Senior Fresh Computational Physics Laboratory Report

Laboratory 2: The Nonlinear Pendulum

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1. Introduction

Aims and overview

The purpose of this computational laboratory was to examine the dynamics of both the linear and nonlinear pendulum. I compared the results of the two systems to deduce when the linear pendulum is a valid approximation and identify the situations where the non-linear term is required. I then introduced damping and driving forces to our pendulum and once again examined the resulting motion.

Principles of methods used

Pendulum:

The pendulum which I studied in this lab was comprised of a mass m coupled with an inextensible rod of length L which is fixed at the opposite end. This setup constrains the motion of the mass to the circle of radius L with its centre at the end of the rod which we will take as the origin. We can derive from Newton's Laws that the equation of motion is:

$$d^2\theta/dt^2 = -(g/L)\sin(\theta),$$

where we take θ as the angle between the pendulum and the vertical line through the origin and g as the acceleration due to gravity. We can make the approximation $\sin(\theta) \sim \theta$ for extremely small values of θ. With this approximation, we can deduce the equations of motion for the linear pendulum as we have removed the nonlinear sine function.

Solving this linear equation for θ gives;

$$\theta = A\sin(\omega t + \delta).$$

This is an analytical solution for the linear pendulum where A is the amplitude of oscillations and δ is the initial phase. Both values are determined by the initial values of angle θ and angular velocity ω. The nonlinear pendulum cannot be solved analytically, and we must use numerical methods to generate a solution. In the more general case of damped, driven oscillations, we introduce a damping force $-k\omega$ and a periodic driving force $A\cos(\phi t)$.

Solving Differential Equations Analytically:

We can calculate the differential equations of the pendulum's motion to be:

$$d\theta/dt = \omega;$$
$$d\omega/dt = f(\theta, \omega, t);$$
$$f(\theta, \omega, t) = -\sin(\theta) - k\omega + A\cos(\phi t).$$

Throughout this lab, I will use two different numerical methods to evaluate these. First; the trapezoidal rule and then the second order Runge-Kutta method.

The Trapezoidal Rule:

This method involves using the approximation of the area under a curve using trapezoids. It has a higher degree of accuracy than Euler's Method as the trapezoids are a better approximation of the required area when compared to rectangular partitions. The expressions used to calculate each consecutive step are:

$$\theta n + 1 = \theta n + \omega n(\Delta t/2) + (\omega n + f(\theta n, \omega n, t)\Delta t)\Delta t/2$$
$$\omega n + 1 = \omega n + f(\theta n, \omega n, t)\Delta t/2 + f(\theta n + 1, \omega n + f(\theta n, \omega n, t)\Delta t, t + \Delta t)\Delta t/2$$

The Runge-Kutta Algorithm:

The Runge-Kutta algorithm differs from the trapezoidal rule because the expansion is taken from the mid-point of each trapezoid, rather than the start. This distinction increases the accuracy of the method from Δt to Δt^2. During this exercise, I used the fourth order Runge-Kutta algorithm. This attained a higher level of accuracy by taking the mean of four increments; one taken from each end-point and two from the mid-point.

This lab is divided in to five main sections which I shall reference throughout the rest of my report.

1. *Solving the linear Pendulum equation - Trapezoidal rule.*
   The opening section involved solving our equations using the trapezoidal rule, and the approximation $Sin(\theta) \approx \theta$. The Equation is analytically solvable with a solution:

$$\theta = ASin(\omega t + \beta);$$

2. *Solving the nonlinear Pendulum equation - Trapezoidal rule.*
   This section is solved similarly to section 1., however I removed the approximation and calculated using the sin term.

3. *Solving the nonlinear Pendulum equation – Runge Katta Method.*
   Here I evaluated the same system as section 2., however the Runge Katta method was used to achieve greater accuracy (accurate to order dt^2).

4. *Solving the nonlinear damped Pendulum equation – Runge Katta Method.*
   This section focused on the damped nonlinear pendulum equation, which had an added friction term:

$$F_{friction} = -mk\omega,$$

   to give an equation of motion:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}Sin(\theta) + k\omega$$

   which was then solved using the same procedure as section 3.

