Data Science | EA 6 – Probability | Patrick Bichiou

1) Probability of rolling a: $1 \to p$

$2 \to 2p$

$3 \to 3p$

$\dots$

$6 \to 6p$

Sum of the probabilities must equal 1:

$$p + 2p + 3p + 4p + 5p + 6p = 1$$

$$21p = 1 \quad |:21$$

$$p = \frac{1}{21}$$

Assign probabilities:

$P(1) = p = \frac{1}{21}$ $\qquad$ $P(4) = 4p = \frac{4}{21}$

$P(2) = 2p = \frac{2}{21}$ $\qquad$ $P(5) = 5p = \frac{5}{21}$

$P(3) = 3p = \frac{3}{21}$ $\qquad$ $P(6) = 6p = \frac{6}{21}$

Expected Value $E(X)$:

$$E(X) = \sum_{x=1}^{6} x \cdot P(x)$$

$$E(x) = 1 \cdot \frac{1}{21} + 2 \cdot \frac{2}{21} + 3 \cdot \frac{3}{21} + 4 \cdot \frac{4}{21} + 5 \cdot \frac{5}{21} + 6 \cdot \frac{6}{21}$$

$$E(x) = \frac{1}{21} + \frac{4}{21} + \frac{9}{21} + \frac{16}{21} + \frac{25}{21} + \frac{36}{21}$$

$$E(x) = \frac{1+4+9+16+25+36}{21} = \frac{91}{21} = \frac{13}{3} \approx \underline{\underline{4{,}\overline{33}}}$$

Variance $Var(X)$:

$$Var(X) = E(X^2) - |E(X)|^2$$

$$E(X^2) = \sum_{x=1}^{6} x^2 \cdot P(x) = 1^2 \cdot \frac{1}{21} + 2^2 \cdot \frac{2}{21} + 3^2 \cdot \frac{3}{21} + 4^2 \cdot \frac{4}{21} + 5^2 \cdot \frac{5}{21} + 6^2 \cdot \frac{6}{21}$$

$$= \frac{1}{21} + \frac{8}{21} + \frac{27}{21} + \frac{64}{21} + \frac{125}{21} + \frac{216}{21} = \frac{1+8+27+64+125+216}{21}$$

$$= \frac{441}{21} = \underline{\underline{21}}$$

$$Var(X) = E(X^2) - |E(X)|^2 = 21 - \left(\frac{13}{3}\right)^2 = 21 - \frac{169}{9}$$

$$= 21 - 18{,}77 = \underline{\underline{2{,}23}}$$

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3) We can use Bayes' theorem: $P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$

A: pulled out the 6-sided die

B: number rolled is 5

1. $P(A)$: Die is pulled at random and there are two dices. So the probality of pulling out the 6-sided die is: $\underline{\underline{P(A) = \frac{1}{2}}}$

2. $P(B|A)$: The probability of rolling a 5 given that the 6-sided dice was pulled: $\underline{\underline{P(B|A) = \frac{1}{6}}}$

3. $P(\bar{A})$: The probability of pulling out the 12-sided die. Complement of $P(A)$: $\underline{\underline{P(\bar{A}) = \frac{1}{2}}}$

4. $P(B|\bar{A})$: The probability of rolling a 5 given that the 12-sided dice was pulled: $\underline{\underline{P(B|\bar{A}) = \frac{1}{12}}}$

5. $P(B)$: The total probability of rolling a 5

$$P(B) = P(B|A) \cdot P(A) + P(B|\bar{A}) \cdot P(\bar{A})$$
$$= \left(\frac{1}{6}\right) \cdot \left(\frac{1}{2}\right) + \left(\frac{1}{12}\right) \cdot \left(\frac{1}{2}\right)$$
$$= \frac{1}{12} + \frac{1}{24} = \frac{2}{24} + \frac{1}{24} = \frac{3}{24} = \underline{\underline{\frac{1}{8}}}$$

6. Apply Bayes' theorem: $P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$

$$= \frac{\left(\frac{1}{6}\right) \cdot \left(\frac{1}{2}\right)}{\frac{1}{8}} = \frac{\frac{1}{12}}{\frac{1}{8}}$$

$$= \frac{1}{12} \cdot \frac{8}{1} = \frac{8}{12} = \underline{\underline{\frac{2}{3}}}$$

=> The probability that the die pulled out was the 6-sided die given that the number rolled is 5 is $\underline{\underline{\frac{2}{3}}}$.

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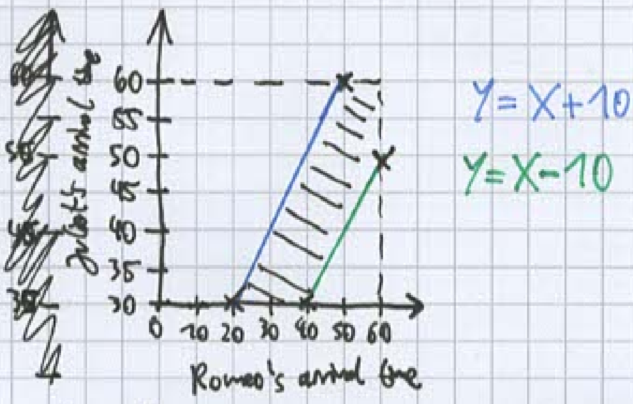
4)
Modelling the problem by using a coordinate system, where X is the time Romeo arrives and Y represent the time Juliet arrives.

Time intervals

- Romeo's arrival time X is uniformly distributed between 9:00 and 10:00 => $X \in [0, 60]$ (where 0 represents 9:00 and 60 represents 10:00)
- Juliet's arrival time Y is uniformly distributed between 9:30 and 10:00 => $Y \in [30, 60]$

Meeting Condition

Romeo and Juliet will meet if the absolute difference between their arrival times is at most 10 minutes => $|X-Y| \leq 10$



Boundaries

1. $Y \geq 30$ (Juliet arrives no earlier than 9:30)
2. $Y \leq 60$ (Juliet arrives no later than 10:00)
3. $X \geq 0$ (Romeo arrives no earlier than 9:00)
4. $X \leq 60$ (Romeo arrives no later than 10:00)

Feasible Region

The feasible region where X and Y satisfy $30 \leq Y \leq 60$ and $0 \leq X \leq 60$ can be divided into segments based on the lines $Y = X+10$ and $Y = X-10$:
- For $0 \leq X \leq 20$, Y can be between 30 and $X+10$
- For $20 \leq X \leq 50$, Y can be between $X-10$ and $X+10$
- For $50 \leq X \leq 60$, Y can be between $X-10$ and 60

Calculating the Area

1. For $0 \leq x \leq 20$

   → The area is a triangle with base 20 and height 10

   $$A_1 = \frac{1}{2} \cdot 20 \cdot 10 = 100$$

2. For $20 \leq x \leq 50$

   → The area is a rectangle with ~~base~~ width 30 and height 20

   $$A_2 = 30 \cdot 20 = 600$$

3. For $50 \leq x \leq 60$

   → The area is a triangle with base 10 and height 10

   $$A_3 = \frac{1}{2} \cdot 10 \cdot 10 = 50$$

Total meeting area:

$$A_{MT} = A_1 + A_2 + A_3 = 100 + 600 + 50 = \underline{750}$$

Probability of meeting:

$$P = \frac{\text{Meeting Area}}{\text{Total Area}} = \frac{750}{1800} = \frac{5}{12} \approx 0{,}4167$$

⇒ The probability that Romeo and Juliet will meet is $\frac{5}{12} \approx 0{,}4167$.