Homework 2 : Convex Optimization

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1. $\Rightarrow C = \{ \underline{x} \in R^n \mid A_0 - (\sum_{i=1}^{n} x_i A_i) \succeq 0 \}$

$\Rightarrow A_i$ are all ~~as~~ symmetric matrices of size m

$\Rightarrow$ For C to be a convex set

if $\underline{x_1}, \underline{x_2} \in C$, $\theta \in [0, 1]$,

$\theta \underline{x_1} + (1-\theta) \underline{x_2} \in C$.

$\Rightarrow \underline{x_1} = [(x_1)_1, (x_1)_2 \; - - - \; (x_1)_n] \in C$

$\underline{x_2} = [(x_2)_1, (x_2)_2 \; - - - \; (x_2)_n] \in C$

$\Rightarrow A_0 - \sum_{i=1}^{n} (x_1)_i A_i \succeq 0$

$A_0 - \sum_{i=1}^{n} (x_2)_i A_i \succeq 0$

$$\Rightarrow A_0 - \sum_{i=1}^{n} \left(\theta(x_1)_i + (1-\theta)(x_2)_i\right) A_i$$

$$= \theta\left[A_0 - \sum_{i=1}^{n} (x_1)_i A_i\right] + (1-\theta)\left[A_0 - \sum_{i=1}^{n} (x_2)_i A_i\right]$$

$$= \theta M + (1-\theta) N$$

where $M = A_0 - \sum_{i=1}^{n} (x_1)_i A_i \in S_+^n$

$N = A_0 - \sum_{i=1}^{n} (x_2)_i A_i \in S_+^n$

M, N are PSD

$\Rightarrow M, N \in S_+^n$, $\theta \in [0, 1]$

$\theta M + (1-\theta) N \in S_+^n$ i.e. $\theta M + (1-\theta) N \succeq 0$

$M \in S_+^n \longrightarrow \underline{z}^T M \underline{z} \geq 0$

$N \in S_+^n \longrightarrow \underline{z}^T N \underline{z} \geq 0$

$\underline{z}^T (\theta M + (1-\theta) N) \underline{z} = \theta \underline{z}^T M \underline{z} + (1-\theta) \underline{z}^T N \underline{z}$

$\Rightarrow \underline{z}^T (\theta M + (1-\theta) N) \underline{z} \geq 0$

$\theta M + (1-\theta) N \in S_+^n$

$\Rightarrow A_0 - \sum_{i=1}^{n} \left(\theta(x_1)_i + (1-\theta)(x_2)_i\right) A_i \succeq 0$

$\Rightarrow \theta \underline{x}_1 + (1-\theta) \underline{x}_2 \in C$

$\Rightarrow$ C is convex

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2. $\Rightarrow$ P = set of probability density functions

$\Rightarrow P = \{ f : \Omega \to \mathbb{R}$ (measurable) $\mid f(x) \geq 0 \ \forall x \in \Omega$

and $\int_\Omega f(x)\, dx = 1 \}$

$\Rightarrow$ Take any $f_1, f_2 \in P$, $\theta \in [0,1]$

$\Rightarrow$ Let $g = \theta f_1 + (1-\theta) f_2$

$\Rightarrow$ We know that $f_1 \geq 0$, $f_2 \geq 0$, $\theta \in [0,1]$

$g = \theta f_1 \geq 0$, $(1-\theta) f_2 \geq 0$

$g(x) = \theta f_1(x) + (1-\theta) f_2(x) \geq 0 \quad \forall x \in \Omega$

$$\Rightarrow \int_\Omega g(x)\,dx = \int_\Omega [\theta f_1(x) + (1-\theta) f_2(x)]\,dx$$

$$= \theta \int_\Omega f_1(x)\,dx + (1-\theta) \int_\Omega f_2(x)\,dx$$

$$= \theta(1) + (1-\theta)1 = 1$$

$\Rightarrow g \in P$, i.e $\theta f_1 + (1-\theta) f_2 \in P$

$\Rightarrow P$ is a convex set

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3. $\Rightarrow K_M = \left\{ \begin{bmatrix} \underline{x} \\ y \end{bmatrix} \middle| \; \|\underline{x}\|_M \leq y, \; \underline{x} \in \mathbb{R}^n, \; y \in \mathbb{R} \right\}$

$\Rightarrow \|\underline{x}\|_M =$ M-norm of $\underline{x}$

$\Rightarrow \|\underline{x}\|_M = \sqrt{\underline{x}^T M \underline{x}}$ if $M > 0$

i.e M is positive definite

$\Rightarrow$ Find dual cone of K_M i.e K_M^*

$\Rightarrow$ Let K be a cone. The set

$K^* = \{ y \mid x^T y \geq 0 \quad \forall x \in K \}$

K^* is called dual cone of K.

$\Rightarrow$ So $K_M^* = \left\{ \begin{bmatrix} \underline{u} \\ v \end{bmatrix} \middle| \; \underline{u} \in \mathbb{R}^n, \; v \in \mathbb{R}, \right.$

$\begin{bmatrix} \underline{x}^T & y \end{bmatrix} \begin{bmatrix} \underline{u} \\ v \end{bmatrix} = \underline{x}^T \underline{u} + yv \geq 0$

for all $\left. \begin{bmatrix} \underline{x} \\ y \end{bmatrix} \in K_M \right\}$,

$\Rightarrow$ By eigenvalue decomposition (EVD) of M

$$M = Q \lambda Q^T$$

Q is orthogonal $\quad \lambda = \text{diag}(\lambda_1, \cdots, \lambda_n)$

$\lambda_i \geq 0$. let $\underline{\tilde{x}} = Q^T \underline{x}$

$$\Rightarrow \|\underline{x}\|_M = \sqrt{\underline{x}^T M \underline{x}} = \sqrt{\underline{x}^T Q \lambda Q^T \underline{x}}$$

$$= \sqrt{\underline{\tilde{x}}^T \lambda \underline{\tilde{x}}} = \sqrt{\underline{\tilde{x}}^T \lambda^{1/2} \lambda^{1/2} \underline{\tilde{x}}}$$

$$\Rightarrow \|\underline{x}\|_M = \|\lambda^{1/2} \underline{\tilde{x}}\|_2$$

$\Rightarrow$ let $\underline{z} = \lambda^{1/2} \underline{\tilde{x}} = \lambda^{1/2} Q^T \underline{x}$

$\lambda^{-1/2} \underline{z} = Q^T \underline{x} = Q^{-1} \underline{x}$ (Q is orthogonal)

