Lecture 1 - 7/1/25

Missed :(

Lecture 2 - 9/1/25

Missed :(

Lecture 3 - 14/1/25

Character theory

Consider $\dim \mathrm{Hom}_G(\rho_i, \rho_j) = 1$ if $i = j$ and 0 if $i \neq j$ (meaning if $\rho_i \ncong \rho_j$)
Recall: Given a representation $\rho : G \to \mathrm{GL}_n(k)$, the character of ρ, χ_ρ, is given by $\chi_\rho : G \to k, g \mapsto \mathrm{tr}(\rho(g))$
For today, G will be finite, $k = \overline{k}$ will be algebraically closed, of characteristic 0.
Basic properties of characters:

1. Suppose $\rho : G \to \mathrm{GL}_n(k)$ is a representation: then $\chi_\rho(e) = n = \dim \rho$.

2. $\chi_\rho(g) = \chi_\rho(hgh^{-1})$ for all $g, h \in G$, i.e. χ_ρ is constant on each conjugacy class of G.

   Definition 0.1. A function $f : G \to k$ which is constant on conjugacy classes is called a class function.

   The ρ_i (isomorphism classes of reps) will form an ONB for the space of class functions.

   Given $\rho_1 : G \to \mathrm{GL}_n(k), \rho_2 : G \to \mathrm{GL}_m(k)$, $\chi_{\rho_1 \oplus \rho_2} = \chi_{\rho_1} + \chi_{\rho_2}$

   $\chi_{\rho_1 \otimes \rho_2} = \chi_{\rho_1}\chi_{\rho_2}$
   To see this, let A, B be diagonalizable (which we have WLOG because the image of any finite group are all diagonalizable over an algebraically closed k of char 0, which follows from Jordan Normal form)

   Then $\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\,\mathrm{tr}(B)$.

   I can't see the board he's writing on very well, and also I am not sure how $A \otimes B$ was defined.

Claim. *$\chi_\rho : G \to k$ always factors through $\mathbb{Q}(\mu_\infty)$, the subfield of k containing $\mathbb{Q}$ (k has char 0) generated by all roots of unity ($k = \overline{k}$)*

Proof. Because G is finite, ρ_G has finite order, hence its eigenvalues are roots of unity, so the trace is the sum of roots of unity.

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Definition 0.2. $\bar{\cdot} : \mathbb{Q}(\mu_\infty) \to \mathbb{Q}(\mu_\infty)$ is the unique field homomorphism with the property that $\overline{\zeta} = \zeta^{-1}$ for all roots of unity $\zeta \in \mathbb{Q}(\mu_\infty)$.

5 $\chi_{\rho^v} = \overline{\chi_\rho}$

Recall ρ^v is defined via the formula $g \cdot f = f(g^{-1} \cdot -)$ where f is a functional. We have

$$\begin{aligned} \chi_{\rho^v}(g) &= \operatorname{tr}(p(g^{-1})) \\ &= \sum_{\zeta \text{ is an eigenvalue of } \rho(G)} \zeta^{-1} \\ &= \sum \overline{\zeta} = \overline{\chi_\rho(g)} \end{aligned}$$

This also follows from the Hom-tensor adjunction because $\operatorname{Hom}_k(\rho_1, \rho_2) = \rho_1^v \otimes \rho_2$.

Definition 0.3. Let $\chi, \psi : G \to \mathbb{Q}(\mu_\infty)$ be class functions. We define their inner product by

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)}$$

This indeed is a positive definite non degenerate.
Let $\rho_1 : G \to \mathrm{GL}_n(k), \rho_2 : G \to \mathrm{GL}_m(k)$. What is $\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle$?

Theorem 0.1.

$$\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle = \dim_k \operatorname{Hom}_G(\rho_1, \rho_2) = \dim_k \operatorname{Hom}(\rho_1, \rho_2)^G$$

Corollary 0.2. *Suppose ρ_1, ρ_2 are irreducible. Then $\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle$ is 0 if ρ_1, ρ_2 are not isomorphic, and 1 if they are. So the ρ_i form an orthonormal basis for the space of class functions.*

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Proof. Let $R_G \in k[G]$ be the element given by

$$R_G = \frac{1}{|G|} \sum_{g \in G} eg$$

We want to show

1. for $v \in V^G, R_G \cdot v = v$.

2. For arbitrary $v \in V, R_G \cdot v \in V^G$

To check:

1. We have

$$
\begin{aligned}
R_G \cdot v &= \frac{1}{|G|} \sum_{g \in G} e_g \cdot v \\
&= \frac{1}{|G|} \sum_{g \in G} v \\
&= v
\end{aligned}
$$

2. Fix $g \in G$. Then

$$
\begin{aligned}
g \cdot R_G \cdot v &= g \cdot (\frac{1}{|G|} \sum_{h \in G} hv) \\
&= \frac{1}{|G|} \sum_{h \in G} gh \cdot v \\
&= \frac{1}{|G|} \sum_{h \in G} h \cdot v \\
&= R_G \cdot v
\end{aligned}
$$

Corollary 0.3. *Let V be a G-representation. Then* $\dim_k V^G = \operatorname{tr}(R_G|V)$

Proof.

Claim. $\operatorname{tr}(\textit{projection}) = \dim_k \operatorname{Im}$

Proof. Claim $\Longrightarrow$ Cor follows from $\operatorname{tr}(R_G) = \dim \operatorname{Im}(R_G|V) = \dim_k V^G$ ∎

We can finally prove the theorem:

Proof.

$$
\begin{aligned}
\dim_k \operatorname{Hom}_G(\rho_1, \rho_2) &= \dim_k \operatorname{Hom}_k(\rho_1, \rho_2)^G \\
&= \operatorname{tr}(R_G | \operatorname{Hom}_k(\rho_1, \rho_2)) \\
&= \operatorname{tr}(\frac{1}{|G|} \sum e_g | \operatorname{Hom}_k(\rho_1, \rho_2)) \\
&= \frac{1}{|G|} \operatorname{tr}(g | \operatorname{hom}_k(\rho_1, \rho_2)) \\
&= \frac{1}{|G|} \sum_{g \in G} \chi_{\operatorname{hom}_k(\rho_1, \rho_2)}(g) \\
&= \frac{1}{|G|} \sum_{g \in G} \overline{\chi_{\rho_1}} \chi_{\rho_2} \\
&= \langle \chi_{\rho_1}, \chi_{\rho_2} \rangle \\
&= \underbrace{\overline{\langle \chi_{\rho_2}, \chi_{\rho_1} \rangle}}_{\in \mathbb{Z}}
\end{aligned}
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