Lecture 4

September 9, 2024

Slope Fields

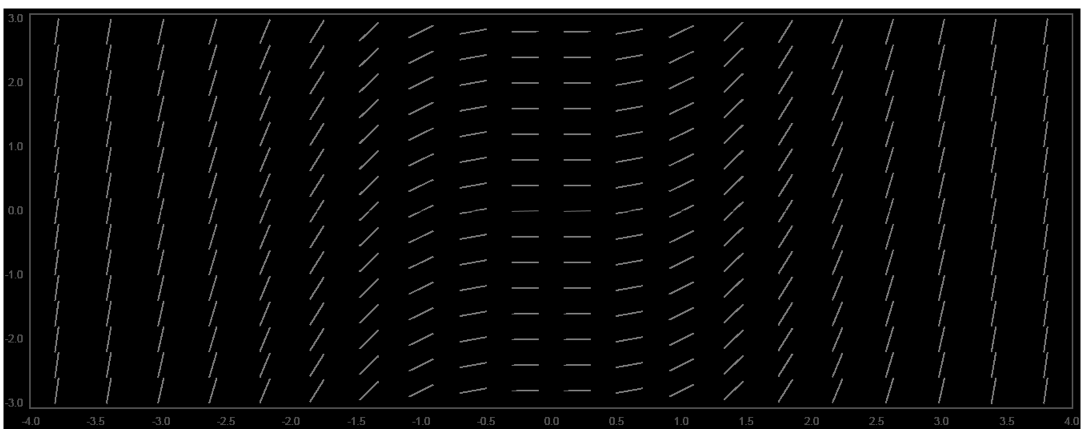
Slope fields give us a method to visualize solutions to differential equations. Suppose we have a differential equation of the form

$$\frac{dy}{dx} = f(x, y)$$

Then we can plot the slope field by drawing a line segment with a slope of $f(x, y)$ for each point (x, y) in the plane. Some examples are given below:

Ex 1)

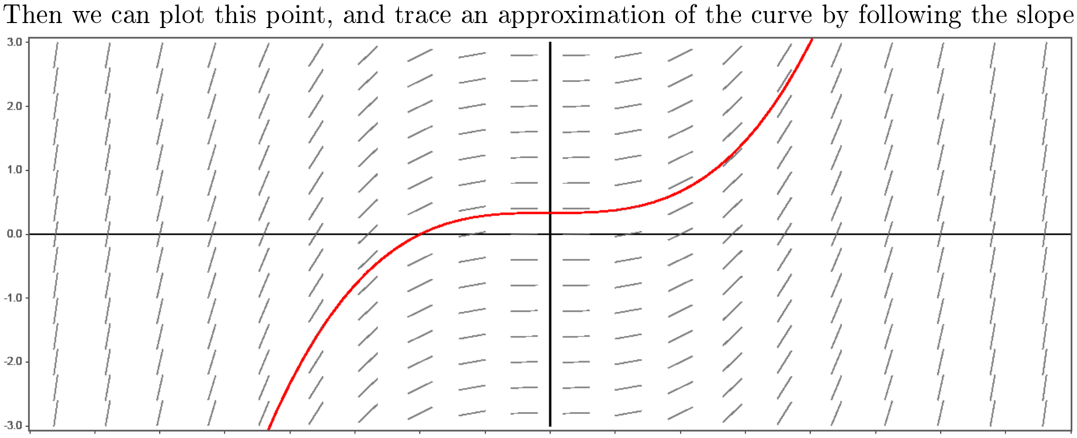
$$\frac{dy}{dx} = x^2$$



If we know a point on the solution curve, such as

$$y(2) = 3$$

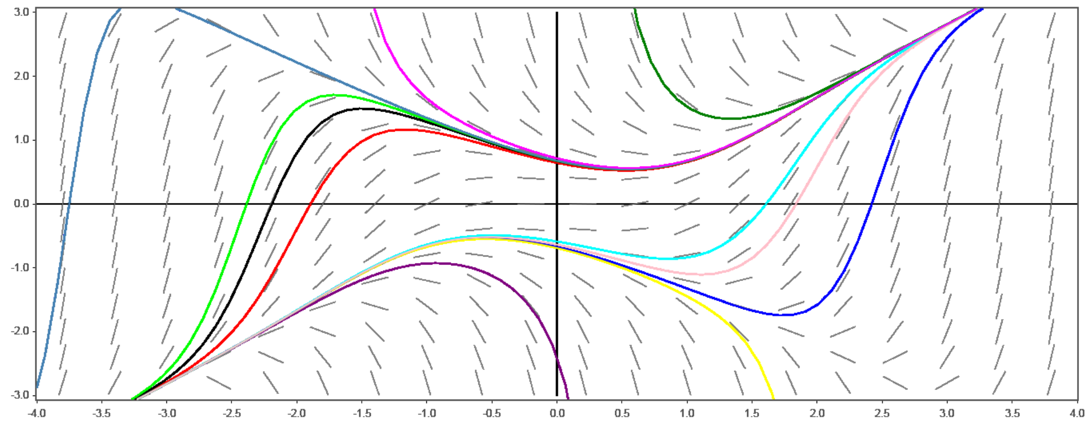
Then we can plot this point, and trace an approximation of the curve by following the slope field.



For the remaining examples, we will plot some slope fields and solution curves.

