \mathfrak{g} \to \mathfrak{gl}(V^*)$ sending $x \mapsto (f \mapsto f(-\rho_V(x)))$
- $\rho_{V \otimes W} : \mathfrak{g} \to \mathfrak{gl}(V \otimes W), x \mapsto (v \otimes w \mapsto \rho_V(x)v \otimes w - v \otimes \rho_W(x)w)$
- $\underline{\mathrm{Hom}}_k(V, W)$ via $x \cdot f = x \cdot f(-) - f(x \cdot -)$
- $V^{\mathfrak{g}} = \{v \in V \mid xv = 0 \forall x \in \mathfrak{g}\}$

  Observation: $\underline{\mathrm{Hom}}_k(V, W)^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(V, W)$

Definition 0.19. A <u>homomorphism of representations</u> is a k-linear $f : V \to W$ such that $f(x \cdot v) = x \cdot f(v)$ for all $x \in \mathfrak{g}$, $v \in V$.

Representations of $\mathfrak{sl}_2(k)$

Let k be a field of characteristic 0. $\mathfrak{sl}_2 \subseteq \mathrm{Mat}_{2\times 2}(k)$ is the set of 2×2 matrices of trace 0. This is $\mathrm{span}(\underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{e}, \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{f}, \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{h})$

Now,

$$[e, f] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -h$$

Similarly,

$$[h, f] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix} = -2f$$

$$[h, e] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 2e$$

A representation of $\mathfrak{sl}_2$ on V is $E, F, H \in \mathfrak{gl}(V)$ such that

$$[E, F] = H$$
$$[H, F] = -2F$$
$$[H, E] = 2E$$

Given these, we can consider $\mathfrak{sl}_2 \to \mathfrak{gl}(V)$ by $e \mapsto E, f \mapsto F, h \mapsto H$.

Lemma 3. *E, F are nilpotent, i.e. some power of them are 0.*

Proof. WLOG, $k = \overline{k}$. Otherwise, we could just lift to the algebraic closure, and check it is nilpotent there, and then it will be nilpotent over k.
Let v be an eigenvector of H with eigenvalue λ. $HFv = -2Fv + FHv = -2Fv + \lambda Fv = (\lambda - 2)Fv$. Similarly, $HEv = (\lambda + 2)Ev$. So $F^n v = E^n v = 0$ for n sufficiently large, because otherwise H would have infinitely many distinct eigenvalues.
Now let $W \subseteq V$ be the span of eigenvectors of H. This is a subrepresentation of $\mathfrak{sl}_2$ (because F and E send eigenvectors to eigenvectors as shown above). Now consider V/W. This is again an $\mathfrak{sl}_2$-representation, so by induction on dimension E, F act nilpotently on it.

∎

Here is another proof:

Proof. Let's compute

$$\begin{aligned} \mathrm{tr}(E^n) &= \mathrm{tr}(\frac{1}{2}E^{n-1}[H, E]) \\ &= \frac{1}{2}\,\mathrm{tr}(E^{n-1}HE - E^n H) \\ &= 0 \end{aligned}$$

Let $\lambda_1, \dots, \lambda_n$ be the generalized eigenvalues of E with multiplicity. Then $0 = \operatorname{tr}(E^n) = \sum \lambda_i^n$.
Why? In characteristic 0, these generate all symmetric polynomials in λ_i.
The characteristic polynomial of E is $x^n - \sum(\lambda_i)x^{n-1} + \sum_{i<j}\lambda_i\lambda_j)x^{n-2} + \cdots + = x^n$.
So the characteristic polynomial is x^n, so it's nilpotent by Cayley-Hamilton.

■

For now, we will use $k = \mathbb{C}$ so we can say things like "maximal real part."

Lemma 4. *Let λ be the eigenvalue of H with maximal real part. Let v be an H-eigenvector with eigenvalue λ. Let n be minimal such that $F^n v = 0$ (there is such an n because F is nilpotent). Then $Ev = 0, \operatorname{span}(v, Fv, \dots, F^{n-1}v)$ is a subrepresentation of V, $\lambda = n - 1$. In particular, the eigenvalues of H are all integers.*

Proof.

$$\begin{aligned}
EFv &= EFv - FEv \\
&= Hv \\
&= \lambda v \\
EF^2 v &= [E, F]Fv + FEFv \\
&= HFv + \lambda Fv \\
&= (\lambda - 2)Fv + \lambda Fv \\
&= (2\lambda - 2)Fv \\
\cdots \\
EF^n v &= (\lambda^2 + (\lambda + 1)n))F^{n-1}v
\end{aligned}$$

So this is a subrep.

$$\begin{aligned}
0 &= EF^N v \\
&= (-N^2 + (\lambda - 1)NF^{N-1}V
\end{aligned}$$

So $(\lambda + 1)N - N^2 = 0$. So either $N = 0$ (which isn't the case), or $\lambda = N - 1$.
<u>Explicit:</u>
Set $v = v_0, v_i = F^i v_0$, so $Fv_i = v_{i+1}$. $Hv_n = (N - 2 - 2n)v_n$, $Ev_n = (-n^2 + Nn)v_{n-1}$.
Let's call this representation V_N.

