Lecture 11

September 13, 2024

Today's lecture is the start of our study of second order linear ODEs. Today we will talk about a lot of general properties of these differential equations. It is okay if you do not have them all memorized right away. We will keep these concepts in mind as we study second order linear ODEs, and they will appear throughout the unit.

Definition

A second order linear ODE is a differential equation of the form

$$A(x)\,y'' + B(x)\,y' + C(x)\,y = F(x)$$

for non-zero $A(x)$. If we divide both sides by $A(x)$, we get an equivalent (and more common) definition.

$$y'' + p(x)\,y' + q(x)\,y = f(x)$$

We say this second order linear ODE is homogeneous if $f(x) = 0$.

Superposition

Suppose y_1 and y_2 are solutions to the homogeneous ODE

$$y'' + p(x)\,y' + q(x)\,y = 0$$

Then so is $y = C_1 y_1 + C_2 y_2$ for any $C_1, C_2 \in \mathbb{R}$. We call this property superposition.

Proof

If y_1, y_2 are solutions to the ODE, then they satisfy the ODE. That means

$$y_1'' + p(x)\,y_1' + q(x)\,y_1 = 0$$

and

$$y_2'' + p(x)\,y_2' + q(x)\,y_2 = 0$$

Let's check y.

$$y = C_1 y_1 + C_2 y_2$$

$$y' = C_1 y_1' + C_2 y_2'$$

$$y'' = C_1 y_1'' + C_2 y_2''$$

Thus

$$y'' + p(x)\,y' + q(x)\,y = \left(C_1 y_1'' + C_2 y_2''\right) + p(x)\left(C_1 y_1' + C_2 y_2'\right) + q(x)\left(C_1 y_1 + C_2 y_2\right)$$

Now, we distribute and group the y_1 and y_2 terms together.

$$= C_1 y_1'' + p(x)\,C_2 y_1' + q(x)\,C_1 y_1 + C_2 y_2'' + p(x)\,C_2 y_2' + q(x)\,C_2 y_2$$

Now, we factor out the constants

$$= C_1 \left(y_1'' + p(x) y_1' + q(x) y_1\right) + C_2 \left(y_2'' + p(x) y_2' + q(x) y_2\right)$$

but as y_1, y_2 are both solutions to the ODE, this means

$$= C_1 (0) + C_2 (0) = 0$$

and thus $y = C_1 y_1 + C_2 y_2$ is also a solution.

Linear Independence of Solutions

Without proof, I claim if y_1, y_2 are two linearly independent solutions to the homogeneous 2nd order linear ODE, then every solution is of the form

$$y = C_1 y_1 + C_2 y_2$$

for $C_1, C_2 \in \mathbb{R}$. We call this the general solution. To check for linear independence, we need to make sure that y_1 and y_2 are not the same function multiplied by a constant. We will show this using the Wronskian, as this generalized for higher order equations. The Wronskian is the determinant of the following matrix:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

If the Wronskian is non-zero, then y_1, y_2 are linearly independent.

Example

Consider

$$y_1 = e^{kx},\ y_2 = e^{-kx}\ k \neq 0$$

Then

$$y_1' = ke^{kx},\ y_2' = -ke^{-kx}$$

Thus

$$W\left(e^{kx}, e^{-kx}\right) = \begin{vmatrix} e^{kx} & e^{-kx} \\ ke^{kx} & -ke^{-kx} \end{vmatrix} = -ke^{kx}e^{-kx} - ke^{kx}e^{-kx} = -k - k = -2k \neq 0$$

as $k \neq 0$. This means y_1, y_2 are linearly independent.

Existence and Uniqueness

For the differential equation

$$y'' + p(x) y' + q(x) y = f(x),\ y(a) = b_0,\ y'(a) = b_1$$

for $b_0, b_1 \in \mathbb{R}$, if there exists an open interval containing a such that $p(x), q(x)$, and $f(x)$ are all continuous, then there exists a unique solution on that open interval.

Example

Consider

$$y'' + \frac{1}{x-2} y' + \ln(x) y = 3,\ y(1) = 1,\ y'(1) = 2$$

Then on the open interval $(0, 2)$ we have $\frac{1}{x-2}$, $\ln(x)$, and 3 are all continuous. We also have $1 \in (0, 2)$, so a unique solution exists on $(0, 2)$. We have to be careful, since we do not have existence and uniqueness outside of this open interval.