Ex 2)

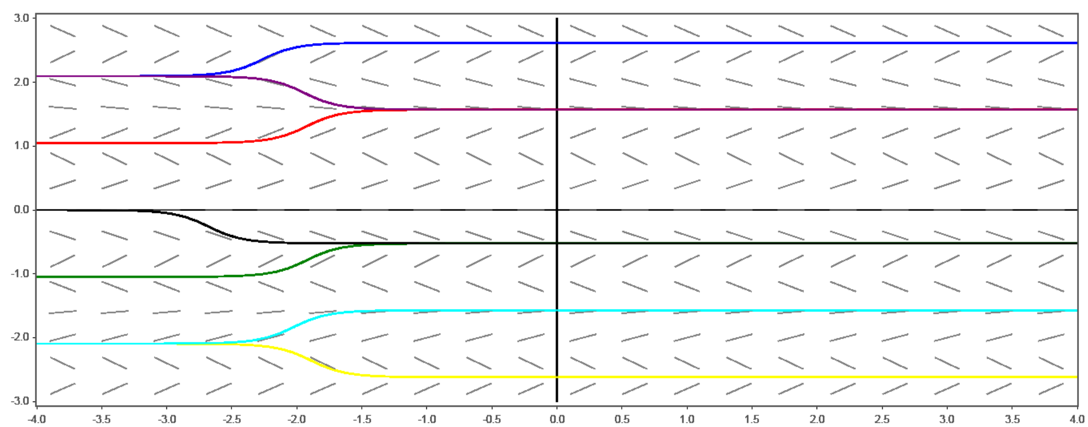
$$\frac{dy}{dx} = x^2 - y^2$$



Note the shape of the solution curve may vary depending on the region the curve lies in. This is why we have to be careful to state the interval of validity in our solutions (when we have initial conditions).

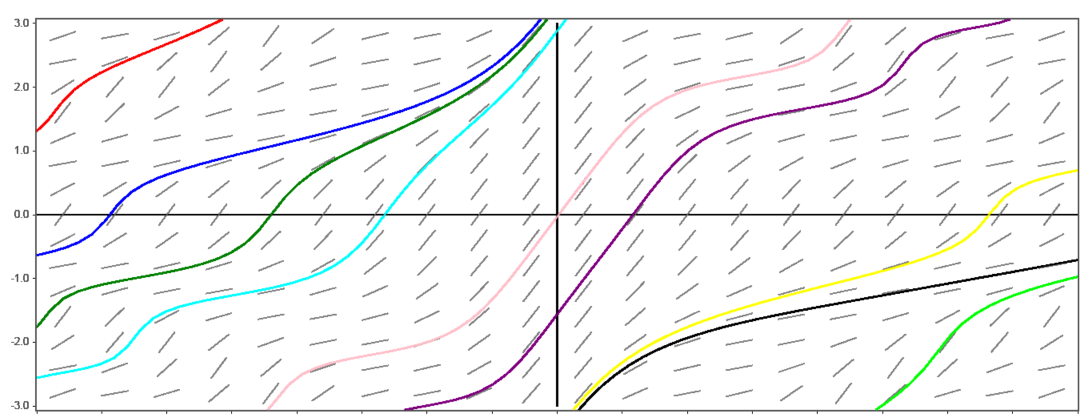
Ex 3)

$$\frac{dy}{dx} = \sin(6y)$$



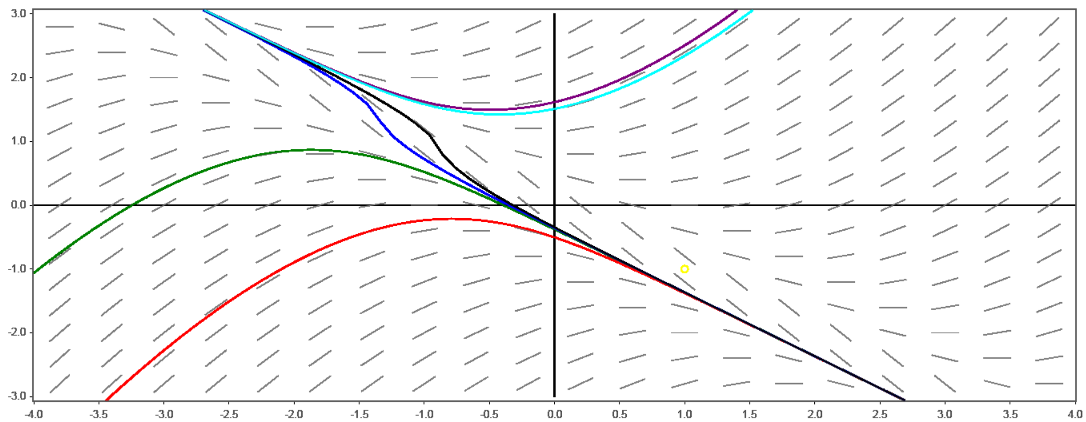
Ex 4)

$$\frac{dy}{dx} = e^{xy}$$



Ex 5)

$$\frac{dy}{dx} = \ln\left(|x + y|\right)$$



Here, we have to be a little bit careful. If we choose a point like $y(1) = -1$ where the derivative is undefined, we may not have a solution. This is shown by the yellow dot in the above slope field at $(1, -1)$.

Picard's Theorem of Existence and Uniqueness

We can know when solutions exist with the following theorem:

Consider the differential equation

$$\frac{dy}{dx} = f(x, y), \ f(x_0) = y_0$$

If $f(x, y)$ is a continuous function near (x_0, y_0) and $\frac{\partial f}{\partial y}$ exists and is continuous near (x_0, y_0) then a solution to

$$\frac{dy}{dx} = f(x, y), \ f(x_0) = y_0$$

exists for some interval around (x_0, y_0) and is unique.

Word of Warning:

This only is true for an interval around the point. Any discontinuity in the derivative or in $f(x, y)$ may give multiple solutions over the entirety of $\mathbb{R}$. Alternatively, a solution may not exist for all real numbers (especially as $y(x)$ may be undefined for some x values).