SOURCES AND SINKS AT THE AXIS OF A ROTATING LIQUID†

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SUMMARY

A solution is obtained for the flow of a rotating, frictionless, incompressible fluid due to a strong source or sink at the axis of rotation. The type of motion is controlled by the value of the Rossby number, *Ro*, a ratio of inertial and Coriolis forces. The solution resembles potential flow if *Ro* is very large. As *Ro* decreases the sink draws more and more from the axis of rotation until, at $Ro = 0{\cdot}261$, the fluid approaches the sink in a jet centred at the axis and with a radius about half that of the cylinder. No solution is obtained for smaller values of *Ro*. If a jet is postulated for small values of *Ro*, it is shown that its radius decreases as the cube root of the flux. Several experimental photographs are shown. They contain some of the features of the theory. The jet type of motion becomes very pronounced for weak sinks.

1. Theory

In a previous paper the author (1) has shown that a solution of the differential equation

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial\rho^2}-\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}+\sigma^2\psi = -\tfrac{1}{2}\sigma^2 u_0\,\rho^2 \tag{1}$$

represents steady-state, axially symmetric flow of a rotating, frictionless, incompressible fluid; ψ is the Stokes's stream-function. The coordinate system is shown in Fig. 1. In the derivation of equation (1) it was assumed that at $x = -\infty$ the fluid has a constant absolute angular velocity Ω about the x-axis and a constant linear velocity u_0 parallel to the axis of rotation. The parameter σ is $2\Omega/u_0$.

The purpose of this note is to apply (1) to the problem of sources and sinks located on the axis of rotation. We suppose first that there is a sink at the point $x = 0$, $\rho = 0$ in an infinitely long cylinder of radius b containing rotating liquid. We require the streamline pattern to be symmetric about the plane $x = 0$ and assume that at $|x| = \infty$ the absolute velocity field is a constant angular velocity Ω and constant velocity u_0 toward the sink. In view of the symmetry, the problem is equivalent to flow from the negative x-axis toward a hole at $x = 0$, $\rho = 0$ in a wall coincident with the plane $x = 0$. An appropriate solution of (1) is

$$\psi = -\frac{u_0\,\rho^2}{2}+\rho\sum_\alpha A_\alpha\, e^{(\alpha^2-\sigma^2)^{\frac{1}{2}}x}J_1(\alpha\rho). \tag{2}$$

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Unless $\alpha = \sigma$ the solution (2) yields uniform flow, u_0, as $x \to -\infty$. It was also shown by the author (2) that w, the component of the absolute velocity tangent to a circle centred at the axis of rotation, is related to ψ by the equation

$$w\rho = -\sigma\psi. \tag{3}$$

Equations (2) and (3) show that $w \to \Omega\rho$ as $x \to -\infty$, or solid rotation.

