MATH-597-Project2

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1 Derivation of Right-hand Side Matrix

1.1 CG

Using the direct stiffness summation (DSS) operator, we can write the global matrix-vector problem for CG as follows

$$
\begin{aligned}
M_{IJ}\frac{dq_J}{dt} + D_{IJ}f_J &= \mathbf{0}, \qquad \text{where } f = uq \\
\Rightarrow M_{IJ}\frac{dq_J}{dt} &= -D_{IJ}f_J \\
\Rightarrow M_{IJ}\frac{dq_J}{dt} &= -uD_{IJ}q_J \\
\Rightarrow \frac{dq_J}{dt} &= M_{IJ}^{-1}(-uD_{IJ})q_J = Rq_J,
\end{aligned}
$$

where M and D are the mass and derivative matrices, respectively.

Therefore the right-hand side matrix $R = M_{IJ}^{-1}(-uD_{IJ})$.

1.2 DG

Using the direct stiffness summation (DSS) operator, we can write the global matrix-vector problem for DG as follows

$$
\begin{aligned}
M_{IJ}\frac{dq_J}{dt} - \tilde{D}_{IJ}f_J + F_{IJ}f_J &= \mathbf{0}, \qquad \text{where } f = uq \\
\Rightarrow M_{IJ}\frac{dq_J}{dt} &= \tilde{D}_{IJ}f_J - F_{IJ}f_J \\
\Rightarrow \frac{dq_J}{dt} &= M_{IJ}^{-1}u(\tilde{D}_{IJ} - F_{IJ})q_J = Rq_J
\end{aligned}
$$

Therefore the right-hand side matrix $R = M_{IJ}^{-1}u(\tilde{D}_{IJ} - F_{IJ})$.

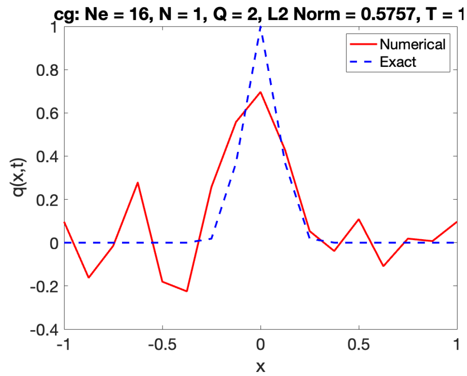
2 Convergence Studies of CG and DG

NOTE: In the simulations below, we used LGL points for both interpolation and integration.

2.1 Convergence Studies of CG

2.1.1 Exact Integration

Carrying out the exact integration for CG, we observe that for lower orders of the polynomial of the basis function, especially for $N \leq 4$, there are apparent differences between the exact and numerical solution as indicated by the L_2 error norm as shown in Figure 1. This is because we are trying to use very few global gridpoints (henceforth referred to as gridpoints), N_p to represent an exponential function, where $N_p = N_eN + 1$, and N and N_e are the polynomial order and total number of elements, respectively. However, as we increase the number of gridpoints by increasing the polynomial order and number of elements, the numerical solution better approximates the exact solution.



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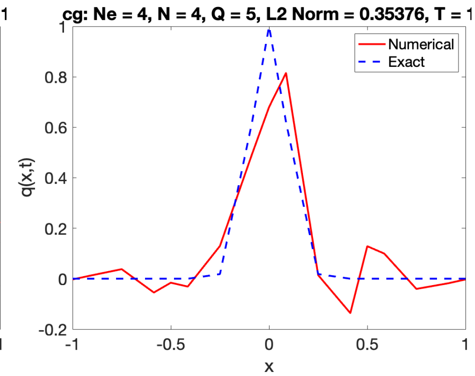

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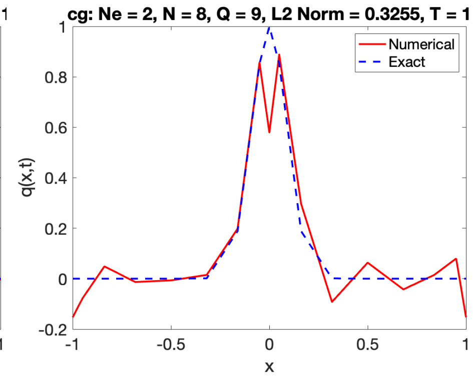

