MATH-597-Project3

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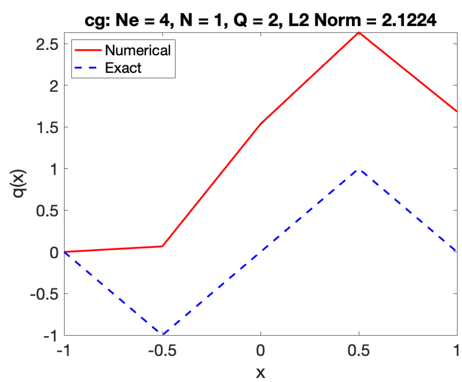
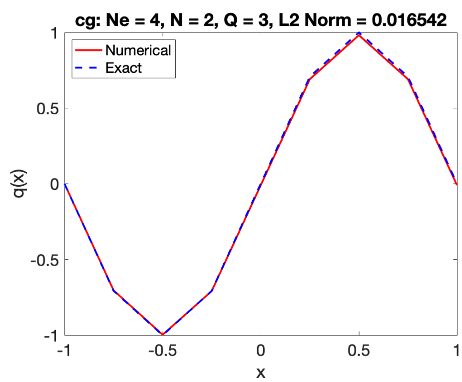
1 Convergence Studies of the CG Method

Before going into the convergence studies, we would like to describe some of the observations noted while carrying out the simulation, especially concerning the correctness of the numerical solution and the accuracy. Below, we present some findings of the CG exact integration.

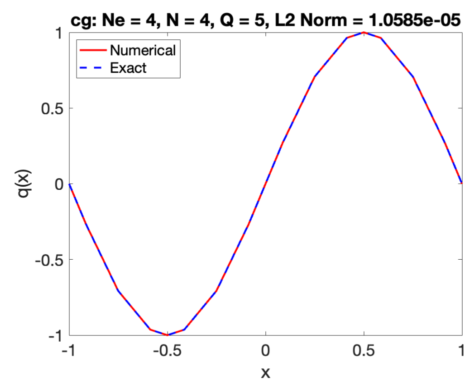
Using a constant number of elements, $Ne = 4$, and increasing the order of the polynomial, N, we observe that for lower orders N, especially for $N = 1$, there are noticeable differences between the exact and CG exact solutions as shown in Figure 1. However, as N increases, the CG solution becomes more accurate and approximates the exact solution better. This is particularly indicated by the L_2 norm, which significantly reduces as the number of gridpoints increases with an increase in the polynomial order. Also, as N_p increases, where $N_p = Ne \cdot N + 1$, the solution becomes smoother.



