

## Lecture 1 - 7/1/25

Missed :(

## Lecture 2 - 9/1/25

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## Lecture 3 - 14/1/25

### Character theory

Consider  $\dim \operatorname{Hom}_G(\rho_i, \rho_j) = 1$  if  $i = j$  and 0 if  $i \neq j$  (meaning if  $\rho_i \not\cong \rho_j$ )

Recall: Given a representation  $\rho : G \rightarrow \operatorname{GL}_n(k)$ , the character of  $\rho$ ,  $\chi_\rho$ , is given by

$$\chi_\rho : G \rightarrow k, g \mapsto \operatorname{tr}(\rho(g))$$

For today,  $G$  will be finite,  $k = \bar{k}$  will be algebraically closed, of characteristic 0.

Basic properties of characters:

1. Suppose  $\rho : G \rightarrow \operatorname{GL}_n(k)$  is a representation: then  $\chi_\rho(e) = n = \dim \rho$ .
2.  $\chi_\rho(g) = \chi_\rho(hgh^{-1})$  for all  $g, h \in G$ , i.e.  $\chi_\rho$  is constant on each conjugacy class of  $G$ .

**Definition 0.1.** A function  $f : G \rightarrow k$  which is constant on conjugacy classes is called a class function.

The  $\rho_i$  (isomorphism classes of reps) will form an ONB for the space of class functions.

Given  $\rho_1 : G \rightarrow \operatorname{GL}_n(k), \rho_2 : G \rightarrow \operatorname{GL}_m(k)$ ,  $\chi_{\rho_1 \oplus \rho_2} = \chi_{\rho_1} + \chi_{\rho_2}$

$$\chi_{\rho_1 \otimes \rho_2} = \chi_{\rho_1} \chi_{\rho_2}$$

To see this, let  $A, B$  be diagonalizable (which we have WLOG because the image of any finite group are all diagonalizable over an algebraically closed  $k$  of char 0, which follows from Jordan Normal form)

$$\text{Then } \operatorname{tr}(A \otimes B) = \operatorname{tr}(A) \operatorname{tr}(B).$$

I can't see the board he's writing on very well, and also I am not sure how  $A \otimes B$  was defined.

**Claim.**  $\chi_\rho : G \rightarrow k$  always factors through  $\mathbb{Q}(\mu_\infty)$ , the subfield of  $k$  containing  $\mathbb{Q}$  ( $k$  has char 0) generated by all roots of unity ( $k = \bar{k}$ )

*Proof.* Because  $G$  is finite,  $\rho_G$  has finite order, hence its eigenvalues are roots of unity, so the trace is the sum of roots of unity. ■

**Definition 0.2.**  $\bar{\cdot} : \mathbb{Q}(\mu_\infty) \rightarrow \mathbb{Q}(\mu_\infty)$  is the unique field homomorphism with the property that  $\bar{\zeta} = \zeta^{-1}$  for all roots of unity  $\zeta \in \mathbb{Q}(\mu_\infty)$ .

$$5 \quad \chi_{\rho^v} = \overline{\chi_\rho}$$

Recall  $\rho^v$  is defined via the formula  $g \cdot f = f(g^{-1} \cdot -)$  where  $f$  is a functional. We have

$$\begin{aligned} \chi_{\rho^v}(g) &= \text{tr}(\rho(g^{-1})) \\ &= \sum_{\zeta \text{ is an eigenvalue of } \rho(G)} \zeta^{-1} \\ &= \sum \bar{\zeta} = \overline{\chi_\rho(g)} \end{aligned}$$

This also follows from the Hom-tensor adjunction because  $\text{Hom}_k(\rho_1, \rho_2) = \rho_1^v \otimes \rho_2$ .

**Definition 0.3.** Let  $\chi, \psi : G \rightarrow \mathbb{Q}(\mu_\infty)$  be class functions. We define their inner product by

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)}$$

This indeed is a positive definite non degenerate.

Let  $\rho_1 : G \rightarrow \text{GL}_n(k), \rho_2 : G \rightarrow \text{GL}_m(k)$ . What is  $\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle$ ?

**Theorem 0.1.**

$$\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle = \dim_k \text{Hom}_G(\rho_1, \rho_2) = \dim_k \text{Hom}(\rho_1, \rho_2)^G$$

**Corollary 0.2.** Suppose  $\rho_1, \rho_2$  are irreducible. Then  $\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle$  is 0 if  $\rho_1, \rho_2$  are not isomorphic, and 1 if they are. So the  $\rho_i$  form an orthonormal basis for the space of class functions. ■

*Proof.* Let  $R_G \in k[G]$  be the element given by

$$R_G = \frac{1}{|G|} \sum_{g \in G} eg$$

We want to show

1. for  $v \in V^G, R_G \cdot v = v$ .

2. For arbitrary  $v \in V$ ,  $R_G \cdot v \in V^G$

To check:

1. We have

$$\begin{aligned} R_G \cdot v &= \frac{1}{|G|} \sum_{g \in G} e_g \cdot v \\ &= \frac{1}{|G|} \sum_{g \in G} v \\ &= v \end{aligned}$$

2. Fix  $g \in G$ . Then

$$\begin{aligned} g \cdot R_G \cdot v &= g \cdot \left( \frac{1}{|G|} \sum_{h \in G} hv \right) \\ &= \frac{1}{|G|} \sum_{h \in G} gh \cdot v \\ &= \frac{1}{|G|} \sum_{h \in G} h \cdot v \\ &= R_G \cdot v \end{aligned}$$

**Corollary 0.3.** *Let  $V$  be a  $G$ -representation. Then  $\dim_k V^G = \text{tr}(R_G|V)$*

*Proof.*

**Claim.**  $\text{tr}(\text{projection}) = \dim_k \text{Im}$

*Proof.* Claim  $\implies$  Cor follows from  $\text{tr}(R_G) = \dim \text{Im}(R_G|V) = \dim_k V^G$

■

We can finally prove the theorem:

*Proof.*

$$\begin{aligned}
 \dim_k \operatorname{Hom}_G(\rho_1, \rho_2) &= \dim_k \operatorname{Hom}_k(\rho_1, \rho_2)^G \\
 &= \operatorname{tr}(R_G | \operatorname{Hom}_k(\rho_1, \rho_2)) \\
 &= \operatorname{tr}\left(\frac{1}{|G|} \sum e_g | \operatorname{Hom}_k(\rho_1, \rho_2)\right) \\
 &= \frac{1}{|G|} \operatorname{tr}(g | \operatorname{hom}_k(\rho_1, \rho_2)) \\
 &= \frac{1}{|G|} \sum_{g \in G} \chi_{\operatorname{hom}_k(\rho_1, \rho_2)}(g) \\
 &= \frac{1}{|G|} \sum_{g \in G} \overline{\chi_{\rho_1}} \chi_{\rho_2} \\
 &= \langle \chi_{\rho_1}, \chi_{\rho_2} \rangle \\
 &= \underbrace{\langle \chi_{\rho_2}, \chi_{\rho_1} \rangle}_{\in \mathbb{Z}}
 \end{aligned}$$

■

## Lecture 4, 16/1/24

As always,  $G$  will be a finite group,  $k = \bar{k}$  is an algebraically closed field of characteristic 0.

$\mathbb{Q}(\mu_\infty)$  is the algebraically closed subfield of  $\mathbb{C}$  which contains all the roots of unity, and this comes with the complex conjugate  $\bar{\cdot}, \zeta \mapsto \zeta^{-1}$ .

Goal: Classify finite dimensional  $G$ -representations over  $k$ .

We have done:

1. Maschke's theorem, which states that any  $G$ -rep in  $V$  over  $k$  is semisimple.
2. Character theory:  $V \sim \chi_V : G \rightarrow \mathbb{Q}(\mu_\infty) \subseteq k, g \mapsto \operatorname{tr}(g|V)$

**Definition 0.4.**  $Cl(G)$  denotes the class functions  $G \mapsto \mathbb{Q}(\mu_\infty)$ , and it is equipped with an inner product,

$$\langle \psi, \varphi \rangle = \frac{1}{|G|} \sum_{g \in G} \psi(g) \overline{\varphi(g)}$$

Remark: There is an isomorphism  $Cl(G) \simeq Z(\mathbb{Q}(\mu_\infty)[G])$ , sending  $\varphi$  to  $\sum_{g \in G} \varphi(g) e_g$

Warning: They come with different ring structures which are not preserved by this isomorphism.

Last time we used the Reynolds operator to show  $\langle \chi_V, \chi_W \rangle = \dim_k \text{Hom}_G(V, W)$ .

If  $\rho_1, \rho_2$  are irreps of  $G$ , then  $\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle$  is 1 if  $\rho_1 \cong \rho_2$ , and 0 otherwise.

**Corollary 0.4.** # of conjugacy classes of irreducible representations of  $G \leq \dim_{\mathbb{Q}(\mu_\infty)} Cl(G) =$   
# of conjugacy classes of  $G$

*Proof.* If  $\chi_{\rho_i}$  are orthonormal, then the number of conjugacy classes of irreps is equal to  $\dim \text{span}(\chi_{\rho_i}) \subseteq Cl(G)$ , so this number is  $\leq \dim Cl(G)$  ■

**Proposition 1.** Let  $V$  be a  $G$ -representation. Then

$$\Phi_V : \bigoplus_{\rho_i \text{ irrep of } G} \rho_i \otimes_k \text{Hom}_G(\rho_i, V) \rightarrow V$$

given by  $v \otimes f \mapsto f(v)$  is an isomorphism.

*Proof.* First, we show it is surjective. By Maschke,  $V = \bigoplus_{\rho_i \text{ reps}} G\rho_i^{n_i}$ .

Let  $v \in \rho_i^{n_i} \subseteq V$ ,  $v = (v_1, \dots, v_{n_i})$ . Let  $f_j : \rho_j \rightarrow \rho_i^{n_i}$  be the inclusion of the  $j$ th coordinate.

Then  $\Phi_v(\sum_j v_j \otimes f_j) = v$ .

Now we show injectivity.

We have

$$\dim_k \bigoplus \rho_i \otimes_k \text{Hom}_G(\rho_i, V) = \dim_k V$$

This follows from

$$\dim_k \text{Hom}_G(\rho_i, V) = n_i$$

This follows from

$$\begin{aligned} \text{Hom}_G(\rho_i, V) &= \text{Hom}_G(\rho_i, \bigoplus \rho_i^{n_i}) \\ &= \bigoplus_j \text{Hom}_G(\rho_i, \rho_j)^{n_i} \\ &= \text{Hom}_G(\rho_i, \rho_i)^{n_i} \end{aligned}$$

Which is  $n_i$ -dimensional

$$\dim_k \bigoplus \rho_i \otimes_k \text{Hom}_G(\rho_i, V) = \sum n_i \dim_k \rho_i = \dim V$$
■

**Corollary 0.5.**

$$V \simeq \bigoplus_{\rho \text{ irreps of } G} \rho^{\langle \chi_{\rho_i}, \chi_V \rangle}$$

*Proof.* Enough to show  $\rho_i^{\langle \rho_i, V \rangle} \simeq \rho_i \otimes_k \text{Hom}_G(\rho_i, V)$ , i.e.  $\dim_k \text{Hom}(\rho_i, V) = \langle \chi_{\rho_i}, \chi_{\rho_j} \rangle$ .  
But that's the theorem. ■

**Corollary 0.6.**

$$V \simeq \bigoplus_{\rho_i \text{ irreps}} \rho_i^{\oplus n_i}$$

, then  $\langle \chi_V, \chi_V \rangle = \sum_i n_i^2$

*Proof.*  $\chi_V = \sum n_i \chi_{\rho_i}$

■

**Corollary 0.7.**  $V \simeq W \iff \chi_V = \chi_W$ **Corollary 0.8.**  $V$  is irreducible if and only if  $\langle \chi_V, \chi_V \rangle = 1$ .

*Proof.* Write  $V = \bigoplus_i \rho_i^{n_i}$ : so  $\langle \chi_V, \chi_V \rangle = \sum_i n_i^2$  is equal to 1 iff exactly 1  $n_i$  is nonzero, and equal to 1.

■

**Example 0.1.** (The regular representation)

Let  $G \curvearrowright k(G)$  via left multiplication.

$\chi_{k[G]}(g) = \text{tr}(g|k[G])$ , which is  $|G|$  if  $g$  is the identity, and 0 otherwise.

Because  $g \cdot e_{g'} = e_{gg'}$ , we have

$$\text{tr}(g|k[G]) = \#\{h \in G \mid gh = g\}$$

Remark: if  $X$  is a  $G$ -set (i.e. a set with a  $G$ -action), then the permutation representation,  $k^X$ , has character

$$\chi_{k^X}(g) = \#\{x \in X \mid g \cdot x = x\}$$

**Corollary 0.9.** As a  $G$ -representation,

$$k[G] \simeq \bigoplus_{\rho_i \text{ irrep}} \rho_i^{\oplus \dim \rho_i}$$

*Proof.*

$$\begin{aligned} \langle \chi_{\rho_i}, \chi_{k[G]} \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_{\rho_i}(g) \overline{\chi_{k[G]}(g)} \\ &= \frac{1}{|G|} \chi_{\rho_i}(e) \overline{\chi_{k[G]}(e)} \\ &= \frac{1}{|G|} \dim \rho_i |G| \\ &= \dim \rho_i \end{aligned}$$

Because this representation is 0 except at the identity.

■

Remark: In fact,  $\text{Hom}_G(k[G], \rho_i) \simeq \rho_i$ , AS A VECTOR SPACE.

*Proof.*  $\text{Hom}_G(k[G], \rho_i) = \text{Hom}_{k[G]}(k[G], \rho_i) \simeq \rho_i$  AS A VECTOR SPACE

■

**Corollary 0.10.** *Let  $\rho_i$  be the (conjugacy classes of) irreps of  $G$ ,  $n_i$  the dimension of  $\rho_i$ .*

*Then  $\sum_i n_i^2 = |G|$ .*

*Proof.*  $|G| = \dim_k k[G] = \dim_k \bigoplus_i \rho_i^{\oplus \dim \rho_i} = \sum n_i^2$

■

**Theorem 0.11.** *Let  $G$  be a finite group,  $k = \bar{k}$  an algebraically closed field of characteristic 0,  $\rho_1, \dots, \rho_n$  the irreps of  $G$ . Then  $\{\chi_{\rho_i}\}$  is an orthonormal basis of  $Cl(G)$ .*

*Proof.* We know it's orthonormal (so in particular linearly independent), so it is left to show that this indeed spans all of  $Cl(G)$ .

What remains to show is that  $\chi_{\rho_i}$  span  $Cl(G)$ .

It is enough to show that if  $\psi \in Cl(G)$  with  $\langle \psi, \chi_{\rho_i} \rangle = 0$  for all  $i$ , then  $\psi = 0$ , i.e. the orthogonal complement of the span of the  $\chi_{\rho_i}$  is trivial.

**Definition 0.5.** If  $\psi : G \rightarrow \mathbb{Q}(\mu_\infty)$  is a class function,

$$\gamma_\psi \stackrel{\text{def}}{=} \sum_{g \in G} \psi(g) e_g \in Z(k[G])$$

**Example 0.2.** If  $\psi : G \rightarrow k$ ,  $g \mapsto \frac{1}{|G|}$ ,  $\gamma_\psi = R_G$ .

We will compute what  $\gamma_\psi$  does to a representation.

**Proposition 2.** *If  $\rho$  is an irreducible representation of  $G$ , then  $\gamma_\psi : \rho \rightarrow \rho$  is multiplication by the scalar  $\frac{|G|}{\dim \rho} \langle \psi, \chi_{\rho^v} \rangle$*

*Proof.*

1. First,  $\gamma_\psi : \rho \rightarrow \rho$  is a homomorphism of  $G$ -representations, which follows from  $\gamma_\psi \cdot g \cdot v = g \cdot \gamma_\psi \cdot v$  for all  $g \in G, v \in \rho$ , as  $\gamma_\psi \in Z(k[G])$ .
2. By Schur,  $\gamma_\psi : \rho \rightarrow \rho$  is a scalar.
3.  $\gamma_\psi = \frac{\text{tr}(\gamma_\psi|_\rho)}{\dim \rho} \cdot \text{Id}_\rho$ , so

$$\text{tr}(\gamma_\psi|_\rho) = \text{tr}\left(\sum_{g \in G} \psi(g) e_g|_\rho\right) = \sum_{g \in G} \psi(g) \chi_\rho(g) = |G| \langle \psi, \overline{\chi_\rho} \rangle = |G| \langle \psi, \chi_{\rho^v} \rangle$$

■

Now, consider  $\gamma_\psi : k[G] \rightarrow k[G]$ . This is zero as  $\gamma_\psi$  acts as zero on every irrep (because it pairs to zero with all the irreps), and because it sends 1 to  $\gamma_\psi$ ,  $\gamma_\psi$  has to be zero.

■

**Corollary 0.12.** (of earlier claim)

$\frac{\dim \rho_i}{|G|} \gamma_{\chi_{\rho_i^v}}$  acts as 1 on  $\rho_i$ , and 0 on  $\rho_j$ , for  $\rho_i \neq \rho_j$  are irreps.

*Proof.* ■

**Corollary 0.13.** Given any  $V = \bigoplus \rho_i^{\oplus n_i}$ ,

$$\frac{\dim \rho_i}{|G|} \gamma_{\chi_{\rho_i^v}}$$

acts as a projection onto  $\rho_i^{n_i} \subseteq V$ , which is called the  $\rho_i$  isotypic part of  $V$ .

**Corollary 0.14.** #irreps of  $G$  = #conjugacy classes of  $G$

*Proof.* Let  $\{\rho_i\}$  be the irreps of  $G$  (up to conjugacy (i.e isomorphism)).

Then  $\{\chi_{\rho_i}\}$  is a basis for  $Cl(G)$ , so # of irreps =  $\dim_k Cl(G)$  = #conjugacy classes of  $G$ .

Remark: These two numbers are equal, but there is no natural or canonical bijection between the two sets in general.

## Classifying rep's

**Theorem 0.15.**  $G$  is abelian iff all irreps of  $G$  are 1-dimensional.

*Proof.* Let  $V$  be an irrep. If  $G$  is commutative, then  $\cdot g : V \rightarrow V$  is a  $G$ -homomorphism for all  $g \in G$ .

By Schur, each  $g \in G$  acts as a scalar. Now every subspace of  $V$  is a subrep, hence  $V$  is 1-dimensional.

Now suppose that all irreps are 1-dimensional. Let  $n_i$  be the dimensions of the irreps  $\rho_i$ , and let  $c$  be the number of conjugacy classes (or equivalently the number of irreps) of  $G$ . Then  $|G| = \sum_i n_i^2$ , but this is at least  $c$ , because we are taking the sum of  $c$  positive numbers, but each  $n_i$  is 1, so each element of  $G$  is its own conjugacy class. ■

**Example 0.3.** Take  $G = \mathbb{Z}/n\mathbb{Z}$

For each element  $\zeta \in \mu_n \stackrel{\text{def}}{=} \text{nth roots of unity}$ , consider  $\chi_\zeta : \mathbb{Z}/n\mathbb{Z} \rightarrow k^*, a \mapsto \zeta^a$

This gives  $n$  distinct reps, which is the number of conjugacy classes, hence we have a complete list.

**Example 0.4.**  $S_3$  has conjugacy classes  $[e], [(12)], [(123)]$ , so there are 3 irreducible representations. We have a trivial representation, whose character sends all conjugacy classes to 1.

We also have  $\text{sgn} : S_3 \rightarrow \{\pm 1\} \subseteq k^*$ , so  $\chi_{\text{sgn}}$  sends  $[e]$  to 1,  $[(12)]$  to -1, and  $[(123)]$  to 1.



At this point we know there must be a third representation,  $std$ , and we can fill in its row in the character table somehow.  $std$  is given by  $S_3 \curvearrowright \mathbb{C}^{\{1,2,3\}} / \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ , with

$$\chi_{std} = \chi_{\mathbb{C}^{\{1,2,3\}}} - \chi_{triv}, \text{ so } \chi_{std}(e) = 2, \chi_{std}(12) = 0, \chi_{std}(123) = -1.$$

We claim that  $\chi_{std}$  is irreducible. To see this, we compute

$$\langle \chi_{std}, \chi_{std} \rangle = \frac{1}{6}(2^2 + 3 * 0^2 + 2(-1)^2) = 1.$$

**Example 0.5.**  $Q_8 = \langle \pm 1, \pm i, \pm j, \pm k \rangle$ , with multiplication given as in the quaternion group,  $i^2 = j^2 = k^2 = ijk = -1$ .

Conjugacy classes:  $(e), -1, \{\pm i\}, \{\pm j\}, \{\pm k\}$ .

$\chi_{triv}$  sends them all to 1, of course.

## Lecture 5, 21/1/25

	1	-1	{i, -i}	{j, -j}	{k, -k}
triv	1	1	1	1	1
i-ker	1	1	1	-1	-1
j-ker	1	1	-1	1	-1
k-ker	1	1	-1	-1	1
?	...	...	...	...	...

Let  $\mathbb{H} = \mathbb{R}\langle 1, i, j, k \rangle$ . Then  $Q_8 \curvearrowright \mathbb{H}$  by left multiplication,  $\mathbb{H} \curvearrowright \mathbb{C}$  by multiplication by  $i$  on the right. This example might be useful to think about for the homework.

Now let's get the character table for  $S_4$ .

conj class	0	(12)	(123)	(12)(134)	(1234)
size	1	6	8	3	6
sgn	1	-1	1	1	-1
$std = \mathbb{C}^4 / \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$	3	1	0	-1	-1
$std \otimes sgn$	3	-1	0	-1	1
$std \circ \pi_{4 \rightarrow 3}$	...	...	...	...	...

If  $S_4$  is the symmetries of a tetrahedron, then  $\pi_{4 \rightarrow 3}$  is the map from  $S_4$  to  $S_3$  furnished by  $S_4$  acting on pairs of sides, of which there are 3.

How does the structure of  $G$  interact with its representation theory?

**Proposition 3.** (Homework)

Let  $G, H$  be groups, (NOT necessarily finite!),  $k = \bar{k}$  algebraically closed.

Then any irrep of  $G \times H$  has the form  $V \boxtimes W$ , where

- $V$  is an irrep of  $G$ ,
- $W$  is an irrep of  $H$
- $(g, h) \cdot v \boxtimes w = (g \cdot v) \boxtimes (h \cdot w)$

This is the same as tensoring the two reps of  $G \times H$  we get from

$$\begin{array}{ccc}
 & & G \longrightarrow \mathrm{GL}(V) \\
 & \nearrow & \\
 G \times H & & \\
 & \searrow & \\
 & & H \longrightarrow \mathrm{GL}(W)
 \end{array}$$

*Proof.* HW

■

We have now classified (modulo the homework) all representations of all finite abelian groups.

