Lecture 1 - 7/1/25

Missed :(

Lecture 2 - 9/1/25

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Lecture 3 - 14/1/25

Character theory

Consider $\dim \mathrm{Hom}_G(\rho_i, \rho_j) = 1$ if $i = j$ and 0 if $i \neq j$ (meaning if $\rho_i \ncong \rho_j$)
Recall: Given a representation $\rho : G \to \mathrm{GL}_n(k)$, the character of ρ, χ_ρ, is given by $\chi_\rho : G \to k, g \mapsto \mathrm{tr}(\rho(g))$
For today, G will be finite, $k = \overline{k}$ will be algebraically closed, of characteristic 0.
Basic properties of characters:

1. Suppose $\rho : G \to \mathrm{GL}_n(k)$ is a representation: then $\chi_\rho(e) = n = \dim \rho$.

2. $\chi_\rho(g) = \chi_\rho(hgh^{-1})$ for all $g, h \in G$, i.e. χ_ρ is constant on each conjugacy class of G.

Definition 0.1. A function $f : G \to k$ which is constant on conjugacy classes is called a class function.

The ρ_i (isomorphism classes of reps) will form an ONB for the space of class functions.

Given $\rho_1 : G \to \mathrm{GL}_n(k), \rho_2 : G \to \mathrm{GL}_m(k)$, $\chi_{\rho_1 \oplus \rho_2} = \chi_{\rho_1} + \chi_{\rho_2}$

$\chi_{\rho_1 \otimes \rho_2} = \chi_{\rho_1}\chi_{\rho_2}$

To see this, let A, B be diagonalizable (which we have WLOG because the image of any finite group are all diagonalizable over an algebraically closed k of char 0, which follows from Jordan Normal form)

Then $\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\,\mathrm{tr}(B)$.

I can't see the board he's writing on very well, and also I am not sure how $A \otimes B$ was defined.

Claim. *$\chi_\rho : G \to k$ always factors through $\mathbb{Q}(\mu_\infty)$, the subfield of k containing $\mathbb{Q}$ (k has char 0) generated by all roots of unity ($k = \overline{k}$)*

Proof. Because G is finite, ρ_G has finite order, hence its eigenvalues are roots of unity, so the trace is the sum of roots of unity.

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Definition 0.2. $\bar{\cdot} : \mathbb{Q}(\mu_\infty) \to \mathbb{Q}(\mu_\infty)$ is the unique field homomorphism with the property that $\overline{\zeta} = \zeta^{-1}$ for all roots of unity $\zeta \in \mathbb{Q}(\mu_\infty)$.

5 $\chi_{\rho^v} = \overline{\chi_\rho}$

Recall ρ^v is defined via the formula $g \cdot f = f(g^{-1} \cdot -)$ where f is a functional. We have

$$\begin{aligned}\chi_{\rho^v}(g) &= \mathrm{tr}(p(g^{-1})) \\ &= \sum_{\zeta \text{ is an eigenvalue of } \rho(G)} \zeta^{-1} \\ &= \sum \overline{\zeta} = \overline{\chi_\rho(g)}\end{aligned}$$

This also follows from the Hom-tensor adjunction because $\mathrm{Hom}_k(\rho_1, \rho_2) = \rho_1^v \otimes \rho_2$.

Definition 0.3. Let $\chi, \psi : G \to \mathbb{Q}(\mu_\infty)$ be class functions. We define their inner product by

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)}$$

This indeed is a positive definite non degenerate.
Let $\rho_1 : G \to \mathrm{GL}_n(k), \rho_2 : G \to \mathrm{GL}_m(k)$. What is $\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle$?

Theorem 0.1.

$$\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle = \dim_k \mathrm{Hom}_G(\rho_1, \rho_2) = \dim_k \mathrm{Hom}(\rho_1, \rho_2)^G$$

Corollary 0.2. *Suppose ρ_1, ρ_2 are irreducible. Then $\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle$ is 0 if ρ_1, ρ_2 are not isomorphic, and 1 if they are. So the ρ_i form an orthonormal basis for the space of class functions.*

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Proof. Let $R_G \in k[G]$ be the element given by

$$R_G = \frac{1}{|G|} \sum_{g \in G} eg$$

We want to show

1. for $v \in V^G, R_G \cdot v = v$.

2. For arbitrary $v \in V, R_G \cdot v \in V^G$

To check:

1. We have

$$\begin{aligned} R_G \cdot v &= \frac{1}{|G|} \sum_{g \in G} e_g \cdot v \\ &= \frac{1}{|G|} \sum_{g \in G} v \\ &= v \end{aligned}$$

2. Fix $g \in G$. Then

$$\begin{aligned} g \cdot R_G \cdot v &= g \cdot (\frac{1}{|G|} \sum_{h \in G} hv) \\ &= \frac{1}{|G|} \sum_{h \in G} gh \cdot v \\ &= \frac{1}{|G|} \sum_{h \in G} h \cdot v \\ &= R_G \cdot v \end{aligned}$$

Corollary 0.3. *Let V be a G-representation. Then* $\dim_k V^G = \operatorname{tr}(R_G|V)$

Proof.

Claim. $\operatorname{tr}(\textit{projection}) = \dim_k \operatorname{Im}$

Proof. Claim $\Longrightarrow$ Cor follows from $\operatorname{tr}(R_G) = \dim \operatorname{Im}(R_G|V) = \dim_k V^G$

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We can finally prove the theorem:

Proof.

$$\begin{aligned}
\dim_k \mathrm{Hom}_G(\rho_1, \rho_2) &= \dim_k \mathrm{Hom}_k(\rho_1, \rho_2)^G \\
&= \mathrm{tr}(R_G | \mathrm{Hom}_k(\rho_1, \rho_2)) \\
&= \mathrm{tr}(\frac{1}{|G|} \sum e_g | \mathrm{Hom}_k(\rho_1, \rho_2)) \\
&= \frac{1}{|G|} \mathrm{tr}(g | \hom_k(\rho_1, \rho_2)) \\
&= \frac{1}{|G|} \sum_{g \in G} \chi_{\hom_k(\rho_1, \rho_2)}(g) \\
&= \frac{1}{|G|} \sum_{g \in G} \overline{\chi_{\rho_1}} \chi_{\rho_2} \\
&= \langle \chi_{\rho_1}, \chi_{\rho_2} \rangle \\
&= \underbrace{\overline{\langle \chi_{\rho_2}, \chi_{\rho_1} \rangle}}_{\in \mathbb{Z}}
\end{aligned}$$

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Lecture 4, 16/1/24

As always, G will be a finite group, $k = \overline{k}$ is an algebraically closed field of characteristic 0.
$\mathbb{Q}(\mu_\infty)$ is the algebraically closed subfield of $\mathbb{C}$ which contains all the roots of unity, and this comes with the complex conjugate $\bar{\cdot}, \zeta \mapsto \zeta^{-1}$.
Goal: Classify finite dimensional G-representations over k.
We have done:

1. Maschke's theorem, which states that any G-rep in V over k is semisimple.

2. Character theory: $V \sim \chi_V : G \to \mathbb{Q}(\mu_\infty) \subseteq k$, $g \mapsto \mathrm{tr}(g|V)$

Definition 0.4. $Cl(G)$ denotes the class functions $G \mapsto \mathbb{Q}(\mu_\infty)$, and it is equipped with an inner product,

$$\langle \psi, \varphi \rangle = \frac{1}{|G|} \sum_{g \in G} \psi(g) \overline{\varphi(g)}$$

Remark: There is an isomorphism $Cl(G) \simeq Z(\mathbb{Q}(\mu_\infty)[G])$, sending φ to $\sum_{g \in G} \phi(g) e_g$
Warning: They come with different ring structures which are not preserved by this isomorphism.

Last time we used the Reynolds operator to show $\langle \chi_V, \chi_W \rangle = \dim_k \mathrm{Hom}_G(V, W)$.
If ρ_1, ρ_2 are irreps of G, then $\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle$ is 1 if $\rho_1 \cong \rho_2$, and 0 otherwise.

