Lecture 8

September 9, 2024

Today we will talk about autonomous equations. These are differential equations of the form below:

$$\frac{dx}{dt} = f(x)$$

These equations are all separable, but it isn't always easy or possible to solve the integrals necessary to solve the equation. We can still learn a lot about the solutions by creating a phase diagram (also known as a phase portrait). To do so, we take the following steps:

1. Set $f(x) = 0$
2. Solve for all roots of $f(x)$. These roots are called critical points.
3. Draw a phase diagram
4. Classify the critical points

When creating a phase diagram, you draw arrows pointing up if $\frac{dx}{dt} > 0$ in the region and downward if $\frac{dx}{dt} < 0$. For a given critical point, if both arrows point toward the critical point, then the solution is a **stable** solution. If both arrows point away from the critical point, then the solution is an **unstable** solution. If some of the arrows point towards the critical point and others point away, we say a solution is **semi-stable.**

Examples

Ex 1)

Draw a phase diagram, classify all critical points, and sketch solution curves for various initial values for

$$\frac{dx}{dt} = (x-1)(x-2)$$

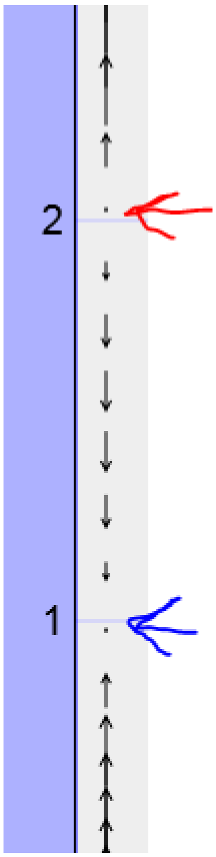
First, we will find the critical points:

$$\frac{dx}{dt} = 0 = (x-1)(x-2)$$

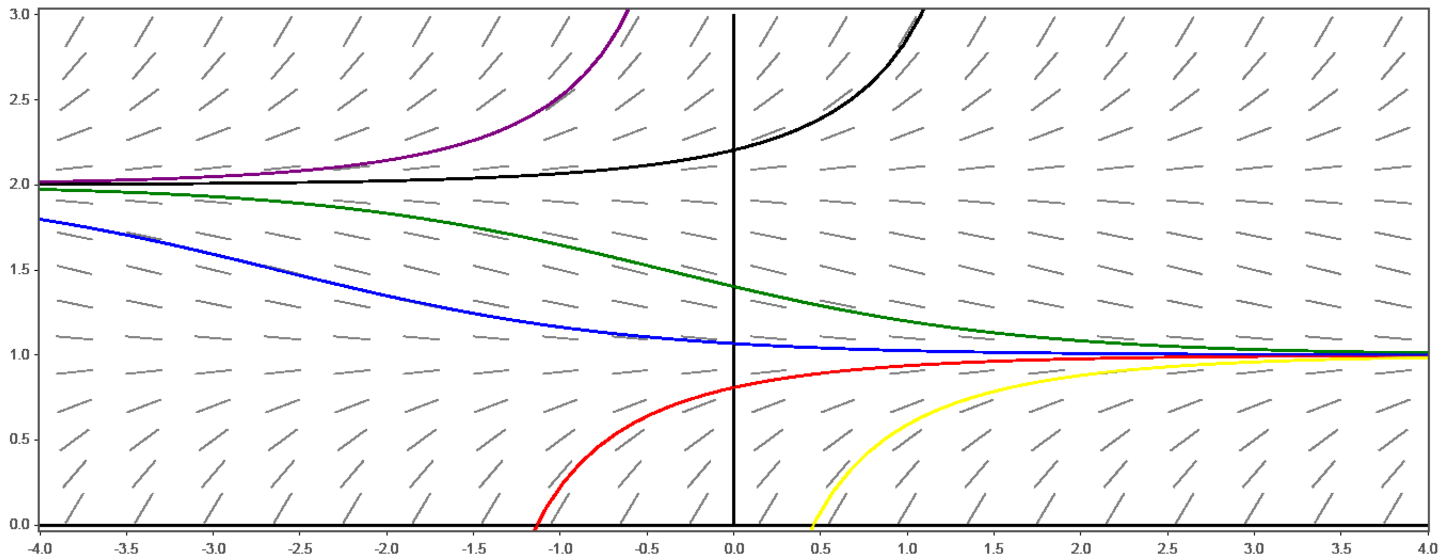
Thus

$$x = 1, 2$$

Now, we can draw a phase diagram.