In some sense, (the sense of Artin's theorem) is that the representation theory of a group is controlled by the rep theory of its abelian subgroups.

### Restriction & induction

Let  $H \subseteq G$  be a subgroup of  $H$ ,  $G$  again finite.

We have a restriction functor  $\mathrm{Res}_H^G : \mathrm{Rep}_G \rightarrow \mathrm{Rep}_H$ ,

$$(\rho : G \rightarrow \mathrm{GL}(W)) \mapsto \rho|_H$$

There is a functor going the other way called induction,  $\mathrm{Ind}_H^G : \mathrm{Rep}_H \rightarrow \mathrm{Rep}_G$ .

**Definition 0.6.** Let  $V$  be an  $H$ -representation. Then

$$\mathrm{Ind}_H^G V \stackrel{\mathrm{def}}{=} k[G] \otimes_{k[H]} V$$

Equivalent descriptions:

$$\mathrm{Ind}_H^G(V) \stackrel{\mathrm{def}}{=} \{ \phi : G \rightarrow V \mid \phi(gh^{-1}) = h\phi(g) \forall g \in G, h \in H \}$$

An element of the former looks like  $\sum_g e_g \otimes v_g$ . Take  $e_g e_h \otimes v = e_g \otimes (h \cdot v)$ ,  $g \cdot \phi = g\phi(g^{-1}-)$ . Think about this and see how this makes the descriptions the same. One more description:

$$\mathrm{Ind}_H^G(V) = \bigoplus_{g \in G/H} g_i \cdot V$$

where  $g \cdot \sum g_i v_i = \sum g_{j(i)} k_i \cdot V$  where  $g_j g_i = g_{j(i)}$  (???)

Exercise: check the above is equivalent to the other two things.

**Example 0.6.**

1.  $\text{Ind}_H^G \text{triv} = k^{G/H}$  follows from second description. By definition,  $\text{Ind}_H^G \text{triv} = \{f : G \rightarrow k \mid f(gh^{-1}) = h \cdot f(g) = f(g)\} = \{f : G/H \rightarrow k\}$
2.  $\text{Ind}_{(1)}^G k = k[G] \otimes_k k = k[G]$
3. Suppose  $\chi : H \rightarrow \mathbb{C}^\times$  is a representation. What is  $\text{Ind}_H^G \chi$ ? To find  $\text{Ind}_H^G \chi(g)$ , pick coset representative  $g_i$  from  $G/H$ , and we get permutation matrix for  $G \curvearrowright G/H$  times the diagonal matrix whose  $i$ th entry is  $\chi(h_i)$ , where  $gh_i^{-1} = g_{j(i)}h_i^{-1}$

**Lecture 6, 23/1/25**

Corrections:

In the homework, problem 4 part a) should include the assumption that the action of  $G$  on  $H$  by conjugation is inner, i.e. for all  $g \in G$ , the map  $(\cdot)^g : H \rightarrow H$  sending  $h \mapsto ghg^{-1}$  is  $(\cdot)^{h'}$  for some  $h' \in H$ .

Remark: An example is if we take  $G = A \times B, H = A \times \{1\}$ . Then  $(\cdot)^{(a,b)} = (\cdot)^{(a,1)}$

Last time:

- We did character tables for  $Q_8, S_4$
- We stated the classification of irreducible representations of a product  $G \times H$
- Classification of irreps of finite abelian groups
- Restriction & induction

Here is more on induction:

$\text{Ind}_H^G(V) \stackrel{\text{def}}{=} k[G] \otimes_{k[H]} V$ , where  $k[G]$  is a right module and  $V$  is a left one. Tensoring a right with a left yields an abelian group (indeed a  $k$ -vector space), and it all works out because  $k[G]$  is a left  $k[H]$  module.

It is also the set  $\{\phi : G \rightarrow V \mid \phi(gh^{-1}) = h \cdot \phi(g) \text{ for all } g \in G, h \in H\}$ , where

$$g \cdot \phi = \phi(g^{-1} \cdot)$$

Explanation

An element of  $k[G] \otimes_{k[H]} V$  is a formal sum  $\sum e_g \otimes v_g$  such that  $e_g e_h \otimes v = e_g \otimes (h \cdot v)$   
How to recognize induced representations:

- Suppose  $V$  is a  $G$ -rep,  $W \subseteq V$  is  $H$ -stable. When is  $V \simeq \text{Ind}_H^G W$ ?
- Consider  $gW \subseteq V$ . Because  $W$  is  $H$ -stable, this only depends on  $[g] \in G/H$

**Proposition 4.**  $V = \text{Ind}_H^G W$  if and only if  $V = \bigoplus_{g \in G/H} gW$

*Proof.* Sketch

Recall the third version,  $\text{Ind}_H^G V = \oplus_{g_i \in G/H} g_i U$

■

**Proposition 5.**

$$\begin{aligned}\chi_{\text{Ind}_H^G \rho}(u) &= \frac{1}{|H|} \sum_{g \in G, g^{-1}ug \in H} \chi_\rho(g^{-1}ug) \\ &= \sum_{x \in G/H} \hat{\chi}_\rho(x^{-1}ux)\end{aligned}$$

$$\text{where } \hat{\chi}_\rho(v) = \begin{cases} \chi_\rho(v) & v \in H \\ 0 & \text{otherwise} \end{cases}$$

*Proof.*

■

**Proposition 6.** *Let  $H \subseteq G$  be a subgroup of finite index. Then*

$$\text{Hom}_G(\text{Ind}_H^G V, W) \simeq \text{Hom}_H(V, \text{Res}_G^H W)$$

*Proof.* This is a special case of the tensor-hom adjunction:

$$\begin{aligned}\text{Hom}_G(\text{Ind}_H^G V, W) &\simeq \text{Hom}_G(k[G] \otimes_{k[H]} V, W) \\ &= \text{Hom}_H(V, \text{Hom}_G(k[G], W)) \\ &= \text{Hom}_H(V, \underbrace{W}_{\text{as an } H\text{-rep}}) \\ &= \text{Hom}_H(V, \text{Res}_G^H W)\end{aligned}$$

■

**Corollary 0.16.** *Let  $V$  be a representation of  $H$ ,  $W$  is a representation of  $G$ , both finite. Then*

$$\langle \chi_{\text{Ind}_H^G V}, \chi_W \rangle = \langle \chi_V, \chi_{\text{Res}_G^H W} \rangle$$

*Proof.* These numbers are the dimensions of the hom-spaces, which are the same by the above.

■

**Theorem 0.17** (Artin). *Let  $G$  be a finite group,  $k = \bar{k}$ ,  $\text{char } k = 0$ . Then the map*

$$\bigoplus_{H \subseteq G \text{ cyclic}} \text{Cl}(H) \twoheadrightarrow \text{Cl}(G)$$

*For each cyclic group  $H$ , it acts on characters linearly, so we can extend that to  $\text{Cl}(H)$ , and we can extend that to  $\bigoplus \text{Cl}(H)$*

*Proof.* Remark: Let  $G$  be a finite group,  $R(G)$  be the “representation ring of  $G$ ”,

$$R(G) = \bigoplus_{\rho_i \text{ irreps of } G} \mathbb{Z}[\rho_i]$$

with  $[\rho_i] \cdot [\rho_j] = [\rho_i \otimes \rho_j]$ , by writing  $\rho_i \otimes \rho_j = \bigoplus_{\rho_k \text{ irreps}} \rho_k^{n_k}$

**Proposition 7.** *There is a map  $R(G) \rightarrow Cl(G)$  sending  $[\rho_i] \rightarrow \chi_{\rho_i}$ . This is a ring homomorphism (because character of tensor product is pointwise product of characters). There is an induced map  $R(G) \otimes_{\mathbb{Z}} k \rightarrow Cl(G)$  which is an isomorphism.*

*Proof.*

1. These are vector spaces of the same dimension
2. The map is surjective because (for example,) characters of irreps span.

■

**Corollary 0.18** (to Artin’s theorem). *The map (linear extension of  $\bigoplus \text{Ind}_H^G$ )*

$$\bigoplus_{H \leq G \text{ cyclic}} R(H)_k \rightarrow R(G)_k$$

*is surjective.*

*I.e. every representation of  $G$  is a “ $k$ -linear combo” of irreps induced from cyclic subgroups.*

**Corollary 0.19.**

1.  $\bigoplus_{H \leq G \text{ cyclic}} R(H)_{\mathbb{Q}} \rightarrow R(G)_{\mathbb{Q}}$  is surjective, i.e. every irreducible character of  $G$  is a  $\mathbb{Q}$ -linear combination of characters induced from cyclic subgroups.
2.  $\bigoplus_{H \leq G \text{ cyclic}} R(H) \rightarrow R(G)$  has finite cokernel.

*Proof.*

- (1)  $\implies$  (2) because the image of  $\text{Ind}$  spans  $R(G)$  rationally by (1), i.e. given  $x \in R(G)$ , there is  $N$  such that  $N \cdot x \in \text{Im}(\text{Ind})$ , so the cokernel is torsion, and torsion finitely generated abelian groups are finite.
- We know (1) by Artin, because  $\text{Ind}_{\mathbb{Q}} \otimes_{\mathbb{Q}} k$  is surjective, as rank  $r$  invariant under extension of scalars?

■

We now prove Artin’s theorem:

*Proof.* It is enough to show that the adjoint map of  $\oplus \text{Ind}_H^G$  is injective. But  $\langle \text{Ind } \chi, \psi \rangle = \langle \chi, \text{Res } \psi \rangle$ , so

$$\bigoplus \text{Res}_G^H : Cl(G) \rightarrow \bigoplus_{H \leq G \text{ cyclic}} Cl(H)$$

is adjoint to  $\text{Ind}$ . Now let  $\psi$  be in the kernel; then  $\text{Res}_G^H \psi \equiv 0$  for all  $H$ , which implies  $\psi \equiv 0$ , so we win. ■

## Loose ends:

- Structure of  $k[G]$
- Integral theory
- Corollary of all this discussion: if  $G$  is a finite group,  $\rho$  an irrep, then  $\dim \rho \mid |G|$

## Structure of $k[G]$ (and more generally, semisimple algebras)

**Definition 0.7.** Let  $k$  be a field,  $R$  a  $k$ -algebra (possibly non-commutative). Then  $R$  is semisimple if

1.  $R$  is finite dimensional as a  $k$ -vector space
2. All left  $R$ -modules which are finite-dimensional  $k$ -vector spaces are semisimple.

**Theorem 0.20.** *Let  $R$  be semisimple  $k$ -algebra. Then*

$$R \simeq \prod \text{Mat}_{n_i}(D_i)$$

where  $D_i$  are division  $k$ -algebras.

*Proof.* (Take  $R = k[G]$ )

Consider  $R$  as a left  $R$ -module;

$$R \simeq \bigoplus M_i^{\oplus n_i}$$

where  $M_i$  is simple, all  $M_j$ s are mutually non-isomorphic left  $R$ -modules.

Note  $\text{Hom}_{R\text{-mod}}(M_i, M_i)$  is a division algebra (otherwise we would have a morphism with a kernel, but  $M_i$  is simple).

Because  $R^{\text{op}} \simeq \text{Hom}_{R\text{-mod}}(R, R)$ , this means

$$R \simeq \text{Hom}_{R\text{-mod}}(\bigoplus M_i^{\oplus n_i}, \bigoplus M_i^{\oplus n_i})$$

Now,  $\text{Hom}_{R\text{-mod}}(M_i, M_j) = 0$  for  $i \neq j$  (again by simplicity and mutual nonisomorphism) so

$$\text{Hom}_{R\text{-mod}}(R, R) \simeq \oplus_i \text{Hom}_{R\text{-mod}}(M_i^{n_i}, M_i^{n_i})$$

So if we take  $D_i^{\text{op}} = \text{Mat}_{n_i}(\text{Hom}(M_i, M_i))$ , we win. ■

**Corollary 0.21.** *Let  $k = \bar{k}$ . Then  $R \simeq \oplus \text{Mat}_{n_i}(k)$*

*Proof.*

1. Finite dimensional central division algebras over an algebraically closed field are the field itself.
2. Or, same proof as in Schur,

$$\text{Hom}_{R\text{-mod}}(M_i, M_i) = k$$
■

Let's specialize to  $R = k[G]$ .

As a  $k[G]$ -module,  $k[G] \simeq \rho_i^{\oplus n_i}$ , so we have a map

$$k[g] \rightarrow \bigoplus_{\rho_i \text{ irrep}} \underline{\text{Hom}}_k(\rho_i, \rho_i) \simeq \bigoplus_{\rho_i \text{ irrep}} \rho_i \boxtimes \rho_i^v \simeq \bigoplus_{\rho_i \text{ irrep}} \rho_i \otimes \text{Hom}(\rho_i, k[G])$$

$$x \mapsto \text{right multiplication by } x$$

Recall: If  $V$  is any  $G$ -rep, then  $V = \bigoplus \rho_i \otimes \text{Hom}_G(\rho_i, V)$ ,  
so we have  $k[G] \rightarrow \bigoplus \text{End}(\text{Hom}(\rho_i, k[G]))$

**Claim.** *This isomorphism of rings is  $G \times G$ -equivariant if we give  $\text{End}(\rho_i^{\dim \rho_i})$  the  $G \times G$  structure  $\rho_i \boxtimes \rho_i^v$*

*Proof.* We need to check  $\text{End}(\rho_i^{\dim \rho_i})$  as a right  $G$ -module it is  $(\rho_i^v)^{\dim \rho_i}$ .

If  $G \hookrightarrow G \times G$  by  $g \mapsto (g, g^{-1})$ , then it has an invariant in  $\text{Hom}_G(\rho_i^{\dim \rho_i}, \rho_i^{\dim \rho_i})$ ,

As  $G$ -reps,  $\text{Hom}(\rho_i, \rho_i) \simeq \rho_i \otimes \rho_i^v$

**Claim.** *Given a rep  $V \boxtimes W$  of  $G \times G$ , the structure of  $V$  and  $V \boxtimes W|_{(g, g^{-1})}$  determines  $W$ .*

*Proof.* ■

## Lecture 7, 28/1/25

Substitute for today: Dr Jacob Tsimerman

Let  $k = \bar{k}$  be an algebraically closed field of characteristic 0,  $G$  a finite group.

Let  $(\rho_1, V_1), \dots, (\rho_n, V_n)$  be the irreducible left representations of  $G$ .

**Theorem 0.22.**

$$k[G] \cong \bigoplus_{i=1}^n \rho_i \boxtimes \rho_i^v = \bigoplus_{i=1}^n V_i \otimes V_i^*$$

as  $G \times G$ -reps  $((g, g') \cdot v \otimes v^* = (g \cdot v) \otimes v^* + v \otimes (g' \cdot v^*))$

*Proof.* Let  $W_i \stackrel{\text{def}}{=} \text{Hom}_G(V_i, k[G])$ . Then

$$k[G] \cong \bigoplus_{i=1}^n V_i \otimes W_i$$

as  $G \times G$ -representations because we get the right  $G$ -action for free.

**Claim.** As right  $G$ -representations,  $W_i \cong V_i^*$

*Proof.*

Convention: Given an element  $x = \sum_{g \in G} a_g(x)g \in k[G]$ , we use  $a_g : k[G] \rightarrow k$  to denote the  $g$ -th coefficient.

This has the property that  $a_g(x \cdot g') = a_{g'g^{-1}}(x)$

Define  $\psi : W_i \rightarrow V_i^*$  by

$$\psi(\phi) \stackrel{\text{def}}{=} a_1 \circ \phi$$

**Claim.**  $\psi$  is an isomorphism

*Proof.* Suppose  $\phi \in W_i$ . For  $g \in G$ ,  $a_g(\phi(v)) = a_1(g^{-1}\phi(v))$ . But  $\phi$  is a map of left  $G$ -modules, so this is  $a_1(\phi(g^{-1}(v))) = \psi(\phi)(g^{-1}v)$ .

So, we can write

$$\phi(v) = \sum_{g \in G} \psi(\phi)(g^{-1}v) \cdot g$$

So  $\phi$  is entirely determined by  $\psi(\phi)$ , or in other words,  $\psi$  is injective.

On the other hand, let  $\ell \in V^*$ .

Consider  $\phi_\ell \in W_i$ ,  $\phi_\ell(v) = \sum_{g \in G} \ell(g^{-1}v) \cdot g$

**Claim.**  $\phi_\ell \in W_i$

*Proof.* Let  $g_0 \in G$ . Then

$$\phi_\ell(g_0v) = \sum_{g \in G} \ell(g^{-1}g_0v) = \sum_{g \in G} \ell(g^{-1}v) \cdot g_0g = g_0 \cdot \phi_\ell(v)$$



This shows that  $\psi$  is surjective. ■

**Claim.**  $\psi$  respects the right  $G$ -action.

*Proof.*

$$\begin{aligned}\psi(\phi^{g_0})(v) &= \psi(\phi)(g_0v) \\ &= a_1(\phi(g_0v)) \\ &= a_1(g_0\phi(v)) \\ &= a_{g_0^{-1}}(\phi(v))\end{aligned}$$

On the other hand,

$$\begin{aligned}\psi(\phi^{g_0}v) &= a_1(\phi^{g_0}(v)) \\ &= a_1(\phi(v)g_0) \\ &= a_{g_0^{-1}}(\phi(v))\end{aligned}$$

So  $\psi(\phi^{g_0}) = \psi(\phi)^{g_0}$  ■

This proves the theorem. ■

### Matrix Coefficients

Let  $\{v_1, \dots, v_n\}$  be a basis for an irreducible representation  $V$ .

Let  $\{v_1^*, \dots, v_n^*\}$  be the dual basis for  $V^*$ .

**Definition 0.8.** Given  $1 \leq i, j \leq m$ , the matrix coefficient  $a_{i,j}$  is given by

$$a_{i,j}(g) = v_i^*(g \cdot v_j)$$

This is a function from  $G$  to  $k$ .

Define  $A_{i,j} \in k[G]$  by

$$A_{i,j} \stackrel{\text{def}}{=} \sum_{g \in G} a_{i,j}(g) \cdot g$$

**Theorem 0.23.**

$$\langle A_{i,j} \rangle_{1 \leq i, j \leq m} = \rho \boxtimes \rho^v$$

where  $(\rho, V)$  is the  $G$ -rep.

*Proof.* ■

**Theorem 0.24.** Let  $G$  be a finite group,  $k = \bar{k}$  an algebraically closed field of characteristic 0.

Let  $(\rho, V)$  be an irreducible representation of  $G$ .

Then  $\dim V \mid |G|$

*Proof.*

**Corollary 0.25.** *If  $d_1, \dots, d_n$  is the dimensions of the irreps of  $G$ , then*

1.  $m = \text{number of conjugacy classes of } G$  (often called  $m$ )
2.  $d_i \mid |G|$  for all  $i$
3.  $\sum_{i=1}^m d_i^2 = |G|$

*Proof.* ■

**Example 0.7.** If  $G = S_3$ ,  $m = 3$ , with conjugacy classes  $[\text{Id}]$ ,  $[(12)]$ ,  $[(123)]$ , then we have  $d_1 = 1$ ,  $1 + d_2^2 + d_3^2 = 6$ ,  $d_2, d_3 \mid 6$ .

So we must have  $d_2 = 1$ ,  $d_3 = 2$ .

Recollections of algebraic integers

**Definition 0.9.** Let  $R$  be a commutative ring.

Then  $x \in R$  is integral, or an algebraic integer, if  $x$  satisfies a monic integer polynomial.

**Example 0.8.**

- 3
- $\sqrt{5}$
- $\frac{1+\sqrt{5}}{2}$

Non-examples include

- $\frac{3}{7}$
- $\frac{1}{\sqrt{2}}$

**Proposition 8.** *The following are equivalent:*

1.  $x$  is integral
2. The subring generated by  $x$  is a finitely generated  $\mathbb{Z}$ -module
3. The subring generated by  $x$  is contained in a finitely generated  $\mathbb{Z}$ -module in  $R$ .

*Proof.* Let's start with (1)  $\implies$  (2).

Suppose  $x^N + \sum_{i=1}^{N-1} a_i x^i = 0$ ,  $a_i \in \mathbb{Z}$ .

Then  $x^N \in \langle 1, x, \dots, x^{N-1} \rangle_{\mathbb{Z}}$ . But then  $x^{N+1} \in \langle 1, x, \dots, x^N \rangle_{\mathbb{Z}}$ , so  $x^{N+1} \in \langle 1, x, \dots, x^{N-1} \rangle_{\mathbb{Z}}$ . So the subring generated by  $x$  equals  $\langle 1, x, \dots, x^{N-1} \rangle_{\mathbb{Z}}$ .

(2)  $\implies$  (3) is clear

So let's see (3)  $\implies$  (1).

Let  $A_N = \langle 1, x, x^{N-1} \rangle_{\mathbb{Z}}$ . By assumption, there exists a finitely generated  $\mathbb{Z}$ -module  $B \subset R$  such that  $A_1 \subseteq A_2 \subseteq \cdots \subseteq B$

By Noetherianity, the sequence stabilizes, so there exists some  $M$  such that  $A_M = A_{M-1}$ , and so  $x^M$  is a finite linear combination of lower powers of  $x$ , so there are  $a_i$  such that

$$x^M + \sum_{i=1}^{M-1} a_i x^i = 0$$

**Corollary 0.26.** *The things on the list of non algebraic integers actually belong on the list!*

*Proof.* ■

## Lecture 8, 30/1/25

Sub Prof: Mathilde Gerbelli-Gauthier

End Goal:  $G$  finite,  $\rho$  irrep of  $G$  over  $k = \bar{k}$  algebraically closed of characteristic 0.

We want to show that  $\dim \rho \mid |G|$

Strategy: Prove that  $\frac{|G|}{\dim \rho}$  is an algebraic integer

As a corollary of the proof of the last prop, we get

**Corollary 0.27.** *Integral elements of  $R$  form a subring.*

*Proof.* ■

## Integrality of characters

As always, let  $G$  be a finite group,  $k = \bar{k}$  algebraically closed of characteristic 0, and  $\rho : G \rightarrow \mathrm{GL}_n(k)$  just any representation (not necessarily irreducible).

**Proposition 9.**

1. *The values of the character of  $\rho$ ,  $\chi_\rho(g)$ , are algebraic integers*
2. *Let  $u = \sum_{g \in G} u(g)g$  be an element of  $Z(k[G])$ . Suppose that  $u(g) \in k$  are algebraic integers. Then  $u$  is integral.*

*At some point in the classes I missed we show that the indicators of conjugacy classes span the center of  $k[G]$ .*

*Proof.*

1.  $\chi_\rho(g)$  is a sum of roots of unity, hence a sum of algebraic integers, hence an algebraic integer.