5. *Solving the driven damped nonlinear Pendulum equation – Runge Katta Method.*
   Lastly, this section involved accounting for an external driving force:

$$F_{driving} = mACos(\varphi t)$$

   in the equation of motion to obtain:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{Lf}Sin(\theta) + k\omega + \mathrm{ACos}(\varphi t).$$

2. Methodology

1. *Solving the linear Pendulum equation - Trapezoidal rule.*

- During the opening exercise, I wrote a python script which defined the function:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta.$$

2. *Solving the nonlinear Pendulum equation - Trapezoidal rule.*

- The variable 'nsteps' was initialised to 0 which functioned as the effective time in the experiment. I inserted the trapezoidal rule to my script using a while loop which was incremented by 'nsteps'. I plotted the points of θ ,and ω against nsteps (time).

- For the nonlinear pendulum, I used a similar method to the linear pendulum, however the definition of the function was altered:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}Sin(\theta).$$

I once again plotted the points θ, and ω against nsteps.

3. *Solving the nonlinear Pendulum equation – Runge Katta Method.*

- For this section, I used the same script as section 2, but I implemented the Runge Katta method with a for loop:

k1a = dt * ω , k1b = dt * f(θ, ω, t)
k2a = dt * (ω+k1b/2), k2b = dt * f(θ+k1a/2, ω+k1b/2, t+dt/2)
k3a = dt * (ω+k2b/2), k3b = dt * f(θ+k2a/2, ω+k2b/2, t+dt/2)
k4a = dt * (ω+k3b), k4b = dt * f(θ+k3a, ω+k3b, t+dt)

θ(t + dt) = θ(t) + (k1a + 2 k2a + 2 k3a + k4a) / 6
ω(t + dt) = ω(t) + (k1b + 2 k2b + 2 k3b + k4b) / 6

I plotted both the Runge Katta and the Trapezoidal method on the same graphs for congruent values in order to compare them.

4. *Solving the nonlinear damped Pendulum equation – Runge Katta Method.*

- The function was altered by adding the damping coefficient:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}Sin(\theta) + k\omega,$$

I then used the Runge Katta algorithm to plot θ and ω against time.

5. *Solving the driven damped nonlinear Pendulum equation – Runge Katta Method.*

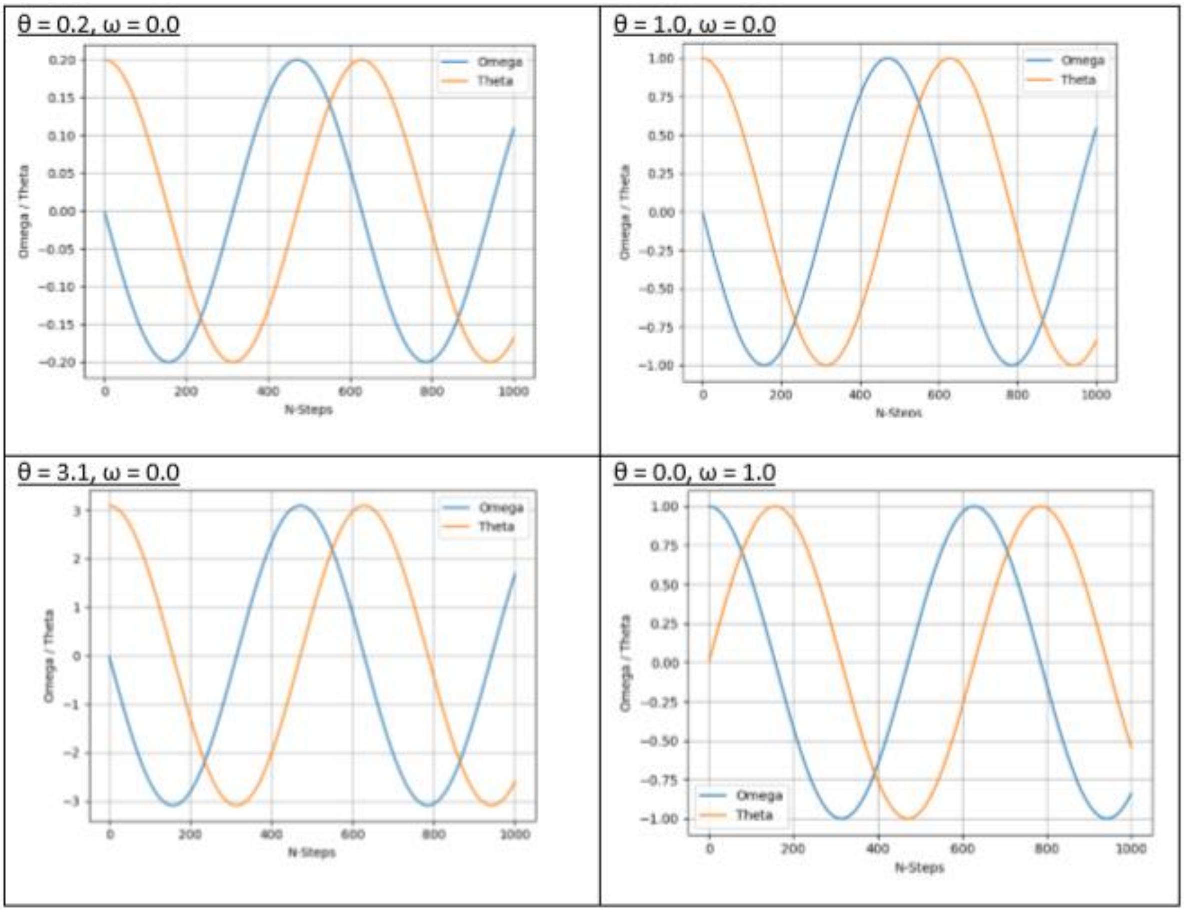
- Finally, I evaluated the equations for the pendulum with the driving force included in the function definition:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{Lf}Sin(\theta) + k\omega + \mathrm{ACos}(\varphi t).$$