Corollary 0.38. *For any non-zero representation W of $\mathfrak{sl}_2$, there exists an $N > 0$ such that V_N is a subrepresentation of W.*

Proof. ■

$V_1 \leq \mathfrak{sl}_2 \to \mathfrak{gl}_2$.
$V_n = Sym^N(V_1)$, where $x \in \mathfrak{sl}_2$ acts on $v_1 \otimes v_2 \otimes \cdots \otimes v_N$ by

$$x \cdot (v_1 \otimes \cdots \otimes v_N) = \sum_{i=1}^{N} v_1 \otimes \cdots \otimes xv_i \otimes \cdots \otimes v_N$$

V_N is the space of homogeneous polynomials in X, Y of degree N.
E acts by $X\frac{\partial}{\partial y}$, F acts by $Y\frac{\partial}{\partial x}$, H by $X\frac{\partial}{\partial x} - Y\frac{\partial}{\partial y}$.

Claim. *V_N are all irreps of $\mathfrak{sl}_2$ (where irrep means the same as in the world of groups, i.e. no nontrivial subrepresentations).*

Proof. Any rep of $\mathfrak{sl}_2$ contains one of these, so it suffices to show that they are irreducibe.
Let $v = \sum a_i v_i \in V_N$ be any element. It suffices to show $\{F^a v, E^b v, a, b \in \mathbb{N}\}$ span.
Acting by some power of E, $E^a v \in \text{span}(v_0)$.
But V_n is the span of $F^a v_0$, so we have shown its irreducible. ■

Definition 0.20. V a representation of $\mathfrak{g}$ is semi-simple if it's the direct sum of irreducibles.

Theorem 0.39. *All finite dimensional representations of $\mathfrak{sl}_2$ in characteristic 0 are semisimple.*

Proof. ■

Lecture 13, 25/2/25

Combinatorics question:
Let $g_{n,k}$ be the number of unlabeled graphs with n vertices and k edges.
Here are the $g_{n,k}$ for various n

1. 1

2. 1, 1

3. 1, 1, 1, 1

4. 1, 1, 2, 3, 2, 1, 1

5. 1, 1, 2, 4, 6, 6, 6, 4, 2, 1

6. 1, 1, 2, 5, 9, 15, 21, 24, 24, 21,15, 9, 5, 2, 1, 1

Observation: $g_{n,k} = g_{n,\binom{n}{2}-k}$

Proposition 14. *The sequence $g_{n,k}$, for a fixed n, is unimodal in k, meaning these sequences are increasing until they peak in the middle, and then go down.*

Proof. We are going to use the representation theory of $\mathfrak{sl}_2$ to prove this unimodality.
Recall: $\mathfrak{sl}_2 = \langle e, f, h \mid [e, f] = h, [h, f] = -2f, [h, e] = 2e\rangle$
We have the standard 2-dimensional representation of $\mathfrak{sl}_2, V$, given by the inclusion $\mathfrak{sl}_2 \hookrightarrow \mathfrak{gl}_2$
Last time, we showed all irreps of $\mathfrak{sl}_2$ are isomorphic to $Sym^n V$ for some nonnegative integer n, where $x \in \mathfrak{sl}_2$ acts on $v_1 \otimes \cdots \otimes v_n$ as

$$(x \cdot v_1) \otimes v_2 \otimes \cdots \otimes v_n + v_1 \otimes (x \cdot v_2) \otimes \cdots \otimes v_n + \cdots + v_1 \otimes \cdots \otimes (x \cdot v_n)$$

We have $h \in \mathfrak{sl}_2$ given by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Fix $v_1, v_{-1} \in V$ such that $hv_1 = v_1, hv_{-1} = -v_{-1}$.
Then h acts by

$$h \cdot (v_1^a v_{-1}^b) = (a + b)v_1 + v_{-1}^b$$

So h acts on $Sym^n V$ with eigenvalues $n, n-2, \cdots, -n+2, -n$
Lemma:
Let V be any finite dimensional $\mathfrak{sl}_2$-representation (not necessarily irreducible).

