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Quantum computing Assignment 1

1)a) Find the inner product of:

$v_1 \quad |\psi\rangle = \sqrt{\frac{2}{3}}|01\rangle + \frac{i}{\sqrt{3}}|10\rangle$

$v_2 \quad |\phi\rangle = \sqrt{\frac{1}{2}}|00\rangle + i\sqrt{\frac{1}{2}}|10\rangle$

$$\langle v_1|v_2\rangle = \cancel{\left(\sqrt{\tfrac{2}{3}}\right)\left(\sqrt{\tfrac{1}{2}}\right)\underbrace{\langle 01|00\rangle}_{=0}} + \cancel{\left(\sqrt{\tfrac{2}{3}}\right)\left(i\sqrt{\tfrac{1}{2}}\right)\underbrace{\langle 01|10\rangle}_{=0}}$$

$$+ \cancel{\left(\frac{i}{\sqrt{3}}\right)\left(\sqrt{\tfrac{1}{2}}\right)\underbrace{\langle 10|00\rangle}_{=0}} + \left(\frac{i}{\sqrt{3}}\right)\left(i\sqrt{\tfrac{1}{2}}\right)\underbrace{\langle 10|10\rangle}_{=1}$$

$$= \left(\frac{i}{\sqrt{3}}\right)\left(i\sqrt{\frac{1}{2}}\right)$$

$$= \frac{i^2}{\sqrt{6}}$$

$$= -\frac{1}{\sqrt{6}}$$

1) b) Find the tensor product of

$$|\psi\rangle = \sqrt{\frac{2}{3}}\,|0 + \frac{i}{\sqrt{3}}|1\rangle$$

$$|\phi\rangle = i\sqrt{\frac{1}{2}}\,|0\rangle + \sqrt{\frac{1}{2}}\,|1\rangle$$

$$|\psi\rangle \otimes |\phi\rangle = \begin{pmatrix} \sqrt{\frac{2}{3}} \\ \frac{i}{\sqrt{3}} \end{pmatrix} \otimes \begin{pmatrix} i\sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} \end{pmatrix}$$

$$|\psi\rangle \otimes |\phi\rangle = \begin{pmatrix} \left(\sqrt{\frac{2}{3}}\right)\left(i\sqrt{\frac{1}{2}}\right) \\ \left(\sqrt{\frac{2}{3}}\right)\left(\sqrt{\frac{1}{2}}\right) \\ \left(\frac{i}{\sqrt{3}}\right)\left(i\sqrt{\frac{1}{2}}\right) \\ \left(\frac{i}{\sqrt{3}}\right)\left(\sqrt{\frac{1}{2}}\right) \end{pmatrix}$$

$$|\psi\rangle \otimes |\phi\rangle = \begin{pmatrix} \frac{i\sqrt{2}}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} \\ \frac{i^2}{\sqrt{6}} \\ \frac{i}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} \frac{i}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} \\ \frac{i}{\sqrt{6}} \end{pmatrix}$$

2) i) Find the operator Z that maps the following

$$\begin{array}{ccc} a & & b \\ |0\rangle & \rightarrow & |0\rangle \\ |1\rangle & \rightarrow & -|1\rangle \end{array}$$

ii) Also, find the matrix form of the operator

$$Z = |a_1\rangle\langle b_1| - |a_2\rangle\langle b_2|$$

i) $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$

ii) $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

3) Consider the operator X which acts as follows on the computational basis states

$X|0\rangle = |1\rangle$
$X|1\rangle = |0\rangle$

Represent the eigen vectors of the operator X in Dirac notation and comment on them

X is a NOT gate

$\therefore \; X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\det(X - \lambda I) = 0$

$X - \lambda I = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} -\lambda & 1 \\ 1 & \lambda \end{bmatrix}$

$\det(X - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}$

$= (-\lambda)(-\lambda) - (1)(1)$

$= \lambda^2 - 1$

$0 = \lambda^2 - 1$

$\lambda^2 = 1$

$\lambda = \pm 1$

Xv

$\cancel{Av} = \lambda v$

$\lambda_1 = 1: \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$0v_1 + v_2 = v_1$ $\qquad v_1 = v_2$

$v_1 + 0v_2 = v_2$

$\lambda_2 = -1 \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -v_1 \\ -v_2 \end{bmatrix}$

$0v_1 + v_2 = -v_1$ $\qquad v_1 = -v_2$

$v_1 + 0v_2 = -v_2$

$v_1 = c \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\qquad c = \dfrac{1}{\cancel{\text{norm}}\ |x|}$

$v_2 = c \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\qquad c = \dfrac{1}{\sqrt{2}}$

$v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \qquad v_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$

Dirac Notation

$|+\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle$

$|-\rangle = \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle$

These are the same eigenvectors as the Hadamard basis

4) Consider the 2-qubit state

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|1\rangle$$

Prove that the state is entangled by proving that there are no possible values of $a_0, a_1, \beta_0, \beta_1$ such that

$$|\psi\rangle = (a_0|0\rangle + a_1|1\rangle)(\beta_0|0\rangle + \beta_1|1\rangle)$$

$$|\psi\rangle = (a_0|0\rangle + a_1|1\rangle)(\beta_0|0\rangle + \beta_1|1\rangle)$$

$$|\psi\rangle = a_0\beta_0|0\rangle|0\rangle + a_0\beta_1|0\rangle|1\rangle + a_1\beta_0|1\rangle|0\rangle + a_1\beta_1|1\rangle|1\rangle$$

$$\frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|1\rangle = a_0\beta_0|0\rangle|0\rangle + a_0\beta_1|0\rangle|1\rangle + a_1\beta_0|1\rangle|0\rangle + a_1\beta_1|1\rangle|1\rangle$$

We get the 4 equations

$$\frac{1}{\sqrt{2}}\cancel{|0\rangle|0\rangle} = a_0\beta_0\cancel{|0\rangle|0\rangle}$$

$$0 = a_0\beta_1\cancel{|0\rangle|1\rangle}$$

$$0 = a_1\beta_0\cancel{|1\rangle|0\rangle}$$

$$\frac{1}{\sqrt{2}}\cancel{|1\rangle|1\rangle} = a_1\beta_1\cancel{|1\rangle|1\rangle}$$

$$\frac{1}{\sqrt{2}} = a_1\beta_1 \quad \frac{1}{\sqrt{2}} = a_0\beta_0 \quad \left[0 = a_0\beta_1\right] \quad \left[0 = a_1\beta_0\right]$$

a_0 or β_1 must be $= 0$

let $a_0 = 0$ $\quad \frac{1}{\sqrt{2}} = 0\beta_0$ cannot be true

let $\beta_1 = 0$ $\quad \frac{1}{\sqrt{2}} = a_0 0$ cannot be true $\quad$ same logic