This was achieved upon addition of a sinusoidal term with the frequency φ , as well as A, the amplitude of the driving force. I then plotted the phase portrait of the system for different initial values. In order to do so, I needed to alter my script to neglect the chaotic initial motion of the pendulum while plotting my graph, and to plot only the transient motion of the pendulum. I achieved this by adding a variable which was incremented along with the Runge Katta algorithm, until it reached the threshold of an 'if' statement which then proceeded to plot the transient phase diagram.

3. Results

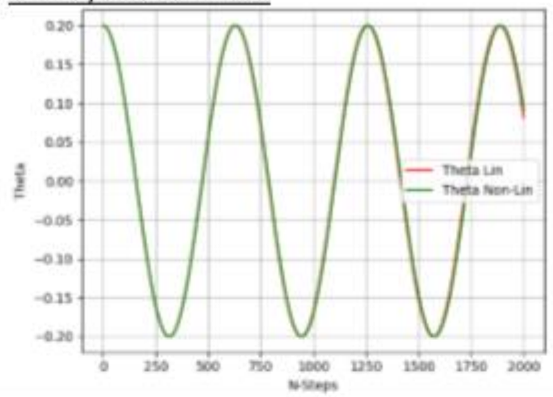
1. *Solving the linear Pendulum equation - trapezoidalTrapezoidal rule.*



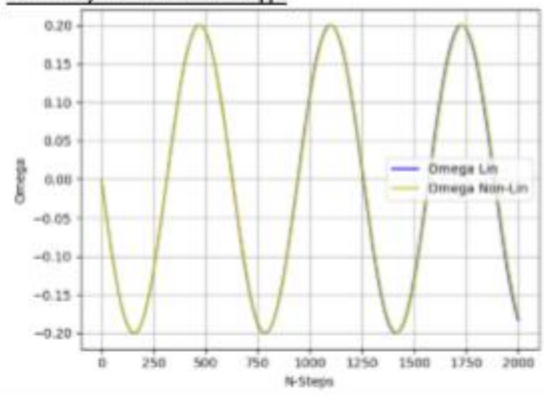
- These graphs of displacement and angular velocity vs time clearly show that a linear pendulum with no damping or driving force, these quantities vary sinusoidally with time and have a phase difference of $\pi/2$ radians.

