Pokemon Go, a Case Study in Mathematical Modeling

Abstract

Third-party models for pokemon and their move-sets in Pokemon Go, by Niantic, are well distributed but fundamentally flawed. Currently, the most accurate predictions are made by full simulations which disagree with the popular, third-party models. We present more accurate mathematical models and heuristics for pokemon, move-sets, and predicting winners of match-ups. Our results find agreement with simulations, where other models have not. We demonstrate these results and rankings based on theorems from game theory, calculus, basic algebra, and linear algebra

Goal#1 is to determine optimal move sets for any Pokémon attacking a gym.

Our calculations can serve as general advice to a high-level player with a max level Pokémon who would to decide on whether they should spend in game resources to change their Pokémon's move set. For our calculations we assume the level of attacker and defender are equal. We use our assumptions to generate the following metric. Note that this does not take into account the defenders type, which should be considered in any individual match up.

$$Damage = floor\left[\tfrac{1}{2} P \left(\frac{A_A}{D_D}\right) \frac{f(L_A)}{f(L_D)} g(t_A, T_A)\, g(t_A, T_{A,2}) h(t_A, T_{D,1})\, h(t_A, T_{D,2})\right] + 1$$

We remove floor+1 since the Pokémon is a high-level species and this has negligible effect on our calculations.

$$Damage \approx \tfrac{1}{2} P \left(\frac{A_A}{D_D}\right) \frac{f(L_A)}{f(L_D)} g(t_A, T_A)\, g(t_A, T_{A,2}) h(t_A, T_{D,1})\, h(t_A, T_{D,2})$$

It is also assumed both attacker and defender are at the same level, so attacker and defender base level terms cancel.

$$Damage \approx \tfrac{1}{2} P \left(\frac{A_A}{D_D}\right) g(t_A, T_A)\, g(t_A, T_{A,2}) h(t_A, T_{D,1})\, h(t_A, T_{D,2})$$

The calculation is without regard to a defender when attacking a gym, so the type effectiveness multiplier is excluded. The defender's defense stat is also removed.

$$Damage \approx \tfrac{1}{2} P \left(\frac{A_A}{D_D}\right) A_A g(t_A, T_A)\, g(t_A, T_{A,2})$$

Finally, we remove constants in comparison for fixed Pokémon.

$$Damage \approx P\, g(t_A, T_A)\, g(t_A, T_{A,2}).$$

We divide by the length of the attack, and we use this approximate damage per second as our metric for determining an optimal moveset for any particular pokemon attacking a gym. $\frac{Damage}{time(s)}$ denoted dps consist of two types,

$$\widehat{dps}_F \approx \frac{P_f\, g(t_f, T_1)\, g(t_f, T_2)}{l_f} \quad \text{and} \quad \widehat{dps}_C \approx \frac{P_c\, g(t_c, T_1)\, g(t_c, T_2)}{l_c}$$

$\widehat{dps}_F$ is the damage per second of a fast move and $\widehat{dps}_C$ is the damage per second of a charge move. From these new quantities, we now can define the difference of the dps types denoted

$$\widehat{dps}_\Delta = \widehat{dps}_C - \widehat{dps}_F \approx \frac{P_c\, g(t_c, T_1)\, g(t_c, T_2)}{l_c} - \frac{P_f\, g(t_f, T_1)\, g(t_f, T_2)}{l_f}$$

Pokémon can inflict $\widehat{dps}_F$ without restriction, but the added $\widehat{dps}_\Delta$ can only occur once N ≥ -nc. We define dps of a pokémon's move set by the weighted sum

$$\widehat{dps} \approx \widehat{dps}_F + \mu\, \widehat{dps}_\Delta$$

$$\widehat{dps} \approx \frac{P_f\, g(t_f, T_1)\, g(t_f, T_2)}{l_f} + \frac{l_c}{x\, l_f + l_c}\left[\frac{P_c\, g(t_c, T_1)\, g(t_c, T_2)}{l_c} - \frac{P_f\, g(t_f, T_1)\, g(t_f, T_2)}{l_f}\right]$$

where $x = \begin{cases} \dfrac{-n_c}{n_f} & \text{if } n_c > -100 \\ \left\lceil \dfrac{-n_c}{n_f} \right\rceil & \text{if } n_c = -100 \end{cases}$ is the number of fast moves necessary to allow for a charged move.

