Lecture 6

September 2, 2024

For this lecture, we are going to talk about some common types of work problems and how to set up/solve them.

1) Simple Kinematics

These word problems are usually of the form:

A car drives at $x'(t) = v(t)$ meters per second, where t is time in seconds and x is position of the car in meters. If the car starts at $x(0)$, where will the car end up in t seconds?

Examples

Ex 1)

A car drives at $x'(t) = t^3 - t$ meters per second, where t is time in seconds and x is position of the car in meters. If the car starts at $x(0) = 2$ meters, where did the car end up in 10 seconds?

$$x(t) = \int x'(t)\,dt = \int t^3 - tdt = \frac{t^4}{4} - \frac{t^2}{2} + C$$

Now plug in initial conditions:

$$x(0) = 2 = \frac{0^4}{4} - \frac{0^2}{2} + C = C$$

So

$$x(t) = \frac{t^4}{4} - \frac{t^2}{2} + 2$$

Thus

$$x(10) = \frac{(10)^4}{4} - \frac{(10)^2}{2} + 2 = 202 \text{ meters}$$

2) Exponential Growth/Decay

These word problems are usually of the form:

A quantity grows at a rate proportional to t (or for decay, growth is at a rate proportional to $-t$). Find the quantity after t seconds.

Examples

Ex 1)

Seaborgium-263 has a half life of about 1s. If there is 12g of Seaborgium-263 at time $t = 0$, how many will remain after 6.3 seconds?

To solve this we have the equation:

$$\frac{dA}{dt} = -kA$$

Where A is the current amount in grams of Seaborgium-263, t is time in seconds, and $k > 0$. We can solve the differential equation by separating:

$$\frac{dA}{A} = -kdt$$

and integrating:

$$\int \frac{dA}{A} = \int -kdt$$

$$\ln(A) = -kt + C$$

(Here we were okay to assume $A \geq 0$ as we cannot have negative mass.

$$e^{\ln(A)} = e^{-kt+C}$$

$$A = Ce^{-kt}$$

We can now solve for C and k based on the conditions from the word problem.

$$A(0) = Ce^{0} = C = 12$$

As we have a half-life of 1s,

$$A(1) = \frac{A(0)}{2} = \frac{12}{2} = 6 = Ce^{-k}$$

We solve for k and plug in C.

$$6 = 12e^{-k}$$

$$\frac{1}{2} = e^{-k}$$

$$\ln\left(\frac{1}{2}\right) = -k$$

$$k = \ln(2)$$

Thus

$$A(t) = 12e^{-\ln(2)t} = 12e^{\ln\left(\frac{1}{2}\right)t} = 12e^{\ln\left(\frac{1}{2}\right)^{t}} = 12\left(\frac{1}{2^{t}}\right)$$

Thus

$$A(6.3) = 12\left(\frac{1}{2^{6.3}}\right) \approx 0.152 \text{ grams}$$

Ex 2)

Suppose the population of a bacteria triples every 15 minutes. If the population starts at 120 organisms, how many will there be after 2.2 hours?

To solve this, we have the equation:

$$\frac{dA}{dt} = kA$$

for some $k > 0$, t in hours, and A as the amount of organisms. We can separate this equation to solve it:

$$\frac{dA}{A} = kdt$$

Now, we integrate

$$\int \frac{dA}{A} = \int k dt$$

$$\ln(A) = kt + C$$

We can safely assume $A \geq 0$, as we cannot have negative organisms. Now we solve for A

$$A = e^{\ln(A)} = e^{kt+C} = Ce^{kt}$$

Now, we can plug in initial conditions:

$$A(0) = 120 = Ce^0 = C$$

and

$$A(0.25) = 3 * A(0) = 3 * 120 = 360 = 120e^{k*0.25}$$

Solving for k:

$$3 = e^{0.25k}$$

$$\ln(3) = 0.25k$$

$$k = \frac{\ln(3)}{0.25}$$

Thus

$$A(t) = 120e^{\frac{\ln(3)}{0.25}t} = 120e^{\ln\left(3^{\frac{t}{0.25}}\right)} = 120\left(3^{\frac{t}{0.25}}\right)$$

So

$$A(2.2) = 120\left(3^{\frac{2.2}{0.25}}\right) \approx 1,896,040 \text{ organisms}$$

3) Compound Interest

These word problems are usually of the form:

A bank account is compounded continuously at a rate of r per time frame. If k dollars are added to the bank account at each time frame, then how much money will be in the account after t time frames? The equation will look like this:

$$\frac{dS}{dt} = rS + k$$

Where S is the amount of money in the account, and if we assume the k dollar deposits are spread evenly throughout the entire time frame (in order to work with a continuous function).

