To approximate the Helmholtz equations $\frac{\delta^2 q}{\delta x^2} + q = f(x)$ using the CG method, we first expand $\frac{\delta^2 q}{\delta x^2}$, q, and $f(x)$ via ~~the~~ the Lagrange polynomial ~~basis~~ basis

$$\frac{\delta^2 q}{\delta x} \approx \frac{\delta^2 q_N^{(e)}}{\delta x^2} = \frac{\delta^2}{\delta x^2} \sum_{j=0}^{N} \psi_j(x)\, q_j^{(e)} \qquad (*)$$

$$q \approx q_N^{(e)} = \sum_{j=0}^{N} \psi_j(x) q_j^{(e)} \; ; \; f(x) \approx f_N^{(e)}(x) = \sum_{j=0}^{N} \psi_j(x) f_j^{(e)}(x) = \sum_{j=0}^{N} \psi_j(x) f_j^{(e)}$$

where ψ_j, $j=0,\dots,N$ are the basis functions and q_j and f_j are the expansion coefficients, respectively.

The multiply the Helmholtz equation by the test function ψ_i, where $\psi_i \in H$ ~~(H is the H~~ and the integrate over the domain Ω_e

$$\Rightarrow \int_{\Omega_e} \psi_i \frac{\delta^2 q}{\delta x^2} d\Omega_e + \int_{\Omega_e} \psi_i q \, d\Omega_e = \int_{\Omega_e} \psi_i f(x) d\Omega_e \quad \text{---} \quad (1)$$

We can write the partial derivative in (1) using the total derivative since q is only a function of x.

$$\Rightarrow \int_{\Omega_e} \psi_i \frac{d^2 q}{dx^2} d\Omega_e + \int_{\Omega_e} \psi_i q \, d\Omega_e = \int_{\Omega_e} \psi_i f(x) d\Omega_e \quad \text{---} \quad (2)$$

but $\frac{d}{dx}\left(\psi_i \frac{dq}{dx}\right) = \frac{d\psi_i}{dx}\frac{dq}{dx} + \psi_i \frac{d^2 q}{dx^2} \Rightarrow \psi_i \frac{d^2 q}{dx^2} = \frac{d}{dx}\left(\psi_i \frac{dq}{dx}\right) - \frac{d\psi_i}{dx}\frac{dq}{dx} \qquad (**)$

Substituting (**) into (2) gives

$$\int_{\Omega_e} \frac{d}{dx}\left(\psi_i \frac{dq}{dx}\right) d\Omega_e - \int_{\Omega_e} \frac{d\psi_i}{dx}\frac{dq}{dx} d\Omega_e + \int_{\Omega_e} \psi_i q \, d\Omega_e = \int_{\Omega_e} \psi_i f(x) d\Omega_e$$

Substituting the expansions in (*) yields

$$\int_{\Omega_e} \frac{d}{dx}\left(\psi_i \frac{dq_N^{(e)}}{dx}\right) d\Omega_e - \int_{\Omega_e} \frac{d\psi_i}{dx}\frac{dq_N^{(e)}}{dx} d\Omega_e + \int_{\Omega_e} \psi_i q_N^{(e)} d\Omega_e = \int_{\Omega_e} \psi_i f_N^{(e)} d\Omega_e$$

$$\Rightarrow \left[\psi_i \frac{dq_N^{(e)}}{dx}\right]_{\Gamma_e} - \int_{\Omega_e} \frac{d\psi_i}{dx}\frac{d}{dx}\sum_{j=0}^{N} \psi_j q_j^{(e)} d\Omega_e + \int_{\Omega_e} \psi_i \sum_{j=0}^{N} \psi_j q_j^{(e)} d\Omega_e = \int_{\Omega_e} \psi_i \sum_{j=0}^{N} \psi_j f_j^{(e)} d\Omega_e$$

$$\Rightarrow \left[\psi_i \frac{dq_N^{(e)}}{dx}\right]_{\Gamma_e} - \sum_{j=0}^{N} \int_{\Omega_e} \left(\frac{d\psi_i}{dx}\frac{d\psi_j}{dx}\right) d\Omega_e \, q_j^{(e)} + \sum_{j=0}^{N} \int_{\Omega_e} (\psi_i \psi_j) d\Omega_e \, q_j^{(e)} = \sum_{j=0}^{N} \int_{\Omega_e} (\psi_i \psi_j) d\Omega_e \, f_j^{(e)}$$

