Lecture 7

September 4, 2024

Substitutions

We will talk about substitutions today. By picking defining a new variable v in terms of x and y, we can often take complicated differential equations and simplify them into nicer equations (i.e. linear or separable equations). As with substitution in calculus 2 for integrals, there may be multiple valid substitutions, and some may yield "easier" equations to solve than others. Here is a table with some common substitutions. These will not work in all cases, but if you are unsure where to start, these are a good starting point:

If you see	Try
yy'	$v = y^2, y^2 + x^2, y^2 - x^2$
y^2y'	$v = y^3$
$\cos(y)\, y'$	$v = \sin(y)$
$\sin(y)\, y'$	$v = \cos(y)$
$e^y y'$	$v = e^y$

When we work through a substitution, the general process is:

1. Pick a v
2. Find v'
3. Substitute v and v' into the equation. All of the y values should be gone after substituting
4. Solve the equation in terms of v
5. Plug v in terms of y
6. Plug in initial conditions and rearrange the implicit solution into an explicit solution (if applicable)

Examples

Ex 1)

Solve

$$yy' + xy^2 = x$$

Let's make the substitution $v = y^2$. Then

$$v' = 2yy'$$

Thus

$$\frac{v'}{2} = yy'$$

Thus

$$\frac{v'}{2} + xv = x$$

Now, we can rearrange to get a linear equation:

$$v' + 2xv = 2x$$

Using integrating factor:

$$\mu(x) = e^{\int 2x dx} = e^{x^2}$$

So

$$v(x) = \frac{\int e^{x^2} * 2x dx + C}{e^{x^2}}$$

To take the integral, we make a u-sub

$$u = x^2, du = 2xdx$$

$$\int e^{x^2} * 2xdx = \int e^u du = e^u + C = e^{x^2}$$

so

$$v(x) = \frac{e^{x^2} + C}{e^{x^2}} = 1 + Ce^{-x^2}$$

Finally, we substitute back in terms of y.

$$y^2 = 1 + Ce^{-x^2}$$

We can leave this as an implicit solution or we can solve for y.

$$y = \pm\sqrt{1 + Ce^{-x^2}}$$

Ex 2)

Solve

$$2yy' = e^{y^2 - x^2} + 2x$$

We can try $v = y^2 - x^2$. Then $v' = 2yy' - 2x$ so if we rearrange the original equation, we get

$$2yy' - 2x = e^{y^2 - x^2}$$

and thus we have the substitution

$$v' = e^v$$

Now, we have a separable equation

$$\frac{dv}{dx} = e^v$$

$$e^{-v} dv = dx$$

Integrate both sides:

$$\int e^{-v} dv = \int dx$$

$$= e^{-v} = x + C$$

$$-v = \ln(x + C)$$

$$v = \ln\left(\frac{1}{x+C}\right)$$

So

$$y^2 - x^2 = \ln\left(\frac{1}{x+C}\right)$$

$$y^2 = \ln\left(\frac{1}{x+C}\right) + x^2$$

or if you prefer

$$y = \pm\sqrt{\ln\left(\frac{1}{x+C}\right) + x^2}$$

Ex 3)

Solve

$$\cos(y)\,y' - 1 = \sin(y) + 2x$$

This equation is not separable, but if we make a substitution $v = \sin(y)$, we get something nicer. We have $v' = \cos(y)\,y'$, so if we substitute, we get

$$v' - 1 = v + 2x$$

We can rearrange to get a linear equation:

$$v' - v = 2x + 1$$

Now we can solve using the integrating factor:

$$\mu(x) = e^{\int -1dx} = e^{-x}$$

and thus

$$v = \frac{\int (2x+1)\,e^{-x}dx + C}{e^{-x}}$$

To take the integral, we can use integration by parts:

$$u = 2x+1,\ dv = e^{-x}dx$$

$$du = 2dx,\ v = -e^{-x}$$

Thus

$$\int (2x+1)\,e^{-x}dx = -(2x+1)\,e^{-x} + \int 2e^{-x}dx$$

$$= -(2x+1)\,e^{-x} - 2e^{-x} + C$$

Plugging this back in yields

$$v = \frac{-(2x+1)\,e^{-x} - 2e^{-x} + C}{e^{-x}} = -(2x+1) - 2 + Ce^{x} = -2x - 3 + Ce^{x}$$

Finally, we substitute back in terms of y.

$$\sin(y) = -2x - 3 + Ce^{x}$$

$$y = \arcsin\left(-2x - 3 + Ce^{x}\right)$$

Without initial conditions, we cannot make any constraints on the domain, as these will depend on C.