Example

Ex 1)

A bank account is compounded continuously at a rate of 8% per year. If \$10,000 is added to the bank account each year, then how long will it take for the bank account to be worth \$2,000,000 million dollars if it starts with \$20,000?

We have the following equation:

$$\frac{dS}{dt} = 0.08S + 10$$

where S is the amount in \$10,000s of the bank account, and t is in years. Then we can solve this equation by separation of variables or by integrating factor. Let's use integrating factor for practice:

$$\frac{dS}{dt} - 0.08S = 10$$

$$\mu(t) = e^{-\int 0.08dt} = e^{-0.08t}$$

Thus

$$S(t) = \frac{\int e^{-0.08t} * 10dt + C}{e^{-0.08t}} = \frac{\frac{10}{-0.08}e^{-0.08t} + C}{e^{-0.08t}} = -\frac{10}{0.08} + Ce^{0.08t} = -125 + Ce^{0.08t}$$

Now, we can plug in the initial condition:

$$S(0) = 20 = -125 + Ce^{0.08*0} = -125 + C$$

$$C = 20 + 125 = 145$$

$$S(t) = -125 + 145e^{0.08t}$$

Now, we can find how long it will take for the bank account to reach the desired amount:

$$S(t) = 2000 = -125 + 145e^{0.08t}$$

$$\frac{2125}{145} = e^{0.08t}$$

$$\ln\left(\frac{2125}{145}\right) = 0.08t$$

$$t = \frac{\ln\left(\frac{2125}{145}\right)}{0.08} \approx 33.56 \text{ years}$$

As you have only paid in

$$\$20,000 + \$10,000 * 33.56 \approx \$355,560$$

this seems like a good return on investment (Assuming you have the time and money to make this investment).

4) Newton's Law of Cooling

These word problems are usually of the form:

A hot object is at temperature T_0 initially and at temperature T_1 after t_1 time units. The ambient temperature is A. How long will it take for the object to cool to a temperature of T?

The equation will be of the form:

$$\frac{dT}{dt} = k(A - T)$$

where $k > 0$, and T is the temperature.

Example

Ex 1)

My morning tea is at a temperature of 150 degrees Fahrenheit initially. After 5 minutes, it is at 120 degrees Fahrenheit. If room temperature is 70 degrees Fahrenheit, how long will it take for my tea to cool to 100 degrees Fahrenheit?

We get the following equation:

$$\frac{dT}{dt} = k\left(70 - T\right)$$

We can solve this either by using separation of variables or by the integrating factor. Let's use the integrating factor:

$$\frac{dT}{dt} = 70k - kT$$

$$\frac{dT}{dt} + kT = 70k$$

Thus

$$\mu\left(t\right) = e^{\int k dt} = e^{kt}$$

$$T\left(t\right) = \frac{\int e^{kt}\left(70k\right)dt + C}{e^{kt}} = \frac{70e^{kt} + C}{e^{kt}} = 70 + Ce^{-kt}$$

Now, we can solve for C and k by plugging in the known values:

$$T\left(0\right) = 70 + Ce^{0} = 70 + C = 150$$

thus

$$C = 80$$

and

$$T\left(5\right) = 120 = 70 + 80e^{-k5}$$

thus

$$50 = 80e^{-5k}$$

$$\frac{5}{8} = e^{-5k}$$

$$\ln\left(\frac{5}{8}\right) = -5k$$

$$\frac{\ln\left(\frac{5}{8}\right)}{5} = -k$$

So we can plug into the original equation to yield

$$T\left(t\right) = 70 + 80e^{\frac{\ln\left(\frac{5}{8}\right)}{5}t} = 70 + 80e^{\ln\left(\frac{5}{8}\right)^{\frac{t}{5}}} = 70 + 80\left(\frac{5}{8}\right)^{\frac{t}{5}}$$

And we can find the amount of time needed for cooling:

$$T\left(t\right) = 100 = 70 + 80\left(\frac{5}{8}\right)^{\frac{t}{5}}$$

$$30 = 80\left(\frac{5}{8}\right)^{\frac{t}{5}}$$

$$\frac{3}{8} = \left(\frac{5}{8}\right)^{\frac{t}{5}}$$

$$\ln\left(\frac{3}{8}\right) = \frac{t}{5}\ln\left(\frac{5}{8}\right)$$

$$t = 5\frac{\ln\left(\frac{3}{8}\right)}{\ln\left(\frac{5}{8}\right)} \approx 10.43 \text{ minutes}$$