Using Einstein's notation

$$\Rightarrow \left[\psi_i \frac{dq_N^{(e)}}{dx}\right]_{\Gamma_e} - \int_{\Omega_e} \frac{d\psi_i}{dx} \cdot \frac{d\psi_j}{dx} d\Omega_e \, q_j^{(e)} + \int_{\Omega_e} \psi_i \psi_j \, d\Omega_e \, q_j^{(e)} = \int_{\Omega_e} \psi_i \psi_j \, d\Omega_e \, f_j^{(e)}$$

Introducing the mapping from the physical element Ω_e to the computational element $\hat{\Omega}$ and invoking numerical integration yields

$$\left[\psi_i \frac{dq_N^{(e)}}{dx}\right]_{\Gamma_e} - \sum_{K=0}^{Q} W_K \left(\frac{d\psi_i(\xi_K)}{d\xi}\frac{d\xi_K}{dx}\right)\left(\frac{d\psi_j(\xi_K)}{d\xi}\frac{d\xi_K}{dx}\right)\frac{dx}{d\xi} q_j^{(e)}$$

$$+ \sum_{K=0}^{Q} W_K \psi_i(\xi_K)\psi_j(\xi_K)\frac{dx}{d\xi} q_j^{(e)} = \sum_{K=0}^{Q} W_K \psi_i(\xi_K)\psi_j(\xi_K)\frac{dx}{d\xi} f_j^{(e)},$$

where W_K, for $K=0,1,\dots,Q$ are the quadrature weights, and $\psi_i(\xi_K)$ and $\psi_j(\xi_K)$ are Lagrange basis functions.

Introducing the metric and Jacobian terms: $\frac{d\xi}{dx} = \frac{2}{\Delta x^{(e)}}$ and $\frac{dx}{d\xi} = \frac{\Delta x^{(e)}}{2}$

$$\Rightarrow \left[\psi_i \frac{dq_N^{(e)}}{dx}\right]_{\Gamma_e} - \sum_{K=0}^{Q} W_K \left(\frac{d\psi_i(\xi_K)}{d\xi}\frac{d\psi_j(\xi_K)}{d\xi}\right)\cdot\left(\frac{2}{\Delta x^{(e)}}\right)^2 \cdot \frac{\Delta x^{(e)}}{2} q_j^{(e)}$$

$$+ \frac{\Delta x^{(e)}}{2}\sum_{K=0}^{Q} W_K \psi_i(\xi_K)\psi_j(\xi_K) q_j^{(e)} = \frac{\Delta x^{(e)}}{2}\sum_{K=0}^{Q} W_K \psi_i(\xi_K)\psi_j(\xi_K) f_j^{(e)}$$

$$\Rightarrow \left[\psi_i \frac{dq_N^{(e)}}{dx}\right]_{\Gamma_e} - \frac{2}{\Delta x^{(e)}}\sum_{K=0}^{Q} W_K \frac{d\psi_i(\xi_K)}{d\xi}\frac{d\psi_j(\xi_K)}{d\xi} q_j^{(e)} + \frac{\Delta x^{(e)}}{2}\sum_{K=0}^{Q} W_K \psi_i(\xi_K)\psi_j(\xi_K) q_j^{(e)}$$

$$= \frac{\Delta x^{(e)}}{2}\sum_{K=0}^{Q} W_K \psi_i(\xi_K)\psi_j(\xi_K) f_j^{(e)} \quad (3)$$