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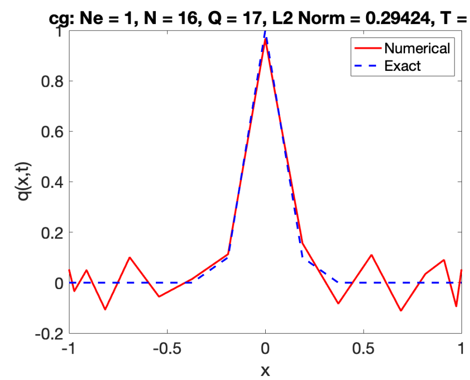
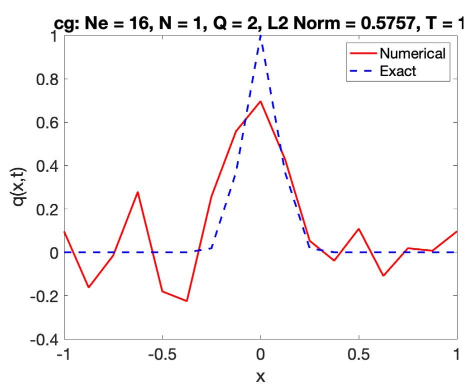


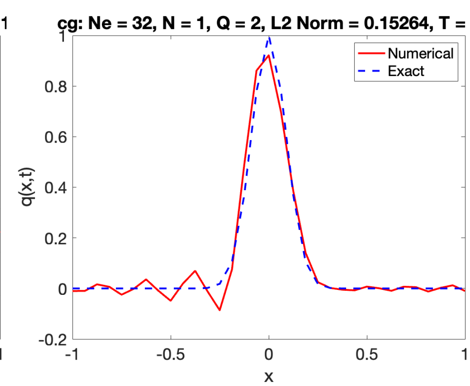
(d)

Figure 1: exact and CG numerical solution for the 1D wave equation using $N = 1, 4, 8$ and 16 with exact integration after one revolution ($t = 1$). Observe that the error norm greatly reduces as the order of the polynomial increases. This is evident in how the numerical solution gets closer to the exact solution.

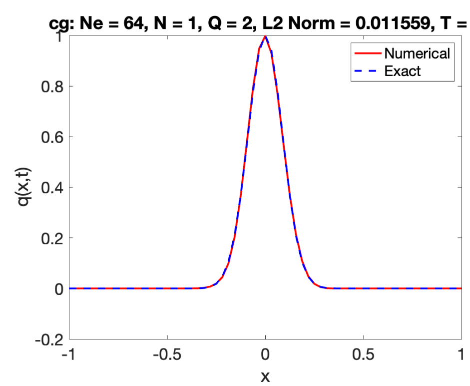
We proceed to show how the number of elements affects the accuracy of the CG exact integration. Figure 2 displays the plot of the exact and CG solutions with exact integration after one revolution ($t = 1$) using polynomial of order $N = 1$, and $Ne = 16, 32$, and 64.



(a) $N_p = 17$



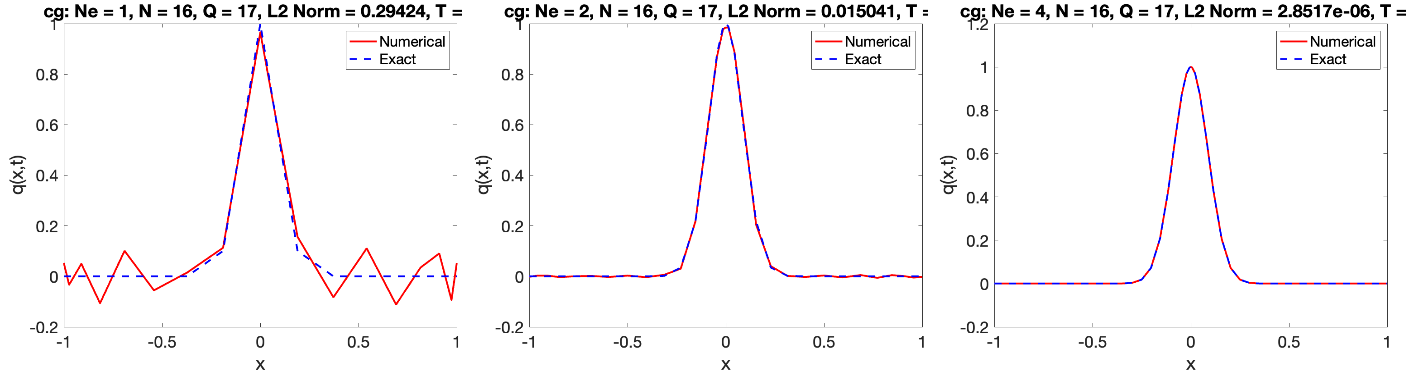
(b) $N_p = 33$



(c) $N_p = 65$

Figure 2: exact and CG numerical solution for the 1D wave equation using $N = 1$ order polynomials with exact integration ($Q = 2$) after one revolution ($t = 1$). Observe that the error norm greatly reduces as the number of gridpoints increases from 17 to 33 and then 65. This decrease in the error indicates that the numerical solution obtained using the CG exact integration becomes more accurate as the number of elements increases.