Lemma 5. *Let d_r be the dimension of the generalized eigenspace of h with generalized eigenvalue r.*
Then $\{d_r\}_{r \text{ even}}, \{d_r\}_{r \text{ odd}}$ are both unimodal and in particular $d_r = d_{-r}$

Proof. By induction on $\dim V$. The base case will be $\dim V = 0$, where it easily holds.
Now, assume the claim holds for $\dim V < n$. Suppose $\dim V = n$. Let $W \subseteq V$ be irreducible. By induction, the claim is true for V/W. By the classification of irreps, it's true for W. But these properties are preserved by addition.

■

The goal now is to write down some generalized eigenspaces of h with exactly this sequence as it's d_r's.
Let V_n be the vector space consisting of formal linear combinations of labeled graphs with n vertices.
Observation: $S_n \curvearrowright V_n$ by permuting the vertices.
Let $V_{n,k}$ be the span of the graphs with k edges. Then

$$V_n = \bigoplus_{k=0}^{\binom{n}{2}} V_{n,k}$$

Observe that $g_{n,k} = \dim V_{n,k}$
Given $i < j \in \{1, \ldots, n\}$, for a labeled graph g, set $a_{ij}(g) = \begin{cases} g \cup (i,j) & (i,j) \text{ not an edge} \\ 0 & \text{otherwise} \end{cases}$.

Similarly, set $b_{ij}(g) = \begin{cases} g \setminus (i,j) & (i,j) \text{ an edge} \\ 0 & \text{otherwise} \end{cases}$.

Let $E = \sum_{i<j} a_{ij}, F = \sum_{i<j} b_{ij}, H = [E, F]$.
$[a_{i,j}, a_{k,\ell}] = 0$, $[b_{i,j}, b_{k,\ell}] = 0$, and $[a_{i,j}, b_{k,\ell}] = 0$ if $(i,j) \neq (k,\ell)$, g if $(i,j) = (k,\ell)$ and g an edge, $-g$ otherwise.
Now

$$\begin{aligned} Hg &= [E, F]g \\ &= \sum_{(i,j)\in g} g - \sum_{(i,j)\notin g} g \\ &= ((2(\text{number of edges of g} - \binom{n}{2})g \end{aligned}$$

Doing this with $[H, F]g$, it turns out this is $-2Fg$.
Next $\mathfrak{sl}_2 \curvearrowright V_n^{S_n}$, and h has eigenspaces $V_{n,k}^{S_n}$ with eigenvalues $2k - \binom{n}{2}$ and hence $g_{n,k} = \dim V_{n,k}^{S_n}$ is unimodal.

Lecture 14, 4/3/25

Let $k = \overline{k}$ be an algebraically closed field (of any characteristic for now), and $\mathfrak{g}$ a Lie algebra over k.

Example 0.17. Recall: Given Lie algebra reps V, W,

- $V \oplus W$ given by $x \cdot (v, w) = (x \cdot v, x \cdot w)$
- $V \otimes W$ given by $x \cdot (v \otimes w) = (x \cdot v) \otimes w + v \otimes (x \cdot w)$
- V^*, given by $(x \cdot f)(v) = f(-x \cdot v)$
- $V^{\mathfrak{g}} = \{v \in V \mid xv = 0 \forall x \in \mathfrak{g}\}$

Exercise: $\underline{\mathrm{Hom}}_k(V, W)^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(V, W)$
We have $\mathfrak{sl}(V) \hookrightarrow \mathfrak{gl}(V)$ is a representation, so $\mathfrak{sl}(V) \curvearrowright V \otimes V = Sym^2 V \oplus \bigwedge^2(V)$ for $char \neq 2$
Pick $q \in Sym^2 V$, $\omega \in \bigwedge^2 CV$, both non-degenerate.
Then $\mathfrak{so}(q) = \{x \in \mathfrak{sl}(V) \mid xq = 0\}$, $\mathfrak{sp}(\omega) = \{x \in \mathfrak{gl}(V) \mid x \cdot \omega = 0\}$
Goal: Study Lie algebras with nice representation theory.
Basic examples will be $\mathfrak{sl}, \mathfrak{so}, \mathfrak{sp}$, and a finite list of exceptional algebras.
Structure:
There are 2 basic classes of Lie algebras:

- Solvable

- Semisimple

Every finite dimensional Lie algebra is built out of these two types.