Writing (3) in the corresponding element matrix form yields

$$B_i^{(e)} - L_{ij}^{(e)} q_j^{(e)} + M_{ij}^{(e)} q_j^{(e)} = M_{ij}^{(e)} f_j^{(e)} \quad \text{---} (4)$$

where

$$B_i^{(e)} = \left[\psi_i \frac{dq_N^{(e)}}{dx}\right]_{\Gamma_e}; \quad L_{ij}^{(e)} = \frac{2}{\Delta x^{(e)}}\sum_{K=0}^{Q} W_K \frac{d\psi_i(\xi_K)}{d\xi}\frac{d\psi_j(\xi_K)}{d\xi}$$

$$M_{ij}^{(e)} = \frac{\Delta x^{(e)}}{2}\sum_{K=0}^{Q} W_K \psi_i(\xi_K)\psi_j(\xi_K)$$

Applying the Direct Stiffness Summation (DSS) operator yields the following global matrix problem

$$B_I - L_{IJ} q_J + M_{IJ} q_J = M_{IJ} f_J .$$

where q_J, f_J are the values of q and f at the global gridpoints $J = 1,\dots,N_p$ and

$$M_{IJ} = \bigwedge_{e=1}^{N_e} M_{ij}^{(e)}, \quad L_{IJ} = \bigwedge_{e=1}^{N_e} L_{ij}^{(e)}, \quad B_I = \bigwedge_{e=1}^{N_e} B_i^{(e)}$$

where the DSS operator performs the summation via the mapping $(i,e) \to (I)$. where $(i,j) = 0,\dots,N$, $e = 1,\dots,N_e$, $(I,J) = 1,\dots,N_p$, and N_e, N_p are the total number of elements and gridpoints.

To enforce Dirichlet boundary conditions on the left boundary, we first build the right-hand side vector $R_I = M_{IJ} f_J$. Then we modify $M_{IJ} - L_{IJ}$ and the vector $R_I - B_I$. Because the Dirichlet boundary condition is zero on the left boundary, we do not necessarily need to create B.

We modify in this manner; ~~we set~~ let $MSL = M_{IJ} - L_{IJ}$. Then set $MSL(1,1) = 1$ and $MSL(1, 2:N_p) = R_I(1) = 0$, which forces $q(1) = 0$.

To enforce the Neumann boundary condition on the right boundary, let us look at $B_i^{(e)}$.

$$
\begin{aligned}
B_i^{(e)} &= \left[\psi_i \frac{dq_N^{(e)}}{dx}\right]_{\Gamma_e} = \left[\cancel{\psi_i \frac{d}{dx}\sum_{j=0}^{N}} \psi_j\right. \\
&= \left[\psi_i \frac{d}{dx}\sum_{j=0}^{N} \psi_j q_j^{(e)}\right]_{\Gamma_e} = [\psi_i \psi_j] \left.\frac{dq}{dx}\right|_j = [\psi_i \psi_j] h_j, \text{ where } h_j = \left.\frac{dq}{dx}\right|_{j} \cancel{= -\pi}
\end{aligned}
$$

$$
\Rightarrow B_i^{(e)} = F_{ij}^{(e)} h_j(x) \quad , \text{ where } \quad F_{ij}^{(e)} = \begin{bmatrix} -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}
$$

After imposing the boundary conditions, we then solve the linear algebra problem

$$
(M_{IJ} - L_{IJ}) q_J = R_I - F_{IJ} h_J \quad , \quad F_{IJ} = \bigwedge_{e=1}^{N_e} F_{ij}^{(e)}
$$

$$
\Rightarrow q_J = (M_{IJ} - L_{IJ})^{-1} (R_I - F_{IJ} h_J)
$$

Here, $h_J = h_{Np} = -\pi$.

To approximate the Helmholtz equation $\frac{\partial^2 q}{\partial x^2} + q = f(x)$ using the DG method, we solve in two steps.