$\underline{x} = Q \lambda^{-1/2} \underline{z}$

$\Rightarrow \|\underline{x}\|_M \leq y \longrightarrow \|\underline{z}\|_2 \leq y$

$\Rightarrow \underline{u}^T \underline{x} + vy \geq 0 \quad (\because \underline{x}^T \underline{u} = \underline{u}^T \underline{x})$

$\underline{u}^T Q \lambda^{-1/2} \underline{z} + vy \geq 0 \quad \forall \underline{z} \in \mathbb{R}^n, \|\underline{z}\|_2 \leq y$

let $\underline{w} = \lambda^{-1/2} Q^T \underline{u}$

$\underline{w}^T \underline{z} + vy \geq 0 \quad \forall \underline{z}, \forall y \ni \|\underline{z}\|_2 \leq y$

$$\boxed{\underline{w}^T \underline{z} + v \|\underline{z}\|_2 \geq 0 \quad \forall \underline{z}}$$

$\Rightarrow$ If $v = 0$, $\underline{w}^T \underline{z} \geq 0 \quad \forall \underline{z}$. So, $\underline{w} = 0$.

$\Rightarrow$ If $v > 0$, $\left(\frac{\underline{w}}{v}\right)^T \underline{z} + \|\underline{z}\|_2 \geq 0 \quad \forall \underline{z}$

$$\Rightarrow \min_{\underline{z}} \left(\frac{\underline{w}}{v}\right)^T \underline{z} = -\left\|\frac{\underline{w}}{v}\right\|_2 \|\underline{z}\|_2$$

$$\Rightarrow -\left\|\frac{\underline{w}}{v}\right\|_2 \|\underline{z}\|_2 + \|\underline{z}\|_2 \geq 0$$

$\Rightarrow \|\underline{w}\|_2 \leq v$

$\Rightarrow \|\underline{w}\|_2 = \sqrt{\underline{w}^T \underline{w}}$

$= \sqrt{(\lambda^{-1/2} Q^T \underline{u})^T \lambda^{-1/2} Q^T \underline{u}}$

$= \sqrt{\underline{u}^T Q \lambda^{-1/2} \lambda^{-1/2} Q^T \underline{u}}$

$= \sqrt{\underline{u}^T Q \lambda^{-1} Q^T \underline{u}}$

$\Rightarrow \|\underline{w}\|_2 = \sqrt{\underline{u}^T M^{-1} \underline{u}} = \|\underline{u}\|_{M^{-1}} \leq v$

$\Rightarrow K_M^* = \left\{ \begin{bmatrix} \underline{u} \\ v \end{bmatrix} \middle| \underline{u} \in \mathbb{R}^n, v \in \mathbb{R}, \sqrt{\underline{u}^T M^{-1} \underline{u}} \leq v \right\}$

$\boxed{K_M^* = K_{M^{-1}}}$

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4. $\Rightarrow K_M = \{ \underline{x} \mid \underline{x}^T M \underline{x} \leq (\underline{c}^T \underline{x})^2, \underline{x} \in \mathbb{R}^n, \underline{c}^T \underline{x} \geq 0 \}$

$\Rightarrow K_M$ is a hyperbolic cone

$\Rightarrow$ Show that K_M is a convex set

$\Rightarrow$ ~~let~~ $\cancel{(\underline{x}^T M \underline{x})^T = \underline{x}^T M^T \underline{x}}$

$\cancel{\underline{x}^T \left(\frac{M + M^T}{2}\right)}$

$\cancel{\underline{x}^T M \underline{x} = \underline{x}^T \left(\frac{M+M^T}{2}\right) \underline{x}}$

$S = M$

M is a PSD matrix

~~As~~ $\cancel{\underline{x}^T (M - M^T) \underline{x} = 0}$ ~~let~~ $\cancel{S = \frac{M+M^T}{2}}$

$\Rightarrow \underline{x}^T S \underline{x} \leq (\underline{c}^T \underline{x})^2, \underline{x} \in \mathbb{R}^n, \underline{c}^T \underline{x} \geq 0$

$\Rightarrow S$ is symmetric PSD matrix.

$\Rightarrow \exists$ matrix L such that $S = LL^T$

(Cholesky decomposition)

$$\Rightarrow K_M = \{\underline{x} \in \mathbb{R}^n \mid \underline{x}^T S \underline{x} = \underline{x}^T L L^T \underline{x}$$

$$= \|L^T \underline{x}\|_2^2 \leq (\underline{c}^T \underline{x})^2$$

where $\underline{x} \in \mathbb{R}^n$, $\underline{c}^T \underline{x} \geq 0\}$

$$\Rightarrow K_M = \{\underline{x} \mid \|L^T \underline{x}\|_2 \leq \underline{c}^T \underline{x},\ \underline{x} \in \mathbb{R}^n,$$

$$\underbrace{\underline{c}^T \underline{x} \geq 0}_{\text{obvious}}\}$$

$$\Rightarrow K_M = \{\underline{x} \in \mathbb{R}^n \mid \|L^T \underline{x}\|_2 \leq \underline{c}^T \underline{x}\}$$

~~$\Rightarrow$ Consider linear mapping~~

$$\cancel{f : \mathbb{R}^n \longrightarrow \mathbb{R} \times \mathbb{R}^k} \quad \cancel{f(\alpha x_1 + \beta x_2)}$$

$$\cancel{f(\underline{x}) = (\underline{c}^T \underline{x},\ L^T \underline{x})} \quad \cancel{= \alpha f(x_1) + \beta f(x_2)}$$

~~k = no. of columns of L~~

$\Rightarrow$ Consider linear mapping

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^k \times \mathbb{R}$$

$$f(\underline{x}) = (L^T \underline{x},\ \underline{c}^T \underline{x})$$

k = no. of columns of L

$\Rightarrow$ $Q = \{(\underline{x}, t) \mid \|\underline{x}\|_2 \leq t\}$ is a 2^{nd} order cone

$$\Rightarrow K_M = \{\underline{x} \in \mathbb{R}^n \mid (L^T \underline{x},\ \underline{c}^T \underline{x}) \in Q\}$$

$$K_M = \{\underline{x} \in \mathbb{R}^n \mid f(\underline{x}) \in Q\}$$

$$\boxed{K_M = f^{-1}(Q)}$$

$\Rightarrow$ Q is convex. Inverse image of convex set under linear map is convex.