We define μ to be the ratio of time that dps_Δ is being applied.
Hence μ is the length of time of a charged attack divided by the length of a full cycle including enough fast charges to make N > $-n_c$ plus the length of the charged attack.

Goal #2 is to determine the best gym raid attackers and their optimal move sets.

As with the previous calculation, we are ignoring the defenders type. Defenders type should be considered for any individual match up. In gym raids, dps is the most important since attackers are not restricted to a fixed number of pokemon. Since we are comparing attackers, we use our approximation from above but include the individuals pokemon A value.

$$\frac{P_f A\, g(t_f, T_1)\, g(t_f, T_2)}{l_f} + \frac{l_c}{x\, l_f + l_c}\left[\frac{P_c A g(t_c, T_1)\, g(t_c, T_2)}{l_c} - \frac{P_f A\, g(t_f, T_1)\, g(t_f, T_2)}{l_f}\right]$$

Goal #3 is to determine the best gym defenders and their optimal move sets.

A defender, player 1, places a Pokémon into a gym to defend with up to 5 other defenders. An attacker, player 2, will choose 6 of their own Pokémon to defeat the 6 defenders. A defender receives an award if they can defeat all the attacker's Pokémon first and vice versa. The attacker has knowledge of what Pokémon are in the gym and therefore, has choice advantage over the defenders. Hence, we decide best defender by a mini-max calculation on the ratio of attackers who die per defenders who die.

We assume that the players are at the same level with high level pokémon and they are using their best pokémon.

Let the number of attackers who die be β and the number of defenders who die be α. Also let the percentage of attacker life lost per second be φ and the percentage of defender life lost per second be λ.

Thus, we obtained:

$$\frac{\beta}{\alpha} = \frac{\varphi}{\lambda} = \frac{\dfrac{dps_D}{S_A}}{\dfrac{dps_A}{2S_D}} = \frac{dps_D\; 2S_D}{dps_A\; S_A} \quad ⬢$$

We define $\widetilde{dps} \approx \frac{P}{l}\left(\frac{A_A}{D_D}\right) g(t_A, T_{A,1}) g(t_A, T_{A,2}) h(t_A, T_{D,1}) h(t_A, T_{D,2})$ *as an approximation with the following reasons.*

$$Damage = floor\left[\tfrac{1}{2} P \left(\frac{A_A}{D_D}\right) \frac{f(L_A)}{f(L_D)} g(t_A, T_A)\, g(t_A, T_{A,2}) h(t_A, T_{D,1})\, h(t_A, T_{D,2})\right] + 1$$

We remove floor+1 since the Pokémon is a high-level species and this has negligible effect on our calculations.

$$Damage \approx \tfrac{1}{2} P \left(\frac{A_A}{D_D}\right) \frac{f(L_A)}{f(L_D)} g(t_A, T_A)\, g(t_A, T_{A,2}) h(t_A, T_{D,1})\, h(t_A, T_{D,2})$$

It is also assumed both attacker and defender are at the same level, so attacker and defender base level terms cancel.

$$Damage \approx \tfrac{1}{2} P \left(\frac{A_A}{D_D}\right) g(t_A, T_A)\, g(t_A, T_{A,2}) h(t_A, T_{D,1})\, h(t_A, T_{D,2}) \quad \bigstar$$

we remove constants such as 1/2

$$\widetilde{Damage} \approx P\left(\frac{A_A}{D_D}\right) g(t_A, T_A)\, g(t_A, T_{A,2}) h(t_A, T_{D,1})\, h(t_A, T_{D,2})$$

From the damage formula we calculated $\widetilde{dps}_F$, $\widetilde{dps}_C$ *,which lead to* $\widetilde{dps}_\Delta$ *and* $\widetilde{dps}$*. Dps is the appropriate metric for these calculations. dps calculations were calculated similar to* $\widehat{dps}_F$, $\widehat{dps}_C$, $\widehat{dps}_\Delta$*, and* $\widehat{dps}$ *calculations.*

For each defender we calculated its worst match up. We then sorted and selected the defenders with best worst-case scenarios as best defenders.