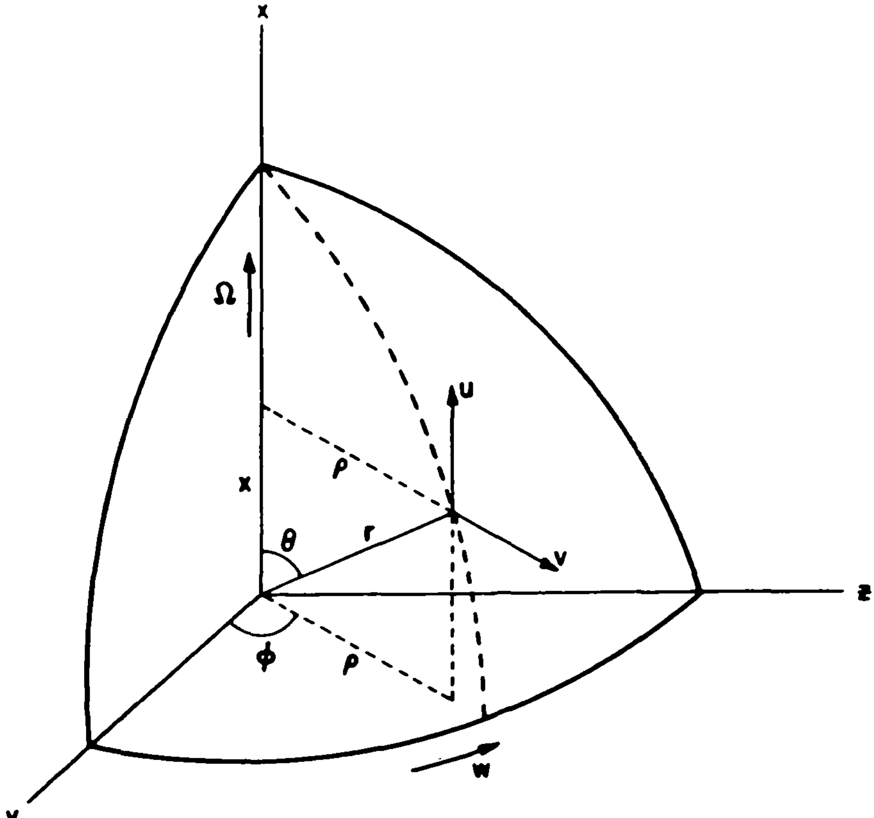

FIG. 1. Coordinate system. The x-axis is taken along the axis of rotation.

At the wall of the cylinder, the uniform flux through a plane perpendicular to the axis is $-2\pi\psi$. Hence at $\rho = b$, $\psi = -u_0 b^2/2$. Therefore, $\alpha b = z_n$, where z_n are the zeros of the Bessel function $J_1(z)$, the first two being $z_1 = 3{\cdot}8317$, $z_2 = 7{\cdot}0156$. The solution (2) becomes

$$\psi = -\frac{u_0\rho^2}{2} + \rho \sum_{n=n_1}^{\infty} A_n e^{(z_n^2 - Ro^{-2})^{\frac{1}{2}}(x/b)} J_1\left(z_n \frac{\rho}{b}\right). \tag{4}$$

We have substituted for $(\sigma b)^{-1}$ the symbol $Ro = u_0/2\Omega b$. This is called the Rossby number in meteorological literature (1). It is a measure of the importance of the rotational or Coriolis forces in rotating systems. The summation in (4) begins at n_1, where n_1 is the first integer for which $Ro > 1/z_n$. For smaller integers we would have oscillating functions of x in (4) and disturbed motion at infinity.

We notice that $\psi = 0$ on the axis $\rho = 0$. We obtain a sink at $x = 0$, $\rho = 0$ by requiring that ψ has the same value at the plane $x = 0$ (except at the point $x = 0$, $\rho = 0$) as it has on the cylinder wall, namely $-u_0 b^2/2$. The resulting jump discontinuity of ψ yields the infinite velocity at $(0,0)$ required in a sink. Hence for $x = 0$, $0 < \rho/b \leqslant 1$,

$$\frac{u_0 b^2}{2}\left(\frac{\rho^2}{b^2}-1\right) = \rho \sum_{n=n_1}^{\infty} A_n J_1\left(z_n \frac{\rho}{b}\right). \tag{5}$$

Multiplying both sides of (5) by $J_1(z_m \rho/b)$ and integrating with respect to ρ/b from 0 to 1, we find

$$A_m = -\frac{u_0 b}{z_m J_0^2(z_m)}. \tag{6}$$

The solution becomes

$$\psi = -\frac{u_0 \rho^2}{2} - u_0 b\rho \sum_{n=1}^{\infty} \frac{\exp[(z_n^2 - Ro^{-2})^{\frac{1}{2}} x/b]}{z_n J_0^2(z_n)} J_1(z_n \rho/b). \tag{7}$$