$\Rightarrow$ $\boxed{K_M \text{ is convex}}$

5. ⇒ Geometric interpretation of PSD cone

⇒ Set of symmetric 2×2 matrices

$$C = \{A \in S^2 \mid A \succeq 0\}.$$

⇒ S^2 = space of 2×2 symmetric matrices

⇒ $A \succeq 0 \longrightarrow$ A is PSD

⇒ A symmetric matrix (2×2) is PSD iff $\det(A) \geq 0$

(a) ⇒ $A = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$, is symmetric 2×2 matrix

⇒ For A to be PSD, $\det(A) \geq 0$.

$$xz - y^2 \geq 0, \quad x \geq 0, \quad z \geq 0$$

(b)

⇒ Boundary of C is given by quadratic surface $xz = y^2$, $x \geq 0$, $z \geq 0$, which forms curved bowl shaped surface opening outward in (x, z) directions, $x \geq 0$, $z \geq 0$.

⇒ This surface curves outward i.e line segments b/w points inside surface remains inside the region.

⇒ C consists of all points lying on or above surface, so it is a solid convex cone, surface does not go inward, interior

⇒ As boundary is filled, any convex combination of 2 PSD matrices is PSD

⇒ Outward curvature of boundary implies convexity of set C

6.

(a) $\Rightarrow C = \text{slab} = \{\underline{x} \in R^n \mid \alpha \leq \underline{a}^T \underline{x} \leq \beta\}$

$\Rightarrow$ Take $\underline{x_1}, \underline{x_2} \in C$. $\theta \in [0,1]$

$\Rightarrow \underline{a}^T \underline{x_1} \geq \alpha, \ \underline{a}^T \underline{x_2} \geq \alpha, \ \underline{a}^T \underline{x_1} \leq \beta, \ \underline{a}^T \underline{x_2} \leq \beta$

$\Rightarrow \underline{a}^T(\theta \underline{x_1} + (1-\theta)\underline{x_2}) = \theta \underline{a}^T \underline{x_1} + (1-\theta)\underline{a}^T \underline{x_2}$.

$\geq \theta\alpha + (1-\theta)\alpha \geq \alpha$

$\Rightarrow \underline{a}^T(\theta \underline{x_1} + (1-\theta)\underline{x_2}) = \theta \underline{a}^T \underline{x_1} + (1-\theta)\underline{a}^T \underline{x_2}$.

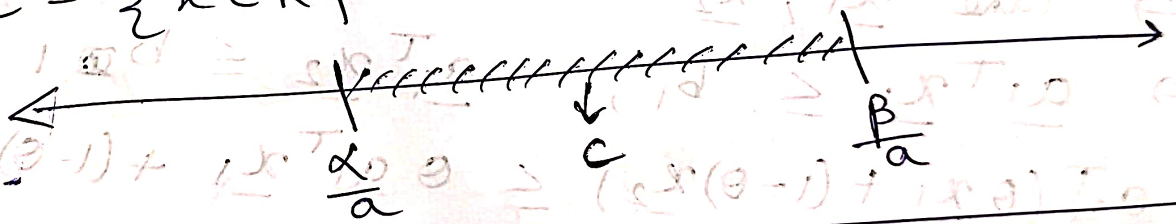
$\leq \theta\beta + (1-\theta)\beta \leq \beta$

$\Rightarrow \alpha \leq \underline{a}^T(\theta \underline{x_1} + (1-\theta)\underline{x_2}) \leq \beta$.

$\Rightarrow \theta \underline{x_1} + (1-\theta)\underline{x_2} \in C$. $\boxed{C \text{ is convex}}$

$\Rightarrow$ Slab is the region sandwiched between 2 hyperplanes $\{\underline{x} \in R^n \mid \underline{a}^T \underline{x} = \alpha\}$, $\{\underline{x} \in R^n \mid \underline{a}^T \underline{x} = \beta\}$

$\Rightarrow$ In R, $C = \{x \in R \mid \alpha \leq ax \leq \beta\}$



6.(b) $\Rightarrow C = $ rectangle (hyper rectangle when $n > 2$)

$\Rightarrow C = \{\underline{x} \in R^n \mid \alpha_i \leq x_i \leq \beta_i, \ i = 1, \ldots, n\}$.

$\Rightarrow$ Take $\underline{x_1}, \underline{x_2} \in C$. $\theta \in [0,1]$

$\Rightarrow \underline{x_1} = [x_{11} \ x_{12} \ \ldots \ x_{1n}]^T$

$\underline{x_2} = [x_{21} \ x_{22} \ \ldots \ x_{2n}]^T$

$\Rightarrow \alpha_i \leq x_{1i} \leq \beta_i, \ \alpha_i \leq x_{2i} \leq \beta_i, \ i = 1 \text{ to } n$

$\cancel{\Rightarrow \theta\alpha_i \leq \theta x_{1i} \leq \theta\beta_i, \ \theta\alpha_i \leq \theta x_{2i} \leq \theta\beta_i}$

$\cancel{(1-\theta)\alpha_i \leq (1-\theta)}$

$\Rightarrow x_{1i} \geq \alpha_i \quad , \quad x_{2i} \geq \alpha_i$

$\theta x_{1i} + (1-\theta) x_{2i} \geq \theta \alpha_i + (1-\theta)\alpha_i \geq \alpha_i$

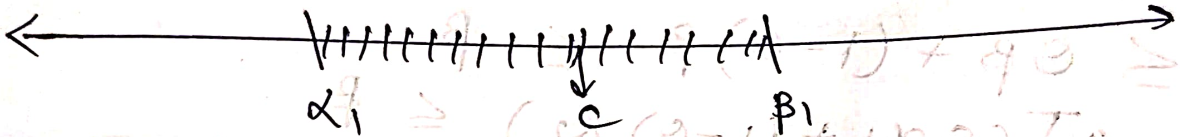
$\Rightarrow x_{1i} \leq \beta_i \quad , \quad x_{2i} \leq \beta_i$

$\theta x_{1i} + (1-\theta) x_{2i} \leq \theta \beta_i + (1-\theta) \beta_i \leq \beta_i$

$\Rightarrow \alpha_i \leq \theta x_{1i} + (1-\theta) x_{2i} \leq \beta_i$

$\Rightarrow \theta \underline{x_1} + (1-\theta) \underline{x_2} \in C$. $\boxed{\text{C is convex}}$

$\Rightarrow$ In R, $C = \{x \in R \mid \alpha_1 \leq x \leq \beta_1\}$



$\Rightarrow R^n$ ——— hyper rectangle.

