Laboratory 4: Fourier Series

Patrick Sinnott

17326757

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1 Introduction

Aims and overview

The purpose of this laboratory session was to introduce and demonstrate the vast applications of Fourier Analysis. Specifically, the aim was to use these methods to decompose periodic and aperiodic signals into sinusoidal components and reconstruct them using these components. Fourier Analysis decomposes a signal into constituent harmonic vibrations and these signals can be used to analyse the signal in extreme detail. For periodic signals, a Fourier series is used to approximate the data however in the case of an aperiodic signal, the Fourier Transform is used. A Fourier transform is essentially a Fourier series for a function with infinite period.

Principles of methods used

The Fourier series is an expansion of a periodic function in terms of an infinite sum of sines and cosines:

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

The coefficients an and bn measure the amount of sine and cosine functions required to present the function f. These coefficients are given by the Euler-Fourier formulae:

$$a_n = \frac{2}{T}\int_0^T f(t)\cos nt\, dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin nt\, dt$$

In order to obtain these results a great deal of integration is required. Therefore, I will need to use a numerical method of integration. For this lab, it would be most effective to use Simpson's rule.

Simpson's rule is a method for approximating the integral of a function f using quadratic polynomials (i.e. parabolic arcs instead of the straight line segments used in the trapezoidal rule); this makes it more accurate than the trapezoidal rule. I will divide the region of integration between the upper and lower limits, a and b, into n (even) subintervals of length $\mathrm{h} = \mathrm{b} - \mathrm{a}/\mathrm{n}$ and this gives the equation:

$$\int_a^b f(x)dx \approx \frac{h}{3}[f(a) + 2\sum_{j=1}^{\frac{n}{2}-1} f(x_{2j}) + 4\sum_{j=1}^{\frac{n}{2}} f(x_{2j-1}) + f(b)]$$

Since I will also be analysing aperiodic functions during this laboratory, I am required to use the Discrete Fourier Transform (DFT). This takes N samples of a function/signal at regular intervals of length h. An approximation of the function is made using these samples and we compute the Fourier Transform:

$$F_n = \sum_{m=1}^{N-1} f_m e^{\frac{2\pi imn}{N}}$$

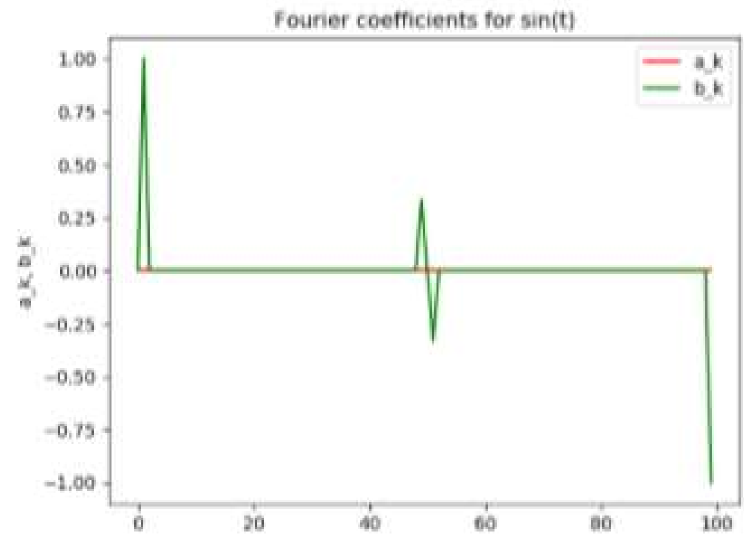
The function can then be reconstructed using the formula:

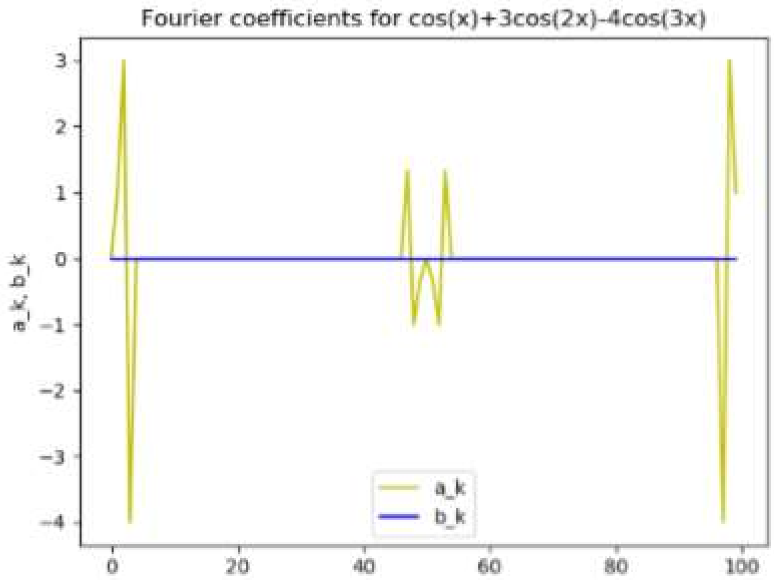
$$f_m = \frac{1}{N}\sum_{n=1}^{N-1} F_n e^{-\frac{2\pi imn}{N}}$$

