Class 13

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Automorphism Group Examples

Ex 1

Let $G = \mathbb{Z}_8$. Let $\phi_a(x)$ be the unique automorphism sending $1 \to a$. For example,

$$\phi_3(1) = 3$$

$$\phi_3(2) = \phi_3(1) + \phi_3(1) = 3 + 3 = 6$$

$$\ldots$$

$$\phi_3(7) = \phi_3(-1) = -\phi_3(1) = 5$$

(As we are in the integers mod 8). Then the automorphism group

$$\mathrm{Aut}(\mathbb{Z}_8) = \{\phi_1, \phi_3, \phi_5, \phi_7\}$$

as $\mathbb{Z}_8$ is cyclic, and any automorphism must send a generator to another generator.

Ex 2

Let $G = \{1, -1, i, -i\}$ under multiplication. Then i is a generator of G, so using the same notation as above,

$$\phi_i(i) = i$$

$$\phi_i(-1) = \phi_i(i * i) = \phi_i(i) * \phi_i(i) = i * i = -1$$

$$\phi_i(-i) = \phi_i(-1 * i) = \phi_i(-1) * \phi_i(i) = -1 * i = -i$$

$$\phi_i(1) = 1$$

as 1 is the identity. Thus $\phi_i(x) = x$. Similarly, we have

$$\phi_{-i}(i) = -i$$

$$\phi_{-i}(-1) = \phi_{-i}(i * i) = \phi_{-i}(i) * \phi_{-i}(i) = (-i)(-i) = -1$$

$$\phi_{-i}(-i) = \phi_{-i}(-1 * i) = \phi_{-i}(-1) * \phi_{-i}(i) = -1 * -i = i$$

$$\phi_{-i}(1) = 1$$

Thus

$$\phi_{-i}(x) = \frac{1}{x}$$

as

$$\frac{1}{i}=\frac{1}{i}*\frac{i}{i}=\frac{i}{-1}=-i$$

Further, as -1 is not a generator of G, then these must be the only two automorphisms, so

$$\text{Aut}\left(G\right)=\left\{\phi_i,\phi_{-i}\right\}\approx\mathbb{Z}_2$$

Inner Automorphism Group Examples

Ex 1

Suppose $G=U\left(5\right)$. Let

$$\phi_a\left(x\right)=axa^{-1}$$

Then

$$\phi_1\left(x\right)=1x1^{-1}=x$$

$$\phi_2\left(x\right)=2*x*2^{-1}=2*x*3=2*3*x=x$$

(As we are working mod 5)

$$\phi_3\left(x\right)=3*x*3^{-1}=3*x*2=3*2*x=x$$

$$\phi_4\left(x\right)=4*x*4^{-1}=4*x*4=4*4*x=x$$

So each $\phi_a\left(x\right)=x$, and we are left with

$$\text{Inn}\left(G\right)=\left\{\phi_1,\phi_2,\phi_3,\phi_4\right\}=\left\{e\right\}$$

as each of these inner automorphisms gives the exact same identity function.

Ex 2

Let $G=A_4$, the group of all even permutations on 4 objects. Then there are at most 12 possible inner automorphisms, as A_4 has 12 elements:

$$A_4=\left\{\left(1\right),\left(123\right),\left(124\right),\left(134\right),\left(234\right),\left(213\right),\left(214\right),\left(314\right),\left(324\right),\left(12\right)\left(34\right),\left(13\right)\left(24\right),\left(14\right)\left(23\right)\right\}$$

So we have the inner automorphisms defined as

$$\phi_{(1)}\left(x\right)=\left(1\right)x\left(1\right)^{-1}=x$$

$$\phi_{(123)}\left(x\right)=\left(123\right)x\left(123\right)^{-1}=\left(123\right)x\left(213\right)$$

$$\phi_{(124)}\left(x\right)=\left(124\right)x\left(124\right)^{-1}=\left(124\right)x\left(214\right)$$

$$\phi_{(134)}\left(x\right)=\left(134\right)x\left(134\right)^{-1}=\left(134\right)x\left(314\right)$$

$$\phi_{(234)}\left(x\right)=\left(234\right)x\left(234\right)^{-1}=\left(234\right)x\left(324\right)$$

$$\phi_{(213)}\left(x\right)=\left(213\right)x\left(213\right)^{-1}=\left(213\right)x\left(123\right)$$

$$\phi_{(214)}\left(x\right)=\left(214\right)x\left(214\right)^{-1}=\left(214\right)x\left(124\right)$$