Definition 0.21. A subspace $I \subseteq \mathfrak{g}$ is an ideal if for all $x \in I, v \in \mathfrak{g}, [x, v] \in I$.
Then $\mathfrak{g}/I$ naturally inherits the structure of a Lie algebra.

Lemma 6. *Let $I_1, I_2 \subseteq \mathfrak{g}$ be ideals. Then*

- *$I_1 + I_2$ is an ideal*
- *$I_2 \cap I_2$ is an ideal*
- *$[I_1, I_2] = \langle [x, y] \mid x \in I_1, y \in I_2 \rangle$ is an ideal*

Proof. Exercise ■

Example 0.18.

- $[\mathfrak{g}, \mathfrak{g}]$ is an ideal (the commutator).
- $[\mathfrak{gl}(V), \mathfrak{gl}(V)] = \mathfrak{sl}(V)$.
- $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian.

Definition 0.22.

- $D^0\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}], D^i\mathfrak{g} = [D^{i-1}\mathfrak{g}, D^{i-1}\mathfrak{g}]$. This is the derived series, or central series
- $C^0\mathfrak{g} = \mathfrak{g}, C^i\mathfrak{g} = [\mathfrak{g}, C^{i-1}\mathfrak{g}]$ This is the lower central series

Definition 0.23. $\mathfrak{g}$ is solvable if $D^i\mathfrak{g} = 0$ for some i, and nilpotent if $C^i\mathfrak{g} = 0$ for some i.
Solvability means if we take $[[[x_1, x_2], [x_3, x_4]], [\cdots] = 0$ if there are enough of them.
Nilpotency essentially means that $[x_1, [x_2, [x_3, \cdots,]]]] = 0$ once you have enough of them.
Unlike in the case of groups, these two notions are extremely closely related.

Example 0.19. $\mathfrak{b}_n = \{ \begin{pmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & * \end{pmatrix} \}$, the upper triangular matrices.

$\mathfrak{n}_n = \{ \begin{pmatrix} 0 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \}$, the nilpotent matrices (strictly upper triangular).
Exercise: $[\mathfrak{b}_n, \mathfrak{n}_n] = \mathfrak{n}_n$

Proposition 15. $\mathfrak{n}_n$ *is nilpotent.*

Proof. We claim $C^i\mathfrak{n}_n$ is the set of upper triangular matrices with the first i superdiagonals are zero. This is because the product of two things of this form has the first $i+1$ superdiagonals zero.

Definition 0.24. The radical of $\mathfrak{g}$ is the maximal solvable ideal in $\mathfrak{g}$.
This is a valid thing to write because there is a unique such ideal, but we must show this:

Lemma 7. *Let* $I_1, I_2 \subseteq \mathfrak{g}$ *be solvable ideals. Then* $I_1 + I_2$ *is solvable.*

Proof. $(I_1 + I_2)/I_1 = I_2/(I_1 \cap I_2)$. These ideals and I_1 are solvable.

Claim. *If* $\mathfrak{g}$ *is a Lie algebra, and* $I \subseteq \mathfrak{g}$, $\mathfrak{g}/I$ *are both solvable, then* $\mathfrak{g}$ *is solvable.*

Proof. $D^i(\mathfrak{g}/I) = 0$ for some i. This implies $D^i\mathfrak{g} \subseteq I$. But iterated commutators are zero in I by assumption.

■

■

Corollary 0.40. *There exists a unique maximal solvable ideal in* $\mathfrak{g}$, *denoted* $rad(\mathfrak{g})$

Proof. We can take the span of all solvable ideals.

■

Definition 0.25. $\mathfrak{g}$ is semisimple if $rad(\mathfrak{g}) = 0$

Proposition 16. *For any* $\mathfrak{g}$, $\mathfrak{g}/rad(\mathfrak{g})$ *is always semisimple.*

Proof. $I \subseteq \mathfrak{g}/rad(\mathfrak{g})$ is a solvable ideal if its preimage under the quotient map is solvable, and this would be an extension of $rad(\mathfrak{g})$.

Example 0.20.

- $\mathfrak{sl}_n$ is simple, meaning there are no non-zero proper ideals
- $\mathfrak{so}(q)$ is simple as well
- $\mathfrak{sp}(\omega)$ is as well

Non-example:
$\mathfrak{gl}(V)$ is NOT semisimple, since scalar matrices make up a solvable ideal.