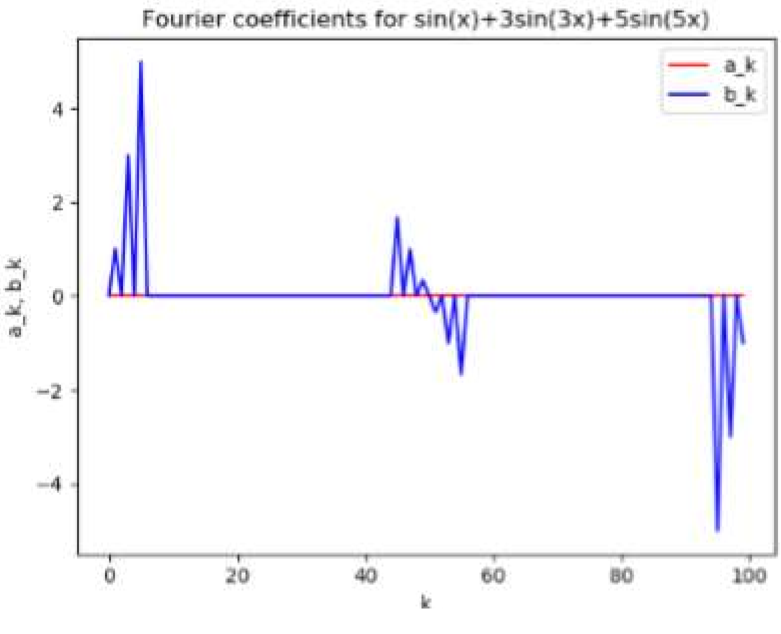
2 Methodology and Results

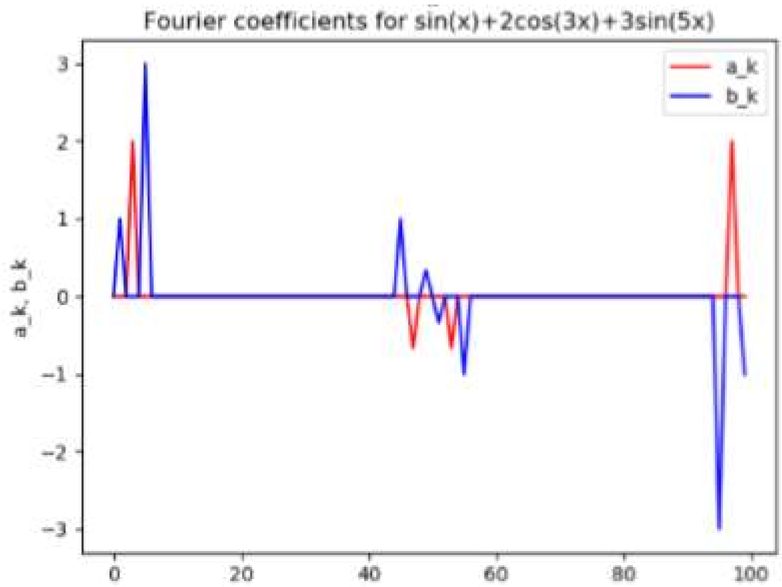
Exercise 1: Simpson's rule Python script for evaluating Fourier series

- I wrote a simple python script which used Simpson's rule to perform definite integrals on functions between two limits: a and b. The function took arguments for the function, the limits and the number of subintervals.
- I then tested this program on the function e^x as I integrated it between 0 and 1. 10 subintervals were used for the purpose of this calculation. I obtained a result of 1.7182827819248236. This value is accurate to the expression; $e - 1$, up to a precision of five decimal places
- This program was then extended to evaluate the Fourier Coefficients: an and bn. I applied this augmented code to a number of functions to analyse them as a combination of sines and cosines.