6.(c)

$\Rightarrow C = \text{wedge} = \{\underline{x} \in R^n \mid \underline{a_1}^T \underline{x} \leq b_1, \ \underline{a_2}^T \underline{x} \leq b_2\}$

$\Rightarrow$ Take $\underline{x_1}, \underline{x_2} \in C$. $\theta \in [0,1]$.

$\Rightarrow \underline{a_1}^T \underline{x_1} \leq b_1 ; \quad \underline{a_1}^T \underline{x_2} \leq b_1$

$\underline{a_1}^T (\theta \underline{x_1} + (1-\theta)\underline{x_2}) \leq \theta \underline{a_1}^T \underline{x_1} + (1-\theta) \underline{a_2}^T \underline{x_2}$

$\leq \theta b_1 + (1-\theta) b_1 \leq b_1$

$\Rightarrow \underline{a_2}^T \underline{x_1} \leq b_2 \quad , \quad \underline{a_2}^T \underline{x_2} \leq b_2$

$\underline{a_2}^T (\theta \underline{x_1} + (1-\theta) \underline{x_2}) \leq \theta \underline{a_2}^T \underline{x_1} + (1-\theta) \underline{a_2}^T \underline{x_2}$

$\leq \theta b_2 + (1-\theta) b_2 \quad \leq b_2$

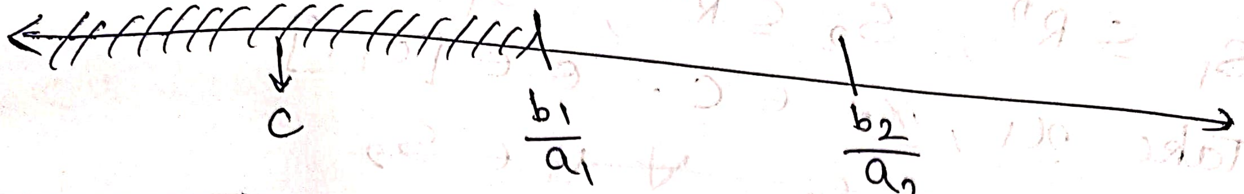
$\Rightarrow \underline{a_1}^T (\theta \underline{x_1} + (1-\theta)\underline{x_2}) \leq b_1, \quad \underline{a_2}^T (\theta \underline{x_1} + (1-\theta)\underline{x_2}) \leq b_2$.

$\Rightarrow \theta \underline{x_1} + (1-\theta) \underline{x_2} \in C$. $\boxed{\text{C is convex}}$

$\Rightarrow$ This is intersection of 2 halfspaces $\rightarrow$ convexity is preserved

$\{\underline{x} \in R^n \mid \underline{a_1}^T \underline{x} \leq b_1\} \cap \{\underline{x} \in R^n \mid \underline{a_2}^T \underline{x} \leq b_2\}$

$\Rightarrow$ In $\mathbb{R}$, $C = \{x \in \mathbb{R} \mid a_1 x \le b_1,\ a_2 x \le b_2\}$



6. (d) $\Rightarrow$ C = set of points closer to given point than given set

$\Rightarrow$ $S \subseteq \mathbb{R}^n$

$\Rightarrow$ $C = \{\underline{x} \in \mathbb{R}^n \mid \|\underline{x} - \underline{x_0}\|_2 \le \|\underline{x} - \underline{y}\|_2\ \forall\, \underline{y} \in S\}$

$\Rightarrow$ Take any $\underline{y} \in S$.

$$\|\underline{x} - \underline{x_0}\|_2 \le \|\underline{x} - \underline{y}\|_2$$

$$\|\underline{x} - \underline{x_0}\|_2^2 \le \|\underline{x} - \underline{y}\|_2^2$$

$$\langle \underline{x} - \underline{x_0}, \underline{x} - \underline{x_0} \rangle \le \langle \underline{x} - \underline{y}, \underline{x} - \underline{y} \rangle$$

$$(\underline{x} - \underline{x_0})^T (\underline{x} - \underline{x_0}) \le (\underline{x} - \underline{y})^T (\underline{x} - \underline{y})$$

$$\underline{x}^T \underline{x} - 2\underline{x}^T \underline{x_0} + \underline{x_0}^T \underline{x_0} \le \underline{x}^T \underline{x} - 2\underline{x}^T \underline{y} + \underline{y}^T \underline{y}$$

$$2\underline{x}^T (\underline{y} - \underline{x_0}) \le \|\underline{y}\|_2^2 - \|\underline{x_0}\|_2^2$$

$$(\underline{y} - \underline{x_0})^T \underline{x} \le \frac{1}{2}\left(\|\underline{y}\|_2^2 - \|\underline{x_0}\|_2^2\right)$$

This is of the form $\underline{a}^T \underline{x} \le b$ so it is a halfspace which is convex set.

$$\Rightarrow C = \bigcap_{\underline{y} \in S} \{\underline{x} \in \mathbb{R}^n \mid \|\underline{x} - \underline{x_0}\|_2 \le \|\underline{x} - \underline{y}\|_2\}$$

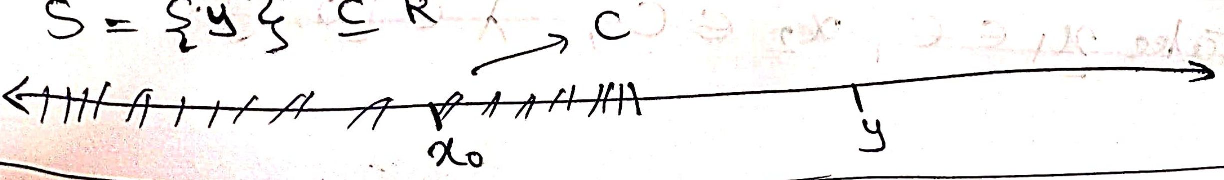
$\Rightarrow$ Intersection is an operation preserving convexity. So, intersection of convex sets is a convex set.