Definition 0.26. (For culture)
$\mathfrak{g}$ is reductive if $rad(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, v] = 0 \forall v \in \mathfrak{g}\}$

Lecture 15, 6/3/25

Recall we were studying the class of nilpotent Lie algebras, which are a subclass of the class of solvable lie algebras.
Recall a lie algebra is semisimple if $rad(\mathfrak{g}) = 0$, that is there are no nonzero solvable ideals.
Standard examples of solvable Lie algebras include $\mathfrak{b}_n$, the $n \times n$ upper triangular matrices, and standard example of a nilpotent Lie algebra is $\mathfrak{n}_n$, the strictly upper triangular matrices.
Standard example of semisimple Lie algebra is $\mathfrak{sl}_n$.

Theorem 0.41. *(Sophus Lie)*
Suppose $\mathfrak{g}$ is solvable. Then any representation of $\mathfrak{g}$ is upper triangularizable, i.e. given a representation $\rho : \mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$, there exists a basis of V in which $\rho(x)$ is upper triangular for all $x \in \mathfrak{g}$, i.e. ρ can be conjugated (by $\mathrm{GL}(V)$) into $\mathfrak{b}$, i.e. there exists a full flag $V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n$, with $\dim V_i = i$, where each V_i is a representation.

Proof.

Lemma 8. *Suppose $\mathfrak{g}$ is solvable, $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ a representation. Then there exists some $v \in V$ so that v is an eigenvector for $\rho(x)$ all $x \in \mathfrak{g}$.*

Proof. We will prove this by induction on dimension. It is clearly true for $\dim \mathfrak{g} = 0$, so this is our base case.
Now for the induction step: $[\mathfrak{g}, \mathfrak{g}] \subsetneq \mathfrak{g}$ because $\mathfrak{g}$ is solvable.
Pick $\mathfrak{g}' \subseteq \mathfrak{g}$ so that

1. $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}'$

2. $\mathrm{codim}(\mathfrak{g}') = 1$

Now observe

(a) $\mathfrak{g}'$ is an ideal because for all $x \in \mathfrak{g}', y \in \mathfrak{g}$, $[x, y] \in [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}'$.

(b) $\mathfrak{g}'$ is solvable.

Pick $x \in \mathfrak{g} \setminus \mathfrak{g}'$.
By the induction hypothesis, there exists some $w \in V$ such that $y \cdot w = \lambda(w)$ for all $y \in \mathfrak{g}'$.
Let $W = \mathrm{span}(w, x \cdot w, x^2 \cdot w, \cdots)$

Claim. *For all $y \in \mathfrak{g}'$, $yx^k w = \lambda(y)x^k w + \sum_{\ell<k} a_{\ell k}(y)x^\ell w$, $k < \dim W - 1$ i.e. $y \in \mathfrak{g}'$ acts on W via an upper triangular matrix with $\lambda(y)$ on the diagonal.*

Proof. We prove this by induction on k. For $k = 0$ it is certainly true, so this is our base case.
For the induction step:

$$\begin{aligned} yx^k w &= xyx^{k-1}w + [y,x]x^{k-1}w \\ &= \lambda(y)x^k w + \text{ lower order terms } + \lambda([y,x])x^{k-1}w + \text{ lower order terms} \end{aligned}$$

So $\operatorname{tr}(y|W) = \dim W \cdot \lambda(y)$ for all $y \in \mathfrak{g}'$
This implies that $\lambda([y_1, y_2]) = 0$ (here we use the assumption $k = \overline{k}$) for all $y_1, y_2 \in \mathfrak{g}$.
The lower order terms are all ultimately built out of commutators, so $\mathfrak{g}'$ acts on W via $\lambda - \mathrm{Id}$.
Now choose $v \in W$ an x-eigenvector.
This proves the claim.

$\blacksquare$

Let's prove the theorem via a full flag.
Given a rep $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$, we can now pick an eigenvector $v \in V$. Now consider the action of $\mathfrak{g}$ on V/V_1, where $V_1 = \operatorname{span}(v)$. But $\dim(V/V_1) < dim(V)$, so by induction we win.

$\blacksquare$

Corollary 0.42. *Any irrep of a solvable Lie algebra is 1-dimensional.*

Proof. For any $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$, there exists a common eigenvector, which means every rep has a 1-dimensional sub representation.

$\blacksquare$

Corollary 0.43. $\mathfrak{g}$ *is solvable if and only if* $[\mathfrak{g}, \mathfrak{g}]$ *is nilpotent.*

Proof. We start with the easy direction: it is enough for $[\mathfrak{g}, \mathfrak{g}]$ to be solvable. Because then $[\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ are solvable, so their product, $\mathfrak{g}$, is solvable.
Now for the other direction.

