Lecture 10

September 12, 2024

Exact Equations

Suppose you have a function

$$F(x,y)$$

We can take the complete derivative using multivariate chain rule

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$

For convenience, we will call

$$\frac{\partial F}{\partial x} = F_x,\ \frac{\partial F}{\partial y} = F_y$$

Thus

$$dF = F_x dx + F_y dy$$

If we set this function equal to a constant, we have the equation

$$F(x,y) = C$$

and if we take the derivative of both sides, we get the differential equation

$$F_x dx + F_y dy = 0$$

or if we rearrange

$$F_x + F_y \frac{dy}{dx} = 0$$

As F_x and F_y come from the same function, we have

$$F_{xy} = F_{yx}$$

(For nice enough $F(x,y)$). We will assume $F(x,y)$ is nice enough to allow this for our purposes. We call $F(x,y)$ the potential function. Any differential equation that can be written by taking the derivative of a potential function is called an exact equation. Equivalently, any equation of the form

$$M + N\frac{dy}{dx} = 0$$

such that

$$M_y = N_x$$

is an exact equation. Here, we replace F_x with M and F_y with N to get the above condition. To solve an exact equation, we have the following steps:

1. Verify that $M_y = N_x$
2. Take the integrals $\int M dx$ and $\int N dy$
3. Set the results of the integrals equal to each other and match the terms. This is $F(x,y)$

We need to do step one to verify we have an exact equation. Otherwise this process will not work. Let's do some examples.

Examples

Ex 1)

$$y\sin(x) = 2e^{2x} + \left(\cos(x) + \frac{2}{y}\right)\frac{dy}{dx}$$

First, let's rearrange this into the proper form.

$$2e^{2x} - y\sin(x) + \left(\cos(x) + \frac{2}{y}\right)\frac{dy}{dx} = 0$$

Thus

$$M = 2e^{2x} - y\sin(x)$$

$$N = \cos(x) + \frac{2}{y}$$

Now, we can do **step 1** and check the partial derivatives.

$$M_y = -\sin(x)$$

$$N_x = -\sin(x)$$

As $M_y = N_x$, we have an exact equation. Now, we can integrate for **step 2**. Since our integration is undoing partial derivatives, we will treat each variable not in the differential as a constant.

$$\int M dx = \int 2e^{2x} - y\sin(x)\, dx = e^{2x} - y\int \sin(x)\, dx = e^{2x} + y\cos(x) + A(y)$$

Where $A(y)$ is some function of y.

$$\int N dy = \int \cos(x) + \frac{2}{y} dy = y\cos(x) + 2\ln|y| + B(x)$$

Where $B(x)$ is some function of x. Now, we set the integrals equal to each other and match terms for **step 3**.

$$e^{2x} + y\cos(x) + A(y) = y\cos(x) + 2\ln|y| + B(x)$$

Here, we have $A(y) = 2\ln|y|$ and $B(x) = e^{2x}$ in order for us to have equality. This means

$$F(x, y) = e^{2x} + y\cos(x) + 2\ln|y| = C$$

is our solution to the differential equation.

Ex 2)

$$2y + 1 + 2x\frac{dy}{dx} = \frac{3}{(xy)^2} + \frac{6}{xy^3}\frac{dy}{dx}$$

First, let's rewrite the equation in the proper form. We will bring all of the terms to the same side.

$$2y + 1 - \frac{3}{(xy)^2} + 2x\frac{dy}{dx} - \frac{6}{xy^3}\frac{dy}{dx} = 0$$

Now, we will group the terms together into M and N.

$$\left(2y + 1 - \frac{3}{(xy)^2}\right) + \left(2x - \frac{6}{xy^3}\right)\frac{dy}{dx} = 0$$

Now, we can do **step 1** and verify this is an exact equation.

$$M_y = 2 + \frac{6}{x^2y^3}$$

$$N_x = 2 + \frac{6}{x^2y^3}$$

As $M_y = N_x$ we have an exact equation and we can integrate both M and N.

$$\int M dx = \int 2y + 1 - \frac{3}{x^2y^2} dx = 2xy + x + \frac{3}{xy^2} + A(y)$$

Where $A(y)$ is some function of y.

$$\int N dy = \int 2x - \frac{6}{xy^3} dy = 2xy + \frac{3}{xy^2} + B(x)$$

Where $B(x)$ is some function of x. Now, we can set the integrals equal to each other and match terms.

$$2xy + x + \frac{3}{xy^2} + A(y) = 2xy + \frac{3}{xy^2} + B(x)$$

Here, $B(x) = x$ and $A(y) = 0$. Thus

$$F(x, y) = 2xy + x + \frac{3}{xy^2} = C$$

is our solution to the differential equation.