Corollary 0.4. *# of conjugacy classes of irreducible representations of* $G \leq \dim_{\mathbb{Q}(\mu_\infty)} Cl(G) =$ *# of conjugacy classes of* G

Proof. If χ_{ρ_i} are orthonormal, then the number of conjugacy classes of irreps is equal to $\dim \mathrm{span}(\chi_{\rho_i}) \subseteq Cl(G)$, so this number is $\leq \dim Cl(G)$

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Proposition 1. *Let V be a G-representation. Then*

$$\Phi_V : \oplus_{\rho_i \text{ irrep of } G} \rho_i \otimes_k \mathrm{Hom}_G(\rho_i, V) \to V$$

given by $v \otimes f \mapsto f(v)$ is an isomorphism.

Proof. First, we show it is surjective. By Maschke, $V = \oplus_{\rho_i \text{ reps } G} \rho_i^{n_i}$.
Let $v \in \rho_i^{n_i} \subseteq V$, $v = (v_1, \ldots, v_{n_i})$. Let $f_j : \rho_j \to \rho_i^{n_i}$ be the inclusion of the jth coordinate.
Then $\Phi_v(\sum_j v_j \otimes f_j) = v$.
Now we show injectivity.
We have

$$\dim_k \oplus \rho_i \otimes_k \mathrm{Hom}_G(\rho_i, V) = \dim_k V$$

This follows from

$$\dim_k \mathrm{Hom}_G(\rho_i, V) = n_i$$

This follows from

$$\begin{aligned} \mathrm{Hom}_G(\rho_i, V) &= \mathrm{Hom}_G(\rho_i, \oplus \rho_i^{n_i}) \\ &= \oplus_j \mathrm{Hom}_G(\rho_i, \rho_j)^{n_i} \\ &= \mathrm{Hom}_G(\rho_i, \rho_j)^{n_i} \end{aligned}$$

Which is n_i-dimensional

$$\dim_k \oplus \rho_i \otimes \mathrm{Hom}_G(\rho_i, V) = \sum n_i \dim_k \rho_i = \dim V$$

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Corollary 0.5.

$$V \simeq \oplus_{\rho \text{ irreps of } G} \rho_i^{\langle \chi_{\rho_i}, \chi_V \rangle}$$

Proof. Enough to show $\rho_i^{\langle \rho_i, V \rangle} \simeq \rho_i \otimes_k \mathrm{Hom}_G(\rho_i, V)$, i.e. $\dim_k \mathrm{Hom}(\rho_i, V) = \langle \chi_{\rho_i}, \chi_{\rho_j} \rangle$.
But that's the theorem.

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Corollary 0.6.

$$V \simeq \oplus_{\rho_i irreps}\rho_i^{\oplus n_i}$$

, then $\langle \chi_V, \chi_V \rangle = \sum_i n_i^2$

Proof. $\chi_V = \sum n_i \chi_{\rho_i}$ ∎

Corollary 0.7. $V \simeq W \iff \chi_V = \chi_W$

Corollary 0.8. *V is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$.*

Proof. Write $V = \oplus_i \rho_i^{n_i}$: so $\langle \chi_V, \chi_V \rangle = \sum_i n_i^2$ is equal to 1 iff exactly 1 n_i is nonzero, and equal to 1. ∎

Example 0.1. (The regular representation)
Let $G \curvearrowright k(G)$ via left multiplication.
$\chi_{k[G]}(g) = \mathrm{tr}(g|k[G])$, which is $|G|$ if g is the identity, and 0 otherwise.
Because $g \cdot e_{g'} = e_{gg'}$, we have

$$\mathrm{tr}(g|k[G]) = \#\{h \in G \mid gh = g\}$$

Remark: if X is a G-set (i.e. a set with a G-action), then the permutation representation, k^X, has character

$$\chi_{k^X}(g) = \#\{x \in X \mid g \cdot x = x\}$$

Corollary 0.9. *As a G-representation,*

$$k[G] \simeq \oplus_{\rho_i\ irrep}\ \rho_i^{\oplus \dim \rho_i}$$

Proof.

$$\begin{aligned}
\langle \chi_{\rho_i}, \chi_{k[G]} \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_{\rho_i}(g) \overline{\chi_{k[G]}(g)} \\
&= \frac{1}{|G|} \chi_{\rho_i}(e) \overline{\chi_{k[G]}(e)} \\
&= \frac{1}{|G|} \dim \rho_i |G| \\
&= \dim \rho_i
\end{aligned}$$

Because this representation is 0 except at the identity. ∎

Remark: In fact, $\mathrm{Hom}_G(k[G], \rho_i) \simeq \rho_i$, AS A VECTOR SPACE.

Proof. $\operatorname{Hom}_G(k[G],\rho_i)=\operatorname{Hom}_{k[G]}(k[G],\rho_i)\simeq\rho_i$ AS A VECTOR SPACE ∎

Corollary 0.10. *Let ρ_i be the (conjugacy classes of) irreps of G, n_i the dimension of ρ_i.*
Then $\sum_i n_i^2=|G|$.

Proof. $|G|=\dim_k k[G]=\dim_k\oplus_i\rho_i^{\oplus\dim\rho_i}=\sum n_i^2$ ∎

Theorem 0.11. *Let G be a finite group, $k=\overline{k}$ an algebraically closed field of characteristic 0, $\rho_1,\dots,\rho_n$ the irreps of G. Then $\{\chi_{\rho_i}\}$ is an orthonormal basis of $Cl(G)$.*

Proof. We know it's orthonormal (so in particular linearly independent), so it is left to show that this indeed spans all of $Cl(G)$.
What remains to show is that χ_{ρ_i} span $Cl(G)$.
It is enough to show that if $\psi\in Cl(G)$ with $\langle\psi,\chi_{\rho_i}\rangle=0$ for all i, then $\psi=0$, i.e. the orthogonal complement of the span of the χ_{ρ_i} is trivial.

Definition 0.5. If $\psi:G\to\mathbb{Q}(\mu_\infty)$ is a class function,

$$\gamma_\psi\overset{\text{def}}{=}\sum_{g\in G}\psi(g)e_g\in Z(k[G])$$

Example 0.2. If $\psi:G\to k$, $g\mapsto\frac{1}{|G|}$, $\gamma_\psi=R_G$.
We will compute what γ_ψ does to a representation.

Proposition 2. *If ρ is an irreducible representation of G, then $\gamma_\psi:\rho\to\rho$ is multiplication by the scalar $\frac{|G|}{\dim\rho}\langle\psi,\chi_{\rho^v}\rangle$*

Proof.

1. First, $\gamma_\psi:\rho\to\rho$ is a homomorphism of G-representations, which follows from $\gamma_\psi\cdot g\cdot v=g\cdot\gamma_\psi\cdot v$ for all $g\in G,v\in\rho$, as $\gamma_\psi\in Z(k[G])$.

2. By Schur, $\gamma_\psi:\rho\to\rho$ is a scalar.

3. $\gamma_\psi=\frac{\operatorname{tr}(\gamma_\psi|\rho)}{\dim\rho}\cdot\operatorname{Id}_\rho$, so

$$\operatorname{tr}(\gamma_\psi|\rho)=\operatorname{tr}(\sum_{g\in G}\psi(g)e_g|\rho)=\sum_{g\in G}\psi(g)\chi_\rho(g)=|G|\langle\psi,\overline{\chi_\rho}\rangle=|G|\langle\psi,\chi_{\rho^v}\rangle$$

∎

Now, consider $\gamma_\psi:k[G]\to k[G]$. This is zero as γ_ψ acts as zero on every irrep (because it pairs to zero with all the irreps), and because it sends 1 to γ_ψ, γ_ψ has to be zero.

