

Lecture 1 - 7/1/25

Missed :(

Lecture 2 - 9/1/25

Missed :(

Lecture 3 - 14/1/25

Character theory

Consider $\dim \operatorname{Hom}_G(\rho_i, \rho_j) = 1$ if $i = j$ and 0 if $i \neq j$ (meaning if $\rho_i \not\cong \rho_j$)

Recall: Given a representation $\rho : G \rightarrow \operatorname{GL}_n(k)$, the character of ρ , χ_ρ , is given by

$$\chi_\rho : G \rightarrow k, g \mapsto \operatorname{tr}(\rho(g))$$

For today, G will be finite, $k = \bar{k}$ will be algebraically closed, of characteristic 0.

Basic properties of characters:

1. Suppose $\rho : G \rightarrow \operatorname{GL}_n(k)$ is a representation: then $\chi_\rho(e) = n = \dim \rho$.
2. $\chi_\rho(g) = \chi_\rho(hgh^{-1})$ for all $g, h \in G$, i.e. χ_ρ is constant on each conjugacy class of G .

Definition 0.1. A function $f : G \rightarrow k$ which is constant on conjugacy classes is called a class function.

The ρ_i (isomorphism classes of reps) will form an ONB for the space of class functions.

Given $\rho_1 : G \rightarrow \operatorname{GL}_n(k), \rho_2 : G \rightarrow \operatorname{GL}_m(k)$, $\chi_{\rho_1 \oplus \rho_2} = \chi_{\rho_1} + \chi_{\rho_2}$

$$\chi_{\rho_1 \otimes \rho_2} = \chi_{\rho_1} \chi_{\rho_2}$$

To see this, let A, B be diagonalizable (which we have WLOG because the image of any finite group are all diagonalizable over an algebraically closed k of char 0, which follows from Jordan Normal form)

$$\text{Then } \operatorname{tr}(A \otimes B) = \operatorname{tr}(A) \operatorname{tr}(B).$$

I can't see the board he's writing on very well, and also I am not sure how $A \otimes B$ was defined.

Claim. $\chi_\rho : G \rightarrow k$ always factors through $\mathbb{Q}(\mu_\infty)$, the subfield of k containing \mathbb{Q} (k has char 0) generated by all roots of unity ($k = \bar{k}$)

Proof. Because G is finite, ρ_G has finite order, hence its eigenvalues are roots of unity, so the trace is the sum of roots of unity. ■

Definition 0.2. $\bar{\cdot} : \mathbb{Q}(\mu_\infty) \rightarrow \mathbb{Q}(\mu_\infty)$ is the unique field homomorphism with the property that $\bar{\zeta} = \zeta^{-1}$ for all roots of unity $\zeta \in \mathbb{Q}(\mu_\infty)$.

$$5 \quad \chi_{\rho^v} = \overline{\chi_\rho}$$

Recall ρ^v is defined via the formula $g \cdot f = f(g^{-1} \cdot -)$ where f is a functional. We have

$$\begin{aligned} \chi_{\rho^v}(g) &= \text{tr}(\rho(g^{-1})) \\ &= \sum_{\zeta \text{ is an eigenvalue of } \rho(G)} \zeta^{-1} \\ &= \sum \bar{\zeta} = \overline{\chi_\rho(g)} \end{aligned}$$

This also follows from the Hom-tensor adjunction because $\text{Hom}_k(\rho_1, \rho_2) = \rho_1^v \otimes \rho_2$.

Definition 0.3. Let $\chi, \psi : G \rightarrow \mathbb{Q}(\mu_\infty)$ be class functions. We define their inner product by

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)}$$

This indeed is a positive definite non degenerate.

Let $\rho_1 : G \rightarrow \text{GL}_n(k), \rho_2 : G \rightarrow \text{GL}_m(k)$. What is $\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle$?

Theorem 0.1.

$$\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle = \dim_k \text{Hom}_G(\rho_1, \rho_2) = \dim_k \text{Hom}(\rho_1, \rho_2)^G$$

Corollary 0.2. Suppose ρ_1, ρ_2 are irreducible. Then $\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle$ is 0 if ρ_1, ρ_2 are not isomorphic, and 1 if they are. So the ρ_i form an orthonormal basis for the space of class functions. ■

Proof. Let $R_G \in k[G]$ be the element given by

$$R_G = \frac{1}{|G|} \sum_{g \in G} eg$$

We want to show

1. for $v \in V^G, R_G \cdot v = v$.

2. For arbitrary $v \in V$, $R_G \cdot v \in V^G$

To check:

1. We have

$$\begin{aligned} R_G \cdot v &= \frac{1}{|G|} \sum_{g \in G} e_g \cdot v \\ &= \frac{1}{|G|} \sum_{g \in G} v \\ &= v \end{aligned}$$

2. Fix $g \in G$. Then

$$\begin{aligned} g \cdot R_G \cdot v &= g \cdot \left(\frac{1}{|G|} \sum_{h \in G} hv \right) \\ &= \frac{1}{|G|} \sum_{h \in G} gh \cdot v \\ &= \frac{1}{|G|} \sum_{h \in G} h \cdot v \\ &= R_G \cdot v \end{aligned}$$

Corollary 0.3. *Let V be a G -representation. Then $\dim_k V^G = \text{tr}(R_G|V)$*

Proof.

Claim. $\text{tr}(\text{projection}) = \dim_k \text{Im}$

Proof. Claim \implies Cor follows from $\text{tr}(R_G) = \dim \text{Im}(R_G|V) = \dim_k V^G$

■

We can finally prove the theorem:

Proof.

$$\begin{aligned}
 \dim_k \operatorname{Hom}_G(\rho_1, \rho_2) &= \dim_k \operatorname{Hom}_k(\rho_1, \rho_2)^G \\
 &= \operatorname{tr}(R_G | \operatorname{Hom}_k(\rho_1, \rho_2)) \\
 &= \operatorname{tr}\left(\frac{1}{|G|} \sum e_g | \operatorname{Hom}_k(\rho_1, \rho_2)\right) \\
 &= \frac{1}{|G|} \operatorname{tr}(g | \operatorname{hom}_k(\rho_1, \rho_2)) \\
 &= \frac{1}{|G|} \sum_{g \in G} \chi_{\operatorname{hom}_k(\rho_1, \rho_2)}(g) \\
 &= \frac{1}{|G|} \sum_{g \in G} \overline{\chi_{\rho_1}} \chi_{\rho_2} \\
 &= \langle \chi_{\rho_1}, \chi_{\rho_2} \rangle \\
 &= \underbrace{\langle \chi_{\rho_2}, \chi_{\rho_1} \rangle}_{\in \mathbb{Z}}
 \end{aligned}$$

■

Lecture 4, 16/1/24

As always, G will be a finite group, $k = \bar{k}$ is an algebraically closed field of characteristic 0.

$\mathbb{Q}(\mu_\infty)$ is the algebraically closed subfield of \mathbb{C} which contains all the roots of unity, and this comes with the complex conjugate $\bar{\cdot}, \zeta \mapsto \zeta^{-1}$.

Goal: Classify finite dimensional G -representations over k .

We have done:

1. Maschke's theorem, which states that any G -rep in V over k is semisimple.
2. Character theory: $V \sim \chi_V : G \rightarrow \mathbb{Q}(\mu_\infty) \subseteq k, g \mapsto \operatorname{tr}(g|V)$

Definition 0.4. $Cl(G)$ denotes the class functions $G \mapsto \mathbb{Q}(\mu_\infty)$, and it is equipped with an inner product,

$$\langle \psi, \varphi \rangle = \frac{1}{|G|} \sum_{g \in G} \psi(g) \overline{\varphi(g)}$$

Remark: There is an isomorphism $Cl(G) \simeq Z(\mathbb{Q}(\mu_\infty)[G])$, sending φ to $\sum_{g \in G} \varphi(g) e_g$

Warning: They come with different ring structures which are not preserved by this isomorphism.

Last time we used the Reynolds operator to show $\langle \chi_V, \chi_W \rangle = \dim_k \text{Hom}_G(V, W)$.

If ρ_1, ρ_2 are irreps of G , then $\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle$ is 1 if $\rho_1 \cong \rho_2$, and 0 otherwise.

Corollary 0.4. # of conjugacy classes of irreducible representations of $G \leq \dim_{\mathbb{Q}(\mu_\infty)} Cl(G) = \#$ of conjugacy classes of G

Proof. If χ_{ρ_i} are orthonormal, then the number of conjugacy classes of irreps is equal to $\dim \text{span}(\chi_{\rho_i}) \subseteq Cl(G)$, so this number is $\leq \dim Cl(G)$ ■

Proposition 1. Let V be a G -representation. Then

$$\Phi_V : \bigoplus_{\rho_i \text{ irrep of } G} \rho_i \otimes_k \text{Hom}_G(\rho_i, V) \rightarrow V$$

given by $v \otimes f \mapsto f(v)$ is an isomorphism.

Proof. First, we show it is surjective. By Maschke, $V = \bigoplus_{\rho_i \text{ reps}} G\rho_i^{n_i}$.

Let $v \in \rho_i^{n_i} \subseteq V$, $v = (v_1, \dots, v_{n_i})$. Let $f_j : \rho_j \rightarrow \rho_i^{n_i}$ be the inclusion of the j th coordinate.

Then $\Phi_v(\sum_j v_j \otimes f_j) = v$.

Now we show injectivity.

We have

$$\dim_k \bigoplus \rho_i \otimes_k \text{Hom}_G(\rho_i, V) = \dim_k V$$

This follows from

$$\dim_k \text{Hom}_G(\rho_i, V) = n_i$$

This follows from

$$\begin{aligned} \text{Hom}_G(\rho_i, V) &= \text{Hom}_G(\rho_i, \bigoplus \rho_i^{n_i}) \\ &= \bigoplus_j \text{Hom}_G(\rho_i, \rho_j)^{n_i} \\ &= \text{Hom}_G(\rho_i, \rho_i)^{n_i} \end{aligned}$$

Which is n_i -dimensional

$$\dim_k \bigoplus \rho_i \otimes_k \text{Hom}_G(\rho_i, V) = \sum n_i \dim_k \rho_i = \dim V$$
■

Corollary 0.5.

$$V \simeq \bigoplus_{\rho \text{ irreps of } G} \rho^{\langle \chi_{\rho_i}, \chi_V \rangle}$$

Proof. Enough to show $\rho_i^{\langle \rho_i, V \rangle} \simeq \rho_i \otimes_k \text{Hom}_G(\rho_i, V)$, i.e. $\dim_k \text{Hom}(\rho_i, V) = \langle \chi_{\rho_i}, \chi_{\rho_j} \rangle$. But that's the theorem. ■

Corollary 0.6.

$$V \simeq \bigoplus_{\rho_i \text{ irreps}} \rho_i^{\oplus n_i}$$

, then $\langle \chi_V, \chi_V \rangle = \sum_i n_i^2$

Proof. $\chi_V = \sum n_i \chi_{\rho_i}$

■

Corollary 0.7. $V \simeq W \iff \chi_V = \chi_W$

Corollary 0.8. V is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$.

Proof. Write $V = \bigoplus_i \rho_i^{n_i}$: so $\langle \chi_V, \chi_V \rangle = \sum_i n_i^2$ is equal to 1 iff exactly 1 n_i is nonzero, and equal to 1.

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Example 0.1. (The regular representation)

Let $G \curvearrowright k(G)$ via left multiplication.

$\chi_{k[G]}(g) = \text{tr}(g|k[G])$, which is $|G|$ if g is the identity, and 0 otherwise.

Because $g \cdot e_{g'} = e_{gg'}$, we have

$$\text{tr}(g|k[G]) = \#\{h \in G \mid gh = g\}$$

Remark: if X is a G -set (i.e. a set with a G -action), then the permutation representation, k^X , has character

$$\chi_{k^X}(g) = \#\{x \in X \mid g \cdot x = x\}$$

Corollary 0.9. As a G -representation,

$$k[G] \simeq \bigoplus_{\rho_i \text{ irrep}} \rho_i^{\oplus \dim \rho_i}$$

Proof.

$$\begin{aligned} \langle \chi_{\rho_i}, \chi_{k[G]} \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_{\rho_i}(g) \overline{\chi_{k[G]}(g)} \\ &= \frac{1}{|G|} \chi_{\rho_i}(e) \overline{\chi_{k[G]}(e)} \\ &= \frac{1}{|G|} \dim \rho_i |G| \\ &= \dim \rho_i \end{aligned}$$

Because this representation is 0 except at the identity.

■

Remark: In fact, $\text{Hom}_G(k[G], \rho_i) \simeq \rho_i$, AS A VECTOR SPACE.

Proof. $\text{Hom}_G(k[G], \rho_i) = \text{Hom}_{k[G]}(k[G], \rho_i) \simeq \rho_i$ AS A VECTOR SPACE

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Corollary 0.10. *Let ρ_i be the (conjugacy classes of) irreps of G , n_i the dimension of ρ_i .*

Then $\sum_i n_i^2 = |G|$.

Proof. $|G| = \dim_k k[G] = \dim_k \bigoplus_i \rho_i^{\oplus \dim \rho_i} = \sum n_i^2$

■

Theorem 0.11. *Let G be a finite group, $k = \bar{k}$ an algebraically closed field of characteristic 0, ρ_1, \dots, ρ_n the irreps of G . Then $\{\chi_{\rho_i}\}$ is an orthonormal basis of $Cl(G)$.*

Proof. We know it's orthonormal (so in particular linearly independent), so it is left to show that this indeed spans all of $Cl(G)$.

What remains to show is that χ_{ρ_i} span $Cl(G)$.

It is enough to show that if $\psi \in Cl(G)$ with $\langle \psi, \chi_{\rho_i} \rangle = 0$ for all i , then $\psi = 0$, i.e. the orthogonal complement of the span of the χ_{ρ_i} is trivial.

Definition 0.5. If $\psi : G \rightarrow \mathbb{Q}(\mu_\infty)$ is a class function,

$$\gamma_\psi \stackrel{\text{def}}{=} \sum_{g \in G} \psi(g) e_g \in Z(k[G])$$

Example 0.2. If $\psi : G \rightarrow k$, $g \mapsto \frac{1}{|G|}$, $\gamma_\psi = R_G$.

We will compute what γ_ψ does to a representation.

Proposition 2. *If ρ is an irreducible representation of G , then $\gamma_\psi : \rho \rightarrow \rho$ is multiplication by the scalar $\frac{|G|}{\dim \rho} \langle \psi, \chi_{\rho^v} \rangle$*

Proof.

1. First, $\gamma_\psi : \rho \rightarrow \rho$ is a homomorphism of G -representations, which follows from $\gamma_\psi \cdot g \cdot v = g \cdot \gamma_\psi \cdot v$ for all $g \in G, v \in \rho$, as $\gamma_\psi \in Z(k[G])$.
2. By Schur, $\gamma_\psi : \rho \rightarrow \rho$ is a scalar.
3. $\gamma_\psi = \frac{\text{tr}(\gamma_\psi|_\rho)}{\dim \rho} \cdot \text{Id}_\rho$, so

$$\text{tr}(\gamma_\psi|_\rho) = \text{tr}\left(\sum_{g \in G} \psi(g) e_g|_\rho\right) = \sum_{g \in G} \psi(g) \chi_\rho(g) = |G| \langle \psi, \overline{\chi_\rho} \rangle = |G| \langle \psi, \chi_{\rho^v} \rangle$$

■

Now, consider $\gamma_\psi : k[G] \rightarrow k[G]$. This is zero as γ_ψ acts as zero on every irrep (because it pairs to zero with all the irreps), and because it sends 1 to γ_ψ , γ_ψ has to be zero.

■

Corollary 0.12. (of earlier claim)

$\frac{\dim \rho_i}{|G|} \gamma_{\chi_{\rho_i^v}}$ acts as 1 on ρ_i , and 0 on ρ_j , for $\rho_i \neq \rho_j$ are irreps.

Proof. ■

Corollary 0.13. Given any $V = \bigoplus \rho_i^{\oplus n_i}$,

$$\frac{\dim \rho_i}{|G|} \gamma_{\chi_{\rho_i^v}}$$

acts as a projection onto $\rho_i^{n_i} \subseteq V$, which is called the ρ_i isotypic part of V .

Corollary 0.14. #irreps of G = #conjugacy classes of G

Proof. Let $\{\rho_i\}$ be the irreps of G (up to conjugacy (i.e isomorphism)).

Then $\{\chi_{\rho_i}\}$ is a basis for $Cl(G)$, so # of irreps = $\dim_k Cl(G)$ = #conjugacy classes of G .

Remark: These two numbers are equal, but there is no natural or canonical bijection between the two sets in general.

Classifying rep's

Theorem 0.15. G is abelian iff all irreps of G are 1-dimensional.

Proof. Let V be an irrep. If G is commutative, then $\cdot g : V \rightarrow V$ is a G -homomorphism for all $g \in G$.

By Schur, each $g \in G$ acts as a scalar. Now every subspace of V is a subrep, hence V is 1-dimensional.

Now suppose that all irreps are 1-dimensional. Let n_i be the dimensions of the irreps ρ_i , and let c be the number of conjugacy classes (or equivalently the number of irreps) of G . Then $|G| = \sum_i n_i^2$, but this is at least c , because we are taking the sum of c positive numbers, but each n_i is 1, so each element of G is its own conjugacy class. ■

Example 0.3. Take $G = \mathbb{Z}/n\mathbb{Z}$

For each element $\zeta \in \mu_n \stackrel{\text{def}}{=} \text{nth roots of unity}$, consider $\chi_\zeta : \mathbb{Z}/n\mathbb{Z} \rightarrow k^*, a \mapsto \zeta^a$

This gives n distinct reps, which is the number of conjugacy classes, hence we have a complete list.

Example 0.4. S_3 has conjugacy classes $[e], [(12)], [(123)]$, so there are 3 irreducible representations. We have a trivial representation, whose character sends all conjugacy classes to 1.

We also have $\text{sgn} : S_3 \rightarrow \{\pm 1\} \subseteq k^*$, so χ_{sgn} sends $[e]$ to 1, $[(12)]$ to -1, and $[(123)]$ to 1.