(Almost) Exact Equations

Sometimes equations are of the form

$$M + N\frac{dy}{dx} = 0$$

but

$$M_y \neq N_x$$

Even so, we can often find a solution if we first multiply all the terms of the equation by an integrating factor. Suppose we have a function $u(x, y)$ such that

$$uM + uN\frac{dy}{dx} = 0$$

That is

$$\frac{\partial (uM)}{\partial y} = u_y M + uM_y$$

and

$$\frac{\partial (uN)}{\partial x} = u_x N + uN_x$$

are equal. Then

$$u_y M + uM_y = u_x N + uN_x$$

If we wish to solve for u, this doesn't seem very helpful. We have turned an ODE into a PDE. To make like a little easier, let's see what happens if u is only a function of x.

$$u_y = 0$$

and

$$u_x = u'$$

thus we have

$$uM_y = u'N + uN_x$$

we can now separate.

$$uM_y - uN_x = u'N$$

$$u'N = u\left(M_y - N_x\right)$$

$$u' = \frac{M_y - N_x}{N}u$$

$$\frac{du}{dx} = \frac{M_y - N_x}{N}u$$

$$\frac{du}{u} = \frac{M_y - N_x}{N}dx$$

In order for these to be equal, we must have

$$\frac{M_y - N_x}{N}$$

as a function only in terms of x. The left side can only contain x terms as u is a function of x, so we must assume the same of the right side. If this condition is met, we can now solve.

$$\int \frac{du}{u} = \int \frac{M_y - N_x}{N}dx$$

$$\ln|u| = \int \frac{M_y - N_x}{N}dx$$

$$u = e^{\int \frac{M_y - N_x}{N}dx}$$

Similarly, we can solve for u if u is a function of y only. If u is a function of y only, then

$$u_x = 0$$

so

$$u'M + uM_y = uN_x$$

We can now separate

$$u'M = uN_x - uM_y$$

$$u' = \frac{N_x - M_y}{M}u$$

$$\frac{u'}{u} = \frac{N_x - M_y}{M}$$

$$\frac{du}{u} = \frac{N_x - M_y}{M}dy$$

As we have all of the u terms on the left side, and u is a function of y only, then we must have $\frac{N_x - M_y}{M}$ be a function of y only as well. Now, we integrate.

$$\int \frac{du}{u} = \int \frac{N_x - M_y}{M} dy$$

$$\ln|u| = \int \frac{N_x - M_y}{M} dy$$

So

$$u = e^{\int \frac{N_x - M_y}{M} dy}$$

In either case, once we know u, we multiply the original differential equation by u to get an exact equation. In summary,

$$u = e^{\int \frac{M_y - N_x}{N} dx}$$

if

$$\frac{M_y - N_x}{N}$$

is a function of x only and

$$u = e^{\int \frac{N_x - M_y}{M} dy}$$

if

$$\frac{N_x - M_y}{M}$$

is a function of y only.

Examples

Ex 3)

$$\cos(x)\, y + (1 + \sin(x)) \frac{dy}{dx} = \sin(x)\, y + y + 1$$

First, let's rearrange this equation into the correct form.

$$(\cos(x)\, y - \sin(x)\, y - y - 1) + (1 + \sin(x)) \frac{dy}{dx} = 0$$

Thus

$$M = \cos(x)\, y - \sin(x)\, y - y - 1$$

$$N = 1 + \sin(x)$$

If we take the partial derivatives,

$$M_y = \cos(x) - \sin(x) - 1$$

and

$$N_x = \cos(x)$$

As $M_y \neq N_x$, we do not have an exact equation. If we take

$$\frac{M_y - N_x}{N} = \frac{(\cos(x) - \sin(x) - 1) - \cos(x)}{1 + \sin(x)} = \frac{-1\,(1 + \sin(x))}{1 + \sin(x)} = -1$$

we have a function purely of x, thus

$$u = e^{\int -1dx} = e^{-x}$$

will be out integrating factor. This turns the original differential equation into the equivalent equation

$$e^{-x}\left(\cos\left(x\right)y - \sin\left(x\right)y - y - 1\right) + e^{-x}\left(1 + \sin\left(x\right)\right)\frac{dy}{dx} = 0$$

Now, we can apply **step 1**, and we see

$$M_y = e^{-x}\left(\cos\left(x\right) - \sin\left(x\right) - 1\right)$$

and

$$N_x = -e^{-x} - e^{-x}\sin\left(x\right) + e^{-x}\cos\left(x\right)$$

and thus

$$M_y = N_x$$

and we have an exact equation. Now, we can apply **step 2**.

$$\int Mdx = \int e^{-x}\left(\cos\left(x\right)y - \sin\left(x\right)y - y - 1\right)dx$$

$$= y\int e^{-x}\left(\cos\left(x\right) - \sin\left(x\right)\right)dx + \left(-y - 1\right)\int e^{-x}dx$$