2. *Solving the nonlinear Pendulum equation - Trapezoidal rule.*

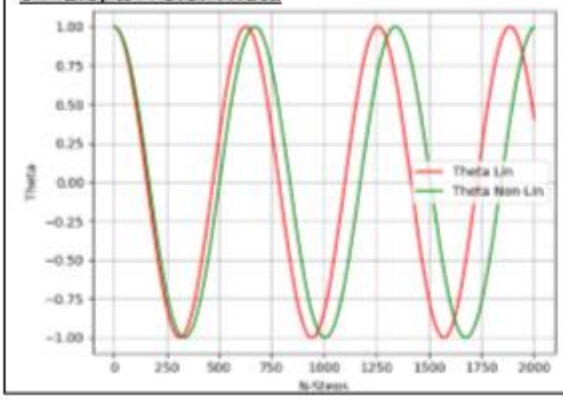
θ = 0.2, ω = 0.0: Theta



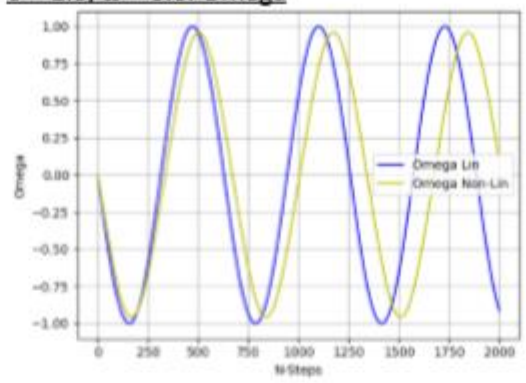
θ = 0.2, ω = 0.0: Omega



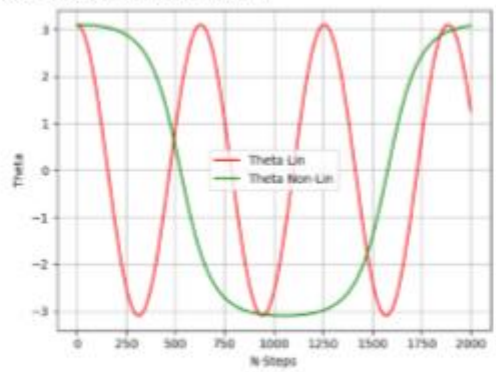
θ = 1.0, ω = 0.0: Theta



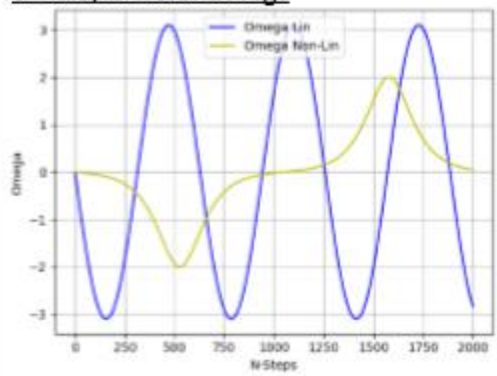
θ = 1.0, ω = 0.0: Omega



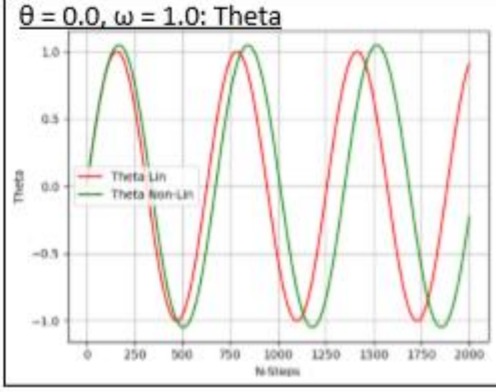
θ = 3.1, ω = 0.0: Theta



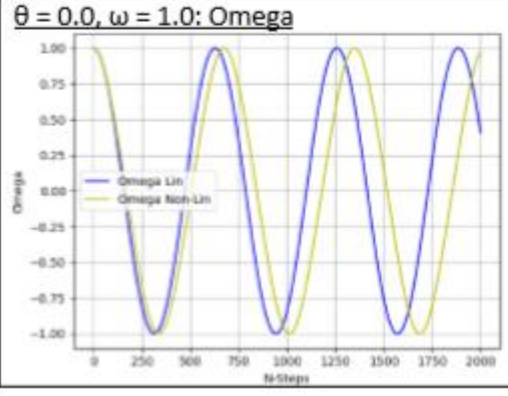
θ = 3.1, ω = 0.0: Omega



θ = 0.0, ω = 1.0: Theta

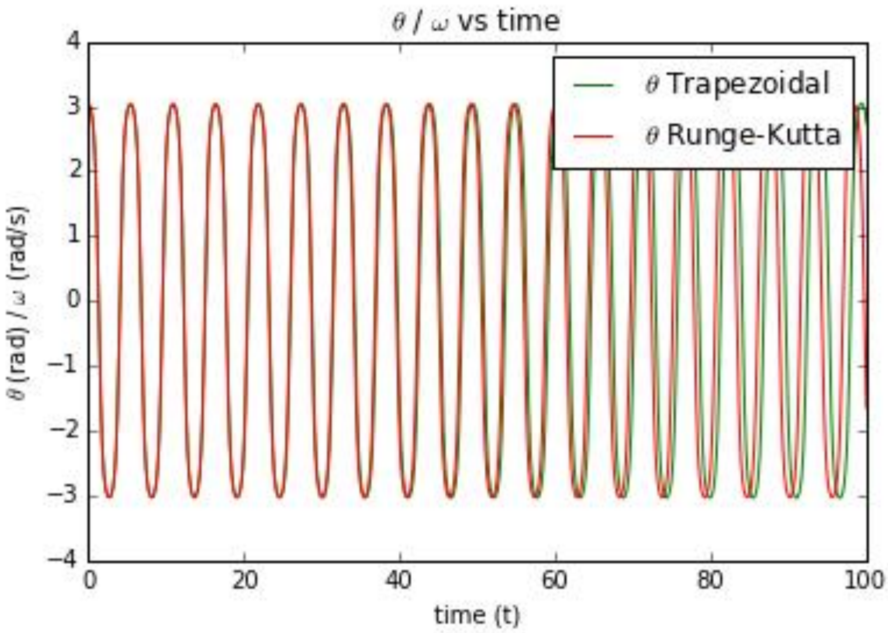


θ = 0.0, ω = 1.0: Omega



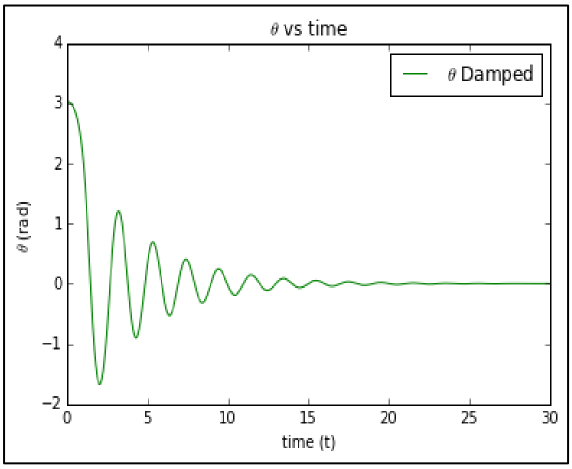
- These plots make a convincing argument for the necessesity of the nonlinear term in the pendulum equation. It can be clearly observed that the linear approximation breaks down for larger initial values of θ or ω. In my first two graphs, there were small oscillations, and