At this point we know there must be a third representation, std , and we can fill in its row in the character table somehow. std is given by $S_3 \curvearrowright \mathbb{C}^{\{1,2,3\}} / \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, with

$$\chi_{std} = \chi_{\mathbb{C}^{\{1,2,3\}}} - \chi_{triv}, \text{ so } \chi_{std}(e) = 2, \chi_{std}(12) = 0, \chi_{std}(123) = -1.$$

We claim that χ_{std} is irreducible. To see this, we compute

$$\langle \chi_{std}, \chi_{std} \rangle = \frac{1}{6}(2^2 + 3 * 0^2 + 2(-1)^2) = 1.$$

Example 0.5. $Q_8 = \langle \pm 1, \pm i, \pm j, \pm k \rangle$, with multiplication given as in the quaternion group, $i^2 = j^2 = k^2 = ijk = -1$.

Conjugacy classes: $(e), -1, \{\pm i\}, \{\pm j\}, \{\pm k\}$.

χ_{triv} sends them all to 1, of course.

Lecture 5, 21/1/25

	1	-1	{i, -i}	{j, -j}	{k, -k}
triv	1	1	1	1	1
i-ker	1	1	1	-1	-1
j-ker	1	1	-1	1	-1
k-ker	1	1	-1	-1	1
?

Let $\mathbb{H} = \mathbb{R}\langle 1, i, j, k \rangle$. Then $Q_8 \curvearrowright \mathbb{H}$ by left multiplication, $\mathbb{H} \curvearrowright \mathbb{C}$ by multiplication by i on the right. This example might be useful to think about for the homework.

Now let's get the character table for S_4 .

conj class	0	(12)	(123)	(12)(134)	(1234)
size	1	6	8	3	6
sgn	1	-1	1	1	-1
$std = \mathbb{C}^4 / \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$	3	1	0	-1	-1
$std \otimes sgn$	3	-1	0	-1	1
$std \circ \pi_{4 \rightarrow 3}$

If S_4 is the symmetries of a tetrahedron, then $\pi_{4 \rightarrow 3}$ is the map from S_4 to S_3 furnished by S_4 acting on pairs of sides, of which there are 3.

How does the structure of G interact with its representation theory?

Proposition 3. (*Homework*)

Let G, H be groups, (NOT necessarily finite!), $k = \bar{k}$ algebraically closed.

Then any irrep of $G \times H$ has the form $V \boxtimes W$, where

- V is an irrep of G ,
- W is an irrep of H
- $(g, h) \cdot v \boxtimes w = (g \cdot v) \boxtimes (h \cdot w)$

This is the same as tensoring the two reps of $G \times H$ we get from

$$\begin{array}{ccc}
 & & G \longrightarrow \mathrm{GL}(V) \\
 & \nearrow & \\
 G \times H & & \\
 & \searrow & \\
 & & H \longrightarrow \mathrm{GL}(W)
 \end{array}$$

Proof. HW

■

We have now classified (modulo the homework) all representations of all finite abelian groups.

In some sense, (the sense of Artin's theorem) is that the representation theory of a group is controlled by the rep theory of its abelian subgroups.

Restriction & induction

Let $H \subseteq G$ be a subgroup of H , G again finite.

We have a restriction functor $\mathrm{Res}_H^G : \mathrm{Rep}_G \rightarrow \mathrm{Rep}_H$,

$$(\rho : G \rightarrow \mathrm{GL}(W)) \mapsto \rho|_H$$

There is a functor going the other way called induction, $\mathrm{Ind}_H^G : \mathrm{Rep}_H \rightarrow \mathrm{Rep}_G$.

Definition 0.6. Let V be an H -representation. Then

$$\mathrm{Ind}_H^G V \stackrel{\mathrm{def}}{=} k[G] \otimes_{k[H]} V$$

Equivalent descriptions:

$$\mathrm{Ind}_H^G(V) \stackrel{\mathrm{def}}{=} \{ \phi : G \rightarrow V \mid \phi(gh^{-1}) = h\phi(g) \forall g \in G, h \in H \}$$

An element of the former looks like $\sum_g e_g \otimes v_g$. Take $e_g e_h \otimes v = e_g \otimes (h \cdot v)$, $g \cdot \phi = g\phi(g^{-1}-)$. Think about this and see how this makes the descriptions the same. One more description:

$$\mathrm{Ind}_H^G(V) = \bigoplus_{g \in G/H} g_i \cdot V$$

where $g \cdot \sum g_i v_i = \sum g_{j(i)} k_i \cdot V$ where $g_j g_i = g_{j(i)}$ (???)

Exercise: check the above is equivalent to the other two things.

Example 0.6.

1. $\text{Ind}_H^G \text{triv} = k^{G/H}$ follows from second description. By definition, $\text{Ind}_H^G \text{triv} = \{f : G \rightarrow k \mid f(gh^{-1}) = h \cdot f(g) = f(g)\} = \{f : G/H \rightarrow k\}$
2. $\text{Ind}_{(1)}^G k = k[G] \otimes_k k = k[G]$
3. Suppose $\chi : H \rightarrow \mathbb{C}^\times$ is a representation. What is $\text{Ind}_H^G \chi$? To find $\text{Ind}_H^G \chi(g)$, pick coset representative g_i from G/H , and we get permutation matrix for $G \curvearrowright G/H$ times the diagonal matrix whose i th entry is $\chi(h_i)$, where $gh_i^{-1} = g_{j(i)}h_i^{-1}$

Lecture 6, 23/1/25

Corrections:

In the homework, problem 4 part a) should include the assumption that the action of G on H by conjugation is inner, i.e. for all $g \in G$, the map $(\cdot)^g : H \rightarrow H$ sending $h \mapsto ghg^{-1}$ is $(\cdot)^{h'}$ for some $h' \in H$.

Remark: An example is if we take $G = A \times B, H = A \times \{1\}$. Then $(\cdot)^{(a,b)} = (\cdot)^{(a,1)}$

Last time:

- We did character tables for Q_8, S_4
- We stated the classification of irreducible representations of a product $G \times H$
- Classification of irreps of finite abelian groups
- Restriction & induction

Here is more on induction:

$\text{Ind}_H^G(V) \stackrel{\text{def}}{=} k[G] \otimes_{k[H]} V$, where $k[G]$ is a right module and V is a left one. Tensoring a right with a left yields an abelian group (indeed a k -vector space), and it all works out because $k[G]$ is a left $k[H]$ module.

It is also the set $\{\phi : G \rightarrow V \mid \phi(gh^{-1}) = h \cdot \phi(g) \text{ for all } g \in G, h \in H\}$, where

$$g \cdot \phi = \phi(g^{-1} \cdot)$$

Explanation

An element of $k[G] \otimes_{k[H]} V$ is a formal sum $\sum e_g \otimes v_g$ such that $e_g e_h \otimes v = e_g \otimes (h \cdot v)$
How to recognize induced representations:

- Suppose V is a G -rep, $W \subseteq V$ is H -stable. When is $V \simeq \text{Ind}_H^G W$?
- Consider $gW \subseteq V$. Because W is H -stable, this only depends on $[g] \in G/H$

Proposition 4. $V = \text{Ind}_H^G W$ if and only if $V = \bigoplus_{g \in G/H} gW$

Proof. Sketch

Recall the third version, $\text{Ind}_H^G V = \oplus_{g_i \in G/H} g_i U$

■

Proposition 5.

$$\begin{aligned}\chi_{\text{Ind}_H^G \rho}(u) &= \frac{1}{|H|} \sum_{g \in G, g^{-1}ug \in H} \chi_\rho(g^{-1}ug) \\ &= \sum_{x \in G/H} \hat{\chi}_\rho(x^{-1}ux)\end{aligned}$$

$$\text{where } \hat{\chi}_\rho(v) = \begin{cases} \chi_\rho(v) & v \in H \\ 0 & \text{otherwise} \end{cases}$$

Proof.

■

Proposition 6. *Let $H \subseteq G$ be a subgroup of finite index. Then*

$$\text{Hom}_G(\text{Ind}_H^G V, W) \simeq \text{Hom}_H(V, \text{Res}_G^H W)$$

Proof. This is a special case of the tensor-hom adjunction:

$$\begin{aligned}\text{Hom}_G(\text{Ind}_H^G V, W) &\simeq \text{Hom}_G(k[G] \otimes_{k[H]} V, W) \\ &= \text{Hom}_H(V, \text{Hom}_G(k[G], W)) \\ &= \text{Hom}_H(V, \underbrace{W}_{\text{as an } H\text{-rep}}) \\ &= \text{Hom}_H(V, \text{Res}_G^H W)\end{aligned}$$

■

Corollary 0.16. *Let V be a representation of H , W is a representation of G , both finite. Then*

$$\langle \chi_{\text{Ind}_H^G V}, \chi_W \rangle = \langle \chi_V, \chi_{\text{Res}_G^H W} \rangle$$

Proof. These numbers are the dimensions of the hom-spaces, which are the same by the above.

■

Theorem 0.17 (Artin). *Let G be a finite group, $k = \bar{k}$, $\text{char } k = 0$. Then the map*

$$\bigoplus_{H \subseteq G \text{ cyclic}} \text{Cl}(H) \twoheadrightarrow \text{Cl}(G)$$

For each cyclic group H , it acts on characters linearly, so we can extend that to $\text{Cl}(H)$, and we can extend that to $\bigoplus \text{Cl}(H)$

Proof. Remark: Let G be a finite group, $R(G)$ be the “representation ring of G ”,

$$R(G) = \bigoplus_{\rho_i \text{ irreps of } G} \mathbb{Z}[\rho_i]$$

with $[\rho_i] \cdot [\rho_j] = [\rho_i \otimes \rho_j]$, by writing $\rho_i \otimes \rho_j = \bigoplus_{\rho_k \text{ irreps}} \rho_k^{n_k}$

Proposition 7. *There is a map $R(G) \rightarrow Cl(G)$ sending $[\rho_i] \rightarrow \chi_{\rho_i}$. This is a ring homomorphism (because character of tensor product is pointwise product of characters). There is an induced map $R(G) \otimes_{\mathbb{Z}} k \rightarrow Cl(G)$ which is an isomorphism.*

Proof.

1. These are vector spaces of the same dimension
2. The map is surjective because (for example,) characters of irreps span.

■

Corollary 0.18 (to Artin’s theorem). *The map (linear extension of $\bigoplus \text{Ind}_H^G$)*

$$\bigoplus_{H \leq G \text{ cyclic}} R(H)_k \rightarrow R(G)_k$$

is surjective.

I.e. every representation of G is a “ k -linear combo” of irreps induced from cyclic subgroups.

Corollary 0.19.

1. $\bigoplus_{H \leq G \text{ cyclic}} R(H)_{\mathbb{Q}} \rightarrow R(G)_{\mathbb{Q}}$ is surjective, i.e. every irreducible character of G is a \mathbb{Q} -linear combination of characters induced from cyclic subgroups.
2. $\bigoplus_{H \leq G \text{ cyclic}} R(H) \rightarrow R(G)$ has finite cokernel.

Proof.

- (1) \implies (2) because the image of Ind spans $R(G)$ rationally by (1), i.e. given $x \in R(G)$, there is N such that $N \cdot x \in \text{Im}(\text{Ind})$, so the cokernel is torsion, and torsion finitely generated abelian groups are finite.
- We know (1) by Artin, because $\text{Ind}_{\mathbb{Q}} \otimes_{\mathbb{Q}} k$ is surjective, as rank r invariant under extension of scalars?

■

We now prove Artin’s theorem:

Proof. It is enough to show that the adjoint map of $\oplus \text{Ind}_H^G$ is injective. But $\langle \text{Ind } \chi, \psi \rangle = \langle \chi, \text{Res } \psi \rangle$, so

$$\bigoplus \text{Res}_G^H : Cl(G) \rightarrow \bigoplus_{H \leq G \text{ cyclic}} Cl(H)$$

is adjoint to Ind . Now let ψ be in the kernel; then $\text{Res}_G^H \psi \equiv 0$ for all H , which implies $\psi \equiv 0$, so we win. ■

Loose ends:

- Structure of $k[G]$
- Integral theory
- Corollary of all this discussion: if G is a finite group, ρ an irrep, then $\dim \rho \mid |G|$

Structure of $k[G]$ (and more generally, semisimple algebras)

Definition 0.7. Let k be a field, R a k -algebra (possibly non-commutative). Then R is semisimple if

1. R is finite dimensional as a k -vector space
2. All left R -modules which are finite-dimensional k -vector spaces are semisimple.

Theorem 0.20. *Let R be semisimple k -algebra. Then*

$$R \simeq \prod \text{Mat}_{n_i}(D_i)$$

where D_i are division k -algebras.

Proof. (Take $R = k[G]$)

Consider R as a left R -module;

$$R \simeq \bigoplus M_i^{\oplus n_i}$$

where M_i is simple, all M_j s are mutually non-isomorphic left R -modules.

Note $\text{Hom}_{R\text{-mod}}(M_i, M_i)$ is a division algebra (otherwise we would have a morphism with a kernel, but M_i is simple).

Because $R^{\text{op}} \simeq \text{Hom}_{R\text{-mod}}(R, R)$, this means

$$R \simeq \text{Hom}_{R\text{-mod}}(\bigoplus M_i^{\oplus n_i}, \bigoplus M_i^{\oplus n_i})$$

Now, $\text{Hom}_{R\text{-mod}}(M_i, M_j) = 0$ for $i \neq j$ (again by simplicity and mutual nonisomorphism) so

$$\text{Hom}_{R\text{-mod}}(R, R) \simeq \oplus_i \text{Hom}_{R\text{-mod}}(M_i^{n_i}, M_i^{n_i})$$

So if we take $D_i^{\text{op}} = \text{Mat}_{n_i}(\text{Hom}(M_i, M_i))$, we win. ■

Corollary 0.21. *Let $k = \bar{k}$. Then $R \simeq \oplus \text{Mat}_{n_i}(k)$*

Proof.

1. Finite dimensional central division algebras over an algebraically closed field are the field itself.
2. Or, same proof as in Schur,

$$\text{Hom}_{R\text{-mod}}(M_i, M_i) = k$$
■

Let's specialize to $R = k[G]$.

As a $k[G]$ -module, $k[G] \simeq \rho_i^{\oplus n_i}$, so we have a map

$$k[g] \rightarrow \bigoplus_{\rho_i \text{ irrep}} \underline{\text{Hom}}_k(\rho_i, \rho_i) \simeq \bigoplus_{\rho_i \text{ irrep}} \rho_i \boxtimes \rho_i^v \simeq \bigoplus_{\rho_i \text{ irrep}} \rho_i \otimes \text{Hom}(\rho_i, k[G])$$

$$x \mapsto \text{right multiplication by } x$$

Recall: If V is any G -rep, then $V = \bigoplus \rho_i \otimes \text{Hom}_G(\rho_i, V)$,
so we have $k[G] \rightarrow \bigoplus \text{End}(\text{Hom}(\rho_i, k[G]))$

Claim. *This isomorphism of rings is $G \times G$ -equivariant if we give $\text{End}(\rho_i^{\dim \rho_i})$ the $G \times G$ structure $\rho_i \boxtimes \rho_i^v$*

Proof. We need to check $\text{End}(\rho_i^{\dim \rho_i})$ as a right G -module it is $(\rho_i^v)^{\dim \rho_i}$.

If $G \hookrightarrow G \times G$ by $g \mapsto (g, g^{-1})$, then it has an invariant in $\text{Hom}_G(\rho_i^{\dim \rho_i}, \rho_i^{\dim \rho_i})$,

As G -reps, $\text{Hom}(\rho_i, \rho_i) \simeq \rho_i \otimes \rho_i^v$

Claim. *Given a rep $V \boxtimes W$ of $G \times G$, the structure of V and $V \boxtimes W|_{(g, g^{-1})}$ determines W .*

Proof. ■

Lecture 7, 28/1/25

Substitute for today: Dr Jacob Tsimerman

Let $k = \bar{k}$ be an algebraically closed field of characteristic 0, G a finite group.

Let $(\rho_1, V_1), \dots, (\rho_n, V_n)$ be the irreducible left representations of G .

Theorem 0.22.

$$k[G] \cong \bigoplus_{i=1}^n \rho_i \boxtimes \rho_i^v = \bigoplus_{i=1}^n V_i \otimes V_i^*$$

as $G \times G$ -reps $((g, g') \cdot v \otimes v^* = (g \cdot v) \otimes v^* + v \otimes (g' \cdot v^*))$

Proof. Let $W_i \stackrel{\text{def}}{=} \text{Hom}_G(V_i, k[G])$. Then

$$k[G] \cong \bigoplus_{i=1}^n V_i \otimes W_i$$

as $G \times G$ -representations because we get the right G -action for free.

Claim. As right G -representations, $W_i \cong V_i^*$

Proof.

Convention: Given an element $x = \sum_{g \in G} a_g(x)g \in k[G]$, we use $a_g : k[G] \rightarrow k$ to denote the g -th coefficient.

This has the property that $a_g(x \cdot g') = a_{g'g^{-1}}(x)$

Define $\psi : W_i \rightarrow V_i^*$ by

$$\psi(\phi) \stackrel{\text{def}}{=} a_1 \circ \phi$$

Claim. ψ is an isomorphism

Proof. Suppose $\phi \in W_i$. For $g \in G$, $a_g(\phi(v)) = a_1(g^{-1}\phi(v))$. But ϕ is a map of left G -modules, so this is $a_1(\phi(g^{-1}(v))) = \psi(\phi)(g^{-1}v)$.

So, we can write

$$\phi(v) = \sum_{g \in G} \psi(\phi)(g^{-1}v) \cdot g$$

So ϕ is entirely determined by $\psi(\phi)$, or in other words, ψ is injective.

On the other hand, let $\ell \in V^*$.

Consider $\phi_\ell \in W_i$, $\phi_\ell(v) = \sum_{g \in G} \ell(g^{-1}v) \cdot g$

Claim. $\phi_\ell \in W_i$

Proof. Let $g_0 \in G$. Then

$$\phi_\ell(g_0v) = \sum_{g \in G} \ell(g^{-1}g_0v) = \sum_{g \in G} \ell(g^{-1}v) \cdot g_0g = g_0 \cdot \phi_\ell(v)$$

This shows that ψ is surjective. ■

Claim. ψ respects the right G -action.

