Lecture 12

September 16, 2024

Today we will talk about second order constant coefficient linear homogeneous ODEs. These are ODEs of the form below:

$$y'' + ay' + by = 0$$

where $a, b \in \mathbb{R}$. To solve an ODE of the form, we take the following steps:

1. "Guess" $y = e^{rt}$ is the solution
2. Plug y into the ODE
3. Solve for r
4. Plug both of the solutions, r_1, r_2 into y to get the final answer *

* The form of the final answers are given below*

If r_1, r_2 are distinct real roots, then

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

If $r_1 = r_2 = r$, then

$$y = (C_1 + C_2 t)\, e^{rt}$$

If $r = a \pm bi$, then

$$y = e^{at}\left(C_1 \cos(bt) + C_2 \sin(bt)\right)$$

Each of these solutions comes from taking a superposition of solutions.

Derivation

Let's show the derivation for the repeated roots and the complex roots.

Repeated Roots

First, suppose $r_1 = r_2 = r$. Then

$$y_1 = e^{rt}$$

and

$$y_2 = u(t)\, e^{rt}$$

for some u. But as we have repeated roots, we know the differential equation will be of the form

$$y'' - 2ry' + r^2 y = 0$$

If we take the derivatives of y_2, we see

$$y_2' = u' e^{rt} + rue^{rt}$$

$$y_2'' = u'' e^{rt} + 2u're^{rt} + r^2 e^{rt}$$

Thus

$$y_2'' - 2ry_2' + r^2 y_2 = 0$$

yields

$$\left(u''e^{rt} + 2u're^{rt} + r^2e^{rt}\right) - 2r\left(u'e^{rt} + rue^{rt}\right) + r^2ue^{rt} = \left(r^2 - 2r^2 + r^2\right)e^{rt} + u''e^{rt} + 2u're^{rt} - 2ru'e^{rt} = 0$$

$$u''e^{rt} = 0$$

Thus

$$u'' = 0$$

And if we use reduction of order by substituting $w = u'$, $w' = u''$, we see

$$w' = 0$$

Thus

$$w = C = u'$$

and thus

$$u = Ct + C'$$

And

$$y_2 = ue^{rt} = (Ct + C')\,e^{rt}$$

But as Ce^{rt} is already a known solution, we see

$$y_2 = Cte^{rt}$$

is another solution.

Complex Roots

Now, for the complex root case, consider $r = a \pm bi$. Then we have Euler's formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Thus

$$y = e^{(a \pm i\theta)t} = e^{at}e^{\pm i\theta t} = e^{at}\left(\cos(\pm\theta t) + i\sin(\pm\theta t)\right)$$

As cos is even and sin is odd:

$$y = e^{at}\left(\cos(\theta t) \pm i\sin(\theta t)\right)$$

This yields a single solution. As we can repeatedly take the derivative of the real part and imaginary part, and they will never interact, then the real part yields one possible solution to the differential equation, and the imaginary part yields another. By superposition, we get

$$y = e^{at}\left(C_1\cos(\theta t) \pm C_2\sin(\theta t)\right) = e^{at}\left(C_1\cos(\theta t) + C_2\sin(\theta t)\right)$$

As C_2 is an arbitrary real constant and can absorb the $\pm$.

Examples

Ex 1)

Suppose

$$y'' + 9y' + 18y = 0$$

Then

$$y = e^{rt}$$

$$y' = re^{rt}$$

$$y'' = r^2 e^{rt}$$

is the form of the solution, thus

$$r^2 e^{rt} + 9re^{rt} + 18e^{rt} = 0$$

Divide out the e^{rt} to yield

$$r^2 + 9r + 18 = 0$$

$$(r + 3)(r + 6) = 0$$

$$r = -3, -6$$

Thus

$$y = C_1 e^{-3t} + C_2 e^{-6t}$$

is our final answer.