Since the summation begins at $n = 1$, the solution exists only if Ro exceeds $1/z_1$, or $Ro \geqslant 0{\cdot}261$. This corresponds to low or moderate values of the angular velocity of the system or to strong sinks. Obviously we may change the sign of u_0 and (7) will still satisfy the differential equation (1). With the proper choice of pressure, the resulting motion will then satisfy the primitive equations of motion and continuity for the steady symmetric flow of a frictionless, incompressible, rotating fluid. The pressure may be found from the equation

$$\frac{p}{q} + \frac{u^2+v^2+w^2}{2} = \frac{p_\infty(\rho_0)}{q} + \frac{u_0^2}{2} + \frac{\Omega^2 \rho_0^2}{2}, \tag{8}$$

where q is the constant density, p_∞ is the pressure at $x = -\infty$, and ρ_0 is the Lagrangian distance of a streamline from the axis. Equation (7), with a negative value of u_0, yields flow due to the emission of fluid from the point $(0,0)$. The emitted fluid possesses vorticity so that this system may be regarded as a source combined with a vortex of a special type. The case of a source emitting irrotational fluid has been considered by Barua (3). In the latter problem the irrotational fluid moves along the axis in the form of a cylindrical jet.

The flow patterns given by (7) are shown for several values of *Ro* in Fig. 2. We note that as *Ro* decreases from ∞ the sink draws more and more from