Goal#4 Considering energy gained from life lost in best defender calculation

Previously in Goal #3, we were calculating the best defenders, and counters to the best defenders. Goal #4 takes in account the energy gained from life lost, which goal # 3 did not. Goal #4 does not take in account of ceiling of x_2 when $n_c = -100$, which goal # 3 did.

Starting with our orginal damage calculation and simplifying using the following assumptions:

- removing floor +1 due to pokemon being a high level species
- both attacker and defender are at same level

We have the above equation denoted by ★ which is

$$Damage \approx \tfrac{1}{2} P \left(\frac{A_A}{D_D}\right) g(t_A, T_A)\, g(t_A, T_{A,2}) h(t_A, T_{D,1})\, h(t_A, T_{D,2})$$

Dividing by the duration we now define a new dps type, $\overline{dps}$.

$$\overline{dps} \approx \tfrac{1}{2} \frac{P}{l}\left(\frac{A_A}{D_D}\right) g(t_A, T_A)\, g(t_A, T_{A,2}) h(t_A, T_{D,1})\, h(t_A, T_{D,2})$$

Similarly in goal#1 we defined x to be the number of fast moves neccesary to generate a charged attack, we now introduce to x_2 which accounts for energy gained from life lost.

x_2 is the difference of energy of a charged attack and the energy gained from life lost times the length of a charge attack divided by the sum of energy of fast attack and the energy gain from life lost times the length of a fast attack.μ

$$x_2 = \frac{-n_c - \left(\dfrac{\overline{dps}}{2} l_c\right)}{n_f + \left(\dfrac{\overline{dps}}{2} l_f\right)} = \frac{-\left(n_c + \dfrac{\overline{dps}}{2} l_c\right)}{n_f + \left(\dfrac{\overline{dps}}{2} l_f\right)} = -\frac{2n_c + \overline{dps}\, l_c}{2n_f + \overline{dps}\, l_f} = -\frac{\dfrac{2n_c}{l_f} + \dfrac{\overline{dps}\, l_c}{l_f}}{\dfrac{2n_f + \overline{dps}}{l_f}}$$

$$= -\frac{\dfrac{2n_c}{l_f} + \dfrac{l_c}{l_f}\left(\overline{dps} + \dfrac{2n_f}{l_f} - \dfrac{2n_f}{l_f}\right)}{\dfrac{2n_f + \overline{dps}}{l_f}} = \frac{-\dfrac{2n_c}{l_f} - \dfrac{l_c \overline{dps}}{l_f} - \dfrac{2l_c n_f}{l_f^2} + \dfrac{2l_c n_f}{l_f^2}}{\dfrac{2n_f + \overline{dps}}{l_f}}$$

$$= \frac{-\dfrac{2n_c}{l_f} - \dfrac{l_c}{l_f}\left(\dfrac{2n_f}{l_f} + \dfrac{\overline{dps}}{l_f}\right) + \dfrac{2l_c n_f}{l_f^2}}{\dfrac{2n_f + \overline{dps}}{l_f}} = \frac{-\dfrac{2n_c}{l_f} + \dfrac{2l_c n_f}{l_f^2}}{\dfrac{2n_f + \overline{dps}}{l_f}} - \frac{l_c}{l_f}$$

previously we defined μ as the length of time of a charged attack divided by the length of a full cycle including enough fast charges to make N > -nc plus the length of the charged attack. We now define μ_2 which consists of our x_2 result shown below.

$$\mu_2 = \frac{l_c}{x_2 l_f + l_c} = \frac{l_c}{\left(\dfrac{-\dfrac{2n_c}{l_f} + \dfrac{2l_c n_f}{l_f^2}}{\dfrac{2n_f + \overline{dps}}{l_f}} - \dfrac{l_c}{l_f}\right) l_f + l_c} = \frac{l_c}{\dfrac{-2n_c + \dfrac{2l_c n_f}{l_f}}{\dfrac{2n_f + \overline{dps}}{l_f}} - l_c + l_c}$$

$$= \frac{l_c\left(\dfrac{2n_f}{l_f} + \overline{dps}\right)}{-2n_c + \dfrac{2l_c n_f}{l_f}} = \frac{2n_f l_c + \overline{dps}\; l_f l_c}{2n_f l_c - 2n_c l_f}$$