∎

Corollary 0.12. *(of earlier claim)*
$\frac{\dim \rho_i}{|G|}\gamma_{\chi_{\rho_i^v}}$ *acts as* 1 *on* ρ_i*, and 0 on* ρ_j*, for* $\rho_i \neq \rho_j$ *are irreps.*

Proof. ■

Corollary 0.13. *Given any* $V = \oplus \rho_i^{\oplus n_i}$,

$$\frac{\dim \rho_i}{|G|}\gamma_{\chi_{\rho_i^v}}$$

acts as a projection onto $\rho_i^{n_i} \subseteq V$*, which is called the* ρ_i *isotypic part of* V*.*

Corollary 0.14. *#irreps of* G = *#conjugacy classes of* G

Proof. Let $\{\rho_i\}$ be the irreps of G (up to conjugacy (i.e isomorhpism)).
Then $\{\chi_{\rho_i}\}$ is a basis for $Cl(G)$, so # of irreps $= \dim_k Cl(G) =$ #conjugacy classes of G.
Remark: These two numbers are equal, but there is no natural or canonical bijection between the two sets in general.

Classifying rep'ns

Theorem 0.15. *G is abelian iff all irreps of G are 1-dimensional.*

Proof. Let V be an irrep. If G is commutative, then $\cdot g : V \to V$ is a G-homomorphism for all $g \in G$.
By Schur, each $g \in G$ acts as a scalar. Now every subspace of V is a subrep, hence V is 1-dimensinoal.
Now suppose that all irreps are 1-dimensional. Let n_i be the dimensions of the irreps ρ_i, and let c be the number of conjugacy classes (or equivalently the number of irreps) of G. Then $|G| = \sum_i n_i^2$, but this is at least c, because we are taking the sum of c positive numbers, but each n_i is 1, so each element of G is its own conjugacy class.

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Example 0.3. Take $G = \mathbb{Z}/n\mathbb{Z}$
For each element $\zeta \in \mu_n \stackrel{\text{def}}{=}$nth roots of unity, consider $\chi_\zeta : \mathbb{Z}/n\mathbb{Z} \to k^*, a \mapsto \zeta^a$
This gives n distinct reps, which is the number of conjugacy classes, hence we have a complete list.

Example 0.4. S_3 has conjugacy classes $[e], [(12)], [(123)]$, so there are 3 irreducible representations. We have a trivial representation, whose character sends all conjugacy classes to 1.
We also have $sgn : S_3 \to \{\pm 1\} \subseteq k^*$, so χ_{sgn} sends $[e]$ to 1, $[(12)]$ to -1, and $[(123)]$ to 1.

At this point we know there must be a third representation, *std*, and we can fill in its row in the character table somehow. *std* is given by $S_3 \curvearrowright \mathbb{C}^{\{1,2,3\}} / \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, with $\chi_{std} = \chi_{\mathbb{C}^{\{1,2,3\}}} - \chi_{triv}$, so $\chi_{st}(e) = 2, \chi_{std}(12) = 0, \chi_{std}(123) = -1$.
We claim that χ_{std} is irreducible. To see this, we compute

$$\langle \chi_{std}, \chi_{std} \rangle = \frac{1}{6}(2^2 + 3 * 0^2 + 2(-1)^2) = 1.$$

Example 0.5. $Q_8 = \langle \pm 1, \pm i, \pm j, \pm k \rangle$, with multiplication given as in the quaternion group, $i^2 = j^2 = k^2 = ijk = -1$.
Conjugacy classes: (e),-1, $\{\pm i\}$,$\{\pm j\}$,$\{\pm k\}$.
χ_{triv} sends them all to 1, of course.

Lecture 5, 21/1/25

	1	-1	{i, -i}	{j,-j}	{k,-k}
triv	1	1	1	1	1
i-ker	1	1	1	-1	-1
j-ker	1	1	-1	1	-1
k-ker	1	1	-1	-1	1
?	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Let $\mathbb{H} = \mathbb{R}\langle 1, i, j, k \rangle$. Then $Q_8 \curvearrowright \mathbb{H}$ by left multiplication, $\mathbb{H} \curvearrowleft \mathbb{C}$ by multiplication by i on the right. This example might be useful to think about for the homework.
Now let's get the character table for S_4.

conj class	0	(12)	(123)	(12)(134)	(1234)
size	1	6	8	3	6
sgn	1	-1	1	1	-1
$std = \mathbb{C}^4 / \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$	3	1	0	-1	- 1
$std \otimes sgn$	3	-1	0	-1	1
$std \circ \pi_{4 \to 3}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

If S_4 is the symmetries of a tetrahedron, then $\pi_{4 \to 3}$ is the map from S_4 to S_3 furnished by S_4 acting on pairs of sides, of which there are 3.
<u>How does the structure of G interact with its representation theory?</u>

Proposition 3. *(Homework)*
Let G, H be groups, (NOT necessarily finite!), $k = \overline{k}$ algebraically closed.
Then any irrep of $G \times H$ has the form $V \boxtimes W$, where

- *V is an irrep of G,*
- *W is an irrep of H*
- $(g, h) \cdot v \boxtimes w = (g \cdot v) \boxtimes (h \cdot w)$
  This is the same as tensoring the two reps of $G \times H$ we get from

$$\begin{tikzcd} & G \arrow[r] & \mathrm{GL}(V) \\ G \times H \arrow[ur] \arrow[dr] & & \\ & H \arrow[r] & \mathrm{GL}(W) \end{tikzcd}$$

Proof. HW

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We have now classified (modulo the homework) all representations of all finite abelian groups.
In some sense, (the sense of Artin's theorem) is that the representation theory of a group is controlled by the rep theory of its abelian subgroups.
<u>Restriction & induction</u>
Let $H \subseteq G$ be a subgroup of H, G again finite.
We have a restriction functor $Res_H^G : Rep_G \to Rep_H$,

$$(\rho : G \to \mathrm{GL}(W)) \mapsto \rho|_H$$

There is a functor going the other way called induction, $Ind_H^G : Rep_H \to Rep_G$.

Definition 0.6. Let V be an H-representation. Then

$$Ind_H^G V \stackrel{\text{def}}{=} k[G] \otimes_{k[H]} V$$

Equivalent descriptions:

$$Ind_H^G(V) \stackrel{\text{def}}{=} \{\phi : G \to V \mid \phi(gh^{-1}) = h\phi(g) \forall g \in G, h \in H\}$$

An element of the former looks like $\sum_g e_g \otimes v_g$. Take $e_g e_h \otimes v = e_g \otimes (h \cdot v)$, $g \cdot \phi = g\phi(g^{-1}-)$. Think about this and see how this makes the descriptions the same. One more description:

$$Ind_H^G(V) = \bigoplus_{g \in G/H} g_i \cdot V$$

where $g \cdot \sum g_i v_i = \sum g_{j(i)} k_i \cdot V$ where $g_j g_i = g_{j(i)}$ (???)
Exercise: check the above is equivalent to the other two things.

Example 0.6.

1. $Ind_H^G triv = k^{G/H}$ follows from second description. By definition, $Ind_H^G triv = \{f : G \to k \mid f(gh^{-1}) = h \cdot f(g) = f(g)\} = \{f : G/H \to k\}$
2. $Ind_{(1)}^G k = k[G] \otimes_k k = k[G]$
3. Suppose $\chi : H \to \mathbb{C}^\times$ is a representation. What is $Ind_H^G \chi$? To find $Ind_H^G \chi(g)$, pick coset representative g_i from G/H, and we get permutation matrix for $G \curvearrowright G/H$ times the diagonal matrix whose ith entry is $\chi(h_i)$, where $gh_i^{-1} = g_{j(i)}h_i^{-1}$

Lecture 6, 23/1/25

Corrections:
In the homework, problem 4 part a) should include the assumption that the action of G on H by conjugation is inner, i.e. for all $g \in G$, the map $(\cdot)^g : H \to H$ sending $h \mapsto ghg^{-1}$ is $(\cdot)^{h'}$ for some $h' \in H$.
Remark: An example is if we take $G = A \times B, H = A \times \{1\}$. Then $(\cdot)^{(a,b)} = (\cdot)^{(a,1)}$
Last time:

- We did character tables for Q_8, S_4
- We stated the classification of irreducible representations of a product $G \times H$
- Classification of irreps of finite abelian groups
- Restriction & induction

