Lecture 9

September 10, 2024

Euler's Method

In general solving differential equations of the form:

$$\frac{dy}{dx} = f(x, y), \ y(x_0) = y_0$$

is difficult. If the differential equation is in one of the nice forms we discussed in class, there might be a nice closed form solution. This usually isn't the case. Even so, we can often numerically approximate a solution. This will give us usable numbers, even if we do not have a closed form solution. One of the simplest methods for approximating differential equations is Euler's method. Here is the process:

Step 0: Check for existence and uniqueness of the solution

We must start by checking for existence and uniqueness of a solution. If we do not check, there is no guarantee that numerical methods will work. In practice, this means that no matter how good of a numerical method you use you might not get a working solution for your model. Most computer tools will not check this, so you should be careful to do so even before using a computer. To check these conditions we can use Picard's Theorem. This is stated below:

Picard's Theorem of Existence and Uniqueness

Consider the differential equation

$$\frac{dy}{dx} = f(x, y), \ f(x_0) = y_0$$

If $f(x, y)$ is a continuous function near (x_0, y_0) and $\frac{\partial f}{\partial y}$ exists and is continuous near (x_0, y_0) then a solution to

$$\frac{dy}{dx} = f(x, y), \ f(x_0) = y_0$$

exists for some interval around (x_0, y_0) and is unique.

Step 1: Get the initial conditions of the approximation

These initial conditions are x_0, y_0 and the step size h. We need these in order to start calculating.

Step 2: Update the current x value

We update the value of x using the formula:

$$x_{i+1} = x_i + h$$

Step 3: Update the current y value

We update the value of y using the formula:

$$y_{i+1} = y_i + hf(x_i, y_i)$$

Step 4: Repeat steps 2 and 3 until you get the desired approximation

This step is pretty self explanatory, but we do have to keep a couple of things in mind. First, smaller step sizes lead to more accurate approximations but longer computation times. Second, the further the value you wish to approximate is from the initial condition, the worse the approximation will get.

Similarity to linearization

In calculus 1, you might have covered linear approximations. That is

$$y(x) \approx y(x_0) + (x - x_0)\, y'(x_0)$$

This is similar to the equation for updating y. Here, we are replacing $x - x_0$ with h, and we find the slope of the line by taking $f(x, y)$ rather than $y'(x)$. This is because $y'(x) = \frac{dy}{dx}$. With Euler's method, the key difference is that we are repeatedly updating the current x and y values. Most of the same considerations with linearization will come into play with Euler's method.

Error of the approximation

If you know the actual solution, you can find the error in the approximation by the formula below.

$$\text{error} = |y_{\text{exact}} - y_{\text{approximation}}|$$

In practice, we do not know the exact solution (if we did we wouldn't need to approximate). In these cases, we have methods for bounding the error in an approximation. These are outside of the scope of this course, but if you wish to learn more a numerical analysis class will cover this.

Examples

Here, we will compute a few examples by hand. A Jupyter notebook with code will also be shared on Canvas. It is good to know how to use code, but it is not expected or required that you use code for any of these problems in this class. If you decide to use code, you still need to provide a table of the calculated values and you are still expected to be able to complete the computations by hand on an exam.

Ex 1

Approximate the solution of

$$\frac{dy}{dx} = y^2 + x,\ y(0) = 1$$

at $x = 1$ with a step size of $h = 0.5$. First, we will verify existence and uniqueness. $y^2 + x$ is continuous at $(0, 1)$. In fact, it is continuous at all points as a sum and product of continuous functions. Now, we will check the partial derivative

$$\frac{\partial f}{\partial y} = 2y$$

As this partial derivative exists and is continuous near $(0, 1)$, then we know a unique solution to the differential equation exists. Now, we can numerically approximate.

$$x_0 = 0, y_0 = 1, h = 0.5$$

For $i = 1$

$$x_1 = x_0 + h = 0 + 0.5 = 0.5$$

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + h\left(y_0^2 + x_0\right) = 1 + 0.5\left(1^2 + 0\right) = 1.5$$

For $i = 2$

$$x_2 = x_1 + h = 0.5 + 0.5 = 1$$

$$y_2 = y_1 + hf(x_1, y_1) = 1.5 + 0.5\left(1.5^2 + 0.5\right) = 2.875$$

So our approximation is $y(1) \approx 2.875$. Our work is summarized in the table below.