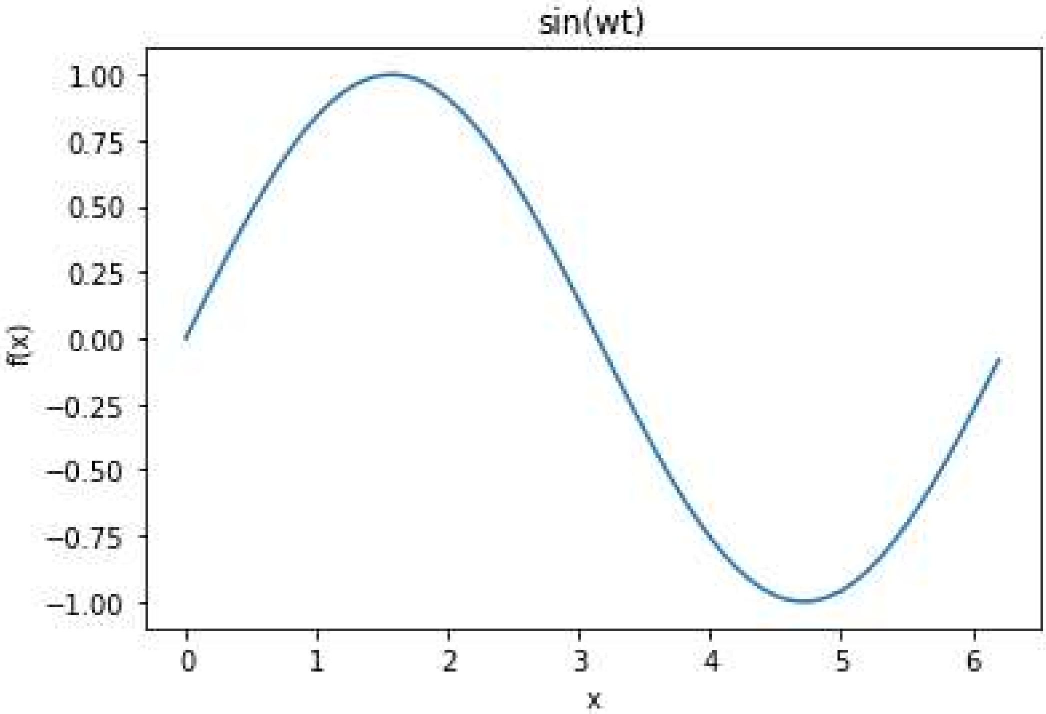

Fourier coefficients for sin(t)
a_k
b_k
a_k, b_k

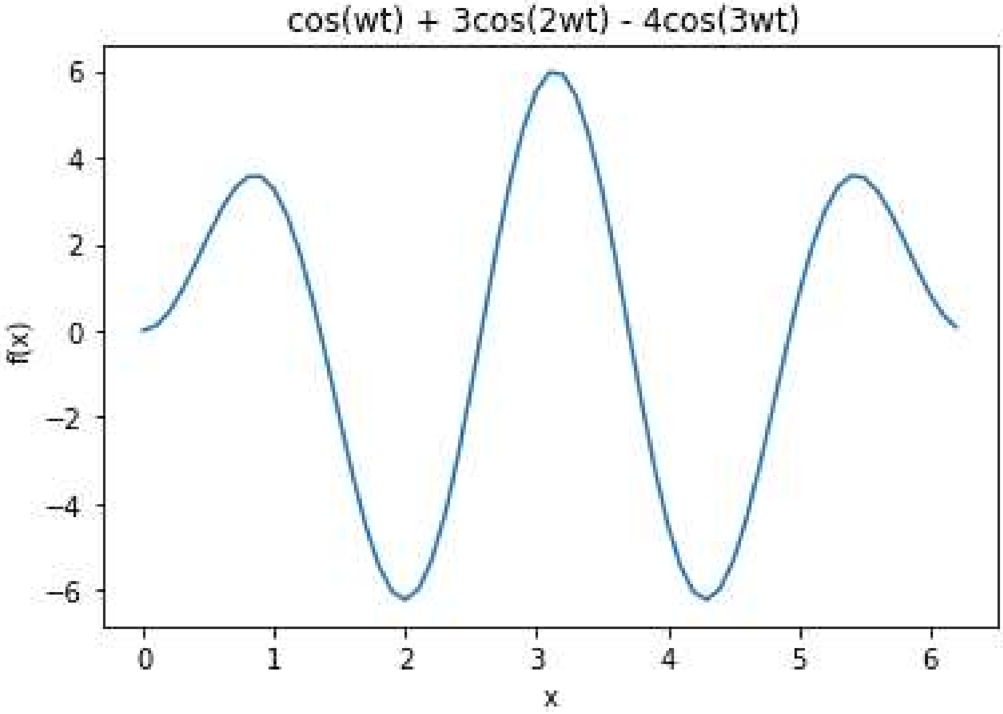

Fourier coefficients for cos(x)+3cos(2x)-4cos(3x)
a_k
b_k
a_k, b_k

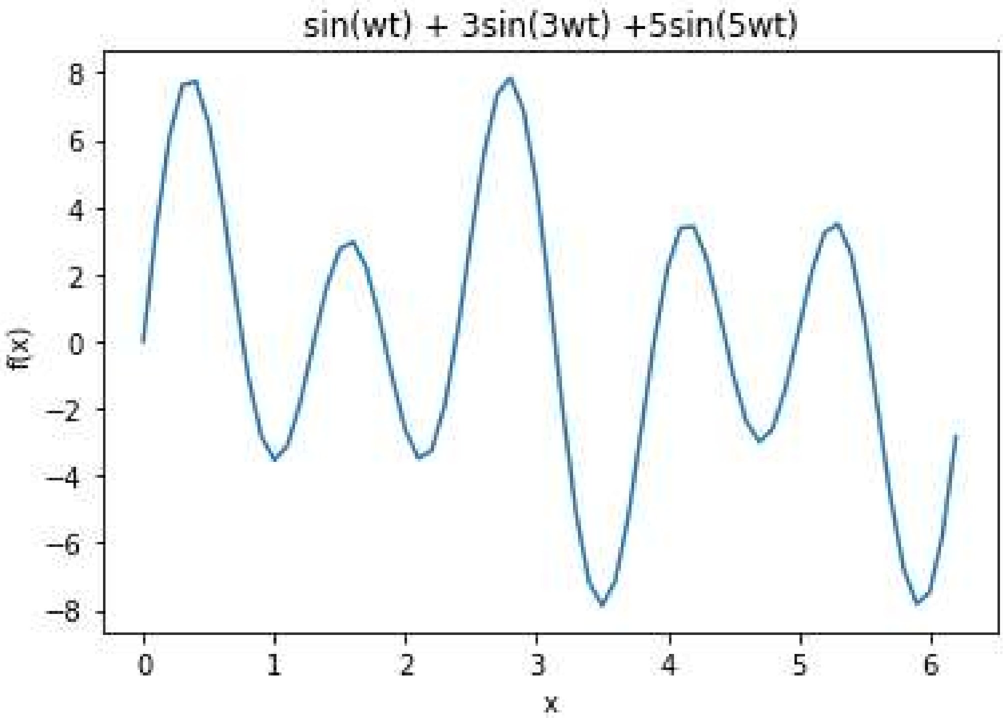

Fourier coefficients for sin(x)+3sin(3x)+5sin(5x)
a_k
b_k
a_k, b_k
k

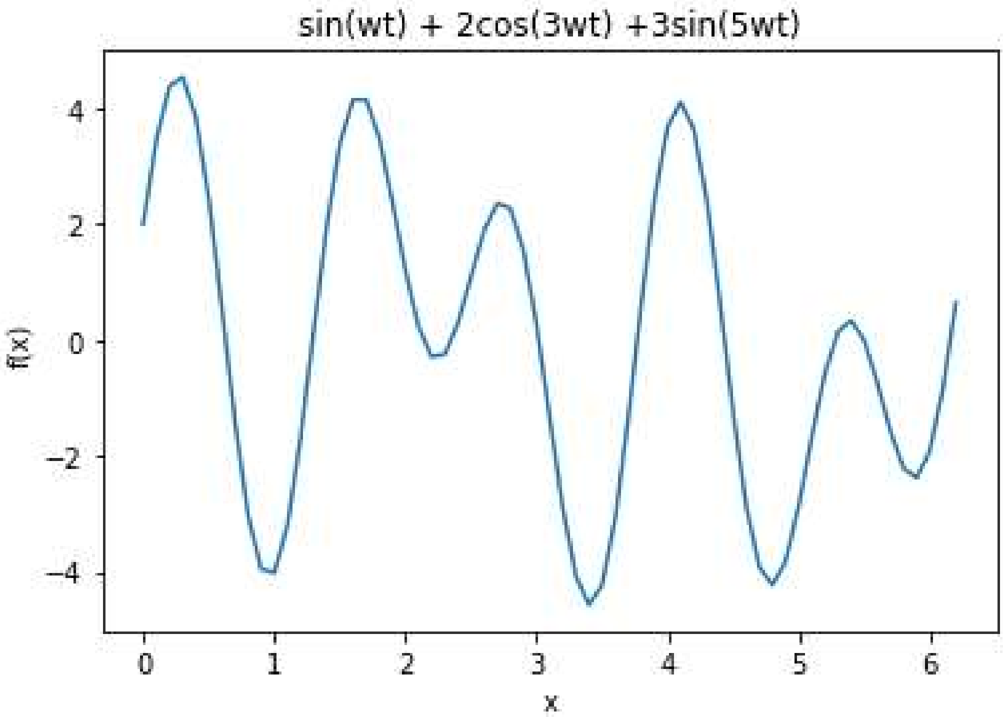


- We can observe that the computation of Fourier coefficients for sinusoidal functions is very straightforward. We can see that the an and bn coefficients very easily match up with the functions we are analysing. For example, with regards to the first function of course each of the an coefficients will be equal to 0 as a decomposition of the sin function will only contain sin elements and no cos elements.