Here is more on induction:
$\mathrm{Ind}_H^G(V) \overset{\text{def}}{=} k[G] \otimes_{k[H]} V$, where $k[G]$ is a right module and V is a left one. Tensoring a right with a left yields an abelian group (indeed a k-vector space), and it all works out because $k[G]$ is a left $k[G]$ module.
It is also the set $\{\phi : G \to V \mid \phi(gh^{-1}) = h \cdot \phi(g) \text{ for all } g \in G, h \in H\}$, where

$$g \cdot \phi = \phi(g^{-1} \cdot)$$

<u>Explanation</u>
An element of $k[G] \otimes_{k[H]} V$ is a formal sum $\sum e_g \otimes v_g$ such that $e_g e_h \otimes v = e_g \otimes (h \cdot v)$
How to recognize induced representations:

- Suppose V is a G-rep, $W \subseteq V$ is H-stable. When is $V \simeq \mathrm{Ind}_H^G W$?
- Consider $gW \subseteq V$. Because W is H-stable, this only depends on $[g] \in G/H$

Proposition 4. *$V = \mathrm{Ind}_H^G W$ if and only if $V = \oplus_{g \in G/H} gW$*

Proof. Sketch
Recall the third version, $\operatorname{Ind}_H^G V = \oplus_{g_i \in G/H} g_i U$

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Proposition 5.

$$\chi_{\operatorname{Ind}_H^G \rho}(u) = \frac{1}{|H|} \sum_{g \in G, g^{-1}ug \in H} \chi_\rho(g^{-1}ug)$$
$$= \sum_{x \in G/H} \hat{\chi}_\rho(x^{-1}ux)$$

where $\hat{\chi}_\rho(v) = \begin{cases} \chi_\rho(v) & v \in H \\ 0 & \textit{otherwise} \end{cases}$

Proof. ■

Proposition 6. *Let $H \subseteq G$ be a subgroup of finite index. Then*

$$\operatorname{Hom}_G(\operatorname{Ind}_H^G V, W) \simeq \operatorname{Hom}_H(V, \operatorname{Res}_G^H W)$$

Proof. This is a special case of the tensor-hom adjunction:

$$\begin{aligned} \operatorname{Hom}_G(\operatorname{Ind}_H^G V, W) &\simeq \operatorname{Hom}_G(k[G] \otimes_{k[H]} V, W) \\ &= \operatorname{Hom}_H(V, \operatorname{Hom}_G(k[G], W)) \\ &= \operatorname{Hom}_H(V, \underbrace{W}_{\text{as an H-rep}}) \\ &= \operatorname{Hom}_H(V, \operatorname{Res}_G^H W) \end{aligned}$$

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Corollary 0.16. *Let V be a representation of H, W is a representation of G, both finite. Then*

$$\langle \chi_{\operatorname{Ind}_H^G V}, \chi_W \rangle = \langle \chi_V, \chi_{\operatorname{Res}_G^H W} \rangle$$

Proof. These numbers are the dimensions of the hom-spaces, which are the same by the above.

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Theorem 0.17 (Artin)**.** *Let G be a finite group, $k = \overline{k}, \operatorname{char} k = 0$. Then the map*

$$\bigoplus_{H \subseteq G cyclic} Cl(H) \twoheadrightarrow Cl(G)$$

For each cyclic group H, it acts on characters linearly, so we can extend that to $Cl(H)$, and we can extend that to $\oplus Cl(H)$

Proof. Remark: Let G be a finite group, $R(G)$ be the "representation ring of G",

$$R(G) = \oplus_{\rho_i \text{ irreps of } G}\mathbb{Z}[\rho_i]$$

with $[\rho_i] \cdot [\rho_j] = [\rho_i \otimes \rho_j]$, by writing $\rho_i \otimes \rho_j = \bigoplus_{\rho_k \text{ irreps}} \rho_k^{n_k}$

Proposition 7. *There is a map $R(G) \to Cl(G)$ sending $[\rho_i] \to \chi_{\rho_i}$. This is a ring homomorphism (because character of tensor product is pointwise product of characters). There is an induced map $R(G) \otimes_{\mathbb{Z}} k \to Cl(G)$ which is an isomorphism.*

Proof.

1. These are vector spaces of the same dimension

2. The map is surjective because (for example,) characters of irreps span.

■

Corollary 0.18 (to Artin's theorem)**.** *The map (linear extension of $\oplus \operatorname{Ind}_H^G$)*

$$\bigoplus_{H \leq G \text{ cyclic}} R(H)_k \to R(G)_k$$

is surjective.
I.e. every representation of G is a "k-linear combo" of irreps induced from cyclic subgroups.

Corollary 0.19.

1. $\oplus_{H\leq G \text{ cyclic}} R(H)_{\mathbb{Q}} \to R(G)_{\mathbb{Q}}$ *is surjective, i.e. every irreducible character of G is a $\mathbb{Q}$-linear combination of characters induced from cyclic subgroups.*

2. $\oplus_{H\leq Q \text{ cyclic}} R(H) \to R(G)$ *has finite cokernel.*

Proof.

- (1) $\implies$ (2) because the image of Ind spans $R(G)$ rationally by (1), i.e. given $x \in R(G)$, there is N such that $N \cdot x \in \operatorname{Im}(\operatorname{Ind})$, so the cokernel is torsion, and torsion finitely generated abelian groups are finite.

- We know (1) by Artin, because $\operatorname{Ind}_{\mathbb{Q}} \otimes_{\mathbb{Q}} k$ is surjective, as rank r invariant under extension of scalars?

■

We now prove Artin's theorem:

Proof. It is enough to show that the adjoint map of $\oplus \operatorname{Ind}_H^G$ is injective. But $\langle \operatorname{Ind} \chi, \psi \rangle = \langle \chi, \operatorname{Res} \psi \rangle$, so

$$\bigoplus \operatorname{Res}_G^H : Cl(G) \to \bigoplus_{H \leq G \text{ cyclic}} Cl(H)$$

is adjoint to Ind. Now let ψ be in the kernel; then $\operatorname{Res}_G^H \psi \equiv 0$ for all H, which implies $\psi \equiv 0$, so we win.

∎

Loose ends:

- Structure of $k[G]$
- Integral theory
- Corollary of all this discussion: if G is a finite group, ρ an irrep, then $\dim \rho \mid |G|$

Structure of $k[G]$ (and more generally, semisimple algebras)

Definition 0.7. Let k be a field, R a k-algebra (possibly non-commutative). Then R is semisimple if

1. R is finite dimensional as a k-vector space

2. All left R-modules which are finite-dimensional k-vector spaces are semisimple.

Theorem 0.20. *Let R be semisimple k-algebra. Then*

$$R \simeq \prod \operatorname{Mat}_{n_i}(D_i)$$

where D_i are division k-algebras.

Proof. (Take $R = k[G]$)
Conside R as a left R-module;

$$R \simeq \oplus M_i^{\oplus n_i}$$

where M_i is simple, all M_js are mutually non-isomorphic left R-modules.
Note $\operatorname{Hom}_{R-\text{mod}}(M_i, M_i)$ is a division algebra (otherwise we would have a morphism with a kernel, but M_i is simple).
Because $R^{op} \simeq \operatorname{Hom}_{R-\text{mod}}(R, R)$, this means

$$R \simeq \operatorname{Hom}_{R-\text{mod}}(\oplus M_i^{\oplus n_i}, \oplus M_i^{\oplus n_i})$$

Now, $\mathrm{Hom}_{R-\mathrm{mod}}(M_i, M_j) = 0$ for $i \neq j$ (again by simplicity and mutual nonisomorphicness) so

$$\mathrm{Hom}_{R-\mathrm{mod}}(R, R) \simeq \oplus_i \mathrm{Hom}_{R-\mathrm{mod}}(M_i^{n_i}, M_i^{n_i})$$

So if we take $D_i^{op} = \mathrm{Mat}_{n_i}(\mathrm{Hom}(M_i, M_i))$, we win.

■

Corollary 0.21. *Let $k = \overline{k}$. Then $R \simeq \oplus \mathrm{Mat}_{n_i}(k)$*

Proof.