Observe that the bell shape of the exponential has a sharp tip that is smoothened when we increase the polynomial order. This is because more quadrature points are used to obtain the solution. This is shown in Figure 3, which is the plot of the exact and CG solution using exact integration with $N = 16$, and $Ne = 1, 2$, and 4.



(a) $N_p = 17$ (b) $N_p = 33$ (c) $N_p = 65$

Figure 3: exact and CG numerical solution for the 1D wave equation using $N = 16$ order polynomials with exact integration ($Q = 17$) after one revolution ($t = 1$). Observe that for $Ne = 1$, the error norm is about half of the error when $N = 1$. As Ne increases, the difference between the exact and numerical solution greatly decreases until there is no noticeable difference between them as shown when $Ne = 4$.

Figure 4 displays the plot of the convergence rates for the exact integration using the CG method. We observe that for each polynomial order, N, as the number of gridpoints increases (corresponding to the number of elements), the solution gets better, and the L_2 norm decreases. Also, increasing the polynomial order gives even more accurate results, as indicated by the decrease in the L_2 norm. We however note that the L_2 norm for $N = 4$ and $Ne = 8$ is higher than the one obtained when $N = 1$ and $Ne = 32$. The next increase in the number of elements decreases the error steeply.

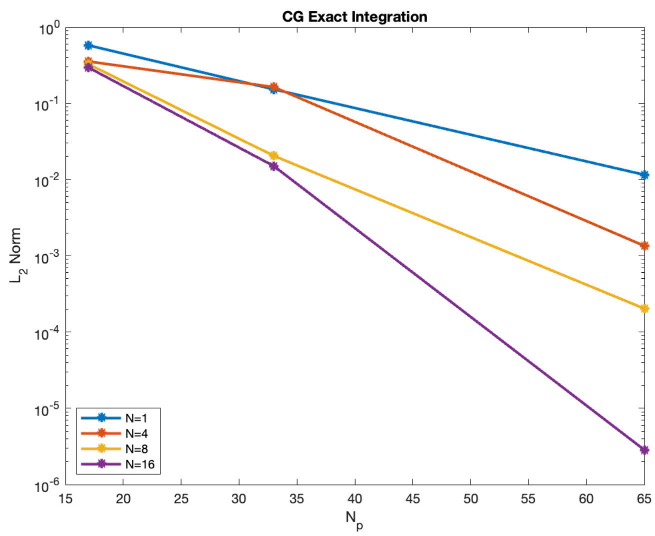
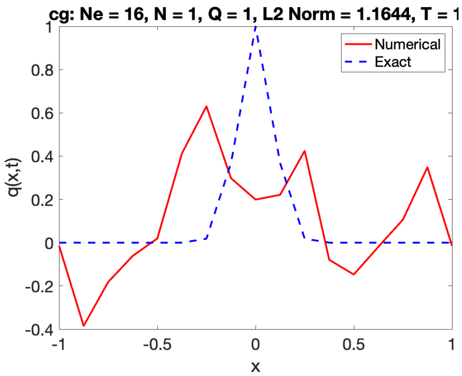


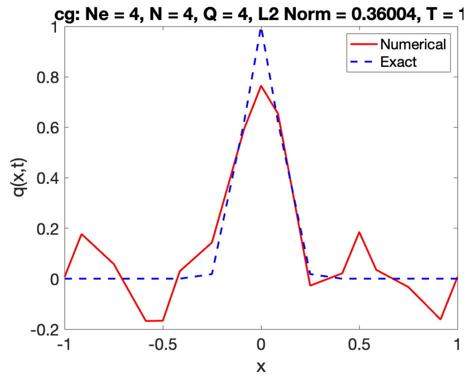
Figure 4: Convergence rates of CG for the 1D wave equation for polynomial orders $N = 1, 4, 8$, and 16 using a total number of gridpoints N_p for exact integration ($Q = N + 1$). Observe that as N increases, the L_2 norm gets smaller. For each N, we obtain the least norm with the largest number of gridpoints.