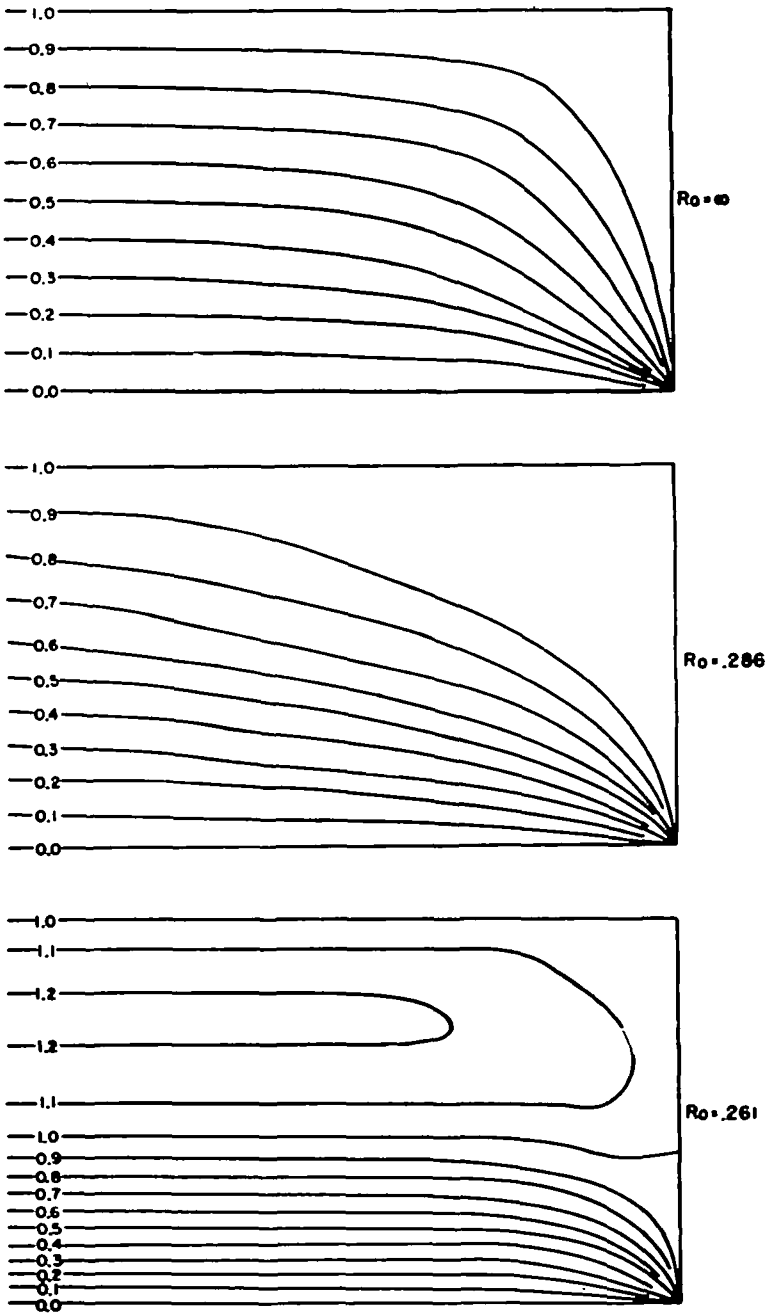

FIG. 2. Flow patterns computed from equation (7). The curves are isolines of $\rho_0/b = (-2\psi/u_0 b^2)^{\frac{1}{2}}$ (see Long, 1953).

the axis of rotation. This tendency is a maximum for the critical value of $Ro = 0{\cdot}261$, in which case the fluid approaches the sink in a 'jet'. As $x \to -\infty$ the velocity tends to

$$u = u_0\left[1+\frac{J_0(z_1\rho/b)}{J_0^2(z_1)}\right]. \tag{9}$$

This is a maximum in the vicinity of the axis, becoming negative near the cylinder wall. Although the condition that $u \to u_0$ at $x = -\infty$ is violated. the difficulty is avoided by choosing Ro slightly larger than 0·261.

If we solve equation (1) for free stationary waves in an infinite cylinder we find, typically,

$$\psi = -\frac{u_0\rho^2}{2}+\rho B_n e^{ik_n x}J_1\left(z_n\frac{\rho}{b}\right), \tag{10}$$

where
$$k_n = \left(\sigma^2-\frac{z_n^2}{b^2}\right)^{\frac{1}{2}}. \tag{11}$$

For a given value of n we may interpret (10) and (11) as a wave moving along the axis of a cylinder at a speed $c = u_0$ relative to the liquid. Then

$$c^2 = \frac{4\Omega^2b^2}{z_n^2+4\pi^2b^2/\lambda_n^2}, \tag{12}$$

where λ_n is the wavelength. The speed decreases with n and increases with wavelength. The maximum speed of any possible wave of this type is a wave of infinite length with no internal nodal surfaces, i.e. $z_n = z_1$. Applying these results to the problem of the sink, if $\sigma b < z_1$, the fluid approaches the sink at great distances with a speed greater than that of any wave. Hence no free wave can advance against this current. If σb exceeds z_1 slightly, a long wave can move upstream against the current. The steady-state solution at $\sigma b = z_1$ shows that this has happened and the wave has altered the upstream velocity distribution to that of equation (9). This reasoning supports the assumption that the flow at $x = -\infty$ is uniform for strong sinks.

Although a steady-state solution exists for all values of Ro greater than 0·261, we find that closed circulations begin to form in some parts of the system when Ro is close to the critical value. These circulations remain weak (Fig. 2) even at $Ro = 0{\cdot}286$. Very near $Ro = 0{\cdot}261$ the closed circulation strengthens, spreads upstream, and develops finally into the pattern of $Ro = 0{\cdot}261$ shown in Fig. 2. The significance of the appearance of negative values of u follows from the equation

$$u = -\frac{1}{\rho}\frac{\partial\psi}{\partial\rho} = \frac{1}{\sigma\rho}\frac{\partial}{\partial\rho}(w\rho). \tag{13}$$