2. Using a previous result, let  $u(g)$  be the indicator function of a conjugacy class. But the sub- $\mathbb{Z}$ -module of  $Z(k[G])$  generated by the indicator functions is a sub-ring (because the product of  $1_{C_1} \cdot 1_{C_2}$  is a linear combination of the indicators of conjugacy classes, and the coefficient in front of each  $g$  is an integer).

Thus each indicator of a conjugacy class is contained in a finitely generated  $\mathbb{Z}$ -module, and is integral. ■

**Corollary 0.28.** *Let  $\rho$  be an irrep of  $G$  and let  $u \in Z(k[G])$  be as before. Then*

$$u_\rho = \frac{1}{\dim \rho} \sum_{g \in G} u(g) \chi_\rho(g) \in k$$

*is an algebraic integer.*

*Proof.*

**Claim.** *Given  $\rho$ ,  $u \mapsto \frac{1}{\dim \rho} \sum u(g) \chi_\rho(g)$  is a ring homomorphism*

*Proof.*

$$u_1 * u_2 \mapsto \left( \frac{1}{\dim \rho} \sum u_1(g) \chi_\rho(g) \right) \left( \frac{1}{\dim \rho} \sum u_2(g) \chi_\rho(g) \right)$$

The goal will be to define a ring-hom from  $Z(k[G])$  to  $k$  sending  $u$  to  $u_\rho$ . Since  $u$  is integral, it maps to an integral element of  $k$ . ■

$$u \mapsto \frac{|G|}{\dim \rho} \langle u, \chi_{\rho^v} \rangle = u_\rho$$

$$\sum u'(g) \chi_\rho(g) = |G| \langle u, \rho^v \rangle$$

Recall that  $Z(k[G]) \curvearrowright \rho$  by  $G$ -homomorphism, that action induces a natural map

$$Z(k[G]) \mapsto \text{Hom}_G(\rho, \rho) = k$$

So

$$u \mapsto \frac{|G|}{\dim \rho} \langle u, \chi_{\rho^v} \rangle$$

The matrix is scalar, so it suffices to compute its trace. Its trace is

$$\sum_{g \in G} u(g) \chi_\rho(g) = |G| \langle u, \chi_{\rho^v} \rangle$$

Dividing by  $\dim \rho$  gives the result. ■?

**Theorem 0.29.** *Let  $G$  be a finite group,  $k = \bar{k}$  an algebraically closed field of characteristic 0,  $V_\rho$  an irrep of  $G$ . Then  $\dim V \mid |G|$*

*Proof.* Set  $u = \sum_{g \in G} \chi_\rho(g^{-1})g$ . By the above, we have

$$\begin{aligned} \frac{1}{\dim \rho} \sum u(g) \chi_\rho(g) &= \frac{|G|}{\dim \rho} \langle \chi_{\rho^v}, \chi_{\rho^v} \rangle \\ &= \frac{|G|}{\dim \rho} \underbrace{\dim \operatorname{Hom}_G(\rho^v, \rho^v)}_{=1} \\ &= \frac{|G|}{\dim \rho} \end{aligned}$$

But the left hand side is an integral element of  $\mathbb{Q}$ , so the right hand side is an integral element of  $\mathbb{Q}$ , hence an integer. ■

## Rep theory of the symmetric group

As always,  $|G| < \infty, \operatorname{Char}(k = \bar{k}) = 0$

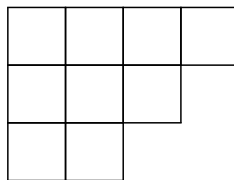
Here are some key facts about the symmetric groups:

1. The number of irreps of  $S_n$  is equal to the number of conjugacy classes in  $S_n$ .
2. The conjugacy classes in  $S_n$  (aka cycle type) are in bijection with partitions of  $n$ .
3. The irreps of  $S_n$  are also indexed by partitions of  $n$ .

**Definition 0.10.** A partition of  $n$  is a sequence  $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r)$  such that  $\sum \lambda_i = n$ .

**Definition 0.11.** The young diagram  $D_\lambda$  has  $\lambda_1$  boxes in the first row,  $\lambda_2$  in the second row, etc.

For example, the corresponding diagram for  $\lambda = (4, 3, 1)$



The conjugate partition  $\lambda'$  is the one such that  $D_{\lambda'}$  is obtained by  $D_\lambda$  by flipping along the diagonal.

If  $\lambda = (4, 3, 1)$ ,  $\lambda' = (3, 2, 2, 1)$ . Then  $D_{\lambda'}$  is


## Projections and young symmetrizers

An algorithm: start with  $\lambda$

1. Number the boxes in your Young diagram  $D_\lambda$  from left to right, top to bottom: you now have a young tableaux.

1	2	3	4
5	6	7	
8			

2. Let  $\cdot P \subseteq S_n$  be the subgroup of all permutations that preserve each row of our Young tableaux. E.g.  $P \simeq S_4 \times S_3 \hookrightarrow S_8$ .
3.  $Q \subseteq S_n$  the subgroup that preserves each column of the same Young tableau e.g.  $Q \simeq S_3 \times S_2 \times S_2 \hookrightarrow S_8$ .

In  $\mathbb{C}[S_n]$ , define  $a = \sum_{p \in P} e_p$ ,  $b = \sum_{q \in Q} \text{sgn}(q) e_q$

4. Suppose that  $V$  is a vector space, and  $S_n \curvearrowright V^{\otimes n}$  by permuting factors.

The element  $a$  symmetrizes along the rows, and projects onto

$$\text{Sym}^{\lambda_1}(V) \otimes \cdots \otimes \text{Sym}^{\lambda_n}(V)$$

up to an isomorphism.

5. The element  $b$  alternates along the columns and projects onto a tensor product of exterior powers indexed by  $\lambda'$ :

$$\bigwedge^{\lambda'_1}(V) \otimes \cdots \otimes \bigwedge^{\lambda'_n}(V)$$

6. Set  $c = ab$ . This is called the Young Symmetrizer

Here are some examples of Young symmetrizers: If  $\lambda = (1, \dots, 1)$ , then  $c$  gives the sign representation.  $\lambda = (n)$  gives the trivial rep.

## Irreducibility and idempotency

**Theorem 0.30.** *A suitable nonzero scalar of  $c = ab$  is an idempotent in  $\mathbb{C}[S_n]$ . Its image, when acting on the regular representation, is irreducible, and denoted  $V_\lambda$ . Distinct partitions give rise to distinct (meaning nonisomorphic) representations and every irep arises from this process for a unique partition.*

**Corollary 0.31.** *Every representation of  $S_n$  is defined over  $\mathbb{Q}$ .*

*Proof.* ■

**Example 0.9.**

- For  $S_3$ ,  $\text{triv} = (4)$ ,  $\text{sgn} = (1, 1, 1)$ ,  $\text{std} = (2, 1)$
- For  $S_4$ ,  $\text{triv} = (4)$ ,  $\text{sgn} = (1, 1, 1, 1)$ ,  $\text{std} = (3, 1)$ ,  $\text{std} \otimes \text{sgn} = (2, 1, 1)$ ,  $S_4 \rightarrow S_3 = (2, 2)$
- In general,  $(d, 1, \dots, 1)$  corresponds to various exterior powers of the standard representation.

**Theorem 0.32.** *(Hook-length formula)*

*Label each box  $b$  in a young diagram (boxes to the right of  $b$ ) + (boxes below).*

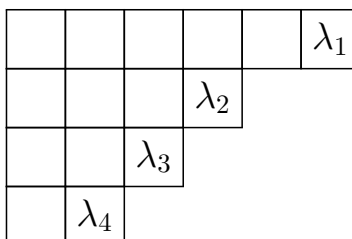
*These are called hook lengths. Then  $\dim V_\lambda = \frac{n!}{\prod(\text{hook lengths of } b)}$*

*Proof.* ■

## Lecture 9, 4/2/25

Let  $n \in \mathbb{Z}_{>0}$ . Our goal is to classify irreps of  $S_n$ . Recall:

**Theorem 0.33.** *For each partition  $\lambda$  of  $n$ , there exists a unique isomorphism class of irrep  $V_\lambda$  of  $S_n$ , constructed as follows:*



where  $\sum \lambda_i = n$ . We let  $R$  be the subgroup of  $S_n$  which preserves the rows,  $Q$  the subgroup preserving the columns. We set

$$a \stackrel{\text{def}}{=} \sum_{g \in P} e_g \in \mathbb{C}[S_n]$$

$$b \stackrel{\text{def}}{=} \sum_{g \in Q} \text{sgn}(g) e_g \in \mathbb{C}[S_n]$$

$$c = ab$$

Then  $V_\lambda \stackrel{\text{def}}{=} \mathbb{C}[S_n]c$  is an irrep of  $S_n$ .

Further, every irrep arises in this way.

*Proof.* Summary: WTS

$$1. \dim \text{Hom}_G(V_\lambda, V_\mu) = \delta_{\mu\lambda}$$

2. Any irrep is some  $V_\lambda$ .

Remark:

1. There is an explicit dimension formula, the hook-length formula

2. There is an explicit formula for the character of  $V_\lambda$  due to Frobenius.

For more, look for Etingof's "Representation theory" notes for a course given at MIT.

We will begin the proof by writing down  $c_\lambda$ .

**Lemma 1.**

$$c_\lambda = \sum_{g = \underbrace{p}_{\in P_\lambda} \underbrace{q}_{\in Q_\lambda}} \text{sgn}(q) e_{pq}$$

*Proof.*

$$\begin{aligned} a_\lambda b_\lambda &= \left( \sum_{g \in P_\lambda} e_g \right) \cdot \left( \sum_{h \in Q_\lambda} \text{sgn}(h) e_h \right) \\ &= \sum_{g \in P_\lambda, h \in Q_\lambda} \text{sgn}(h) \underbrace{e_g e_h}_{e_{gh}} \end{aligned}$$

■

Goal: Compute  $c_\lambda^2 = a_\lambda b_\lambda a_\lambda b_\lambda$

**Lemma 2.** For all  $x \in \mathbb{C}[S_n]$ ,  $a_\lambda x b_\lambda = \ell_\lambda(x) c_\lambda$ , where  $\ell_\lambda : \mathbb{C}[S_n] \rightarrow \mathbb{C}$  is some linear map.

**Corollary 0.34.**  $c_\lambda^2 = \ell_\lambda(b_\lambda a_\lambda) c_\lambda$

*Proof.* Check this on each  $e_g \in \mathbb{C}[S_n]$ ,  $g \in S_n$ .



Case 1  $g \in P_\lambda Q_\lambda$

We have  $g = pq, e_g = e_p e_q$ .

$$\begin{aligned}
 a_\lambda e_g b_\lambda &= \left( \sum_{h \in P_\lambda} e_h \right) e_g \left( \sum_{u \in Q_\lambda} \text{sgn}(u) e_u \right) \\
 &= \underbrace{\left( \sum_{h \in P_\lambda} e_h e_p \right)}_{a_\lambda} \underbrace{\sum_{u \in Q_\lambda} \text{sgn}(u) e_q e_u}_{\text{sgn}(q) b_\lambda} \\
 &= \text{sgn}(q) c_\lambda b_\lambda \\
 &= \text{sgn}(q) c_\lambda
 \end{aligned}$$

Case 2  $g \notin P_\lambda Q_\lambda$

In this case,  $a_\lambda e_g b_\lambda = 0$ . To see this, it is enough to show that there exists a transposition  $t \in P_\lambda$  such that  $g^{-1}tg \in Q_\lambda$ , i.e.  $g$  sends two elements of  $\{1, \dots, n\}$  in the same row of the Young diagram for  $\lambda$ , to two elements of the same column.

It is enough to show this because

$$\begin{aligned}
 a_\lambda g b_\lambda &= a_\lambda t g b_\lambda \\
 &= a_\lambda g \overbrace{(g^{-1}tg)}^{\text{sgn}=-1} b_\lambda \\
 &= -a_\lambda g b_\lambda
 \end{aligned}$$

This implies  $a_\lambda g b_\lambda = 0$ .

Now, suppose there do not exist 2 elements in the same row of  $\lambda$  sent to the same column of  $\lambda$  by  $g$ .

Then  $g \in P_\lambda Q_\lambda$ .

To see this, let  $T$  be the standard Young Tableau for  $\lambda$ ,  $T' = gT$ ,  $P'$  the stabilizer of rows of  $T'$ ,  $Q'$  the stabilizers of columns.

- (i) By assumption, any two numbers in the first row of  $T$  lie in different columns of  $T'$ .
- (ii) Then there exists  $q'_1 \in Q'$  such that  $q'_1 T'$  has the same elements in first row (perhaps in a different order).
- (iii) Choose  $p'_1 \in P_\lambda$  such that  $p'_1 q'_1 T'$  has the first row as  $T$ .
- (iv) Likewise with the 2nd row and so on.

**Corollary 0.35.**

$$\ell_\lambda(b_\lambda a_\lambda) = \frac{n!}{\dim V_\lambda}$$

*Proof.* later

## Lecture 10, 6/2/25

Note: For the finite group stuff we are using “Linear reps of finite groups” by Serre (first 3rd is for chemists apparently which is amusing). Specifically chapters 1-3, 6, 9. Other stuff is also on the quercus.

To finish the proof of the theorem, we have to show that the  $V_\lambda$  are irreducible and mutually non-isomorphic. Then, from a bijection between conjugacy classes and partitions, we will be done.

Last time we showed that  $a_\lambda x b_\lambda = \ell_\lambda(x) c_\lambda$ , and its corollary, that  $c_\lambda^2 = \ell_\lambda(b_\lambda a_\lambda) c_\lambda$

**Corollary 0.36.**

$$\ell_\lambda(b_\lambda a_\lambda) = \frac{n!}{\dim V_\lambda}$$

*Proof.* We know that  $c_\lambda = \alpha \cdot p_\lambda$ , where  $p_\lambda$  is an idempotent.

$$\begin{aligned} c_\lambda^2 &= \alpha^2 p_\lambda^2 \\ &= \alpha^2 p_\lambda \\ &= \alpha c_\lambda \end{aligned}$$

So  $\alpha = \ell_\lambda(b_\lambda a_\lambda)$  so we calculate the trace of  $c_\lambda$ : Trace of an idempotent is dim of its image, and  $c_\lambda$  has the same image as  $p_\lambda$

$$\begin{aligned} \text{tr}(c_\lambda) &= \alpha \cdot \dim \text{Im}(c_\lambda) \\ &= \ell_\lambda(b_\lambda a_\lambda) \cdot \dim \text{Im}(c_\lambda) \\ &= \ell_\lambda(b_\lambda a_\lambda) \cdot \dim V_\lambda \end{aligned}$$

Now, if this number is not zero, then we get an idempotent by dividing  $c_\lambda$  by this number. We calculate

$$\begin{aligned} \text{tr}(c_\lambda) &= \sum_{pq \in P_\lambda Q_\lambda} \text{tr}(\cdot e_{pq}) \text{sgn}(q) \\ &= \text{tr}(\cdot \text{Id}) \\ &= n! \end{aligned}$$

Goal: Compute  $\dim_{\mathbb{C}} \text{Hom}_{S_n}(V_{\lambda}, V_{\mu}) = \begin{cases} 1 & \lambda = \mu \\ 0 & \text{otherwise} \end{cases}$

We know  $\text{Hom}_{S_n}(V_{\lambda}, V_{\mu}) = \text{Hom}_{S_n}(\mathbb{C}[S_n]c_{\lambda}, \mathbb{C}[S_n]c_{\mu})$

**Proposition 10.** *Let  $A$  be a  $\mathbb{C}$ -algebra,  $e \in A$  an idempotent,  $M$  an  $A$ -module. Then  $\text{Hom}_A(Ae, M) \simeq eM$  naturally.*

*Proof.* For  $x \in eM$ , we have a morphism  $x \mapsto (a \mapsto ax)$ , and  $f \mapsto f(e)$ .  $e$  is an idempotent, so  $1 - e$  is also an idempotent, so  $1 = e + (1 - e)$ , so  $A \simeq Ae \oplus A(1 - e)$ , so  $\text{Hom}(Ae, M) \simeq \text{Hom}(A/A(1 - e), M) = \{f : A \rightarrow M \mid f(e) = f(1)\} = \{x \in M \mid x \in eM\} = eM$

Now let's prove the main theorem.

**Proposition 11.**

$$\dim_{\mathbb{C}} \text{Hom}_{S_n}(V_{\lambda}, V_{\lambda}) = 1$$

Thus,  $V_{\lambda}$  is irreducible

*Proof.*

$$\begin{aligned} \text{Hom}_{S_n}(V_{\lambda}, V_{\lambda}) &= c_{\lambda} \mathbb{C}[S_n] c_{\lambda} \\ &\subseteq a_{\lambda} \mathbb{C}[S_n] b_{\lambda} \\ &\subseteq \text{span}_{\mathbb{C}}(c_{\lambda}) \end{aligned}$$

So the dimension is at most 1. To see it is exactly 1, this space has  $c_{\lambda} \cdot 1 \cdot c_{\lambda} \neq 0$   
So  $\dim = 1$ , so  $V_{\lambda}$  is irreducible.

Now let  $\lambda, \mu$  be two partitions of  $n$ . Say  $\lambda > \mu$  if the first  $\lambda_i \neq \mu_i$  has  $\lambda_i > \mu_i$ , i.e. the lexicographical ordering. This is a total ordering, i.e. for any pair  $(\lambda, \mu)$ , exactly one of  $\lambda = \mu, \lambda > \mu, \lambda < \mu$  is true.

**Proposition 12.** *If  $\lambda > \mu$ , then  $a_{\lambda} \mathbb{C}[S_n] b_{\mu} = 0$ .*

*Proof.* In a bit

Assuming this, then, if  $\lambda \neq \mu$ , we want to show that  $\dim \text{Hom}_{S_n}(V_{\lambda}, V_{\mu}) = 0$ .

*Proof.* We have

$$\begin{aligned} \text{Hom}_{S_n}(V_{\lambda}, V_{\mu}) &= c_{\lambda} \mathbb{C}[S_n] c_{\mu} \\ &= a_{\lambda} b_{\lambda} \mathbb{C}[S_n] a_{\mu} b_{\mu} \\ &\subseteq a_{\lambda} \mathbb{C}[S_n] b_{\mu} \\ &= 0 \end{aligned}$$

if  $\lambda > \mu$ . But  $\dim \operatorname{Hom}_{S_n}(V_\lambda, V_\mu) = \dim \operatorname{Hom}_{S_n}(V_\mu, V_\lambda)$ , so one, hence both, are 0. ■

Now we prove the proposition

*Proof.* We will verify it on  $e_g \in \mathbb{C}[S_n]$ .

**Claim.** *There exist two numbers on the same row of the standard Young tableaux for  $\lambda$ , same column for  $g \cdot$  (standard Young tableaux of  $\mu$ )*

*Proof.* Homework ■

**Example 0.10.** If  $g = \operatorname{Id}$ ,  $\lambda_1 > \mu_1$ ,

1	2	3	4		1	2	3
				,	4	5	

Let  $t$  be the transposition for these two numbers. Then

$$\begin{aligned}
 a_\lambda g b_\lambda &= c_\lambda t g b_\mu \\
 &= a_\lambda g g^{-1} t g b_\lambda \\
 &= -a_\lambda g b_\mu
 \end{aligned}$$

## The rep theory of $\operatorname{GL}_2(\mathbb{F}_p)$

Goal: Understand the irreps of  $\operatorname{GL}_n(\mathbb{F}_q)$

What is the size of this group?

$$|\operatorname{GL}_2(\mathbb{F}_q)| = (q^2 - 1)(q^2 - q) = q(q^2 - 1)(q - 1)$$

*Proof.*

$$\operatorname{GL}_2(\mathbb{F}_q) = \{(v, w) \mid v, w \in (\mathbb{F}_q)^2 \text{ linearly independent} \}$$

So we can pick any  $v$  a nonzero vector, and any  $w$  not in the span of  $v$ . The number of such possible choices is  $(q^2 - 1)(q^2 - q)$  ■

Conjugacy classes:

What are the conjugacy classes of  $\operatorname{GL}_2(\mathbb{F}_q)$ ?

What are the reps of  $\operatorname{GL}_2(\mathbb{F}_q)$  over  $\mathbb{C}$ ? Besides the trivial one, we also have  $P^1(\mathbb{F}_1) = \{1 - \dim \text{subspaces of } \mathbb{F}_q^2\}$ .

This gives a permutation representation  $\mathbb{C}^{P^1(\mathbb{F}_q)}$ .

We have  $\operatorname{std} = \mathbb{C}^{P^1(\mathbb{F}_q)} / \mathbb{C}$  has dimension  $q$ . Let's compute the character of this representation. Let's call the first set of conjugacy classes above  $z_x$ , the second  $d_{x,y}$ ,  $u_x$ ,  $t_{x,y}$

	$z_x$	$d_{x,y}$	$u_x$	$t_{x,y}$
triv	1	1	1	1
std	$q$	1	0	-1

We have

$$\begin{aligned}\langle std, std \rangle &= \frac{1}{q(q-1)^2(q+1)} \left( (q-1)q^2 + \frac{q(q-1)(q-2)(q+1)}{2} + 0 + \frac{q^2(q-1)^2}{2} \right) \\ &= 1\end{aligned}$$

What other representation are there?

Choose  $\chi : \mathbb{F}_q^\times \rightarrow \mathbb{C}$ , and then  $(\chi \circ \det)^n$ , for  $n = 1, \dots, q-2$ .

To construct more reps, we will examine some induces reps.

**Definition 0.12.** Let  $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subseteq \mathrm{GL}_2(\mathbb{F}_q)$  ( $B$  is for Borel)

$|B| = q(q-1)^2$ . Let  $U$  be all the matrices of the form  $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ .

What is  $B/U$ ? It is  $\mathbb{F}_q^\times \times \mathbb{F}_q^\times$ . We will take reps of this and view them as reps of  $B$  via the quotient map and induced reps.

For each  $\psi : \mathbb{F}_q^\times \times \mathbb{F}_q^\times \rightarrow \mathbb{C}^\times$ , we can consider the induction  $\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{F}_1)}(\psi|_B)$ . These are indexed by  $\psi(\epsilon, 1)$  and  $\psi(1, \epsilon)$ .

Then  $\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{F}_q)}(\psi_{a,b}|_B)$  has dimension  $q+1$  and has character  $(q+1)\psi(x)^2$

For  $d_{x,y}$  we have  $\psi(x, 1) + \psi(1, x) + \psi(y, 1) \cdot \psi(1, y)$

I have kind of lost the plot at this point I'm sorry.

**Proposition 13.** Let  $\chi = \sum n_i \rho_i \in R(G)$  be a virtual character of a finite group  $G$ . Then  $\chi$  is the character of an honest irrep iff  $\langle \chi, \chi \rangle = 1$ , and  $\chi(1) > 0$ .