2.1.2 Inexact Integration

Figure 5 shows the results of the inexact integration with the CG method. We observe significant differences between the numerical solution (inexact CG) and the exact solution. As the order of the polynomial increases, the numerical solution approaches the exact solution, and the L_2 norm decreases. We, however, observed that as the order increases to 8, the L_2 norm increases. We also note that when compared with the L_2 norm of the exact CG, the L_2 norm of the inexact CG is larger, except where $N = 16$ and $Ne = 1$, at which the norms are equal.



(a)



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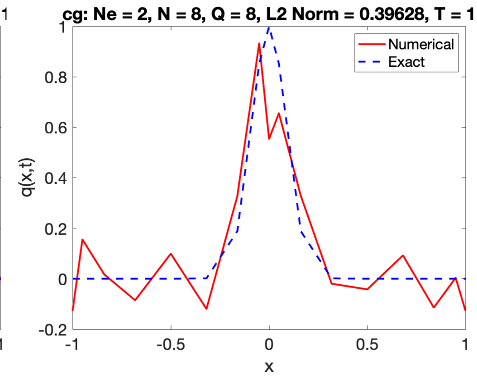

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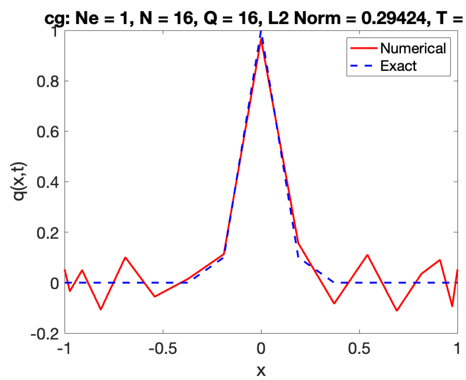
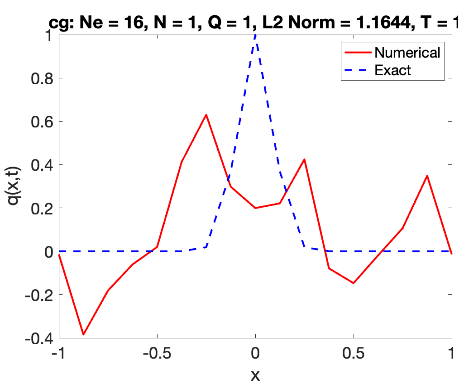


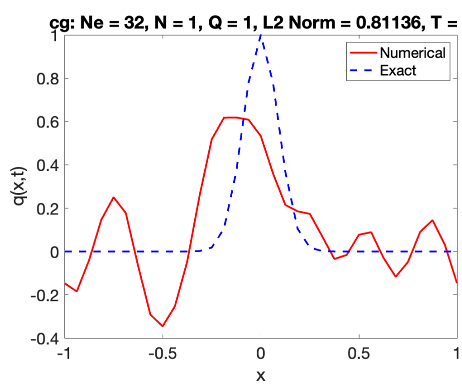
(d)

Figure 5: exact and CG numerical solution for the 1D wave equation using $N = 1, 4, 8$ and 16 with inexact integration ($Q = N$) after one revolution ($t = 1$).

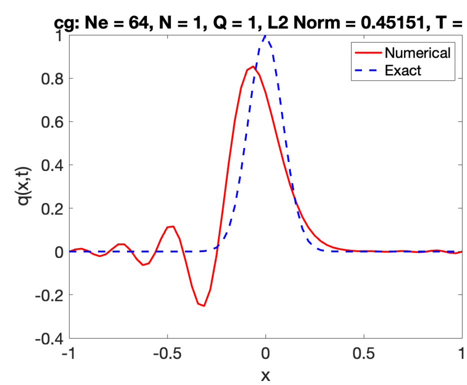
We proceed to show how the number of elements affects the accuracy of the CG inexact integration. Figure 6 displays the plot of the exact and CG solutions with inexact integration after one revolution ($t = 1$) using polynomial of order $N = 1$, and $Ne = 16, 32$, and 64.