- Similar results can be observed in the following graphs; each of the functions composed only of sin elements can be entirely expressed with bn coefficients and the same applies to cos functions and an coefficients.
- It can also be observed that the values of the Fourier coefficients align perfectly with the coefficients of the sin and cos terms of the functions. For example; b1 = 1 for $f(x) = \sin(x)$ and a1 =**1**, a2 = **3**, a3 = **-4** for $f(x) = \mathbf{1}\cos(x) + \mathbf{3}\cos(2x) - \mathbf{4}\cos(3x)$
- These coeffiecients were then used to construct a Fourier series for each of these periodic functions.


sin(wt)
f(x)
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
0
1
2
3
4
5
6
x


cos(wt) + 3cos(2wt) - 4cos(3wt)
f(x)
6
4
2
0
−2
−4
−6
0
1
2
3
4
5
6
x





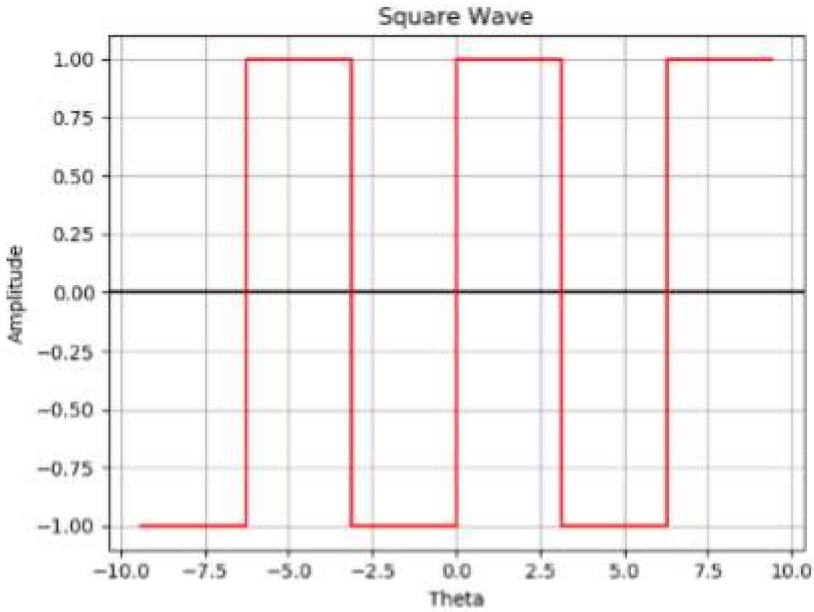
- We can clearly see that the Fourier Series very efficiently recreates the functions in to a legible and useful format.

Exercise 2: Python script for Fourier series of square and rectangular waves

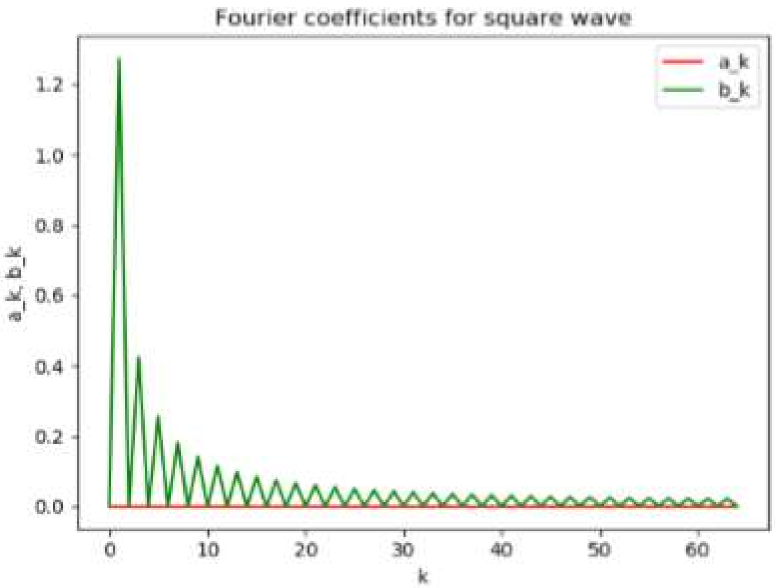
- The purpose of this exercise was to demonstrate how useful the Fourier Series is for analysing more irregular functions such as the square or rectangular wave functions.