Proof.

$$\begin{aligned}\psi(\phi^{g_0})(v) &= \psi(\phi)(g_0v) \\ &= a_1(\phi(g_0v)) \\ &= a_1(g_0\phi(v)) \\ &= a_{g_0^{-1}}(\phi(v))\end{aligned}$$

On the other hand,

$$\begin{aligned}\psi(\phi^{g_0}v) &= a_1(\phi^{g_0}(v)) \\ &= a_1(\phi(v)g_0) \\ &= a_{g_0^{-1}}(\phi(v))\end{aligned}$$

So $\psi(\phi^{g_0}) = \psi(\phi)^{g_0}$ ■

This proves the theorem. ■

Matrix Coefficients

Let $\{v_1, \dots, v_n\}$ be a basis for an irreducible representation V .

Let $\{v_1^*, \dots, v_n^*\}$ be the dual basis for V^* .

Definition 0.8. Given $1 \leq i, j \leq m$, the matrix coefficient $a_{i,j}$ is given by

$$a_{i,j}(g) = v_i^*(g \cdot v_j)$$

This is a function from G to k .

Define $A_{i,j} \in k[G]$ by

$$A_{i,j} \stackrel{\text{def}}{=} \sum_{g \in G} a_{i,j}(g) \cdot g$$

Theorem 0.23.

$$\langle A_{i,j} \rangle_{1 \leq i, j \leq m} = \rho \boxtimes \rho^v$$

where (ρ, V) is the G -rep.

Proof. ■

Theorem 0.24. Let G be a finite group, $k = \bar{k}$ an algebraically closed field of characteristic 0.

Let (ρ, V) be an irreducible representation of G .

Then $\dim V \mid |G|$

Proof.

Corollary 0.25. *If d_1, \dots, d_n is the dimensions of the irreps of G , then*

1. $m = \text{number of conjugacy classes of } G$ (often called m)
2. $d_i \mid |G|$ for all i
3. $\sum_{i=1}^m d_i^2 = |G|$

Proof. ■

Example 0.7. If $G = S_3$, $m = 3$, with conjugacy classes $[\text{Id}]$, $[(12)]$, $[(123)]$, then we have $d_1 = 1$, $1 + d_2^2 + d_3^2 = 6$, $d_2, d_3 \mid 6$.

So we must have $d_2 = 1$, $d_3 = 2$.

Recollections of algebraic integers

Definition 0.9. Let R be a commutative ring.

Then $x \in R$ is integral, or an algebraic integer, if x satisfies a monic integer polynomial.

Example 0.8.

- 3
- $\sqrt{5}$
- $\frac{1+\sqrt{5}}{2}$

Non-examples include

- $\frac{3}{7}$
- $\frac{1}{\sqrt{2}}$

Proposition 8. *The following are equivalent:*

1. x is integral
2. The subring generated by x is a finitely generated \mathbb{Z} -module
3. The subring generated by x is contained in a finitely generated \mathbb{Z} -module in R .

Proof. Let's start with (1) \implies (2).

Suppose $x^N + \sum_{i=1}^{N-1} a_i x^i = 0$, $a_i \in \mathbb{Z}$.

Then $x^N \in \langle 1, x, \dots, x^{N-1} \rangle_{\mathbb{Z}}$. But then $x^{N+1} \in \langle 1, x, \dots, x^N \rangle_{\mathbb{Z}}$, so $x^{N+1} \in \langle 1, x, \dots, x^{N-1} \rangle_{\mathbb{Z}}$. So the subring generated by x equals $\langle 1, x, \dots, x^{N-1} \rangle_{\mathbb{Z}}$.

(2) \implies (3) is clear

So let's see (3) \implies (1).

Let $A_N = \langle 1, x, x^{N-1} \rangle_{\mathbb{Z}}$. By assumption, there exists a finitely generated \mathbb{Z} -module $B \subset R$ such that $A_1 \subseteq A_2 \subseteq \cdots \subseteq B$

By Noetherianity, the sequence stabilizes, so there exists some M such that $A_M = A_{M-1}$, and so x^M is a finite linear combination of lower powers of x , so there are a_i such that

$$x^M + \sum_{i=1}^{M-1} a_i x^i = 0$$

Corollary 0.26. *The things on the list of non algebraic integers actually belong on the list!*

Proof. ■

Lecture 8, 30/1/25

Sub Prof: Mathilde Gerbelli-Gauthier

End Goal: G finite, ρ irrep of G over $k = \bar{k}$ algebraically closed of characteristic 0.

We want to show that $\dim \rho \mid |G|$

Strategy: Prove that $\frac{|G|}{\dim \rho}$ is an algebraic integer

As a corollary of the proof of the last prop, we get

Corollary 0.27. *Integral elements of R form a subring.*

Proof. ■

Integrality of characters

As always, let G be a finite group, $k = \bar{k}$ algebraically closed of characteristic 0, and $\rho : G \rightarrow \mathrm{GL}_n(k)$ just any representation (not necessarily irreducible).

Proposition 9.

1. *The values of the character of ρ , $\chi_\rho(g)$, are algebraic integers*
2. *Let $u = \sum_{g \in G} u(g)g$ be an element of $Z(k[G])$. Suppose that $u(g) \in k$ are algebraic integers. Then u is integral.*

At some point in the classes I missed we show that the indicators of conjugacy classes span the center of $k[G]$.

Proof.

1. $\chi_\rho(g)$ is a sum of roots of unity, hence a sum of algebraic integers, hence an algebraic integer.

2. Using a previous result, let $u(g)$ be the indicator function of a conjugacy class. But the sub- \mathbb{Z} -module of $Z(k[G])$ generated by the indicator functions is a sub-ring (because the product of $1_{C_1} \cdot 1_{C_2}$ is a linear combination of the indicators of conjugacy classes, and the coefficient in front of each g is an integer).

Thus each indicator of a conjugacy class is contained in a finitely generated \mathbb{Z} -module, and is integral. ■

Corollary 0.28. *Let ρ be an irrep of G and let $u \in Z(k[G])$ be as before. Then*

$$u_\rho = \frac{1}{\dim \rho} \sum_{g \in G} u(g) \chi_\rho(g) \in k$$

is an algebraic integer.

Proof.

Claim. *Given ρ , $u \mapsto \frac{1}{\dim \rho} \sum u(g) \chi_\rho(g)$ is a ring homomorphism*

Proof.

$$u_1 * u_2 \mapsto \left(\frac{1}{\dim \rho} \sum u_1(g) \chi_\rho(g) \right) \left(\frac{1}{\dim \rho} \sum u_2(g) \chi_\rho(g) \right)$$

The goal will be to define a ring-hom from $Z(k[G])$ to k sending u to u_ρ . Since u is integral, it maps to an integral element of k . ■

$$u \mapsto \frac{|G|}{\dim \rho} \langle u, \chi_{\rho^v} \rangle = u_\rho$$

$$\sum u'(g) \chi_\rho(g) = |G| \langle u, \rho^v \rangle$$

Recall that $Z(k[G]) \curvearrowright \rho$ by G -homomorphism, that action induces a natural map

$$Z(k[G]) \mapsto \text{Hom}_G(\rho, \rho) = k$$

So

$$u \mapsto \frac{|G|}{\dim \rho} \langle u, \chi_{\rho^v} \rangle$$

The matrix is scalar, so it suffices to compute its trace. Its trace is

$$\sum_{g \in G} u(g) \chi_\rho(g) = |G| \langle u, \chi_{\rho^v} \rangle$$

Dividing by $\dim \rho$ gives the result. ■?

Theorem 0.29. *Let G be a finite group, $k = \bar{k}$ an algebraically closed field of characteristic 0, V_ρ an irrep of G . Then $\dim V \mid |G|$*

Proof. Set $u = \sum_{g \in G} \chi_\rho(g^{-1})g$. By the above, we have

$$\begin{aligned} \frac{1}{\dim \rho} \sum u(g) \chi_\rho(g) &= \frac{|G|}{\dim \rho} \langle \chi_{\rho^v}, \chi_{\rho^v} \rangle \\ &= \frac{|G|}{\dim \rho} \underbrace{\dim \operatorname{Hom}_G(\rho^v, \rho^v)}_{=1} \\ &= \frac{|G|}{\dim \rho} \end{aligned}$$

But the left hand side is an integral element of \mathbb{Q} , so the right hand side is an integral element of \mathbb{Q} , hence an integer. ■

Rep theory of the symmetric group

As always, $|G| < \infty$, $\operatorname{Char}(k = \bar{k}) = 0$

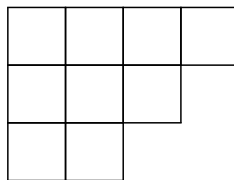
Here are some key facts about the symmetric groups:

1. The number of irreps of S_n is equal to the number of conjugacy classes in S_n .
2. The conjugacy classes in S_n (aka cycle type) are in bijection with partitions of n .
3. The irreps of S_n are also indexed by partitions of n .

Definition 0.10. A partition of n is a sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r)$ such that $\sum \lambda_i = n$.

Definition 0.11. The young diagram D_λ has λ_1 boxes in the first row, λ_2 in the second row, etc.

For example, the corresponding diagram for $\lambda = (4, 3, 1)$



The conjugate partition λ' is the one such that $D_{\lambda'}$ is obtained by D_λ by flipping along the diagonal.

If $\lambda = (4, 3, 1)$, $\lambda' = (3, 2, 2, 1)$. Then $D_{\lambda'}$ is

Projections and young symmetrizers

An algorithm: start with λ

1. Number the boxes in your Young diagram D_λ from left to right, top to bottom: you now have a young tableaux.

1	2	3	4
5	6	7	
8			

2. Let $\cdot P \subseteq S_n$ be the subgroup of all permutations that preserve each row of our Young tableaux. E.g. $P \simeq S_4 \times S_3 \hookrightarrow S_8$.
3. $Q \subseteq S_n$ the subgroup that preserves each column of the same Young tableau e.g. $Q \simeq S_3 \times S_2 \times S_2 \hookrightarrow S_8$.

In $\mathbb{C}[S_n]$, define $a = \sum_{p \in P} e_p$, $b = \sum_{q \in Q} \text{sgn}(q)e_q$

4. Suppose that V is a vector space, and $S_n \curvearrowright V^{\otimes n}$ by permuting factors.

The element a symmetrizes along the rows, and projects onto

$$\text{Sym}^{\lambda_1}(V) \otimes \cdots \otimes \text{Sym}^{\lambda_n}(V)$$

up to an isomorphism.

5. The element b alternates along the columns and projects onto a tensor product of exterior powers indexed by λ' :

$$\bigwedge^{\lambda'_1}(V) \otimes \cdots \otimes \bigwedge^{\lambda'_n}(V)$$

6. Set $c = ab$. This is called the Young Symmetrizer

Here are some examples of Young symmetrizers: If $\lambda = (1, \dots, 1)$, then c gives the sign representation. $\lambda = (n)$ gives the trivial rep.

Irreducibility and idempotency

Theorem 0.30. *A suitable nonzero scalar of $c = ab$ is an idempotent in $\mathbb{C}[S_n]$. Its image, when acting on the regular representation, is irreducible, and denoted V_λ . Distinct partitions give rise to distinct (meaning nonisomorphic) representations and every irep arises from this process for a unique partition.*

Corollary 0.31. *Every representation of S_n is defined over \mathbb{Q} .*

Proof. ■

Example 0.9.

- For S_3 , $\text{triv} = (4)$, $\text{sgn} = (1, 1, 1)$, $\text{std} = (2, 1)$
- For S_4 , $\text{triv} = (4)$, $\text{sgn} = (1, 1, 1, 1)$, $\text{std} = (3, 1)$, $\text{std} \otimes \text{sgn} = (2, 1, 1)$, $S_4 \rightarrow S_3 = (2, 2)$
- In general, $(d, 1, \dots, 1)$ corresponds to various exterior powers of the standard representation.

Theorem 0.32. *(Hook-length formula)*

Label each box b in a young diagram (boxes to the right of b) + (boxes below).

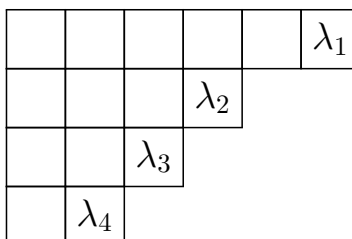
These are called hook lengths. Then $\dim V_\lambda = \frac{n!}{\prod(\text{hook lengths of } b)}$

Proof. ■

Lecture 9, 4/2/25

Let $n \in \mathbb{Z}_{>0}$. Our goal is to classify irreps of S_n . Recall:

Theorem 0.33. *For each partition λ of n , there exists a unique isomorphism class of irrep V_λ of S_n , constructed as follows:*



where $\sum \lambda_i = n$. We let R be the subgroup of S_n which preserves the rows, Q the subgroup preserving the columns. We set

$$a \stackrel{\text{def}}{=} \sum_{g \in P} e_g \in \mathbb{C}[S_n]$$

$$b \stackrel{\text{def}}{=} \sum_{g \in Q} \text{sgn}(g) e_g \in \mathbb{C}[S_n]$$

$$c = ab$$

Then $V_\lambda \stackrel{\text{def}}{=} \mathbb{C}[S_n]c$ is an irrep of S_n .

Further, every irrep arises in this way.

Proof. Summary: WTS

$$1. \dim \text{Hom}_G(V_\lambda, V_\mu) = \delta_{\mu\lambda}$$

2. Any irrep is some V_λ .

Remark:

1. There is an explicit dimension formula, the hook-length formula

2. There is an explicit formula for the character of V_λ due to Frobenius.

For more, look for Etingof's "Representation theory" notes for a course given at MIT.

We will begin the proof by writing down c_λ .

Lemma 1.

$$c_\lambda = \sum_{g = \underbrace{p}_{\in P_\lambda} \underbrace{q}_{\in Q_\lambda}} \text{sgn}(q) e_{pq}$$

Proof.

$$\begin{aligned} a_\lambda b_\lambda &= \left(\sum_{g \in P_\lambda} e_g \right) \cdot \left(\sum_{h \in Q_\lambda} \text{sgn}(h) e_h \right) \\ &= \sum_{g \in P_\lambda, h \in Q_\lambda} \text{sgn}(h) \underbrace{e_g e_h}_{e_{gh}} \end{aligned}$$

■

Goal: Compute $c_\lambda^2 = a_\lambda b_\lambda a_\lambda b_\lambda$

Lemma 2. For all $x \in \mathbb{C}[S_n]$, $a_\lambda x b_\lambda = \ell_\lambda(x) c_\lambda$, where $\ell_\lambda : \mathbb{C}[S_n] \rightarrow \mathbb{C}$ is some linear map.

Corollary 0.34. $c_\lambda^2 = \ell_\lambda(b_\lambda a_\lambda) c_\lambda$

Proof. Check this on each $e_g \in \mathbb{C}[S_n]$, $g \in S_n$.

Case 1 $g \in P_\lambda Q_\lambda$

We have $g = pq, e_g = e_p e_q$.

$$\begin{aligned}
 a_\lambda e_g b_\lambda &= \left(\sum_{h \in P_\lambda} e_h \right) e_g \left(\sum_{u \in Q_\lambda} \text{sgn}(u) e_u \right) \\
 &= \underbrace{\left(\sum_{h \in P_\lambda} e_h e_p \right)}_{a_\lambda} \underbrace{\sum_{u \in Q_\lambda} \text{sgn}(u) e_q e_u}_{\text{sgn}(q) b_\lambda} \\
 &= \text{sgn}(q) c_\lambda b_\lambda \\
 &= \text{sgn}(q) c_\lambda
 \end{aligned}$$

Case 2 $g \notin P_\lambda Q_\lambda$

In this case, $a_\lambda e_g b_\lambda = 0$. To see this, it is enough to show that there exists a transposition $t \in P_\lambda$ such that $g^{-1}tg \in Q_\lambda$, i.e. g sends two elements of $\{1, \dots, n\}$ in the same row of the Young diagram for λ , to two elements of the same column.

It is enough to show this because

$$\begin{aligned}
 a_\lambda g b_\lambda &= a_\lambda t g b_\lambda \\
 &= a_\lambda g \overbrace{(g^{-1}tg)}^{\text{sgn}=-1} b_\lambda \\
 &= -a_\lambda g b_\lambda
 \end{aligned}$$

This implies $a_\lambda g b_\lambda = 0$.

Now, suppose there do not exist 2 elements in the same row of λ sent to the same column of λ by g .

Then $g \in P_\lambda Q_\lambda$.

To see this, let T be the standard Young Tableau for λ , $T' = gT$, P' the stabilizer of rows of T' , Q' the stabilizers of columns.

- (i) By assumption, any two numbers in the first row of T lie in different columns of T' .
- (ii) Then there exists $q'_1 \in Q'$ such that $q'_1 T'$ has the same elements in first row (perhaps in a different order).
- (iii) Choose $p'_1 \in P_\lambda$ such that $p'_1 q'_1 T'$ has the first row as T .
- (iv) Likewise with the 2nd row and so on.

Corollary 0.35.

$$\ell_\lambda(b_\lambda a_\lambda) = \frac{n!}{\dim V_\lambda}$$

Proof. later

Lecture 10, 6/2/25

Note: For the finite group stuff we are using “Linear reps of finite groups” by Serre (first 3rd is for chemists apparently which is amusing). Specifically chapters 1-3, 6, 9. Other stuff is also on the quercus.

To finish the proof of the theorem, we have to show that the V_λ are irreducible and mutually non-isomorphic. Then, from a bijection between conjugacy classes and partitions, we will be done.