In goal#1 we defined the dps of a pokemon's moveset by the weighted sum, $\widehat{dps} \approx \widehat{dps}_F + \mu\, \widehat{dps}_\Delta$, we now revisit the dps of a pokemon's moveset which will consist of the above μ_2 term using $\overline{dps}$.

$$\overline{dps} = \overline{dps}_f + \mu_2 \overline{dps}_\Delta$$

The attacker gains energy when it receives damage from its opponent, μ_2 will contain the dps of the opposing pokemon to reflect this change in our model.

$$\overline{dps}_A = \overline{dps}_{f,A} + \left(\frac{2n_{f,A} l_{c,A} + \overline{dps}_D l_{f,A} l_{c,A}}{2n_{f,A} l_{c,A} - 2n_{c,A} l_{f,A}}\right) \overline{dps}_{\Delta,A}$$

$$\overline{dps}_D = \overline{dps}_{f,D} + \left(\frac{2n_{f,D} l_{c,D} + \overline{dps}_A l_{f,D} l_{c,D}}{2n_{f,D} l_{c,D} - 2n_{c,D} l_{f,D}}\right) \overline{dps}_{\Delta,D}$$

Our new dps formula has two different $\overline{dps}$ types and to solve it in terms of one $\overline{dps}$ type, we make the following substitution.

calculating $\overline{dps}_A$ we have,

$$\overline{dps}_A = \overline{dps}_{f,A} + \left(\frac{2n_{f,A} l_{c,A} + \left[\overline{dps}_{f,D} + \left(\dfrac{2n_{f,D} l_{c,D} + dps_A l_{f,D} l_{c,D}}{2n_{f,D} l_{c,D} - 2n_{c,D} l_{f,D}}\right)\overline{dps}_{\Delta,D}\right] l_{f,A} l_{c,A}}{2n_{f,A} l_{c,A} - 2n_{c,A} l_{f,A}}\right) \overline{dps}_{\Delta,A}.$$

$$\overline{dps}_A = \frac{A_A P_{f,A}}{2D_D l_{f,A}} + \frac{n_{f,A} A_A\left(P_{c,A} l_{f,a} - P_{f,A} l_{c,A}\right)}{2D_D l_{f,A}(n_{f,A} l_{c,A} - n_{c,A} l_{c,A})} + \frac{A_A A_D P_{f,D}(l_{f,a} P_{c,A} - l_{c,A} P_{f,A})}{8D_D D_A l_{f,A}(n_{f,A} l_{c,A} - n_{c,A} l_{f,A})} + \frac{A_A n_{f,D} A_D (l_{f,A} P_{c,A} - l_{c,A} P_{f,A})(l_{f,D} P_{c,D} - l_{c,D} P_{f,D})}{8D_D D_A l_{f,D}(n_{f,A} l_{c,A} - n_{c,A} l_{f,A})(l_{c,D} n_{f,D} - l_{f,D} n_{c,D})} + \frac{A_A A_D (l_{f,A} P_{c,A} - l_{c,A} P_{f,A})(l_{f,D} P_{c,D} - l_{c,D} P_{f,D})}{16D_D D_A (n_{f,A} l_{c,A} - n_{c,A} l_{f,A})(l_{c,D} n_{f,D} - l_{f,D} n_{c,D})} \overline{dps}_A$$