*Proof.* If we write  $\chi = \sum n'_i \rho'_i$ , where  $\rho'_i$  are irreps, then  $\langle \chi, \chi \rangle = \sum_i (n'_i)^2 = 1$  by assumption. So at most one of the  $n'_i$  are nonzero, and it must be  $\pm 1$ . So  $\chi = \pm \rho$  for some irrep  $\rho$ . If  $\chi(1) > 0$ , then  $\chi(1) = \pm \dim \rho > 0$

## Lecture 11, 11/2/25

For simplicity, we will assume  $q$  is odd (even is similar, but annoying to do uniformly)

Recall that  $|\mathrm{GL}_2(\mathbb{F}_q)| = (q^2 - 1)(q^2 - q)$

Conjugacy class	number of such conjugacy classes	size of each
$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, x \in \mathbb{F}_q^\times$	$q - 1$	1
$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, x \neq y \in \mathbb{F}_q^\times$	$\frac{(q-1)(q-2)}{2}$	$q(q+1)$
$\begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}, x \in \mathbb{F}_q^\times$	$q - 1$	$q^2 - 1$
$\begin{pmatrix} x & \epsilon y \\ y & x \end{pmatrix}, \epsilon \text{ a generator of } \mathbb{F}_q^\times$	$\frac{q(q-1)}{2}$	$q^2 - q$

For the last one,  $\text{char} \neq 2$

	$z_x$	$d_{x,y}$	$u_x$	$t_{x,y}$
triv	1	1	1	1
std	$q$	1	0	-1

We denote by  $B$  all matrices of the form  $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ , and by  $T$  the span of all matrices of the form  $\begin{pmatrix} x & \epsilon y \\ y & x \end{pmatrix}$

Let  $U$  denote the matrices of the form  $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ . Then  $B/U \cong (\mathbb{F}_q^\times)^2$  (picks out two diagonal entries of  $B$ ).

Given  $\alpha, \beta : \mathbb{F}_q^\times \rightarrow \mathbb{C}^\times$  reps of  $\mathbb{F}_q^\times$ , then we have a rep  $\psi_{\alpha,\beta} : B \rightarrow B/U \simeq (\mathbb{F}_q^\times)^2 \rightarrow \mathbb{C}^\times$ , where this second morphism is by  $\alpha \boxtimes \beta$ .

	$z_x$	$d_{x,y}$	$u_x$	$t_{x,y}$
triv	1	1	1	1
std	$q$	1	0	-1
$\text{Ind}_B^{\text{GL}_2(\mathbb{F}_q)} \psi_{\alpha,\beta}$	$(q+1)\alpha(x)\beta(x)$	$\alpha(x)\beta(y) + \alpha(y)\beta(x)$	$\alpha(x)\beta(x)$	0

This third line is irreducible if  $\alpha \neq \beta$ . Note  $\text{Ind}_B^{\text{GL}_2} \psi_{\alpha,\beta} = \text{Ind}_B^{\text{GL}_2} \psi_{\beta,\alpha} \text{Ind}_B^{\text{GL}_2} \psi_{\alpha,\alpha} = \alpha \circ \det \oplus (\alpha \circ \det) \otimes (\alpha)(?)$

Recall we proved last time that if  $G$  is a finite group,  $R(G)$  the representation ring. As a group,  $R(G)$  is generated by the irreps of  $G$ . Multiplication is given by expressing the tensor of two irreps as a sum of two irreps. Given  $V \in R(G)$ ,  $V$  is the class of an irrep if and only if  $\langle \chi_V, \chi_V \rangle = 1$ , and  $\chi_V(\text{Id}_G) > 0$ .

Now, we can think of  $T$  as being isomorphic to  $\{x + \sqrt{\epsilon}y \in \mathbb{F}_{q^2}^\times\}$ . Given a  $\varphi : \mathbb{F}_{q^2}^\times \rightarrow \mathbb{C}^\times$ , we have

	$z_x$	$d_{x,y}$	$u_x$	$t_{x,y}$
triv	1	1	1	1
std	$q$	1	0	-1
$\text{Ind}_B^{\text{GL}_2(\mathbb{F}_q)} \psi_{\alpha,\beta}$	$(q+1)\alpha(x)\beta(x)$	$\alpha(x)\beta(y) + \alpha(y)\beta(x)$	$\alpha(x)\beta(x)$	0
$\text{Ind}_T^{\text{GL}_2(\mathbb{F}_q)} \varphi$	$q(q-1)\varphi(x)$	0	0	$\varphi(\zeta) + \varphi(\zeta)^q(?)$

where  $\zeta = x + \sqrt{2}y(?)$ . There's another very convoluted row I didn't quite catch you get by putting the others together in various ways, but that's all of them.

For more: wikipedia Deligne-Lustztig(sp?) theory.

The story of  $\text{SL}_2(\mathbb{F}_q)$  is similar: restrict reps to  $\text{SL}_2(\mathbb{F}_q)$ , some of the  $\alpha, \beta$  break up (into at most two pieces), there's some redundancies, and every rep of  $\text{SL}_2(\mathbb{F}_q)$  is a restriction.

Fun exercise: Show  $\text{PSL}_2(\mathbb{F}_q)$  is simple for  $q > 3$  odd.

## Lie Algebras

Let  $k$  be an arbitrary field,  $\mathfrak{g}$  a finite dimensional  $k$ -vector space.

**Definition 0.13.** A Lie bracket  $[-, -] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$

- is Bilinear, meaning  $[-, *] : \mathfrak{g} \rightarrow \mathfrak{g}$  is linear, as is  $[*, -]$  for any  $* \in \mathfrak{g}$
- is Alternating, meaning  $[x, x] = 0$  for all  $x \in \mathfrak{g}$ . This implies  $[x, y] = -[y, x]$ . In characteristic 2, this is stronger!
- satisfies the Jacobi Identity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

A Lie Algebra is a finite dimensional  $k$ -vector space  $\mathfrak{g}$  with a Lie bracket.

**Corollary 0.37.**  $[x, y] = -[y, x]$

*Proof.*  $[x + y, x + y] = [x, x] + [y, y] + [x, y] + [y, x] = 0$

■

**Example 0.11.**

- Let  $R$  be an associative  $k$ -algebra. Then  $[x, y] = xy - yx$  will be a Lie bracket.
- $\text{Mat}_{n \times n}(k)$  is a Lie algebra with

$$[A, B] = AB - BA$$

- We have  $\mathfrak{sl}_n(k) \subseteq \text{Mat}_{n \times n}(k)$  the set of all matrices with trace 0. Then  $[A, B] = AB - BA$  is again a Lie bracket.

- A Lie algebra is abelian if  $[-, -] = 0$  for all vectors.

Suppose  $R$  is a commutative  $k$ -algebra. A  $k$ -derivation  $\delta : R \rightarrow R$  is a  $k$ -linear map such that  $\delta(a) = 0$  for  $a \in k$ , and  $\delta(xy) = \delta(x)y + x\delta(y)$ .

**Example 0.12.**

We can take  $R = k[t]$ ,  $\delta = \frac{\partial}{\partial t}$ .

Fact:

$Der_k(R)$ , the set of all  $k$ -derivations on  $R$ , is a Lie algebra, with  $[\delta, \gamma] = \delta \circ \gamma - \gamma \circ \delta$ .

## Lecture 12, 13/2/25

Let's prove the above fact.

**Claim.**  $Der_k(R)$  is a Lie algebra, with bracket  $[\delta, \gamma] = \delta \circ \gamma - \gamma \circ \delta$ .

*Proof.* We have

$$\begin{aligned}\delta \circ \gamma(xy) &= \delta(\gamma(x)y + x\gamma(y)) \\ &= \delta\gamma(x)y + \gamma(x)\delta(y) + \delta(x)\gamma(y) + x\delta\gamma(y) \\ \gamma \circ \delta(xy) &= \dots \\ (\delta \circ \gamma - \gamma \circ \delta)(xy) &= \delta\gamma(x)y + x\delta\gamma(y) - \gamma\delta(x)y - x\gamma\delta(y)\end{aligned}$$

■

**Example 0.13.**

- If  $M$  is a smooth manifold,  $Der_{\mathbb{R}}(C^\infty(M)) = C^\infty$  vector fields on  $M$ .
- If  $G$  is a Lie group (i.e. a  $C^\infty$  manifold equipped with a  $C^\infty$  group structure). Define  $Lie(G)$  to be the space of left  $G$ -invariant vector fields. This has another description as the tangent space of the identity,  $T_e G$ .

**Definition 0.14.** Let  $\mathfrak{g}$  be a Lie algebra over  $k$ . A representation of  $\mathfrak{g}$  is a  $k$ -linear map  $\rho : \mathfrak{g} \rightarrow \text{Mat}_{n \times n}(k)$  which respects the Lie bracket, i.e.  $\rho([x, y]) = [\rho(x), \rho(y)]$ , where on the right hand side it's just the commutator of matrices.

**Definition 0.15.** Given Lie algebras  $\mathfrak{g}, \mathfrak{h}$  over a field  $k$ , a homomorphism  $f : \mathfrak{g} \rightarrow \mathfrak{h}$  is a  $k$ -linear map such that  $f([x, y]) = [f(x), f(y)]$  for all  $x, y \in \mathfrak{g}$ .

Equivalently, a representation is a Lie algebra morphism  $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}_n$ . Equivalently,  $\mathfrak{gl}(V) = \text{End}(V)$ , so a rep is a morphism  $\mathfrak{g} \rightarrow \mathfrak{gl}(V)$

**Definition 0.16.** The Universal Enveloping Algebra  $U\mathfrak{g}$  is defined by

$$U\mathfrak{g} \stackrel{\text{def}}{=} \frac{\bigoplus_{n \geq 0} \mathfrak{g}^{\otimes n}}{\langle x \otimes y - y \otimes x - [x, y] \forall x, y \in \mathfrak{g} \rangle}$$



Main property:

$$\mathrm{Hom}_{\mathrm{Assoc } k\text{-alg}}(U\mathfrak{g}, R) = \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, R)$$

**Definition 0.17.** A (finite-dimensional)  $\mathfrak{g}$ -representation is the same as a left  $U\mathfrak{g}$ -module which is finite dimensional as a  $k$ -vector space.

**Example 0.14.** If  $\mathfrak{g}$  is the Lie algebra of a Lie group, then  $U\mathfrak{g}$  is the left-invariant differential operators.

**Example 0.15.** of representations

- $\mathrm{Id} : \mathfrak{g} = \mathfrak{gl}_n \rightarrow \mathfrak{gl}_n$ . Inside of  $\mathfrak{gl}_n$  is  $\mathfrak{sl}_n$ , and this gives a rep of it.
- $k, [-, -] = 0$  is repped by sending  $*$  to  $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$ . The span of  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  is a subrepresentation.
- $\rho_A : k \rightarrow \mathfrak{gl}_n, * \mapsto *A$

**Definition 0.18.** Let  $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  be a representation.  $W \subseteq V$  is a subrepresentation if for all  $x \in \mathfrak{g}, w \in W, \rho(x)w \in W$ . This is the same as saying  $W$  is a  $U\mathfrak{g}$ -submodule of  $V$ .

**Example 0.16.** Let  $\mathfrak{b}_n$  be the Lie algebra of upper triangular matrices. It contains  $\mathfrak{N}_n$ , the strictly upper triangular matrices. The former is solvable, the latter nilpotent.

Given some representations, how can we make new ones?

## Operations on representations

Let  $\mathfrak{g}$  be a Lie algebra over  $k$ ,  $V, W$  representations of  $\mathfrak{g}$ , i.e. are equipped with  $\rho_V : \mathfrak{g} \rightarrow \mathfrak{gl}(V), \rho_W : \mathfrak{g} \rightarrow \mathfrak{gl}(W)$ .

- $\rho_V \oplus \rho_W : \mathfrak{g} \rightarrow \mathfrak{gl}(V) \oplus \mathfrak{gl}(W) \rightarrow \mathfrak{gl}(V \oplus W)$  via the block matrix  $\begin{pmatrix} \rho_V(x) & 0 \\ 0 & \rho_W(x) \end{pmatrix}$
- $V^*$  is a rep via  $\rho_{V^*} : \mathfrak{g} \rightarrow \mathfrak{gl}(V^*)$  sending  $x \mapsto (f \mapsto f(-\rho_V(x)))$
- $\rho_{V \otimes W} : \mathfrak{g} \rightarrow \mathfrak{gl}(V \otimes W), x \mapsto (v \otimes w \mapsto \rho_V(x)v \otimes w - v \otimes \rho_W(x)w)$
- $\underline{\mathrm{Hom}}_k(V, W)$  via  $x \cdot f = x \cdot f(-) - f(x \cdot -)$
- $V^{\mathfrak{g}} = \{v \in V \mid xv = 0 \forall x \in \mathfrak{g}\}$

Observation:  $\underline{\mathrm{Hom}}_k(V, W)^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(V, W)$

**Definition 0.19.** A homomorphism of representations is a  $k$ -linear  $f : V \rightarrow W$  such that  $f(x \cdot v) = x \cdot f(v)$  for all  $x \in \mathfrak{g}, v \in V$ .

## Representations of $\mathfrak{sl}_2(k)$

Let  $k$  be a field of characteristic 0.  $\mathfrak{sl}_2 \subseteq \text{Mat}_{2 \times 2}(k)$  is the set of  $2 \times 2$  matrices of trace 0. This is  $\text{span}(\underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_e, \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_f, \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_h)$

Now,

$$[e, f] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -h$$

Similarly,

$$[h, f] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix} = -2f$$

$$[h, e] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 2e$$

A representation of  $\mathfrak{sl}_2$  on  $V$  is  $E, F, H \in \mathfrak{gl}(V)$  such that

$$\begin{aligned} [E, F] &= H \\ [H, F] &= -2F \\ [H, E] &= 2E \end{aligned}$$

Given these, we can consider  $\mathfrak{sl}_2 \rightarrow \mathfrak{gl}(V)$  by  $e \mapsto E, f \mapsto F, h \mapsto H$ .

**Lemma 3.**  $E, F$  are nilpotent, i.e. some power of them are 0.

*Proof.* WLOG,  $k = \bar{k}$ . Otherwise, we could just lift to the algebraic closure, and check it is nilpotent there, and then it will be nilpotent over  $k$ .

Let  $v$  be an eigenvector of  $H$  with eigenvalue  $\lambda$ .  $HFv = -2Fv + FHv = -2Fv + \lambda Fv = (\lambda - 2)Fv$ . Similarly,  $HEv = (\lambda + 2)Ev$ . So  $F^n v = E^n v = 0$  for  $n$  sufficiently large, because otherwise  $H$  would have infinitely many distinct eigenvalues.

Now let  $W \subseteq V$  be the span of eigenvectors of  $H$ . This is a subrepresentation of  $\mathfrak{sl}_2$  (because  $F$  and  $E$  send eigenvectors to eigenvectors as shown above). Now consider  $V/W$ . This is again an  $\mathfrak{sl}_2$ -representation, so by induction on dimension  $E, F$  act nilpotently on it. ■

Here is another proof:

*Proof.* Let's compute

$$\begin{aligned} \text{tr}(E^n) &= \text{tr}\left(\frac{1}{2}E^{n-1}[H, E]\right) \\ &= \frac{1}{2}\text{tr}(E^{n-1}HE - E^nH) \\ &= 0 \end{aligned}$$

Let  $\lambda_1, \dots, \lambda_n$  be the generalized eigenvalues of  $E$  with multiplicity. Then  $0 = \text{tr}(E^n) = \sum \lambda_i^n$ .

Why? In characteristic 0, these generate all symmetric polynomials in  $\lambda_i$ .

The characteristic polynomial of  $E$  is  $x^n - \sum(\lambda_i)x^{n-1} + \sum_{i < j} \lambda_i \lambda_j x^{n-2} + \dots + = x^n$ . So the characteristic polynomial is  $x^n$ , so it's nilpotent by Cayley-Hamilton. ■

For now, we will use  $k = \mathbb{C}$  so we can say things like “maximal real part.”

**Lemma 4.** *Let  $\lambda$  be the eigenvalue of  $H$  with maximal real part. Let  $v$  be an  $H$ -eigenvector with eigenvalue  $\lambda$ . Let  $n$  be minimal such that  $F^n v = 0$  (there is such an  $n$  because  $F$  is nilpotent). Then  $Ev = 0$ ,  $\text{span}(v, Fv, \dots, F^{n-1}v)$  is a subrepresentation of  $V$ ,  $\lambda = n - 1$ . In particular, the eigenvalues of  $H$  are all integers.*

*Proof.*

$$\begin{aligned}
 EFv &= EFv - FEv \\
 &= Hv \\
 &= \lambda v \\
 EF^2v &= [E, F]Fv + FEFv \\
 &= HFv + \lambda Fv \\
 &= (\lambda - 2)Fv + \lambda Fv \\
 &= (2\lambda - 2)Fv \\
 &\dots \\
 EF^n v &= (\lambda^2 + (\lambda + 1)n)F^{n-1}v
 \end{aligned}$$

So this is a subrep.

$$\begin{aligned}
 0 &= EF^N v \\
 &= (-N^2 + (\lambda + 1)N)F^{N-1}v
 \end{aligned}$$

So  $(\lambda + 1)N - N^2 = 0$ . So either  $N = 0$  (which isn't the case), or  $\lambda = N - 1$ .

Explicit:

Set  $v = v_0, v_i = F^i v_0$ , so  $Fv_i = v_{i+1}$ .  $Hv_n = (N - 2 - 2n)v_n$ ,  $Ev_n = (-n^2 + Nn)v_{n-1}$ . Let's call this representation  $V_N$ .

**Corollary 0.38.** *For any non-zero representation  $W$  of  $\mathfrak{sl}_2$ , there exists an  $N > 0$  such that  $V_N$  is a subrepresentation of  $W$ .*

*Proof.* ■

$V_1 \leq \mathfrak{sl}_2 \rightarrow \mathfrak{gl}_2$ .

$V_n = \text{Sym}^N(V_1)$ , where  $x \in \mathfrak{sl}_2$  acts on  $v_1 \otimes v_2 \otimes \dots \otimes v_N$  by

$$x \cdot (v_1 \otimes \dots \otimes v_N) = \sum_{i=1}^N v_1 \otimes \dots \otimes xv_i \otimes \dots \otimes v_N$$

$V_N$  is the space of homogeneous polynomials in  $X, Y$  of degree  $N$ .

$E$  acts by  $X \frac{\partial}{\partial y}$ ,  $F$  acts by  $Y \frac{\partial}{\partial x}$ ,  $H$  by  $X \frac{\partial}{\partial x} - Y \frac{\partial}{\partial y}$ .

**Claim.**  $V_N$  are all irreps of  $\mathfrak{sl}_2$  (where irrep means the same as in the world of groups, i.e. no nontrivial subrepresentations).

*Proof.* Any rep of  $\mathfrak{sl}_2$  contains one of these, so it suffices to show that they are irreducible.

Let  $v = \sum a_i v_i \in V_N$  be any element. It suffices to show  $\{F^a v, E^b v, a, b \in \mathbb{N}\}$  span. Acting by some power of  $E$ ,  $E^a v \in \text{span}(v_0)$ .

But  $V_n$  is the span of  $F^a v_0$ , so we have shown its irreducible. ■

**Definition 0.20.**  $V$  a representation of  $\mathfrak{g}$  is semi-simple if it's the direct sum of irreducibles.

**Theorem 0.39.** All finite dimensional representations of  $\mathfrak{sl}_2$  in characteristic 0 are semisimple.

*Proof.* ■

## Lecture 13, 25/2/25

Combinatorics question:

Let  $g_{n,k}$  be the number of unlabeled graphs with  $n$  vertices and  $k$  edges.

Here are the  $g_{n,k}$  for various  $n$

1. 1
2. 1, 1
3. 1, 1, 1, 1
4. 1, 1, 2, 3, 2, 1, 1
5. 1, 1, 2, 4, 6, 6, 4, 2, 1
6. 1, 1, 2, 5, 9, 15, 21, 24, 24, 21, 15, 9, 5, 2, 1, 1

Observation:  $g_{n,k} = g_{n, \binom{n}{2} - k}$

**Proposition 14.** The sequence  $g_{n,k}$ , for a fixed  $n$ , is unimodal in  $k$ , meaning these sequences are increasing until they peak in the middle, and then go down.

*Proof.* We are going to use the representation theory of  $\mathfrak{sl}_2$  to prove this unimodality. Recall:  $\mathfrak{sl}_2 = \langle e, f, h \mid [e, f] = h, [h, f] = -2f, [h, e] = 2e \rangle$

We have the standard 2-dimensional representation of  $\mathfrak{sl}_2$ ,  $V$ , given by the inclusion  $\mathfrak{sl}_2 \hookrightarrow \mathfrak{gl}_2$

Last time, we showed all irreps of  $\mathfrak{sl}_2$  are isomorphic to  $Sym^n V$  for some nonnegative integer  $n$ , where  $x \in \mathfrak{sl}_2$  acts on  $v_1 \otimes \cdots \otimes v_n$  as

$$(x \cdot v_1) \otimes v_2 \otimes \cdots \otimes v_n + v_1 \otimes (x \cdot v_2) \otimes \cdots \otimes v_n + \cdots + v_1 \otimes \cdots \otimes (x \cdot v_n)$$

We have  $h \in \mathfrak{sl}_2$  given by  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ . Fix  $v_1, v_{-1} \in V$  such that  $h v_1 = v_1, h v_{-1} = -v_{-1}$ .

Then  $h$  acts by

$$h \cdot (v_1^a v_{-1}^b) = (a + b)v_1 + v_{-1}^b$$

So  $h$  acts on  $Sym^n V$  with eigenvalues  $n, n - 2, \dots, -n + 2, -n$

Lemma:

Let  $V$  be any finite dimensional  $\mathfrak{sl}_2$ -representation (not necessarily irreducible).

**Lemma 5.** *Let  $d_r$  be the dimension of the generalized eigenspace of  $h$  with generalized eigenvalue  $r$ .*

*Then  $\{d_r\}_{r \text{ even}}, \{d_r\}_{r \text{ odd}}$  are both unimodal and in particular  $d_r = d_{-r}$*

*Proof.* By induction on  $\dim V$ . The base case will be  $\dim V = 0$ , where it easily holds.

Now, assume the claim holds for  $\dim V < n$ . Suppose  $\dim V = n$ . Let  $W \subseteq V$  be irreducible. By induction, the claim is true for  $V/W$ . By the classification of irreps, it's true for  $W$ . But these properties are preserved by addition. ■

The goal now is to write down some generalized eigenspaces of  $h$  with exactly this sequence as it's  $d_r$ 's.

Let  $V_n$  be the vector space consisting of formal linear combinations of labeled graphs with  $n$  vertices.

Observation:  $S_n \curvearrowright V_n$  by permuting the vertices.