(a) $N_p = 17$



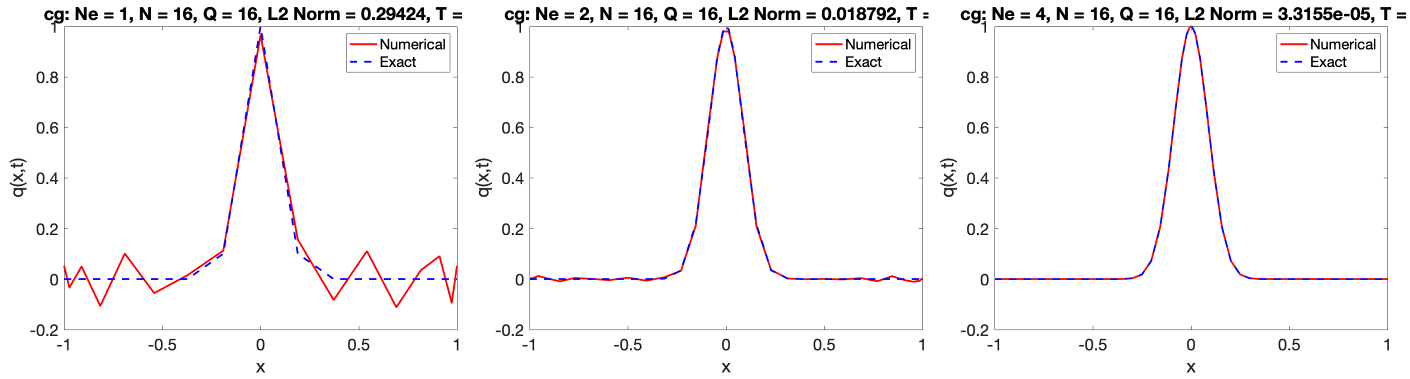
(b) $N_p = 33$



(c) $N_p = 65$

Figure 6: exact and CG numerical solution for the 1D wave equation using $N = 1$ order polynomials with inexact integration ($Q = 1$) after one revolution ($t = 1$). Observe that the error norm decreases by about a half as the number of gridpoints is increased from 17 to 33, and then 65. This is an indication that the numerical solution obtained using the CG exact integration becomes more accurate as the number of elements increase.

When we further increased the order of the polynomial to 16, we obtained a higher accuracy. The result displayed in Figure 7.



(a) $N_p = 17$

(b) $N_p = 33$

(c) $N_p = 65$

Figure 7: exact and CG numerical solution for the 1D wave equation using $N = 16$ order polynomials with inexact integration ($Q = 16$) after one revolution ($t = 1$).

Figure 8 shows the plot of the convergence rates for the inexact integration using the CG method. Similarly to the exact integration, we observe that as the order of the polynomial, N, increases along with the number of elements, the L_2 norm decreases. The only exception to this occurs when $N = 8$. We also note that for almost all values of N and Ne, the L_2 norm of the inexact integration is higher than that of the exact integration. For instance, for $N = 16$ and $Ne = 4$; for the inexact integration, we have the norm to be 3.3154×10^{-5}. On the other hand, for the same values of N and Ne, the norm of the exact integration is 2.8517×10^{-6}, which is significantly lower than that of the inexact integration. This alludes to the fact that the exact integration is more accurate than the inexact integration.

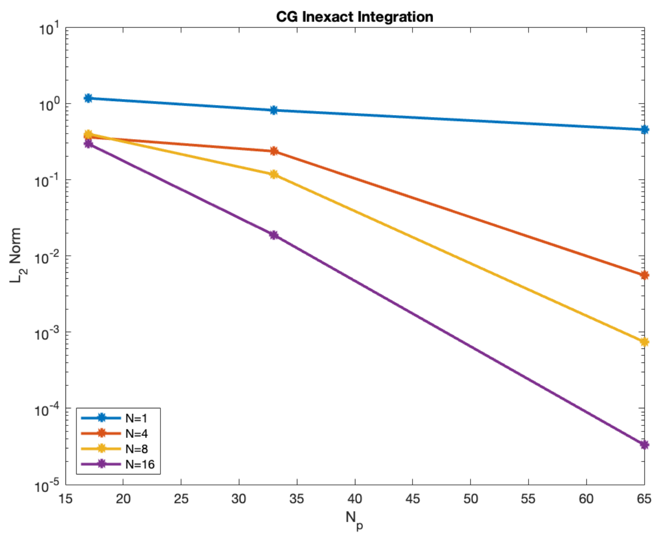


Figure 8: Convergence rates of CG for the 1D wave equation for polynomial orders $N = 1, 4, 8$, and 16 using a total number of gridpoints N_p for inexact integration ($Q = N$). Observe that as N increases, the L_2 norm gets smaller. For each N, we obtain the smallest error norm with the largest number of gridpoints.

