Lecture 1

August 19, 2024

In a nutshell, differential equations are equations that contain derivatives. They appear in many different disciplines such as physics, biology, engineering, finance, chemistry, ect. In this class you might hope to get one or more of the following skills:

1. Building Good Models with Differential Equations
2. Developing Intuition Around Differential Equations
3. Learning Solving Techniques for Common Class of Differential Equations
4. Learning Numerical Methods for Approximate Solutions to Differential Equations

Examples of Differential Equations

Here are some examples of differential equations:

Ex 1)

$$\frac{dy}{dx} = -2y$$

I claim that $y = e^{-2x}$ is a solution to this differential equation. To verify this, observe:

$$\frac{dy}{dx} = -2e^{-2x}$$

thus

$$\frac{dy}{dx} = -2e^{-2x} = -2\left(e^{-2x}\right) = -2y$$

This solution is not unique. In fact, for any real number C, we have

$$y = Ce^{-2x}$$

as a solution.

Ex 2)

$$\frac{d^2y}{dx^2} = -4y$$

I claim

$$y = C_1 \sin(2x) + C_2 \cos(2x)$$

is a solution for any $C_1, C_2 \in \mathbb{R}$. To verify this, note:

$$\frac{dy}{dx} = 2C_1 \cos(2x) - 2C_2 \sin(2x)$$

$$\frac{d^2y}{dx^2} = -4C_1 \sin(2x) - 4C_2 \cos(2x)$$

Thus

$$\frac{d^2y}{dx^2} = -4C_1 \sin(2x) - 4C_2 \cos(2x) = -4\left(C_1 \sin(2x) + C_2 \cos(2x)\right) = -4y$$

as expected.

Ex 3)

Sometimes differential equations have initial conditions. Consider the following modification to the first example:

$$\frac{dy}{dx} = -2y,\ y(0) = 4$$

We already know the differential equation has a general solution in the form

$$y = Ce^{-2x}$$

but now we also require $y(0) = 4$, thus

$$4 = y(0) = Ce^{-2(0)} = C$$

So our solution to the differential equation with the given initial condition is:

$$y = 4e^{-2x}$$

Ex 4)

Some differential equations may have multiple initial conditions. Consider the following modification to the differential equation from the second example:

$$\frac{d^2y}{dx^2} = -4y,\ y(0) = 3,\ y'(0) = 0$$

As given in the second example, our solution will be of the form

$$y = C_1 \sin(2x) + C_2 \cos(2x)$$

So we can plug in the initial condition to show:

$$3 = y(0) = C_1 \sin(0) + C_2 \cos(0) = C_2$$

and

$$y' = 2C_1 \cos(2x) - 2C_2 \sin(2x)$$

thus

$$0 = y'(0) = 2C_1 \cos(0) - 2C_2 \sin(0) = 2C_1$$

Thus $C_1 = 0, C_2 = 3$ and

$$y = 3\cos(2x)$$

Preliminaries

In this class, we have some knowledge that will be assumed. If anything is unfamiliar, I suggest reviewing the concepts. In addition, I will try to give illustrative examples in class, so we can review any less familiar concepts as we use them.

Calculus 1

Derivative and Integral Rules (Chain Rule, Product Rule, Quotient Rule, ect.)

Calculus 2

Integral Rules (trig integrals, partial fractions, trig subs, u-subs), Improper Integrals, and Power Series (If time permits us to cover series solutions at the end of the semester)

Calculus 3

Partial Derivatives (Just the basics. This will not be as important as the calc 2 integral tricks)

Linear Algebra

Setting up and solving linear systems of equations, determinants, eigenvalues, eigenvectors (we will use this at the end of the class with coupled linear systems. If you are taking linear algebra concurrently, do not worry)

Other Concepts

In addition, we will use Euler's Formula and Hyperbolic trig functions. If you have not seen these before, no worries. A brief explanation will be given below.

Euler's Formula

Euler's Formula gives us a nice way of converting between complex exponentials and complex numbers. It is stated as

$$Re^{i\theta} = R\left(\cos\left(\theta\right) + i\sin\left(\theta\right)\right)$$

This looks similar to polar coordinates (with the real part, $\cos(\theta)$ corresponding to the x value and the imaginary part $\sin(\theta)$ corresponding to the y value). This is no coincidence. The benefit of working with the form $Re^{i\theta}$ is that we can avoid having to memorize countless trig identities.

Hyperbolic Trig Functions

The hyperbolic trig functions are defined as

$$\sinh\left(x\right) = \frac{e^x - e^{-x}}{2}, \ \cosh\left(x\right) = \frac{e^x + e^{-x}}{2}$$

For practice, you should verify the following derivatives:

$$\frac{d}{dx}\sinh\left(x\right) = \cosh\left(x\right)$$

$$\frac{d}{dx}\cosh\left(x\right) = \sinh\left(x\right)$$

If you accept these, then all of the other hyperbolic trig derivatives can be derived using product, quotient, and chain rule (although we will probably not need to do so for this class). You should also verify the following useful hyperbolic trig identities:

$$\sinh\left(0\right) = 0$$

$$\cosh\left(0\right) = 1$$

and

$$\cosh^2\left(x\right) - \sinh^2\left(x\right) = 1$$

Next class we will start classification of differential equations.