Let  $V_{n,k}$  be the span of the graphs with  $k$  edges. Then

$$V_n = \bigoplus_{k=0}^{\binom{n}{2}} V_{n,k}$$

Observe that  $g_{n,k} = \dim V_{n,k}$

Given  $i < j \in \{1, \dots, n\}$ , for a labeled graph  $g$ , set  $a_{ij}(g) = \begin{cases} g \cup (i, j) & (i, j) \text{ not an edge} \\ 0 & \text{otherwise} \end{cases}$ .

Similarly, set  $b_{ij}(g) = \begin{cases} g \setminus (i, j) & (i, j) \text{ an edge} \\ 0 & \text{otherwise} \end{cases}$ .

Let  $E = \sum_{i < j} a_{ij}$ ,  $F = \sum_{i < j} b_{ij}$ ,  $H = [E, F]$ .

$[a_{i,j}, a_{k,\ell}] = 0$ ,  $[b_{i,j}, b_{k,\ell}] = 0$ , and  $[a_{i,j}, b_{k,\ell}] = 0$  if  $(i, j) \neq (k, \ell)$ ,  $g$  if  $(i, j) = (k, \ell)$  and  $g$  an edge,  $-g$  otherwise.

Now

$$\begin{aligned} Hg &= [E, F]g \\ &= \sum_{(i,j) \in g} g - \sum_{(i,j) \notin g} g \\ &= ((2(\text{number of edges of } g) - \binom{n}{2}))g \end{aligned}$$

Doing this with  $[H, F]g$ , it turns out this is  $-2Fg$ .

Next  $\mathfrak{sl}_2 \curvearrowright V_n^{S_n}$ , and  $h$  has eigenspaces  $V_{n,k}^{S_n}$  with eigenvalues  $2k - \binom{n}{2}$  and hence  $g_{n,k} = \dim V_{n,k}^{S_n}$  is unimodal.

## Lecture 14, 4/3/25

Let  $k = \bar{k}$  be an algebraically closed field (of any characteristic for now), and  $\mathfrak{g}$  a Lie algebra over  $k$ .

**Example 0.17.** Recall: Given Lie algebra reps  $V, W$ ,

- $V \oplus W$  given by  $x \cdot (v, w) = (x \cdot v, x \cdot w)$
- $V \otimes W$  given by  $x \cdot (v \otimes w) = (x \cdot v) \otimes w + v \otimes (x \cdot w)$
- $V^*$ , given by  $(x \cdot f)(v) = f(-x \cdot v)$
- $V^{\mathfrak{g}} = \{v \in V \mid xv = 0 \forall x \in \mathfrak{g}\}$

Exercise:  $\underline{\text{Hom}}_k(V, W)^{\mathfrak{g}} = \text{Hom}_{\mathfrak{g}}(V, W)$

We have  $\mathfrak{sl}(V) \hookrightarrow \mathfrak{gl}(V)$  is a representation, so  $\mathfrak{sl}(V) \curvearrowright V \otimes V = \text{Sym}^2 V \oplus \wedge^2(V)$  for  $\text{char} \neq 2$

Pick  $q \in \text{Sym}^2 V$ ,  $\omega \in \wedge^2 CV$ , both non-degenerate.

Then  $\mathfrak{so}(q) = \{x \in \mathfrak{sl}(V) \mid xq = 0\}$ ,  $\mathfrak{sp}(\omega) = \{x \in \mathfrak{gl}(V) \mid x \cdot \omega = 0\}$

Goal: Study Lie algebras with nice representation theory.

Basic examples will be  $\mathfrak{sl}$ ,  $\mathfrak{so}$ ,  $\mathfrak{sp}$ , and a finite list of exceptional algebras.

Structure:

There are 2 basic classes of Lie algebras:

- Solvable

- Semisimple

Every finite dimensional Lie algebra is built out of these two types.

**Definition 0.21.** A subspace  $I \subseteq \mathfrak{g}$  is an ideal if for all  $x \in I, v \in \mathfrak{g}, [x, v] \in I$ . Then  $\mathfrak{g}/I$  naturally inherits the structure of a Lie algebra.

**Lemma 6.** Let  $I_1, I_2 \subseteq \mathfrak{g}$  be ideals. Then

- $I_1 + I_2$  is an ideal
- $I_1 \cap I_2$  is an ideal
- $[I_1, I_2] = \langle [x, y] \mid x \in I_1, y \in I_2 \rangle$  is an ideal

*Proof.* Exercise ■

**Example 0.18.**

- $[\mathfrak{g}, \mathfrak{g}]$  is an ideal (the commutator).
- $[\mathfrak{gl}(V), \mathfrak{gl}(V)] = \mathfrak{sl}(V)$ .
- $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$  is abelian.

**Definition 0.22.**

- $D^0 \mathfrak{g} = [\mathfrak{g}, \mathfrak{g}], D^i \mathfrak{g} = [D^{i-1} \mathfrak{g}, D^{i-1} \mathfrak{g}]$ . This is the derived series, or central series
- $C^0 \mathfrak{g} = \mathfrak{g}, C^i \mathfrak{g} = [\mathfrak{g}, C^{i-1} \mathfrak{g}]$  This is the lower central series

**Definition 0.23.**  $\mathfrak{g}$  is solvable if  $D^i \mathfrak{g} = 0$  for some  $i$ , and nilpotent if  $C^i \mathfrak{g} = 0$  for some  $i$ .

Solvability means if we take  $[[[x_1, x_2], [x_3, x_4]], [\dots]] = 0$  if there are enough of them. Nilpotency essentially means that  $[x_1, [x_2, [x_3, \dots, ]]] = 0$  once you have enough of them.

Unlike in the case of groups, these two notions are extremely closely related.

**Example 0.19.**  $\mathfrak{b}_n = \left\{ \begin{pmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & * \end{pmatrix} \right\}$ , the upper triangular matrices.

$\mathfrak{n}_n = \left\{ \begin{pmatrix} 0 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \right\}$ , the nilpotent matrices (strictly upper triangular).

Exercise:  $[\mathfrak{b}_n, \mathfrak{n}_n] = \mathfrak{n}_n$

**Proposition 15.**  $\mathfrak{n}_n$  is nilpotent.

*Proof.* We claim  $C^i \mathfrak{n}_n$  is the set of upper triangular matrices with the first  $i$  super-diagonals are zero. This is because the product of two things of this form has the first  $i + 1$  superdiagonals zero.

**Definition 0.24.** The radical of  $\mathfrak{g}$  is the maximal solvable ideal in  $\mathfrak{g}$ .

This is a valid thing to write because there is a unique such ideal, but we must show this:

**Lemma 7.** Let  $I_1, I_2 \subseteq \mathfrak{g}$  be solvable ideals. Then  $I_1 + I_2$  is solvable.

*Proof.*  $(I_1 + I_2)/I_1 = I_2/(I_1 \cap I_2)$ . These ideals and  $I_1$  are solvable.

**Claim.** If  $\mathfrak{g}$  is a Lie algebra, and  $I \subseteq \mathfrak{g}$ ,  $\mathfrak{g}/I$  are both solvable, then  $\mathfrak{g}$  is solvable.

*Proof.*  $D^i(\mathfrak{g}/I) = 0$  for some  $i$ . This implies  $D^i \mathfrak{g} \subseteq I$ . But iterated commutators are zero in  $I$  by assumption. ■

**Corollary 0.40.** There exists a unique maximal solvable ideal in  $\mathfrak{g}$ , denoted  $\text{rad}(\mathfrak{g})$  ■

*Proof.* We can take the span of all solvable ideals. ■

**Definition 0.25.**  $\mathfrak{g}$  is semisimple if  $\text{rad}(\mathfrak{g}) = 0$

**Proposition 16.** For any  $\mathfrak{g}$ ,  $\mathfrak{g}/\text{rad}(\mathfrak{g})$  is always semisimple.

*Proof.*  $I \subseteq \mathfrak{g}/\text{rad}(\mathfrak{g})$  is a solvable ideal if its preimage under the quotient map is solvable, and this would be an extension of  $\text{rad}(\mathfrak{g})$ .

**Example 0.20.**

- $\mathfrak{sl}_n$  is simple, meaning there are no non-zero proper ideals
- $\mathfrak{so}(q)$  is simple as well
- $\mathfrak{sp}(\omega)$  is as well

Non-example:

$\mathfrak{gl}(V)$  is NOT semisimple, since scalar matrices make up a solvable ideal.

**Definition 0.26.** (For culture)

$\mathfrak{g}$  is reductive if  $\text{rad}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, v] = 0 \forall v \in \mathfrak{g}\}$



## Lecture 15, 6/3/25

Recall we were studying the class of nilpotent Lie algebras, which are a subclass of the class of solvable lie algebras.

Recall a lie algebra is semisimple if  $\text{rad}(\mathfrak{g}) = 0$ , that is there are no nonzero solvable ideals.

Standard examples of solvable Lie algebras include  $\mathfrak{b}_n$ , the  $n \times n$  upper triangular matrices, and standard example of a nilpotent Lie algebra is  $\mathfrak{n}_n$ , the strictly upper triangular matrices.

Standard example of semisimple Lie algebra is  $\mathfrak{sl}_n$ .

### Theorem 0.41. (Sophus Lie)

Suppose  $\mathfrak{g}$  is solvable. Then any representation of  $\mathfrak{g}$  is upper triangularizable, i.e. given a representation  $\rho : \mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$ , there exists a basis of  $V$  in which  $\rho(x)$  is upper triangular for all  $x \in \mathfrak{g}$ , i.e.  $\rho$  can be conjugated (by  $\text{GL}(V)$ ) into  $\mathfrak{b}$ , i.e. there exists a full flag  $V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n$ , with  $\dim V_i = i$ , where each  $V_i$  is a representation.

*Proof.*

**Lemma 8.** Suppose  $\mathfrak{g}$  is solvable,  $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  a representation. Then there exists some  $v \in V$  so that  $v$  is an eigenvector for  $\rho(x)$  all  $x \in \mathfrak{g}$ .

*Proof.* We will prove this by induction on dimension. It is clearly true for  $\dim \mathfrak{g} = 0$ , so this is our base case.

Now for the induction step:  $[\mathfrak{g}, \mathfrak{g}] \subsetneq \mathfrak{g}$  because  $\mathfrak{g}$  is solvable.

Pick  $\mathfrak{g}' \subseteq \mathfrak{g}$  so that

1.  $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}'$
2.  $\text{codim}(\mathfrak{g}') = 1$

Now observe

- (a)  $\mathfrak{g}'$  is an ideal because for all  $x \in \mathfrak{g}', y \in \mathfrak{g}$ ,  $[x, y] \in [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}'$ .
- (b)  $\mathfrak{g}'$  is solvable.

Pick  $x \in \mathfrak{g} \setminus \mathfrak{g}'$ .

By the induction hypothesis, there exists some  $w \in V$  such that  $y \cdot w = \lambda(y)w$  for all  $y \in \mathfrak{g}'$ .

Let  $W = \text{span}(w, x \cdot w, x^2 \cdot w, \dots)$

**Claim.** For all  $y \in \mathfrak{g}'$ ,  $yx^k w = \lambda(y)x^k w + \sum_{\ell < k} a_{\ell k}(y)x^\ell w$ ,  $k < \dim W - 1$  i.e.  $y \in \mathfrak{g}'$  acts on  $W$  via an upper triangular matrix with  $\lambda(y)$  on the diagonal.

*Proof.* We prove this by induction on  $k$ . For  $k = 0$  it is certainly true, so this is our base case.

For the induction step:

$$\begin{aligned} yx^k w &= xyx^{k-1}w + [y, x]x^{k-1}w \\ &= \lambda(y)x^k w + \text{lower order terms} + \lambda([y, x])x^{k-1}w + \text{lower order terms} \end{aligned}$$

So  $\text{tr}(y|W) = \dim W \cdot \lambda(y)$  for all  $y \in \mathfrak{g}'$

This implies that  $\lambda([y_1, y_2]) = 0$  (here we use the assumption  $k = \bar{k}$ ) for all  $y_1, y_2 \in \mathfrak{g}$ . The lower order terms are all ultimately built out of commutators, so  $\mathfrak{g}'$  acts on  $W$  via  $\lambda - \text{Id}$ .

Now choose  $v \in W$  an  $x$ -eigenvector.

This proves the claim. ■

Let's prove the theorem via a full flag.

Given a rep  $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$ , we can now pick an eigenvector  $v \in V$ . Now consider the action of  $\mathfrak{g}$  on  $V/V_1$ , where  $V_1 = \text{span}(v)$ . But  $\dim(V/V_1) < \dim(V)$ , so by induction we win. ■

**Corollary 0.42.** *Any irrep of a solvable Lie algebra is 1-dimensional.*

*Proof.* For any  $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$ , there exists a common eigenvector, which means every rep has a 1-dimensional sub representation. ■

**Corollary 0.43.**  *$\mathfrak{g}$  is solvable if and only if  $[\mathfrak{g}, \mathfrak{g}]$  is nilpotent.*

*Proof.* We start with the easy direction: it is enough for  $[\mathfrak{g}, \mathfrak{g}]$  to be solvable. Because then  $[\mathfrak{g}, \mathfrak{g}]$  and  $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$  are solvable, so their product,  $\mathfrak{g}$ , is solvable.

Now for the other direction.

**Definition 0.27.** Let  $\mathfrak{g}$  be a Lie algebra. The adjoint representation is the map  $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ , given by  $x \mapsto [x, -]$ .

Observe  $\ker \text{ad} = \mathfrak{z}(\mathfrak{g})$ , the center.

Suppose  $\mathfrak{g}$  is solvable. By classification of representations, the adjoint representation factors through  $\mathfrak{b}$ :

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\text{ad}} & \mathfrak{gl}(\mathfrak{g}) \\ & \searrow & \nearrow \\ & \mathfrak{b} & \end{array}$$

Observe

- (i)  $[\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{n}$ , and  $\mathfrak{n}$  is nilpotent
- (ii)  $[\mathfrak{g}/\mathfrak{z}(\mathfrak{g}), \mathfrak{g}/\mathfrak{z}(\mathfrak{g})]$  is nilpotent
- (iii)  $[\mathfrak{g}, \mathfrak{g}]$  is nilpotent by definition of center.

■

Recall  $\mathfrak{g}$  is semisimple if  $\text{rad}(\mathfrak{g}) = 0$ .

**Example 0.21.** Let  $\mathfrak{g} = \mathfrak{sl}_2$ . We have  $\text{ad} : \mathfrak{sl}_2 \rightarrow \mathfrak{gl}(\mathfrak{sl}_2)$ .

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow [h, -]$$

Eigenvalues of 0, 2, -2, with eigenvectors  $h, e_v, f$ , and  $f, e, h$  (?), respectively.

Suppose  $I \subseteq \mathfrak{sl}_2$  is an ideal. Then  $\text{ad}(h)$  preserves  $I$ .

So this means that none of the possible subalgebras are ideals (can check each of them 1 by 1 because there are only finitely many).

So in particular there are no nontrivial solvable ideals.

**Corollary 0.44.**  $\mathfrak{sl}_2$  is simple (meaning no nontrivial ideals).

**Theorem 0.45.** Let  $V$  be an irrep of a Lie algebra  $\mathfrak{g}$ .

Then for all  $x \in \text{rad}(\mathfrak{g})$ ,  $x$  acts on  $V$  via scalars. For all  $x \in [\text{rad}(\mathfrak{g}), \mathfrak{g}]$ ,  $x$  acts on  $V$  by 0

*Proof.* (sketch)

Let  $v \in V$  be an eigenvector for  $\text{rad}(\mathfrak{g})$ , i.e. for all  $x \in \text{rad}(\mathfrak{g})$ ,  $x \cdot v = \lambda(x) \cdot v$

Let  $V_\lambda = \{w \in V \mid x \cdot w = \lambda(x) \cdot w \forall x \in \text{rad}(\mathfrak{g})\}$

Fix  $y \in \mathfrak{g}, x \in \text{rad}(\mathfrak{g})$ . Let  $w \in V_\lambda$ . Then  $xyw = yxw + [x, y]w = \lambda(x)yw + \lambda([x, y])w$

We can set  $W = \text{span}(w, y \cdot w, y^2 \cdot w, \dots)$ . This is certainly preserved by  $y$ , and to show it's preserved by the radical you do something similar as the previous theorem.

■

Look at theorem 6.16 in the book for this course (Kiralov (sp?))

## Bilinear forms

**Definition 0.28.** Let  $\mathfrak{g}$  be a Lie algebra. A bilinear form  $B$  on  $\mathfrak{g}$  is invariant if

$$B([x, y], z) + B(y, [x, z]) = 0$$

for all  $x, y, z \in \mathfrak{g}$ .

**Proposition 17.** Suppose  $\mathfrak{g}$  is a Lie algebra,  $B$  an invariant bilinear form on  $\mathfrak{g}$ ,  $I \subseteq \mathfrak{g}$  an ideal.

Then  $I^\perp = \{x \mid B(x, y) = 0 \forall y \in I\}$  is an ideal.

*Proof.* Let  $x \in I, y \in \mathfrak{g}, z \in I^\perp$ . We want to show  $[y, z] \in I^\perp$ .  
 We have  $B([y, z], x) = -B(z, \underbrace{[y, x]}_{\in I}) = 0$

■

**Corollary 0.46.** *Let  $\mathfrak{g}, B$  as before. Then  $\mathfrak{g}^\perp$  is an ideal.*

■

**Example 0.22.** Take  $\mathfrak{g} = \mathfrak{gl}_n, B(x, y) = \text{tr}(xy)$ . Then

$$\begin{aligned} \text{tr}([x, y], z) + \text{tr}(y, [x, z]) &= \text{tr}(xyz - yxz) + \text{tr}(yxz - yzx) \\ &= 0 \end{aligned}$$

**Example 0.23.** Let  $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  be a representation. Then we may define

$$B_\rho(x, y) = \text{tr}(\rho(x)\rho(y))$$

How to check if  $B$  is nondegenerate:

We have  $B : V \times V \rightarrow k$

We have a  $\psi_B : V \rightarrow V^*$  given by  $x \mapsto B(x, -)$ .

Then  $B$  is non degenerate if  $\psi_B$  is an isomorphism. So we can pick a basis  $e_1, \dots, e_n$ , and show  $\det(B(e_i, e_j)) \neq 0$ .

**Theorem 0.47.** *Suppose there exists  $\rho$  with  $B_\rho$  non-degenerate. Then  $\mathfrak{g}$  is reductive (i.e.  $\text{rad}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$ ).*

*Proof.* It is enough to show  $[\mathfrak{g}, \text{rad}(\mathfrak{g})] = 0$

But  $\rho([\mathfrak{g}, \text{rad}(\mathfrak{g})])$  acts by zero on any irreducible representation of  $\mathfrak{g}$

**Claim.** *This implies  $[\mathfrak{g}, \text{rad}(\mathfrak{g})] \subseteq \ker B_\rho$*

*Proof.* By induction on  $\dim \rho$ ,  $\rho$  irreducible (otherwise we take a irreducible subrep  $\psi \subseteq \rho$ ).

It is enough to show  $B_\rho = B_\psi + B_{\rho/\psi}$

$B_\rho$  being nondegenerate means  $\ker B_\rho = 0$ , so  $\text{rad}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$   
 (?)

■

**Corollary 0.48.**  $\mathfrak{gl}_n, \mathfrak{sl}_n, \mathfrak{so}(n), \mathfrak{sp}(2n)$  are all reductive.

*Proof.* We will do the proof for  $\mathfrak{gl}_n$ .

Take  $\rho = \text{Id} : \mathfrak{gl}_n \rightarrow \mathfrak{gl}_n, B_\rho(x, y) = \text{tr}(xy)$ .

Let  $e_{ij}$  be the matrix with  $(e_{ij})_{kl} = \delta_{kl}^{ij}$ .

We see  $B(e_{ij}, e_{kl}) = \delta_{il}\delta_{jk}$

■

## Lecture 16, 11/3/25

**Definition 0.29.** Let  $\mathfrak{g}$  be a Lie algebra.

The Killing form  $K$  is defined as  $K = B_{ad}$ , so  $K(x, y) = \text{tr}(\text{ad}(x) \cdot \text{ad}(y))$

**Theorem 0.49.**

- (a)  $\mathfrak{g}$  is semisimple iff  $K$  is non-degenerate
- (b)  $\mathfrak{g}$  is solvable if and only if  $K([\mathfrak{g}, \mathfrak{g}], \mathfrak{g}) = 0$ .

*Proof.* Next time.

**Definition 0.30.** Let  $k = \bar{k}$ ,  $\text{char } k = 0$ ,  $\mathfrak{g}$  a finite dimensional  $k$ -Lie algebra. A bilinear form  $B : \mathfrak{g} \times \mathfrak{g} \rightarrow k$  is invariant if

$$B([x, y], z) + B(y, [x, z]) = 0$$

Given a representation  $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ , there is a canonical form

$$B_\rho(x, y) = \text{tr}(\rho(x)\rho(y))$$

**Definition 0.31.** The Killing form is given by  $K = B_{ad}$ , so  $K(x, y) = \text{tr}(\text{ad}(x) \text{ad}(y))$

**Theorem 0.50** (Cartan's criterion).

- (a)  $\mathfrak{g}$  is semisimple if and only if  $K$  is non-degenerate
- (b)  $\mathfrak{g}$  is solvable if and only if  $K([\mathfrak{g}, \mathfrak{g}], \mathfrak{g}) = 0$

**Corollary 0.51.** Let  $\mathfrak{g}$  be semisimple,  $I \subseteq \mathfrak{g}$  an ideal.

Then there is an  $I' \subseteq \mathfrak{g}$  such that  $\mathfrak{g} = I \oplus I'$  (as a Lie algebra).

*Proof.*  $\mathfrak{g}$  is semisimple, so by the theorem (yet to be proven), we can take  $I'$  to be the perpendicular subspace under  $K$ , and  $\dim I + \dim I' = \dim \mathfrak{g}$  simply because  $K$  is non degenerate.

We want to show  $I \cap I' = 0$ .

**Claim.**  $K|_I$  is the killing form for  $I$ .

*Proof.* Let  $x, y \in I$ . Then

$$\begin{aligned} K(x, y) &= \text{tr}(\text{ad}(x) \text{ad}(y)|_{\mathfrak{g}}) \\ &= \text{tr}(\text{ad}(x) \text{ad}(y)|_I) + \underbrace{\text{tr}(\text{ad}(x) \text{ad}(y)|_{\mathfrak{g}/I})}_{=0} \end{aligned}$$

because  $\text{ad}(x), \text{ad}(y)$  act on  $\mathfrak{g}/I$  as 0.



Now to prove the theorem.

We wish to show  $K|_I$  is non-degenerate. Let  $x \in I$ . We want to show there is a  $y \in I$  with  $K(x, y) \neq 0$ .

We know there is some  $z \in \mathfrak{g}$  such that  $K(x, z) \neq 0$ .