- I began by graphing the function that was to be examined The square wave function:

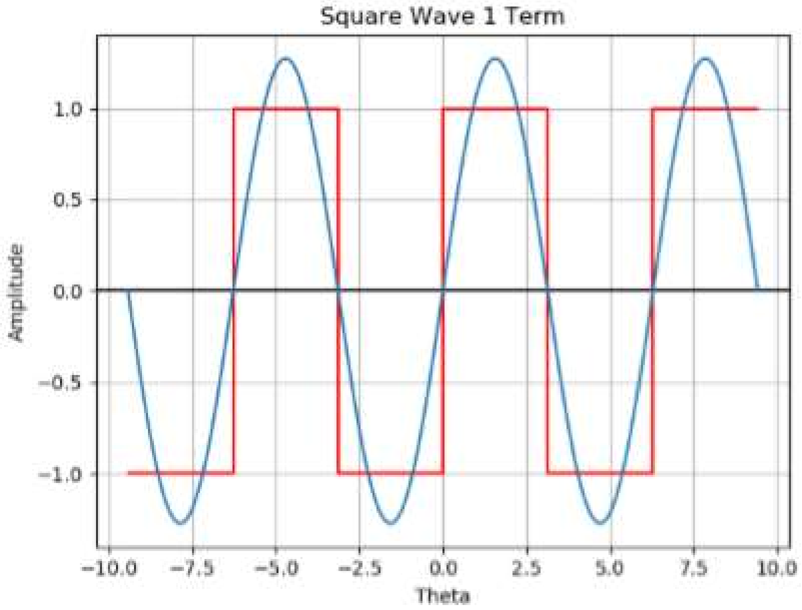
$$f(\theta) = \begin{cases} 1 & 0 \leq \theta \leq \pi \\ -1 & \pi < \theta \leq 2\pi \end{cases}$$

