Lecture 2

August 21, 2024

Today we will talk about classification of differential equations.

Checklist

As we classify differential equations and systems of differential equations, we have a "checklist" of possible properties to test for. This list is not all of the possible properties, but for our purposes, it is large enough. Let's define each of these properties and give examples.

System/Equation

If the equation is a single differential equation, it is an equation. It is also possible to have systems of differential equations.

Ex 1)

This is an equation.

$$\frac{dy}{dx} + 2x = 0$$

Ex 2)

This is an equation.

$$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = \sin(y)$$

Ex 3)

This is a system.

$$\begin{cases} \frac{dy}{dt} = 2x \\ \frac{dx}{dt} = 2y \end{cases}$$

Ex 4)

This is a system.

$$\begin{cases} \frac{d^2y}{dt^2} = \frac{dx}{dt} \\ \frac{d^2x}{dt^2} = \frac{dy}{dt} + 3xy \end{cases}$$

ODE/PDE

An Ordinary Differential Equation (ODE) is a differential equation that only contains ordinary derivatives. A PDE (Partial Differential Equation) is a differential equation that contains partial derivatives.

Ex 1)

This is an ODE.

$$\frac{dy}{dx} + 2x = 0$$

Ex 2)

This is an ODE.

$$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = \sin(y)$$

Ex 3)

This is a PDE.

$$\frac{\partial y}{\partial t} + \frac{\partial y}{\partial x} = 0$$

Ex 4)

This is a PDE.

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + \frac{\partial^3 u}{\partial x \partial y \partial t}$$

Order

The order of a differential equation/system is the highest order derivative taken in the differential equation. We only care about the term with the highest number of derivatives.

Ex 1)

This is first order.

$$\frac{dy}{dx} + 2x = 0$$

Ex 2)

This is a second order.

$$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = \sin(y)$$

Ex 3)

This is first order.

$$\frac{\partial y}{\partial t} + \frac{\partial y}{\partial x} = 0$$

Ex 4)

This is third order, as the last term has 3 partial derivatives (even though they are split between multiple independent variables).

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + \frac{\partial^3 u}{\partial x \partial y \partial t}$$

Autonomous

An ODE is autonomous if it has no explicit dependence on the independent variable. Below are some examples and non-examples.

Ex 1)

This equation is autonomous. The independent variable only appears in the expression for the derivative.

$$\frac{dy}{dx} + y = 0$$

Ex 2)

This equation is autonomous.

$$\frac{d^2y}{dt^2} + 3y\frac{dy}{dt} = \sin\left(y^2\right)$$

Ex 3)

This equation is not autonomous since x appears explicitly.

$$\frac{dy}{dx} + xy = 0$$

Ex 4)

This equation is not autonomous since x appears explicitly.

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = x$$

Linear/Non-Linear

An ordinary differential equation is said to be linear if it can be written in the following form:

$$f_n(x)\frac{d^ny}{dx^n} + f_{n-1}(x)\frac{d^{n-1}y}{dx^n} + \cdots + f_0(x)\,y = g(x)$$

Where each $f_i(x)$ is a function of x and $g(x)$ is a function of x.

Ex 1)

This equation is linear, as it is in the right form (note some of the $f_i(x)$ could be 0).

$$\left(x^2 - 4\right)\frac{d^2y}{dx^2} + \cos(x)\,y = 0$$

Ex 2)

This equation is linear (but we will have to rearrange the equation to show this).

$$\ln\left(\frac{dy}{dx}\right) = x^2 + 1$$

Take e to the power of both sides:

$$e^{\ln\left(\frac{dy}{dx}\right)} = e^{x^2+1}$$

$$\frac{dy}{dx} = e^{x^2+1}$$

Ex 3)

This equation is non-linear as we have a $y\frac{dy}{dx}$ term.

$$y\frac{dy}{dx} + y = \sin(x)$$

Ex 4)

This equation is non-linear. It is not possible to remove the power of y by algebraic manipulation.

$$\frac{dy}{dx} + y^2 = 2x$$

Homogeneous/Non-Homogeneous

Any linear ODE with $g(x) = 0$ is said to be homogeneous.

Ex 1)

This equation is homogeneous.

$$\left(x^2 - 4\right)\frac{d^2y}{dx^2} + \cos(x)\, y = 0$$

Ex 2)

This equation is not homogeneous.

$$\frac{dy}{dx} = e^{x^2+1}$$

Ex 3)

Sometimes you might have to rearrange the equation to check for homogeneity. The equation below is not homogeneous.

$$\frac{dy}{dx} + 2x - y = 0$$

If we rearrange into the proper form, we see:

$$\frac{dy}{dx} - y = -2x$$

Constant Coefficient

A linear differential equation is said to be constant coefficient if every $f_i(x)$ is constant.

