Math 320 HW 2

Due: Jan. 31, 2023 at 11:59 PM

1)

None of these sets with binary operation are groups. Each of these almost groups fails at least one of the group axioms. For each of these almost groups, show which axioms fail and which axioms hold.

1. $\mathbb{Z}$ under $\diamond$, where $a \diamond b = |a - b|$
2. $\mathcal{F}(\mathbb{R}, \mathbb{R})$ under $\circ$, where $\mathcal{F}(\mathbb{R}, \mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R}\}$ and $(f \circ g)(x) = f(g(x))$

2)

The group of invertible $n \times n$ real matrices under matrix multiplication is called The General Linear Group over $\mathbb{R}$ and denoted as $GL_n(\mathbb{R})$. Show

a)

$GL_n(\mathbb{R})$ is a group (Hint: For $n \times n$ matrices A, B, $\det(AB) = \det(A)\det(B)$)

b)

The set of $n \times n$ invertible matrices does not form a group under matrix addition.

3)

The shoe-sock property states for all $a, b \in G$, that

$$(ab)^{-1} = b^{-1}a^{-1}$$

a)

Prove the shoe-sock property

b)

This can be generalized to the following:

$$(a_1 a_2 \ldots a_n)^{-1} = a_n^{-1} a_{n-1}^{-1} \ldots a_2^{-1} a_1^{-1}$$

Prove this.

4)

Determine if the set H is a subgroup of the given group G under the same binary operation as G. If so, prove it. Otherwise, explain why it isn't.

1. $H = \{3n | n \in \mathbb{Z}\}$, $G = (\mathbb{Z}, +)$
2. $H = \{1, 2, 5\}$,$G = (Z_6, +)$
3. $H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in \mathbb{Z} \right\}$, $G = \left(\left\{\mathbb{R}^{2\times 2}\right\}, +\right)$

5)

Prove that a group G is abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}\ \forall a, b \in G$.

6)

Consider the cyclic group $\mathbb{Z}_6$.

a)

Find all of the subgroups of $\mathbb{Z}_6$

b)

Draw multiplication tables for each of these subgroups and color each element in the table a distinct color. Remember, even though this is a multiplication table, the binary operation is addition mod 6.

c)

What patterns do you notice in these multiplication tables?

7)

Solve the following equations for x in the given cyclic group. If multiple solutions exist, be sure to state all solutions.

a)

$x + 4 + 1 \equiv_{11} 7$, $G = \mathbb{Z}_{11}$

b)

$3 * x \equiv_{13} 1$, $G = U(13)$

c)

$2 * x \equiv_{11} 3$, $G = U(11)$