When u is negative the absolute angular momentum *decreases outward* locally, and we would expect this to be unstable. This suggests the possibility that weak sinks may lead to flows that are essentially turbulent and unsteady, and that no physically meaningful steady-state solution may exist. Since we may write

$$Ro = \frac{F}{2\Omega\pi b^3}, \tag{14}$$

where F is the volume flux through the sink, we are led to the further possibility that no steady-state flow exists for a sink of any finite strength in a completely unbounded rotating fluid ($b = \infty$). In this case, however, we may postulate the existence of a quasi-steady jet of radius b_0 moving toward the sink. If so, the only pertinent quantities, F, Ω, b_0, lead to only one non-dimensional number, say

$$Ro' = \frac{F}{2\pi\Omega b_0^3}, \tag{15}$$

which must therefore be an absolute constant. The magnitude of the constant may be inferred from a consideration of the jet existing theoretically at $Ro = 0{\cdot}261$. Taking the boundary of the jet at the value of ρ where $|du/d\rho|$ is a maximum, $b_0/b \simeq 0{\cdot}48$ and

$$Ro' = Ro\frac{b^3}{b_0^3} \simeq 2{\cdot}3. \tag{16}$$

This is probably a crude estimate since the theoretical jet has a radius that is not small compared to that of the cylinder and therefore does not resemble a jet in an unlimited fluid.†

2. Experimental results

An experimental study was made of the effect of a sink at the axis of rotation of a tall cylinder. A cylinder of water of radius $b = 5{\cdot}7$ cm. and height 125 cm. is positioned on a rotating turntable and brought up to solid rotation. Water is then extracted at the bottom at the axis of rotation through a hole of radius 0·66 cm. The flux and rotation rate may be varied to obtain different values of $Ro = F/2\Omega\pi b^3$. The flow pattern is recorded by streak photography, using a camera rotating with the turntable, with line of sight perpendicular to a plane of light passed through the axis of rotation. Aluminium particles are used as tracers.

The experimental photographs are shown in Figs. 3–7 in order of decreasing Rossby number. Figs. 3 and 4, with Ro values of ∞ and $0{\cdot}281$ respectively, show the effect of a moderate amount of rotation. The sink draws more strongly from regions near the axis of rotation in Fig. 4, and there is

† Sir Geoffrey Taylor, in a private communication to the author, remarked that Barua obtained $Ro' = 0{\cdot}31$ for a jet due to a source at the axis of a rotating fluid.

a definite indication of the eddies near the walls, predicted by the theory and shown in the 0·286 drawing of Fig. 2. No abrupt change of flow pattern occurs in the experiments at the critical value of the Rossby number.

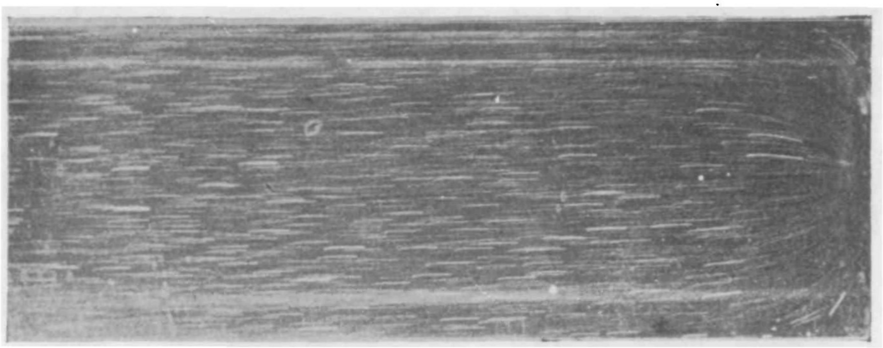

FIG. 3. Flow toward a sink. No rotation, $Ro = \infty$.