Step 1: Solve $Q = \frac{\partial q}{\partial x}$

Multiply both sides by the test function and integrate.

$$\Rightarrow \int_{\Omega_e} \psi_i Q \, d\Omega_e = \int_{\Omega_e} \psi_i \frac{\partial q}{\partial x} \, d\Omega_e$$

$$\Rightarrow \int_{\Omega_e} \psi_i Q \, d\Omega_e = \int_{\Omega_e} \psi_i \frac{dq}{dx} \, d\Omega_e$$ (we can write this using the total derivative since q is only a function of x)

$\Rightarrow$ ~~$\int$~~ Expanding the functions with the Lagrange basis and applying the idea from (**) i.e. the fundamental theorem of Calculus, we have

$$\int_{\Omega_e} \psi_i \sum_{j=0}^{N} \psi_j Q_j^{(e)} d\Omega_e = \left[\psi_i \sum_{j=0}^{N} \psi_j q_j^{(e)}\right]_{\Gamma_e} - \int_{\Omega_e} \frac{d\psi_i}{dx} \sum_{j=0}^{N} \psi_j q_j^{(e)} d\Omega_e$$

$$\Rightarrow \sum_{j=0}^{N} \int_{\Omega_e} \psi_i \psi_j \, d\Omega_e \, Q_j^{(e)} = \sum_{j=0}^{N} \left[\psi_i \psi_j q_j^{(e)}\right]_{\Gamma_e} - \sum_{j=0}^{N} \int_{\Omega_e} \frac{d\psi_i}{dx} \psi_j \, d\Omega_e \, q_j^{(e)}.$$

$$\Rightarrow \int_{\Omega_e} \psi_i \psi_j \, d\Omega_e \, Q_j^{(e)} = \left[\psi_i \psi_j q_j^{(e)}\right]_{\Gamma_e} - \int_{\Omega_e} \frac{d\psi_i}{dx} \psi_j \, d\Omega_e \, q_j^{(e)}$$ (Einstein's notation).

$$\Rightarrow M_{ij}^{(e)} Q_j^{(e)} = F_{ij}^{(e)} q_j^{(*,e)} - \tilde{D}_{ij} q_j^{(e)}$$; where $q_j^{(*,e)}$ is the central numerical flux and $\tilde{D}_{ij}$ is the differentiation matrix.

$$\Rightarrow M_{ij}^{(e)} Q_j^{(e)} = F_{ij}^{(*)} q_j^{(e)} - \tilde{D}_{ij} q_j^{(e)} \quad \text{--- (a).}$$ (here, the flux is shifted to F_{ij}).

Step 2: Solve $\frac{dQ}{dx} + q = f(x)$.

$$\Rightarrow \int_{\Omega_e} \psi_i \frac{dQ}{dx} d\Omega_e + \int_{\Omega_e} \psi_i q \, d\Omega_e = \int_{\Omega_e} \psi_i f(x) \, d\Omega_e.$$

$$\Rightarrow \left[\psi_i Q_N^{(e)}\right]_{\Gamma_e} - \int_{\Omega_e} \frac{d\psi_i}{dx} Q_N^{(e)} d\Omega_e + \int_{\Omega_e} \psi_i q_N^{(e)} d\Omega_e = \int_{\Omega_e} \psi_i f_N^{(e)} d\Omega_e$$

$$\Rightarrow \sum_{j=0}^{N} \left[\psi_i \psi_j Q_j^{(e)}\right]_{\Gamma_e} - \sum_{j=0}^{N} \int_{\Omega_e} \frac{d\psi_i}{dx} \psi_j \, d\Omega_e \, Q_j^{(e)} + \sum_{j=0}^{N} \int_{\Omega_e} \psi_i \psi_j \, d\Omega_e \, q_j^{(e)} = \sum_{j=0}^{N} \int_{\Omega_e} \psi_i \psi_j \, d\Omega_e \, f_j^{(e)}$$

$$\Rightarrow F_{ij}^{(e)} Q_j^{(*,e)} - \tilde{D}_{ij}^{(e)} Q_j^{(e)} + M_{ij}^{(e)} q_j^{(e)} = M_{ij}^{(e)} f_j^{(e)} \quad \text{--- (b)};$$ $Q^{(*,e)}$ is the central numerical flux.

$$\Rightarrow F_{ij}^{(*)} Q_j^{(e)} - \tilde{D}_{ij}^{(e)} Q_j^{(e)} + M_{ij}^{(e)} q_j^{(e)} = M_{ij}^{(e)} f_j^{(e)} \quad \text{--- (b)}$$ (like in (a), the flux is shifted to F_{ij}).