$$\phi_{(314)}(x) = (314)\,x\,(314)^{-1} = (314)\,x\,(134)$$

$$\phi_{(324)}(x) = (324)\,x\,(324)^{-1} = (324)\,x\,(234)$$

$$\phi_{(12)(34)}(x) = (12)(34)\,x\,((12)(34))^{-1} = (12)(34)\,x\,(12)(34)$$

$$\phi_{(13)(24)}(x) = (13)(24)\,x\,((13)(24))^{-1} = (13)(24)\,x\,(13)(24)$$

$$\phi_{(14)(23)}(x) = (14)(23)\,x\,((14)(23))^{-1} = (14)(23)\,x\,(14)(23)$$

While we listed all of these out, some of these might describe the same function. This can be checked either by plugging in each x into each of these functions (to see which functions are the same), or by noticing particular patterns to see which functions cannot possibly be the same. Checking this can be tedious for non-abelian groups.

Homomorphisms

A function $\phi : G \to H$ is a homomorphism if $\phi(ab) = \phi(a)\phi(b)\ \forall a, b \in G$. Notice this is a less restrictive function than an isomorphism, as we no longer require ϕ be a bijection. Even without a bijection, homomorphisms share many of the same properties that isomorphisms do, such as:

1. $\phi(e_G) = e_H$
2. $\phi(a^n) = \phi(a)^n\ \ \forall n \in \mathbb{Z}$
3. The composition of homomorphisms is a homomorphism
4. ...

If you check the properties of isomorphisms, very few of the properties required us to use that ϕ be a bijection. As such, most of them still apply to homomorphisms.

Ex 1

Let $G = \mathbb{Z}_6$, $H = \mathbb{Z}_3$. Take $\phi(x) = x$ mod3. So

$$\phi(0) = 0$$

$$\phi(1) = 1$$

$$\phi(2) = 2$$

$$\phi(3) = 0$$

$$\phi(4) = 1$$

$$\phi(5) = 2$$

To show this is a homomorphism, let $x, y \in \mathbb{Z}_6$. Then

$$\phi(x+y) = (x+y) \text{ mod3} = x \mod 3 + y \mod 3 = \phi(x) + \phi(y)$$

In particular, note that homomorphism ϕ need not be 1-to-1.

Ex 2

If $K \leq G$, then $\phi : K \to G$ given by $\phi(x) = x$ is a homomorphism. This is the case, as

$$\phi(xy) = xy = \phi(x)\phi(y) \ \ \forall x, y \in K$$

In particular, note that homomorphism ϕ need not be onto.

Ex 3

Let $G = \left(\mathbb{R}^3, +\right)$, $H = \left(\mathbb{R}^2, +\right)$. Then $\phi : G \to H$ given by

$$\phi(\vec{x}) = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{pmatrix} \vec{x} = A\vec{x}$$

is a homomorphism as

$$\phi(\vec{x} + \vec{y}) = A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \phi(\vec{x}) + \phi(\vec{y})$$

Kernel

For a homomorphism ϕ,

$$\mathrm{Ker}(\phi) = \{x \in G | \phi(x) = e_H\}$$

That is, the kernel is the set of all elements in G that map to the identity in H. Let's find the kernel for each of the homomorphisms from the previous examples:

Ex 1

As $\phi(0) = \phi(3) = 0$ are the times when $\phi(x) = 0$, then

$$\mathrm{Ker}(\phi) = \{0, 3\}$$

Ex 2

As each element in K maps to itself with ϕ, then

$$\phi(e) = e$$

only for e, so

$$\mathrm{Ker}(\phi) = \{e\}$$

Ex 3

In linear algebra, the kernel and the null space are the same. We can find the null space of the transformation by finding all vectors $\vec{x}$ such that $A\vec{x} = \vec{0}$.

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

using row reduction:

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus

$$x + 2y + z = 0$$

and we have

$$\mathrm{Ker}(\phi) = \left\{ s \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} | s, t \in \mathbb{R} \right\} = \mathrm{Span}\left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

Thm

For homomorphism $\phi : G \to H$, $\mathrm{Ker}\,(\phi) \leq G$.

Proof

Let $a, b \in \mathrm{Ker}\,(\phi)$. First, to show $\mathrm{Ker}\,(\phi)$ is non-empty:

$$\phi\,(e) = e \in \mathrm{Ker}\,(\phi)$$

Now, suppose we wish to use the first subgroup test to show this:

$$\phi\left(ab^{-1}\right) = \phi\,(a)\,\phi\left(b^{-1}\right) = \phi\,(a)\,\phi\,(b)^{-1} = e\,(e)^{-1} = ee = e$$

so

$$ab^{-1} \in \mathrm{Ker}\,(\phi)$$