Lecture 5

August 28, 2024

A first order linear differential equation is a differential equation of the following form:

$$y' + p(x)\,y = q(x)$$

This is equivalent to the definition given in the classification lecture as

$$f_1(x)\,y' + f_0(x)\,y = g(x)$$

can be divided by $f_1(x)$

$$y' + \frac{f_0(x)}{f_1(x)}y = \frac{g(x)}{f_1(x)}$$

with

$$p(x) = \frac{f_0(x)}{f_1(x)}, q(x) = \frac{g(x)}{f_1(x)}$$

We can solve linear equations by using the integrating factor. Below, we will derive the solution. Once this solution has been derived, we can use the result for any linear equations we wish to solve.

Derivation of Integrating Factor

Consider the following:

$$\frac{d}{dx}(y\mu(x)) = y'\mu(x) + y\mu'(x)$$

By product rule. If we divide both sides by $\mu(x)$, we get

$$\frac{\frac{d}{dx}(y\mu(x))}{\mu(x)} = \frac{y'\mu(x) + y\mu'(x)}{\mu(x)} = y' + \frac{\mu'(x)}{\mu(x)}y$$

If we set $p(x) = \frac{\mu'(x)}{\mu(x)}$, then we have the left side of our linear differential equation. Thus

$$\frac{\frac{d}{dx}(y\mu(x))}{\mu(x)} = \frac{y'\mu(x) + y\mu'(x)}{\mu(x)} = y' + \frac{\mu'(x)}{\mu(x)}y = y' + p(x)\,y = q(x)$$

We now can solve for $\mu(x)$, as we have a separable equation.

$$\frac{d\mu}{dx}\frac{1}{\mu} = p(x)$$

Separate and integrate

$$\frac{d\mu}{\mu} = p(x)\,dx$$

$$\int \frac{d\mu}{\mu} = \int p(x)\,dx$$

$$\ln|\mu| = \int p(x)\,dx + C$$

$$|\mu| = e^{\int p(x)dx + C}$$

$$\mu = Ce^{\int p(x)dx}$$

Usually we do not bother including the C in the integrating factor. We will see why later. We can now solve for y. Recall

$$\frac{\frac{d}{dx}(y\mu(x))}{\mu(x)} = q(x)$$

We can rearrange to yield

$$\frac{d}{dx}(y\mu(x)) = q(x)\,\mu(x)$$

and integrate

$$y\mu(x) = \int q(x)\,\mu(x)\,dx + C$$

And now, we solve for y.

$$y(x) = \frac{\int q(x)\,\mu(x)\,dx + C}{\mu(x)}$$

Now, to show why we are okay to drop the constant in front of the integrating factor, suppose

$$\mu(x) = C_1 e^{\int p(x)dx}$$

Then

$$y(x) = \frac{\int q(x)\,C_1 e^{\int p(x)dx}dx + C}{C_1 e^{\int p(x)dx}}$$

We can pull a C_1 out of the numerator

$$y(x) = \frac{C_1\left(\int q(x)\,e^{\int p(x)dx}dx + \frac{C}{C_1}\right)}{C_1 e^{\int p(x)dx}}$$

and cancel it from the numerator and denominator

$$y(x) = \frac{\int q(x)\,e^{\int p(x)dx}dx + \frac{C}{C_1}}{e^{\int p(x)dx}}$$

As C is an arbitrary real constant, then it can absorb the $\frac{1}{C_1}$, thus

$$y(x) = \frac{\int q(x)\,e^{\int p(x)dx}dx + C}{e^{\int p(x)dx}}$$

In summary, to solve a differential equation of the form

$$y' + p(x)\,y = q(x)$$

we take

$$y(x) = \frac{\int q(x)\,\mu(x)\,dx + C}{\mu(x)}$$

where

$$\mu(x) = e^{\int p(x)dx}$$

Examples

Now that we have our formula for integrating factor, we are able to use it freely. Let's solve some linear equations using the integrating factor.

Ex 1)

$$y' + \cos(x)\, y = 2\cos(x)$$

First we find $\mu(x)$:

$$\mu(x) = e^{\int \cos(x) dx} = e^{\sin(x)}$$

(Recall we usually drop C when finding the integrating factor). Now, we plug this into the solution for y.

$$y = \frac{\int e^{\sin(x)} \left(2\cos(x)\right) dx + C}{e^{\sin(x)}}$$

We can solve the top integral using a u-sub. Thus

$$\int e^{\sin(x)} \left(2\cos(x)\right) dx = 2\int e^{\sin(x)} \cos(x)\, dx$$

$$u = \sin(x) \;\; du = \cos(x)$$

So

$$2\int e^{\sin(x)} \cos(x)\, dx = 2\int e^{u} du = 2e^{u} = 2e^{\sin(x)}$$

(We usually drop the $+C$ here, since it is already accounted for in the solution for y). We can plug this in to yield:

$$y = \frac{\int e^{\sin(x)} \left(2\cos(x)\right) dx + C}{e^{\sin(x)}} = y = \frac{2e^{\sin(x)} + C}{e^{\sin(x)}} = 2 + Ce^{-\sin(x)}$$

Ex 2)

Solve

$$y' + 3y = e^{2x}$$

First we find the integrating factor:

$$\mu(x) = e^{\int 3dx} = e^{3x}$$

Now, we plug in for y.