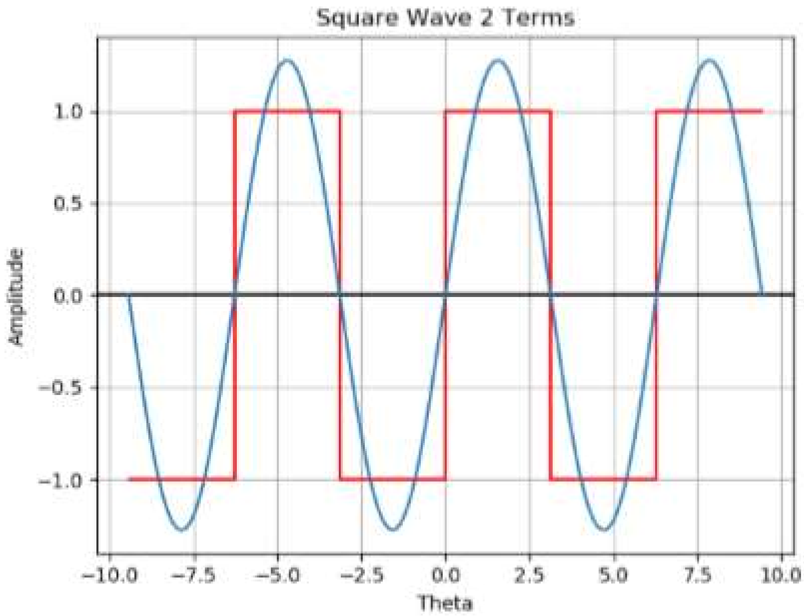


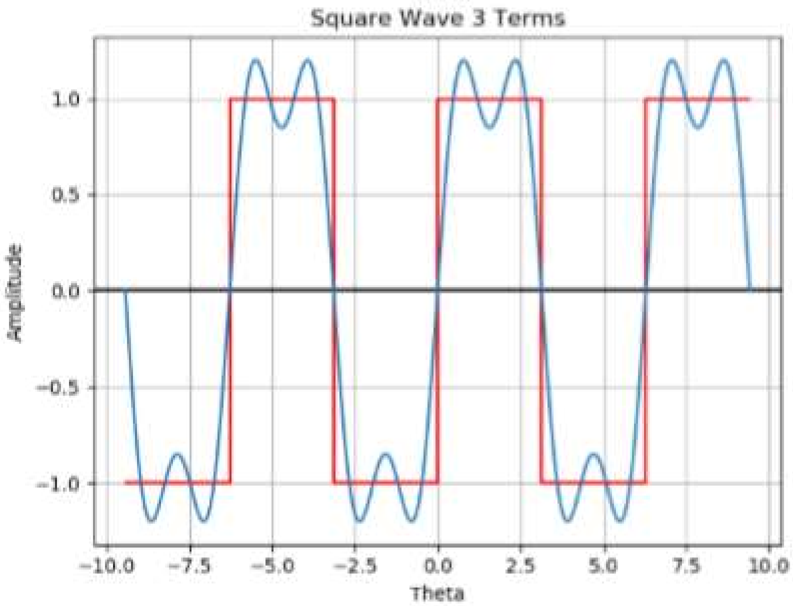
- The next step was to slowly reconstruct the square wave in the form of Fourier Coefficients. I began by calculating the Fourier coefficients.

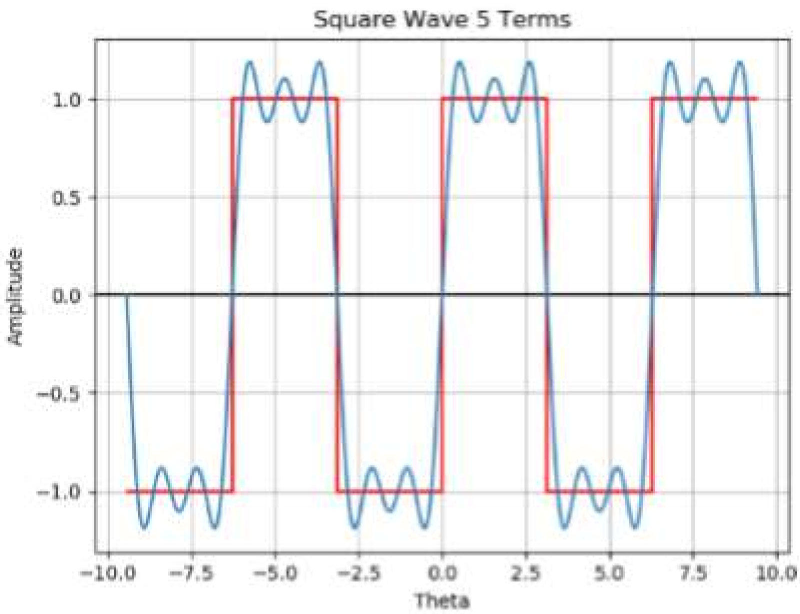


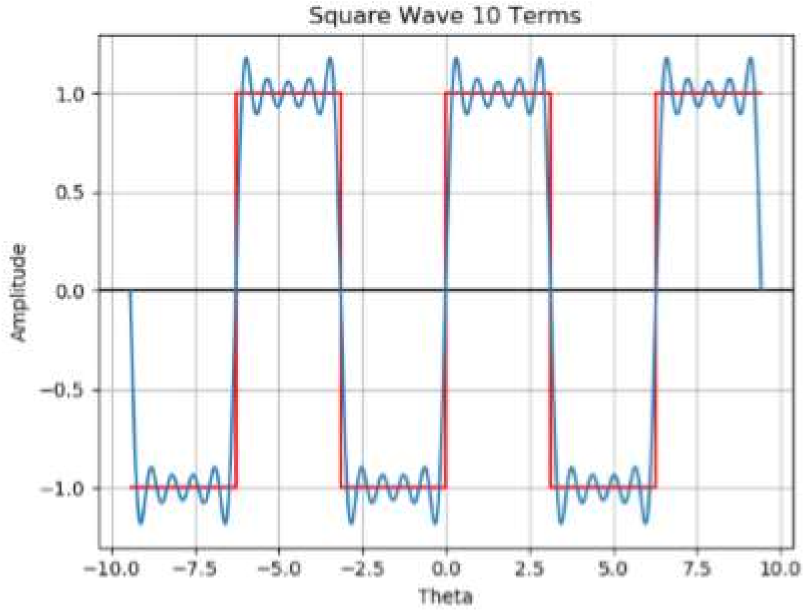
- I began with limited sets of an and bn coefficients and slowly increased them to achieve a more and more accurate representation of the function in question.


