Math 320 HW 9

April 13, 2024

1)

For each of the following sets with the given binary operations, show R is a ring.

a)

Let $R = \{a + bi | a, b \in \mathbb{Z}\} = \mathbb{Z}[i]$ be the set of Gaussian integers (the complex numbers with integer coefficients) under the usual definition of addition and multiplication of complex numbers.

b)

Let X be a set. Let $R = P(X)$ the power set of X. Let $A + B = A \cup B - A \cap B$ (recall this is the symmetric difference) and $AB = A \cap B$.

2)

For each of the rings in **1)**, determine

a)

Is R commutative?

b)

Does R have any zero divisors?

c)

Does R have a unity? If so, which elements of R are units?

d)

Is R an integral domain?

3)

Consider the ring of polynomials $R = \mathbb{Z}[x]$.

a)

Show $\langle x^2 + 1 \rangle = \left\{ r\left(x^2 + 1\right) | r \in \mathbb{Z}[x] \right\}$ is an ideal of R.

b)

Show $\mathbb{Z}[x] / \langle x^2 + 1 \rangle \approx \mathbb{Z}[i]$. (Hint, the mapping $\phi\left(ax + b + \langle x^2 + 1 \rangle\right) = ai + b$ is one possible isomorphism).

c)

Show $\langle x^2 + 1 \rangle$ is a prime ideal but not a maximal ideal. (Hint: $\mathbb{Z}[x]$ is a commutative ring with unity).

4)

For $f(x), g(x)$ in polynomial ring R, find $q(x), r(x)$ such that $f(x) = g(x)q(x) + r(x)$ and such that $\deg(r(x)) < \deg(g(x))$

a)

$g(x) = x^2 + x,\ f(x) = x^5 + 4x^2 + 1,\ R = \mathbb{Z}_5[x]$

b)

$g(x) = x + 1,\ f(x) = x^5 + x^3 + x^2 + 1,\ R = \mathbb{Z}_2[x]$

5)

Show $x^2 + 3$ is irreducible over $\mathbb{Z}_5$ but is reducible over $\mathbb{Z}_7$.

6)

Let R be a ring and let $a, b \in R$. Show the difference of squares formula $(a-b)(a+b) = a*a - b*b$ holds if and only if R is a commutative ring.

7)

Let $r \in R$. Let $S_r = \{x \in R | xr = 0\}$. Show S_r is a subring of R.

8)

Show that if I_1, I_2 are ideals of R, then $I_1 \cap I_2$ is an ideal of R.

9)

If R is a commutative ring, show $\mathrm{Char}(R[x]) = \mathrm{Char}(R)$.