$$y = \frac{\int e^{3x} e^{2x} dx + C}{e^{3x}} = \frac{\int e^{5x} dx + C}{e^{3x}} = \frac{\frac{e^{5x}}{5} + C}{e^{3x}} = \frac{e^{2x}}{5} + Ce^{-3x}$$

Ex 3)

Sometimes you will run into an integral that cannot be computed in terms of elementary functions (polynomials, radicals, exponentials, logs, and trig functions). Here is an example of this and how to solve the equation in this case:

$$y' + \left(x^3\right) y = x^2 + x, y(1) = 2$$

First, we find the integrating factor:

$$\mu(x) = e^{\int x^3 dx} = e^{\frac{x^4}{4}}$$

Now, we plug into the solution for y.

$$y=\frac{\int e^{\frac{x^4}{4}}\left(x^2+x\right)dx+C}{e^{\frac{x^4}{4}}}$$

I claim the integral $\int e^{\frac{x^4}{4}}\left(x^2+x\right)dx$ cannot be solved with elementary functions (at least I do not know how to). We can still get a solution by rewriting the integral with a dummy variable t. We take $\int f(x)\,dx=\int_{x_0}^{x}f(t)\,dt+C$ where x_0 is the given initial value of x. For our problem this means

$$y=\frac{\int e^{\frac{x^4}{4}}\left(x^2+x\right)dx+C}{e^{\frac{x^4}{4}}}=\frac{\int_1^x e^{\frac{t^4}{4}}\left(t^2+t\right)dt+C}{e^{\frac{x^4}{4}}}$$

And we can now plug in the initial condition to solve for C.

$$y(1)=\frac{\int_1^1 e^{\frac{t^4}{4}}\left(t^2+t\right)dt+C}{e^{\frac{1^4}{4}}}=\frac{0+C}{e^{\frac{1}{4}}}=2$$

Thus

$$C=2e^{\frac{1}{4}}$$

and we get a final answer of

$$y=\frac{\int_1^x e^{\frac{t^4}{4}}\left(t^2+t\right)dt+2e^{\frac{1}{4}}}{e^{\frac{x^4}{4}}}$$

Ex 4)

$$y'+\cos(x)\,y=x,\,y(\pi)=3$$

First we find the integrating factor:

$$\mu(x)=e^{\int\cos(x)dx}=e^{\sin(x)}$$

Now we plug in for y.

$$y=\frac{\int e^{\sin(x)}xdx+C}{e^{\sin(x)}}$$

The integral is not nice, so we will use a dummy variable t, in our expression of y.

$$y=\frac{\int_\pi^x e^{\sin(t)}tdt+C}{e^{\sin(x)}}$$

Now, we can plug in our initial conditions.

$$y(\pi)=\frac{\int_\pi^\pi e^{\sin(t)}tdt+C}{e^{\sin(\pi)}}=3$$

$$\frac{C}{e^0}=C=3$$

Thus

$$y(x)=\frac{\int_\pi^x e^{\sin(t)}tdt+3}{e^{\sin(x)}}$$

Ex 5)

This last example will let us review an integral trick from calculus 2. Consider

$$y' + y = \sin(3x)$$

First we can find the integrating factor:

$$\mu(x) = e^{\int dx} = e^x$$

Thus

$$y = \frac{\int e^x \sin(3x)\,dx + C}{e^x}$$

We can find the integral

$$\int e^x \sin(3x)\,dx$$

by using integration by parts twice.

$$u = \sin(3x)\,,\;\; dv = e^x dx$$

$$du = 3\cos(3x)\,dx,\;\; v = e^x$$

So

$$\int e^x \sin(3x)\,dx = \sin(3x)\,e^x - 3\int \cos(3x)\,e^x dx$$

If we apply integration by parts once more, we see

$$u = \cos(3x)\,,\;\; dv = e^x dx$$

$$du = -3\sin(3x)\,,\;\; v = e^x$$

$$\sin(3x)\,e^x - 3\int \cos(3x)\,e^x dx = \sin(3x)\,e^x - 3\left(\cos(3x)\,e^x + 3\int e^x \sin(3x)\right)$$

$$= \sin(3x)\,e^x - 3\cos(3x)\,e^x - 9\int e^x \sin(3x)$$

But the original integral has appeared again on the right hand side of the equation. As

$$\int e^x \sin(3x)\,dx = \sin(3x)\,e^x - 3\cos(3x)\,e^x - 9\int e^x \sin(3x)$$

We can solve for the original integral algebraically:

$$10\int e^x \sin(3x)\,dx = \sin(3x)\,e^x - 3\cos(3x)\,e^x$$

$$\int e^x \sin(3x)\,dx = \frac{\sin(3x)\,e^x - 3\cos(3x)\,e^x}{10}$$

Note the $+C$ was dropped, as it is already accounted for in the solution of y. Thus

$$y = \frac{\int e^x \sin(3x)\,dx + C}{e^x} = \frac{\frac{\sin(3x)e^x - 3\cos(3x)e^x}{10} + C}{e^x} = \frac{\sin(3x)\,e^x - 3\cos(3x)\,e^x}{10e^x} + Ce^{-x}$$

$$y = \frac{\sin(3x) - 3\cos(3x)}{10} + Ce^{-x}$$