Ex 1)

This equation is constant coefficient.

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 0$$

Ex 2)

This equation is constant coefficient.

$$\frac{dy}{dx} = e^{x^2+1}$$

Ex 3)

This equation is not constant coefficient.

$$\left(x^2 - 4\right)\frac{d^2y}{dx^2} + \cos(x)\, y = 0$$

Ex 4)

Sometimes you have to rearrange an equation to show it is constant coefficient.

$$e^{-x}\frac{dy}{dx} = 3x$$

Multiply both sides by e^{-x}

$$\frac{dy}{dx} = 3xe^{x}$$

Classification Examples

We will do a few examples of full classifications. After this, we will classify many famous differential equations (Note, some of the later examples will be messier than what is expected for the homework, but you might see these differential equations in later classes, depending on your major).

Some Examples

Ex 1)

$$\frac{dy}{dx} = y + x$$

This is an ordinary 1st order linear non-homogeneous constant coefficient equation, as we can rearrange the equation to the form below.

$$\frac{dy}{dx} - y = x$$

Ex 2)

$$\begin{cases} \frac{d^2y}{dt} + y - tx = 0 \\ \frac{dx}{dt} - tx + 3y = 0 \end{cases}$$

This is an ordinary 2nd order homogeneous linear system of equations. It is not constant coefficient due to the tx term.

Ex 3)

$$\frac{\partial^3 y}{\partial x_1 \partial x_2^2} = y - x_1^2 + x_2$$

This is a 3rd order linear PDE. It is linear as the only y terms (dependent variable terms) are in derivatives (without any products of derivatives of y).

Ex 4)

$$\frac{dy}{dx} + y^3 - 2y = e^y$$

This is a 1st order non-linear Autonomous ODE. It is non-linear as we have a y^3 term that we cannot simplify.

Ex 5)

$$4dx = 5dy$$

We can rearrange this equation by bringing the dx to the right side. This is not the same as division, but the algebra is very similar to division. This gives the equivalent equation:

$$\frac{dy}{dx} = \frac{4}{5}$$

As such, the equation is a constant coefficient non-homogeneous linear 1st order homogeneous ODE.

Ex 6)

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = u^2 - tx$$

This is a non-linear 1st order PDE.

Some Famous Equations

Below we are going to classify a lot of famous differential equations. Solving some of these is outside of the scope of the class, but hopefully these examples will illustrate the wide variety of uses of differential equations.

Ex 1) Heat Equation

This equation describes the temperature in a surface over time.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

This is a 2nd order linear PDE.

Ex 2) Bernoulli Differential Equation

This is a special class of differential equations we will learn to solve in this course.

$$y' + p(x)\, y = q(x)\, y^n$$

Unless $n = 0, 1$, then this equation is non-linear. Regardless of the n value, this is an ordinary 1st order differential equation. It is only autonomous in the special case where $p(x), q(x)$ are both constants.

Ex 3) Velocity and Acceleration

You might have seen these in a physics course before. The velocity and acceleration of an object describe its motion.

$$\begin{cases} v = \frac{dr}{dt} \\ a = \frac{dv}{dt} \end{cases}$$

This is a 1st order linear constant coefficient system of ODES. If velocity and acceleration are functions of time, we cannot assume the systems are autonomous. If v, a are both constants, then we have autonomous linear equations.

Ex 4) Newton's Second Law of Motion

This describes the relationship between force and acceleration of an object.

$$F = \frac{d}{dt}\left(m\frac{dx}{dt}\right)$$

If m is a constant, we can pull it out. If mass is a function of time, we can rewrite this equation using product rule.

$$F = m'\frac{dx}{dt} + m\frac{d^2x}{dt^2}$$

If F is only a function of time, this gives a second order linear non-homogeneous ODE (that is not constant coefficient in general). If F is a function of both x and t, then this equation may be non-linear.

Ex 5) Wave Equation

This equation describes waves passing through 3D space.

$$\frac{\partial^2 u}{\partial t^2} = k^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

This is a second order linear constant coefficient homogeneous PDE.