1. Finite dimensional central division algebras over an algebraically closed field are the field itself.

2. Or, same proof as in Schur,

$$\mathrm{Hom}_{R-\mathrm{mod}}(M_i, M_i) = k$$

■

Let's specialize to $R = k[G]$.
As a $k[G]$-module, $k[G] \simeq \rho_i^{\oplus n_i}$, so we have a map

$$k[g] \to \bigoplus_{\rho_i \text{ irrep}} \underline{\mathrm{Hom}}_k(\rho_i, \rho_i) \simeq \oplus_{\rho_i \text{ irrep}} \rho_i \boxtimes \rho_i^v \simeq \oplus_{\rho_i \text{ irrep}} \rho_i \otimes \mathrm{Hom}(\rho_i, k[G])$$

$$x \mapsto \text{right multiplication by } x$$

Recall: If V is any G-rep, then $V = \oplus \rho_i \otimes \mathrm{Hom}_G(\rho_i, V)$,
so we have $k[G] \to \oplus \mathrm{End}(\mathrm{Hom}(\rho_i, k[G]))$

Claim. *This isomorphism of rings is $G \times G$-equivariant if we give* $\mathrm{End}(\rho_i^{\dim \rho_i})$ *the $G \times G$ structure $\rho_i \boxtimes \rho_i^v$*

Proof. We need to check $\mathrm{End}(\rho_i^{\dim_i})$ as a right G-module it is $(\rho_i^v)^{\dim \rho_i}$.
If $G \hookrightarrow G \times G$ by $g \mapsto (g, g^{-1})$, then it has an invariant in $\mathrm{Hom}_G(\rho_i^{\dim \rho_i}, \rho_i^{\dim \rho_i})$,
As G-reps, $\mathrm{Hom}(\rho_i, \rho_i) \simeq \rho_i \otimes \rho_i^v$

Claim. *Given a rep $V \boxtimes W$ of $G \times G$, the structure of V and $V \boxtimes W|_{(g,g^{-1})}$ determines W.*

Proof. ■

Lecture 7, 28/1/25

Substitute for today: Dr Jacob Tsimerman
Let $k = \overline{k}$ be an algebraically closed field of characteristic 0, G a finite group.
Let $(\rho_1, V_1), \ldots, (\rho_n, V_n)$ be the irreducible left representations of G.

Theorem 0.22.

$$k[G] \cong \bigoplus_{i=1}^{n} \rho_i \boxtimes \rho_i^v = \bigoplus_{i=1}^{n} V_i \otimes V_i^*$$

as $G \times G$-reps $((g, g') \cdot v \otimes v^* = (g \cdot v) \otimes v^* + v \otimes (g' \cdot v^*))$

Proof. Let $W_i \stackrel{\text{def}}{=} \text{Hom}_G(V_i, k[G])$. Then

$$k[G] \cong \bigoplus_{i=1}^{n} V_i \otimes W_i$$

as $G \times G$-representations because we get the right G-action for free.

Claim. *As right G-representations, $W_i \cong V_i^*$*

Proof.
<u>Convention:</u> Given an element $x = \sum_{g \in G} a_g(x) g \in k[G]$, we use $a_g : k[G] \to k$ to denote the g-th coefficient.
This has the property that $a_g(x \cdot g') = a_{g'g^{-1}}(x)$
Define $\psi : W_i \to V_i^*$ by

$$\psi(\phi) \stackrel{\text{def}}{=} a_1 \circ \phi$$

Claim. *ψ is an isomorphism*

Proof. Suppose $\phi \in W_i$. For $g \in G$, $a_g(\phi(v)) = a_1(g^{-1}\phi(v))$. But ϕ is a map of left G-modules, so this is $a_1(\phi(g^{-1}(v))) = \psi(\phi)(g^{-1}v)$.
So, we can write

$$\phi(v) = \sum_{g \in G} \psi(\phi)(g^{-1}v) \cdot v$$

So ϕ is entirely determined by $\psi(\phi)$, or in other words, ψ is injective.
On the other hand, let $\ell \in V^*$.
Consider $\phi_\ell \in W_i$, $\phi_\ell(v) = \sum_{g \in G} \ell(g^{-1}v) \cdot g$

Claim. $\phi_\ell \in W_i$

Proof. Let $g_0 \in G$. Then

$$\phi_\ell(g_0 v) = \sum_{g \in G} \ell(g^{-1} g_0 v) = \sum_{g \in G} \ell(g^{-1}v) \cdot g_0 g = g_0 \cdot \phi_\ell(v)$$

This shows that ψ is surjective. ■

Claim. *ψ respects the right G-action.*

Proof.

$$\begin{aligned}\psi(\phi^{g_0})(v) &= \psi(\phi)(g_0 v)\\ &= a_1(\phi(g_0 v))\\ &= a_1(g_0\phi(v))\\ &= a_{g_0^{-1}}(\phi(v))\end{aligned}$$

On the other hand,

$$\begin{aligned}\psi(\phi^{g_0} v) &= a_1(\phi^{g_0}(v))\\ &= a_1(\phi(v)g_0)\\ &= a_{g_0^{-1}}(\phi(v))\end{aligned}$$

So $\psi(\phi^{g_0}) = \psi(\phi)^{g_0}$ ■
This proves the theorem. ■
Matrix Coefficients
Let $\{v_1, \ldots, v_n\}$ be a basis for an irreducible representation V.
Let $\{v_1^*, \ldots, v_n^*\}$ be the dual basis for V^*.

Definition 0.8. Given $1 \le i, j \le m$, the matrix coefficient $a_{i,j}$ is given by

$$a_{i,j}(g) = v_i^*(g \cdot v_j)$$

This is a function from G to k.
Define $A_{i,j} \in k[G]$ by

$$A_{i,j} \overset{\text{def}}{=} \sum_{g\in G} a_{i,j}(g) \cdot g$$

Theorem 0.23.

$$\langle A_{i,j}\rangle_{1\le i,j\le m} = \rho \boxtimes \rho^v$$

where (ρ, V) is the G-rep.

Proof. ■

Theorem 0.24. *Let G be a finite group, $k = \overline{k}$ an algebraically closed field of characteristic 0.*
Let (ρ, V) be an irreducible representation of G.
Then $\dim V \mid |G|$

Proof.

Corollary 0.25. *If $d_1, \ldots, d_n$ is the dimensions of the irreps of G, then*

1. *m =number of conjugacy classes of G (often called m)*

2. *$d_i || G|$ for all i*

3. $\sum_{i=1}^m d_i^2 = |G|$

Proof. ■

Example 0.7. If $G = S_3$, $m = 3$, with conjugacy classes $[\mathrm{Id}], [(12)], [(123)]$, then we have $d_1 = 1, 1 + d_2^2 + d_3^2 = 6, d_2, d_3 \mid 6$.
So we must have $d_2 = 1, d_3 = 2$.
Recollections of algebraic integers

Definition 0.9. Let R be a commutative ring.
Then $x \in R$ is integral, or an algebraic integer, if x satisfies a monic integer polynomial.

Example 0.8.

- 3
- $\sqrt{5}$
- $\frac{1+\sqrt{5}}{2}$

Non-examples include

- $\frac{3}{7}$
- $\frac{1}{\sqrt{2}}$

Proposition 8. *The following are equivalent:*

1. *x is integral*

2. *The subring generated by x is a finitely generated $\mathbb{Z}$-module*

3. *The subring generated by x is contained in a finitely generated $\mathbb{Z}$-module in R.*

Proof. Let's start with (1) $\implies$ (2).
Suppose $x^N + \sum_{i=1}^{N-1} a_i x^i = 0$, $a_i \in \mathbb{Z}$.
Then $x^N \in \langle 1, x, \ldots, x^{N-1}\rangle_{\mathbb{Z}}$. But then $x^{N+1} \in \langle 1, x, \ldots, x^N\rangle_{\mathbb{Z}}$, so $x^{N+1} \in \langle 1, x, \ldots, x^{N-1}\rangle_{\mathbb{Z}}$.
So the subring generated by x equals $\langle 1, x, \ldots, x^{N-1}\rangle_{\mathbb{Z}}$.
(2) $\implies$ (3) is clear
So let's see (3) $\implies$ (1).