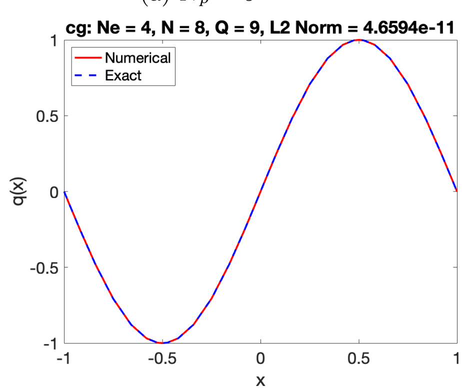
(a) $N_p = 5$



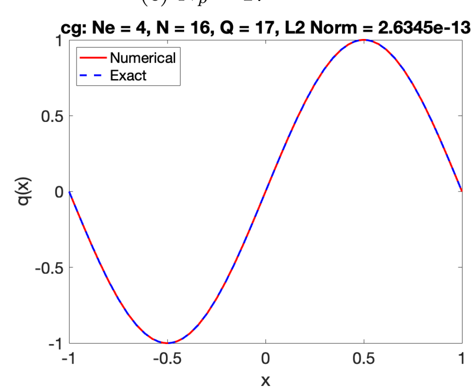
(b) $N_p = 9$



(c) $N_p = 17$



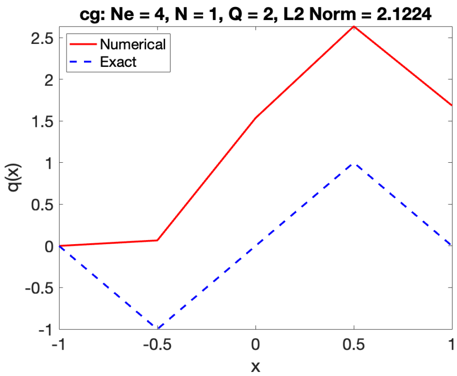
(d) $N_p = 33$



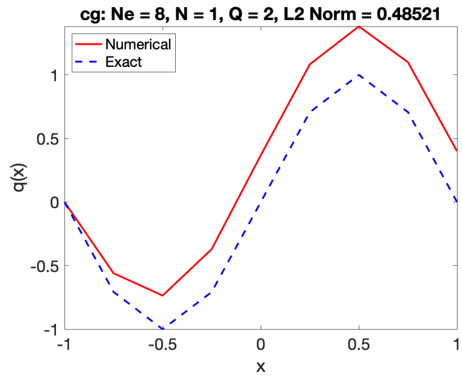
(e) $N_p = 65$

Figure 1: exact and CG numerical solution for the 1D Helmholtz equation using $N = 1, 2, 4, 8$ and 16 with exact integration. Observe that the error norm greatly reduces as the order of the polynomial increases, with a corresponding increase in N_p. This is evident in how the numerical solution gets closer to the exact solution.

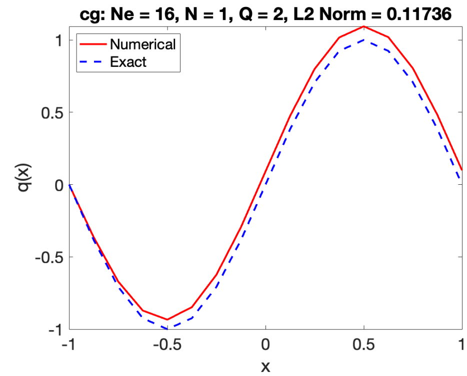
We also show how the number of elements affects the accuracy of the CG exact integration in Figure 2. Using $N = 1$, observe that with the least number of elements, *i.e.*, $Ne = 4$, the CG exact solution is quite different from the exact solution of the Helmholtz equation. In fact, the numerical value of $q(x)$ goes as high as 2.5, whereas the maximum of the analytical solution is 1. As we increase Ne, the total number of gridpoints, N_p, also increases, thereby increasing the accuracy of the solution. This is reflected in the L_2 norm, which reduces with each increase in Ne.



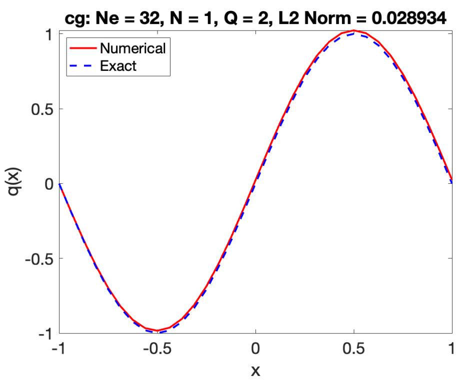
(a) $N_p = 5$



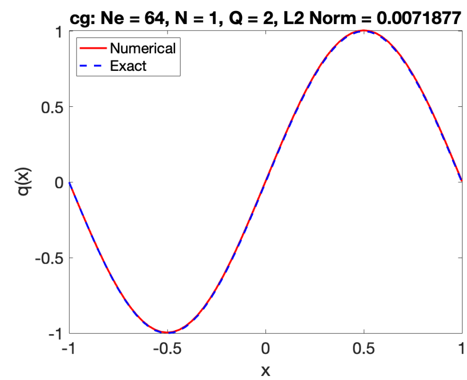
(b) $N_p = 9$



(c) $N_p = 17$



(d) $N_p = 33$



(e) $N_p = 65$

Figure 2: exact and CG solutions with exact integration using polynomial of order $N = 1$, and $Ne = 4, 8, 16, 32$, and 64 for the 1D Helmoltz equation. Observe that the error norm greatly reduces as the order of the polynomial increases. This is evident in how the numerical solution gets closer to the exact solution.