$\Rightarrow$ $\boxed{C \text{ is convex}}$

$\Rightarrow$ In $\mathbb{R}$, $C = \{x \in \mathbb{R} \mid |x - x_0| \le |x - y|,\ y \in \mathbb{R}\}$

WLOG, let $x_0 < y$

$S = \{y\} \subseteq \mathbb{R}$



6.(e) $\Rightarrow C = \{ \underline{x} \in \mathbb{R}^n \mid \underline{x} + S_2 \subseteq S_1 \}$

$\Rightarrow S_1 \subseteq \mathbb{R}^n$, $S_2 \subseteq \mathbb{R}^n$, S_1 is convex

$\Rightarrow$ Take $\underline{x_1}, \underline{x_2} \in C$. $\theta \in [0,1]$

$\cancel{\Rightarrow \underline{x_1} + \underline{y} \in S_1 \ \forall \underline{y} \in S_2}$

$\cancel{\underline{x_2} + \underline{y} \in S}$

$\Rightarrow \underline{x_1} + \underline{y} \in S_1 \quad \forall \underline{y} \in S_2$

$\underline{x_2} + \underline{y} \in S_1 \quad \forall \underline{y} \in \underline{S_2}$

$\cancel{\Rightarrow \theta \underline{x_1} + \theta \underline{y} \in S_1 \ \forall \underline{y} \in S_2}$

$\cancel{(1-\theta)\underline{x_2} + \theta(1-\theta)\underline{y} \in S_1 \ \forall \underline{y} \in S_2}$

$\Rightarrow \theta(\underline{x_1} + \underline{y}) + (1-\theta)(\underline{x_2} + \underline{y}) \in S_1 \ \forall \underline{y} \in S_2$

since S_1 is convex

$\Rightarrow \theta \underline{x_1} + (1-\theta)\underline{x_2} + \underline{y} \in S_1 \ \forall \underline{y} \in S_2$

$\Rightarrow \theta \underline{x_1} + (1-\theta)\underline{x_2} \in C$

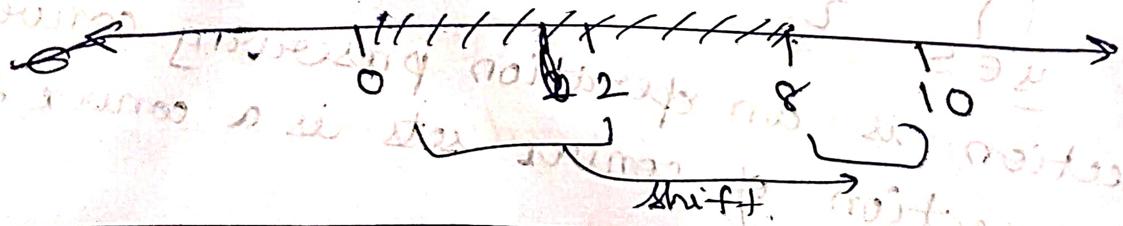
$\Rightarrow \boxed{C \text{ is convex}}$

$\Rightarrow$ In $\mathbb{R}$, $C = \{ x \in \mathbb{R} \mid x + S_2 \subseteq S_1 \}$

$S_1 \subseteq \mathbb{R}$, $S_2 \subseteq \mathbb{R}$, S_1 is convex.

Eg: $S_1 = [0, 10]$, $S_2 = [0, 2]$

$C = [0, 8]$



6.(f) $\Rightarrow C = \{ \underline{x} \in \mathbb{R}^n \mid \|\underline{x} - \underline{a}\|_2 \le \theta \|\underline{x} - \underline{b}\|_2 \}$

$\Rightarrow \underline{a} \ne \underline{b}$, $\theta \in [0,1]$.

Case 1: $0 < \theta < 1$.

$\Rightarrow \|\underline{x} - \underline{a}\|_2^2 \le \theta^2 \|\underline{x} - \underline{b}\|_2^2$

$\Rightarrow \langle \underline{x} - \underline{a}, \underline{x} - \underline{a} \rangle \le \theta^2 \langle \underline{x} - \underline{b}, \underline{x} - \underline{b} \rangle$

$\Rightarrow \|\underline{x}\|_2^2 - 2\underline{x}^T\underline{a} + \|\underline{a}\|^2 \le \theta^2 \|\underline{x}\|_2^2 - 2\theta^2 \underline{x}^T \underline{b} + \theta^2 \|\underline{b}\|_2^2$

$\Rightarrow (1-\theta^2)\|\underline{x}\|_2^2 - 2\underline{x}^T(\underline{a} - \theta^2 \underline{b}) + (\|\underline{a}\|_2^2 - \theta^2\|\underline{b}\|_2^2) \le 0$ — (1)

$\Rightarrow 1-\theta^2 > 0$. So, this can be written

as $\left\| \underline{x} - \dfrac{\underline{a} - \theta^2 \underline{b}}{1-\theta^2} \right\|_2 \le \dfrac{\theta}{1-\theta^2} \|\underline{a} - \underline{b}\|_2$

This is closed ball $\rightarrow$ $\boxed{\text{Convex}}$.

Case 2: $\theta = 0$

$\Rightarrow \|\underline{x} - \underline{a}\|_2 \le 0 \Rightarrow \underline{x} = \underline{a}$. $\boxed{C \text{ is convex}}$

Case 3: $\theta = 1$

$\Rightarrow -2\underline{x}^T(\underline{a} - \underline{b}) + (\|\underline{a}\|_2^2 - \|\underline{b}\|_2^2) \le 0$ in (1)

$\Rightarrow 2(\underline{b} - \underline{a})^T \underline{x} + \|\underline{a}\|_2^2 - \|\underline{b}\|_2^2 \le 0$

$\Rightarrow$ Closed halfspace — $\boxed{\text{convex}}$

$\Rightarrow \boxed{C \text{ is convex}}$

$\Rightarrow$ In $\mathbb{R}$, $C = \{x \in \mathbb{R} \mid |x-a| \le \theta |x-b|\}$. WLOG, $a < b$.

$a \ne b$, $\theta \in [0,1]$

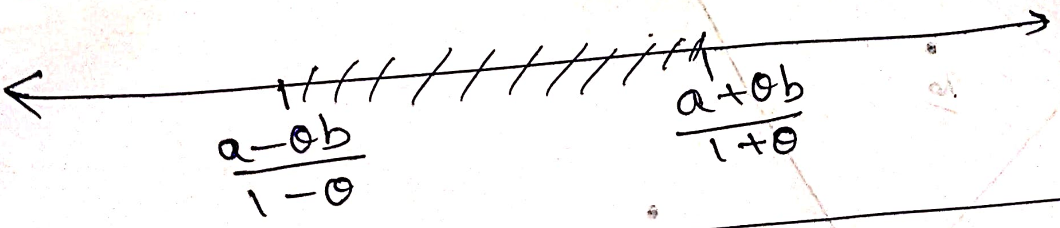
$(1-\theta^2)x^2 - 2(a - \theta^2 b)x + (a^2 - \theta^2 b^2) \le 0$

$|x-a| = \theta|x-b|$

$(x-a)^2 = \theta^2(x-b)^2$.