He put it on the homework :(

■

**Corollary 0.52.** *If  $\mathfrak{g}$  is semisimple, then  $\mathfrak{g}$  is a direct sum of simple non-abelian Lie algebras.*

*Proof.* Given by induction on dimension:

Choose any ideal  $I \subseteq \mathfrak{g}$ . If it doesn't exist you're done. Otherwise  $\mathfrak{g} = I \oplus I'$ .  $I, I'$  are quotients of  $\mathfrak{g}$ , hence semisimple, so we are done by the inductive hypothesis.

■

**Corollary 0.53.** *If  $\mathfrak{g}$  is semisimple, then  $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ .*

*Proof.* It is enough to check for  $\mathfrak{g}$  simple and non-abelian (because  $\mathfrak{g}$  is a direct sum of such).

But  $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}$  is an ideal, hence everything, and  $\mathfrak{g}$  is nonabelian by assumption.

■

**Corollary 0.54.** *Let  $\mathfrak{g}$  be semisimple. Then all finite dimensional  $\mathfrak{g}$ -representations are semisimple (meaning a direct sum of irreps).*

*Proof.*

■

Now to prove Cartan's criterion.

For the rest of today,  $k = \mathbb{C}$  (this is a convenience and it will turn out we can reduce to this case).

**Theorem 0.55.** *Let  $V$  be a finite dimensional vector space,  $A \in \text{End}(V)$ .*

1.  $A = A_s + A_n$ , with  $A_s$  diagonalizable and  $A_n$  nilpotent, with  $[A_n, A_s] = 0$ .
2.  $\text{ad}(A_s) = \text{ad}(A)_s$ . Moreover,  $\text{ad}(A)_s = P(\text{ad}(A))$  for some polynomial with vanishing constant term  $P \in \mathbb{C}[t]$ .
3. Let  $\overline{A_s}$  be the complex conjugate matrix. Then  $\text{ad}(\overline{A_s}) = Q(\text{ad } A_s)$  for some polynomial  $Q \in \mathbb{C}[t]$

*sketch.* 1. First, write  $A$  in Jordan normal form by choosing an appropriate basis.

We can take  $A_s$  to be the diagonal of this, and  $A_n = A - A_s$ .

In basis free (and thus choice free (and thus canonical)) language,

Let  $V_\lambda = \ker((A - \lambda \text{Id})^N)$   $N \gg 0$ .  $V = \bigoplus_\lambda V_\lambda$ . We can take  $A_s \stackrel{\text{def}}{=} (v_\lambda)_{\lambda \in \mathbb{C}} \rightarrow (\lambda v_\lambda)_{\lambda \in \mathbb{C}}$ , and then  $A_n = A - A_s$ .

2. We want to show  $A_s = P(A)$  for some  $P \in \mathbb{C}[t]$ . Let  $n_\lambda = \dim V_\lambda$ . Choose  $p \in \mathbb{C}[t]$  such that

$$P \equiv \lambda \pmod{t - \lambda}^{n_\lambda} \forall \lambda$$

This exists by the chinese remainder theorem.

Then  $P(A) = \lambda$  on  $V_\lambda$  because  $(A - \lambda \text{Id})^{n_\lambda} = 0$  on  $V_\lambda$

Remark: This tells us that  $A_n$  is also a polynomial in  $A$  ( $A - P(A)$ )

Now,

$$\begin{aligned} \text{ad}(A) &= \text{ad}(A_s + A_n) \\ &= \text{ad}(A_s) + \text{ad}(A_n) \end{aligned}$$

To see that  $\text{ad}(A_s) = \text{ad}(A)_s$ , it is enough to show  $\text{ad}(A_s)$  is diagonalizable,  $\text{ad}(A_n)$  is nilpotent (we can do this by computation), and  $[\text{ad}(A_s), \text{ad}(A_n)] = 0$ .

Computation for  $\text{ad}(A_s)$ :

$\text{ad}(A_s)$  acts on  $V_\lambda \otimes V_{\lambda'}^v$  as  $\lambda - \lambda'$ ,

$$\text{End}(V) = \bigoplus_{\lambda, \lambda'} V_\lambda \otimes V_{\lambda'}^v$$

We know  $P \in t\mathbb{C}[t]$  because 0 is a generalized eigenvalue so  $P \equiv 0 \pmod{t}^{n_0}$

3. Choose  $f$  so that  $f(\lambda_i - \lambda_j) = \overline{\lambda_i} - \overline{\lambda_j}$  for all  $i, j$  (Lagrange). Then  $f(P(\text{ad}(A))) = \text{ad } \overline{A_s}$

## Lecture 17, 6/3/25

We finish the proof. Let  $\mathfrak{g}$  be a Lie algebra over  $\mathbb{C}$ , and let  $K(x, y) = \text{tr}(\text{ad}(x) \text{ad}(y))$  be the Killing form.

**Theorem 0.56** (Cartan).

(a)  $\mathfrak{g}$  is semisimple iff  $K$  is non-degenerate

(b)  $\mathfrak{g}$  is solvable iff  $K(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$ .

Last time we showed that if  $I$  is an ideal (hence a sub-algebra), then the Killing form for  $I$  is the Killing form for  $\mathfrak{g}$  restricted to  $I$ .

**Lemma 9.** Suppose  $\mathfrak{g} \subseteq \mathfrak{gl}(V)$  is a Lie algebra such that for all  $x \in \mathfrak{g}$ ,  $x$  is nilpotent. Then  $\mathfrak{g}$  is itself nilpotent.

*Sketch.* It is enough to show that there exists  $v \in V$  such that  $xv = 0$  for all  $x \in \mathfrak{g}$ . By induction on  $\dim V$ : consider  $V/\text{span}(v)$ . Lift strict upper triangularizing basis to  $V$ , adjoin  $v$  to it.

To show that there exists such a  $v$ , use induction on  $\dim \mathfrak{g}$ .

For  $\dim \mathfrak{g} = 1$  note that nilpotent matrices have nonzero kernel.

For the inductive step, let  $\mathfrak{g}' \subseteq \mathfrak{g}$  be a maximal proper sub-algebra.

Find  $v$  with  $\mathfrak{g}'v = 0$ , now copy the proof for solvable. ■

**Theorem 0.57.** (*Jordan*)

Let  $V$  be a finite dimensional  $\mathbb{C}$ -vector space,  $A \in \text{End}(V)$ . Then

1.  $A = A_s + A_n$  with  $[A_s, A_n] = 0$ , with  $A_s$  diagonalizable and  $A_n$  nilpotent.
2.  $\text{ad}(A)_s = \text{ad}(A_s) = P(\text{ad}(A)), P \in t\mathbb{C}[t]$  (i.e.  $P$  has zero constant term).
3.  $\overline{\text{ad } A_s} = Q(\text{ad } A)$  for some  $Q \in t\mathbb{C}[t]$

*Proof of Cartan.* Most of the content in (a) is proving (b) so we'll prove (b) first.

We have  $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{b} \subseteq \mathfrak{gl}(\mathfrak{g}), [\mathfrak{g}, \mathfrak{g}] \rightarrow \mathfrak{n} \subseteq \mathfrak{gl}(\mathfrak{g})$ ,

$$K(x, y) = \text{tr}(\underbrace{\text{ad}(x)}_{\in \mathfrak{b}} \underbrace{\text{ad}(y)}_{\in \mathfrak{n}}) = \text{tr}(\text{strictly upper triangular matrix}) = 0$$

To show that  $K(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0 \implies \mathfrak{g}$  solvable, reduce to  $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ . Then  $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  has kernel  $\mathfrak{z}(\mathfrak{g})$

It is enough to show the image of the adjoint is solvable, because then  $\mathfrak{g}$  is an extension of a solvable by a solvable.

Now,  $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ ,  $K(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$ , and we want to show it's solvable.

Pick  $x \in [\mathfrak{g}, \mathfrak{g}], y \in \mathfrak{g}$ . Take  $x = x_s + x_n$ . To show  $x$  is nilpotent, we want to show all its eigenvalues are zero.

We will show  $\text{tr}(x_s \overline{x_s}) = \sum_{\lambda \in \text{Spec}(x_s)} |\lambda|^2$

By hypothesis we can write  $x = \sum_i [y_i, z_i]$ .

$$\begin{aligned} \text{tr}(x_s \overline{x_s}) &= \text{tr}\left(\sum [y_i, z_i] \overline{x_s}\right) \\ &= \pm \sum \text{tr}(y_i [z_i, \overline{x_s}]) \\ &= \pm \sum \text{tr}(y_i \text{ad}(x_s) z_i) \\ &= \pm \sum \text{tr}(y_i Q(\text{ad}(x) z_i)) \end{aligned}$$

But  $y_i \in \mathfrak{g}$ , and  $\text{ad}(x) z_i \in [\mathfrak{g}, \mathfrak{g}]$  by definition, and  $Q$  has no constant term, so this sum is 0.



Finally, by assumption  $K(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$ .

So all eigenvalues are zero, so  $x$  is nilpotent.

By the lemma,  $[\mathfrak{g}, \mathfrak{g}]$  is nilpotent, so  $\mathfrak{g}$  is solvable.

We can reduce to the case of  $\mathbb{C}$  by the “Lefschetz principal”.

So now we prove part (a).

First, suppose that  $K$  is nondegenerate.

We already know  $\mathfrak{g}$  is reductive, i.e.  $\text{rad}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$ .

Let  $x \in \mathfrak{z}(\mathfrak{g})$ . Then  $K(x, y) = \text{tr}(\text{ad}(x) \text{ad}(y)) = 0$ , i.e.  $x \in \ker K$ . But  $K$  is nondegenerate, so  $x = 0$ , so  $\text{rad}(\mathfrak{g}) = 0$ .

Now suppose  $\mathfrak{g}$  is semisimple.

Set  $I = \ker K = \mathfrak{g}^\perp$ , which is an ideal.

But by the earlier lemma,  $K|_I \equiv 0$ , so by part (b),  $I$  itself is solvable.

But  $\mathfrak{g}$  is semisimple, so  $I = 0$ . ■

**Corollary 0.58** (Corollaries on structure of semisimple Lie algebras).

**Theorem 0.59.** *Let  $\mathfrak{g}$  be semisimple,  $I \subseteq \mathfrak{g}$  an ideal. Then there exists an  $I'$  with  $\mathfrak{g} = I \oplus I'$  as Lie algebras.*

*Proof.* Let  $K$  be the killing form of  $\mathfrak{g}$ . By Cartan, it is nondegenerate. Let  $I' = I^\perp$ . It is enough to show  $I \cap I' = 0$ .

$I \cap I'$  is an ideal such that  $K|_{I \cap I'} = 0$ . Now by Cartan, it is solvable, hence zero. ■

**Corollary 0.60.** *Semisimple Lie algebras are direct sums of simple Lie algebras.*

**Corollary 0.61.** *If  $\mathfrak{g}$  is semisimple then  $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ .*

*Proof.* We did these last time. ■

**Theorem 0.62.** *Suppose  $\mathfrak{g}$  is semisimple. Then every representation of  $\mathfrak{g}$  is a direct sum of irreducible representations.*

*In other words, any representation of a semisimple Lie algebras is semisimple.*

*Proof.* First Daniel wants to sketch a proof using the theory of Lie groups.

Sketch:

1. Reduce to the case  $k = \mathbb{R}$ .
2.  $\mathfrak{g}$  is the Lie algebra of a compact Lie group  $G$  such that
3. All representations of  $\mathfrak{g}$  come from representations of  $G$ , in the sense that the derivative of a Lie group representation is a Lie algebra representation.
4. There exists a left-invariant Haar measure  $\mu$  on  $G$

5. It is now enough to show all reps of  $G$  are semisimple

6. Given  $G \rightarrow \mathrm{GL}(V)$ ,  $W \subseteq V$  invariant, choose a projection  $V \rightarrow W$ , and take

$$\frac{1}{\mu(G)} \int_G \rho(g, -) d\mu$$

as in the proof of Maschke

Here is a proof using stuff from this class:

Idea: Use the Laplacian on  $G$ .

Recall:  $U\mathfrak{g}$  is the universal enveloping algebra of  $\mathfrak{g}$ , defined by

$$U\mathfrak{g} = \bigoplus_{i \geq 0} \mathfrak{g}^{\otimes i} / \langle x \otimes y - y \otimes x - px \rangle$$

When  $\mathfrak{g}$  is the Lie algebra of a Lie group we can identify this with the left-invariant differential operators on  $G$ .

**Definition 0.32** (Casimir operator). Suppose  $\mathfrak{g}$  is a Lie algebra,  $B$  an invariant nondegenerate bilinear form  $B : \mathfrak{g} \times \mathfrak{g} \rightarrow k$ .

Choose a basis  $(e_i)$  of  $\mathfrak{g}$ , and let  $(e^i)$  be the dual basis, i.e.

$$B(e_i, e^j) = \delta_{ij}$$

Then  $C_B = \sum e_i e^i \in U\mathfrak{g}$ .

**Proposition 18.**  $C_B$  is central and independent of the choice of basis.

*Proof.* We think of  $B \in \mathfrak{g}^* \otimes \mathfrak{g}^*$ . We can also think of it as  $B \in \underline{\mathrm{Hom}}_k(\mathfrak{g}, \mathfrak{g}^*)^{\mathfrak{g}}$ .  $B$  furnishes an isomorphism between  $\mathfrak{g}$  and  $\mathfrak{g}^*$ .

We would like an element of  $\mathfrak{g} \otimes \mathfrak{g}$ .

We have an isomorphism  $(\mathfrak{g}^* \otimes \mathfrak{g}^*) \rightarrow (\mathfrak{g} \otimes \mathfrak{g})$  given by  $B^{-1} \otimes B^{-1}$ .

The claim is that this sends  $B$  to  $C_B$ . But  $B$  was  $\mathfrak{g}$  invariant on the left, so it is  $\mathfrak{g}$ -invariant on the right, and that's what it means to be central. ■

**Proposition 19.** Suppose  $\mathfrak{g}$  is semisimple,  $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  is a nontrivial irreducible representation.

Then there exists  $C_\rho \in Z(U\mathfrak{g})$  such that  $C_\rho$  acts on the trivial representation by 0 and on  $\rho$  by a nonzero scalar.

*Proof.* Let  $B_\rho(x, y) = \mathrm{tr}(\rho(x)\rho(y))$

Set  $I = \ker B_\rho$ . This is an ideal, so  $\mathfrak{g} = \mathfrak{g}' \oplus I$  where  $B_\rho|_{\mathfrak{g}'}$  is invariant and nondegenerate.

We have  $C = C_{B_\rho|_{\mathfrak{g}'}} \in Z(U\mathfrak{g}') \hookrightarrow U\mathfrak{g}$ .

1.  $C$  acts trivially by 0: true for any element in  $U\mathfrak{g}_{>0}$
2.  $C$  acts on  $\rho$  by some scalar by Schur's lemma. We must check that this scalar is nonzero.

We have

$$\begin{aligned}
 \mathrm{tr}(C|\rho) &= \mathrm{tr}\left(\sum e_i e^i | \rho\right) \\
 &= \sum \mathrm{tr}(e_i e^i | \rho) \\
 &= \sum B_\rho(e_i, e^i) \\
 &= \sum 1 \\
 &= \dim \mathfrak{g}' \\
 &\neq 0
 \end{aligned}$$

So in fact  $\lambda = \frac{\dim \mathfrak{g}'}{\dim V}$

*Proof.*

**Example 0.24.** Let  $\mathfrak{g} = \mathfrak{sl}_2$ ,  $B = K$ .

Take a basis  $\mathfrak{sl}_2 = \langle e, f, h \rangle$ . We must compute a dual basis.

We have  $\mathrm{ad}(e) : e \mapsto 0, f \mapsto h, h \mapsto -2e$

$\mathrm{ad}(f) : e \mapsto -h, f \mapsto 0, h \mapsto 2f$

$\mathrm{ad}(h) : e \mapsto 2e, f \mapsto -2f, h \mapsto 0$

We have  $B(e, e) = 0, B(e, f) = 4, B(e, h) = 0$ . So keep going, then we solve the system of equations

$$\begin{cases} B(e, ae + bf + ch) &= 1 \\ B(f, \dots) &= 0 \\ B(h, \dots) &= 0 \end{cases}$$

We wind up getting  $C = \frac{1}{2}h^2 + ef + fe$

## Lecture 18, 18/3/25

**Theorem 0.63.** *Let  $\mathfrak{g}$  be a semisimple (in the Lie algebra sense) Lie algebra over  $\mathbb{C}$ . Then any finite dimensional representation of  $\mathfrak{g}$  is itself semisimple (in the representation of a Lie algebra sense).*

*Proof.*

**Proposition 20.** *If  $\mathfrak{g}$  is a semisimple Lie algebra (always over  $\mathbb{C}$ ), and  $V$  is a non-trivial irreducible (always finite-dimensional) representation of  $\mathfrak{g}$ , then there is a  $C_V \in Z(U\mathfrak{g})$  such that  $C_V$  acts on  $V$  via a non-zero scalar and on the trivial representation by zero.*

*Proof.* Last time ■

*Proof.* Now let's prove the theorem. We will do this in three cases, each of which will reduce to the previous case.

**Case 1:** Let  $W$  be a representation of  $\mathfrak{g}$ , and suppose that all  $C_V$  act on  $W$  nilpotently. Then  $W$  is trivial.

*Proof.*

By induction on dimension of  $W$ : If  $W$  is 1-dimensional then  $W$  is trivial.

Now suppose  $\dim W > 1$ . By assumption,  $W$  is not irreducible. Let  $U \subseteq W$  be a nontrivial proper subrepresentation.

By induction,  $U, W/U$  are trivial. But then the morphism  $\mathfrak{g} \rightarrow \mathfrak{gl}(W)$  factors through a group of block upper triangular matrices which corresponds to an abelian Lie algebra.

So  $\mathfrak{g}$  has an abelian quotient if representation is nontrivial.

So  $\mathfrak{g}$  has an abelian, hence solvable, ideal if the representation is non-trivial.

**Case 2:** Let  $W$  be a representation of  $\mathfrak{g}$ ,  $U \subseteq W$  a subrep. We want a section of  $W \twoheadrightarrow W/U$ .

(i) Assume  $W/U = \mathbb{C}$

(ii) Consider the action  $\langle C_V \rangle_V \curvearrowright W$

Then  $W = \bigoplus W_\lambda$ ,  $\lambda : \langle C_V \rangle \rightarrow \mathbb{C}$ , where  $W_\lambda$  are the generalized eigenspaces of  $\langle C_V \rangle$ .

Then  $p = \bigoplus p_\lambda : W_\lambda \rightarrow \mathbb{C}$ .  $p_\lambda = 0$  for  $\lambda \neq 0$ , and  $p_0$  is surjective.

It is enough to find a splitting  $s$  of  $p_0 : W_0 \twoheadrightarrow \mathbb{C}$ .

But  $W_0$  is trivial by case 1, so any  $s$  works.

(iii) Suppose  $W/U$  is arbitrary. We have a SES

$$0 \longrightarrow U \xrightarrow{\iota} W \xrightarrow{\pi} W/U \longrightarrow 0$$

Hitting this SES with the functor  $\mathrm{Hom}_{\mathbb{C}}(W/U, -)$ , we get

$$0 \longrightarrow \operatorname{Hom}_{\mathbb{C}}(W/U, U) \xrightarrow{\iota^*} \operatorname{Hom}_{\mathbb{C}}(W/U, W) \xrightarrow{\pi^*} \operatorname{Hom}_{\mathbb{C}}(W/U, W/U) \longrightarrow 0$$

It is enough to split this sequence. We can use the  $s'$

**Case 3:** Do induction.

Proof in Kirilov:

The SES's  $U \hookrightarrow W \twoheadrightarrow W/U$  is controlled by  $H^1(\mathfrak{g}, \operatorname{Hom}_{\mathbb{C}}(W/U, U)) = 0$ . But this group is  $\operatorname{Ext}_{U\mathfrak{g}}^1(W/U, U)$ .

So  $\operatorname{Ext}^1$  vanishes, and thus every every such extension splits.

$H^1(\mathfrak{g}, V) = 0$  for a vector space  $V$ , and  $\operatorname{Ext}^1(A, B)$  is  $H^1(\mathfrak{g}, \operatorname{Hom}(A, B))$ .

**Corollary 0.64.** *Let  $\mathfrak{g}$  be reductive (i.e.  $\operatorname{rad}\mathfrak{g} = \mathfrak{z}(\mathfrak{g})$ ). Then*

$$\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$$

*Proof.* Consider

$$0 \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{z}(\mathfrak{g}) \xrightarrow{\operatorname{ad}} \mathfrak{gl}(\mathfrak{g}) \longrightarrow 0$$

Given by the adjoint rep.  $\mathfrak{g}$  is semisimple, and by the theorem  $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ .  
So  $\mathfrak{g} \subset \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$  as Lie algebras. ■

Goal for the rest of the course:

Understand the analogue of

$$\langle \text{diagonal matrices} \rangle \hookrightarrow \mathfrak{sl}_n$$

For arbitrary semisimple Lie algebras.

**Example 0.25.** Consider  $h \in \mathfrak{sl}_2$ . Suppose  $h$   $\curvearrowright$  representations of  $\mathfrak{sl}_2$ .

Then  $h$  is always diagonalizable and irreps were classified in terms of largest eigenvalues of  $h$ .

**Definition 0.33.** Let  $\mathfrak{g}$  be a semisimple Lie algebra.  $x \in \mathfrak{g}$  semisimple if  $\operatorname{ad}(x) \in \mathfrak{gl}(\mathfrak{g})$  is diagonalizable.

$x \in \mathfrak{g}$  is nilpotent if  $\operatorname{ad}(x) \in \mathfrak{gl}(\mathfrak{g})$  is nilpotent.

**Example 0.26.** In  $\mathfrak{sl}_2 = \langle h, e, f \rangle$ ,  $h$  is semisimple and  $e, f$  are nilpotent.

**Theorem 0.65.** *Let  $\mathfrak{g}$  be semisimple over  $\mathbb{C}$ ,  $x \in \mathfrak{g}$ . Then there is a unique way to write  $x = x_s + x_n$ ,  $x_s$  semisimple and  $x_n$  nilpotent, with  $[x_s, x_n] = 0$ , and  $\operatorname{ad}(x_s) = P(\operatorname{ad}(x))$ ,  $P \in \mathbb{C}[t]$ .*

**Corollary 0.66.** *If  $\mathfrak{g}$  is semisimple then there are semisimple elements.*

*Proof.* ■

## Lecture 19, 20/3/25

Last time, we proved that if  $\mathfrak{g}$  is a semisimple Lie algebra over  $\mathbb{C}$ , then all finite dimensional representations of  $\mathfrak{g}$  are themselves semisimple, i.e. direct sums of irreducibles.

**Example 0.27.** Consider  $\mathfrak{sl}_n = \{x \in \mathfrak{gl}_n \mid \text{tr } x = 0\}$

We will denote by  $\mathfrak{t}_n \subseteq \mathfrak{sl}_n$  the diagonal matrices.