Introducing the mapping from the physical element Ω_e to the reference element $\hat{\Omega}$ with the following metric and Jacobian terms

$$\frac{d\xi}{dx} = \frac{2}{\Delta x} \quad \text{and} \quad \frac{dx}{d\xi} = \frac{\Delta x}{2}$$

We can write the element matrices as

$$M_{ij}^{(e)} = \frac{\Delta x^{(e)}}{2} \sum_{k=0}^{Q} w_k \psi_i(\xi_k) \psi_j(\xi_k),$$

$$F_{ij}^{(e)} = [\psi_i(\xi)\psi_j(\xi)]_{\Gamma}, \quad \text{and} \quad \tilde{D}_{ij}^{(e)} = \sum_{k=0}^{Q} w_k \frac{d\psi_i(\xi_k)}{d\xi} \psi_j(\xi_k).$$

Then we define the numerical fluxes $Q^{(*,e)}$ and $q^{(*,e)}$) using the centered fluxes

$$Q^{(*,e)} = \tfrac{1}{2}\left(Q^{(e)} + Q^{(k)}\right); \quad q^{(*,e)} = \tfrac{1}{2}\left(q^{(e)} + q^{(k)}\right)$$

where the superscript (k) denotes the neighbor of the element (e).

Equations (a) and (b) can be viewed as a global matrix problem because the fluxes connect adjacent elements. Here, we can use the direct stiffness summation to build the global matrix

$$B_I^{(q)} + F_{IJ} q_J - \tilde{D}_{IJ} q_J = M_{IJ} Q_J \quad \text{--- (c)}$$

$$B_I^{(Q)} + F_{IJ} Q_J - \tilde{D}_{IJ} Q_J + M_{IJ} q_J = M_{IJ} f_J \quad \text{--- (d)}$$

where the vectors B_I are due to the Dirichlet and Neumann boundary conditions with respect to Q and q, respectively, and $I, J = 1, \ldots, N_p$. where $N_p = N$

$$B^{(Q)} = \tfrac{1}{2} n_\Gamma \frac{dq}{dx}(x)\Big|_\Gamma \quad \text{and} \quad B^{(q)} = \tfrac{1}{2} n_\Gamma q(x)\Big|_\Gamma,$$

where n_Γ is the outward pointing unit normal vector of the physical boundary Γ, and the $\tfrac{1}{2}$ comes from the fact that the numerical flux being implemented is the average value.

$$B^{(Q)}_{(end)} = -\tfrac{1}{2}\pi \quad \text{and} \quad B^{(q)} = \underline{0}. \qquad \left(n_\Gamma = 1, \ \frac{dq}{dx}\Big|_{x=1} = -\pi, \ q|_{x=-1} = 0\right)$$

Rearranging (c) ~~and (d)~~

$$\therefore Q_J = M_{IJ}^{-1}\left(B_I^{(q)} + D^{DG} q_J\right), \text{ where } D^{DG} = F_{IJ} - \tilde{D}_{IJ}.$$

Substituting into (d)

$$B_I^{(Q)} + D^{DG} M_{IJ}^{-1}\left(B_I^{(q)} + D^{DG} q_J\right) + M_{IJ} q_J = M_{IJ} f_J$$

$$\Rightarrow \left(D^{DG} M_{IJ}^{-1} D^{DG} + M_{IJ}\right) q_J = M_{IJ} f_J - B_I^{(Q)} - D^{DG} M_{IJ}^{-1} B_I^{(q)}$$