(1) $x - a = \theta(x-b) \longrightarrow x = \dfrac{a-\theta b}{1-\theta}$

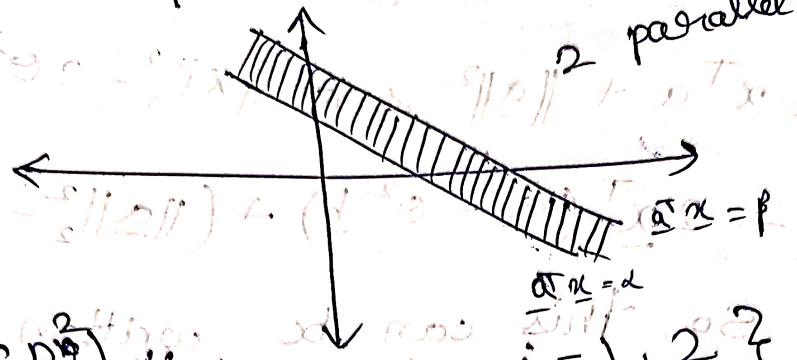
(2) $x - a = -\theta(x-b) \longrightarrow x = \dfrac{a+\theta b}{1+\theta}$.



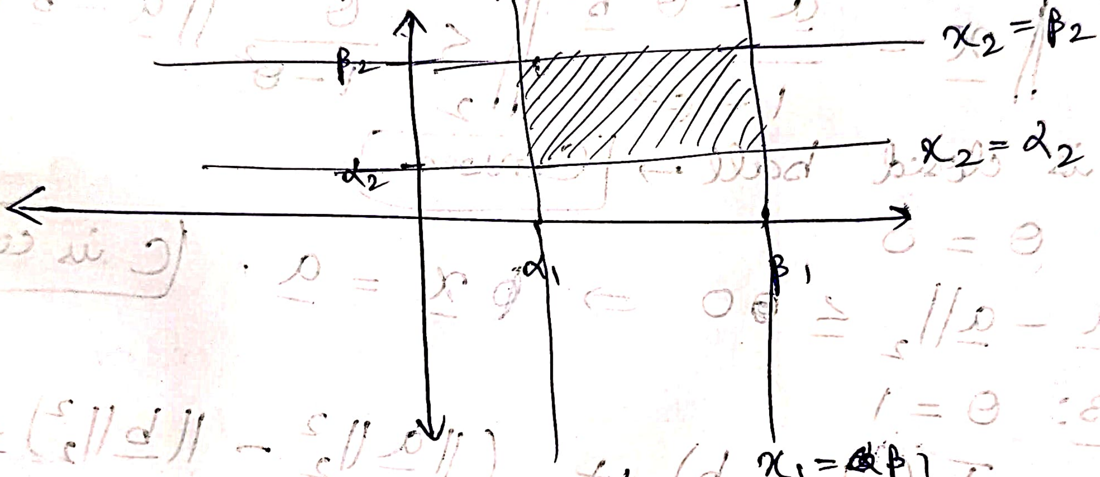
6. Figures drawn in Cartesian plane: ($\mathbb{R}^2$)

(a) Slab $\{\underline{x} \in \mathbb{R}^2 \mid \underline{a}^T \underline{x} \geq \alpha,\ \underline{a}^T \underline{x} \leq \beta\}$

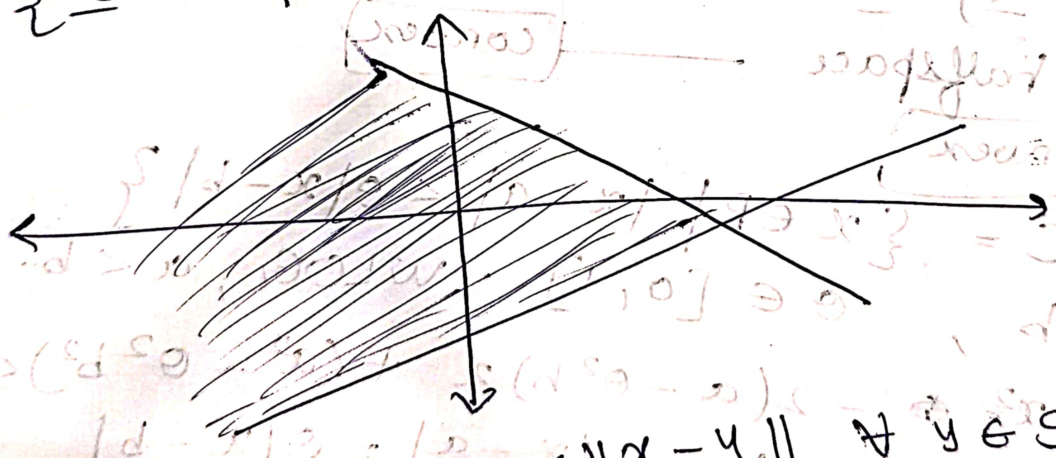
2 parallel lines



(b) $\{\underline{x} \in \mathbb{R}^2 \mid \alpha_i \leq x_i \leq \beta_i,\ i = 1, 2\}$

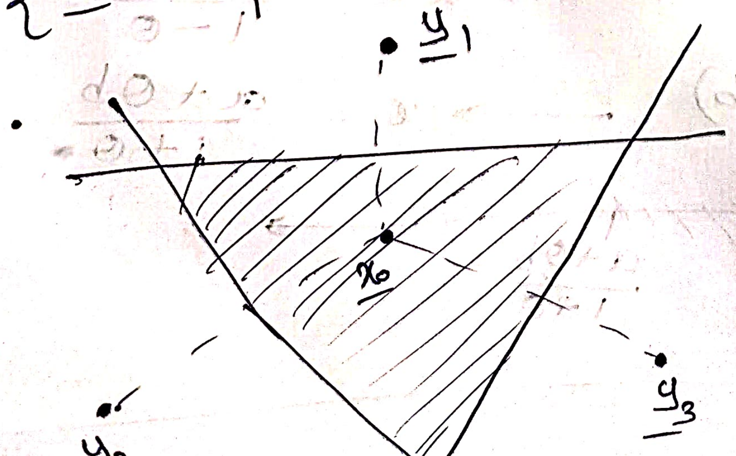


(c) $\{\underline{x} \in \mathbb{R}^2 \mid \underline{a}_1^T \underline{x} \leq b_1,\ \underline{a}_2^T \underline{x}_2 \leq b_2\}$