i	x_i	y_i
0	0	1
1	0.5	1.5
2	1	2.875

Ex 2

Approximate the solution of

$$\frac{dy}{dx} = \frac{xy}{x+y},\ y(2) = 2$$

at $x = 2.3$ with a step size of $h = 0.1$. First, let's check for existence and uniqueness. At $(2,2)$ we have continuity for $f(x,y) = \frac{xy}{x+y}$, as we are taking the quotient of two continuous functions. The point $(0,0)$ is the only discontinuity. If we are near enough to $(2,2)$, we can avoid this discontinuity. We also have the partial derivative

$$\frac{\partial f}{\partial y} = \frac{x(x+y) - xy(1)}{(x+y)^2} = \frac{x^2 + xy - xy}{(x+y)^2} = \frac{x^2}{(x+y)^2}$$

This partial derivative exists and will be continuous as long as we not at the point $(0,0)$. As such, a unique solution will exist near $(2,2)$. Now, we can numerically approximate.

$$x_0 = 2, y_0 = 2, h = 0.1$$

For $i = 1$

$$x_1 = x_0 + h = 2 + 0.1 = 2.1$$

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + hf(2,2) = 2 + 0.1\left(\frac{2*2}{2+2}\right) = 2.1$$

For $i = 2$

$$x_2 = x_1 + h = 2.1 + 0.1 = 2.2$$

$$y_2 = y_1 + hf(x_1, y_1) = 2.1 + 0.1f(2.1, 2.1) = 2.1 + 0.1\left(\frac{2.1*2.1}{2.1+2.1}\right) = 2.205$$

For $i = 3$

$$x_3 = x_2 + h = 2.2 + 0.1 = 2.3$$

$$y_3 = y_2 + hf(x_2, y_2) = 2.205 + 0.1f(2.2, 2.205) = 2.205 + 0.1\left(\frac{2.2*2.205}{2.2+2.205}\right) \approx 2.31512485$$

So our approximation is

$$y(2.3) \approx 2.31512485$$

The results are summarized in the table below.

i	x_i	y_i
0	2	2
1	2.1	2.1
2	2.2	2.205
3	2.3	2.31512485

Ex 3

Approximate the solution of

$$\frac{dy}{dx} = xy + y, \ \ y(0) = 1$$

at $x = 4$ with a step size of $h = 1$. First, let check for existence and uniqueness of the solution. $f(x, y) = xy + y$ is continuous as a sum and product of continuous functions. The partial derivative

$$\frac{\partial f}{\partial y} = x + 1$$

exists for all $x \in \mathbb{R}$ and is continuous for all $x \in \mathbb{R}$, thus a solution exists and is unique. Now, we can numerically approximate. The initial conditions are

$$x_0 = 0, y_0 = 1, h = 1$$

For $i = 1$

$$x_1 = x_0 + h = 0 + 1 = 1$$

$$y_1 = y_0 + hf(x_0, y_0) = 0 + 1f(0, 1) = 0 + 1(0 + 1) = 1$$

For $i = 2$

$$x_2 = x_1 + h = 1 + 1 = 2$$

$$y_2 = y_1 + hf(x_1, y_1) = 1 + 1f(1, 1) = 1 + (1 + 1) = 3$$

For $i = 3$

$$x_3 = x_2 + h = 2 + 1 = 3$$

$$y_3 = y_2 + hf(x_2, y_2) = 3 + 1f(3, 3) = 3 + 1(3 * 3 + 3) = 15$$

For $i = 4$

$$x_4 = x_3 + h = 3 + 1 = 4$$

$$y_4 = y_3 + hf(x_3, y_3) = 15 + 1f(4, 15) = 15 + (4 * 15 + 15) = 90$$

Thus $y(4) \approx 90$. The results are summarized in the table below.

i	x_i	y_i
0	0	1
1	1	1
2	2	3
3	3	15
4	4	90

Since this is a separable equation, we can find an exact solution and compare the approximation to the exact solution (to find the error in our approximation). Let's solve the differential equation!

$$\frac{dy}{dx} = xy + y = y(x + 1)$$

Separate and integrate

$$\frac{dy}{y} = (x + 1)\,dx$$

$$\int \frac{dy}{y} = \int (x+1)\,dx$$

$$\ln|y| = \frac{x^2}{2} + x + C$$

$$y = \pm e^{\frac{x^2}{2}+x+C} = Ce^{\frac{x^2}{2}+x}$$

With the initial condition

$$y(0) = 1$$

we have

$$y(0) = Ce^{\frac{0^2}{2}+0} = C = 1$$

So

$$y = e^{\frac{x^2}{2}+x}$$

and

$$y(4) \approx 162754.79$$

This means the error is

$$\text{error} = |y_{\text{exact}} - y_{\text{approximation}}| \approx |162754.79 - 90| = 162664.79$$

The error is very large since we used a very large step size despite the exponential growth in the slope.