Grouping our $\overline{dps}_A$ terms gives,

$$\overline{dps}_A - \frac{A_A A_D (l_{f,A} P_{c,A} - l_{c,A} P_{f,A})(l_{f,D} P_{c,D} - l_{c,D} P_{f,D})}{16D_D D_A (n_{f,A} l_{c,A} - n_{c,A} l_{f,A})(l_{c,D} n_{f,D} - l_{f,D} n_{c,D})} \overline{dps}_A = \frac{A_A P_{f,A}}{2D_D l_{f,A}} + \frac{n_{f,A} A_A\left(P_{c,A} l_{f,a} - P_{f,A} l_{c,A}\right)}{2D_D l_{f,A}(n_{f,A} l_{c,A} - n_{c,A} l_{c,A})} + \frac{A_A A_D P_{f,D}(l_{f,a} P_{c,A} - l_{c,A} P_{f,A})}{8D_D D_A l_{f,A}(n_{f,A} l_{c,A} - n_{c,A} l_{f,A})} + \frac{A_A n_{f,D} A_D (l_{f,A} P_{c,A} - l_{c,A} P_{f,A})(l_{f,D} P_{c,D} - l_{c,D} P_{f,D})}{8D_D D_A l_{f,D}(n_{f,A} l_{c,A} - n_{c,A} l_{f,A})(l_{c,D} n_{f,D} - l_{f,D} n_{c,D})}.$$

Solving for $\overline{dps}_A$, our equation is,

$$\overline{dps}_A = \frac{\dfrac{A_A P_{f,A}}{2D_D l_{f,A}} + \dfrac{n_{f,A} A_A\left(P_{c,A} l_{f,a} - P_{f,A} l_{c,A}\right)}{2D_D l_{f,A}(n_{f,A} l_{c,A} - n_{c,A} l_{c,A})} + \dfrac{A_A A_D P_{f,D}(l_{f,a} P_{c,A} - l_{c,A} P_{f,A})}{8D_D D_A l_{f,A}(n_{f,A} l_{c,A} - n_{c,A} l_{f,A})} + \dfrac{A_A n_{f,D} A_D (l_{f,A} P_{c,A} - l_{c,A} P_{f,A})(l_{f,D} P_{c,D} - l_{c,D} P_{f,D})}{8D_D D_A l_{f,D}(n_{f,A} l_{c,A} - n_{c,A} l_{f,A})(l_{c,D} n_{f,D} - l_{f,D} n_{c,D})}}{1 - \dfrac{A_A A_D (l_{f,A} P_{c,A} - l_{c,A} P_{f,A})(l_{f,D} P_{c,D} - l_{c,D} P_{f,D})}{16D_D D_A (n_{f,A} l_{c,A} - n_{c,A} l_{f,A})(l_{c,D} n_{f,D} - l_{f,D} n_{c,D})}}.$$

By Multiplying the right side of the equation by $\frac{16D_D D_A l_{f,A} l_{f,D}(n_{f,A} l_{c,A} - n_{c,A} l_{f,A})(l_{c,D} n_{f,D} - l_{f,D} n_{c,D})}{16D_D D_A l_{f,A} l_{f,D}(n_{f,A} l_{c,A} - n_{c,A} l_{f,A})(l_{c,D} n_{f,D} - l_{f,D} n_{c,D})}$ we obtain our final simplified $\overline{dps}_A$ formula,

$$\overline{dps}_A = \frac{8A_A P_{f,A} D_A l_{f,D}(n_{f,A} l_{c,A} - n_{c,A} l_{f,A})(n_{f,D} l_{c,D} - n_{c,D} l_{f,D}) + 8n_{f,a} A_A D_A l_{f,D}(P_{c,A} l_{f,A} - P_{f,A} l_{c,A})(n_{f,D} l_{c,D} - n_{c,D} l_{f,D}) + 2A_A A_D P_{f,D} l_{f,D}(P_{c,A} l_{f,A} - P_{f,A} l_{c,A})(n_{f,D} l_{c,D} - n_{c,D} l_{f,D}) + 2A_A A_D n_{f,D} l_{f,A}(P_{c,A} l_{f,A} - P_{f,A} l_{c,A})(l_{f,D} P_{c,D} - l_{c,D} P_{f,D})}{16D_D D_A l_{f,A} l_{f,D}(n_{f,A} l_{c,A} - n_{c,A} l_{f,A})(l_{c,D} n_{f,D} - l_{f,D} n_{c,D}) - A_A A_D l_{f,A} l_{f,D}(l_{f,A} P_{c,A} - l_{c,A} P_{f,A})(l_{f,D} P_{c,D} - l_{c,D} P_{f,D})}.$$

For readibility, we defined $P_{f,A}$ to be $P_{f,A}\, g(t_{f,A}, T_{A,1}) g(t_{f,A}, T_{A,2}) h(t_{f,A}, T_{D,1}) h(t_{f,A}, T_{D,2})$, and $P_{c,A}$, $P_{f,D}$, and $P_{c,D}$ similarly.