Last time we showed that $a_\lambda x b_\lambda = \ell_\lambda(x) c_\lambda$, and its corollary, that $c_\lambda^2 = \ell_\lambda(b_\lambda a_\lambda) c_\lambda$

Corollary 0.36.

$$\ell_\lambda(b_\lambda a_\lambda) = \frac{n!}{\dim V_\lambda}$$

Proof. We know that $c_\lambda = \alpha \cdot p_\lambda$, where p_λ is an idempotent.

$$\begin{aligned} c_\lambda^2 &= \alpha^2 p_\lambda^2 \\ &= \alpha^2 p_\lambda \\ &= \alpha c_\lambda \end{aligned}$$

So $\alpha = \ell_\lambda(b_\lambda a_\lambda)$ so we calculate the trace of c_λ : Trace of an idempotent is dim of its image, and c_λ has the same image as p_λ

$$\begin{aligned} \text{tr}(c_\lambda) &= \alpha \cdot \dim \text{Im}(c_\lambda) \\ &= \ell_\lambda(b_\lambda a_\lambda) \cdot \dim \text{Im}(c_\lambda) \\ &= \ell_\lambda(b_\lambda a_\lambda) \cdot \dim V_\lambda \end{aligned}$$

Now, if this number is not zero, then we get an idempotent by dividing c_λ by this number. We calculate

$$\begin{aligned} \text{tr}(c_\lambda) &= \sum_{pq \in P_\lambda Q_\lambda} \text{tr}(\cdot e_{pq}) \text{sgn}(q) \\ &= \text{tr}(\cdot \text{Id}) \\ &= n! \end{aligned}$$

Goal: Compute $\dim_{\mathbb{C}} \text{Hom}_{S_n}(V_{\lambda}, V_{\mu}) = \begin{cases} 1 & \lambda = \mu \\ 0 & \text{otherwise} \end{cases}$

We know $\text{Hom}_{S_n}(V_{\lambda}, V_{\mu}) = \text{Hom}_{S_n}(\mathbb{C}[S_n]c_{\lambda}, \mathbb{C}[S_n]c_{\mu})$

Proposition 10. *Let A be a \mathbb{C} -algebra, $e \in A$ an idempotent, M an A -module. Then $\text{Hom}_A(Ae, M) \simeq eM$ naturally.*

Proof. For $x \in eM$, we have a morphism $x \mapsto (a \mapsto ax)$, and $f \mapsto f(e)$. e is an idempotent, so $1 - e$ is also an idempotent, so $1 = e + (1 - e)$, so $A \simeq Ae \oplus A(1 - e)$, so $\text{Hom}(Ae, M) \simeq \text{Hom}(A/A(1 - e), M) = \{f : A \rightarrow M \mid f(e) = f(1)\} = \{x \in M \mid x \in eM\} = eM$

Now let's prove the main theorem.

Proposition 11.

$$\dim_{\mathbb{C}} \text{Hom}_{S_n}(V_{\lambda}, V_{\lambda}) = 1$$

Thus, V_{λ} is irreducible

Proof.

$$\begin{aligned} \text{Hom}_{S_n}(V_{\lambda}, V_{\lambda}) &= c_{\lambda} \mathbb{C}[S_n] c_{\lambda} \\ &\subseteq a_{\lambda} \mathbb{C}[S_n] b_{\lambda} \\ &\subseteq \text{span}_{\mathbb{C}}(c_{\lambda}) \end{aligned}$$

So the dimension is at most 1. To see it is exactly 1, this space has $c_{\lambda} \cdot 1 \cdot c_{\lambda} \neq 0$
So $\dim = 1$, so V_{λ} is irreducible.

Now let λ, μ be two partitions of n . Say $\lambda > \mu$ if the first $\lambda_i \neq \mu_i$ has $\lambda_i > \mu_i$, i.e. the lexicographical ordering. This is a total ordering, i.e. for any pair (λ, μ) , exactly one of $\lambda = \mu, \lambda > \mu, \lambda < \mu$ is true.

Proposition 12. *If $\lambda > \mu$, then $a_{\lambda} \mathbb{C}[S_n] b_{\mu} = 0$.*

Proof. In a bit

Assuming this, then, if $\lambda \neq \mu$, we want to show that $\dim \text{Hom}_{S_n}(V_{\lambda}, V_{\mu}) = 0$.

Proof. We have

$$\begin{aligned} \text{Hom}_{S_n}(V_{\lambda}, V_{\mu}) &= c_{\lambda} \mathbb{C}[S_n] c_{\mu} \\ &= a_{\lambda} b_{\lambda} \mathbb{C}[S_n] a_{\mu} b_{\mu} \\ &\subseteq a_{\lambda} \mathbb{C}[S_n] b_{\mu} \\ &= 0 \end{aligned}$$

if $\lambda > \mu$. But $\dim \operatorname{Hom}_{S_n}(V_\lambda, V_\mu) = \dim \operatorname{Hom}_{S_n}(V_\mu, V_\lambda)$, so one, hence both, are 0. ■

Now we prove the proposition

Proof. We will verify it on $e_g \in \mathbb{C}[S_n]$.

Claim. *There exist two numbers on the same row of the standard Young tableaux for λ , same column for $g \cdot$ (standard Young tableaux of μ)*

Proof. Homework ■

Example 0.10. If $g = \operatorname{Id}$, $\lambda_1 > \mu_1$,

1	2	3	4		1	2	3
				,	4	5	

Let t be the transposition for these two numbers. Then

$$\begin{aligned}
 a_\lambda g b_\lambda &= c_\lambda t g b_\mu \\
 &= a_\lambda g g^{-1} t g b_\lambda \\
 &= -a_\lambda g b_\mu
 \end{aligned}$$

The rep theory of $\operatorname{GL}_2(\mathbb{F}_p)$

Goal: Understand the irreps of $\operatorname{GL}_n(\mathbb{F}_q)$

What is the size of this group?

$$|\operatorname{GL}_2(\mathbb{F}_q)| = (q^2 - 1)(q^2 - q) = q(q^2 - 1)(q - 1)$$

Proof.

$$\operatorname{GL}_2(\mathbb{F}_q) = \{(v, w) \mid v, w \in (\mathbb{F}_q)^2 \text{ linearly independent} \}$$

So we can pick any v a nonzero vector, and any w not in the span of v . The number of such possible choices is $(q^2 - 1)(q^2 - q)$ ■

Conjugacy classes:

What are the conjugacy classes of $\operatorname{GL}_2(\mathbb{F}_q)$?

What are the reps of $\operatorname{GL}_2(\mathbb{F}_q)$ over \mathbb{C} ? Besides the trivial one, we also have $P^1(\mathbb{F}_1) = \{1 - \dim \text{subspaces of } \mathbb{F}_q^2\}$.

This gives a permutation representation $\mathbb{C}^{P^1(\mathbb{F}_q)}$.

We have $\operatorname{std} = \mathbb{C}^{P^1(\mathbb{F}_q)} / \mathbb{C}$ has dimension q . Let's compute the character of this representation. Let's call the first set of conjugacy classes above z_x , the second $d_{x,y}$, u_x , $t_{x,y}$

	z_x	$d_{x,y}$	u_x	$t_{x,y}$
triv	1	1	1	1
std	q	1	0	-1

We have

$$\begin{aligned}\langle std, std \rangle &= \frac{1}{q(q-1)^2(q+1)} \left((q-1)q^2 + \frac{q(q-1)(q-2)(q+1)}{2} + 0 + \frac{q^2(q-1)^2}{2} \right) \\ &= 1\end{aligned}$$

What other representation are there?

Choose $\chi : \mathbb{F}_q^\times \rightarrow \mathbb{C}$, and then $(\chi \circ \det)^n$, for $n = 1, \dots, q-2$.

To construct more reps, we will examine some induces reps.

Definition 0.12. Let $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subseteq \mathrm{GL}_2(\mathbb{F}_q)$ (B is for Borel)

$|B| = q(q-1)^2$. Let U be all the matrices of the form $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$.

What is B/U ? It is $\mathbb{F}_q^\times \times \mathbb{F}_q^\times$. We will take reps of this and view them as reps of B via the quotient map and induced reps.

For each $\psi : \mathbb{F}_q^\times \times \mathbb{F}_q^\times \rightarrow \mathbb{C}^\times$, we can consider the induction $\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{F}_1)}(\psi|_B)$. These are indexed by $\psi(\epsilon, 1)$ and $\psi(1, \epsilon)$.

Then $\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{F}_q)}(\psi_{a,b}|_B)$ has dimension $q+1$ and has character $(q+1)\psi(x)^2$

For $d_{x,y}$ we have $\psi(x, 1) + \psi(1, x) + \psi(y, 1) \cdot \psi(1, y)$

I have kind of lost the plot at this point I'm sorry.

Proposition 13. Let $\chi = \sum n_i \rho_i \in R(G)$ be a virtual character of a finite group G . Then χ is the character of an honest irrep iff $\langle \chi, \chi \rangle = 1$, and $\chi(1) > 0$.

Proof. If we write $\chi = \sum n'_i \rho'_i$, where ρ'_i are irreps, then $\langle \chi, \chi \rangle = \sum_i (n'_i)^2 = 1$ by assumption. So at most one of the n'_i are nonzero, and it must be ± 1 . So $\chi = \pm \rho$ for some irrep ρ . If $\chi(1) > 0$, then $\chi(1) = \pm \dim \rho > 0$

Lecture 11, 11/2/25

For simplicity, we will assume q is odd (even is similar, but annoying to do uniformly)

Recall that $|\mathrm{GL}_2(\mathbb{F}_q)| = (q^2 - 1)(q^2 - q)$

Conjugacy class	number of such conjugacy classes	size of each
$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, x \in \mathbb{F}_q^\times$	$q - 1$	1
$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, x \neq y \in \mathbb{F}_q^\times$	$\frac{(q-1)(q-2)}{2}$	$q(q+1)$
$\begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}, x \in \mathbb{F}_q^\times$	$q - 1$	$q^2 - 1$
$\begin{pmatrix} x & \epsilon y \\ y & x \end{pmatrix}, \epsilon \text{ a generator of } \mathbb{F}_q^\times$	$\frac{q(q-1)}{2}$	$q^2 - q$

For the last one, $\text{char} \neq 2$

	z_x	$d_{x,y}$	u_x	$t_{x,y}$
triv	1	1	1	1
std	q	1	0	-1

We denote by B all matrices of the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, and by T the span of all matrices of the form $\begin{pmatrix} x & \epsilon y \\ y & x \end{pmatrix}$

Let U denote the matrices of the form $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$. Then $B/U \cong (\mathbb{F}_q^\times)^2$ (picks out two diagonal entries of B).

Given $\alpha, \beta : \mathbb{F}_q^\times \rightarrow \mathbb{C}^\times$ reps of \mathbb{F}_q^\times , then we have a rep $\psi_{\alpha,\beta} : B \rightarrow B/U \simeq (\mathbb{F}_q^\times)^2 \rightarrow \mathbb{C}^\times$, where this second morphism is by $\alpha \boxtimes \beta$.

	z_x	$d_{x,y}$	u_x	$t_{x,y}$
triv	1	1	1	1
std	q	1	0	-1
$\text{Ind}_B^{\text{GL}_2(\mathbb{F}_q)} \psi_{\alpha,\beta}$	$(q+1)\alpha(x)\beta(x)$	$\alpha(x)\beta(y) + \alpha(y)\beta(x)$	$\alpha(x)\beta(x)$	0

This third line is irreducible if $\alpha \neq \beta$. Note $\text{Ind}_B^{\text{GL}_2} \psi_{\alpha,\beta} = \text{Ind}_B^{\text{GL}_2} \psi_{\beta,\alpha} \text{Ind}_B^{\text{GL}_2} \psi_{\alpha,\alpha} = \alpha \circ \det \oplus (\alpha \circ \det) \otimes (\alpha)(?)$

Recall we proved last time that if G is a finite group, $R(G)$ the representation ring. As a group, $R(G)$ is generated by the irreps of G . Multiplication is given by expressing the tensor of two irreps as a sum of two irreps. Given $V \in R(G)$, V is the class of an irrep if and only if $\langle \chi_V, \chi_V \rangle = 1$, and $\chi_V(\text{Id}_G) > 0$.

Now, we can think of T as being isomorphic to $\{x + \sqrt{\epsilon}y \in \mathbb{F}_{q^2}^\times\}$. Given a $\varphi : \mathbb{F}_{q^2}^\times \rightarrow \mathbb{C}^\times$, we have

	z_x	$d_{x,y}$	u_x	$t_{x,y}$
triv	1	1	1	1
std	q	1	0	-1
$\text{Ind}_B^{\text{GL}_2(\mathbb{F}_q)} \psi_{\alpha,\beta}$	$(q+1)\alpha(x)\beta(x)$	$\alpha(x)\beta(y) + \alpha(y)\beta(x)$	$\alpha(x)\beta(x)$	0
$\text{Ind}_T^{\text{GL}_2(\mathbb{F}_q)} \varphi$	$q(q-1)\varphi(x)$	0	0	$\varphi(\zeta) + \varphi(\zeta)^q(?)$

where $\zeta = x + \sqrt{2}y(?)$. There's another very convoluted row I didn't quite catch you get by putting the others together in various ways, but that's all of them.

For more: wikipedia Deligne-Lustztig(sp?) theory.

The story of $\text{SL}_2(\mathbb{F}_q)$ is similar: restrict reps to $\text{SL}_2(\mathbb{F}_q)$, some of the α, β break up (into at most two pieces), there's some redundancies, and every rep of $\text{SL}_2(\mathbb{F}_q)$ is a restriction.

Fun exercise: Show $\text{PSL}_2(\mathbb{F}_q)$ is simple for $q > 3$ odd.

Lie Algebras

Let k be an arbitrary field, \mathfrak{g} a finite dimensional k -vector space.

Definition 0.13. A Lie bracket $[-, -] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$

- is Bilinear, meaning $[-, *] : \mathfrak{g} \rightarrow \mathfrak{g}$ is linear, as is $[*, -]$ for any $* \in \mathfrak{g}$
- is Alternating, meaning $[x, x] = 0$ for all $x \in \mathfrak{g}$. This implies $[x, y] = -[y, x]$. In characteristic 2, this is stronger!
- satisfies the Jacobi Identity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

A Lie Algebra is a finite dimensional k -vector space \mathfrak{g} with a Lie bracket.

Corollary 0.37. $[x, y] = -[y, x]$

Proof. $[x + y, x + y] = [x, x] + [y, y] + [x, y] + [y, x] = 0$

■

Example 0.11.

- Let R be an associative k -algebra. Then $[x, y] = xy - yx$ will be a Lie bracket.
- $\text{Mat}_{n \times n}(k)$ is a Lie algebra with

$$[A, B] = AB - BA$$

- We have $\mathfrak{sl}_n(k) \subseteq \text{Mat}_{n \times n}(k)$ the set of all matrices with trace 0. Then $[A, B] = AB - BA$ is again a Lie bracket.

- A Lie algebra is abelian if $[-, -] = 0$ for all vectors.

Suppose R is a commutative k -algebra. A k -derivation $\delta : R \rightarrow R$ is a k -linear map such that $\delta(a) = 0$ for $a \in k$, and $\delta(xy) = \delta(x)y + x\delta(y)$.

Example 0.12.

We can take $R = k[t]$, $\delta = \frac{\partial}{\partial t}$.

Fact:

$Der_k(R)$, the set of all k -derivations on R , is a Lie algebra, with $[\delta, \gamma] = \delta \circ \gamma - \gamma \circ \delta$.

Lecture 12, 13/2/25

Let's prove the above fact.

Claim. $Der_k(R)$ is a Lie algebra, with bracket $[\delta, \gamma] = \delta \circ \gamma - \gamma \circ \delta$.

Proof. We have

$$\begin{aligned} \delta \circ \gamma(xy) &= \delta(\gamma(x)y + x\gamma(y)) \\ &= \delta\gamma(x)y + \gamma(x)\delta(y) + \delta(x)\gamma(y) + x\delta\gamma(y) \\ \gamma \circ \delta(xy) &= \dots \\ (\delta \circ \gamma - \gamma \circ \delta)(xy) &= \delta\gamma(x)y + x\delta\gamma(y) - \gamma\delta(x)y - x\gamma\delta(y) \end{aligned}$$

■

Example 0.13.

- If M is a smooth manifold, $Der_{\mathbb{R}}(C^\infty(M)) = C^\infty$ vector fields on M .
- If G is a Lie group (i.e. a C^∞ manifold equipped with a C^∞ group structure). Define $Lie(G)$ to be the space of left G -invariant vector fields. This has another description as the tangent space of the identity, $T_e G$.

Definition 0.14. Let \mathfrak{g} be a Lie algebra over k . A representation of \mathfrak{g} is a k -linear map $\rho : \mathfrak{g} \rightarrow \text{Mat}_{n \times n}(k)$ which respects the Lie bracket, i.e. $\rho([x, y]) = [\rho(x), \rho(y)]$, where on the right hand side it's just the commutator of matrices.

Definition 0.15. Given Lie algebras $\mathfrak{g}, \mathfrak{h}$ over a field k , a homomorphism $f : \mathfrak{g} \rightarrow \mathfrak{h}$ is a k -linear map such that $f([x, y]) = [f(x), f(y)]$ for all $x, y \in \mathfrak{g}$.

Equivalently, a representation is a Lie algebra morphism $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}_n$. Equivalently, $\mathfrak{gl}(V) = \text{End}(V)$, so a rep is a morphism $\mathfrak{g} \rightarrow \mathfrak{gl}(V)$

Definition 0.16. The Universal Enveloping Algebra $U\mathfrak{g}$ is defined by

$$U\mathfrak{g} \stackrel{\text{def}}{=} \frac{\bigoplus_{n \geq 0} \mathfrak{g}^{\otimes n}}{\langle x \otimes y - y \otimes x - [x, y] \forall x, y \in \mathfrak{g} \rangle}$$

Main property:

$$\mathrm{Hom}_{\mathrm{Assoc } k\text{-alg}}(U\mathfrak{g}, R) = \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, R)$$

Definition 0.17. A (finite-dimensional) \mathfrak{g} -representation is the same as a left $U\mathfrak{g}$ -module which is finite dimensional as a k -vector space.

Example 0.14. If \mathfrak{g} is the Lie algebra of a Lie group, then $U\mathfrak{g}$ is the left-invariant differential operators.

Example 0.15. of representations

- $\mathrm{Id} : \mathfrak{g} = \mathfrak{gl}_n \rightarrow \mathfrak{gl}_n$. Inside of \mathfrak{gl}_n is \mathfrak{sl}_n , and this gives a rep of it.
- $k, [-, -] = 0$ is repped by sending $*$ to $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$. The span of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a subrepresentation.
- $\rho_A : k \rightarrow \mathfrak{gl}_n, * \mapsto *A$

Definition 0.18. Let $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ be a representation. $W \subseteq V$ is a subrepresentation if for all $x \in \mathfrak{g}, w \in W, \rho(x)w \in W$. This is the same as saying W is a $U\mathfrak{g}$ -submodule of V .

