Lecture 13

September 19, 2024

We are a little bit ahead of schedule, so we will take an aside to learn about operators. This will help us prove concepts for higher order linear ODEs. We are not testing over operators, but it is still good to know as they will appear in later courses (such as PDEs, Quantum Mechanics, Electromagnetism, etc.).

Operators

Definition

Informally, an operator L takes a real (differentiable) function and outputs a real function. In general, the functions need not be differentiable, but we will assume they are in this course. In general, operators need not commute. Here are some examples of operators:

Ex 1

$$L = \frac{d}{dx}$$

The derivative is an operator, since it takes any differentiable real function and outputs its derivative

$$L\left(f\right)\left(x\right) = \frac{df}{dx}\left(x\right)$$

Ex 2

$$L\left(f\right)\left(x\right) = \int_0^x f\left(t\right)dt$$

The integral in this form is an operator, since it takes any real differentiable function and outputs it's integral (technically all integrable functions could be used as inputs).

Ex 3

$$L\left(f\right) = f$$

This is the identity operator. It takes a function and outputs the same function.

Ex 4

$$L = 3$$

This is an operator as we can take the input function f and return another function $3f$

$$Lf = 3f$$

In some sense, operators generalize real functions.

Definition

An operator L is said to be a linear operator if for all functions f, g it "operates" on and for any real constant C, we have

$$L(Cf) = CL(f)$$

and

$$L(f+g) = L(f) + L(g)$$

Ex 1

Our first operator

$$L = \frac{d}{dx}$$

is linear as

$$L(Cf) = \frac{d}{dx}(Cf) = C\frac{d}{dx}f = CL(f)$$

and

$$L(f+g) = \frac{d}{dx}(f+g) = \frac{d}{dx}f + \frac{d}{dx}g = L(f) + L(g)$$

These properties are assumed from calculus 1, but they could also be shown from the limit definition of the derivative (I showed these using the other limit definition in lecture. The proofs are very similar).

$$L(Cf)(x) = \frac{d}{dx}(Cf)(x) = \lim_{h\to 0}\frac{Cf(x+h) - Cf(x)}{h} = \lim_{h\to 0}\frac{C(f(x+h) - f(x))}{h} = C\lim_{h\to 0}\frac{(f(x+h) - f(x))}{h} = C\frac{d}{dx}(f)(x) = CL(f$$

and

$$L(f+g)(x) = \frac{d}{dx}(f+g)(x) = \lim_{h\to 0}\frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} + \lim_{h\to 0}\frac{g(x+h) - g(x)}{h} = \frac{d}{dx}(f)(x) + \frac{d}{dx}(g)(x) = L(f)(x) + L(g)(x)$$

Ex 2

The second operator we saw

$$L(f)(x) = \int_0^x f(t)\,dt$$

is also linear. As

$$L(Cf)(x) = \int_0^x Cf(t)\,dt = C\int_0^x f(t)\,dt = CL(f)(x)$$

and

$$L(f+g)(x) = \int_0^x f(t) + g(t)\,dt = \int_0^x f(t)\,dt + \int_0^x g(t)\,dt = L(f)(x) + L(g)(x)$$

Ex 3

The identity operator

$$L(f) = f$$

is linear as

$$L(Cf) = Cf = CL(f)$$

and

$$L(f+g) = f + g = L(f) + L(g)$$

Non Ex 4

The fourth operator we saw

$$L(f) = 3$$

is not linear as

$$L(Cf) = 3 \neq C * 3 = CL(f)$$

for arbitrary real C.

Properties

Now, we can show some useful properties of linear operators.

Property 1

If you apply a linear operator to itself multiple times, the result is still a linear operator. Suppose L is a linear operator. Then

$$L(L(Cf)) = L(CL(f)) = C(L(L(f)))$$

and

$$L(L(f+g)) = L(L(f) + L(g)) = L(L(f)) + L(L(g))$$

In general, we use $L^n(f)$ to be the linear operator applied to itself n times. This is analogous to $f^{(n)}(x)$ being used as the nth derivative of f. This implies that repeated differentiation is linear.

Property 2

The sum of linear operators is still linear. Suppose L_1 and L_2 are linear operators. Then if

$$(L_1 + L_2)(f) = L_1(f) + L_2(f)$$

We have

$$(L_1 + L_2)(Cf) = L_1(Cf) + L_2(Cf) = C(L_1(f) + L_2(f)) = C(L_1 + L_2)(f)$$

and

$$(L_1 + L_2)(f+g) = L_1(f+g) + L_2(f+g) = L_1(f) + L_1(g) + L_2(f) + L_2(g)$$

$$= L_1(f) + L_2(f) + L_1(g) + L_2(g) = (L_1 + L_2)(f) + (L_1 + L_2)(g)$$

Property 3

If we multiply a linear operator by a function, this still gives a linear operator. Suppose L is a linear operator and $h(x)$ is an arbitrary function of x. Then if

$$(hL)(f) = hL(f)$$

$$(hL)(Cf) = hCL(f) = ChL(f) = C(hL)(f)$$

and

$$(hL)(f+g) = hL(f+g) = h(L(f) + L(g)) = hL(f) + hL(g) = (hL)(f) + (hL)(g)$$

Higher Order Linear ODEs

These linear operators give us a concise way of writing linear ODEs. Suppose you have a linear ODE of order n. That is an ODE of the following form:

$$\sum_{i=0}^{n} f_i(x)\, y^{(i)}(x) = h(x)$$

for some real functions $f_i(x)$ and $h(x)$. We can instead write this as

$$L(y)(x) = h(x)$$

where

$$L(y)(x) = \sum_{i=0}^{n} f_i(x)\, y^{(i)}(x)$$

This gives us a tool for proving some properties about higher order linear ODEs. First, let's prove superposition.

Superposition

Suppose $y_1, y_2, \ldots, y_n$ are all solutions to a linear homogeneous ODE. Then so is $y = \sum_{i=1}^{n} C_i y_i$. We can always take linear combinations of solutions to the ODE and get a solution to the ODE. To prove this, suppose the linear homogeneous ODE is of the form

$$L(y)(x) = 0$$

for some linear operator L. Then

$$L(y)(x) = L\left(\sum_{i=1}^{n} C_i y_i\right)(x) = \sum_{i=1}^{n} L(C_i y_i)(x) = \sum_{i=1}^{n} C_i L(y_i)(x)$$

but as each y_i is a solution to the linear homogeneous ODE, then

$$L(y_i)(x) = 0$$

thus

$$\sum_{i=1}^{n} C_i L(y_i)(x) = \sum_{i=1}^{n} C_i (0) = 0$$

and thus the superposition of solutions is still a solution.

Existence and Uniqueness

Without proof, we also have an existence and uniqueness theorem for higher order linear ODEs. If we have the linear ODE

$$\sum_{i=0}^{n} f_i(x)\, y^{(i)}(x) = h(x),\ \ y^{(i)}(x_0) = b_i \ \ 0 \le i \le n-1$$

and if $f_i(x), h(x)$ are all continuous on some open interval I containing x_0, then a unique solution to the differential equation exists on I. Next lecture, we will talk about solving constant coefficient higher order linear ODEs.