Similarly, we use the same approach for solving for the $\overline{dps}$ for the defender. The equation is identical except defender and attacker subscripts are switched.

Using our mini max result denoted by ⬢, we are able to calulate our best defenders and counters using our newly defined $\overline{dps}$.

Goal#5 Determining an optimal IV stat

There are three main stats for a pokemon: attack, defense, and stamina represented as variables A, D, and S respectfully. Each main stat consists of two parts, its base value and its individual value(IV). Every pokemon of a specific species will have the same base value, however; not all pokemon of a specific species will have the same individual value(IV). The total value of main stat is the sum of its base value and its individual value(IV):

$$\text{Attack} = A_B + A_i \qquad \text{Defense} = D_B + D_i \qquad \text{Stamina} = S_B + S_i$$

Since base values are fixed but IV's are not we can determine the most optimal stats by determining the best IV's. IV's can range from 0 to 15.

$$A_i, D_i, S_i \in [0, 15]$$

The current standard of determining the optimal IV's of a specific pokemon is calculated using the formula,

$$\frac{A_i + D_i + S_i}{45}$$

The above formula shows that as the value approaches 1, the better the pokemon will be, however; this is not necessarily true. A better way to determine an optimal IV is to create a balance among the total stats of a pokemon. We show this optimal condition by using a lagrange multiplier. We define our contraint equation,g(A,D,S), as the inherent power of a pokemon which is the product of the total attack, defense, and stamina stats. We define the function that we wish to optimize as $f(A_i, D_i, S_i)$ to be the sum of the individual values which is constant.

$$f(A_i, D_i, S_i) = A_i + D_i + S_i = C \qquad g(A,D,S) = (A_B + A_i)(D_B + D_i)(S_B + S_i)$$

The gradient of the above functions are:

$$\Delta f = \langle 1, 1, 1 \rangle \text{ and } \Delta g = \langle (D_B + D_i)(S_B + S_i), (A_B + A_i)(S_B + S_i), (A_B + A_i)(D_B + D_i) \rangle$$

Our Lagrange multiplier equation is:

$$\langle 1, 1, 1 \rangle = \lambda \langle (D_B + D_i)(S_B + S_i), (A_B + A_i)(S_B + S_i), (A_B + A_i)(D_B + D_i) \rangle$$

Solving for *λ in terms of the total attack stats yields:*

$$\lambda = \frac{1}{(D_B + D_i)(S_B + S_i)} = \frac{1}{(A_B + A_i)(S_B + S_i)} = \frac{1}{(A_B + A_i)(D_B + D_i)}$$

Then by setting our solutions for *λ* equal to eachother, we find that:

$$\frac{1}{(D_B + D_i)(S_B + S_i)} = \frac{1}{(A_B + A_i)(S_B + S_i)} \quad \& \quad \frac{1}{(A_B + A_i)(S_B + S_i)} = \frac{1}{(A_B + A_i)(D_B + D_i)}$$

$$(D_B + D_i)(S_B + S_i) = (A_B + A_i)(S_B + S_i) \quad \& \quad (A_B + A_i)(S_B + S_i) = (A_B + A_i)(D_B + D_i)$$

$$D_B + D_i = A_B + A_i \quad \& \quad S_B + S_i = D_B + D_i$$

and thus, the total defense stat = total attack stat = total stamina stat is an optimal condition for a pokemo -n's IV.

Goal#5 continued

Computing the first order partial derivatives of g(A,D,S) gives

$$\frac{\partial g}{\partial A} = SD, \quad \frac{\partial g}{\partial D} = AS, \quad \text{and} \quad \frac{\partial g}{\partial S} = AD.$$

A change in one paticular stat reflects a change in the other two stats. If a paticular stat decreases it will increase the other two and vise versa. This consequence is a factor that effects many of our calculations because the product of the total attack, total defense, & total stamina occur frequently in many of our results. An example of this product appearing in our calculations is in our mini max ratio where we calculate best defenders and counters.