Example 0.16. Let \mathfrak{b}_n be the Lie algebra of upper triangular matrices. It contains \mathfrak{N}_n , the strictly upper triangular matrices. The former is solvable, the latter nilpotent.

Given some representations, how can we make new ones?

Operations on representations

Let \mathfrak{g} be a Lie algebra over k , V, W representations of \mathfrak{g} , i.e. are equipped with $\rho_V : \mathfrak{g} \rightarrow \mathfrak{gl}(V), \rho_W : \mathfrak{g} \rightarrow \mathfrak{gl}(W)$.

- $\rho_V \oplus \rho_W : \mathfrak{g} \rightarrow \mathfrak{gl}(V) \oplus \mathfrak{gl}(W) \rightarrow \mathfrak{gl}(V \oplus W)$ via the block matrix $\begin{pmatrix} \rho_V(x) & 0 \\ 0 & \rho_W(x) \end{pmatrix}$
- V^* is a rep via $\rho_{V^*} : \mathfrak{g} \rightarrow \mathfrak{gl}(V^*)$ sending $x \mapsto (f \mapsto f(-\rho_V(x)))$
- $\rho_{V \otimes W} : \mathfrak{g} \rightarrow \mathfrak{gl}(V \otimes W), x \mapsto (v \otimes w \mapsto \rho_V(x)v \otimes w - v \otimes \rho_W(x)w)$
- $\underline{\mathrm{Hom}}_k(V, W)$ via $x \cdot f = x \cdot f(-) - f(x \cdot -)$
- $V^{\mathfrak{g}} = \{v \in V \mid xv = 0 \forall x \in \mathfrak{g}\}$

Observation: $\underline{\mathrm{Hom}}_k(V, W)^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(V, W)$

Definition 0.19. A homomorphism of representations is a k -linear $f : V \rightarrow W$ such that $f(x \cdot v) = x \cdot f(v)$ for all $x \in \mathfrak{g}, v \in V$.

Representations of $\mathfrak{sl}_2(k)$

Let k be a field of characteristic 0. $\mathfrak{sl}_2 \subseteq \text{Mat}_{2 \times 2}(k)$ is the set of 2×2 matrices of trace 0. This is $\text{span}(\underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_e, \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_f, \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_h)$

Now,

$$[e, f] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -h$$

Similarly,

$$[h, f] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix} = -2f$$

$$[h, e] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 2e$$

A representation of \mathfrak{sl}_2 on V is $E, F, H \in \mathfrak{gl}(V)$ such that

$$\begin{aligned} [E, F] &= H \\ [H, F] &= -2F \\ [H, E] &= 2E \end{aligned}$$

Given these, we can consider $\mathfrak{sl}_2 \rightarrow \mathfrak{gl}(V)$ by $e \mapsto E, f \mapsto F, h \mapsto H$.

Lemma 3. E, F are nilpotent, i.e. some power of them are 0.

Proof. WLOG, $k = \bar{k}$. Otherwise, we could just lift to the algebraic closure, and check it is nilpotent there, and then it will be nilpotent over k .

Let v be an eigenvector of H with eigenvalue λ . $HFv = -2Fv + FHv = -2Fv + \lambda Fv = (\lambda - 2)Fv$. Similarly, $HEv = (\lambda + 2)Ev$. So $F^n v = E^n v = 0$ for n sufficiently large, because otherwise H would have infinitely many distinct eigenvalues.

Now let $W \subseteq V$ be the span of eigenvectors of H . This is a subrepresentation of \mathfrak{sl}_2 (because F and E send eigenvectors to eigenvectors as shown above). Now consider V/W . This is again an \mathfrak{sl}_2 -representation, so by induction on dimension E, F act nilpotently on it. ■

Here is another proof:

Proof. Let's compute

$$\begin{aligned} \text{tr}(E^n) &= \text{tr}\left(\frac{1}{2}E^{n-1}[H, E]\right) \\ &= \frac{1}{2}\text{tr}(E^{n-1}HE - E^nH) \\ &= 0 \end{aligned}$$

Let $\lambda_1, \dots, \lambda_n$ be the generalized eigenvalues of E with multiplicity. Then $0 = \text{tr}(E^n) = \sum \lambda_i^n$.

Why? In characteristic 0, these generate all symmetric polynomials in λ_i .

The characteristic polynomial of E is $x^n - \sum(\lambda_i)x^{n-1} + \sum_{i < j} \lambda_i \lambda_j x^{n-2} + \dots + = x^n$. So the characteristic polynomial is x^n , so it's nilpotent by Cayley-Hamilton. ■

For now, we will use $k = \mathbb{C}$ so we can say things like “maximal real part.”

Lemma 4. *Let λ be the eigenvalue of H with maximal real part. Let v be an H -eigenvector with eigenvalue λ . Let n be minimal such that $F^n v = 0$ (there is such an n because F is nilpotent). Then $Ev = 0$, $\text{span}(v, Fv, \dots, F^{n-1}v)$ is a subrepresentation of V , $\lambda = n - 1$. In particular, the eigenvalues of H are all integers.*

Proof.

$$\begin{aligned}
 EFv &= EFv - FEv \\
 &= Hv \\
 &= \lambda v \\
 EF^2v &= [E, F]Fv + FEFv \\
 &= HFv + \lambda Fv \\
 &= (\lambda - 2)Fv + \lambda Fv \\
 &= (2\lambda - 2)Fv \\
 &\dots \\
 EF^n v &= (\lambda^2 + (\lambda + 1)n)F^{n-1}v
 \end{aligned}$$

So this is a subrep.

$$\begin{aligned}
 0 &= EF^N v \\
 &= (-N^2 + (\lambda + 1)N)F^{N-1}v
 \end{aligned}$$

So $(\lambda + 1)N - N^2 = 0$. So either $N = 0$ (which isn't the case), or $\lambda = N - 1$.

Explicit:

Set $v = v_0, v_i = F^i v_0$, so $Fv_i = v_{i+1}$. $Hv_n = (N - 2 - 2n)v_n$, $Ev_n = (-n^2 + Nn)v_{n-1}$. Let's call this representation V_N .

Corollary 0.38. *For any non-zero representation W of \mathfrak{sl}_2 , there exists an $N > 0$ such that V_N is a subrepresentation of W .*

Proof. ■

$V_1 \leq \mathfrak{sl}_2 \rightarrow \mathfrak{gl}_2$.

$V_n = \text{Sym}^N(V_1)$, where $x \in \mathfrak{sl}_2$ acts on $v_1 \otimes v_2 \otimes \dots \otimes v_N$ by

$$x \cdot (v_1 \otimes \dots \otimes v_N) = \sum_{i=1}^N v_1 \otimes \dots \otimes xv_i \otimes \dots \otimes v_N$$

V_N is the space of homogeneous polynomials in X, Y of degree N .

E acts by $X \frac{\partial}{\partial y}$, F acts by $Y \frac{\partial}{\partial x}$, H by $X \frac{\partial}{\partial x} - Y \frac{\partial}{\partial y}$.

Claim. V_N are all irreps of \mathfrak{sl}_2 (where irrep means the same as in the world of groups, i.e. no nontrivial subrepresentations).

Proof. Any rep of \mathfrak{sl}_2 contains one of these, so it suffices to show that they are irreducible.

Let $v = \sum a_i v_i \in V_N$ be any element. It suffices to show $\{F^a v, E^b v, a, b \in \mathbb{N}\}$ span. Acting by some power of E , $E^a v \in \text{span}(v_0)$.

But V_n is the span of $F^a v_0$, so we have shown its irreducible. ■

Definition 0.20. V a representation of \mathfrak{g} is semi-simple if it's the direct sum of irreducibles.

Theorem 0.39. All finite dimensional representations of \mathfrak{sl}_2 in characteristic 0 are semisimple.

Proof. ■

Lecture 13, 25/2/25

Combinatorics question:

Let $g_{n,k}$ be the number of unlabeled graphs with n vertices and k edges.

Here are the $g_{n,k}$ for various n

1. 1
2. 1, 1
3. 1, 1, 1, 1
4. 1, 1, 2, 3, 2, 1, 1
5. 1, 1, 2, 4, 6, 6, 4, 2, 1
6. 1, 1, 2, 5, 9, 15, 21, 24, 24, 21, 15, 9, 5, 2, 1, 1

Observation: $g_{n,k} = g_{n, \binom{n}{2} - k}$

Proposition 14. The sequence $g_{n,k}$, for a fixed n , is unimodal in k , meaning these sequences are increasing until they peak in the middle, and then go down.

Proof. We are going to use the representation theory of \mathfrak{sl}_2 to prove this unimodality. Recall: $\mathfrak{sl}_2 = \langle e, f, h \mid [e, f] = h, [h, f] = -2f, [h, e] = 2e \rangle$

We have the standard 2-dimensional representation of \mathfrak{sl}_2 , V , given by the inclusion $\mathfrak{sl}_2 \hookrightarrow \mathfrak{gl}_2$

Last time, we showed all irreps of \mathfrak{sl}_2 are isomorphic to $Sym^n V$ for some nonnegative integer n , where $x \in \mathfrak{sl}_2$ acts on $v_1 \otimes \cdots \otimes v_n$ as

$$(x \cdot v_1) \otimes v_2 \otimes \cdots \otimes v_n + v_1 \otimes (x \cdot v_2) \otimes \cdots \otimes v_n + \cdots + v_1 \otimes \cdots \otimes (x \cdot v_n)$$

We have $h \in \mathfrak{sl}_2$ given by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Fix $v_1, v_{-1} \in V$ such that $h v_1 = v_1, h v_{-1} = -v_{-1}$.

Then h acts by

$$h \cdot (v_1^a v_{-1}^b) = (a + b)v_1 + v_{-1}^b$$

So h acts on $Sym^n V$ with eigenvalues $n, n - 2, \dots, -n + 2, -n$

Lemma:

Let V be any finite dimensional \mathfrak{sl}_2 -representation (not necessarily irreducible).

Lemma 5. *Let d_r be the dimension of the generalized eigenspace of h with generalized eigenvalue r .*

Then $\{d_r\}_{r \text{ even}}, \{d_r\}_{r \text{ odd}}$ are both unimodal and in particular $d_r = d_{-r}$

Proof. By induction on $\dim V$. The base case will be $\dim V = 0$, where it easily holds.

Now, assume the claim holds for $\dim V < n$. Suppose $\dim V = n$. Let $W \subseteq V$ be irreducible. By induction, the claim is true for V/W . By the classification of irreps, it's true for W . But these properties are preserved by addition. ■

The goal now is to write down some generalized eigenspaces of h with exactly this sequence as it's d_r 's.

Let V_n be the vector space consisting of formal linear combinations of labeled graphs with n vertices.

Observation: $S_n \curvearrowright V_n$ by permuting the vertices.

Let $V_{n,k}$ be the span of the graphs with k edges. Then

$$V_n = \bigoplus_{k=0}^{\binom{n}{2}} V_{n,k}$$

Observe that $g_{n,k} = \dim V_{n,k}$

Given $i < j \in \{1, \dots, n\}$, for a labeled graph g , set $a_{ij}(g) = \begin{cases} g \cup (i, j) & (i, j) \text{ not an edge} \\ 0 & \text{otherwise} \end{cases}$.

Similarly, set $b_{ij}(g) = \begin{cases} g \setminus (i, j) & (i, j) \text{ an edge} \\ 0 & \text{otherwise} \end{cases}$.

Let $E = \sum_{i < j} a_{ij}$, $F = \sum_{i < j} b_{ij}$, $H = [E, F]$.

$[a_{i,j}, a_{k,\ell}] = 0$, $[b_{i,j}, b_{k,\ell}] = 0$, and $[a_{i,j}, b_{k,\ell}] = 0$ if $(i, j) \neq (k, \ell)$, g if $(i, j) = (k, \ell)$ and g an edge, $-g$ otherwise.

Now

$$\begin{aligned} Hg &= [E, F]g \\ &= \sum_{(i,j) \in g} g - \sum_{(i,j) \notin g} g \\ &= ((2(\text{number of edges of } g) - \binom{n}{2}))g \end{aligned}$$

Doing this with $[H, F]g$, it turns out this is $-2Fg$.

Next $\mathfrak{sl}_2 \curvearrowright V_n^{S_n}$, and h has eigenspaces $V_{n,k}^{S_n}$ with eigenvalues $2k - \binom{n}{2}$ and hence $g_{n,k} = \dim V_{n,k}^{S_n}$ is unimodal.

Lecture 14, 4/3/25

Let $k = \bar{k}$ be an algebraically closed field (of any characteristic for now), and \mathfrak{g} a Lie algebra over k .

Example 0.17. Recall: Given Lie algebra reps V, W ,

- $V \oplus W$ given by $x \cdot (v, w) = (x \cdot v, x \cdot w)$
- $V \otimes W$ given by $x \cdot (v \otimes w) = (x \cdot v) \otimes w + v \otimes (x \cdot w)$
- V^* , given by $(x \cdot f)(v) = f(-x \cdot v)$
- $V^{\mathfrak{g}} = \{v \in V \mid xv = 0 \forall x \in \mathfrak{g}\}$

Exercise: $\underline{\text{Hom}}_k(V, W)^{\mathfrak{g}} = \text{Hom}_{\mathfrak{g}}(V, W)$

We have $\mathfrak{sl}(V) \hookrightarrow \mathfrak{gl}(V)$ is a representation, so $\mathfrak{sl}(V) \curvearrowright V \otimes V = \text{Sym}^2 V \oplus \wedge^2(V)$ for $\text{char} \neq 2$

Pick $q \in \text{Sym}^2 V$, $\omega \in \wedge^2 CV$, both non-degenerate.

Then $\mathfrak{so}(q) = \{x \in \mathfrak{sl}(V) \mid xq = 0\}$, $\mathfrak{sp}(\omega) = \{x \in \mathfrak{gl}(V) \mid x \cdot \omega = 0\}$

Goal: Study Lie algebras with nice representation theory.

Basic examples will be \mathfrak{sl} , \mathfrak{so} , \mathfrak{sp} , and a finite list of exceptional algebras.

Structure:

There are 2 basic classes of Lie algebras:

- Solvable

- Semisimple

Every finite dimensional Lie algebra is built out of these two types.

Definition 0.21. A subspace $I \subseteq \mathfrak{g}$ is an ideal if for all $x \in I, v \in \mathfrak{g}, [x, v] \in I$. Then \mathfrak{g}/I naturally inherits the structure of a Lie algebra.

Lemma 6. Let $I_1, I_2 \subseteq \mathfrak{g}$ be ideals. Then

- $I_1 + I_2$ is an ideal
- $I_1 \cap I_2$ is an ideal
- $[I_1, I_2] = \langle [x, y] \mid x \in I_1, y \in I_2 \rangle$ is an ideal

Proof. Exercise ■

Example 0.18.

- $[\mathfrak{g}, \mathfrak{g}]$ is an ideal (the commutator).
- $[\mathfrak{gl}(V), \mathfrak{gl}(V)] = \mathfrak{sl}(V)$.
- $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian.

Definition 0.22.

- $D^0 \mathfrak{g} = [\mathfrak{g}, \mathfrak{g}], D^i \mathfrak{g} = [D^{i-1} \mathfrak{g}, D^{i-1} \mathfrak{g}]$. This is the derived series, or central series
- $C^0 \mathfrak{g} = \mathfrak{g}, C^i \mathfrak{g} = [\mathfrak{g}, C^{i-1} \mathfrak{g}]$ This is the lower central series

Definition 0.23. \mathfrak{g} is solvable if $D^i \mathfrak{g} = 0$ for some i , and nilpotent if $C^i \mathfrak{g} = 0$ for some i .

Solvability means if we take $[[[x_1, x_2], [x_3, x_4]], [\dots]] = 0$ if there are enough of them. Nilpotency essentially means that $[x_1, [x_2, [x_3, \dots,]]] = 0$ once you have enough of them.

Unlike in the case of groups, these two notions are extremely closely related.

Example 0.19. $\mathfrak{b}_n = \left\{ \begin{pmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & * \end{pmatrix} \right\}$, the upper triangular matrices.

$\mathfrak{n}_n = \left\{ \begin{pmatrix} 0 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \right\}$, the nilpotent matrices (strictly upper triangular).

Exercise: $[\mathfrak{b}_n, \mathfrak{n}_n] = \mathfrak{n}_n$

Proposition 15. \mathfrak{n}_n is nilpotent.

Proof. We claim $C^i \mathfrak{n}_n$ is the set of upper triangular matrices with the first i super-diagonals are zero. This is because the product of two things of this form has the first $i + 1$ superdiagonals zero.

Definition 0.24. The radical of \mathfrak{g} is the maximal solvable ideal in \mathfrak{g} .

This is a valid thing to write because there is a unique such ideal, but we must show this:

Lemma 7. Let $I_1, I_2 \subseteq \mathfrak{g}$ be solvable ideals. Then $I_1 + I_2$ is solvable.

Proof. $(I_1 + I_2)/I_1 = I_2/(I_1 \cap I_2)$. These ideals and I_1 are solvable.

Claim. If \mathfrak{g} is a Lie algebra, and $I \subseteq \mathfrak{g}$, \mathfrak{g}/I are both solvable, then \mathfrak{g} is solvable.

Proof. $D^i(\mathfrak{g}/I) = 0$ for some i . This implies $D^i \mathfrak{g} \subseteq I$. But iterated commutators are zero in I by assumption. ■

Corollary 0.40. There exists a unique maximal solvable ideal in \mathfrak{g} , denoted $\text{rad}(\mathfrak{g})$ ■

Proof. We can take the span of all solvable ideals. ■

Definition 0.25. \mathfrak{g} is semisimple if $\text{rad}(\mathfrak{g}) = 0$

Proposition 16. For any \mathfrak{g} , $\mathfrak{g}/\text{rad}(\mathfrak{g})$ is always semisimple.

Proof. $I \subseteq \mathfrak{g}/\text{rad}(\mathfrak{g})$ is a solvable ideal if its preimage under the quotient map is solvable, and this would be an extension of $\text{rad}(\mathfrak{g})$.

Example 0.20.