If we let $f(x) = (x-1)(x-2)$, then $f(x) > 0$ for $x > 2$, $f(x) < 0$ for $1 < x < 2$, and $f(x) > 0$ for $x < 1$. This gives the directions of the arrows in the diagram. As all arrows at $x = 2$ point away from 2 and the arrows at $x = 1$ all point towards 1, then $x = 2$ is an unstable critical point and $x = 1$ is a stable critical point. A typical solution curve will move away from unstable critical points and towards stable critical points as $t \to \infty$. Below is a plot of the slope field with some typical solution curves.



Ex 2)

Draw a phase diagram, classify all critical points, and sketch solution curves for various initial values for

$$\frac{dx}{dt} = x^2(3-x)$$

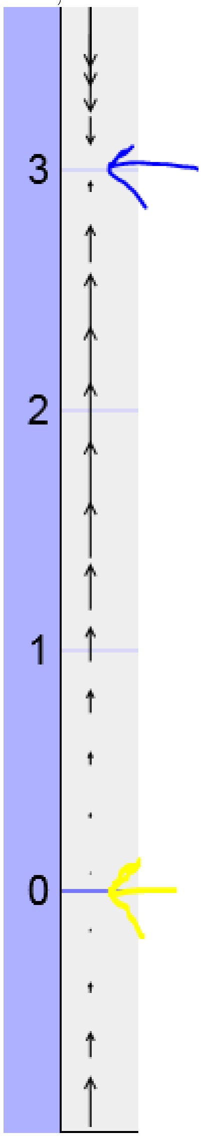
First, we will find the critical points:

$$\frac{dx}{dt} = 0 = x^2(3-x)$$

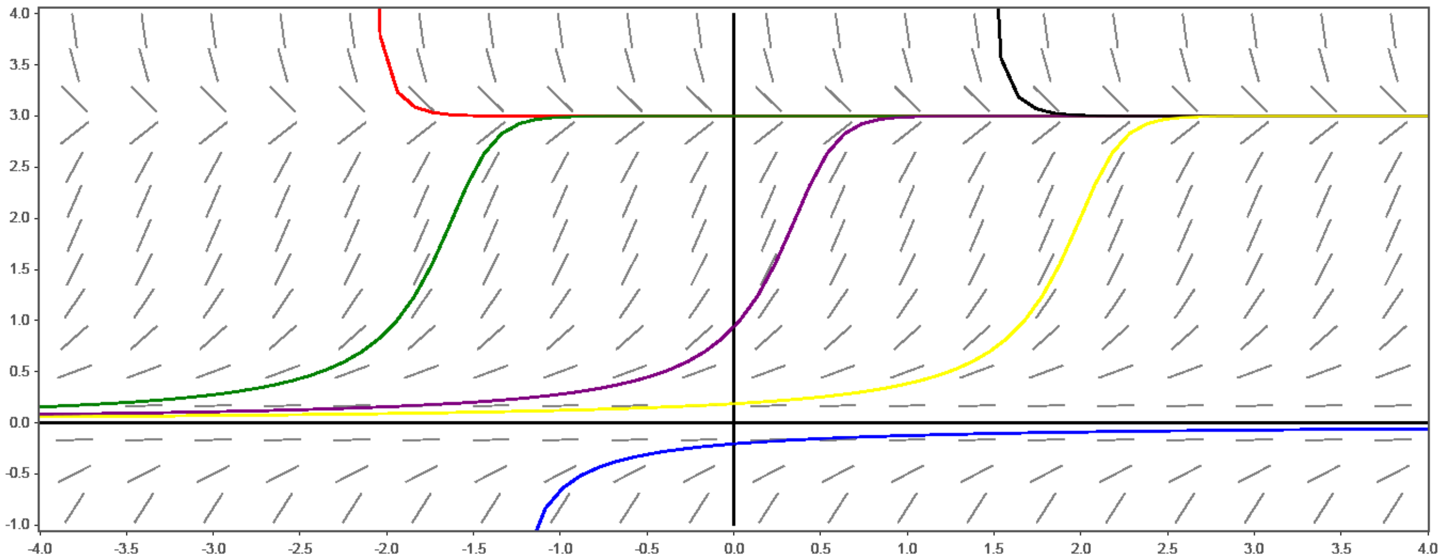
Thus

$$x = 0, 3$$

Now, we can draw a phase diagram.



If we let $f(x) = x^2(3-x)$, then $f(x) < 0$ for $x > 3$, $f(x) > 0$ for $0 < x < 3$, and $f(x) > 0$ for $x < 0$. This gives the directions of the arrows in the diagram. As all arrows at $x = 3$ point towards 3 and some of the arrows at $x = 0$ point towards and away from 0, then $x = 3$ is an stable critical point and $x = 0$ is a semi-stable critical point. A typical solution curve will move away from unstable critical points and towards stable critical points as $t \to \infty$. For a semi-stable critical point, the solution curve will only move towards the critical point on the stable side. Below is a plot of the slope field with some typical solution curves.



Ex 3)

The logistic equation models population growth. Logistic equations are of the form:

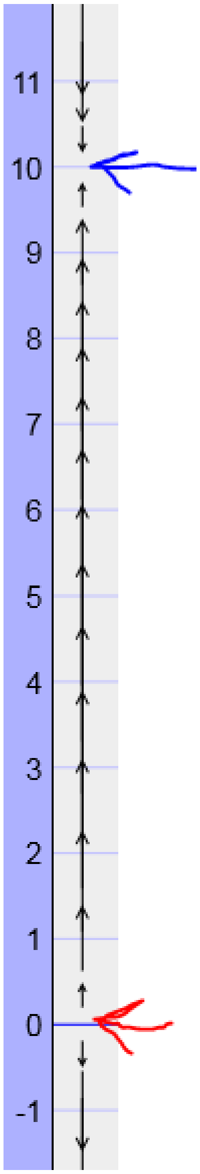
$$\frac{dP}{dt} = kP\,(M - P)$$

where $k, M > 0$. k is a constant related to how quickly a population grows or dies. M is the carrying capacity of a population. This is the largest population that an ecosystem can support. It is possible to solve this equation using separation of variables, but the integral involves fairly gross partial fraction decomposition. Even without solving the equation, we can find a fair bit about the solution by making a phase diagram. Suppose $k = 1$ and $M = 10$. Then we get