the linear plot functions as an almost perfect approximation to the nonlinear plot. However, for the following set of figures, the increased initial value of θ = 3.1, there is a far greater amplitude and the linear approximation breaks down extremely quickly. As well as this, in my final set of figures we can observe the graphs quickly going out of phase due to the increased initial value of ω = 1.0.

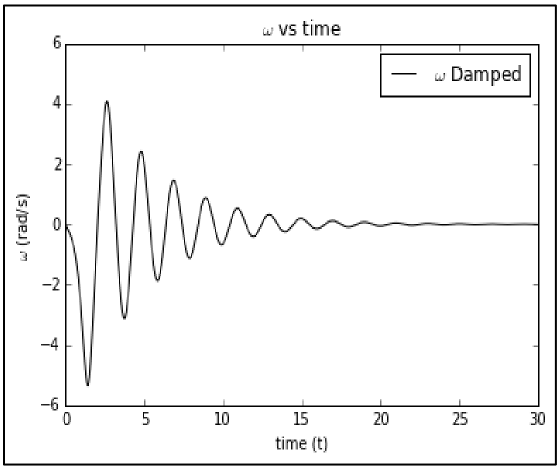
3. *<u>Solving the nonlinear Pendulum equation – Runge Katta Method.</u>*



- We can see from the graph above that our two methods of numerical analysis give very similar approximations for the motion of the pendulum. However, on close inspection we can see that over time they shift out of phase. We can deduce that the Runge Kutta algorithm is the more accurate of the methods and it was therefore used for the remainder of the lab.