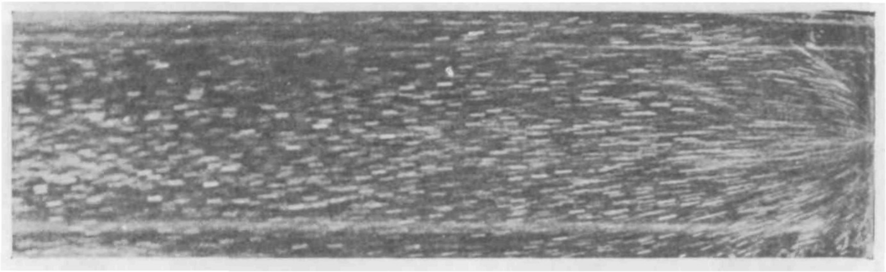

FIG. 4. Flow toward a sink. $Ro = 0{\cdot}281$.

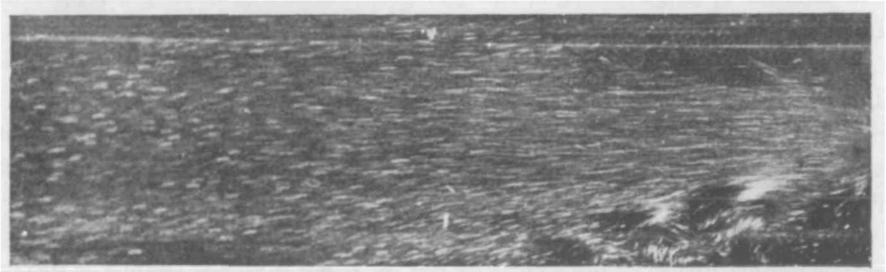

FIG. 5. Flow toward a sink. $Ro = 0{\cdot}203$.

Instead, the tendency for a jet formation seems to increase in a continuous manner. This is not surprising in view of the fact that the experiment is only quasi-steady. This point is illustrated in Fig. 5. The Rossby number, 0·203, is well below the critical value. A well-defined jet is being formed in the vicinity of the sink, apparently by upstream propagation of the waves seen in the photograph. These waves have progressed a short distance

FIG. 6. Flow toward a sink. $Ro = 0{\cdot}100$.

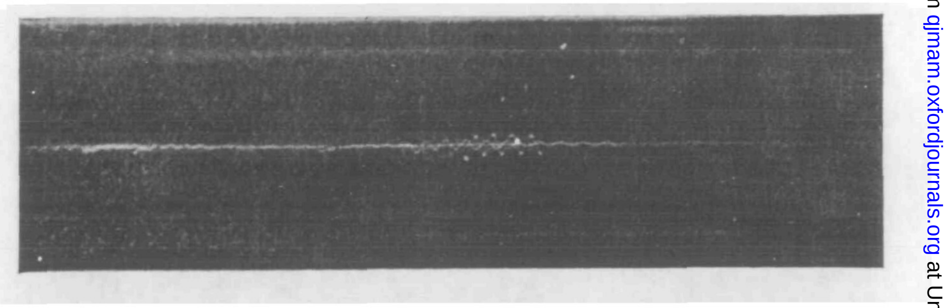

FIG. 7. Flow toward a sink. $Ro = 0{\cdot}006$.

ahead of the sink and the approaching flow is only affected in this vicinity. For smaller Rossby numbers the formation of the jet is more rapid. In Fig. 6 it exists far ahead of the sink.

Fig. 7 illustrates the flow at an extremely low value of $Ro = 0{\cdot}0061$. The jet is concentrated very near the axis of rotation and is spinning very rapidly. A single tracer particle has performed five revolutions about the axis in the $\frac{1}{6}$ sec. time exposure of the photograph, yielding an angular velocity about 60 times the basic rotation (27 r.p.m.).

An effort may be made to recompute the constant in equation (16) from experimental observation. Thus in Fig. 6, if we take the diameter of the jet as, perhaps, $\frac{1}{4}$ the diameter of the vessel, equation (16) yields $Ro' \cong 1{\cdot}0$. Similar calculations for experiments with lower Rossby numbers also lead to values around 1·0. In all cases, however, the diameter of the jet is difficult to estimate in the photographs and Ro' is very sensitive to variations in this estimate.

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