Bernoulli Differential Equations

There is a special class of differential equations called, " Bernoulli Equations". These are not to be confused with Bernoulli's Equation for fluids (or one of the many other things called Bernoulli's equation). These are the differential equations of the form:

$$y' + p(x)\,y = q(x)\,y^n$$

Where n can be any real number except 0 or 1. To solve these, we make the substitution

$$v = y^{1-n}$$

Thus

$$v' = (1-n)\,y^{-n}y'$$

$$\frac{v'}{(1-n)}y^n = y'$$

So if we substitute, we get

$$\frac{v'}{(1-n)}y^n + p(x)\,y = q(x)\,y^n$$

Divide by y^n

$$\frac{v'}{(1-n)} + p(x)\,y^{1-n} = q(x)$$

and finally substitute in for y^{1-n} and rearrange

$$\frac{v'}{(1-n)} + p(x)\,v = q(x)$$

$$v' + (1-n)\,p(x)\,v = (1-n)\,q(x)$$

This is a linear equation so we can use the integrating factor!

Examples

Ex 4)

Solve

$$y' + \cos(x)\,y = \cos(x)\,y^5$$

We make the substitution

$$v = y^{1-5} = y^{-4}$$

Thus

$$v' = -4y^{-5}y'$$

$$\frac{y^5}{-4}v' = y'$$

and now, we substitute:

$$\frac{y^5}{-4}v' + \cos(x)\,y = \cos(x)\,y^5$$

We multiply both sides by $\frac{-4}{y^5}$ to yield:

$$v' + -4\cos(x)\, y^{-4} = -4\cos(x)$$

But as $y^{-4} = v$, this gives

$$v' + -4\cos(x)\, v = -4\cos(x)$$

and now we can use integrating factor:

$$\mu(x) = e^{\int -4\cos(x)dx} = e^{-4\sin(x)}$$

Thus

$$v = \frac{\int e^{-4\sin(x)}\left(-4\cos(x)\right)dx + C}{e^{-4\sin(x)}}$$

And we can solve the integral using a u-sub.

$$u = -4\sin(x)\,,\ du = -4\cos(x)\,dx$$

$$\int e^{-4\sin(x)}\left(-4\cos(x)\right)dx = \int e^{u}du = e^{u} + C = e^{-4\sin(x)} + C$$

Thus

$$v = \frac{e^{-4\sin(x)} + C}{e^{-4\sin(x)}} = 1 + Ce^{-4\sin(x)}$$

We now substitute back in terms of y.

$$y^{-4} = 1 + Ce^{-4\sin(x)}$$

$$y = \pm\frac{1}{\left(1 + Ce^{-4\sin(x)}\right)^{\frac{1}{4}}}$$

Ex 5)

Solve

$$y' + \frac{1}{x}y = \sqrt{y},\ y(1) = 1$$

As $\sqrt{y} = y^{\frac{1}{2}}$, then this is a Bernoulli equation. We can make the substitution

$$v = y^{1-\frac{1}{2}} = y^{\frac{1}{2}}$$

$$v' = \frac{1}{2}y^{\frac{-1}{2}}y' = \frac{1}{2\sqrt{y}}y'$$

Thus

$$y' = 2\sqrt{y}v'$$

and thus

$$2\sqrt{y}v' + \frac{1}{x}y = \sqrt{y}$$

Now, we divide both sides by $2\sqrt{y}$.

$$v' + \frac{1}{2x}\sqrt{y} = \frac{1}{2}$$

and we plug back in $v = \sqrt{y}$.