Let $A_N = \langle 1, x, x^{N-1} \rangle_{\mathbb{Z}}$. By assumption, there exists a finitely generated $\mathbb{Z}$-module $B \subset R$ such that $A_1 \subseteq A_2 \subseteq \cdots \subseteq B$
By Noetheriality, the sequence stabilizes, so there exists some M such that $A_M = A_{M-1}$, and so x^M is a finite linear combination of lower powers of x, so there are a_i such that

$$x^M + \sum_{i=1}^{M-1} a_i x^i = 0$$

Corollary 0.26. *The things on the list of non algebraic integers actually belong on the list!*

Proof. ■

Lecture 8, 30/1/25

Sub Prof: Mathilde Gerbelli-Gauthier
End Goal: G finite, ρ irrep of G over $k = \overline{k}$ algebraically closed of characteristic 0.
We want to show that $\dim \rho \mid |G|$
Strategy: Prove that $\frac{|G|}{\dim \rho}$ is an algebraic integer
As a corollary of the proof of the last prop, we get

Corollary 0.27. *Integral elements of R form a subring.*

Proof. ■

Integrality of characters

As always, let G be a finite group, $k = \overline{k}$ algebraically closed of characteristic 0, and $\rho : G \to \mathrm{GL}_n(k)$ just any representation (not necessarily irreducible).

Proposition 9.

1. *The values of the character of $\rho, \chi_\rho(g)$, are algebraic integers*
2. *Let $u = \sum_{g \in G} u(g)g$ be an element of $Z(k[G])$. Suppose that $u(g) \in k$ are algebraic integers. Then u is integral.*

At some point in the classes I missed we show that the indicators of conjugacy classes span the center of $k[G]$.

Proof.

1. $\chi_\rho(g)$ is a sum of roots of unity, hence a sum of algebraic integers, hence an algebraic integer.

2. Using a previous result, let $u(g)$ be the indicator function of a conjugacy class. But the sub-$\mathbb{Z}$-module of $Z(k[G])$ generated by the indicator functions is a sub-ring (because the product of $1_{C_1} \cdot 1_{C_2}$ is a linear combination of the indicators of conjugacy classes, and the coefficient in front of each g is an integer).

Thus each indicator of a conjugacy class is contained in a finitely generated $\mathbb{Z}$-module, and is integral.

$\blacksquare$

Corollary 0.28. *Let ρ be an irrep of G and let $u \in Z(k[G])$ be as before. Then*

$$u_\rho = \frac{1}{\dim \rho} \sum_{g \in G} u(g) \chi_\rho(g) \in k$$

is an algebraic integer.

Proof.

Claim. *Given ρ, $u \mapsto \frac{1}{\dim \rho} \sum u(g)\chi_\rho(g)$ is a ring homomorphism*

Proof.

$$u_1 * u_2 \mapsto \left(\frac{1}{\dim \rho} \sum u_1(g) \chi_\rho(g) \right) \left(\frac{1}{\dim \rho} \sum u_2(g) \chi_\rho(g) \right)$$

$\blacksquare$

The goal will be to define a ring-hom from $Z(k[G])$ to k sending u to u_ρ. Since u is integral , it maps to to an integral element of k.

$$u \mapsto \frac{|G|}{\dim \rho} \langle u, \chi_{\rho^v} \rangle = u_\rho$$

$$\sum u'(g) \chi_\rho(g) = |G| \langle u, \rho^v \rangle$$

Recall that $Z(k[G]) \curvearrowright \rho$ by G-homomorphism, that action induces a natural map

$$Z(k[G]) \mapsto \mathrm{Hom}_G(\rho, \rho) = k$$

So

$$u \mapsto \frac{|G|}{\dim \rho} \langle u, \chi_{\rho^v} \rangle$$

The matrix is scalar, so it suffices to compute its trace. Its trace is

$$\sum_{g \in G} u(g) \chi_\rho(g) = |G| \langle u, \chi_{\rho^v} \rangle$$

Dividing by $\dim \rho$ gives the result.

$\blacksquare$?

Theorem 0.29. *Let G be a finite group, $k = \overline{k}$ an algebraically closed field of characteristic 0, V_ρ an irrep of G. Then* $\dim V \mid \|G|$

Proof. Set $u = \sum_{g\in G} \chi_\rho(g^{-1})g$. By the above, we have

$$\begin{aligned}\frac{1}{\dim\rho}\sum u(g)\chi_\rho(g) &= \frac{|G|}{\dim\rho}\langle\chi_{\rho^v},\chi_{\rho^v}\rangle\\ &= \frac{|G|}{\dim\rho}\underbrace{\dim\mathrm{Hom}_G(\rho^v,\rho^v)}_{=1}\\ &= \frac{|G|}{\dim\rho}\end{aligned}$$

But the left hand side is an integral element of $\mathbb{Q}$, so the right hand side is an integral element of $\mathbb{Q}$, hence an integer.

■

Rep theory of the symmetric group

As always, $|G| < \infty, Char(k = \overline{k}) = 0$
Here are some key facts about the symmetric groups:

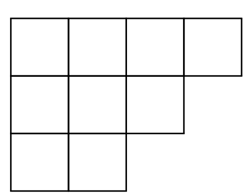
1. The number of irreps of S_n is equal to the number of conjugacy classes in S_n.

2. The conjugacy classes in S_n (aka cycle type) are in bijection with partitions of n.

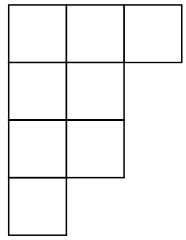
3. The irreps of S_n are also indexed by partitions of n.

Definition 0.10. A partition of n is a sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r)$ such that $\sum \lambda_i = n$.

Definition 0.11. The young diagram D_λ has λ_1 boxes in the first row, λ_2 in the second row, etc.
For example, the corresponding diagram for $\lambda = (4,3,1)$



The conjugate partition λ' is the one such that $D_{\lambda'}$ is obtained by D_λ by flipping along the diagonal.
If $\lambda = (4,3,1), \lambda' = (3,2,2,1)$. Then $D_{\lambda'}$ is



Proejctions and young symmetrizers

An algorithm: start with λ

1. Number the booxes in your Young diagram D_λ from left to right, top to bottom: you now have a young tableaux.

1	2	3	4
5	6	7	
8			

2. Let $\cdot P \subseteq S_n$ be the subgroup of all permutations that preserve each row of our Young tableaux. E.g. $P \simeq S_4 \times S_3 \hookrightarrow S_8$.

3. $Q \subseteq S_n$ the subgroup that preserves each column of the same Young tableau e.g. $Q \simeq S_3 \times S_2 \times S_2 \hookrightarrow S_8$.

In $\mathbb{C}[S_n]$, define $a = \sum_{p \in P} e_p, b = \sum_{q \in Q} sgn(q) e_q$

4. Suppose that V is a vector space, and $S_n \curvearrowright V^{\otimes n}$ by permuting factors.

The element a symmetrizes along the rows, and projects onto

$$Sym^{\lambda_1}(V) \otimes \cdots \otimes Sym^{\lambda_n}(V)$$

up to an isomorphism.

5. The element b alternates along the columns and projects onto a tensor product of exterior powers indexed by λ':

$$\bigwedge^{\lambda_1'}(V) \otimes \cdots \otimes \bigwedge^{\lambda_n'}(V)$$

6. Set $c = ab$. This is called the Young Symmetrizer

Here are some examples of Young symmetrizers: If $\lambda = (1, \dots, 1)$, then c gives the sign representation. $\lambda = (n)$ gives the trivial rep.

Irreducibility and idempotency

Theorem 0.30. *A suitable nonzero scalar of $c = ab$ is an idempotent in $\mathbb{C}[S_n]$. Its image, when acting on the regular representation, is irreducible, and denoted V_λ. Distinct partitions give rise to distinct (meaning nonisomorphic) representations and every irep arises from this process for a unique partition.*

Corollary 0.31. *Every representation of S_n is defined over $\mathbb{Q}$.*

Proof. ■

Example 0.9.