Definition 0.27. Let $\mathfrak{g}$ be a Lie algebra. The <u>adjoint representation</u> is the map $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, given by $x \mapsto [x, -]$.
Observe $\ker \mathrm{ad} = \mathfrak{z}(\mathfrak{g})$, the center.
Suppose $\mathfrak{g}$ is solvable. By classification of representations, the adjoint representation factors through $\mathfrak{b}$:

$$\begin{tikzcd} \mathfrak{g} \arrow[rr, "\mathrm{ad}"] \arrow[dr] & & \mathfrak{gl}(\mathfrak{g}) \\ & \mathfrak{b} \arrow[ur] & \end{tikzcd}$$

Observe

(i) $[\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{n}$, and $\mathfrak{n}$ is nilpotent

(ii) $[\mathfrak{g}/\mathfrak{z}(\mathfrak{g}), \mathfrak{g}/\mathfrak{z}(\mathfrak{g})]$ is nilpotent

(iii) $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent by definition of center.

■

Recall $\mathfrak{g}$ is semisimple if $rad(\mathfrak{g}) = 0$.

Example 0.21. Let $\mathfrak{g} = \mathfrak{sl}_2$. We have ad : $\mathfrak{sl}_2 \to \mathfrak{gl}(\mathfrak{sl}_2)$.
$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \to [h, -]$
Eigenvalues of $0, 2, -2$, with eigenvectors h, ev_f, and fwe (?), respectively.
Suppose $I \subseteq \mathfrak{sl}_2$ is an ideal. Then $\mathrm{ad}(h)$ preserves I.
So this means that none of the possible subalgebras are ideals (can check each of them 1 by 1 because there are only finitely many).
So in particular there are no nontrivial solvable ideals.

Corollary 0.44. *$\mathfrak{sl}_2$ is simple (meaning no nontrivial ideals).*

Theorem 0.45. *Let V be an irrep of a Lie algebra $\mathfrak{g}$.*
Then for all $x \in rad(\mathfrak{g})$, x acts on V via scalars. For all $x \in [rad(\mathfrak{g}), \mathfrak{g}]$, x acts on V by 0

Proof. (sketch)
Let $v \in V$ be an eigenvector for $rad(\mathfrak{g})$, i.e. for all $x \in rad(\mathfrak{g})$, $x \cdot v = \lambda(x) \cdot v$
Let $V_\lambda = \{w \in V \mid x \cdot w = \lambda(x) \cdot w \forall x \in rad(\mathfrak{g})\}$
Fix $y \in \mathfrak{g}, x \in rad(\mathfrak{g})$. Let $w \in V_\lambda$. Then $xyw = yxw + [x, y]w = \lambda(x)yw + \lambda([x, y])w$
We can set $W = \mathrm{span}(w, y \cdot w, y^2 \cdot w, \cdots)$. This is certainly preserved by y, and to show it's preserved by the radical you do something similar as the previous theorem.

■

Look at theorem 6.16 in the book for this course (Kiralov (sp?))

Bilinear forms

Definition 0.28. Let $\mathfrak{g}$ be a Lie algebra. A bilinear form B on $\mathfrak{g}$ is invariant if

$$B([x, y], z) + B(y, [x, z]) = 0$$

for all $x, y, z \in \mathfrak{g}$.

Proposition 17. *Suppose $\mathfrak{g}$ is a Lie algebra, B an invariant bilinear form on $\mathfrak{g}$, $I \subseteq \mathfrak{g}$ an ideal.*
Then $I^\perp = \{x \mid B(x, y) = 0 \forall y \in I\}$ is an ideal.

Proof. Let $x \in I, y \in \mathfrak{g}, z \in I^\perp$. We want to show $[y,z] \in I^\perp$.
We have $B([y,z],x) = -B(z, \underbrace{[y,x]}_{\in I}) = 0$

■

Corollary 0.46. *Let $\mathfrak{g}, B$ as before. Then $\mathfrak{g}^\perp$ is an ideal.*

■

Example 0.22. Take $\mathfrak{g} = \mathfrak{gl}_n$, $B(x,y) = \mathrm{tr}(xy)$. Then

$$\begin{aligned}\mathrm{tr}([x,y],z) + \mathrm{tr}(y,[x,z]) &= \mathrm{tr}(xyz - yxz) + \mathrm{tr}(yxz - yzx) \\ &= 0\end{aligned}$$

Example 0.23. Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation. Then we may define

$$B_\rho(x,y) = \mathrm{tr}(\rho(x)\rho(y))$$

How to check if B is nondegenerate:
We have $B : V \times V \to k$
We have a $\psi_B : V \to V^*$ given by $x \mapsto B(x,-)$.
Then B is non degenerate if ψ_B is an isomorphism. So we can pick a basis $e_1, \dots, e_n$, and show $\det(B(e_i,e_j)) \neq 0$.