Square Wave 1 Term
Amplitude
Theta
1.0
0.5
0.0
−0.5
−1.0
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0

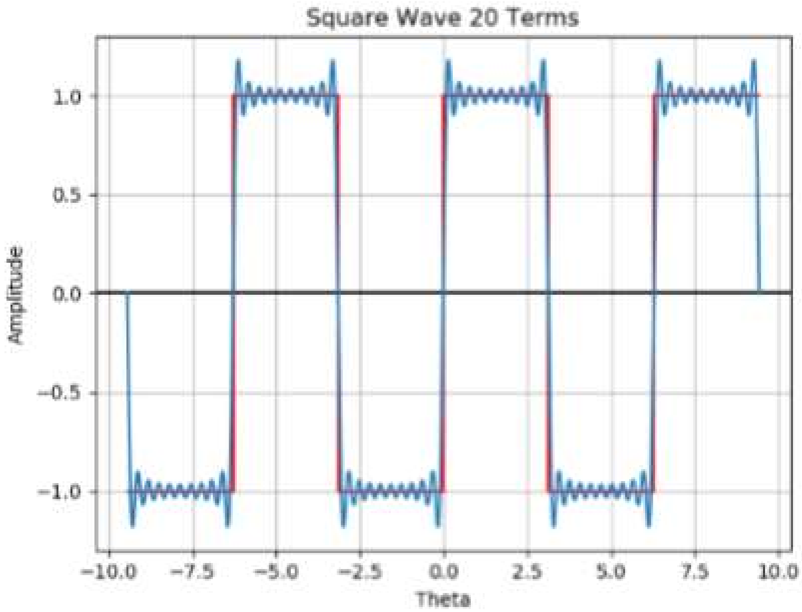

Square Wave 2 Terms
Amplitude
Theta
1.0
0.5
0.0
−0.5
−1.0
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0

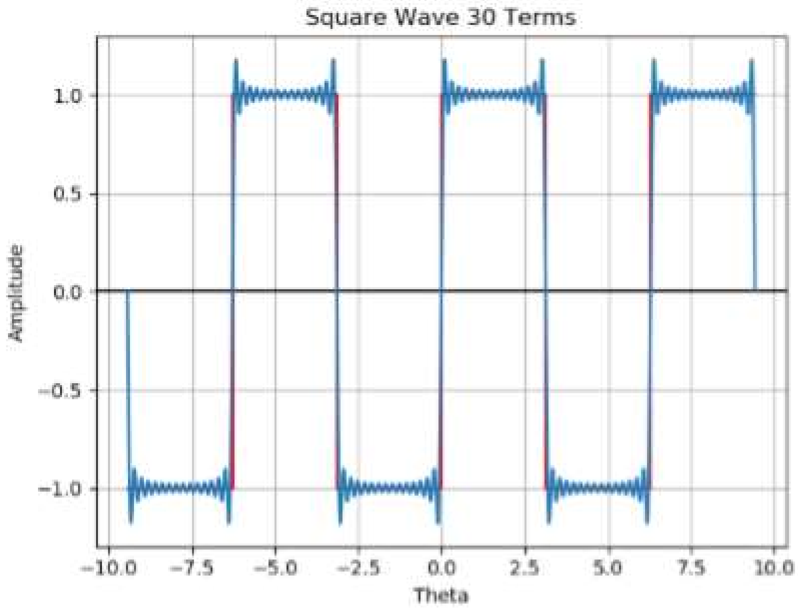

Square Wave 3 Terms
Amplitude
Theta
1.0
0.5
0.0
−0.5
−1.0
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0


Square Wave 5 Terms
Amplitude
Theta
1.0
0.5
0.0
−0.5
−1.0
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0



Square Wave 10 Terms
Amplitude
Theta
1.0
0.5
0.0
−0.5
−1.0
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0





- We can see the Fourier series giving only a very loose outline of the function in the first graph as we only have one term but as progressively more and more terms are added, the approximation becomes rapidly much more accurate. I noted that the approximation with 30 terms was highly accurate, however if we were to double this number again it would be an almost perfect approximation of the function.
- I performed an identical analysis on the rectangular wave and the results were very similar.

3 Conclusions

- Overall, this laboratory was very effective foe demonstrating the usefulness of Fourier Analysis. Throughout the exercises I developed an intuitive feel for the applications of this powerful approximation.
- In the first exercise, I began by developing a program which could effectively perform definite integrals. It functioned as expected and performed an integration to a degree of accuracy of 5 decimal places.
  This function was then extended to calculate the an and bn coefficients for a Fourier Series. It did so very successfully. This then allowed me to graph the Fourier Series for various sinusoidal function
- In the following section these methods were developed and applied to functions that were periodic but not sinusoidal. I slowly added more Fourier coefficients to the Fourier approximation of the square wave function and clearly saw each approximation become subsequently more accurate. This idea can be clearly extended to show that if the number of coefficients are summed to infinity (Which is what normally happens with a Fourier Series) the approximation will be virtually flawless.