Lecture 3

August 23, 2024

An equation is separable if it can be written in the following form:

$$\frac{dy}{dx} = f(x)\, g(y)$$

To solve this equation (and get an implicit solution), move all of the x terms to one side and all of the y terms to the other.

$$\frac{dy}{g(y)} = f(x)\, dx$$

After this, integrate.

$$\int \frac{dy}{g(y)} = \int f(x)\, dx$$

In lecture, we gave an equivalent formulation. If we instead have

$$\frac{dy}{dx} f(x)\, g(y) = h(x)\, u(y)$$

then we still can bring all of the x terms to one side, and all of the y terms to the other, then integrate.

$$\frac{g(y)}{u(y)} dy = \frac{h(x)}{f(x)} dx$$

$$\int \frac{g(y)}{u(y)} dy = \int \frac{h(x)}{f(x)} dx$$

In either case, if you have initial conditions, the final steps are to plug in the initial conditions to solve for C and to find the interval of validity for x. We will work through some examples below.

Examples

Ex 1)

$$\frac{dy}{dx} = xy + y, y(0) = 3$$

We can factor this expression to get the right form:

$$\frac{dy}{dx} = y(x+1)$$

Bring all of the x terms to one side and y terms to the other.

$$\frac{dy}{y} = (x+1)\, dx$$

Integrate

$$\int \frac{dy}{y} = \int (x+1)\, dx$$

$$\ln|y| = x^2 + x + C$$

Now, we can rearrange to get an explicit function of y.

$$e^{\ln|y|} = e^{x^2+x+C} = e^{x^2+x}e^C$$

$$|y| = e^C e^{x^2+x}$$

As e^C is an arbitrary positive constant, we can replace it with $C_+ > 0$.

$$|y| = C_+ e^{x^2+x}$$

But if we get rid of the absolute value, we get

$$y = \pm C_+ e^{x^2+x} = Ce^{x^2+x}$$

where C is an arbitrary real constant. Now, we can plug in the initial condition:

$$y(0) = 3 = Ce^{0^2+0} = C$$

So

$$y = 3e^{x^2+x}$$

Ex 2)

$$\frac{dy}{dx}\sec(x) = y\ln(y)$$

First we separate the variables:

$$\frac{dy}{y\ln(y)} = \frac{1}{\sec(x)}dx$$

Now we integrate both sides:

$$\int \frac{dy}{y\ln(y)} = \int \frac{1}{\sec(x)}dx = \int \cos(x)\,dx = \sin(x) + C$$

For the left integral, we can use a u-sub.

$$u = \ln(y)\,,\ \ du = \frac{1}{y}dy$$

$$\int \frac{dy}{y\ln(y)} = \int \frac{1}{u}du = \ln|u| + C = \ln|\ln(y)| + C$$

This gives us an implicit solution to the differential equation.

$$\ln|\ln(y)| = \sin(x) + C$$

As there are not any initial conditions, we are not going to check for an interval of validity. We are okay to leave an implicit solution, since we are not told to solve for y explicitly.

Ex 3)

$$\frac{dy}{dx} = y^2 x \sin(x), \ y(0) = 1$$

First we separate the equation.

$$\frac{dy}{y^2} = x \sin(x)\, dx$$

Then we integrate both sides:

$$\int \frac{dy}{y^2} = \int x \sin(x)\, dx$$

$$\frac{-1}{y} = \int x \sin(x)\, dx$$

For the right integral we will use integration by parts. Let

$$u = x, \ dv = \sin(x)$$

$$du = dx, \ v = -\cos(x)$$

So

$$\int x \sin(x)\, dx = -x\cos(x) - \int -\cos(x)\, dx = -x\cos(x) + \sin(x) + C$$

Thus

$$\frac{-1}{y} = -x\cos(x) + \sin(x) + C$$

Now, we can use the initial conditions to solve for C.

$$\frac{-1}{1} = 0\cos(0) + \sin(0) + C = C$$

so $C = -1$

$$\frac{-1}{y} = -x\cos(x) + \sin(x) - 1$$

We can rearrange to isolate y

$$\frac{-1}{-x\cos(x) + \sin(x) - 1} = y$$

$$y = \frac{1}{x\cos(x) - \sin(x) + 1}$$

As $y > 0$, we need to make the assumption

$$\frac{1}{x\cos(x) - \sin(x) + 1} > 0$$

as we must assume our solution is continuous and contains the initial condition. This means we have the added assumption

$$x\cos(x) - \sin(x) + 1 > 0$$

(This cannot easily be solved further).

Ex 4)

Let's practice Trig subs.

$$\frac{dy}{dx}x^2 = (x-1)\sqrt{1-y^2},\; y(1) = 0$$

First we separate the equation

$$\frac{1}{\sqrt{1-y^2}}dy = \frac{x-1}{x^2}dx$$

Now we integrate both sides

$$\int \frac{1}{\sqrt{1-y^2}}dy = \int \frac{x-1}{x^2}dx$$

First, for the left integral we will use a trig sub.

$$\int \frac{1}{\sqrt{1-y^2}}dy$$

We can set

$$y = \sin(\theta),\; dy = \cos(\theta)\, d\theta$$

Thus

$$\int \frac{1}{\sqrt{1-\sin^2(\theta)}}\cos(\theta) = \int \frac{\cos(\theta)}{\sqrt{\cos^2(\theta)}}d\theta = \int d\theta = \theta + C$$

Now, we substitute back in terms of the original variable:

$$\theta + C = \arcsin(y) + C$$

Now, we can do the second integral:

$$\int \frac{x-1}{x^2}dx = \int \frac{1}{x} - \frac{1}{x^2}dx = \ln|x| + \frac{1}{x} + C$$

Now, we can combine these results to show

$$\int \frac{1}{\sqrt{1-y^2}}dy = \int \frac{x-1}{x^2}dx$$

$$\arcsin(y) = \ln|x| + \frac{1}{x} + C$$

$$y = \sin\left(\ln|x| + \frac{1}{x} + C\right)$$

Now, we can plug in the initial condition. As $x > 0$ in the initial condition,

$$y(1) = 0 = \sin\left(\ln(1) + \frac{1}{1} + C\right) = \sin(1 + C)$$

Thus

$$C = -1$$

and we have

$$y = \sin\left(\ln(x) + \frac{1}{x} - 1\right)$$

with $x > 0$.

Ex 5)

$$\frac{dy}{dx} = 2 + 2y$$

When separating, we always divide. A common mistake is for students to subtract the 2 or the $2y$ when separating. Let's separate the equation here:

$$\frac{dy}{2+2y} = dx, \;\; y(0) = 0$$

Now we integrate both sides:

$$\frac{1}{2}\int \frac{dy}{1+y} = \int dx$$

$$\frac{1}{2}\ln|1+y| = x + C$$

We can now rearrange to solve for y.

$$\ln|1+y| = 2x + C$$

Here, we note that $2C$ is still an arbitrary real constant, so the C "absorbs" the 2. Take e to both sides:

$$|1+y| = e^{2x+C} = C_+ e^{2x}$$

$$1 + y = \pm C_+ e^{2x} = Ce^{2x}$$

$$y = Ce^{2x} - 1$$

Now, we can plug in the initial conditions

$$y(0) = 0 = Ce^0 - 1 = C - 1$$

$$C = 1$$

So

$$y(x) = e^{2x} - 1$$