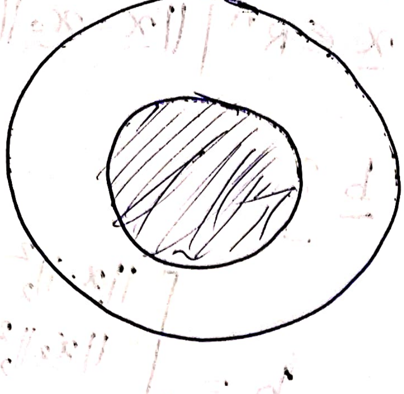


(d) $\{\underline{x} \in \mathbb{R}^2 \mid \|\underline{x} - \underline{x}_0\|_2 \leq \|\underline{x} - \underline{y}\|,\ \forall\ \underline{y} \in S\}$

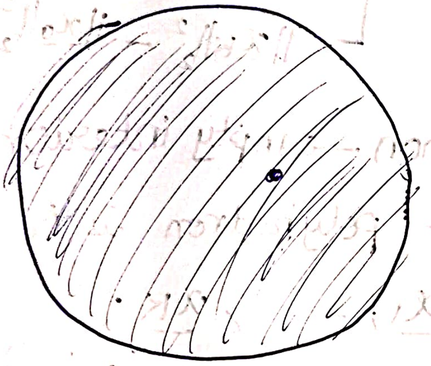
all are $\perp^r$ bisectors



e) $\{\underline{x} \mid \underline{x} + S_2 \subseteq S_1\}$, $S_1, S_2 \subseteq R^n$, S_1 convex



f) $\{\underline{x} \mid \|\underline{x} - \underline{a}\|_2 \leq \theta \|\underline{x} - \underline{b}\|_2\}$, $\underline{a} \neq \underline{b}$, $0 \leq \theta \leq 1$



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7. ⇒ Voronoi sets and Polyhedral decomposition

⇒ $\underline{x_0}, \underline{x_1}, \ldots, \underline{x_K} \in R^n$ be distinct

⇒ Set of points that are closer in Euclidean norm to $\underline{x_0}$ than to other $\underline{x_i}$ i.e

$$V = \{\underline{x} \in R^n \mid \|\underline{x} - \underline{x_0}\|_2 \leq \|\underline{x} - \underline{x_i}\|_2, i = 1, \ldots, K\}$$

⇒ V = Voronoi region around $\underline{x_0}$ wrt $\underline{x_1}, \ldots, \underline{x_K}$

(a) ⇒ $\|\underline{x} - \underline{x_0}\|_2 \leq \|\underline{x} - \underline{x_i}\|_2$ i = 1 to K

⇒ $\|\underline{x} - \underline{x_0}\|_2^2 \leq \|\underline{x} - \underline{x_i}\|_2^2$, i = 1 to K

⇒ $\langle \underline{x} - \underline{x_0}, \underline{x} - \underline{x_0} \rangle \leq \langle \underline{x} - \underline{x_i}, \underline{x} - \underline{x_i} \rangle$, i = 1 to K

⇒ $\underline{x}^T\underline{x} - 2\underline{x_0}^T\underline{x} + \underline{x_0}^T\underline{x_0} \leq \underline{x}^T\underline{x} - 2\underline{x_i}^T\underline{x} + \underline{x_i}^T\underline{x_i}$ i = 1 to K

⇒ $2(\underline{x_i} - \underline{x_0})^T\underline{x} \leq \|\underline{x_i}\|_2^2 - \|\underline{x_0}\|_2^2$, i = 1 to K

$\Rightarrow V$ is intersection of K halfspaces (polyhedron)

$$V = \bigcap_{i \in \{1,2,\dots,K\}} \{\underline{x} \in \mathbb{R}^n \mid \|\underline{x} - \underline{x_0}\|_2 \leq \|\underline{x} - \underline{x_i}\|_2\}$$

$$\Rightarrow V = \{\underline{x} \mid A\underline{x} \preceq \underline{b}\}$$

$$A = \begin{bmatrix} 2(\underline{x_1} - \underline{x_0})^T \\ 2(\underline{x_2} - \underline{x_0})^T \\ \vdots \\ 2(\underline{x_K} - \underline{x_0})^T \end{bmatrix} \qquad b = \begin{bmatrix} \|\underline{x_1}\|_2^2 - \|\underline{x_0}\|_2^2 \\ \|\underline{x_2}\|_2^2 - \|\underline{x_0}\|_2^2 \\ \vdots \\ \|\underline{x_K}\|_2^2 - \|\underline{x_0}\|_2^2 \end{bmatrix}$$

(b) $\Rightarrow$ Given a polyhedron P with non-empty interior

$\Rightarrow$ Find $x_0, \dots, x_K$ such that polyhedron is the Voronoi region of $\underline{x_0}$ wrt $\underline{x_1}, \dots, \underline{x_K}$.

$\Rightarrow P \subset \mathbb{R}^n$. Let $P = \{\underline{x} \mid \underline{a_i}^T \underline{x} \leq b_i,\ i = 1 \text{ to } m\}$

with each $a_i \neq 0$.

$\Rightarrow$ int $P \neq \phi$. Pick $x_0 \in$ int P.

$$\underline{a_i}^T \underline{x_0} < b \ \forall i$$

$$\Rightarrow \forall i;\ \text{define } \alpha_i = \frac{4(b_i - \underline{a_i}^T x_0)}{\|\underline{a_i}\|_2^2},\ \alpha_i > 0$$

$$\Rightarrow \underline{x_i} = \underline{x_0} + \frac{\alpha_i}{2} \underline{a_i}$$

Justification: Choose $\underline{x_i} = \underline{x_0} + t_i \underline{a_i}$

Solve for t_i so that perpendicular bisector equals $\underline{a_i}^T \underline{x} = b_i$. $\quad t_i = \frac{\alpha_i}{2}$.