- For S_3, $triv = (4), sgn = (1,1,1), std = (2,1)$
- For S_4, $triv = (4), sgn = (1,1,1,1), std = (3,1), std \otimes sgn = (2,1,1), S_4 \to S_3 = (2,2)$
- In general, $(d, 1, \cdots, 1)$ corresponds to various exterior powers of the standard representation.

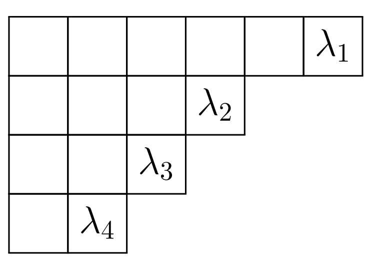
Theorem 0.32. *(Hook-length formula)*
Label each box b in a young diagram (boxes to the right of b) + (boxes below). These are called hook lengths. Then $\dim V_\lambda = \frac{n!}{\prod(\text{hook lengths of } b)}$

Proof. ■

Lecture 9, 4/2/25

Let $n \in \mathbb{Z}_{>0}$. Our goal is to classify irreps of S_n. Recall:

Theorem 0.33. *For each partition λ of n, there exists a unique isomorphism class of irrep V_λ of S_n, constructed as follows:*



where $\sum \lambda_i = n$ We let R be the subgroup of S_n which preserves the rows, Q the subgroup preserving the columns. We set

$$a \stackrel{\text{def}}{=} \sum_{g \in P} e_g \in \mathbb{C}[S_n]$$

$$b \stackrel{\text{def}}{=} \sum_{g \in Q} sgn(g) e_g \in \mathbb{C}[S_n]$$

$$c = ab$$

Then $V_\lambda \stackrel{\text{def}}{=} \mathbb{C}[S_n]c$ *is an irrep of* S_n.
Further, every irrep arises in this way.

Proof. Summary: WTS

1. $\dim \mathrm{Hom}_G(V_\lambda, V_\mu) = \delta_{\mu\lambda}$

2. Any irrep is some V_λ.

Remark:

1. There is an explicit dimension formula, the hook-length formula

2. There is an explicit formula for the character of V_λ due to Frobenius.

For more, look for Etingof's "Representation theory" notes for a course given at MIT.
We will begin the proof by writing down c_λ.

Lemma 1.

$$c_\lambda = \sum_{g = \underbrace{p}_{\in P_\lambda} \underbrace{q}_{\in Q_\lambda}} sgn(q) e_{pq}$$

Proof.

$$\begin{aligned} a_\lambda b_\lambda &= \left(\sum_{g \in P_\lambda} e_g \right) \cdot \left(\sum_{h \in Q_\lambda} sgn(h) e_h \right) \\ &= \sum_{g \in P_\lambda, h \in Q_\lambda} sgn(h) \underbrace{e_g e_h}_{e_{gh}} \end{aligned}$$

∎

Goal: Compute $c_\lambda^2 = a_\lambda b_\lambda a_\lambda b_\lambda$

Lemma 2. *For all* $x \in \mathbb{C}[S_n]$, $a_\lambda x b_\lambda = \ell_\lambda(x) c_\lambda$, *where* $\ell_\lambda : \mathbb{C}[S_n] \to \mathbb{C}$ *is some linear map.*

Corollary 0.34. $c_\lambda^2 = \ell_\lambda(b_\lambda a_\lambda) c_\lambda$

Proof. Check this on each $e_g \in \mathbb{C}[S_n]$, $g \in S_n$.

Case 1 $g \in P_\lambda Q_\lambda$

We have $g = pq, e_g = e_p e_q$.

$$\begin{aligned}
a_\lambda e_g b_\lambda &= \left(\sum_{h \in P_\lambda} e_h\right) e_g \left(\sum_{u \in Q_\lambda} sgn(u) e_u\right) \\
&= \underbrace{\left(\sum_{h \in P_\lambda} e_h e_p\right)}_{a_\lambda} \underbrace{\sum_{u \in Q_\lambda} sgn(u) e_q e_u}_{sgn(q) b_\lambda} \\
&= sgn(q) c_\lambda b_\lambda \\
&= sgn(q) c_\lambda
\end{aligned}$$

Case 2 $g \notin P_\lambda Q_\lambda$

In this case, $a_\lambda e_g b_\lambda = 0$. To see this, it is enough to show that there exists a transposition $t \in P_\lambda$ such that $g^{-1} t g \in Q_\lambda$, i.e. g sends two elements of $\{1, \ldots, n\}$ in the same row of the Young diagram for λ, to two elements of the same column.

It is enough to show this because

$$\begin{aligned}
a_\lambda g b_\lambda &= a_\lambda t g b_\lambda \\
&= a_\lambda g \overbrace{\left(g^{-1} t g\right)}^{sgn=-1} b_\lambda \\
&= -a_\lambda g b_\lambda
\end{aligned}$$

This implies $a_\lambda g b_\lambda = 0$.

Now, suppose there do not exist 2 elements in the same row of λ sent to the same column of λ by g.

Then $g \in P_\lambda Q_\lambda$.

To see this, let T be the standard Young Tableau for $\lambda, T' = gT$, P' the stabilizer of rows of T', Q' the stabilizers of columns.

(i) By assumption, any two numbers in the first row of T lie in different columns of T'.

(ii) Then there exists $q_1' \in Q'$ such that $q'T'$ has the same elements in first row (perhaps in a different order).

(iii) Choose $p_1' \in P_\lambda$ such that $p_1' q_1' T'$ has the first row as T.

(iv) Likewise with the 2nd row and so on.

■

Corollary 0.35.

$$\ell_\lambda(b_\lambda a_\lambda) = \frac{n!}{\dim V_\lambda}$$

Proof. later

■

Lecture 10, 6/2/25

Note: For the finite group stuff we are using "Linear reps of finite groups" by Serre (first 3rd is for chemists apparently which is amusing). Specifically chapters 1-3, 6, 9
Other stuff is also on the quercus.
To finish the proof of the theorem, we have to show that the V_λ are irreducible and mutually non-isomorphic. Then, from a bijection between conjugacy classes and partitions, we will be done.
Last time we showed that $a_\lambda x b_\lambda = \ell_\lambda(x)c_\lambda$, and its corrolary, that $c_\lambda^2 = \ell_\lambda(b_\lambda a_\lambda)c_\lambda$

Corollary 0.36.

$$\ell_\lambda(b_\lambda a_\lambda) = \frac{n!}{\dim V_\lambda}$$

Proof. We know that $c_\lambda = \alpha \cdot p_\lambda$, where p_λ is an idempotent.

$$\begin{aligned} c_\lambda^2 &= \alpha^2 p_\lambda^2 \\ &= \alpha^2 p_\lambda \\ &= \alpha c_\lambda \end{aligned}$$

So $\alpha = \ell_\lambda(b_\lambda a_\lambda)$ so we calculate the trace of c_λ : Trace of an idempotent is dim of its image, and c_λ has the same image as p_λ

$$\begin{aligned} \operatorname{tr}(c_\lambda) &= \alpha \cdot \dim \operatorname{Im}(c_\lambda) \\ &= \ell(b_\lambda a_\lambda) \cdot \dim \operatorname{Im}(c_\lambda) \\ &= \ell(b_\lambda a_\lambda) \cdot \dim V_\lambda \end{aligned}$$

Now, if this number is not zero, then we get an idempotent by dividing c_λ by this number. We calculate

$$\begin{aligned} \operatorname{tr}(c_\lambda) &= \sum_{pq \in P_\lambda Q_\lambda} \operatorname{tr}(\cdot e_{pg}) sgn(q) \\ &= \operatorname{tr}(\cdot \operatorname{Id}) \\ &= n! \end{aligned}$$

■

Goal: Compute $\dim_{\mathbb{C}} \operatorname{Hom}_{S_n}(V_\lambda, V_\mu) = \begin{cases} 1 & \lambda = \mu \\ 0 & \text{otherwise} \end{cases}$

We know $\operatorname{Hom}_{S_n}(V_\lambda, V_\mu) = \operatorname{Hom}_{S_n}(\mathbb{C}[S_n]c_\lambda, \mathbb{C}[S_n]c_\mu)$

Proposition 10. *Let A be a $\mathbb{C}$-algebra, $e \in A$ an idempotent, M an A-module. Then $\operatorname{Hom}_A(Ae, M) \simeq eM$ naturally.*

Proof. For $x \in eM$, we have a morphism $x \mapsto (a \mapsto ax)$, and $f \mapsto f(e)$.
e is an idempotent, so $1 - e$ is also an idempotent, so $1 = e + (1 - e)$, so $A \simeq Ae \oplus A(1 - e)$, so $\operatorname{Hom}(Ae, M) \simeq \operatorname{Hom}(A/A(1 - e), M) = \{f : A \to M \mid f(e) = f(1)\} = \{x \in M \mid x \in eM\} = eM$

■

Now let's prove the main theorem.