Ex 4

Let's try one last example. Approximate the solution of

$$y' + \frac{1}{x}y = x, \ y(1) = 1$$

at $x = 1.2$ with $h = 0.05$. First, we need to rewrite the differential equation into the form appropriate for Euler's method.

$$y' = x - \frac{1}{x}y$$

Now we can check for existence and uniqueness of the solution. $f(x,y) = x - \frac{1}{x}y$ is continuous near $(1,1)$. We do need to be careful to stay near enough to $(1,1)$ to avoid $x = 0$ as this is a discontinuity and could cause trouble. The partial derivative

$$\frac{\partial f}{\partial y} = -\frac{1}{x}$$

exists and is continuous near $(1,1)$. Once again, we need to be careful to avoid $x = 0$ as this is a discontinuity and could cause us trouble. Now, let's approximate. The initial conditions are

$$x_0 = 1, y_0 = 1, h = 0.05$$

For $i = 1$

$$x_1 = x_0 + h = 1 + 0.05 = 1.05$$

$$y_1 = y_0 + hf(x_0, y_0) = 1 + 0.05f(1,1) = 1 + 0.05\left(1 - \frac{1}{1}1\right) = 1$$

For $i = 2$

$$x_2 = x_1 + h = 1.05 + 0.05 = 1.1$$

$$y_2 = y_1 + hf\left(x_1, y_1\right) = 1 + 0.05f\left(1.05, 1\right) = 1 + 0.05\left(1.05 - \frac{1}{1.05}1\right) \approx 1.00488095$$

For $i = 3$

$$x_3 = x_2 + h = 1.1 + 0.05 = 1.15$$

$$y_3 = y_2 + hf\left(x_2, y_2\right) = 1.00488095 + 0.05f\left(1.15, 1.00488095\right)$$

$$= 1.00488095 + 0.05\left(1.1 - \frac{1}{1.1} * 1.00488095\right) \approx 1.014294545$$

For $i = 4$

$$x_4 = x_3 + h = 1.15 + 0.05 = 1.2$$

$$y_4 = y_3 + hf\left(x_3, y_3\right) = 1.014294545 + 0.05f\left(1.15, 1.014294545\right)$$

$$= 1.014294545 + 0.05\left(1.15 - \frac{1}{1.15} * 1.014294545\right) \approx 1.027608696$$

So

$$y\left(1.2\right) \approx 1.027608696$$

Our work is summarized in the table below:

i	x_i	y_i
0	1	1
1	1.05	1
2	1.1	1.00488095
3	1.15	1.014294545
4	1.2	1.027608696

As this is a linear equation, we can also find an exact solution for comparison. Let's solve this!

$$y' + \frac{1}{x}y = x$$

Using the integrating factor:

$$\mu\left(x\right) = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = |x|$$

But as $x > 0$ from our initial conditions,

$$\mu\left(x\right) = x$$

So

$$y = \frac{\int x * xdx + C}{x} = \frac{\frac{x^3}{3} + C}{x} = \frac{x^2}{3} + \frac{C}{x}$$

using the initial condition

$$y\left(1\right) = 1 = \frac{1}{3} + \frac{C}{1}$$

$$C = \frac{2}{3}$$

Thus

$$y = \frac{x^2}{3} + \frac{2}{3x}$$

So

$$y\left(1.2\right) = \frac{1.2^2}{3} + \frac{2}{3 * 1.2} \approx 1.035555555$$

So the error is

$$\text{error} = \left|y_{\text{exact}} - y_{\text{approximation}}\right| = \left|1.035555555 - 1.027608696\right| \approx 0.00794686$$

This is much more reasonable for a first order approximation (Note, the Jupyter notebook was used to help with these calculations. The code will be shared in Canvas. You could do this by hand, but the decimals are long and the process is a little tedious).