2.2 Convergence Studies of DG

2.2.1 Exact Integration

Figure 9 displays the plot of the convergence rates for the exact integration using the DG method together with the Rusanov (upwind) flux. We see that as the number of elements increases, causing a corresponding increase in the number of gridpoints, which is calculated in the DG method as $N_p = (N+1)N_e$, the solution improves, and the L_2 norm decreases for each order of the polynomial, N. Also, increasing the polynomial order gives even more accurate results, as indicated by the decrease in the L_2 norm.

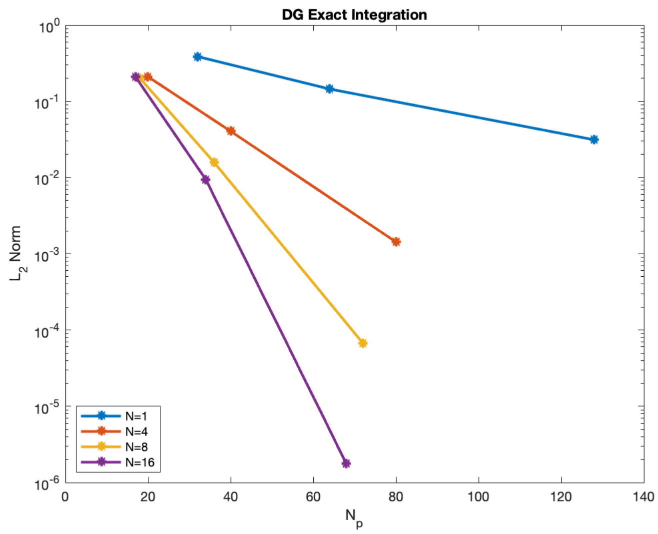


Figure 9: Convergence rates of DG for the 1D wave equation for polynomial orders $N = 1, 4, 8$, and 16 using a total number of gridpoints N_p for exact integration ($Q = N + 1$). For each N, we obtain the least norm with the largest number of gridpoints.

Using the centered flux, we observe that increasing the order of the polynomials as well as the number of elements gives accurate results, with a corresponding decrease in the L_2 nor. However, we noted that the L_2 norm for $N = 8$ was higher than that of $N = 4$. The plot is displayed below in Figure 10.

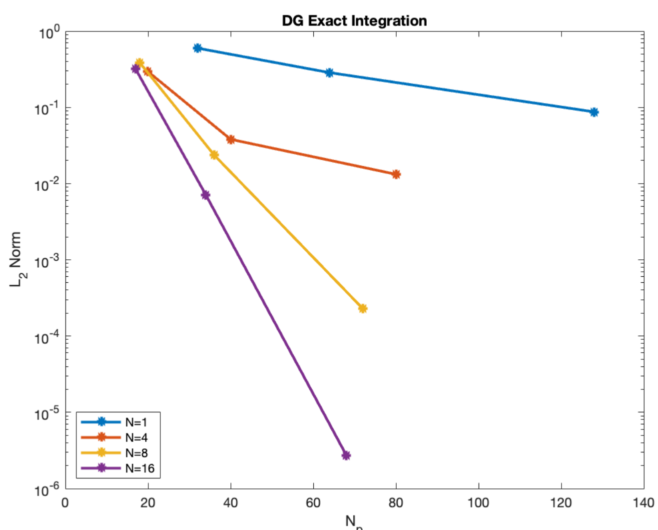
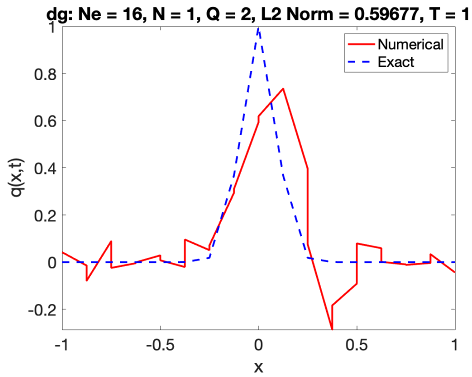
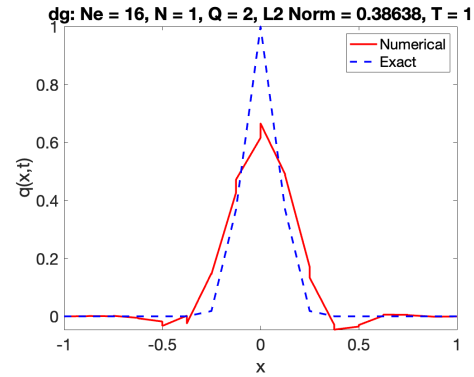


Figure 10: Convergence rates of DG for the 1D wave equation for polynomial orders $N = 1, 4, 8$, and 16 using a total number of gridpoints N_p for exact integration ($Q = N + 1$).

