Pendulum System

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1 System Description and Objective

A bob with mass m is attached to a massless, rigid string with length l. This string is attached to a pivot joint fixed to the ceiling. Acceleration due to gravity is purely in the vertical direction, perpendicular to the ceiling. The bob with mass m is represented as a point mass and at times will be referred to as a point mass. From the given information the objective is to write a differential equation $F = ma$ and find the solution of this differential equation. This solution is the position of the point mass as a function of time. If an object has circular motion with a constant radius, the object will always move in the direction of $\hat{\theta}$, which is tangential to the circumference of the circle.

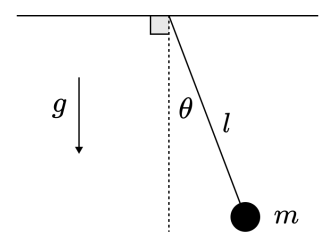

Figure 1: Diagram of the pendulum system.

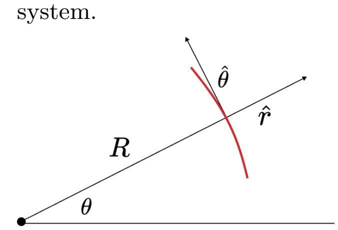

Figure 2: Polar coordinates and circular trajectory.

2 Coordinate System

The path of the point mass is a circular arc with an invariant radius in a two dimensional plane. If the state of the system is represented by polar coordinates (r, θ) there is no calculation needed for r, it will always be l the length of the string. With polar coordinates only one dynamical variable θ is needed to describe the state of the system. Where θ is the angle between the string and a vertical line parallel to the direction of gravitational acceleration.

3 Writing expressions for F and ma

The velocity and acceleration of the point mass are purely in the $\hat{\theta}$ direction which is tangent to the circular arc. During a small time interval Δt the point mass travels a distance $l\Delta\theta$. The force $mg\sin\theta$ is parallel to $\hat{\theta}$, the direction of motion of the mass.

$$l\Delta\theta = l\big[\theta(t+\Delta t) - \theta(t)\big] \tag{1}$$

If this small distance is divided by Δt and $\Delta t \to 0$ then (1) is the expression for the velocity of the point mass, it can be stated more compactly as $l\dot{\theta}$.

$$l\left[\lim_{\Delta t \to 0} \frac{\theta(t+\Delta t) - \theta(t)}{\Delta t}\right] \tag{2}$$

To get an expression for acceleration, take the time derivative of the velocity $l\dot{\theta}$.

$$\frac{d}{dt} l\dot{\theta} = l\ddot{\theta} \tag{3}$$

The expression for ma is

$$ml\ddot{\theta}. \tag{4}$$

Half of the problem for setting up $F = ma$ is done, now the expression for F needs to be found. In the free body diagram the total force in the $\hat{\theta}$ direction is $-mg\sin\theta$ which is the expression for F. The differential equation $F = ma$ can now be written. Forces in the $\hat{r}$ direction sum to zero, so the magnitude of the tension in the string is $mg\cos\theta$. However, this information is not needed for the equation of motion.

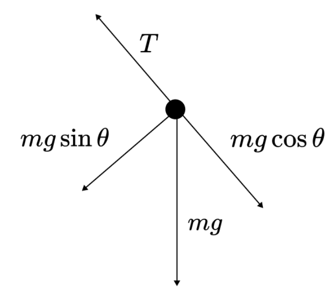

Figure 3: Free body diagram for the pendulum.

$$ml\ddot{\theta} = -mg\sin\theta \tag{5}$$

$$\ddot{\theta} = -\frac{g}{l}\sin\theta \tag{6}$$

For small angles θ, $\sin\theta$ is approximated by θ. Using this approximation the differential equation is now

$$\ddot{\theta} = -\frac{g}{l}\theta, \tag{7}$$

where the series representation of $\sin\theta$ is

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots.$$

If $\omega = \sqrt{g/l}$, then equation (7) can be written as

$$\ddot{\theta} = -\omega^2\theta. \tag{8}$$

4 Qualitative Analysis

The differential equation (6) expresses that the acceleration of the point mass is proportional to the sine of the angle between the string and the vertical line. In a sense the mass is getting a stronger "push" as $\sin\theta$ increases. When $\theta = \pi/4$, $\sin\theta$ is at its maximum. At this angle the force due to gravity is completely lined up or parallel with the $\hat{\theta}$ direction. When $\theta = 0$, there is no force applied to the mass in the $\hat{\theta}$ direction, all the force that was applied to the mass prior to this point accumulated into kinetic energy.

5 Solving the Differential Equation

$$\begin{gathered}\frac{d^2}{dt^2}\cos\omega t = -\omega^2\cos\omega t \\ \\ \frac{d^2}{dt^2}\sin\omega t = -\omega^2\sin\omega t\end{gathered} \tag{9}$$

There is a common pattern that the equations (9) share, that indicates the solutions of (8). The general form of the solution to (8) is

$$\theta(t) = A\cos\sqrt{\frac{g}{l}}t + B\sin\sqrt{\frac{g}{l}}t \tag{10}$$

where A and B are dimensionless constants that are determined from $\theta(0)$ and $\dot{\theta}(0)$. If $\theta(0) = A$ and $\dot{\theta}(0) = 0$ then the particular solution is

$$\theta(t) = A\cos\sqrt{\frac{g}{l}}t. \tag{11}$$

6 Dimensional Analysis

Verification of dimensions is a good way to check for errors. The arguments of the sine and cosine functions are radians, a dimensionless quantity, which means $\sqrt{\frac{g}{l}}$ needs to have dimensions of T^{-1}.

$$\left[\sqrt{\frac{g}{l}}\right] = [L^1T^{-2}L^{-1}]^{1/2} = T^{-1} \tag{12}$$

7 Lagrangian Method

Motion of the simple pendulum can also be derived using the Lagrangian method. Once the Lagrangian formula (13) is found, the Euler-Lagrange equation (14) can be written and solved. The solution of (14) will be equivalent to (10).

$$\mathcal{L} = T - V \tag{13}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q} \tag{14}$$

$\mathcal{L}$ is the kinetic energy T minus the potential energy V.

$$T = \frac{1}{2}ml^2\dot{\theta}^2 \tag{15}$$

$$V = -mgl\cos\theta \tag{16}$$

$$\mathcal{L} = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta \tag{17}$$

$$\tag{18}$$

This specific Lagrangian plugged into (14) yields the following differential equation, which is the same as equation (5).

$$ml^2\ddot{\theta} = -mgl\sin\theta$$