We can solve each integral separately. The first integral will require us to use integration by parts twice

$$\int e^{-x}\left(\cos\left(x\right) - \sin\left(x\right)\right)dx$$

$$u = \cos\left(x\right) - \sin\left(x\right),\ dv = e^{-x}dx$$

$$du = -\sin\left(x\right) - \cos\left(x\right)dx\ dv = -e^{-x}$$

$$\int e^{-x}\left(\cos\left(x\right) - \sin\left(x\right)\right)dx = e^{-x}\left(\sin\left(x\right) - \cos\left(x\right)\right) + \int -e^{-x}\left(\sin\left(x\right) + \cos\left(x\right)\right)dx$$

$$u = \sin\left(x\right) + \cos\left(x\right)\ \ dv = -e^{-x}dx$$

$$du = \cos\left(x\right) - \sin\left(x\right)dx\ v = e^{-x}$$

$$e^{-x}\left(\sin\left(x\right) - \cos\left(x\right)\right) + \int -e^{-x}\left(\sin\left(x\right) + \cos\left(x\right)\right)dx = e^{-x}\left(\sin\left(x\right) - \cos\left(x\right)\right) + e^{-x}\left(\sin\left(x\right) + \cos\left(x\right)\right) - \int e^{x}\left(\cos\left(x\right) - \sin\left(x\right)\right)$$

$$= 2e^{-x}\sin\left(x\right) - \int e^{x}\left(\cos\left(x\right) - \sin\left(x\right)\right)$$

On the right side, we have the original integral again, so we have

$$\int e^{-x}\left(\cos\left(x\right) - \sin\left(x\right)\right)dx = 2e^{-x}\sin\left(x\right) - \int e^{x}\left(\cos\left(x\right) - \sin\left(x\right)\right)$$

Thus

$$2\int e^{-x}\left(\cos\left(x\right)-\sin\left(x\right)\right)dx=2e^{-x}\sin\left(x\right)$$

$$\int e^{-x}\left(\cos\left(x\right)-\sin\left(x\right)\right)dx=e^{-x}\sin\left(x\right)$$

Thus

$$\int Mdx=ye^{-x}\sin\left(x\right)+\left(-y-1\right)\int e^{-x}dx=ye^{-x}\sin\left(x\right)+\left(y+1\right)e^{-x}+A\left(y\right)$$

Where $A\left(y\right)$ is some function of y. Now, we can take the second integral.

$$\int Ndy=\int e^{-x}\left(1+\sin\left(x\right)\right)dy=e^{-x}y+e^{-x}\sin\left(x\right)y+B\left(x\right)$$

where $B\left(x\right)$ is some function of x. Now, we match the terms.

$$ye^{-x}\sin\left(x\right)+\left(y+1\right)e^{-x}+A\left(y\right)=e^{-x}y+e^{-x}\sin\left(x\right)y+B\left(x\right)$$

Here, we see $A\left(y\right)=0$ and $B\left(x\right)=e^{-x}$, thus

$$F\left(x,y\right)=ye^{-x}\sin\left(x\right)+\left(y+1\right)e^{-x}=C$$

is the solution to our differential equation.

Ex 4)

Consider

$$\left(3y-y^{2}\sin\left(x\right)\right)+\left(2+6x+3y\cos\left(x\right)\right)\frac{dy}{dx}=0$$

Here

$$M=3y-y^{2}\sin\left(x\right)$$

and

$$N=2+6x+3y\cos\left(x\right)$$

Thus

$$M_{y}=3-2y\sin\left(x\right)$$

and

$$N_{x}=6-3y\sin\left(x\right)$$

As

$$M_{y}\neq N_{x}$$

we do not have an exact equation. Even so, suppose we take

$$\frac{N_{x}-M_{y}}{M}=\frac{\left(6-3y\sin\left(x\right)\right)-\left(3-2y\sin\left(x\right)\right)}{3y-y^{2}\sin\left(x\right)}=\frac{3-y\sin\left(x\right)}{\left(3-y\sin\left(x\right)\right)y}=\frac{1}{y}$$

So we have $\frac{N_{x}-M_{y}}{M}$ as a function of y only, thus we have an integrating factor

$$u=e^{\int\frac{N_{x}-M_{y}}{M}dy}=e^{\int\frac{1}{y}dy}=e^{\ln|y|}=y$$

So we can solve the equivalent equation

$$y\left(3y - y^2 \sin(x)\right) + y\left(2 + 6x + 3y\cos(x)\right)\frac{dy}{dx} = 0$$

$$3y^2 - y^3 \sin(x) + \left(2y + 6xy + 3y^2 \cos(x)\right)\frac{dy}{dx} = 0$$

If we check M, N we have

$$M_y = 6y - 3y^2 \sin(x)$$

and

$$N_x = 6y - 3y^2 \sin(x)$$

So

$$M_y = N_x$$

and the above equation is exact. As **step 1** is satisfied, we can now move on to **step 2** and integrate.

$$\int M dx = \int 3y^2 - y^3 \sin(x)\, dx = 3xy^2 + y^3 \cos(x) + A(y)$$

where $A(y)$ is some function of y. For the second integral, we have

$$\int N dy = \int 2y + 6xy + 3y^2 \cos(x)\, dy = y^2 + 3xy^2 + y^3 \cos(x) + B(x)$$

where $B(x)$ is some function of x. Finally, we match terms and solve our differential equation.

$$3xy^2 + y^3 \cos(x) + A(y) = y^2 + 3xy^2 + y^3 \cos(x) + B(x)$$

By matching terms, we see $A(y) = y^2$ and $B(x) = 0$, thus

$$F(x, y) = y^2 + 3xy^2 + y^3 \cos(x) = C$$