1.1 Convergence of CG: exact and inexact

Figure 3 displays the convergence rates of CG exact and inexact solutions, with the convergence rates of the CG exact solution for the various values of the polynomial order, N, given by the black lines. By merely looking at the plot, we observe that for all values of N, except $N = 16$, the error norms of the CG exact and inexact solutions decrease as the number of elements increases with a corresponding increase in the number of gridpoints.

For $N = 1$ and $N = 8$, for most values of Ne, we observe that the L_2 norm of the CG exact integration is greater than that of the inexact integration. While for $N = 2$ and $N = 4$, the L_2 norm of the CG exact integration is less than that of the inexact. The most striking thing in Figure 3 is the convergence rates of both methods at $N = 16$. As Ne increases from 1 to 2, we observe that the L_2 norm of the CG exact method is less than that of the inexact method. When Ne increases further, the L_2 norm of the CG exact integration rises steadily until it finally surpasses that of the inexact integration. It also surpasses the norm of both integration types at $N = 8$.

On the other hand, the error norm for $N = 16$ of the inexact method increases a little but remains relatively constant and stays below the norm of $N = 8$ throughout the simulation. Typically, one would expect that increasing the number of elements of a higher-order polynomial like $N = 16$ would decrease the error norm and improve the solution. However, this is not the case because the accuracy of the solution was already established for a lower number of elements (as indicated by the very low error norms), and as such, the solution cannot get any better.

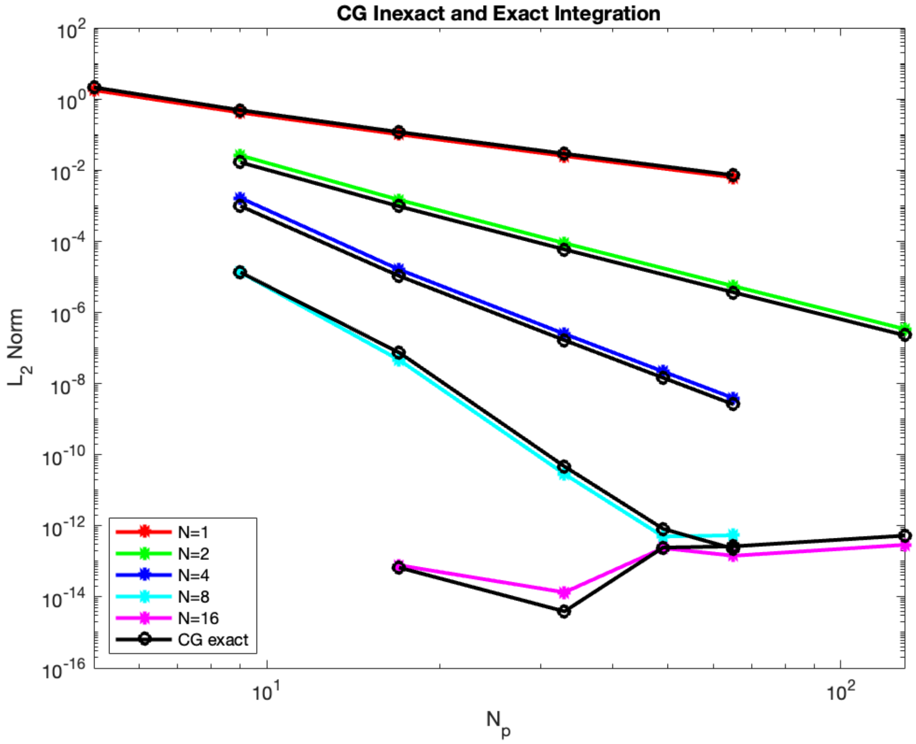


Figure 3: Convergence rates of CG exact ($Q = N + 1$) and inexact ($Q = N$) integration for the 1D Helmholtz equation for polynomial orders $N = 1, 2, 4, 8$, and 16 using a total number of gridpoints, N_p. Generally speaking, for both exact and inexact integration, the error norm decreases as the polynomial order increases. Furthermore, the error norm also decreases as the number of elements, Ne, for each N increases with a corresponding increase in N_p.

1.2 Convergence rates

The slopes, r, of the log-log plot of the convergence of CG exact and inexact solutions are given in Table 1.