- \mathfrak{sl}_n is simple, meaning there are no non-zero proper ideals
- $\mathfrak{so}(q)$ is simple as well
- $\mathfrak{sp}(\omega)$ is as well

Non-example:

$\mathfrak{gl}(V)$ is NOT semisimple, since scalar matrices make up a solvable ideal.

Definition 0.26. (For culture)

\mathfrak{g} is reductive if $\text{rad}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, v] = 0 \forall v \in \mathfrak{g}\}$

Lecture 15, 6/3/25

Recall we were studying the class of nilpotent Lie algebras, which are a subclass of the class of solvable lie algebras.

Recall a lie algebra is semisimple if $\text{rad}(\mathfrak{g}) = 0$, that is there are no nonzero solvable ideals.

Standard examples of solvable Lie algebras include \mathfrak{b}_n , the $n \times n$ upper triangular matrices, and standard example of a nilpotent Lie algebra is \mathfrak{n}_n , the strictly upper triangular matrices.

Standard example of semisimple Lie algebra is \mathfrak{sl}_n .

Theorem 0.41. (Sophus Lie)

Suppose \mathfrak{g} is solvable. Then any representation of \mathfrak{g} is upper triangularizable, i.e. given a representation $\rho : \mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$, there exists a basis of V in which $\rho(x)$ is upper triangular for all $x \in \mathfrak{g}$, i.e. ρ can be conjugated (by $\text{GL}(V)$) into \mathfrak{b} , i.e. there exists a full flag $V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n$, with $\dim V_i = i$, where each V_i is a representation.

Proof.

Lemma 8. Suppose \mathfrak{g} is solvable, $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ a representation. Then there exists some $v \in V$ so that v is an eigenvector for $\rho(x)$ all $x \in \mathfrak{g}$.

Proof. We will prove this by induction on dimension. It is clearly true for $\dim \mathfrak{g} = 0$, so this is our base case.

Now for the induction step: $[\mathfrak{g}, \mathfrak{g}] \subsetneq \mathfrak{g}$ because \mathfrak{g} is solvable.

Pick $\mathfrak{g}' \subseteq \mathfrak{g}$ so that

1. $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}'$
2. $\text{codim}(\mathfrak{g}') = 1$

Now observe

- (a) \mathfrak{g}' is an ideal because for all $x \in \mathfrak{g}', y \in \mathfrak{g}$, $[x, y] \in [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}'$.
- (b) \mathfrak{g}' is solvable.

Pick $x \in \mathfrak{g} \setminus \mathfrak{g}'$.

By the induction hypothesis, there exists some $w \in V$ such that $y \cdot w = \lambda(y)w$ for all $y \in \mathfrak{g}'$.

Let $W = \text{span}(w, x \cdot w, x^2 \cdot w, \dots)$

Claim. For all $y \in \mathfrak{g}'$, $yx^k w = \lambda(y)x^k w + \sum_{\ell < k} a_{\ell k}(y)x^\ell w$, $k < \dim W - 1$ i.e. $y \in \mathfrak{g}'$ acts on W via an upper triangular matrix with $\lambda(y)$ on the diagonal.

Proof. We prove this by induction on k . For $k = 0$ it is certainly true, so this is our base case.

For the induction step:

$$\begin{aligned} yx^k w &= xyx^{k-1}w + [y, x]x^{k-1}w \\ &= \lambda(y)x^k w + \text{lower order terms} + \lambda([y, x])x^{k-1}w + \text{lower order terms} \end{aligned}$$

So $\text{tr}(y|W) = \dim W \cdot \lambda(y)$ for all $y \in \mathfrak{g}'$

This implies that $\lambda([y_1, y_2]) = 0$ (here we use the assumption $k = \bar{k}$) for all $y_1, y_2 \in \mathfrak{g}$. The lower order terms are all ultimately built out of commutators, so \mathfrak{g}' acts on W via $\lambda - \text{Id}$.

Now choose $v \in W$ an x -eigenvector.

This proves the claim. ■

Let's prove the theorem via a full flag.

Given a rep $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$, we can now pick an eigenvector $v \in V$. Now consider the action of \mathfrak{g} on V/V_1 , where $V_1 = \text{span}(v)$. But $\dim(V/V_1) < \dim(V)$, so by induction we win. ■

Corollary 0.42. *Any irrep of a solvable Lie algebra is 1-dimensional.*

Proof. For any $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$, there exists a common eigenvector, which means every rep has a 1-dimensional sub representation. ■

Corollary 0.43. *\mathfrak{g} is solvable if and only if $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.*

Proof. We start with the easy direction: it is enough for $[\mathfrak{g}, \mathfrak{g}]$ to be solvable. Because then $[\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ are solvable, so their product, \mathfrak{g} , is solvable.

Now for the other direction.

Definition 0.27. Let \mathfrak{g} be a Lie algebra. The adjoint representation is the map $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$, given by $x \mapsto [x, -]$.

Observe $\ker \text{ad} = \mathfrak{z}(\mathfrak{g})$, the center.

Suppose \mathfrak{g} is solvable. By classification of representations, the adjoint representation factors through \mathfrak{b} :

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\text{ad}} & \mathfrak{gl}(\mathfrak{g}) \\ & \searrow & \nearrow \\ & \mathfrak{b} & \end{array}$$

Observe

- (i) $[\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{n}$, and \mathfrak{n} is nilpotent
- (ii) $[\mathfrak{g}/\mathfrak{z}(\mathfrak{g}), \mathfrak{g}/\mathfrak{z}(\mathfrak{g})]$ is nilpotent
- (iii) $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent by definition of center.

■

Recall \mathfrak{g} is semisimple if $\text{rad}(\mathfrak{g}) = 0$.

Example 0.21. Let $\mathfrak{g} = \mathfrak{sl}_2$. We have $\text{ad} : \mathfrak{sl}_2 \rightarrow \mathfrak{gl}(\mathfrak{sl}_2)$.

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow [h, -]$$

Eigenvalues of 0, 2, -2, with eigenvectors h, e_v, f , and f, e, h (?), respectively.

Suppose $I \subseteq \mathfrak{sl}_2$ is an ideal. Then $\text{ad}(h)$ preserves I .

So this means that none of the possible subalgebras are ideals (can check each of them 1 by 1 because there are only finitely many).

So in particular there are no nontrivial solvable ideals.

Corollary 0.44. \mathfrak{sl}_2 is simple (meaning no nontrivial ideals).

Theorem 0.45. Let V be an irrep of a Lie algebra \mathfrak{g} .

Then for all $x \in \text{rad}(\mathfrak{g})$, x acts on V via scalars. For all $x \in [\text{rad}(\mathfrak{g}), \mathfrak{g}]$, x acts on V by 0

Proof. (sketch)

Let $v \in V$ be an eigenvector for $\text{rad}(\mathfrak{g})$, i.e. for all $x \in \text{rad}(\mathfrak{g})$, $x \cdot v = \lambda(x) \cdot v$

Let $V_\lambda = \{w \in V \mid x \cdot w = \lambda(x) \cdot w \forall x \in \text{rad}(\mathfrak{g})\}$

Fix $y \in \mathfrak{g}, x \in \text{rad}(\mathfrak{g})$. Let $w \in V_\lambda$. Then $xyw = yxw + [x, y]w = \lambda(x)yw + \lambda([x, y])w$

We can set $W = \text{span}(w, y \cdot w, y^2 \cdot w, \dots)$. This is certainly preserved by y , and to show it's preserved by the radical you do something similar as the previous theorem.

■

Look at theorem 6.16 in the book for this course (Kiralov (sp?))

Bilinear forms

Definition 0.28. Let \mathfrak{g} be a Lie algebra. A bilinear form B on \mathfrak{g} is invariant if

$$B([x, y], z) + B(y, [x, z]) = 0$$

for all $x, y, z \in \mathfrak{g}$.

Proposition 17. Suppose \mathfrak{g} is a Lie algebra, B an invariant bilinear form on \mathfrak{g} , $I \subseteq \mathfrak{g}$ an ideal.

Then $I^\perp = \{x \mid B(x, y) = 0 \forall y \in I\}$ is an ideal.

Proof. Let $x \in I, y \in \mathfrak{g}, z \in I^\perp$. We want to show $[y, z] \in I^\perp$.
 We have $B([y, z], x) = -B(z, \underbrace{[y, x]}_{\in I}) = 0$

■

Corollary 0.46. *Let \mathfrak{g}, B as before. Then \mathfrak{g}^\perp is an ideal.*

■

Example 0.22. Take $\mathfrak{g} = \mathfrak{gl}_n$, $B(x, y) = \text{tr}(xy)$. Then

$$\begin{aligned} \text{tr}([x, y], z) + \text{tr}(y, [x, z]) &= \text{tr}(xyz - yxz) + \text{tr}(yxz - yzx) \\ &= 0 \end{aligned}$$

Example 0.23. Let $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ be a representation. Then we may define

$$B_\rho(x, y) = \text{tr}(\rho(x)\rho(y))$$

How to check if B is nondegenerate:

We have $B : V \times V \rightarrow k$

We have a $\psi_B : V \rightarrow V^*$ given by $x \mapsto B(x, -)$.

Then B is non degenerate if ψ_B is an isomorphism. So we can pick a basis e_1, \dots, e_n , and show $\det(B(e_i, e_j)) \neq 0$.

Theorem 0.47. *Suppose there exists ρ with B_ρ non-degenerate. Then \mathfrak{g} is reductive (i.e. $\text{rad}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$).*

Proof. It is enough to show $[\mathfrak{g}, \text{rad}(\mathfrak{g})] = 0$

But $\rho([\mathfrak{g}, \text{rad}(\mathfrak{g})])$ acts by zero on any irreducible representation of \mathfrak{g}

Claim. *This implies $[\mathfrak{g}, \text{rad}(\mathfrak{g})] \subseteq \ker B_\rho$*

Proof. By induction on $\dim \rho$, ρ irreducible (otherwise we take a irreducible subrep $\psi \subseteq \rho$).

It is enough to show $B_\rho = B_\psi + B_{\rho/\psi}$

B_ρ being nondegenerate means $\ker B_\rho = 0$, so $\text{rad}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$
 (?)

■

Corollary 0.48. $\mathfrak{gl}_n, \mathfrak{sl}_n, \mathfrak{so}(n), \mathfrak{sp}(2n)$ are all reductive.

Proof. We will do the proof for \mathfrak{gl}_n .

Take $\rho = \text{Id} : \mathfrak{gl}_n \rightarrow \mathfrak{gl}_n$, $B_\rho(x, y) = \text{tr}(xy)$.

Let e_{ij} be the matrix with $(e_{ij})_{kl} = \delta_{kl}^{ij}$.

We see $B(e_{ij}, e_{kl}) = \delta_{il}\delta_{jk}$

■

Lecture 16, 11/3/25

Definition 0.29. Let \mathfrak{g} be a Lie algebra.

The Killing form K is defined as $K = B_{ad}$, so $K(x, y) = \text{tr}(\text{ad}(x) \cdot \text{ad}(y))$

Theorem 0.49.

- (a) \mathfrak{g} is semisimple iff K is non-degenerate
- (b) \mathfrak{g} is solvable if and only if $K([\mathfrak{g}, \mathfrak{g}], \mathfrak{g}) = 0$.

Proof. Next time.

Definition 0.30. Let $k = \bar{k}$, $\text{char } k = 0$, \mathfrak{g} a finite dimensional k -Lie algebra. A bilinear form $B : \mathfrak{g} \times \mathfrak{g} \rightarrow k$ is invariant if

$$B([x, y], z) + B(y, [x, z]) = 0$$

Given a representation $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$, there is a canonical form

$$B_\rho(x, y) = \text{tr}(\rho(x)\rho(y))$$

Definition 0.31. The Killing form is given by $K = B_{ad}$, so $K(x, y) = \text{tr}(\text{ad}(x) \text{ad}(y))$

Theorem 0.50 (Cartan's criterion).

- (a) \mathfrak{g} is semisimple if and only if K is non-degenerate
- (b) \mathfrak{g} is solvable if and only if $K([\mathfrak{g}, \mathfrak{g}], \mathfrak{g}) = 0$

Corollary 0.51. Let \mathfrak{g} be semisimple, $I \subseteq \mathfrak{g}$ an ideal.

Then there is an $I' \subseteq \mathfrak{g}$ such that $\mathfrak{g} = I \oplus I'$ (as a Lie algebra).

Proof. \mathfrak{g} is semisimple, so by the theorem (yet to be proven), we can take I' to be the perpendicular subspace under K , and $\dim I + \dim I' = \dim \mathfrak{g}$ simply because K is non degenerate.

We want to show $I \cap I' = 0$.

Claim. $K|_I$ is the killing form for I .

Proof. Let $x, y \in I$. Then

$$\begin{aligned} K(x, y) &= \text{tr}(\text{ad}(x) \text{ad}(y)|_{\mathfrak{g}}) \\ &= \text{tr}(\text{ad}(x) \text{ad}(y)|_I) + \underbrace{\text{tr}(\text{ad}(x) \text{ad}(y)|_{\mathfrak{g}/I})}_{=0} \end{aligned}$$

because $\text{ad}(x), \text{ad}(y)$ act on \mathfrak{g}/I as 0.



Now to prove the theorem.

We wish to show $K|_I$ is non-degenerate. Let $x \in I$. We want to show there is a $y \in I$ with $K(x, y) \neq 0$.

We know there is some $z \in \mathfrak{g}$ such that $K(x, z) \neq 0$.

He put it on the homework :(

■

Corollary 0.52. *If \mathfrak{g} is semisimple, then \mathfrak{g} is a direct sum of simple non-abelian Lie algebras.*

Proof. Given by induction on dimension:

Choose any ideal $I \subseteq \mathfrak{g}$. If it doesn't exist you're done. Otherwise $\mathfrak{g} = I \oplus I'$. I, I' are quotients of \mathfrak{g} , hence semisimple, so we are done by the inductive hypothesis.

■

Corollary 0.53. *If \mathfrak{g} is semisimple, then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.*

Proof. It is enough to check for \mathfrak{g} simple and non-abelian (because \mathfrak{g} is a direct sum of such).

But $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}$ is an ideal, hence everything, and \mathfrak{g} is nonabelian by assumption.

■

Corollary 0.54. *Let \mathfrak{g} be semisimple. Then all finite dimensional \mathfrak{g} -representations are semisimple (meaning a direct sum of irreps).*

Proof.

■

Now to prove Cartan's criterion.

For the rest of today, $k = \mathbb{C}$ (this is a convenience and it will turn out we can reduce to this case).

Theorem 0.55. *Let V be a finite dimensional vector space, $A \in \text{End}(V)$.*

1. $A = A_s + A_n$, with A_s diagonalizable and A_n nilpotent, with $[A_n, A_s] = 0$.
2. $\text{ad}(A_s) = \text{ad}(A)_s$. Moreover, $\text{ad}(A)_s = P(\text{ad}(A))$ for some polynomial with vanishing constant term $P \in \mathbb{C}[t]$.
3. Let $\overline{A_s}$ be the complex conjugate matrix. Then $\text{ad}(\overline{A_s}) = Q(\text{ad } A_s)$ for some polynomial $Q \in \mathbb{C}[t]$

sketch. 1. First, write A in Jordan normal form by choosing an appropriate basis.

We can take A_s to be the diagonal of this, and $A_n = A - A_s$.

In basis free (and thus choice free (and thus canonical)) language,

Let $V_\lambda = \ker((A - \lambda \text{Id})^N)$ $N \gg 0$. $V = \bigoplus_\lambda V_\lambda$. We can take $A_s \stackrel{\text{def}}{=} (v_\lambda)_{\lambda \in \mathbb{C}} \rightarrow (\lambda v_\lambda)_{\lambda \in \mathbb{C}}$, and then $A_n = A - A_s$.

2. We want to show $A_s = P(A)$ for some $P \in \mathbb{C}[t]$. Let $n_\lambda = \dim V_\lambda$. Choose $p \in \mathbb{C}[t]$ such that

$$P \equiv \lambda \pmod{t - \lambda}^{n_\lambda} \forall \lambda$$

This exists by the chinese remainder theorem.

Then $P(A) = \lambda$ on V_λ because $(A - \lambda \text{Id})^{n_\lambda} = 0$ on V_λ

Remark: This tells us that A_n is also a polynomial in A ($A - P(A)$)

Now,

$$\begin{aligned} \text{ad}(A) &= \text{ad}(A_s + A_n) \\ &= \text{ad}(A_s) + \text{ad}(A_n) \end{aligned}$$

To see that $\text{ad}(A_s) = \text{ad}(A)_s$, it is enough to show $\text{ad}(A_s)$ is diagonalizable, $\text{ad}(A_n)$ is nilpotent (we can do this by computation), and $[\text{ad}(A_s), \text{ad}(A_n)] = 0$.

Computation for $\text{ad}(A_s)$:

$\text{ad}(A_s)$ acts on $V_\lambda \otimes V_{\lambda'}^v$ as $\lambda - \lambda'$,

$$\text{End}(V) = \bigoplus_{\lambda, \lambda'} V_\lambda \otimes V_{\lambda'}^v$$

We know $P \in t\mathbb{C}[t]$ because 0 is a generalized eigenvalue so $P \equiv 0 \pmod{t}^{n_0}$

3. Choose f so that $f(\lambda_i - \lambda_j) = \overline{\lambda_i} - \overline{\lambda_j}$ for all i, j (Lagrange). Then $f(P(\text{ad}(A))) = \text{ad } \overline{A_s}$

Lecture 17, 6/3/25

We finish the proof. Let \mathfrak{g} be a Lie algebra over \mathbb{C} , and let $K(x, y) = \text{tr}(\text{ad}(x) \text{ad}(y))$ be the Killing form.

Theorem 0.56 (Cartan).

(a) \mathfrak{g} is semisimple iff K is non-degenerate

(b) \mathfrak{g} is solvable iff $K(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$.

Last time we showed that if I is an ideal (hence a sub-algebra), then the Killing form for I is the Killing form for \mathfrak{g} restricted to I .

Lemma 9. Suppose $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ is a Lie algebra such that for all $x \in \mathfrak{g}$, x is nilpotent. Then \mathfrak{g} is itself nilpotent.

Sketch. It is enough to show that there exists $v \in V$ such that $xv = 0$ for all $x \in \mathfrak{g}$. By induction on $\dim V$: consider $V/\text{span}(v)$. Lift strict upper triangularizing basis to V , adjoin v to it.