$\alpha_i > 0,\ \underline{a_i} \neq 0$

$$\underline{x_i} \neq \underline{x_0}, \quad (\underline{x_i} - \underline{x_0})^T \underline{x} \leq \frac{\|\underline{x_i}\|_2^2 - \|\underline{x_0}\|_2^2}{2}$$

$$\approx$$

$$\underline{a_i}^T \underline{x} \leq b_i$$

Choose $\underline{x_i} = \underline{x_0} + t_i \underline{a_i}$

Taking bisector gives $t_i = \frac{\alpha_i}{2} = 2\left(\frac{b_i - \underline{a_i}^T \underline{x_0}}{\|\underline{a_i}\|_2^2}\right)$

$\|\underline{x_i}\|_2^2 - \|\underline{x_0}\|_2^2 = \|\underline{x_0} + \frac{\alpha_i}{2}\underline{a_i}\|^2 - \|\underline{x_0}\|_2^2$

$= \alpha_i \underline{a_i}^T \underline{x_0} + \frac{\alpha_i^2}{4}\|\underline{a_i}\|_2^2$

$= \frac{4(b_i - \underline{a_i}^T \underline{x_0})}{\|\underline{a_i}\|_2^2} \underline{a_i}^T \underline{x_0} + \frac{1}{4}\left(\frac{16(b_i - \underline{a_i}^T \underline{x_0})^2}{\|\underline{a_i}\|_2^4}\right) \|\underline{a_i}\|_2^2$

$= \alpha_i b_i$

$\Rightarrow \alpha_i \underline{a_i}^T \underline{x_0} \le \alpha_i b_i \iff \underline{a_i}^T \underline{x} \le b_i$

$\Rightarrow V = \{\underline{x} \in \mathbb{R}^n \mid \|\underline{x} - \underline{x_0}\|_2 \le \|\underline{x} - \underline{x_{0i}}\|_2,\ i = 1 \text{ to } m\}$

$\boxed{V = \{\underline{x} \in \mathbb{R}^n \mid \underline{a_i}^T \underline{x_0} \le b_i,\ i = 1 \text{ to } m\} = P}$

Hence polyhedron equals Voronoi region of $\underline{x_0}$ wrt. finitely many points $\underline{x_i}$.

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8. $\Rightarrow S = \{\underline{x} \in \mathbb{R}^n \mid |p(t)| \le 1 \quad \forall\ |t| \le \frac{\pi}{3}\}$

$\Rightarrow p(t) =$ trigonometric polynomial $= \sum_{k=1}^{m} x_k \cos(kt)$

$\Rightarrow$ We know that S can be expressed as intersection of infinitely many slabs. $S = \bigcap_{|t| \le \frac{\pi}{3}} S_t$

$\Rightarrow$ Each slab is defined by

$S_t = \{\underline{x} \in \mathbb{R}^m \mid -1 \le (\cos t, \cos 2t, \ldots, \cos mt)^T \underline{x} \le 1\}$

(a)(i) $\Rightarrow m = 2$

$\Rightarrow \boxed{p(t) = x_1 \cos t + x_2 \cos 2t}$

(a)(ii) $\Rightarrow S_t = \{(x_1, x_2) \in R^2 \mid$

$-1 \leq (\cos t, \cos 2t)^T (x_1, x_2) \leq 1\}$

$\Rightarrow S_t = \{(x_1, x_2) \in R^2 \mid -1 \leq x_1 \cos t + x_2 \cos 2t \leq 1\}$

$\Rightarrow S = \{(x_1, x_2) \in R^2 \mid |x_1 \cos t + x_2 \cos 2t| \leq 1$

$\forall\ |t| \leq \frac{\pi}{3}\}$

$$S = \bigcap_{|t| \leq \frac{\pi}{3}} S_t$$

$\Rightarrow$ Let $u = \cos t \in [\frac{1}{2}, 1]\ \forall\ |t| \leq \frac{\pi}{3}$

$p(t) = x_1 u + x_2 (2u^2 - 1)$

$|x_1 u + x_2(2u^2 - 1)| \leq 1\ \forall\ u \in [\frac{1}{2}, 1]$

$\Rightarrow$ As S_t is a slab, S_t is convex, and intersection is convexity preserving, S is convex.

$S = \{x \mid (u, 2u^2-1)^T (x_1, x_2) \leq 1\ \forall\ u \in [\frac{1}{2}, 1]\}$

Intersection of infinitely many supporting hyperplanes.

$\Rightarrow S = \{(x_1, x_2) \in R^2 \mid \sup_{u \in [\frac{1}{2}, 1]} |x_1 u + x_2(2u^2 - 1)| \leq 1\}$

$\Rightarrow S$ is in finite dimensional space.

$t = 0 \rightarrow |x_1 + x_2| \leq 1$ $\quad$ S is bounded

$t = \frac{\pi}{3} \rightarrow |x_1 - x_2| \leq 2$ $\quad$ and closed.

S is compact.

(c) $\Rightarrow S = \{(x_1, x_2) \in R^2 \mid |x_1 \cos t + x_2 \cos 2t| \leq 1$

$\forall\ |t| \leq \frac{\pi}{3}\}$

$\Rightarrow$ Smoothness of boundary:

$\Rightarrow |(\cos t, \cos 2t)^T (x_1, x_2)| \leq 1$

$\Rightarrow$ Defines slab whose boundary consists of 2 parallel lines

$\Rightarrow x_1 \cos t + x_2 \cos 2t = \pm 1$

$\Rightarrow$ Let $a(t) = (\cos t, \cos 2t)$ (normal vector)

$\Rightarrow a(t)$ continuously varies with $t \in [-\frac{\pi}{3}, \frac{\pi}{3}]$

Supporting lines rotate all the time

$\Rightarrow S =$ intersection of infinitely many slabs.

$\Rightarrow$ Smooth boundary, continuously curved, no corners, no flat segments

Symmetries of S:

$\Rightarrow$ If $\underline{x} \in S$, $\forall t$, $|a(t)^T \underline{x}| \le 1$

$|a(t)^T(-\underline{x})| = |-a(t)^T \underline{x}| = |a(t)^T \underline{x}|$

$-\underline{x} \in S$

$\Rightarrow$ S is centrally symmetric about origin.

$\Rightarrow$ No independent symmetry wrt x_1, x_2 axes in general.

Average curve:

$\Rightarrow$ Let $\underline{x}_1, \underline{x}_2 \in S$. Then $\forall\ |t| \le \frac{\pi}{3}$

$|p^{(1)}(t)| \le 1, \quad |p^{(2)}(t)| \le 1$

$p_{avg}(t) = \frac{1}{2}(p^{(1)}(t) + p^{(2)}(t))$

$\Rightarrow |p_{avg}(t)| = |\frac{1}{2}p^{(1)}(t) + \frac{1}{2}p^{(2)}(t)|$

$\le \frac{1}{2}|p^{(1)}(t)| + \frac{1}{2}|p^{(2)}(t)| \le \frac{1}{2}(1+1)$

$\Rightarrow |p_{avg}(t)| \le 1 \quad \forall\ |t| \le \frac{\pi}{3}$