$$v' + \frac{1}{2x}v = \frac{1}{2}$$

Now, we have a linear equation we can solve using integrating factor!

$$\mu(x) = e^{\int \frac{1}{2x}dx} = e^{\frac{1}{2}\int \frac{1}{x}} = e^{\frac{1}{2}\ln|x|} = e^{\ln|x|^{\frac{1}{2}}} = \sqrt{|x|}$$

As we are assuming $x > 0$ by the initial condition, we have

$$\mu(x) = \sqrt{|x|} = \sqrt{x}$$

Thus

$$v(x) = \frac{\int \frac{\sqrt{x}}{2}dx + C}{\sqrt{x}} = \frac{\frac{x^{\frac{3}{2}}}{3} + C}{\sqrt{x}} = \frac{x}{3} + Cx^{\frac{-1}{2}}$$

Finally, we substitute back in terms of y.

$$\sqrt{y} = \frac{x}{3} + Cx^{\frac{-1}{2}}$$

And using the initial condition $y(1) = 1$:

$$\sqrt{1} = 1 = \frac{1}{3} + C(1)^{\frac{-1}{2}} = \frac{1}{3} + C$$

So $C = \frac{2}{3}$ and thus

$$\sqrt{y} = \frac{x}{3} + \left(\frac{2}{3}\right)x^{\frac{-1}{2}}$$

Or

$$y = \left(\frac{x}{3} + \left(\frac{2}{3}\right)x^{\frac{-1}{2}}\right)^2$$

with $x > 0$ is our final answer.

Homogeneous Equations

Some differential equations can be written in the form

$$y' = F\left(\frac{y}{x}\right)$$

for some function F. When this is the case, try the substitution

$$v = \frac{y}{x}$$

when making the substitution, we have

$$xv = y$$

$$\frac{d}{dx}(xv) = xv' + v = y'$$

So the equation becomes

$$xv' + v = F(v)$$

And as we can rewrite the equation as

$$xv' = F(v) - v$$

$$\frac{v'}{F(v) - v} = \frac{1}{x}$$

The substitution will yield a separable equation to solve in terms of v.

Examples

Ex 6)

Solve

$$x^2y' + 3xy = \frac{y^3}{x}$$

We might have to rearrange to get the equation in the proper form.

$$x^2y' = -3xy + \frac{y^3}{x}$$

$$y' = -3\frac{y}{x} + \frac{y^3}{x^3}$$

Now, we can make the substitution

$$v = \frac{y}{x}$$

Thus

$$xv = y$$

and

$$xv' + v = y'$$

So if we substitute:

$$xv' + v = -3v + v^3$$

Now, we can separate:

$$xv' = -4v + v^3$$

We bring all of the v terms to the left side and all of the x terms to the right side:

$$x\frac{dv}{dx} = v\left(v^2 - 4\right)$$

$$\frac{dv}{v\left(v^2-4\right)} = \frac{dx}{x}$$

Now, we integrate both sides:

$$\int \frac{dv}{v\left(v^2-4\right)} = \int \frac{dx}{x} = \ln|x| + C$$

For the right side, we will use partial fraction decomposition. The integral is computed below:

$$\int \frac{dv}{v\left(v^2-4\right)} = \int \frac{dv}{v\left(v-2\right)\left(v+2\right)} = \int \frac{A}{v} + \frac{B}{v-2} + \frac{C}{v+2}dv$$

Let's find A, B, C

$$1 = A\left(v-2\right)\left(v+2\right) + Bv\left(v+2\right) + Cv\left(v-2\right)$$

So if $v = 2$:

$$1 = B\left(2\right)\left(4\right) = 8B$$

$$B = \frac{1}{8}$$

If $v=0$:

$$1=A\left(-2\right)\left(2\right)=-4A$$

$$A=\frac{-1}{4}$$

If $v=-2$

$$1=C\left(-2\right)\left(-4\right)$$

$$C=\frac{1}{8}$$

Thus

$$\int\frac{A}{v}+\frac{B}{v-2}+\frac{C}{v+2}dv=\int\frac{-1}{4}\frac{1}{v}+\frac{1}{8}\frac{1}{v-2}+\frac{1}{8}\frac{1}{v+2}dv$$

$$=-\frac{1}{4}\ln\left|v\right|+\frac{1}{8}\ln\left|v-2\right|+\frac{1}{8}\ln\left|v+2\right|+C$$