Ex 2)

Suppose

$$y'' - 4y' + 4y = 0, \; y(0) = 1, \; y'(0) = 0$$

Then

$$y = e^{rt}$$

$$y' = re^{rt}$$

$$y'' = r^2 e^{rt}$$

is the form of the solutions, thus

$$r^2 e^{rt} - 4re^{rt} + 4e^{rt} = 0$$

Divide out the e^{rt}:

$$r^2 - 4r + 4 = 0$$

$$(r - 2)(r - 2) = 0$$

$$r = 2m2$$

So

$$y = (C_1 + C_2 t)\, e^{2t}$$

Now, we can find the constants C_1, C_2 be using the initial conditions.

$$y(0) = 1 = (C_1 + 0)\, e^0 = C_1$$

and

$$y' = C_2 e^{2t} + 2\,(C_1 + C_2 t)\, e^{2t}$$

Thus

$$y'(0) = C_2 + 2C_1 = 0$$

So

$$C_2 = -2C_1 = -2$$

and we get the final answer

$$y = (1 - 2t)\, e^{2t}$$

Ex 3)

$$y'' - 2y' + 2y = 0,\ y(0) = 1,\ y\left(\frac{\pi}{2}\right) = -1$$

We "guess" the solution takes the form

$$y = e^{rt}$$

$$y' = re^{rt}$$

$$y'' = r^2 e^{rt}$$

So if we plug into the differential equation, we see

$$r^2 e^{rt} - 2re^{rt} + 2e^{rt} = 0$$

Divide by e^{rt} to see:

$$r^2 - 2r + 2 = 0$$

Using quadratic formula:

$$\frac{2 \pm \sqrt{(-2)^2 - 4 * 2}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

Thus

$$y = e^t\,(C_1 \cos(t) + C_2 \sin(t))$$

is the general solution to the homogeneous equation. Now, we consider the initial conditions.

$$y(0) = 1\,(C_1 \cos(0) + C_2 \sin(0)) = C_1 = 1$$

and

$$y\left(\frac{\pi}{2}\right) = e^{\frac{\pi}{2}}\left(C_1 \cos\left(\frac{\pi}{2}\right) + C_2 \sin\left(\frac{\pi}{2}\right)\right) = e^{\frac{\pi}{2}}\,(C_2) = -1$$

so

$$C_2 = -e^{\frac{-\pi}{2}}$$

Thus

$$y = e^t\left(\cos(t) - e^{-\frac{\pi}{2}} \sin(t)\right)$$

is our final answer.

Ex 4)

Consider

$$y'' + y' - 12y = 0,\ y(0) = 3,\ y'(0) = -5$$

We "guess" the solution takes the form

$$y = e^{rt}$$

$$y' = re^{rt}$$

$$y'' = r^2e^{rt}$$

Now, we can plug this into the equation to see

$$r^2e^{rt} + re^{rt} - 12e^{rt} = 0$$

We can divide out the e^{rt} to yield

$$r^2 + r - 12 = 0$$

$$(r+4)(r-3) = 0$$

So

$$r = 3, -4$$

Thus

$$y = C_1e^{3t} + C_2e^{-4t}$$

is the general solution to the homogeneous ODE. Now, we can plug in the initial conditions to see

$$y(0) = C_1 + C_2 = 3$$

and

$$y'(t) = 3C_1e^{3t} - 4C_2e^{-4t}$$

$$y'(0) = 3C_1 - 4C_2 = -5$$

And now, we have a system of equations we can solve for C_1 and C_2. We can use elimination. First, let's multiply the top equation by 4. Then

$$4C_1 + 4C_2 = 12$$

Add this to the bottom equation to see

$$(4C_1 + 4C_2) + (3C_1 - 4C_2) = 12 - 5$$

$$7C_1 = 7$$

$$C_1 = 1$$

and if we go back to the first equation

$$1 + C_2 = 3$$

$$C_2 = 2$$

So we get the final answer

$$y = e^{3t} + 2e^{-4t}$$