Comparing the results we obtained using the centred flux with that obtained using the Rusanov flux, we see that the solutions obtained using the Rusanov flux are more accurate and closer to the exact solution than those gotten with the centered flux. This can be easily seen from the L_2 norms of the results for the exact integration with $N = 1$ and $Ne = 16$ using both fluxes in Figure 11 below.



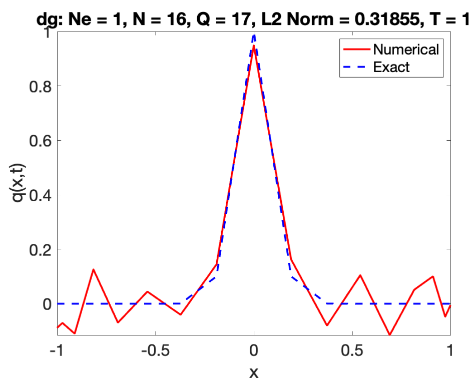
(a) Centered flux



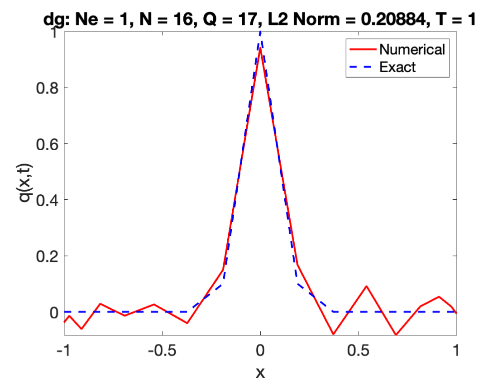
(b) Rusanov flux

Figure 11: exact and DG numerical solution for the 1D wave equation using $N = 1$ order polynomials with exact integration ($Q = 2$) using both fluxes after one revolution ($t = 1$). Observe how the numerical solution using the centered flux deviates from the exact solution at several points, whereas the numerical solution using the Rusanov flux tries to follow the exact solution closely. Notice also that the solution obtained using the Rusanov flux is more accurate than that obtained using the centred flux, with its L_2 norm being almost half of that of the centered flux.

We go further to display the results for $N = 16$ in Figure 12



(a) Centered flux



(b) Rusanov flux

Figure 12: exact and DG numerical solution for the 1D wave equation using $N = 16$ order polynomials with exact integration ($Q = 17$) using both fluxes after one revolution ($t = 1$). Although the results using both fluxes are almost similar, observe that the L_2 norm for the DG exact integration using Rusanov flux is once more less than that gotten with the centered flux.

Using these results (Figures 11 and 12), as well as several results which are omitted, we conclude that the results obtained with Rusanov flux are more accurate than those obtained with the centered flux.

2.2.2 Inexact Integration

For inexact integration with the DG method using the upwind flux, we observe that increasing the order of the polynomial and increasing the number of elements gives more accurate results, and the L_2 norm decreases. The only exception is the value of the L_2 norm when $N = 16$, which is higher than the norm when $N = 8$. The convergence plot is displayed in Figure 13.

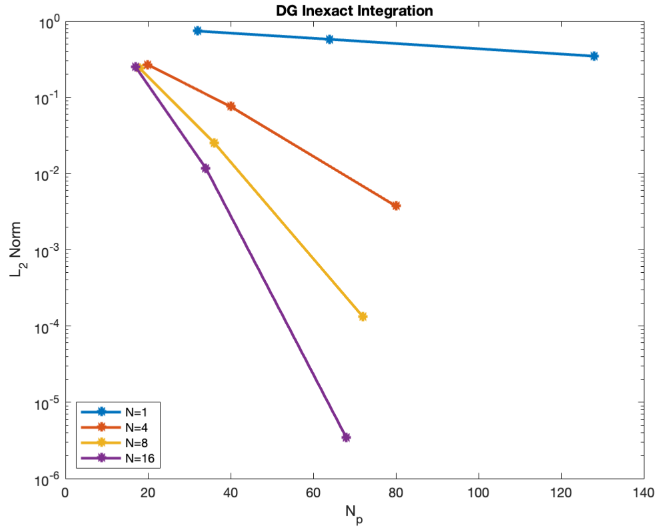


Figure 13: Convergence rates of DG for the 1D wave equation for polynomial orders $N = 1, 4, 8$, and 16 using a total number of gridpoints N_p for inexact integration ($Q = N$). Observe that as N increases, the L_2 norm gets smaller.

