Problem 1:

设置插值函数为：$S_i(x) = y_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$，同时按照插值函数的性质可得：

$$S_i(x_{i+1}) = S_{i+1}(x_{i+1})$$
$$S_i'(x_{i+1}) = S_{i+1}'(x_{i+1})$$
$$S_i''(x_{i+1}) = S_{i+1}''(x_{i+1})$$

再结合自然插值条件：

$$S_1''(x_1) = 0$$
$$S_{n-1}''(x_n) = 0$$

为简化运算，引入额外的未知变量$c_n = \frac{S_{n-1}''(x_n)}{2}$；同时引入速记表示法：$\delta_i = x_{i+1} - x_i$，$\Delta_i = y_{i+1} - y_i$.

将其余两个参数都用c_i表示：

$$d_i = \frac{c_{i+1} - c_i}{3\delta_i}$$
$$b_i = \frac{\Delta_i}{\delta_i} - \frac{\delta_i}{3}(2c_i + c_{i+1})$$

结合边值条件，可得以下矩阵方程：

$$\begin{bmatrix} 1 & 0 & 0 & & & \\ \delta_1 & 2\delta_1 + \delta_2 & \delta_2 & \ddots & & \\ 0 & \delta_2 & 2\delta_2 + 2\delta_3 & \delta_3 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & & \delta_{n-2} & 2\delta_{n-2} + 2\delta_{n-1} & \delta_{n-1} \\ & & & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 3\left(\frac{\Delta_2}{\delta_2} - \frac{\Delta_1}{\delta_1}\right) \\ 3\left(\frac{\Delta_{n-1}}{\delta_{n-1}} - \frac{\Delta_{n-2}}{\delta_{n-2}}\right) \\ 0 \end{bmatrix}$$

解出来c_i后反解出b_i，c_i，得到插值函数。

如果使用钳制边值条件：

$$S_1'(x_1) = v_1$$
$$S_{n-1}'(x_n) = v_n$$

则有：

$$2\delta_1 c_1 + \delta_1 c_2 = 3\left(\frac{\Delta_1}{\delta_1} - v_1\right)$$
$$\delta_{n-1} + 2\delta_{n-1} c_n = 3\left(v_n - \frac{\Delta_{n-1}}{\delta_{n-1}}\right)$$

矩阵也要改写为：

$$\left[\begin{array}{cccccccc|c} 2\delta_1 & \delta_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 3(\Delta_1/\delta_1 - v_1) \\ 0 & 0 & 0 & 0 & \cdots & 0 & \delta_{n-1} & 2\delta_{n-1} & 3(v_n - \Delta_{n-1}/\delta_{n-1}) \end{array}\right]$$

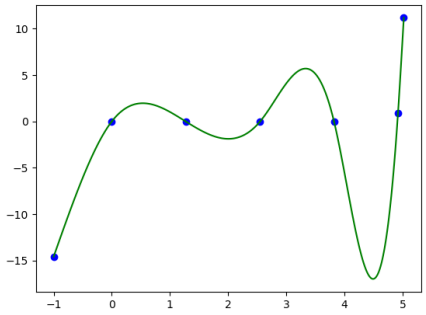

Figure 1 **自然边值条件**

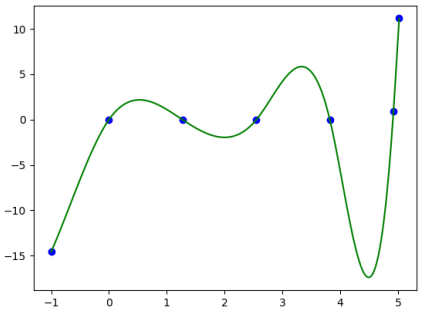

Figure 2 **钳制边值条件**

Problem 2:

随着插值点的增加，我们观察到插值函数曲线（绿色）与真实函数（红色）的误差逐渐减少，但是可以观察到初始部分与真实函数总是有一定的误差。

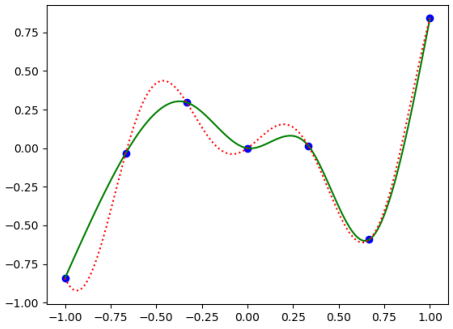

Figure 3 **插值 7 个点**

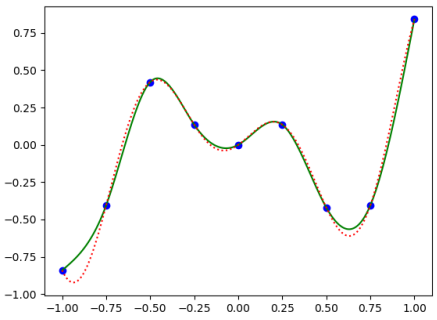

Figure 4 **插值 9 个点**

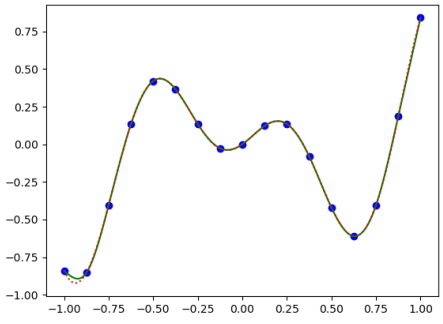

Figure 5 **插值 17 个点**

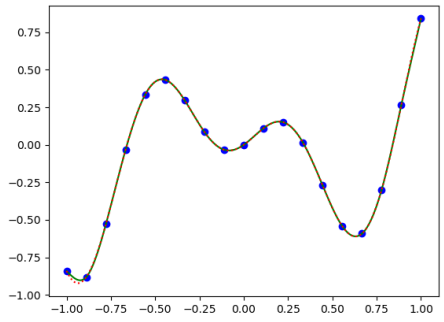

Figure 6 **插值 19 个点**

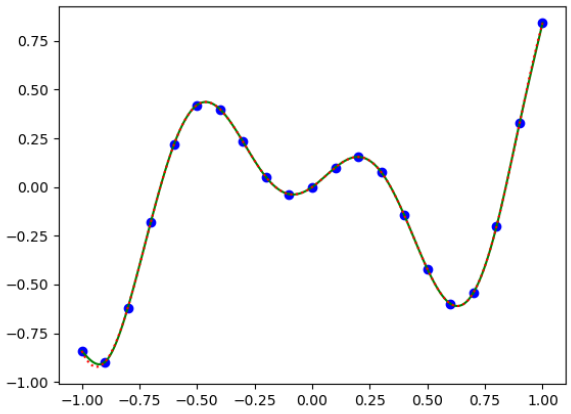

Figure 7 **插值** 21 **个点**

Problem 3:

使用钳制插值，斜率值设定为第一二两个点的斜率和最后两个点的斜率尝试减少与插值函数的误差：

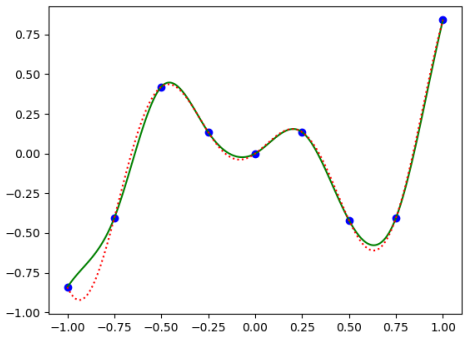

Figure 8 插值 7 个点

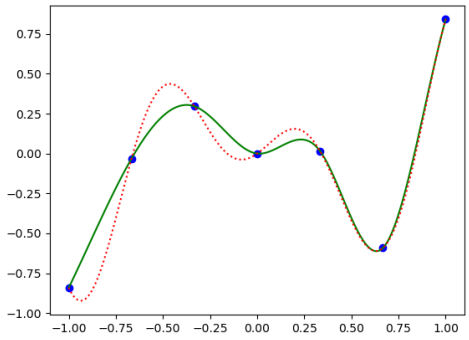

Figure 9 插值 9 个点

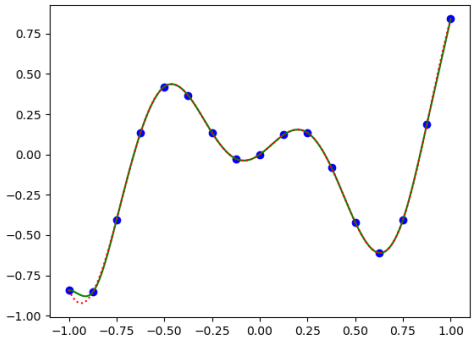

Figure 10 插值 17 个点

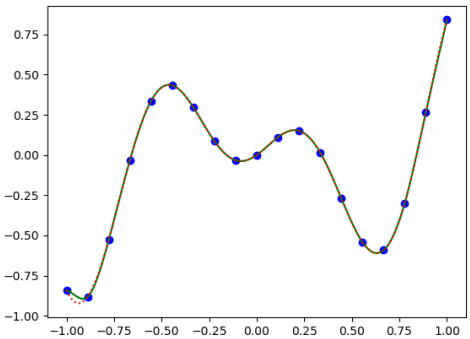

Figure 11 插值 19 个点

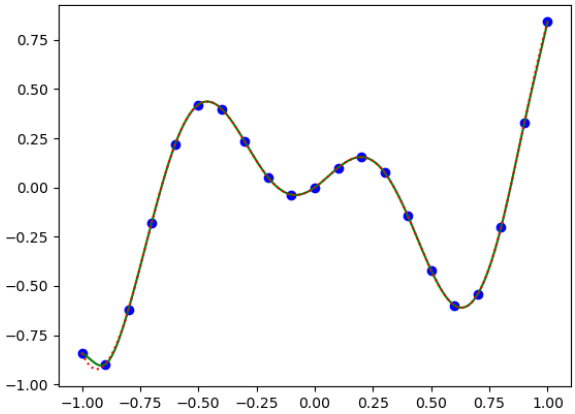

Figure 12 **插值 21 个点**

Problem 4:

打包插值函数（natural_cubic_splines.py），先对$x-t$和$y-t$分别进行插值，返回插值函数值，然后使用参数函数绘图。

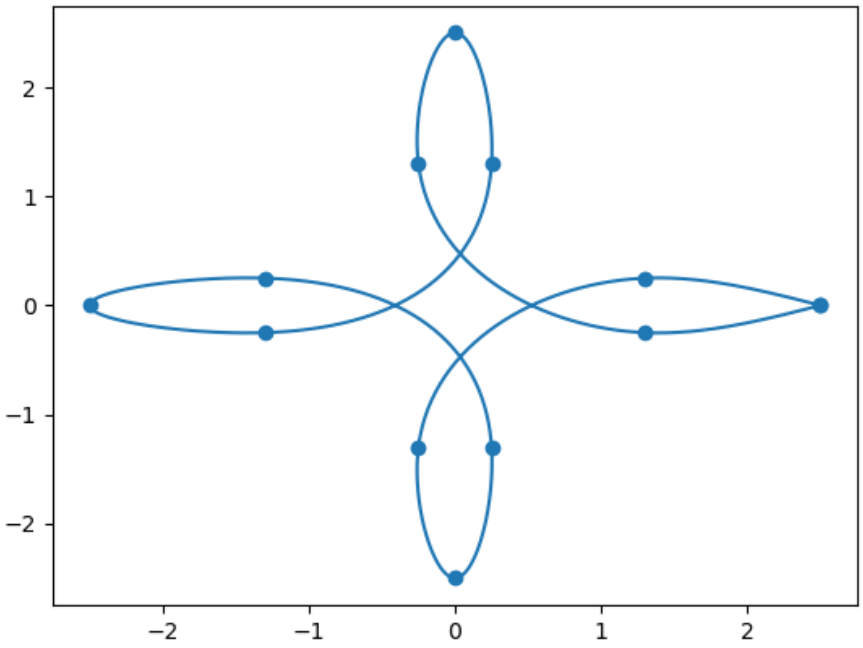

Figure 13 **Problem 4 图像**