N	r-exact	r-inexact
1	-2.2056	-2.1925
2	-4.1918	-4.2070
4	-6.4624	-6.5162
8	-9.4947	-9.2597
16	-4.2782	-2.6442

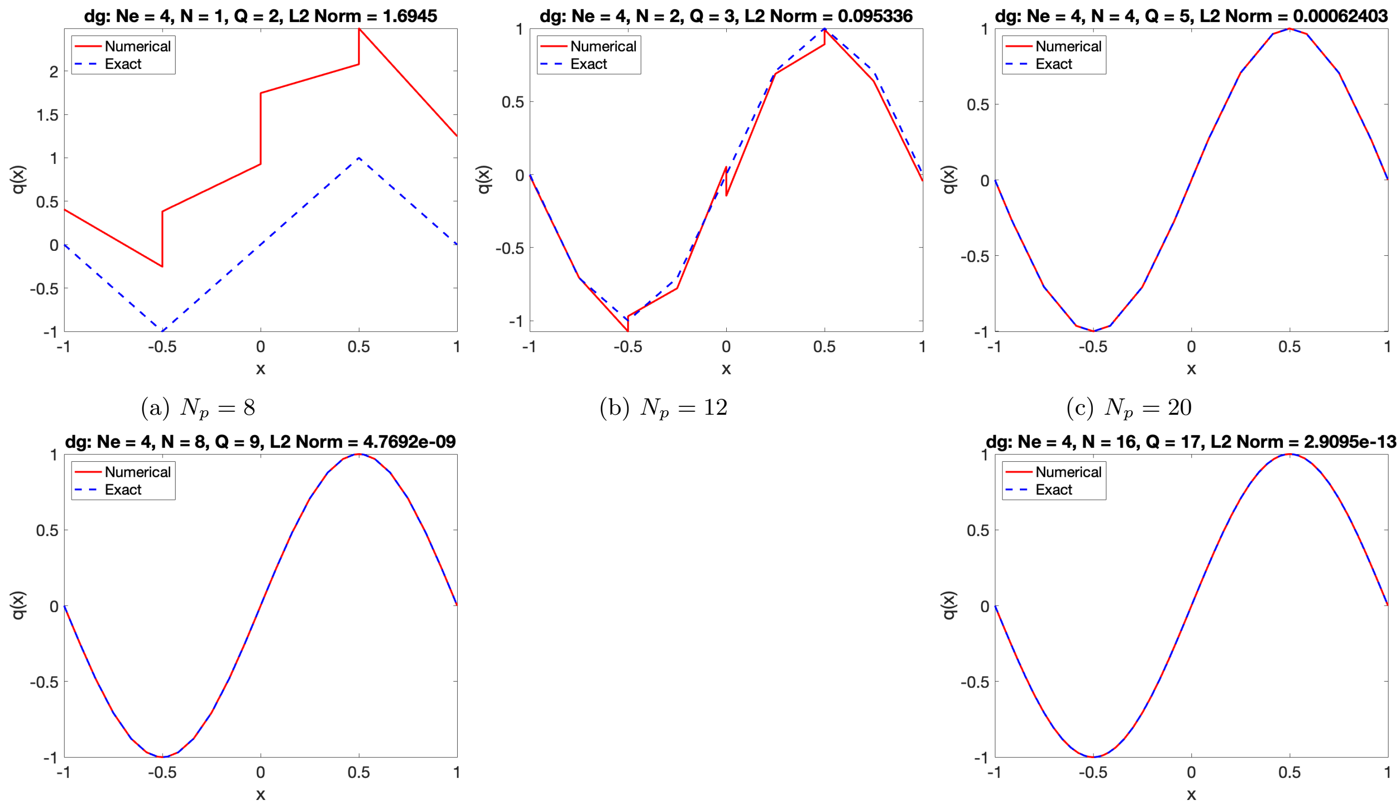
Table 1: Convergence rates of CG exact and inexact integration

Neglecting the signs, observe that both CG's exact and inexact integration convergence rates fall within the same range. More critically, for $N = 2$, and $N = 4$, we observe that the convergence rates of the CG inexact integration are higher than that of the exact integration. As a result, the CG inexact integration converges faster to the analytical (exact) solution than its exact counterpart, whereas, for $N = 1, 8$, and 16, the exact integration converges faster than the inexact. Comparing r of the CG exact and inexact integration with the expected value of $r = N+1$, we note that for $N = 1$ and $N = 8$, the slopes of the CG exact and inexact integration are approximately the expected value. For $N = 2$ and $N = 4$, the slopes obtained for the CG exact and inexact integration are approximately $N + 2$, which is one more than the expected slope.