4. *<u>Solving the nonlinear damped Pendulum equation – Runge Katta Method.</u>*

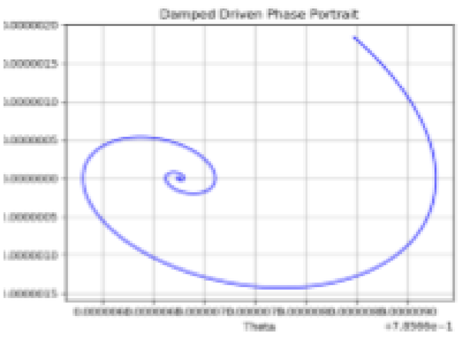




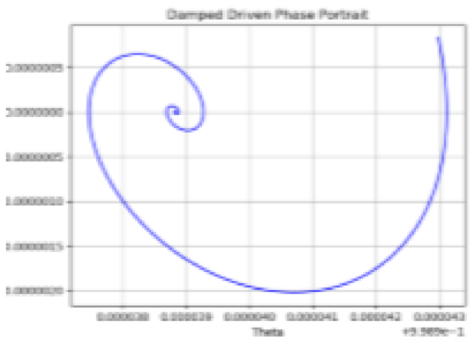
- My graphs for this section show the effect of damping over time. They show the gradual decrease of the amplitude of oscillations over time. This degree of damping is commonly known as light damping. I experimented with increasing the value of k and found that if it was raised high enough, θ tended towards 0 with no oscillations taking place. This is referred to as critical damping.

5. *Solving the driven damped nonlinear Pendulum equation – Runge Katta Method.*

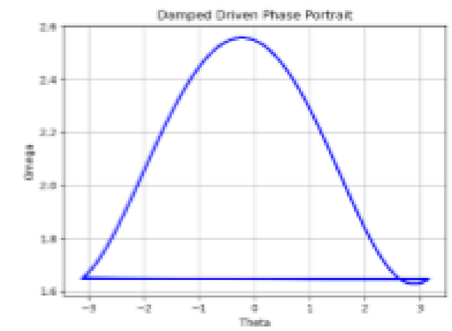
1: A = 0.9



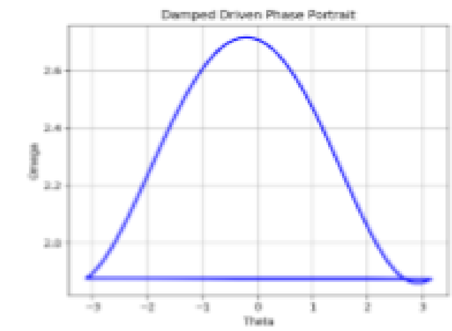
2: A = 1.07



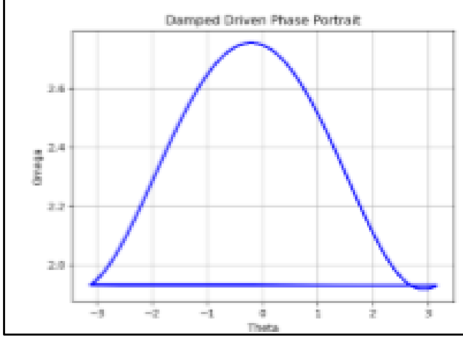
3: A = 1.35



4: A = 1.47



5: A = 1.5



- In figures 1 and 2, it is observed that both θ and ω tend to 0 as with increasing time. We can evaluate from this that the damping force is large enough to overcome the driving force and the oscillations decrease in amplitude over time and tend to 0.
- In figures 3, 4 and 5, the driving force is large enough to overcome the damping force and oscillations continue at constant amplitude and frequency once steady state motion has been achieved. When A is increased, the phase portrait is translated in the positive ω direction.

4. Conclusions

In conclusion, I feel that the main aims of this laboratory were completed successfully.

- My initial program demonstrated the effectiveness of the linear pendulum approximation for small values of θ and ω.

- However, in the second exercise where I compared the linear and non-linear pendulums; the flaws in the small angle approximation: $Sin(\theta) \approx \theta$, became clear. We need to add the sinusoidal term to achieve a reasonable degree of accuracy for any realistic simulation of a pendulum.

- It was then made clear from exercise 3 that the Runge Kutta algorithm was a measurably more accurate way of simulating our pendulum and we continued using this method for the remainder of the lab. I used this algorithm to observe the properties of both the damped and damped driven pendulums. I observed both light and critical damping; as well as observing the driven pendulum tend towards chaotic motion with increasing external force.