With the centered flux, we observe that increasing the order of the polynomial and increasing the number of elements gives more accurate results, and the L_2 norm decreases. The plot is presented in Figure 14.

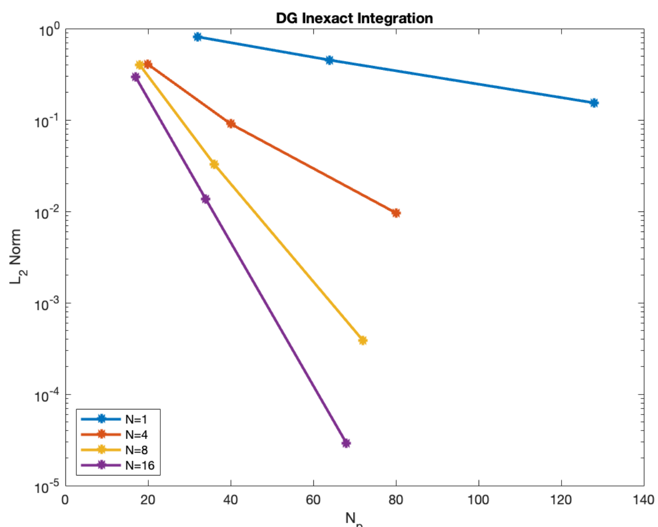
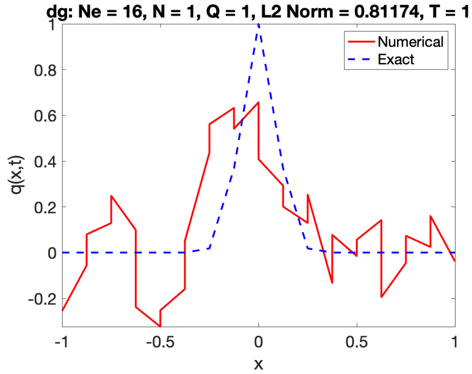
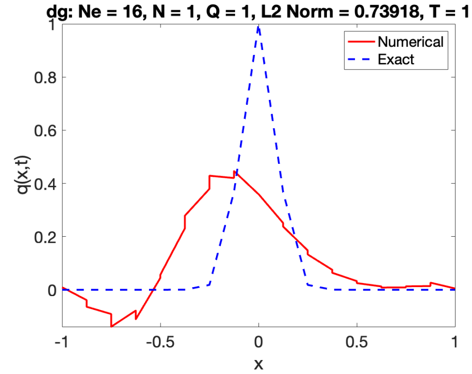


Figure 14: Convergence rates of DG for the 1D wave equation for polynomial orders $N = 1, 4, 8$, and 16 using a total number of gridpoints N_p for inexact integration ($Q = N$) with centered flux. Observe that as N increases, the L_2 norm gets smaller.

Like we did in the DG exact integration, we also compare the results obtained using centered and Rusanov fluxes in the DG inexact integration. We observe that the L_2 norm of the DG solution using Rusanov flux is less than that of the centered flux, thus implying that the numerical solution obtained using the Rusanov flux is more accurate than the one obtained using the centered flux. To back our results, we present in Figure 15, the plots of the exact and numerical solution of the DG inexact integration with both fluxes for $N = 1$.



(a) Centered flux



(b) Rusanov flux

Figure 15: exact and DG numerical solution for the 1D wave equation using $N = 1$ order polynomials with inexact integration ($Q = 1$) using both fluxes after one revolution ($t = 1$).

2.3 Findings

Going through the various simulations, we observe that the polynomial order and the number of elements chosen play a significant role on the accuracy of the solution. By increasing the polynomial order, as well as increasing the number of standard elements, the number of gridpoints over which the solution is defined increases, thereby increasing the accuracy of the numerical solution.

To comment on the comparison of DG and CG, it suffices to compare apples with apples. In order to do that, we shall compare CG exact with DG exact and CG inexact with DG inexact. Regardless of the flux being implemented, we get more accurate results with DG exact integration than with the CG exact integration. In fact, on all occasions, the L_2 norms of the solutions obtained using DG exact integration are significantly lower than those of the CG exact integration.

On the other hand, comparing CG inexact integration with DG inexact integration, we observe that regardless of the type of flux used, the solution obtained using the DG inexact method is more accurate than that of the CG inexact method, with the L_2 norms of the DG inexact integration being less than those of the CG inexact integration at all the values of N.

We conclude that although the CG exact integration is better than its inexact integration, the DG exact integration beats it to the task. The DG inexact integration performs better than the CG inexact integration when either the Rusanov or centered flux is used.