Let  $E_{ij}$  be the matrix with 1 in the  $i, j$ th entry and 0 elsewhere. If  $i \neq j$  then  $E_{ij} \in \mathfrak{sl}_n$ .

$\mathfrak{t}_n$  is commutative with a basis given by  $E_{i,i} - E_{i+1,i+1}$ .

**Proposition 21.** For every  $x \in \mathfrak{t}_n$  and representation  $\rho : \mathfrak{sl}_n \rightarrow \mathfrak{gl}_n$ ,  $\rho(x)$  is diagonalizable.

*Proof.* It is enough to show that  $\rho(E_{i,i} - E_{i+1,i+1})$  is diagonalizable.

But this is a copy of  $h$  in a subalgebra isomorphic to  $\mathfrak{sl}_2$ , and this is diagonalizable because of facts we know about the rep theory of  $\mathfrak{sl}_2$ .

We have a distinguished class of elements  $\{E_{i,i} - E_{j,j} \mid i \neq j\} \subseteq \mathfrak{t}_n$

Such that

$$\langle E_{i,j}, E_{i,j} - E_{j,i}, E_{ji} \rangle \simeq \mathfrak{sl}_2$$

We can break  $\mathfrak{sl}_n$  into eigenspaces for the  $\mathfrak{t}_n$ -action (which is by  $\text{ad}$ ).

Then  $\mathfrak{sl}_n = \mathfrak{t}_n \oplus \bigoplus_{i \neq j} \text{span}(E_{ij})$ , because the centralizer of  $\mathfrak{t}_n$  is  $\mathfrak{t}_n$ .

The direct sum is indexed by some functionals on  $\mathfrak{t}_n$  (namely the eigenvalues). But the killing form  $K(x, y) = \text{tr}(\text{ad}(x) \text{ad}(y)|_{\mathfrak{sl}_n})$  is a nondegenerate pairing on  $\mathfrak{t}_n$ , so furnishes a canonical isomorphism between  $\mathfrak{t}_n$  and its dual.

So each  $\lambda$  corresponds to  $K(x_\lambda, -)$  for some  $x_\lambda \in \mathfrak{t}_n$ , and these turn out to be the  $E_{i,j}$ 's. (?) ■

**Remark:** There's a more general story for Cartan subalgebras of  $\mathfrak{sl}_n$ .

Now we have to come up with a way to say “diagonal matrix” for an abstract Lie algebra.

**Definition 0.34.** Let  $\mathfrak{g}$  be a semisimple Lie algebra. We say  $x \in \mathfrak{g}$  is semisimple if  $\text{ad}(x)$  is diagonalizable, and we say  $x \in \mathfrak{g}$  is nilpotent if  $\text{ad}(x)$  is nilpotent.

**Definition 0.35.**  $\text{Der}(\mathfrak{g}) \subseteq \mathfrak{gl}(\mathfrak{g})$  is  $\{x \in \mathfrak{gl}(\mathfrak{g}) \mid [ax, b] + [a, xb] = x[a, b]\}$ .

This is the space of derivations on  $\mathfrak{g}$ . One can check as an exercise that it is in fact a subalgebra of  $\mathfrak{gl}(\mathfrak{g})$ .

**Theorem 0.67.** Let  $\mathfrak{g}$  be semisimple,  $x \in \mathfrak{g}$ . Then  $x$  can be written uniquely as  $x = x_s + x_n$  with  $x_s$  semisimple,  $x_n$  nilpotent, and  $\text{ad}(x_s) = P(\text{ad}(x)) \in t\mathbb{C}[t]$ .

*Proof.* What we need is a characterization of the image of  $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ .

**Proposition 22.** *If  $\mathfrak{g}$  is semisimple, then  $\text{ad} : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{g})$  is an isomorphism.*

*Proof.* First,  $\ker \text{ad} = \mathfrak{z}(\mathfrak{g})$ , which is zero because  $\mathfrak{g}$  is semisimple, thus this is injective.

Extend the killing form  $K$  from  $\mathfrak{g}$  to  $\text{Der}(\mathfrak{g})$  (which we can do because this map is injective) by saying  $K(\delta_1, \delta_2) = \text{tr}(\delta_1 \delta_2 | \mathfrak{g})$ .

Observe  $K|_{\mathfrak{g}}$  is the Killing form of  $\mathfrak{g}$ , which is non-degenerate by Cartan's criterion. Hence  $\text{Der}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^\perp$ , because  $\mathfrak{g} \cap \mathfrak{g}^\perp = 0$  and  $\dim \mathfrak{g} + \dim \mathfrak{g}^\perp \geq \dim \text{Der}(\mathfrak{g})$ .

**Claim.**  $\mathfrak{g}$  is an ideal.

*Proof.*

$$[\delta, \text{ad}(x)] = \text{ad } \delta(x)$$

So  $\mathfrak{g}$  is an ideal, so  $\text{Der}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^\perp$ , not just as vector spaces, but as Lie algebras. So then  $\mathfrak{g}^\perp$  and  $\mathfrak{g}$  commute, i.e. for all  $\delta \in \mathfrak{g}^\perp$ ,  $x \in \mathfrak{g}$ ,  $\delta(x) = 0$ . So then  $\mathfrak{g}^\perp = 0$ . This proves the claim. ■

Now we can prove the theorem. ■

*Proof.* Uniqueness is easy, and follows from the uniqueness of the usual Jordan decomposition on  $\mathfrak{gl}(\mathfrak{g})$ .

For existence, write  $\text{ad}(x) = \text{ad}(x)_s + \text{ad}(x)_n$ .

It is enough to show  $\text{ad}(x)_s = \text{ad}(x_s)$  for some  $x_s \in \mathfrak{g}$ , because then we can set  $x_n = x - x_s$ , and we get the usual Jordan decomposition.

By the proposition, to show this, it is enough to show that  $\text{ad}(x)_s$  is a derivation.

$\mathfrak{g} = \bigoplus \mathfrak{g}_\lambda$ , where  $\mathfrak{g}_\lambda$  are the generalized eigenspaces of  $\text{ad}(x)$ .

$\text{ad}(x)$  acts on  $\mathfrak{g}_\lambda$  by  $\lambda \cdot \text{Id}$ .

**Claim.**

$$[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}$$

*Proof.* Compute ■

**Corollary 0.68.** *If  $\mathfrak{g} \neq 0$  is a semisimple Lie algebra, then there exists a semisimple element in  $\mathfrak{g}$ .*

*Proof.* If this is not the case, then  $x = x_n$  for all  $x \in \mathfrak{g}$ , i.e. the image of  $\text{ad}$  consists of nilpotent matrices, which implies  $\mathfrak{g}$  is nilpotent by a problem on homework 1.

**Proposition 23.** *Let  $\mathfrak{g}$  be semisimple,  $x \in \mathfrak{g}$  semisimple,  $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}_n$  a representation. Then  $\rho(x)$  is diagonalizable.*

*Proof.* WLOG,  $\rho$  is irreducible.

Let  $\mathbb{C}^N \supseteq \bigoplus_{\lambda} V_{\lambda}$ , where  $V_{\lambda}$  is the honest (meaning NOT generalized) eigenspace of  $\rho(x)$ .

It is enough to show  $\bigoplus_{\lambda} V_{\lambda}$  is a subrepresentation, because  $\rho$  is irreducible.

Because  $\rho$  is semisimple,  $\mathfrak{g} = \bigoplus \mathfrak{g}_{\alpha}$ , where  $\mathfrak{g}_{\alpha}$  is the  $\alpha$ -eigenspace of  $\text{ad}(x)$ .

To show  $V_{\lambda}$  is a subrepresentation it's enough to show that acting by an element of  $\mathfrak{g}$  stays in the sum, so it's enough to do it for a basis, so it's enough to do it for an eigenvector.

Fix  $y \in \mathfrak{g}_{\alpha}, v \in V_{\lambda}$ . Then

$$xyv = yxv + [x, y]v = \lambda yv + \text{ad}(x)(y) \cdot v = \lambda yv + \alpha yv = (\lambda + \alpha)yv$$

So  $yv \in V_{\alpha+\lambda}$ . ■

This completes the proof of the theorem. ■

**Definition 0.36.** Let  $\mathfrak{g}$  be a semisimple Lie algebra over  $\mathbb{C}$ . Then

- $\mathfrak{t} \subseteq \mathfrak{g}$  is toroidal if it is commutative and consists entirely of semisimple elements
- $\mathfrak{h} \subseteq \mathfrak{g}$  is Cartan if it is toroidal, and  $\{x \in \mathfrak{g} \mid [x, y] = 0 \forall y \in \mathfrak{h}\}$ . That is,  $\mathfrak{h}$  is its own centralizer.

**Theorem 0.69.** Let  $\mathfrak{g}$  be semisimple,  $\mathfrak{t} \subseteq \mathfrak{g}$  toroidal.

1.  $\mathfrak{g} = \bigoplus \mathfrak{g}_{\alpha}$ ,  $\alpha \in \mathfrak{t}^*$ ,  $\mathfrak{g}_{\alpha} = \{x \in \mathfrak{g} \mid [y, x] = \text{ad}(y)x \forall y \in \mathfrak{t}\}$ .

*That is, the adjoint action is diagonalizable.*

2.  $[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}] \subseteq \mathfrak{g}_{\alpha+\beta}$ .
3. If  $\alpha + \beta \neq 0$ , then  $\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}$  are orthogonal with respect to  $K$ .
4. The Killing form induces a non-degenerate pairing  $\mathfrak{g}_{\alpha} \times \mathfrak{g}_{-\alpha} \rightarrow \mathbb{C}$ . In particular,  $K|_{\mathfrak{g}_0}$  is non-degenerate.

*Proof.*

1. Simultaneously diagonalizable
2. Special case of computation from before
3. Take  $x \in \mathfrak{g}_{\alpha}, y \in \mathfrak{g}_{\beta}$ . Goal: compute  $\text{tr}(\text{ad}(x) \text{ad}(y)|_{\mathfrak{g}})$ .

$\text{ad}(x) \text{ad}(y)|_{(\mathfrak{g}_{\lambda})} \subseteq \mathfrak{g}_{\lambda+\alpha+\beta}$  which has zero intersection  $\mathfrak{g}_{\lambda}$ .

So the matrix of  $\text{ad}(x) \text{ad}(y)$  is zero on the diagonal, so  $\text{tr}(\text{ad}(x) \text{ad}(y)|_{\mathfrak{g}}) = 0$ .



4. The Killing form is non-degenerate.

**Theorem 0.70.** *Let  $\mathfrak{g}$  be a semisimple Lie algebra over  $\mathbb{C}$ , and suppose  $\mathfrak{t} \subseteq \mathfrak{g}$  is a maximal toroidal subalgebra. Then  $\mathfrak{t}$  is a Cartan subalgebra.*

**Corollary 0.71.** *Cartan subalgebras exist*

*Proof.* First we must show there exists a non-zero toroidal sub-algebra of  $\mathfrak{g}$ . Then take a maximal via Zorn's lemma or the like. ■

*Proof.* Now let's prove the theorem.

Let  $\mathfrak{t}$  be a maximal toroidal subalgebra.

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{t}^*} \mathfrak{g}_\alpha$$

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [y, x] = \alpha(y)x \forall y \in \mathfrak{t}\}$$

Now note  $\mathfrak{g}_0 = C(\mathfrak{t})$ , the centralizer of  $\mathfrak{t}$ .

We want to show  $\mathfrak{t} = \mathfrak{g}_0$ .

Note from what we've just shown that  $K|_{\mathfrak{g}_0}$  is non-degenerate.

**Claim.** *Let  $x \in C(\mathfrak{t}) = \mathfrak{g}_0$ . Then  $x_s, x_n \in C(\mathfrak{t})$ .*

*Proof.* We have  $\text{ad } x_s = P(\text{ad } x)$ , so for all  $y \in \mathfrak{t}$ ,  $(\text{ad } x_s)(y) = P(\text{ad } x)(y) = 0$ , because  $P \in t\mathbb{C}[t]$ .

But this implies  $x_s \in C(\mathfrak{t})$ , which implies that  $x_n = x - x_s \in C(\mathfrak{t})$  ■

Now note that  $C(\mathfrak{t})$  is reductive because  $K|_{C(\mathfrak{t})}$  is non-degenerate.

**Claim.**  *$C(\mathfrak{t})$  is commutative*

*Proof.* First note  $C(\mathfrak{t}) = \mathfrak{z}(C(\mathfrak{t})) \oplus \mathfrak{g}'$ , with  $\mathfrak{g}'$  semisimple.  $\mathfrak{t} \subseteq \mathfrak{z}(C(\mathfrak{t}))$

Let  $x_s \in \mathfrak{g}'$  be semisimple.

Then  $\langle \mathfrak{t}, x_s \rangle$  contradicts maximality of  $\mathfrak{t}$ . ■

**Claim.**  *$C(\mathfrak{t})$  contains no nilpotents.*

*Proof.* Fix some  $x \in C(\mathfrak{t})$  nilpotent, and fix  $y \in C(\mathfrak{t})$  arbitrary. Then

$$\begin{aligned} K(x, y) &= \text{tr}(\text{ad}(x) \text{ad}(y)|_{\mathfrak{g}}) \\ &= \text{tr}(\text{ad}(y) \text{ad}(x)|_{\mathfrak{g}}) \end{aligned}$$

By previous point,  $\text{ad}(x), \text{ad}(y)$  commute, and so their product is nilpotent, and hence the trace is zero.

So then  $x$  has to be zero by nondegeneracy of the Killing form on  $C(\mathfrak{t})$

Finally,  $C(\mathfrak{t})$  is toroidal because it is commutative and for all  $x \in C(\mathfrak{t})$ ,  $x_s, x_n \in C(\mathfrak{t})$ , so  $x_n = 0$  by the previous point. So  $x = x_s$ , i.e.  $x$  is semisimple. Hence  $\mathfrak{t} = C(\mathfrak{t})$ , because  $\mathfrak{t}$  is maximal toroidal. ■

**Fact:**  $\text{Aut}(\mathfrak{g})$  acts transitively on the Cartan subalgebras.

## Root system

Let  $\mathfrak{g}$  be semisimple,  $\mathfrak{h} \subseteq \mathfrak{g}$  a Cartan subalgebra (i.e. a commutative subalgebra consisting of semisimple elements which is its own centralizer).

**Theorem 0.72.**

1.  $\mathfrak{g} \simeq \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$ , where  $R$  is the set  $\{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$ , and  $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [y, x] = \alpha(y)x \forall y \in \mathfrak{h}\}$

$R$  is called the set of roots

2.  $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$
3. If  $\alpha + \beta \neq 0$ , then  $\mathfrak{g}_\alpha, \mathfrak{g}_\beta$  are orthogonal wrt the Killing form
4.  $K : \mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \rightarrow \mathbb{C}$  is non-degenerate. In particular,  $K|_{\mathfrak{h}}$  is non-degenerate.

*Proof.*  $\mathfrak{h} = \mathfrak{g}_0$ . ■

**Observation:**  $K|_{\mathfrak{h}}$  is non-degenerate, thus furnishes an isomorphism  $\mathfrak{h} \simeq \mathfrak{h}^*$  sending  $y$  to  $K(y, -)$ .

So given  $\alpha \in R$ , there exists a unique  $H_\alpha \in \mathfrak{h}$  such that  $\alpha = K(H_\alpha, -)$ .

**Example 0.28.** Consider  $\mathfrak{sl}_3$ . Then  $\mathfrak{h}$  is the diagonal matrices with trace 0.

Then  $\mathfrak{h}^* = \langle e_1, e_2, e_3 \rangle / (\sum e_i = 0)$  with  $e_i(M) = (M)_{ii}$ .

Then  $R = \{\pm(e_1 - e_2) \pm (e_1 - e_3) \pm (e_2 - e_3)\}$

## Lecture 20, 25/3/25

**Example 0.29.** Let  $\mathfrak{g} = \mathfrak{sl}_n$ ,  $\mathfrak{h}$  the diagonal matrices.

We have shown there is a decomposition  $\mathfrak{sl}_n = \mathfrak{h} \oplus \bigoplus \text{span}(E_{ij})$ , which we can use to find many copies of  $\mathfrak{sl}_2$  inside  $\mathfrak{sl}_n$ .

**Theorem 0.73.** Let  $\mathfrak{g}$  be a Lie algebra,  $\mathfrak{h}$  a Cartan subalgebra.

1.  $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$ , where  $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \forall h \in \mathfrak{h}\}$ ,  $\alpha \in \mathfrak{h}^*$ , where  $R = \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$ , and  $R$  is called the set of roots.
2.  $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$
3. If  $\alpha + \beta \neq 0$ , then  $\mathfrak{g}_\alpha, \mathfrak{g}_\beta$  are orthogonal with respect to  $K$ .
4.  $K : \mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \rightarrow \mathbb{C}$  is nondegenerate; in particular,  $K|_{\mathfrak{h}}$  is non-degenerate.

*Proof.* ■

**Observation:** There is a canonical isomorphism between  $\mathfrak{h}$  and  $\mathfrak{h}^*$  given by  $x \mapsto K(x, -)$ .

So given  $\alpha \in R$ , there is a unique  $H_\alpha \in \mathfrak{h}$  such that  $K(H_\alpha, -) = \alpha$ .

**Notation:** Given  $\alpha, \beta \in \mathfrak{h}$ , we write  $(\alpha, \beta) \stackrel{\text{def}}{=} K(\alpha, \beta)$ .

Given  $\alpha \in \mathfrak{h}^*, \beta \in \mathfrak{h}$ , we write  $\langle \alpha, \beta \rangle \stackrel{\text{def}}{=} \alpha(\beta)$ .

**Goal:** Construct many copies of  $\mathfrak{sl}_2$  inside of  $\mathfrak{g}$ .

The idea will be to take  $e \in \mathfrak{g}_\alpha, f \in \mathfrak{g}_{-\alpha}$  such that  $(e, f) \neq 0$ , and then consider the subalgebra generated by  $e, f$ .

**Lemma 10.** *Let  $e \in \mathfrak{g}_\alpha, f \in \mathfrak{g}_{-\alpha}$ . Then  $[e, f] = (e, f)H_\alpha$ .*

*Proof.*

- First, observe  $[e, f] \in \mathfrak{h}$ .
- Let  $x \in \mathfrak{h}$ . What is  $([e, f], x)$ ? Well,  $K$  is an invariant bilinear form, so this is equal to

$$\begin{aligned}
 (e, [f, x]) &= -(e, [x, f]) \\
 &= -(e, \alpha(x)f) \\
 &= \alpha(x)(e, f) \\
 &= (H_\alpha, x)(e, f) \\
 &= ((e, f)H_\alpha, x)
 \end{aligned}$$

So  $H_{[e, f]}(x) = (e, f)H_\alpha(x)$

**Lemma 11.**

1. Let  $\alpha \in R$ . Then  $(H_\alpha, H_\alpha) \neq 0$ .

2. If  $e \in \mathfrak{g}_\alpha, f \in \mathfrak{g}_{-\alpha}$  such that  $(e, f) = \frac{2}{(\alpha, \alpha)}$ , set  $h_\alpha = \frac{2H_\alpha}{(\alpha, \alpha)}$

Then  $\langle e, f, h_\alpha \rangle \simeq \mathfrak{sl}_2$ .

Further,  $\langle h_\alpha, \alpha \rangle = 2$ .

*Proof.*

1. Suppose  $(\alpha, \alpha) \stackrel{\text{def}}{=} (H_\alpha, H_\alpha) = 0$ . Pick  $e \in \mathfrak{g}_\alpha, f \in \mathfrak{g}_{-\alpha}$  with  $(e, f) \neq 0$  (possible as  $K$  is non-degenerate).

Consider the sub-algebra  $\langle e, f, \underbrace{[e, f]}_{=x} \rangle \stackrel{\text{def}}{=} \mathfrak{s}$ .

We have  $[x, e] = [(e, f)H_\alpha, e] = (e, f)\alpha(H_\alpha)e = 0$ . Same argument shows  $[x, f] = 0$ .

So  $x \in \mathfrak{z}(\langle e, f, x \rangle)$ , so  $\mathfrak{s}$  is solvable.

Now, we have the upper triangularizable representation  $\text{ad} : \mathfrak{s} \rightarrow \mathfrak{gl}(\mathfrak{g})$ , so  $\text{ad}(x) = \text{ad}([e, f])$  is nilpotent. But  $x$  is a multiple of  $H_\alpha$ , which is semisimple, so  $x = 0$ .

So  $x = (e, f)H_\alpha$  with  $(e, f)$  nonzero.

2. Directly compute  $[e, f], [h, f], [h, e]$ .

■

**Definition 0.37.** For each  $\alpha \in R$ , choose  $e, f$  as above, and set  $(\mathfrak{sl}_2)_\alpha \subseteq \mathfrak{g}$  to be the sub-algebra generated by  $e, f, [e, f]$ .

It will turn out later that this is independent of the choice of  $e, f$ .

**Lemma 12.** Fix  $(\mathfrak{sl}_2)_\alpha \subseteq \mathfrak{g}$  as before. Then  $V \stackrel{\text{def}}{=} \mathbb{C}h_\alpha \oplus \bigoplus_{k \in \mathbb{Z}_{\neq 0}} \mathfrak{g}_{k\alpha}$  is irreducible as a  $(\mathfrak{sl}_2)_\alpha$ -representation.

*Proof.* First, this decomposition is preserved by  $\mathfrak{sl}_2$  :, because  $\text{ad}(e) : \mathfrak{g}_{k\alpha} \rightarrow \mathfrak{g}_{(k+1)\alpha}$ , and  $\text{ad}(f) : \mathfrak{g}_{k\alpha} \rightarrow \mathfrak{g}_{(k-1)\alpha}$ , and those follow from the fact that  $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ .

To see that it is irreducible, we compute the eigenvalues of  $h_\alpha \curvearrowright V$ . Indeed,  $\text{ad}(h_\alpha)|_{\mathfrak{g}_\alpha} = 2 \cdot \text{Id}$ , and  $\text{ad}(h_\alpha)|_{\mathfrak{g}_{k\alpha}} = 2k$ .

This implies  $V$  is a direct sum of  $\text{Sym}^n V_{\text{std}}$  for  $n$  even, and 0 appears as an eigenvalue with multiplicity 1, so this is irreducible.