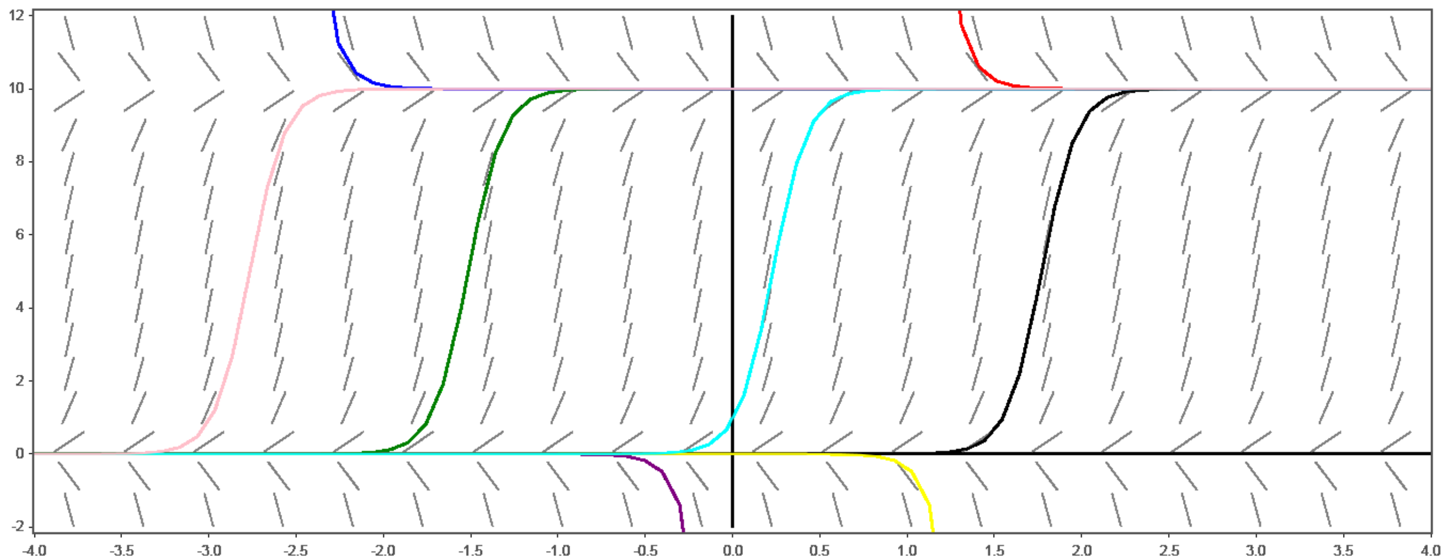
$$\frac{dP}{dt} = P\,(10 - P)$$

therefore we have critical points at

$$P = 0, 10$$



This phase diagram shows a stable solution at $P = 10$ and an unstable solution at $P = 0$. Physically, this means the population will grow to 10 if it is less than 10 and will shrink to 10 if it is less than 10. The model also suggests that a negative population will decrease to negative infinity. This isn't a physical solution, so we are okay with allowing this. We can never have negative population in practice. For many species, a population of 1 will not allow the population to grow. We are okay with this slight inaccuracy in practice, since the model does a good job of estimating population for most realistic cases. We also see this in the solution curves in the slope field.



Ex 4)

Sometimes we might modify the logistic equation. Suppose the population of fish in a pond is currently modeled by

$$\frac{dP}{dt} = 2P\left(100 - P\right)$$

where P is in fish and t is in months. Suppose a scientist wishes to add 5 additional fish to the pond each month. Then the new model becomes:

$$\frac{dP}{dt} = 2P\left(100 - P\right) + 5$$

We can find the stable population of the pond by finding and classifying the critical points of this differential equation. First, let's find the critical points.

$$\frac{dP}{dt} = 2P\left(100 - P\right) + 5 = 0$$

$$200P - 2P^2 + 5 = 0$$

Thus

$$2P^2 - 200P - 5 = 0$$

and

$$P = \frac{200 \pm \sqrt{\left(-200\right)^2 + 4 * 2 * 5}}{2 * 2} = \frac{200 \pm \sqrt{40000 + 40}}{4} = \frac{200 \pm \sqrt{40040}}{4} = 50 \pm \frac{\sqrt{10010}}{2}$$

We will call these two critical points

$$P_{\pm} = 50 \pm \frac{\sqrt{10010}}{2}$$

Where P_+ is when we take the $+$ of the $\pm$ and P_- is when we take the $-$ of the $\pm$. We can see that for $P < P_-$, $\frac{dP}{dt} < 0$, for $P_- < P < P_+$ that $\frac{dP}{dt} > 0$ and for $P > P_+$ that $\frac{dP}{dt} < 0$, thus P_- is unstable and P^+ is a stable. This agrees with our intuition. We cannot have a negative population of fish.