Theorem 0.47. *Suppose there exists ρ with B_ρ non-degenerate. Then $\mathfrak{g}$ is reductive (i.e. $rad(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$).*

Proof. It is enough to show $[\mathfrak{g}, rad(\mathfrak{g})] = 0$
But $\rho([\mathfrak{g}, rad(\mathfrak{g})])$ acts by zero on any irreducible representation of $\mathfrak{g}$

Claim. *This implies $[\mathfrak{g}, rad(\mathfrak{g})] \subseteq \ker B_\rho$*

Proof. By induction on $\dim \rho$, ρ irreducible (otherwise we take a irreducible subrep $\psi \subseteq \rho$).
It is enough to show $B_\rho = B_\psi + B_{\rho/\psi}$
B_ρ being nondegenerate means $\ker B_\rho = 0$, so $rad(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$
(?)

■

Corollary 0.48. $\mathfrak{gl}_n, \mathfrak{sl}_n, \mathfrak{so}(n), \mathfrak{sp}(2n)$ *are all reductive.*

Proof. We will do the proof for $\mathfrak{gl}_n$.
Take $\rho = \mathrm{Id} : \mathfrak{gl}_n \to \mathfrak{gl}_n$, $B_\rho(x,y) = \mathrm{tr}(xy)$.
Let e_{ij} be the matrix with $(e_{ij})_{k\ell} = \delta^{ij}_{k\ell}$.
We see $B(e_{ij}, e_{k\ell}) = \delta_{i\ell}\delta jk$

■

Definition 0.29. Let $\mathfrak{g}$ be a Lie algebra.
The Killing form K is defined as $K = B_{ad}$, so $K(x,y) = \mathrm{tr}(\mathrm{ad}(x) \cdot \mathrm{ad}(y))$

Theorem 0.49.

(a) $\mathfrak{g}$ *is semisimple iff* K *is non-degenerate*

(b) $\mathfrak{g}$ *is solvable if and only if* $K([\mathfrak{g},\mathfrak{g}],\mathfrak{g}) = 0$.

Proof. Next time.

Definition 0.30. Let $k = \overline{k}, char k = 0$, $\mathfrak{g}$ a finite dimensional k-Lie algebra.
A bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to k$ is invariant if

$$B([x,y],z) + B(y,[x,z]) = 0$$

Given a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, there is a canonical form

$$B_\rho(x,y) = \operatorname{tr}(\rho(x)\rho(y))$$

Definition 0.31. The Killing form is given by $K = B_{\mathrm{ad}}$, so $K(x,y) = \operatorname{tr}(\operatorname{ad}(x)\operatorname{ad}(y))$

Theorem 0.50 (Cartan's criterion)**.**

(a) $\mathfrak{g}$ *is semisimple if and only if* K *is non-degenerate*

(b) $\mathfrak{g}$ *is solvable if and only if* $K([\mathfrak{g},\mathfrak{g}],\mathfrak{g}) = 0$

Corollary 0.51. *Let* $\mathfrak{g}$ *be semisimple,* $I \subseteq \mathfrak{g}$ *an ideal.*
Then there is an $I' \subseteq \mathfrak{g}$ *such that* $\mathfrak{g} = I \oplus I'$ *(as a Lie algebra).*

Proof. $\mathfrak{g}$ is semisimple, so by the theorem (yet to be proven), we can take I' to be the perpendicular subspace under K, and $\dim I + \dim I' = \dim \mathfrak{g}$ simply because K is non degenerate.
We want to show $I \cap I' = 0$.

Claim. $K|_I$ *is the killing form for* I.

Proof. Let $x, y \in I$. Then

$$\begin{aligned} K(x,y) &= \operatorname{tr}(\operatorname{ad}(x)\operatorname{ad}(y)|\mathfrak{g}) \\ &= \operatorname{tr}(\operatorname{ad}(x)\operatorname{ad}(y)|I) + \underbrace{\operatorname{tr}(\operatorname{ad}(x)\operatorname{ad}(y)|\mathfrak{g}/I)}_{=0} \end{aligned}$$

because $\operatorname{ad}(x), \operatorname{ad}(y)$ act on $\mathfrak{g}/I$ as 0. ∎

Now to prove the theorem.
We wish to show $K|_I$ is non-degenerate. Let $x \in I$. We want to show there is a $y \in I$ with $K(x,y) \neq 0$.
We know there is some $z \in \mathfrak{g}$ such that $K(x,z) \neq 0$.
He put it on the homework :(∎