To show that there exists such a v , use induction on $\dim \mathfrak{g}$.

For $\dim \mathfrak{g} = 1$ note that nilpotent matrices have nonzero kernel.

For the inductive step, let $\mathfrak{g}' \subseteq \mathfrak{g}$ be a maximal proper sub-algebra.

Find v with $\mathfrak{g}'v = 0$, now copy the proof for solvable. ■

Theorem 0.57. (*Jordan*)

Let V be a finite dimensional \mathbb{C} -vector space, $A \in \text{End}(V)$. Then

1. $A = A_s + A_n$ with $[A_s, A_n] = 0$, with A_s diagonalizable and A_n nilpotent.
2. $\text{ad}(A)_s = \text{ad}(A_s) = P(\text{ad}(A)), P \in t\mathbb{C}[t]$ (i.e. P has zero constant term).
3. $\overline{\text{ad } A_s} = Q(\text{ad } A)$ for some $Q \in t\mathbb{C}[t]$

Proof of Cartan. Most of the content in (a) is proving (b) so we'll prove (b) first.

We have $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{b} \subseteq \mathfrak{gl}(\mathfrak{g}), [\mathfrak{g}, \mathfrak{g}] \rightarrow \mathfrak{n} \subseteq \mathfrak{gl}(\mathfrak{g})$,

$$K(x, y) = \text{tr}(\underbrace{\text{ad}(x)}_{\in \mathfrak{b}} \underbrace{\text{ad}(y)}_{\in \mathfrak{n}}) = \text{tr}(\text{strictly upper triangular matrix}) = 0$$

To show that $K(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0 \implies \mathfrak{g}$ solvable, reduce to $\mathfrak{g} \subseteq \mathfrak{gl}(V)$. Then $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ has kernel $\mathfrak{z}(\mathfrak{g})$

It is enough to show the image of the adjoint is solvable, because then \mathfrak{g} is an extension of a solvable by a solvable.

Now, $\mathfrak{g} \subseteq \mathfrak{gl}(V)$, $K(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$, and we want to show it's solvable.

Pick $x \in [\mathfrak{g}, \mathfrak{g}], y \in \mathfrak{g}$. Take $x = x_s + x_n$. To show x is nilpotent, we want to show all its eigenvalues are zero.

We will show $\text{tr}(x_s \overline{x_s}) = \sum_{\lambda \in \text{Spec}(x_s)} |\lambda|^2$

By hypothesis we can write $x = \sum_i [y_i, z_i]$.

$$\begin{aligned} \text{tr}(x_s \overline{x_s}) &= \text{tr}\left(\sum [y_i, z_i] \overline{x_s}\right) \\ &= \pm \sum \text{tr}(y_i [z_i, \overline{x_s}]) \\ &= \pm \sum \text{tr}(y_i \text{ad}(x_s) z_i) \\ &= \pm \sum \text{tr}(y_i Q(\text{ad}(x) z_i)) \end{aligned}$$

But $y_i \in \mathfrak{g}$, and $\text{ad}(x) z_i \in [\mathfrak{g}, \mathfrak{g}]$ by definition, and Q has no constant term, so this sum is 0.

Finally, by assumption $K(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$.

So all eigenvalues are zero, so x is nilpotent.

By the lemma, $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent, so \mathfrak{g} is solvable.

We can reduce to the case of \mathbb{C} by the “Lefschetz principal”.

So now we prove part (a).

First, suppose that K is nondegenerate.

We already know \mathfrak{g} is reductive, i.e. $\text{rad}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$.

Let $x \in \mathfrak{z}(\mathfrak{g})$. Then $K(x, y) = \text{tr}(\text{ad}(x) \text{ad}(y)) = 0$, i.e. $x \in \ker K$. But K is nondegenerate, so $x = 0$, so $\text{rad}(\mathfrak{g}) = 0$.

Now suppose \mathfrak{g} is semisimple.

Set $I = \ker K = \mathfrak{g}^\perp$, which is an ideal.

But by the earlier lemma, $K|_I \equiv 0$, so by part (b), I itself is solvable.

But \mathfrak{g} is semisimple, so $I = 0$. ■

Corollary 0.58 (Corollaries on structure of semisimple Lie algebras).

Theorem 0.59. *Let \mathfrak{g} be semisimple, $I \subseteq \mathfrak{g}$ an ideal. Then there exists an I' with $\mathfrak{g} = I \oplus I'$ as Lie algebras.*

Proof. Let K be the killing form of \mathfrak{g} . By Cartan, it is nondegenerate. Let $I' = I^\perp$. It is enough to show $I \cap I' = 0$.

$I \cap I'$ is an ideal such that $K|_{I \cap I'} = 0$. Now by Cartan, it is solvable, hence zero. ■

Corollary 0.60. *Semisimple Lie algebras are direct sums of simple Lie algebras.*

Corollary 0.61. *If \mathfrak{g} is semisimple then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.*

Proof. We did these last time. ■

Theorem 0.62. *Suppose \mathfrak{g} is semisimple. Then every representation of \mathfrak{g} is a direct sum of irreducible representations.*

In other words, any representation of a semisimple Lie algebras is semisimple.

Proof. First Daniel wants to sketch a proof using the theory of Lie groups.

Sketch:

1. Reduce to the case $k = \mathbb{R}$.
2. \mathfrak{g} is the Lie algebra of a compact Lie group G such that
3. All representations of \mathfrak{g} come from representations of G , in the sense that the derivative of a Lie group representation is a Lie algebra representation.
4. There exists a left-invariant Haar measure μ on G

5. It is now enough to show all reps of G are semisimple
6. Given $G \rightarrow \mathrm{GL}(V)$, $W \subseteq V$ invariant, choose a projection $V \rightarrow W$, and take

$$\frac{1}{\mu(G)} \int_G \rho(g, -) d\mu$$

as in the proof of Maschke

Here is a proof using stuff from this class:

Idea: Use the Laplacian on G .

Recall: $U\mathfrak{g}$ is the universal enveloping algebra of \mathfrak{g} , defined by

$$U\mathfrak{g} = \bigoplus_{i \geq 0} \mathfrak{g}^{\otimes i} / \langle x \otimes y - y \otimes x - px \rangle$$

When \mathfrak{g} is the Lie algebra of a Lie group we can identify this with the left-invariant differential operators on G .

Definition 0.32 (Casimir operator). Suppose \mathfrak{g} is a Lie algebra, B an invariant nondegenerate bilinear form $B : \mathfrak{g} \times \mathfrak{g} \rightarrow k$.

Choose a basis (e_i) of \mathfrak{g} , and let (e^i) be the dual basis, i.e.

$$B(e_i, e^j) = \delta_{ij}$$

Then $C_B = \sum e_i e^i \in U\mathfrak{g}$.

Proposition 18. C_B is central and independent of the choice of basis.

Proof. We think of $B \in \mathfrak{g}^* \otimes \mathfrak{g}^*$. We can also think of it as $B \in \underline{\mathrm{Hom}}_k(\mathfrak{g}, \mathfrak{g}^*)^{\mathfrak{g}}$. B furnishes an isomorphism between \mathfrak{g} and \mathfrak{g}^* .

We would like an element of $\mathfrak{g} \otimes \mathfrak{g}$.

We have an isomorphism $(\mathfrak{g}^* \otimes \mathfrak{g}^*) \rightarrow (\mathfrak{g} \otimes \mathfrak{g})$ given by $B^{-1} \otimes B^{-1}$.

The claim is that this sends B to C_B . But B was \mathfrak{g} invariant on the left, so it is \mathfrak{g} -invariant on the right, and that's what it means to be central. ■

Proposition 19. Suppose \mathfrak{g} is semisimple, $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is a nontrivial irreducible representation.

Then there exists $C_\rho \in Z(U\mathfrak{g})$ such that C_ρ acts on the trivial representation by 0 and on ρ by a nonzero scalar.

Proof. Let $B_\rho(x, y) = \mathrm{tr}(\rho(x)\rho(y))$

Set $I = \ker B_\rho$. This is an ideal, so $\mathfrak{g} = \mathfrak{g}' \oplus I$ where $B_\rho|_{\mathfrak{g}'}$ is invariant and nondegenerate.

We have $C = C_{B_\rho|_{\mathfrak{g}'}} \in Z(U\mathfrak{g}') \hookrightarrow U\mathfrak{g}$.

1. C acts trivially by 0: true for any element in $U\mathfrak{g}_{>0}$
2. C acts on ρ by some scalar by Schur's lemma. We must check that this scalar is nonzero.

We have

$$\begin{aligned}
 \mathrm{tr}(C|\rho) &= \mathrm{tr}\left(\sum e_i e^i | \rho\right) \\
 &= \sum \mathrm{tr}(e_i e^i | \rho) \\
 &= \sum B_\rho(e_i, e^i) \\
 &= \sum 1 \\
 &= \dim \mathfrak{g}' \\
 &\neq 0
 \end{aligned}$$

So in fact $\lambda = \frac{\dim \mathfrak{g}'}{\dim V}$

Proof.

Example 0.24. Let $\mathfrak{g} = \mathfrak{sl}_2$, $B = K$.

Take a basis $\mathfrak{sl}_2 = \langle e, f, h \rangle$. We must compute a dual basis.

We have $\mathrm{ad}(e) : e \mapsto 0, f \mapsto h, h \mapsto -2e$

$\mathrm{ad}(f) : e \mapsto -h, f \mapsto 0, h \mapsto 2f$

$\mathrm{ad}(h) : e \mapsto 2e, f \mapsto -2f, h \mapsto 0$

We have $B(e, e) = 0, B(e, f) = 4, B(e, h) = 0$. So keep going, then we solve the system of equations

$$\begin{cases} B(e, ae + bf + ch) &= 1 \\ B(f, \dots) &= 0 \\ B(h, \dots) &= 0 \end{cases}$$

We wind up getting $C = \frac{1}{2}h^2 + ef + fe$

Lecture 18, 18/3/25

Theorem 0.63. *Let \mathfrak{g} be a semisimple (in the Lie algebra sense) Lie algebra over \mathbb{C} . Then any finite dimensional representation of \mathfrak{g} is itself semisimple (in the representation of a Lie algebra sense).*

Proof.

Proposition 20. *If \mathfrak{g} is a semisimple Lie algebra (always over \mathbb{C}), and V is a non-trivial irreducible (always finite-dimensional) representation of \mathfrak{g} , then there is a $C_V \in Z(U\mathfrak{g})$ such that C_V acts on V via a non-zero scalar and on the trivial representation by zero.*

Proof. Last time ■

Proof. Now let's prove the theorem. We will do this in three cases, each of which will reduce to the previous case.

Case 1: Let W be a representation of \mathfrak{g} , and suppose that all C_V act on W nilpotently. Then W is trivial.

Proof.

By induction on dimension of W : If W is 1-dimensional then W is trivial.

Now suppose $\dim W > 1$. By assumption, W is not irreducible. Let $U \subseteq W$ be a nontrivial proper subrepresentation.

By induction, $U, W/U$ are trivial. But then the morphism $\mathfrak{g} \rightarrow \mathfrak{gl}(W)$ factors through a group of block upper triangular matrices which corresponds to an abelian Lie algebra.

So \mathfrak{g} has an abelian quotient if representation is nontrivial.

So \mathfrak{g} has an abelian, hence solvable, ideal if the representation is non-trivial.

Case 2: Let W be a representation of \mathfrak{g} , $U \subseteq W$ a subrep. We want a section of $W \twoheadrightarrow W/U$.

(i) Assume $W/U = \mathbb{C}$

(ii) Consider the action $\langle C_V \rangle_V \curvearrowright W$

Then $W = \bigoplus W_\lambda$, $\lambda : \langle C_V \rangle \rightarrow \mathbb{C}$, where W_λ are the generalized eigenspaces of $\langle C_V \rangle$.

Then $p = \bigoplus p_\lambda : W_\lambda \rightarrow \mathbb{C}$. $p_\lambda = 0$ for $\lambda \neq 0$, and p_0 is surjective.

It is enough to find a splitting s of $p_0 : W_0 \twoheadrightarrow \mathbb{C}$.

But W_0 is trivial by case 1, so any s works.

(iii) Suppose W/U is arbitrary. We have a SES

$$0 \longrightarrow U \xrightarrow{\iota} W \xrightarrow{\pi} W/U \longrightarrow 0$$

Hitting this SES with the functor $\mathrm{Hom}_{\mathbb{C}}(W/U, -)$, we get

$$0 \longrightarrow \operatorname{Hom}_{\mathbb{C}}(W/U, U) \xrightarrow{\iota^*} \operatorname{Hom}_{\mathbb{C}}(W/U, W) \xrightarrow{\pi^*} \operatorname{Hom}_{\mathbb{C}}(W/U, W/U) \longrightarrow 0$$

It is enough to split this sequence. We can use the s'

Case 3: Do induction.

Proof in Kirilov:

The SES's $U \hookrightarrow W \twoheadrightarrow W/U$ is controlled by $H^1(\mathfrak{g}, \operatorname{Hom}_{\mathbb{C}}(W/U, U)) = 0$. But this group is $\operatorname{Ext}_{U\mathfrak{g}}^1(W/U, U)$.

So Ext^1 vanishes, and thus every every such extension splits.

$H^1(\mathfrak{g}, V) = 0$ for a vector space V , and $\operatorname{Ext}^1(A, B)$ is $H^1(\mathfrak{g}, \operatorname{Hom}(A, B))$.

Corollary 0.64. *Let \mathfrak{g} be reductive (i.e. $\operatorname{rad}\mathfrak{g} = \mathfrak{z}(\mathfrak{g})$). Then*

$$\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$$

Proof. Consider

$$0 \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{z}(\mathfrak{g}) \xrightarrow{\operatorname{ad}} \mathfrak{gl}(\mathfrak{g}) \longrightarrow 0$$

Given by the adjoint rep. \mathfrak{g} is semisimple, and by the theorem $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$.
So $\mathfrak{g} \subset \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ as Lie algebras. ■

Goal for the rest of the course:

Understand the analogue of

$$\langle \text{diagonal matrices} \rangle \hookrightarrow \mathfrak{sl}_n$$

For arbitrary semisimple Lie algebras.

Example 0.25. Consider $h \in \mathfrak{sl}_2$. Suppose h \curvearrowright representations of \mathfrak{sl}_2 .

Then h is always diagonalizable and irreps were classified in terms of largest eigenvalues of h .

Definition 0.33. Let \mathfrak{g} be a semisimple Lie algebra. $x \in \mathfrak{g}$ semisimple if $\operatorname{ad}(x) \in \mathfrak{gl}(\mathfrak{g})$ is diagonalizable.

$x \in \mathfrak{g}$ is nilpotent if $\operatorname{ad}(x) \in \mathfrak{gl}(\mathfrak{g})$ is nilpotent.

Example 0.26. In $\mathfrak{sl}_2 = \langle h, e, f \rangle$, h is semisimple and e, f are nilpotent.

Theorem 0.65. *Let \mathfrak{g} be semisimple over \mathbb{C} , $x \in \mathfrak{g}$. Then there is a unique way to write $x = x_s + x_n$, x_s semisimple and x_n nilpotent, with $[x_s, x_n] = 0$, and $\operatorname{ad}(x_s) = P(\operatorname{ad}(x))$, $P \in \mathbb{C}[t]$.*

Corollary 0.66. *If \mathfrak{g} is semisimple then there are semisimple elements.*

Proof. ■

Lecture 19, 20/3/25

Last time, we proved that if \mathfrak{g} is a semisimple Lie algebra over \mathbb{C} , then all finite dimensional representations of \mathfrak{g} are themselves semisimple, i.e. direct sums of irreducibles.

Example 0.27. Consider $\mathfrak{sl}_n = \{x \in \mathfrak{gl}_n \mid \text{tr } x = 0\}$

We will denote by $\mathfrak{t}_n \subseteq \mathfrak{sl}_n$ the diagonal matrices.

Let E_{ij} be the matrix with 1 in the i, j th entry and 0 elsewhere. If $i \neq j$ then $E_{ij} \in \mathfrak{sl}_n$.

\mathfrak{t}_n is commutative with a basis given by $E_{i,i} - E_{i+1,i+1}$.

Proposition 21. For every $x \in \mathfrak{t}_n$ and representation $\rho : \mathfrak{sl}_n \rightarrow \mathfrak{gl}_n$, $\rho(x)$ is diagonalizable.

Proof. It is enough to show that $\rho(E_{i,i} - E_{i+1,i+1})$ is diagonalizable.

But this is a copy of h in a subalgebra isomorphic to \mathfrak{sl}_2 , and this is diagonalizable because of facts we know about the rep theory of \mathfrak{sl}_2 .

We have a distinguished class of elements $\{E_{i,i} - E_{j,j} \mid i \neq j\} \subseteq \mathfrak{t}_n$

Such that

$$\langle E_{i,j}, E_{i,j} - E_{j,i}, E_{ji} \rangle \simeq \mathfrak{sl}_2$$

We can break \mathfrak{sl}_n into eigenspaces for the \mathfrak{t}_n -action (which is by ad).

Then $\mathfrak{sl}_n = \mathfrak{t}_n \oplus \bigoplus_{i \neq j} \text{span}(E_{ij})$, because the centralizer of \mathfrak{t}_n is \mathfrak{t}_n .

The direct sum is indexed by some functionals on \mathfrak{t}_n (namely the eigenvalues). But the killing form $K(x, y) = \text{tr}(\text{ad}(x) \text{ad}(y)|_{\mathfrak{sl}_n})$ is a nondegenerate pairing on \mathfrak{t}_n , so furnishes a canonical isomorphism between \mathfrak{t}_n and its dual.

So each λ corresponds to $K(x_\lambda, -)$ for some $x_\lambda \in \mathfrak{t}_n$, and these turn out to be the $E_{i,j}$'s. (?) ■

Remark: There's a more general story for Cartan subalgebras of \mathfrak{sl}_n .

Now we have to come up with a way to say “diagonal matrix” for an abstract Lie algebra.

Definition 0.34. Let \mathfrak{g} be a semisimple Lie algebra. We say $x \in \mathfrak{g}$ is semisimple if $\text{ad}(x)$ is diagonalizable, and we say $x \in \mathfrak{g}$ is nilpotent if $\text{ad}(x)$ is nilpotent.