■

**Corollary 0.74.**  $\dim \mathfrak{g}_\alpha = 1$  for all  $\alpha$ .

*Proof.*

■

**Fact:**  $V \simeq (\mathfrak{sl}_2)_\alpha$ .

## Lecture 21, 27/3/25

**Definition 0.38** (Root system). Let  $E$  be a real vector space with pairing  $(-, -)$ ,  $R \subseteq E \setminus \{0\}$  such that

R1  $R$  spans  $E$

R2 The quantity  $\eta_{\alpha\beta} \stackrel{\text{def}}{=} \frac{2(\alpha, \beta)}{(\beta, \beta)} \in \mathbb{Z}$  for all  $\alpha, \beta \in R$

R3 The function  $s_\alpha : E \rightarrow E$ , sending  $\lambda \mapsto \lambda - \frac{2(\alpha, \lambda)}{(\alpha, \alpha)}\alpha$  preserves  $R$  for all  $\alpha \in R$ .

Reduced If  $\alpha, c\alpha \in R$ , then  $c = \pm 1$ .

We can rephrase R3 by saying  $s_\alpha$  is the reflection across  $\{\lambda \mid (\lambda, \alpha) = 0\}$ .

We can rephrase R2 by saying the projection of  $\beta$  onto  $R_\alpha$  has half-integer length.

**Last time:** Let  $\mathfrak{g}$  be a semisimple Lie algebra over  $\mathbb{C}$ ,  $\mathfrak{h} \subseteq \mathfrak{g}$  a Cartan subalgebra.

Then  $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$ .

We showed this by constructing copies of  $\mathfrak{sl}_2$  for each  $\alpha$ ,  $(\mathfrak{sl}_2)_\alpha \subseteq \mathfrak{g}$ , given by  $\langle e \in \mathfrak{g}_\alpha, t \in \mathfrak{g}_{-\alpha}, h_\alpha \in \mathfrak{h} \rangle$ .

Then  $\mathbb{C}h_\alpha \oplus \bigoplus_{k \in \mathbb{Z} \setminus \{0\}} \mathfrak{g}_{k\alpha}$  is irreducible as a  $(\mathfrak{sl}_2)_\alpha$ -representation, where  $h + \alpha \curvearrowright \mathfrak{g}_{k\alpha}$  by  $\cdot 2k \text{ Id}$ .

**Corollary 0.75.**

- $\mathfrak{g}_\alpha$  is 1-dimensional
- $(\mathfrak{sl}_2)_\alpha$  is independent of choices.
- If  $\alpha, \beta$  are roots with  $\beta = k\alpha$ , then  $k = \pm 1$ .

*Proof.* We will prove the third corollary (we did the other two last time).

We have  $(\mathfrak{sl}_2)_\alpha \subseteq V_\alpha$ , so  $(\mathfrak{sl}_2)_\alpha = V_\alpha$  by irreducibility. ■

**Theorem 0.76.** Let  $\mathfrak{g}, \mathfrak{h}$  be as before,  $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$ .

1.  $R$  spans  $\mathfrak{h}^*$  as a  $\mathbb{C}$ -vector space.
2. For all  $\alpha$ ,  $\dim \mathfrak{g}_\alpha = 1$
3. Setting  $h_\alpha = \frac{2H_\alpha}{(\alpha, \alpha)}$ , with  $(H_\alpha, -) = \alpha(-)$ , we have

$$\beta(h_\alpha) = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}$$

with  $\alpha, \beta \in R$ .

4.  $s_\alpha : \mathfrak{h}^* \rightarrow \mathfrak{h}^*$  given by

$$s_\alpha(\lambda) = \lambda - \lambda(h_\alpha)\alpha = \lambda - \frac{2(\alpha, \lambda)}{(\alpha, \alpha)}\alpha$$

sends roots to roots.

5. If  $\alpha$  is a root,  $c\alpha$  a root, then  $c = \pm 1$

6. Let  $\alpha, \beta \neq \pm\alpha$ . Then  $V_{\alpha, \beta} = \bigoplus \mathfrak{g}_{\beta+k\alpha}$  is irreducible as  $(\mathfrak{sl}_2)_\alpha$ -representations

7. Let  $\alpha, \beta, \alpha + \beta$  be roots. Then  $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ .

*Proof.*

1. Suppose there exists some  $x \in \mathfrak{h}$  such that  $\alpha(x) = 0$  for all roots  $\alpha$ . So  $x \in \mathfrak{z}(\mathfrak{g})$ , so  $x = 0$ .

2. Follows from irreducibility of  $V_\alpha$

3.  $\beta(h_\alpha)$  = eigenvalue of  $h_\alpha \curvearrowright \mathfrak{g}_\beta$ . But  $h_\alpha \in (\mathfrak{sl}_2)_\alpha$ , and acts on  $V_{\beta+k\alpha}$ . This is an  $\mathfrak{sl}_2$ -representation, so the eigenvalues must be integers.

4. Consider the action  $(\mathfrak{sl}_2)_\alpha \curvearrowright V_{\alpha, \beta}$ . We want to show  $s_\alpha(\beta)$  is a root, i.e.  $g_{s_\alpha(\beta)} \neq 0$

Case 1  $n \geq 0$ . Fix  $0 \neq x \in \mathfrak{g}_\alpha$ . Then  $f^n x \in \mathfrak{g}_{\beta-n\alpha}$ , where  $f^n x$  denotes the action of  $x$   $n$  times.

Case 2 If  $n \leq 0$ , replace  $f$  by  $e$ .

5. We have ruled out every case but  $c = \pm 1, 2, \frac{1}{2}$ . But we can assume  $c \in \mathbb{Z}$ . We ruled out 2 but I missed it :(

6. Eigenspaces of  $h_\alpha$  are 1-dimensional.

7. Follows from 6 - if  $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$ , then  $\mathfrak{g}_\beta$ -generates a sub- $(\mathfrak{sl}_2)_\alpha$ -representation, which contradicts irreducibility.

■

## Lecture 22, 4/1/25

**Goal:** Explain the correspondence between semisimple Lie algebras and root systems. We've proven that given a semisimple Lie algebra  $\mathfrak{g}$  with Cartan subalgebra  $\mathfrak{h}$ ,  $R \in \mathfrak{h}^*$  roots, and killing form  $K$ . Then if  $E = \text{span}(\mathbb{R})$ , then  $K|_E$  is a root system. We finish the proof of the theorem.

*Proof.* We want to show that  $K|_E$  is positive definite.  
Let  $h_\alpha$  span  $\mathfrak{h}^*$  (we know this to be true).

**Claim.** For  $\beta \in R$ ,  $\beta(h_\alpha) \in \mathbb{Z}$

*Proof.* Follows from being an  $\mathfrak{sl}_2$ -rep ■

So if  $\sum c_\alpha h_\alpha \in E$ , i.e.  $c_\alpha \in \mathbb{R}$ ,  $\beta(\sum c_\alpha h_\alpha) \in \mathbb{R}$ .  
So

$$\begin{aligned} K\left(\sum c_\alpha h_\alpha, \sum c_\alpha h_\alpha\right) &= \text{tr}(\text{ad}(\sum c_\alpha h_\alpha)^2) &= \sum_\beta \beta\left(\sum c_\alpha h_\alpha\right)^2 \\ &\geq 0 \end{aligned}$$

This is strictly positive, because if this is zero, then  $\sum c_\alpha h_\alpha \in \mathfrak{z}(\mathfrak{g}) = 0$ . ■

Now we want to see how to get a Lie algebra from a root system.

**Idea:** Imitate upper triangular matrices.

**Choose:**  $t \in E$  such that  $(t, \alpha) \neq 0$  for all roots  $\alpha$ .  
Then  $R = R^+ \cup R^-$ , where

$$\begin{aligned} R^+ &\stackrel{\text{def}}{=} \{\alpha \in R \mid (t, \alpha) > 0\} \\ R^- &\stackrel{\text{def}}{=} \{\alpha \in R \mid (t, \alpha) < 0\} \end{aligned}$$

We call  $R^+$  the positive roots and  $R^-$  the negative roots

**Observe:** If a root system is associated to some  $(\mathfrak{g}, \mathfrak{h})$ , then

$$\mathfrak{g} = \mathfrak{h} \oplus \underbrace{\bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha}_{\eta_+} \oplus \underbrace{\bigoplus_{\alpha \in R^-} \mathfrak{g}_\alpha}_{\eta_-}$$

**Example 0.30.**  $\eta_+, \eta_-$  are nilpotent sub-algebras.

*Proof.* A picture ■

**Definition 0.39.**  $\Phi \subseteq R^+$  is  $\alpha \in R^+$  such that  $\alpha \neq \beta + \gamma$  for any  $\gamma, \beta \in R^+$ .  
This is the set of simple roots.

**Proposition 24.** Any positive root is a positive integer linear combination of simple roots.

*Proof.* Follows from  $\Phi$  being a basis (exercise (easy)). And any  $\alpha \in R$  is a unique  $\mathbb{Z}$ -linear combination of simple roots. ■

**Fact:** If  $\alpha, \beta \in R$   $|\alpha| > |\beta|$ ,  $\alpha \neq t\beta$ . Then there's a finite set of possible angles between  $\alpha, \beta$ , and  $\frac{|\alpha|}{|\beta|}$ . That set is

$$\left\{ \frac{\pi}{2}, \frac{2\pi}{3}, \frac{\pi}{3}, \frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{6}, \frac{\pi}{6} \right\}$$

**Fact:** If  $\alpha, \beta$  are simple roots,  $(\alpha, \beta) \leq 0$ .

**Corollary 0.77.**  $\mathfrak{g}$  is determined, up to isomorphism, by its root system.

*Proof.* Let  $\mathfrak{h} = E \otimes \mathbb{C}$ , and let  $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_{\alpha}$ , where  $\mathfrak{g}_{\alpha}$  are 1-dimensional.

Relations:  $\mathfrak{h} \curvearrowright \mathfrak{g}_{\alpha}$  via  $\alpha_i$ ,  $\mathfrak{h}$  commutative.

For each  $\alpha \in \Phi$ , choose  $e_{\alpha} \in \mathfrak{g}_{\alpha}$ .

Then  $[e_{\alpha}, e_{\alpha'}] \in \mathfrak{g}_{\alpha+\alpha'}$ .

For each  $\beta \in R^+$ , there is a unique element of  $\mathfrak{g}_{\beta}$  obtained as nested commutator of  $e_{\alpha}$ 's.

So there is a unique  $f_{\beta}$ ,  $\beta \in R$ , such that  $[e_{-\beta}, f_{\beta}] = h_{\beta} \in \mathfrak{h}$ , where  $h_{\beta} = \frac{2H_{\beta}}{(\beta, \beta)}$ ,  $(H_{\beta}, -) = \beta$ .

For a complete set of relations: Kirillov Chapter 8. ■

**Observation** Given root systems  $R_1, R_2$ ,  $R_1 \oplus R_2$  is a root systems.

**Definition 0.40.**  $R$  is irreducible if it is not a direct sum of root systems.

**Lemma 13.** If  $\mathfrak{g}$  is simple, its root system is irreducible.

*Proof.* ■

**Definition 0.41.** If  $R$  is a root system with a set of simple roots  $\Phi$ , we call the 2-tuple  $(R, \Phi)$  a polarized root system.

Cartan matrix:

$$\left( \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \right)_{\alpha_i, \alpha_j \in \Phi}$$

The diagonal entries are 2, the off-diagonal are all nonpositive integers.

**Definition 0.42.** A Dynkin diagram is a graph with vertex set  $\Phi$ , and for  $\alpha, \beta \in \Phi$ , we connect by 1 edge of the angle between them if the angle is  $\frac{2\pi}{3}$ , 2 edges if  $\frac{3\pi}{4}$ , 3 edges if  $\frac{5\pi}{6}$ , and 0 edges if  $\frac{\pi}{2}$ .

If the root system is irreducible, the graph is connected-otherwise we could find an irreducible subsystem.

We make it a digraph by having an arrow from the longer to the shorter root.



**Classification:** I'm not fucking drawing these root systems again...

## Lecture 23, 4/3/25

Let  $\mathfrak{g}$  be a semisimple Lie algebra,  $\mathfrak{h} \subseteq \mathfrak{g}$  a Cartan subalgebra.

We have a set of roots  $R \subseteq \mathfrak{h}^*$ . We can write

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_{\alpha}$$

We define  $(\mathfrak{sl}_2)_{\alpha}$  as the sub algebra generated by  $\mathfrak{g}_{\alpha}, \mathfrak{g}_{-\alpha}$ , with its canonical element  $h_{\alpha} = \frac{2H_{\alpha}}{(\alpha, \alpha)}$ , where  $(H_{\alpha}, -) = \alpha(-)$ .

We let  $E = \text{span}_{\mathbb{R}}(R) \subseteq \mathfrak{h}^*$ . This is a root system. Choosing a  $t \in E$  (last choice) such that  $(t, \alpha) \neq 0$  for all  $\alpha \in R$ . We call  $R^+$  the roots that pair positively with  $t$  (i.e.  $(\alpha, t) > 0$ ), and  $R^-$  the roots that pair negatively.

We call  $n_+ = \bigoplus_{\alpha \in R^+} \mathfrak{g}_{\alpha}$ , and  $n_- = \bigoplus_{\alpha \in R^-} \mathfrak{g}_{-\alpha}$ . Then the above equation can be written

$$\mathfrak{g} = \mathfrak{h} \oplus n_+ \oplus n_-$$

Suppose  $V$  is an irrep of  $\mathfrak{g}$ . For  $\lambda \in \mathfrak{h}^*$ , we define  $V[\lambda] = \{v \in V \mid xv = \lambda(x)v \forall x \in \mathfrak{h}\}$ .

**Proposition 25.**  $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V[\lambda]$

*Proof.*

**Proposition 26.** If  $V$  is any finite dimensional irrep of  $\mathfrak{g}$ , and  $V[\lambda] \neq 0$ , then  $\lambda(h_{\alpha}) \in \mathbb{Z}$ .

*Proof.*  $V|_{(\mathfrak{sl}_2)_{\alpha}}$  is an  $\mathfrak{sl}_2$ -representation. ■  
■

**Definition 0.43.**

1. The Root lattice  $Q$  is the  $\mathbb{Z}$ -span of  $R$
2.  $Q^*$  is the  $\mathbb{Z}$ -span of the  $h_{\alpha}$ , the coroot lattice.
3.  $P = \{\lambda \in \mathfrak{h}^* \mid \lambda(h_{\alpha}) \in \mathbb{Z}\}$ . The weight lattice.

Note  $Q$  and  $Q^*$  are not dual lattices despite the notation;  $Q$  is dual to  $P$ .

**Definition 0.44.** The character of  $V$  is  $f : P \rightarrow \mathbb{Z}$ ,  $\lambda \mapsto \dim V[\lambda]$ .

This will turn out to classify irreps, but we won't prove that here.

**Definition 0.45.**  $v \in V[\lambda]$  is a highest weight vector if it is nonzero and  $e_{\alpha} \cdot v = 0$  for all  $\alpha \in R_+$ .

**Computation:**  $\mathfrak{g}_\alpha \otimes V[\lambda] \rightarrow V[\lambda + \alpha]$ .

**Definition 0.46.** A highest weight representation is one generated by a highest weight vector.

**Proposition 27.** Any finite dimensional irrep of  $\mathfrak{g}$  is a highest weight representation.

*Proof.* Choose  $\lambda$  with  $V[\lambda] \neq 0$  such that  $(\lambda, t)$  is maximal. Then  $v \in V[\lambda]$  generates  $V$  by irreducibility. ■

**Proposition 28.** Suppose  $V[\lambda] \neq 0$ . Then for all roots  $\alpha$ ,  $V[s_\alpha(\lambda)] \neq 0$ .

*Proof.*  $(\mathfrak{sl}_2)_\alpha \curvearrowright \oplus_k V[\lambda + k\alpha]$ . This implies  $\dim V[\lambda + k\alpha] = \dim V[s_\alpha(\lambda)]$ . ■

**Definition 0.47.** Let  $\lambda \in \mathfrak{h}^*$ . Define  $I_\lambda \subseteq U\mathfrak{g}$  to be the left ideal generated by  $n_+$  and  $(x - \lambda(x))_{x \in \mathfrak{h}}$ . Then define

$$M_\lambda \stackrel{\text{def}}{=} U\mathfrak{g}/I_\lambda$$

We define  $v_\lambda \in M_\lambda$  to be the image of 1 in  $U\mathfrak{g}$ .

Observe:  $v_\lambda$  generates  $M_\lambda$ ,  $n_+ \cdot v_\lambda = 0$ , and  $x \cdot v_\lambda = \lambda(x)v_\lambda$  for all  $x \in \mathfrak{h}$ .

This is called a Verma module, and it is infinite dimensional.

This is the same as  $U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}\lambda$ , where  $\mathfrak{b} = \mathfrak{h} \oplus n_+$ , and  $\mathfrak{b}$  acts on  $\mathbb{C}\lambda$  via  $n_+ \equiv 0$  and  $x \in \mathfrak{h}$  acts by  $\lambda(x)$ .

**Proposition 29.** If  $V$  has highest weight  $\lambda$ , then  $V = M_\lambda/W$  for some  $W \subseteq M_\lambda$ .

*Proof.* Pick  $w \in V$  a highest weight generator, and consider the map  $M_\lambda \rightarrow V$  given by  $v_\lambda \mapsto w$ . This is surjective because  $w$  generates. ■

Basic properties of  $M_\lambda$  :

There is an isomorphism  $U_{n_-} \rightarrow M_\lambda$  sending  $x$  to  $xv_\lambda$ .

**Corollary 0.78.**  $M_\lambda[\lambda']$  is finite dimensional for all  $\lambda'$ , and  $\dim M_\lambda[\lambda] = 1$ .

*Proof.* ■

So classifying the finite dimensional irreps of  $\mathfrak{g}$  is the same as classifying maximal proper submodules of  $M_\lambda$ .

**Theorem 0.79.** Suppose  $\lambda \in \mathfrak{h}^*$ . Then there exists at most one finite dimensional irrep of  $\mathfrak{g}$  of highest weight  $\lambda$ .

*Proof.* It is enough to show  $M_\lambda$  has a unique maximal proper submodule.

Let

$$W = \sum_{J \subseteq M_\lambda, J[\lambda]=0} J$$

It is maximal because if  $J$  is a proper submodule, then  $J[\lambda] = 0$  (otherwise it would contain  $v_\lambda$ ), and so  $J[\lambda] = 0$  implies  $J \subseteq W$ .

It is proper for the same reason;  $W[\lambda] = 0$ .

■

**Definition 0.48.** Let  $L_\lambda = M_\lambda/W_{max}$ , where  $W_{max}$  is the unique maximal submodule.

**Definition 0.49.**  $\lambda \in P \subseteq \mathfrak{h}^*$  is dominant integral if  $\lambda(h_\alpha) \in \mathbb{Z}_{\geq 0}$  for all  $\alpha \in R^+$ , i.e.  $(\lambda, \alpha) \in \mathbb{Z}_{\geq 0}$  for all  $\alpha \in R^+$ .

**Theorem 0.80.**  $L_\lambda$  is finite dimensional if and only if  $\lambda$  is dominant integral. So this is the classification of semisimple Lie algebras.

**Corollary 0.81.** There is a natural bijection between finite dimensional irreps of  $\mathfrak{g}$  and dominant integral weights, given by sending  $V$  to the highest weight of  $V$ .

*Proof.* Suppose  $L_\lambda$  is finite dimensional. We want to show that  $\lambda$  is dominant integral.

Given  $\alpha \in R^+$ ,  $L_\lambda|_{(\mathfrak{sl}_2)_\alpha}$ ,  $\lambda(\alpha)$  is a highest weight of this  $\mathfrak{sl}_2$ -rep in  $\mathbb{Z}_{\geq 0}(?)$

Now suppose  $\lambda$  is dominant integral.

We want to show  $L_\lambda$  is finite dimensional, where  $L_\lambda = M_\lambda/\text{unique maximal proper submodule}$ .

Fix  $v_\lambda \in M_\lambda[\lambda]$  a highest weight vector (we argued earlier such a thing is unique up to scaling).

Let  $n_i = (\lambda, \alpha_i)$ ,  $\alpha_i \in R^+$ .

$$(\mathfrak{sl}_2)_{\alpha_i} = \langle e_i, f_i, h_i \rangle$$

Let  $v_{\lambda_i} = f_i^{n_i+1}v_\lambda$ .

**Claim.**  $v_{\lambda_i}$  is annihilated by all  $e_j$  (this is what it means to be highest weight)

*Proof.* It is enough to check this for  $e_j$  coming from simple roots. If  $i \neq j$  then  $e_i, e_j$  commute:  $e_j f_i^{n_i+1} e_j v_\lambda = 0$ .

If  $i = j$  this is a statement coming from  $\mathfrak{sl}_2$ -reps (?).

Set  $M_i = U\mathfrak{g} \cdot v_{\lambda_i} \subseteq M_\lambda$ .

So then each  $M_i$  is a proper submodule of  $M_\lambda$ , as  $M_i[\lambda] = 0$ .

It is enough to show that  $M_\lambda/(\sum M_i)$  is finite dimensional.

**Observe:** In this quotient, the action of  $(\mathfrak{sl}_2)_\alpha$  on each quotient for each  $\alpha$  is a finite dimensional vector space.

$(\mathfrak{sl}_2)_\alpha v_\lambda \subseteq M_\lambda/(\sum M_i)$  is finite dimensional.

**Definition 0.50.**  $V$  is integrable if for all  $v \in V$ ,  $U(\mathfrak{sl}_2)_{\alpha_i} \cdot v$  is finite dimensional.

**Lemma 14.** Given any representation  $V$  of  $\mathfrak{g}$ , the integrable elements,  $V^{int}$ , is an integral subrepresentation.

**Lemma 15.** Any highest weight integrable subrepresentation is finite dimensional.

*Proof.* Given the lemmas,  $L_\lambda$  is a quotient of  $M_\lambda/(\sum M_i)$ , hence  $(L_\lambda)^{int}$  is nonzero.  $L_\lambda$  we've shown is irreducible, but by irreducibility,  $L_\lambda = (L_\lambda)^{int}$ , hence by lemma 2,  $\dim L_\lambda < \infty$ . ■

Let's prove the first lemma now.

*Proof.* Let  $x \in \mathfrak{g}, v \in V^{int}$ . We want to show that  $xv \in V^{int}$ . hen  $xU(\mathfrak{sl}_2)_\alpha v \subseteq \mathfrak{g}U(\mathfrak{sl}_2)_\alpha v$ . But this is finite dimensional. But  $\mathfrak{g}U(\mathfrak{sl}_2)_\alpha v$  is stable under the action of  $U(\mathfrak{sl}_2)_\alpha$  ■

Let's prove the second.

*Proof.* Let  $V$  be an integrable highest weight representation of highest weight  $w$ . Consider  $\lambda$  such that  $V[\lambda] \neq 0$ .

This set is stable under reflections  $s_\alpha$  for  $\alpha \in R$ .

This implies the support of  $w(V)$  is contained in a bounded set.

So  $w(V)$  is finite. But  $\dim M_\lambda[\lambda']$  is finite. So finitely many weight spaces, each of which is finite dimensional, implise  $\dim V < \infty$ . ■