Now, we can go back to the original substitution to find

$$\int\frac{dv}{v\left(v^2-4\right)}=-\frac{1}{4}\ln\left|v\right|+\frac{1}{8}\ln\left|v-2\right|+\frac{1}{8}\ln\left|v+2\right|=\ln\left|x\right|+C$$

We can multiply both sides by -8 to simplify

$$2\ln\left|v\right|+\ln\left|v-2\right|+\ln\left|v+2\right|=-8\ln\left|x\right|+C$$

Combine the logarithms on the left side:

$$\ln\left|\frac{v^2}{\left(v-2\right)\left(v+2\right)}\right|=-8\ln\left|x\right|+C=\ln\left|x^{-8}\right|+C$$

And take e to the power of both sides:

$$e^{\ln\left|\frac{v^2}{\left(v-2\right)\left(v+2\right)}\right|}=e^{\ln\left|x^{-8}\right|+C}$$

$$\frac{v^2}{\left(v-2\right)\left(v+2\right)}=C\left(x^{-8}\right)=\frac{C}{x^8}$$

Finally, we can plug back in $\frac{y}{x}$ to get an implicit solution

$$\frac{\left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}-2\right)\left(\frac{y}{x}+2\right)}=\frac{C}{x^8}$$

Ex 7)

Solve

$$y'-e^{\frac{y}{x}}\left(\frac{x}{y}\right)=\frac{y}{x},\ y\left(1\right)=-\ln\left(2\right)$$

We can make the substitution

$$v=\frac{y}{x}$$

Thus

$$xv=y$$

$$xv' + v = y'$$

Now, we substitute:

$$xv' + v - e^v\left(\frac{1}{v}\right) = v$$

So

$$xv' = \frac{e^v}{v}$$

Now, we can separate:

$$x\frac{dv}{dx} = \frac{e^v}{v}$$

$$\frac{v}{e^v}dv = \frac{1}{x}dx$$

and integrate:

$$\int \frac{v}{e^v}dv = \int \frac{1}{x}dx$$

For the left integral, we use integration by parts (we will use w rather than v in integration by parts to avoid using the same variable name twice):

$$u = v,\ dw = e^{-v}dv$$

$$du = dv\ w = -e^{-v}$$

Thus

$$\int \frac{v}{e^v}dv = -ve^{-v} + \int e^{-v}dv = -ve^{-v} + -e^{-v} + C$$

So

$$\int \frac{v}{e^v}dv = \int \frac{1}{x}dx$$

Gives

$$-ve^{-v} + -e^{-v} = \ln|x| + C$$

Now, we plug back in terms of y and x:

$$-\left(\frac{y}{x}\right)e^{-\left(\frac{y}{x}\right)} - e^{-\frac{y}{x}} = \ln|x| + C$$

As our initial conditions specify $x > 0$, then we are okay to drop the absolute value in the natural log.

$$-\left(\frac{y}{x}\right)e^{-\left(\frac{y}{x}\right)} - e^{-\frac{y}{x}} = \ln x + C$$

Now, we can plug in the initial conditions:

$$-\frac{-\ln(2)}{1}e^{-\left(\frac{-\ln(2)}{1}\right)} - e^{-\left(\frac{-\ln(2)}{1}\right)} = \ln(1) + C$$

$$\ln(2)\,e^{\ln(2)} - e^{\ln(2)} = C$$

$$C = 2\ln(2) - 2$$

Thus

$$-\left(\frac{y}{x}\right)e^{-\left(\frac{y}{x}\right)} - e^{-\frac{y}{x}} = \ln x + 2\ln(2) - 2$$

with $x > 0$ is our solution.