Definition 0.35. $\text{Der}(\mathfrak{g}) \subseteq \mathfrak{gl}(\mathfrak{g})$ is $\{x \in \mathfrak{gl}(\mathfrak{g}) \mid [ax, b] + [a, xb] = x[a, b]\}$.

This is the space of derivations on \mathfrak{g} . One can check as an exercise that it is in fact a subalgebra of $\mathfrak{gl}(\mathfrak{g})$.

Theorem 0.67. Let \mathfrak{g} be semisimple, $x \in \mathfrak{g}$. Then x can be written uniquely as $x = x_s + x_n$ with x_s semisimple, x_n nilpotent, and $\text{ad}(x_s) = P(\text{ad}(x)) \in t\mathbb{C}[t]$.

Proof. What we need is a characterization of the image of $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$.

Proposition 22. *If \mathfrak{g} is semisimple, then $\text{ad} : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{g})$ is an isomorphism.*

Proof. First, $\ker \text{ad} = \mathfrak{z}(\mathfrak{g})$, which is zero because \mathfrak{g} is semisimple, thus this is injective.

Extend the killing form K from \mathfrak{g} to $\text{Der}(\mathfrak{g})$ (which we can do because this map is injective) by saying $K(\delta_1, \delta_2) = \text{tr}(\delta_1 \delta_2 | \mathfrak{g})$.

Observe $K|_{\mathfrak{g}}$ is the Killing form of \mathfrak{g} , which is non-degenerate by Cartan's criterion. Hence $\text{Der}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^\perp$, because $\mathfrak{g} \cap \mathfrak{g}^\perp = 0$ and $\dim \mathfrak{g} + \dim \mathfrak{g}^\perp \geq \dim \text{Der}(\mathfrak{g})$.

Claim. \mathfrak{g} is an ideal.

Proof.

$$[\delta, \text{ad}(x)] = \text{ad } \delta(x)$$

So \mathfrak{g} is an ideal, so $\text{Der}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^\perp$, not just as vector spaces, but as Lie algebras. So then \mathfrak{g}^\perp and \mathfrak{g} commute, i.e. for all $\delta \in \mathfrak{g}^\perp$, $x \in \mathfrak{g}$, $\delta(x) = 0$. So then $\mathfrak{g}^\perp = 0$. This proves the claim. ■

Now we can prove the theorem. ■

Proof. Uniqueness is easy, and follows from the uniqueness of the usual Jordan decomposition on $\mathfrak{gl}(\mathfrak{g})$.

For existence, write $\text{ad}(x) = \text{ad}(x)_s + \text{ad}(x)_n$.

It is enough to show $\text{ad}(x)_s = \text{ad}(x_s)$ for some $x_s \in \mathfrak{g}$, because then we can set $x_n = x - x_s$, and we get the usual Jordan decomposition.

By the proposition, to show this, it is enough to show that $\text{ad}(x)_s$ is a derivation.

$\mathfrak{g} = \bigoplus \mathfrak{g}_\lambda$, where \mathfrak{g}_λ are the generalized eigenspaces of $\text{ad}(x)$.

$\text{ad}(x)$ acts on \mathfrak{g}_λ by $\lambda \cdot \text{Id}$.

Claim.

$$[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}$$

Proof. Compute ■

Corollary 0.68. *If $\mathfrak{g} \neq 0$ is a semisimple Lie algebra, then there exists a semisimple element in \mathfrak{g} .*

Proof. If this is not the case, then $x = x_n$ for all $x \in \mathfrak{g}$, i.e. the image of ad consists of nilpotent matrices, which implies \mathfrak{g} is nilpotent by a problem on homework 1.

Proposition 23. *Let \mathfrak{g} be semisimple, $x \in \mathfrak{g}$ semisimple, $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}_n$ a representation. Then $\rho(x)$ is diagonalizable.*

Proof. WLOG, ρ is irreducible.

Let $\mathbb{C}^N \supseteq \bigoplus_{\lambda} V_{\lambda}$, where V_{λ} is the honest (meaning NOT generalized) eigenspace of $\rho(x)$.

It is enough to show $\bigoplus_{\lambda} V_{\lambda}$ is a subrepresentation, because ρ is irreducible.

Because ρ is semisimple, $\mathfrak{g} = \bigoplus \mathfrak{g}_{\alpha}$, where \mathfrak{g}_{α} is the α -eigenspace of $\text{ad}(x)$.

To show V_{λ} is a subrepresentation it's enough to show that acting by an element of \mathfrak{g} stays in the sum, so it's enough to do it for a basis, so it's enough to do it for an eigenvector.

Fix $y \in \mathfrak{g}_{\alpha}, v \in V_{\lambda}$. Then

$$xyv = yxv + [x, y]v = \lambda yv + \text{ad}(x)(y) \cdot v = \lambda yv + \alpha yv = (\lambda + \alpha)yv$$

So $yv \in V_{\alpha+\lambda}$. ■

This completes the proof of the theorem. ■

Definition 0.36. Let \mathfrak{g} be a semisimple Lie algebra over \mathbb{C} . Then

- $\mathfrak{t} \subseteq \mathfrak{g}$ is toroidal if it is commutative and consists entirely of semisimple elements
- $\mathfrak{h} \subseteq \mathfrak{g}$ is Cartan if it is toroidal, and $\{x \in \mathfrak{g} \mid [x, y] = 0 \forall y \in \mathfrak{h}\}$. That is, \mathfrak{h} is its own centralizer.

Theorem 0.69. Let \mathfrak{g} be semisimple, $\mathfrak{t} \subseteq \mathfrak{g}$ toroidal.

1. $\mathfrak{g} = \bigoplus \mathfrak{g}_{\alpha}$, $\alpha \in \mathfrak{t}^*$, $\mathfrak{g}_{\alpha} = \{x \in \mathfrak{g} \mid [y, x] = \text{ad}(y)x \forall y \in \mathfrak{t}\}$.

That is, the adjoint action is diagonalizable.

2. $[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}] \subseteq \mathfrak{g}_{\alpha+\beta}$.
3. If $\alpha + \beta \neq 0$, then $\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}$ are orthogonal with respect to K .
4. The Killing form induces a non-degenerate pairing $\mathfrak{g}_{\alpha} \times \mathfrak{g}_{-\alpha} \rightarrow \mathbb{C}$. In particular, $K|_{\mathfrak{g}_0}$ is non-degenerate.

Proof.

1. Simultaneously diagonalizable
2. Special case of computation from before
3. Take $x \in \mathfrak{g}_{\alpha}, y \in \mathfrak{g}_{\beta}$. Goal: compute $\text{tr}(\text{ad}(x) \text{ad}(y)|_{\mathfrak{g}})$.

$\text{ad}(x) \text{ad}(y)|_{(\mathfrak{g}_{\lambda})} \subseteq \mathfrak{g}_{\lambda+\alpha+\beta}$ which has zero intersection \mathfrak{g}_{λ} .

So the matrix of $\text{ad}(x) \text{ad}(y)$ is zero on the diagonal, so $\text{tr}(\text{ad}(x) \text{ad}(y)|_{\mathfrak{g}}) = 0$.

4. The Killing form is non-degenerate.

Theorem 0.70. *Let \mathfrak{g} be a semisimple Lie algebra over \mathbb{C} , and suppose $\mathfrak{t} \subseteq \mathfrak{g}$ is a maximal toroidal subalgebra. Then \mathfrak{t} is a Cartan subalgebra.*

Corollary 0.71. *Cartan subalgebras exist*

Proof. First we must show there exists a non-zero toroidal sub-algebra of \mathfrak{g} . Then take a maximal via Zorn's lemma or the like. ■

Proof. Now let's prove the theorem.
Let \mathfrak{t} be a maximal toroidal subalgebra.

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{t}^*} \mathfrak{g}_\alpha$$

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [y, x] = \alpha(y)x \forall y \in \mathfrak{t}\}$$

Now note $\mathfrak{g}_0 = C(\mathfrak{t})$, the centralizer of \mathfrak{t} .

We want to show $\mathfrak{t} = \mathfrak{g}_0$.

Note from what we've just shown that $K|_{\mathfrak{g}_0}$ is non-degenerate.

Claim. *Let $x \in C(\mathfrak{t}) = \mathfrak{g}_0$. Then $x_s, x_n \in C(\mathfrak{t})$.*

Proof. We have $\text{ad } x_s = P(\text{ad } x)$, so for all $y \in \mathfrak{t}$, $(\text{ad } x_s)(y) = P(\text{ad } x)(y) = 0$, because $P \in t\mathbb{C}[t]$.

But this implies $x_s \in C(\mathfrak{t})$, which implies that $x_n = x - x_s \in C(\mathfrak{t})$ ■

Now note that $C(\mathfrak{t})$ is reductive because $K|_{C(\mathfrak{t})}$ is non-degenerate.

Claim. *$C(\mathfrak{t})$ is commutative*

Proof. First note $C(\mathfrak{t}) = \mathfrak{z}(C(\mathfrak{t})) \oplus \mathfrak{g}'$, with \mathfrak{g}' semisimple. $\mathfrak{t} \subseteq \mathfrak{z}(C(\mathfrak{t}))$

Let $x_s \in \mathfrak{g}'$ be semisimple.

Then $\langle \mathfrak{t}, x_s \rangle$ contradicts maximality of \mathfrak{t} . ■

Claim. *$C(\mathfrak{t})$ contains no nilpotents.*

Proof. Fix some $x \in C(\mathfrak{t})$ nilpotent, and fix $y \in C(\mathfrak{t})$ arbitrary. Then

$$\begin{aligned} K(x, y) &= \text{tr}(\text{ad}(x) \text{ad}(y)|_{\mathfrak{g}}) \\ &= \text{tr}(\text{ad}(y) \text{ad}(x)|_{\mathfrak{g}}) \end{aligned}$$

By previous point, $\text{ad}(x), \text{ad}(y)$ commute, and so their product is nilpotent, and hence the trace is zero.

So then x has to be zero by nondegeneracy of the Killing form on $C(\mathfrak{t})$

Finally, $C(\mathfrak{t})$ is toroidal because it is commutative and for all $x \in C(\mathfrak{t})$, $x_s, x_n \in C(\mathfrak{t})$, so $x_n = 0$ by the previous point. So $x = x_s$, i.e. x is semisimple. Hence $\mathfrak{t} = C(\mathfrak{t})$, because \mathfrak{t} is maximal toroidal. ■

Fact: $\text{Aut}(\mathfrak{g})$ acts transitively on the Cartan subalgebras.

Root system

Let \mathfrak{g} be semisimple, $\mathfrak{h} \subseteq \mathfrak{g}$ a Cartan subalgebra (i.e. a commutative subalgebra consisting of semisimple elements which is its own centralizer).

Theorem 0.72.

1. $\mathfrak{g} \simeq \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$, where R is the set $\{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$, and $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [y, x] = \alpha(y)x \forall y \in \mathfrak{h}\}$

R is called the set of roots

2. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$
3. If $\alpha + \beta \neq 0$, then $\mathfrak{g}_\alpha, \mathfrak{g}_\beta$ are orthogonal wrt the Killing form
4. $K : \mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \rightarrow \mathbb{C}$ is non-degenerate. In particular, $K|_{\mathfrak{h}}$ is non-degenerate.

Proof. $\mathfrak{h} = \mathfrak{g}_0$. ■

Observation: $K|_{\mathfrak{h}}$ is non-degenerate, thus furnishes an isomorphism $\mathfrak{h} \simeq \mathfrak{h}^*$ sending y to $K(y, -)$.

So given $\alpha \in R$, there exists a unique $H_\alpha \in \mathfrak{h}$ such that $\alpha = K(H_\alpha, -)$.

Example 0.28. Consider \mathfrak{sl}_3 . Then \mathfrak{h} is the diagonal matrices with trace 0.

Then $\mathfrak{h}^* = \langle e_1, e_2, e_3 \rangle / (\sum e_i = 0)$ with $e_i(M) = (M)_{ii}$.

Then $R = \{\pm(e_1 - e_2) \pm (e_1 - e_3) \pm (e_2 - e_3)\}$

Lecture 20, 25/3/25

Example 0.29. Let $\mathfrak{g} = \mathfrak{sl}_n$, \mathfrak{h} the diagonal matrices.

We have shown there is a decomposition $\mathfrak{sl}_n = \mathfrak{h} \oplus \bigoplus \text{span}(E_{ij})$, which we can use to find many copies of \mathfrak{sl}_2 inside \mathfrak{sl}_n .

Theorem 0.73. Let \mathfrak{g} be a Lie algebra, \mathfrak{h} a Cartan subalgebra.

1. $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \forall h \in \mathfrak{h}\}$, $\alpha \in \mathfrak{h}^*$, where $R = \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$, and R is called the set of roots.
2. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$
3. If $\alpha + \beta \neq 0$, then $\mathfrak{g}_\alpha, \mathfrak{g}_\beta$ are orthogonal with respect to K .
4. $K : \mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \rightarrow \mathbb{C}$ is nondegenerate; in particular, $K|_{\mathfrak{h}}$ is non-degenerate.

Proof. ■

Observation: There is a canonical isomorphism between \mathfrak{h} and \mathfrak{h}^* given by $x \mapsto K(x, -)$.

So given $\alpha \in R$, there is a unique $H_\alpha \in \mathfrak{h}$ such that $K(H_\alpha, -) = \alpha$.

Notation: Given $\alpha, \beta \in \mathfrak{h}$, we write $(\alpha, \beta) \stackrel{\text{def}}{=} K(\alpha, \beta)$.

Given $\alpha \in \mathfrak{h}^*, \beta \in \mathfrak{h}$, we write $\langle \alpha, \beta \rangle \stackrel{\text{def}}{=} \alpha(\beta)$.

Goal: Construct many copies of \mathfrak{sl}_2 inside of \mathfrak{g} .

The idea will be to take $e \in \mathfrak{g}_\alpha, f \in \mathfrak{g}_{-\alpha}$ such that $(e, f) \neq 0$, and then consider the subalgebra generated by e, f .

Lemma 10. *Let $e \in \mathfrak{g}_\alpha, f \in \mathfrak{g}_{-\alpha}$. Then $[e, f] = (e, f)H_\alpha$.*

Proof.

- First, observe $[e, f] \in \mathfrak{h}$.
- Let $x \in \mathfrak{h}$. What is $([e, f], x)$? Well, K is an invariant bilinear form, so this is equal to

$$\begin{aligned}
 (e, [f, x]) &= -(e, [x, f]) \\
 &= -(e, \alpha(x)f) \\
 &= \alpha(x)(e, f) \\
 &= (H_\alpha, x)(e, f) \\
 &= ((e, f)H_\alpha, x)
 \end{aligned}$$

So $H_{[e, f]}(x) = (e, f)H_\alpha(x)$

Lemma 11.

1. Let $\alpha \in R$. Then $(H_\alpha, H_\alpha) \neq 0$.

2. If $e \in \mathfrak{g}_\alpha, f \in \mathfrak{g}_{-\alpha}$ such that $(e, f) = \frac{2}{(\alpha, \alpha)}$, set $h_\alpha = \frac{2H_\alpha}{(\alpha, \alpha)}$

Then $\langle e, f, h_\alpha \rangle \simeq \mathfrak{sl}_2$.

Further, $\langle h_\alpha, \alpha \rangle = 2$.

Proof.

1. Suppose $(\alpha, \alpha) \stackrel{\text{def}}{=} (H_\alpha, H_\alpha) = 0$. Pick $e \in \mathfrak{g}_\alpha, f \in \mathfrak{g}_{-\alpha}$ with $(e, f) \neq 0$ (possible as K is non-degenerate).

Consider the sub-algebra $\langle e, f, \underbrace{[e, f]}_{=x} \rangle \stackrel{\text{def}}{=} \mathfrak{s}$.

We have $[x, e] = [(e, f)H_\alpha, e] = (e, f)\alpha(H_\alpha)e = 0$. Same argument shows $[x, f] = 0$.

So $x \in \mathfrak{z}(\langle e, f, x \rangle)$, so \mathfrak{s} is solvable.

Now, we have the upper triangularizable representation $\text{ad} : \mathfrak{s} \rightarrow \mathfrak{gl}(\mathfrak{g})$, so $\text{ad}(x) = \text{ad}([e, f])$ is nilpotent. But x is a multiple of H_α , which is semisimple, so $x = 0$.

So $x = (e, f)H_\alpha$ with (e, f) nonzero.

2. Directly compute $[e, f], [h, f], [h, e]$.

■

Definition 0.37. For each $\alpha \in R$, choose e, f as above, and set $(\mathfrak{sl}_2)_\alpha \subseteq \mathfrak{g}$ to be the sub-algebra generated by $e, f, [e, f]$.

It will turn out later that this is independent of the choice of e, f .

Lemma 12. Fix $(\mathfrak{sl}_2)_\alpha \subseteq \mathfrak{g}$ as before. Then $V \stackrel{\text{def}}{=} \mathbb{C}h_\alpha \oplus \bigoplus_{k \in \mathbb{Z}_{\neq 0}} \mathfrak{g}_{k\alpha}$ is irreducible as a $(\mathfrak{sl}_2)_\alpha$ -representation.

Proof. First, this decomposition is preserved by \mathfrak{sl}_2 :, because $\text{ad}(e) : \mathfrak{g}_{k\alpha} \rightarrow \mathfrak{g}_{(k+1)\alpha}$, and $\text{ad}(f) : \mathfrak{g}_{k\alpha} \rightarrow \mathfrak{g}_{(k-1)\alpha}$, and those follow from the fact that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.

To see that it is irreducible, we compute the eigenvalues of $h_\alpha \curvearrowright V$. Indeed, $\text{ad}(h_\alpha)|_{\mathfrak{g}_\alpha} = 2 \cdot \text{Id}$, and $\text{ad}(h_\alpha)|_{\mathfrak{g}_{k\alpha}} = 2k$.

This implies V is a direct sum of $\text{Sym}^n V_{\text{std}}$ for n even, and 0 appears as an eigenvalue with multiplicity 1, so this is irreducible.

■

Corollary 0.74. $\dim \mathfrak{g}_\alpha = 1$ for all α .

Proof.

■

Fact: $V \simeq (\mathfrak{sl}_2)_\alpha$.