Ex 6) Friedmann Equations

These equations give a model for the expansion of the universe within cosmology. In the simplest model (which we will assume below), every variable except a and t is a constant.

$$\begin{cases} \frac{\left(\frac{da}{dt}\right)^2+kc^2}{a^2} = \frac{8\pi G\rho+\Lambda c^2}{3} \\ \frac{\left(\frac{d^2a}{dt^2}\right)}{a} = -\frac{4\pi G}{3}\left(\rho+\frac{3P}{c^2}\right)+\frac{\Lambda c^2}{3} \end{cases}$$

This is a second order autonomous non-linear system of ODEs. The non-linearity comes from the top equation in its current form. Even though the bottom equation is linear (and can be shown by multiplying through by a), the top equation is non-linear. This make the entire system non-linear.

Ex 7) Radioactive Decay

A radioactive isotope will lose some proportion of its mass with time. The equation below describes this.

$$-\frac{dN}{N} = \lambda dt$$

We can rearrange this to get

$$\frac{dN}{dt} = -\lambda N$$

$$\frac{dN}{dt} + \lambda N = 0$$

This is a linear constant coefficient homogeneous 1st order autonomous ODE.

Ex 8) RLC Circuit

A circuit with resistors (R), inductors (L), and capacitors (C) is a RLC circuit. It is governed by the equation below.

$$\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{I}{LC} = f(t)$$

As the resistance, inductance, and capacitance (R,L,C respectively) are assumed to be constant, then this is a constant coefficient 2nd order linear non-homogeneous ODE. It is not autonomous as there is a $f(t)$ on the right side (which is not constant in general).

Ex 9) Euler-Bernoulli Beam Equation

This equation describes the deflection of a beam as a function of position, given the elastic modulus E, 2nd moment of area I, and the distributed load q. In nicer forms, E, I are constant. In the most general form given below, we assume E, I are functions of x.

$$\frac{d^2}{dx^2}\left(E(x)\, I[x]\, \frac{d^2w}{dx^2}\right) = q(x)$$

If we use product rule, we get

$$\frac{d}{dx}\left(E'(x)\, I'(x)\, \frac{d^2w}{dx^2} + E(x)\, I(x)\, \frac{d^3w}{dx^3}\right) = q(x)$$

$$E''(x)\, I''(x)\, \frac{d^2w}{dx^2} + E'(x)\, I'(x)\, \frac{d^3w}{dx^3} + E'(x)\, I'(x)\, \frac{d^3w}{dx^3} + E(x)\, I(x)\, \frac{d^4w}{dx^4} = q(x)$$

$$E''(x)\, I''(x)\, \frac{d^2w}{dx^2} + 2E'(x)\, I'(x)\, \frac{d^3w}{dx^3} + E(x)\, I(x)\, \frac{d^4w}{dx^4} = q(x)$$

This is a fourth order linear non-homogeneous ODE.

Ex 10) Rate Equation

In chemistry, $[A]$ represents the concentration of molecule A as a function of time. The rate equation describes how the concentration of various molecules in a reaction change.

$$\frac{-d\,[R]}{dt} = k\,[A]^x\,[B]^y$$

where x, y are the partial reaction orders (we can treat them as constants). This equation is a linear non-homogeneous 1st order constant coefficient ODE. It is not homogeneous as $[A], [B]$ are both functions of time. The notation is a little unusual, but this is due to the chemistry conventions.

Ex 11) Logistic Equation

This equation models the population of a given species of animal as a function of time.

$$\frac{dP}{dt} = P\,(1-P)$$

It is a first order non-linear autonomous ODE.

Ex 12) Lotka-Volterra Equations

This system of equations models the population of a predator and prey in a biological system.

$$\begin{cases} \frac{dx}{dt} = \alpha x - \beta xy \\ \frac{dy}{dt} = -\gamma y + \delta xy \end{cases}$$

This is a nonlinear first order system of ODEs.

Ex 13) SIR Model

The Susceptible, Infected, Recovered (SIR) Model is used in epidemiology to model the proportion of the population in various stages of an illness. It was used recently with modeling COVID.

$$\begin{cases} \frac{dS}{dt} = -\beta IS \\ \frac{dI}{dt} = \beta IS - \gamma I \\ \frac{dR}{dt} = \gamma I \end{cases}$$

This is a first order non-linear system of ODEs.

Ex 14) Black-Scholes Equation

This equation models the prices of European Options. This equation led to a revolution in quantitative finance in the 70s.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} = rv - rs\frac{\partial v}{\partial s}$$

It is a second order linear non-homogeneous PDE.

Ex 15) Solow-Swan Model

The Solow-Swan Model models economic growth.

$$\frac{dk}{dt} = sk^\alpha - \delta k$$

This equation is non-linear (for all α except $\alpha = 0, 1$). It is a 1st order ODE.