Proposition 11.

$$\dim_C \operatorname{Hom}_{S_n}(V_\lambda, V_\lambda) = 1$$

Thus, V_λ is irreducible

Proof.

$$\begin{aligned} \operatorname{Hom}_{S_n}(V_\lambda, V_\lambda) &= c_\lambda \mathbb{C}[S_n] c_\lambda \\ &\subseteq a_\lambda \mathbb{C}[S_n] b_\lambda \\ &\subseteq \operatorname{span}_{\mathbb{C}}(c_\lambda) \end{aligned}$$

So the dimension is at most 1. To see it is exactly 1, this space has $c_\lambda \cdot 1 \cdot c_\lambda \neq 0$
So $\dim = 1$, so V_λ is irreducible.

■

Now let λ, μ be two partitions of n. Sat $\lambda > \mu$ if the first $\lambda_i \neq \mu_i$ has $\lambda_i > \mu_i$, i.e. the lexicographical ordering. This is a total ordering, i.e. for any pair (λ, μ), exactly one of $\lambda = \mu, \lambda > \mu, \lambda < \mu$ is true.

Proposition 12. *If $\lambda > \mu$, then $a_\lambda \mathbb{C}[S_n] b_\mu = 0$.*

Proof. In a bit

■

Assuming this, then, if $\lambda \neq \mu$, we want to show that $\dim \operatorname{Hom}_{S_n}(V_\lambda, V_\mu) = 0$.

Proof. We have

$$\begin{aligned} \operatorname{Hom}_{S_n}(V_\lambda, V_\lambda) &= c_\lambda \mathbb{C}[S_n] c_\mu \\ &= a_\lambda b_\lambda \mathbb{C}[S_n] a_\mu b_\mu \\ &\subseteq a_\lambda \mathbb{C}[S_n] b_\mu \\ &= 0 \end{aligned}$$

if $\lambda > \mu$. But $\dim \operatorname{Hom}_{S_n}(V_\lambda, V_\mu) = \dim \operatorname{Hom}_{S_n}(V_\mu, V_\lambda)$, so one, hence both, are 0. ■

Now we prove the proposition

Proof. We will verify it on $e_g \in \mathbb{C}[S_n]$.

Claim. *There exist two numbers on the same row of the standard Young tableaux for λ, same column for $g \cdot$ (standard Young tableaux of μ)*

Proof. Homework ■

Example 0.10. If $g = \mathrm{Id}, \lambda_1 > \mu_1$,

$$\begin{ytableau} 1 & 2 & 3 & 4 \\ \; & \; & \; \\ \; & \; \end{ytableau}\ ,\ \begin{ytableau} 1 & 2 & 3 \\ 4 & 5 \\ \; & \; \end{ytableau}$$

Let t be the transpotion for these two numbers. Then

$$\begin{aligned} a_\lambda g b_\lambda &= c_\lambda t g b_\mu \\ &= a_\lambda g g^{-1} t g b_\lambda \\ &= -a_\lambda g b_\mu \end{aligned}$$

The rep theory of $\mathrm{GL}_2(\mathbb{F}_p)$

<u>Goal:</u> Understand the irreps of $\mathrm{GL}_n(\mathbb{F}_q)$
What is the size of this group?
$|\mathrm{GL}_2(\mathbb{F}_q)| = (q^2-1)(q^2-q) = q(q^2-1)(q-1)$

Proof.

$$\mathrm{GL}_2(\mathbb{F}_q) = \{(v,w) \mid v, w \in (\mathbb{F}_q)^2 \text{ linearly independent } \}$$

So we can pick any v a nonzero vector, and any w not in the span of v. The number of such possible choices is $(q^2-1)(q^2-q)$ ■

<u>Conjugacy classes:</u>
What are the conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_q)$?

Conjugacy class	number of such conjugacy classes	size of each
$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, x \in \mathbb{F}_q^\times$	$q-1$	1
$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, x \neq y \in \mathbb{F}_q^\times$	$\frac{(q-1)(q-2)}{2}$	$q(q+1)$
$\begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}, x \in \mathbb{F}_q^\times$	$q-1$	q^2-1
$\begin{pmatrix} x & \epsilon y \\ y & x \end{pmatrix}$, ϵ a generator of $\mathbb{F}_q^\times$	$\frac{q(q-1)}{2}$	q^2-q

For the last one, $char \neq 2$

What are the reps of $\mathrm{GL}_2(\mathbb{F}_q)$ over $\mathbb{C}$? Besides the trivial one, we also have $P^1(\mathbb{F}_1) = \{1-dim \text{ subspaces of } \mathbb{F}_q^2\}$.

This gives a permutation representation $\mathbb{C}^{P^1(\mathbb{F}_q)}$.

We have $std = \mathbb{C}^{P^1(\mathbb{F}_q)}/\mathbb{C}$ has dimension q. Let's compute the character of this representation. Let's call the first set of conjugacy classes above z_x, the second $d_{x,y}$, u_x, $t_{x,y}$

	z_x	$d_{x,y}$	u_x	$t_{x,y}$
triv	1	1	1	1
std	q	1	0	-1

We have

$$\begin{aligned}\langle std, std\rangle &= \frac{1}{q(q-1)^2(q+1)}\left((q-1)q^2 + \frac{q(q-1)(q-2)(q+1)}{2} + 0 + \frac{q^2(q-1)^2}{2}\right) \\ &= 1\end{aligned}$$

What other representation are there?

Choose $\chi : \mathbb{F}_q^\times \to \mathbb{C}$, and then $(\chi \circ \det)^n$, for $n = 1, \ldots, q-2$.

To construct more reps, we will examine some induces reps.

Definition 0.12. Let $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subseteq \mathrm{GL}_2(\mathbb{F}_q)$ (B is for Borel)

$|B| = q(q-1)^2$. Let U be all the matrices of the form $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$.

What is B/U? It is $\mathbb{F}_q^\times \times \mathbb{F}_q^\times$. We will take reps of this and view them as reps of B via the quotient map and induced reps.

For each $\psi : \mathbb{F}_q^\times \times \mathbb{F}_q^\times \to \mathbb{C}^\times$, we can consider the induction $\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{F}_1)}(\psi|_B)$. These are indexed by $\psi(\epsilon, 1)$ and $\psi(1, \epsilon)$.

Then $\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{F}_q)}(\psi_{a,b}|_B)$ has dimension $q+1$ and has character $(q+1)\psi(x)^2$

For $d_{x,y}$ we have $\psi(x,1) + \psi(1,x) + \psi(y,1) \cdot \psi(1,y)$

I have kind of lost the plot at this point I'm sorry.

Proposition 13. *Let $\chi = \sum n_i \rho_i \in R(G)$ be a virtual character of a finite group G. Then χ is the character of an honest irrep iff $\langle \chi, \chi \rangle = 1$, and $\chi(1) > 0$.*

Proof. If we write $\chi = \sum n_i' \rho_i'$, where ρ_i' are irreps, then $\langle \chi, \chi \rangle = \sum_i (n_i)^2 = 1$ by assumption. So at most one of the n_i are nonzero, and it must be ± 1. So $\chi = \pm \rho$ for some irrep ρ. If $\chi(1) > 0$, then $\chi(1) = \pm \dim \rho > 0$