For $N = 16$, we used only $Ne = 1$ and $Ne = 2$ to calculate the slopes. As shown in Table 1, the slopes of the two integration types fall short of the expected slope of 17. The rate of convergence for $N = 16$ for both integration types is smaller than the expected value because their errors start out very low, increase, and then plateau because the numerical solution cannot get any better than it already is.

2 Convergence Studies of the DG Method

Similar to what we did in the convergence studies of the CG, we first present some findings of the DG exact integration, most especially to show the correctness of the numerical solution and the accuracy. Keeping the number of elements, $Ne = 4$ fixed and increasing the polynomial order N, we present the analytical and DG exact solution in Figure 4. We observe that for low orders of N, especially $N = 1$ and $N = 2$, there are some noticeable differences between the exact and DG solution. As N increases, causing a corresponding increase in the number of gridpoints, the accuracy of the DG solution increases, as indicated by the decrease in the L_2 norms. Also, the sharp points in the exact and DG solutions become smoother because more nodal points are used to represent the solution.



(a) $N_p = 8$ (b) $N_p = 12$ (c) $N_p = 20$

(d) $N_p = 36$ (e) $N_p = 68$

Figure 4: Exact and DG numerical solution for the 1D Helmholtz equation using $N = 1, 2, 4, 8$ and 16 with exact integration. Observe that the error norm greatly reduces as the order of the polynomial increases, with a corresponding increase in N_p. This is evident in how the numerical solution gets closer to the exact solution.

2.1 Convergence of DG: exact and inexact

Figure 5 displays the convergence plot of the DG exact and inexact integration. Generally speaking, for both exact and inexact integration, the L_2 error norm decreases as the polynomial order increases, except for the case when $N = 16$. For $N = 16$, as Ne increases from 1 to 2, the error norms for both the exact and inexact integration decrease. However, as Ne increases to 3 and further, the error norm of the DG exact integration increases. On the other hand for the inexact integration, as Ne increases to 3, the error norm decreases further before increasing with a subsequent increase in Ne.

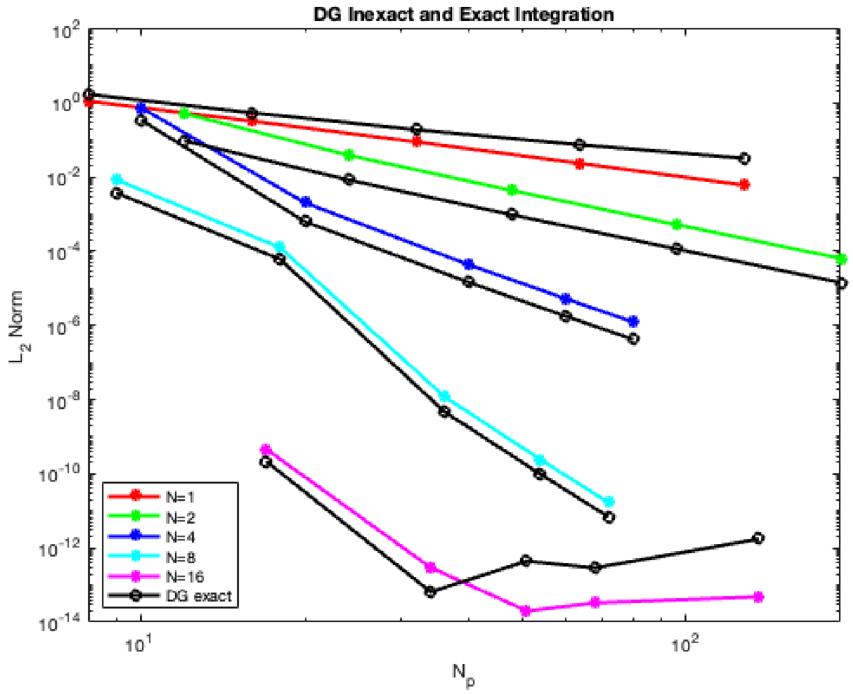


Figure 5: Convergence rates of DG exact ($Q = N + 1$) and inexact ($Q = N$) integration for the 1D Helmholtz equation for polynomial orders $N = 1, 2, 4, 8,$ and 16 using a total number of gridpoints, N_p. Observe that, in general, for each N, as Ne increases, with a corresponding increase in N_p, the L_2 norm decreases.