Reduction of Order

Suppose you have the differential equation

$$y'' + p(x) y' + q(x) y = 0$$

and you know that y_1 is one solution to the ODE. If you want to find a second linearly independent solution, then $y_2(x) = y_1(x) u(x)$ for some non-constant $u(x)$. This allows us to rewrite the ODE. As

$$y_2' = y_1' u + y_1 u'$$

$$y_2'' = y_1'' u + y_1' u' + y_1' u' + y_1 u'' = y_1'' u + 2y_1' u' + y_1 u''$$

Thus

$$(y_1'' u + 2y_1' u' + y_1 u'') + p(x)(y_1' u + y_1 u') + q(x)(y_1(x) u(x)) = 0$$

If we group all of the u terms together, we see

$$(y_1'' + p(x) y_1' + q(x) y_1) u + 2y_1' u' + y_1 u'' + p(x) y_1 u' = 0$$

But as y_1 is a solution to the ODE, then $(y_1'' + p(x) y_1' + q(x) y_1) = 0$, thus

$$y_1 u'' + p(x) y_1 u' + 2y_1' u' = 0$$

If we make the substitution

$$w = u', \; w' = u''$$

then we get the following first order linear ODE:

$$y_1 w' + (p(x) y_1 + 2y_1) w = 0$$

Then, we can solve for w, substitute back to solve for u, then plug this into our initial expression for y_2 to find y_2.

Example

Consider

$$2t^2 y'' + ty' - 3y = 0 \; t > 0$$

Suppose $y_1 = \frac{1}{t}$ is a solution. Find another linearly independent solution. We can start by taking

$$y_2 = \frac{1}{t} u$$

Thus

$$y_2' = \frac{-1}{t^2} u + \frac{1}{t} u'$$

$$y_2'' = \frac{2}{t^3} u + \frac{-1}{t^2} u' + \frac{-1}{t^2} u' + \frac{1}{t} u'' = \frac{2}{t^3} u + \frac{-2}{t^2} u' + \frac{1}{t} u''$$

Thus

$$2t^2 y_2'' + ty_2' - 3y_2 = 0$$

$$2t^2 \left(\frac{2}{t^3} u + \frac{-2}{t^2} u' + \frac{1}{t} u'' \right) + t \left(\frac{-1}{t^2} u + \frac{1}{t} u' \right) - 3\frac{1}{t} u = 0$$

$$= \frac{4u}{t} - 2u' + 2tu'' - \frac{u}{t} + u' - \frac{3u}{t} = 0$$

$$= 2tu'' - 3u' = 0$$

Now, we make the substitution $w = u'$, $w' = u''$.

$$2tw' - 3w = 0$$

And now, we can separate and solve.

$$2t\frac{dw}{dt} = 3w$$

$$2\frac{dw}{w} = 3\frac{dt}{t}$$

Integrate

$$2\int \frac{dw}{w} = \int 3\frac{dt}{t}$$

$$2\ln|w| = 3\ln|t| + C$$

$$w^2 = Ct^3$$

$$w = Ct^{\frac{3}{2}}$$

Now, we can go back to equation for u and integrate.

$$u' = Ct^{\frac{3}{2}}$$

$$u = C\frac{2}{5}t^{\frac{5}{2}} + C_1 = Ct^{\frac{5}{2}} + C_1$$

Thus

$$y_2 = y_1 u = \frac{1}{t}\left(Ct^{\frac{5}{2}} + C_1\right) = Ct^{\frac{3}{2}} + \frac{C_1}{t}$$

But as $\frac{1}{t} = y_1$, we usually drop this and the constant C to get a linearly independent. This leaves us with

$$y_2 = t^{\frac{3}{2}}$$

and the general solution

$$y = C_1 y_1 + C_2 y_2 = C_1 t^{-1} + C_2 t^{\frac{3}{2}}$$

Hyperbolic Trig Review

We can also review the definition and some properties of hyperbolic trig functions.

Definition

Recall

$$\cosh(x) = \frac{e^x + e^{-x}}{2},\ \sinh(x) = \frac{e^x - e^{-x}}{2}$$

This gives us the following properties

1. $\cosh(0) = 1$
2. $\sinh(0) = 0$
3. $\frac{d}{dx}(\cosh(x)) = \sinh(x)$
4. $\frac{d}{dx}(\sinh(x)) = \cosh(x)$
5. $\cosh^2(x) - \sinh^2(x) = 1$

Proofs

1)

$$\cosh\left(0\right)=\frac{e^{0}+e^{-0}}{2}=\frac{1+1}{2}=1$$

2)

$$\sinh\left(0\right)=\frac{e^{0}-e^{-0}}{2}=\frac{1-1}{2}=0$$

3)

$$\frac{d}{dx}\left(\frac{e^{x}+e^{-x}}{2}\right)=\frac{e^{x}-e^{-x}}{2}=\sinh\left(x\right)$$

4)

$$\frac{d}{dx}\left(\frac{e^{x}-e^{-x}}{2}\right)=\frac{e^{x}+e^{-x}}{2}=\cosh\left(x\right)$$

5)

$$\cosh^{2}\left(x\right)-\sinh^{2}\left(x\right)=\left(\frac{e^{x}+e^{-x}}{2}\right)^{2}-\left(\frac{e^{x}-e^{-x}}{2}\right)^{2}$$

$$=\frac{e^{2x}+2e^{x}e^{-x}+e^{-2x}}{4}-\frac{e^{2x}-2e^{x}e^{-x}+e^{-2x}}{4}=\frac{e^{2x}+2+e^{-2x}}{4}-\frac{e^{2x}-2+e^{-2x}}{4}=\frac{2}{4}-\frac{-2}{4}=1$$