Corollary 0.52. *If $\mathfrak{g}$ is semisimple, then $\mathfrak{g}$ is a direct sum of simple non-abelian Lie algebras.*

Proof. Given by induction on dimension:
Choose any ideal $I \subseteq \mathfrak{g}$. If it doesn't exist you're done. Otherwise $\mathfrak{g} = I \oplus I'$. I, I' are quotients of $\mathfrak{g}$, hence semisimple, so we are done by the inductive hypothesis. ∎

Corollary 0.53. *If $\mathfrak{g}$ is semisimple, then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.*

Proof. It is enough to check for $\mathfrak{g}$ simple and non-abelian (because $\mathfrak{g}$ is a direct sum of such).
But $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}$ is an ideal, hence everything, and $\mathfrak{g}$ is nonabelian by assumption. ∎

Corollary 0.54. *Let $\mathfrak{g}$ be semisimple. Then all finite dimensional $\mathfrak{g}$-representations are semisimple (meaning a direct sum of irreps).*

Proof. ∎

Now to prove Cartan's criterion.
For the rest of today, $k = \mathbb{C}$ (this is a convenience and it will turn out we can reduce to this case).

Theorem 0.55. *Let V be a finite dimensional vector space, $A \in \operatorname{End}(V)$.*

1. $A = A_s + A_n$, *with A_s diagonalizable and A_n nilpotent, with* $[A_n, A_s] = 0$.

2. $\operatorname{ad}(A_s) = \operatorname{ad}(A)_s$. *Moreover,* $\operatorname{ad}(A)_s = P(\operatorname{ad}(A))$ *for some polynomial with vanishing constant term $P \in C[t]$.*

3. *Let $\overline{A_s}$ be the complex conjugate matrix. Then* $\operatorname{ad}(\overline{A_s}) = Q(\operatorname{ad} A_s)$ *for some polynomial $Q \in \mathbb{C}[t]$*

sketch. **1.** First, write A in Jordan normal form by choosing an appropriate basis. We can take A_s to be the diagonal of this, and $A_n = A - A_s$.

In basis free (and thus choice free (and thus canonical)) language,

Let $V_\lambda = \ker((A - \lambda \operatorname{Id})^N) N >> 0$. $V = \bigoplus_\lambda V_\lambda$. We can take $A_s \stackrel{\text{def}}{=} (v_\lambda)_{\lambda \in \mathbb{C}} \to (\lambda v_\lambda)_{\lambda \in \mathbb{C}}$, and then $A_n = A - A_s$.

2. We want to show $A_s = P(A)$ for some $P \in \mathbb{C}[t]$. Let $n_\lambda = \dim V_\lambda$. Choose $p \in \mathbb{C}[t]$ such that

$$P \equiv \lambda \ (\text{mod } t - \lambda)^{n_\lambda}) \ \forall \lambda$$

This exists by the chinese remainder theorem.

Then $P(A) = \lambda$ on V_λ because $(A - \lambda \operatorname{Id})^{n_\lambda} = 0$ on V_λ

Remark: This tells us that A_n is also a polynomial in A $(A - P(A))$

Now,

$$\begin{aligned}\mathrm{ad}(A) &= \mathrm{ad}(A_s + A_n)\\ &= \mathrm{ad}(A_s) + \mathrm{ad}(A_n)\end{aligned}$$

To see that $\mathrm{ad}(A_s) = \mathrm{ad}(A)_s$, it is enough to show $\mathrm{ad}(A_s)$ is diagonalizable, $\mathrm{ad}(A_n)$ is nilpotent (we can do this by computation), and $[\mathrm{ad}(A_s), \mathrm{ad}(A_n)] = 0$.

Computation for $\mathrm{ad}(A_s)$:

$\mathrm{ad}(A_s)$ acts on $V_\lambda \otimes V_{\lambda'}^v$ as $\lambda - \lambda'$,

$$\mathrm{End}(V) = \bigoplus_{\lambda,\lambda'} V_\lambda \otimes V_{\lambda'}^v$$

We know $P \in t\mathbb{C}[t]$ because 0 is a generalized eigenvalue so $P \equiv 0 \pmod{t}^{n_0}$

3. Choose f so that $f(\lambda_i - \lambda_j) = \overline{\lambda_i} - \overline{\lambda_j}$ for all i, j (Lagrange). Then $f(P(\mathrm{ad}(A)) = \mathrm{ad}\,\overline{A_s}$