2.2 Convergence rates

The slopes, r, of the log-log plot of the convergence of DG exact and inexact solutions are given in Table 2.

N	r-exact	r-inexact
1	-1.4337	-1.8850
2	-3.1588	-3.2152
4	-6.4036	-6.2867
8	-10.1574	-10.0920
16	-11.6685	-10.5851

Table 2: Convergence rates of DG exact and inexact integration

Neglecting the signs, we observe that for $N = 1$ and $N = 2$, the convergence rates of the DG inexact integration are higher than that of the exact integration. However, for $N = 4, 8$ and 16, the convergence rates of the exact integration are higher than that of the inexact integration. This implies that the DG inexact integration converged faster to the analytic solution and outperformed the exact integration for lower orders of N, while the exact integration stood out for its better performance when higher orders of N were implemented.

Comparing the slopes from both integration types with the expected slope, $r = N + 1$, we note that the convergence rate of the DG inexact is closer to the expected slope at $N = 1$ than the convergence rate of the exact integration. For $N = 2$, the slopes of both integration types were close to the expected slope of 3. Although the convergence rate of the DG exact for $N = 16$ is higher than that of the inexact, they are both less than the expected rate of 17.

3 Findings

Throughout this exercise, we observe that for lower orders of the polynomial, N, both the CG and DG inexact integration methods converge faster than their exact counterparts, respectively. The only exception was the convergence rate of CG exact method when $N = 1$, which was higher than the convergence of the CG inexact. In order to compare the results of the CG and DG methods, we make use of the convergence rates, r.

To compare the convergence rates of the CG exact and DG exact, we present their slopes side by side in Table 3.

N	CG exact	DG exact
1	-2.2056	-1.4337
2	-4.1918	-3.1588
4	-6.4624	-6.4036
8	-9.4947	-10.1574
16	-4.2782	-11.6685

Table 3: Convergence rates of CG exact and DG exact integration

By neglecting the signs, we observe that for lower orders of N, *i.e.*, for $N = 1, 2$, and 4, the CG exact integration converges faster than the DG exact integration. However, for $N = 8$ and $N = 16$, the convergence rate of the DG exact integration is faster than that of the CG exact. We can infer from this that for the Helmholtz equation, the DG exact integration performs better than the CG exact for higher orders of N, while the reverse is the case for lower orders of N.

Similarly, we shall compare the inexact integration of the CG and DG methods. Table 4 displays the comparison.

N	CG inexact	DG inexact
1	-2.1925	-1.8850
2	-4.2070	-3.2152
4	-6.5162	-6.2867
8	-9.2597	-10.0920
16	-2.6442	-10.5851

Table 4: Convergence rates of CG exact and inexact integration

As usual, we neglect the signs of the slopes. We note again that for $N = 1, 2$, and 4, the CG inexact integration converges faster to the analytical solution than the DG inexact integration, whereas for $N = 8$ and $N = 16$, the DG inexact converges to the analytical solution faster than the CG inexact. This implies that for the given Helmholtz equation, for lower orders of N, the order of convergence of the CG inexact integration is higher than that of the DG inexact integration. On the other hand, for higher orders of N, the order of convergence of the DG inexact integration is higher than that of its CG counterpart.

In conclusion, for lower orders of the polynomial, N, *i.e.*, $N = 1, 2, 4$, the CG method has a higher convergence rate and performs better than the DG method. On the contrary, for higher orders of N, the DG method has a higher convergence rate and outperforms the CG method. The somewhat low performance of the DG can be attributed to the enforcement of the boundary conditions. Despite the repercussions associated with it, as seen in the low convergence rate of lower orders, it was able to perform better than the CG for orders $N = 8$ and $N = 16$.