Math 320 Final Key

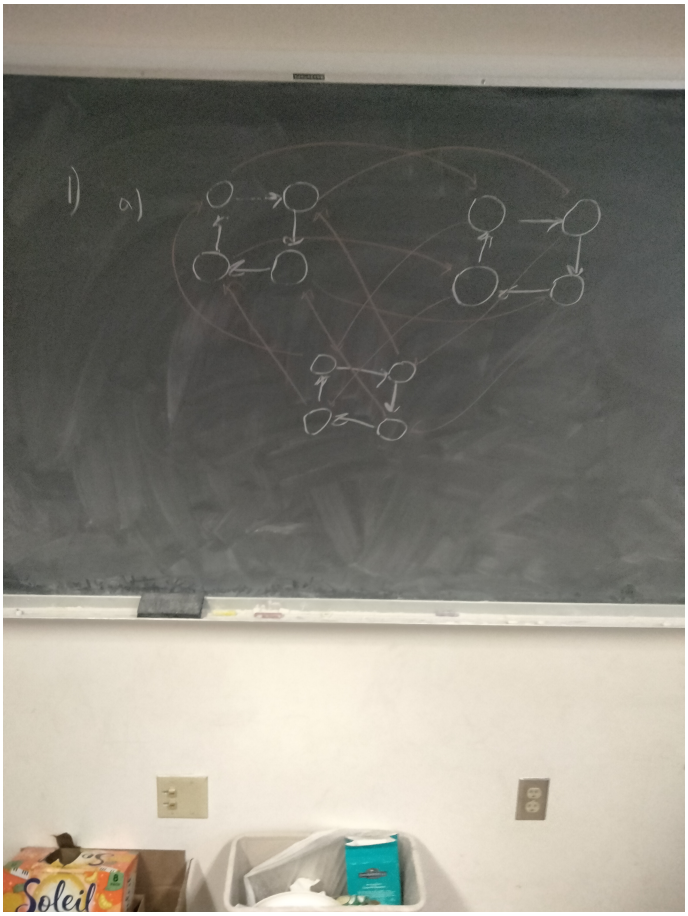
May 5, 2023

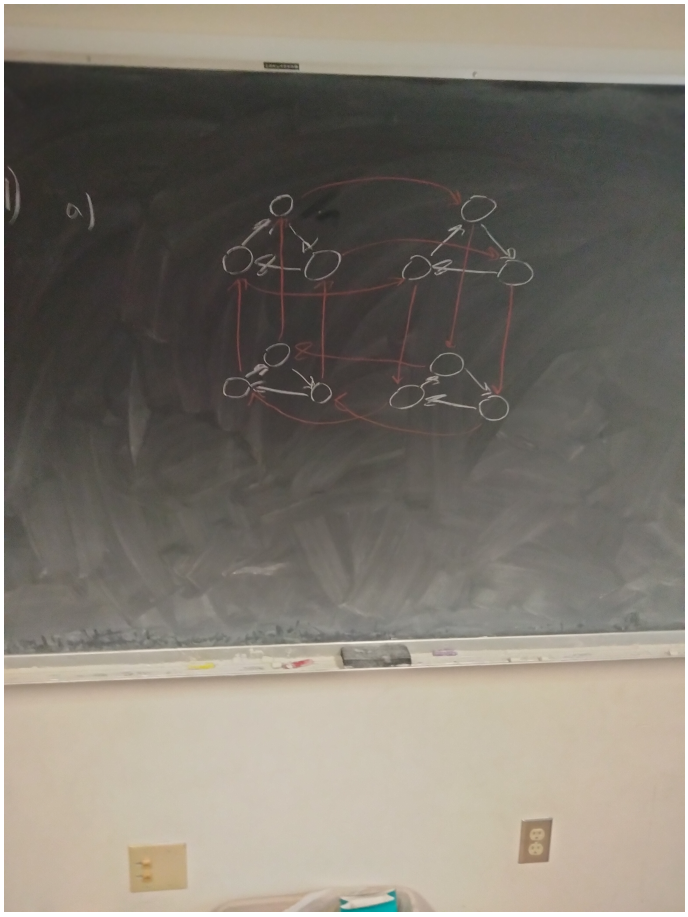
Computation

1.

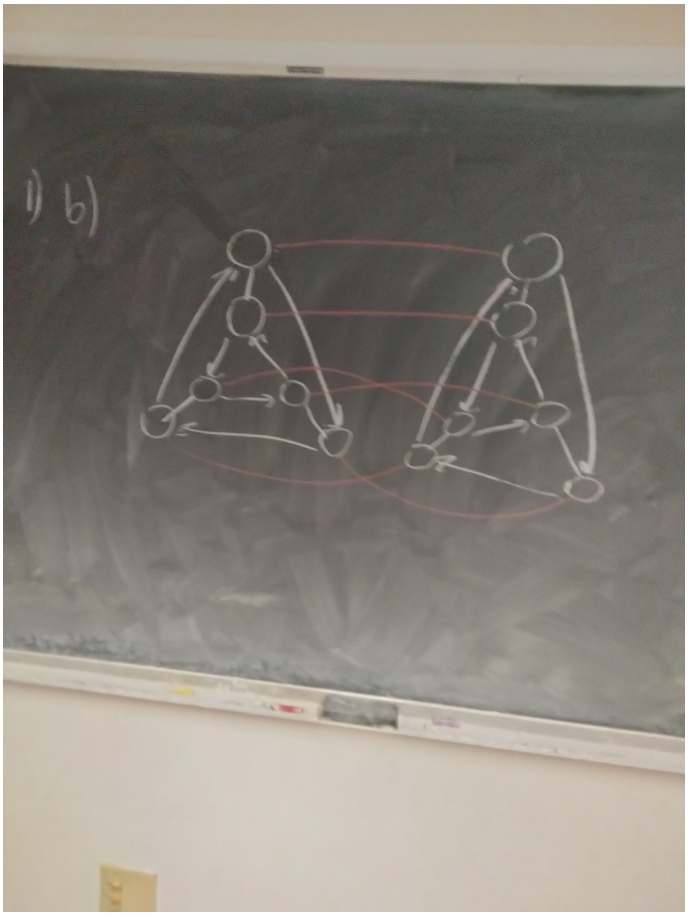
For each of these, there are many correct answers. It is important each generator has a different arrow type, and any generator that is not it's own inverse must have an arrow (but generators that are their own inverses can be lines. Some possible answers are shown below:

a)



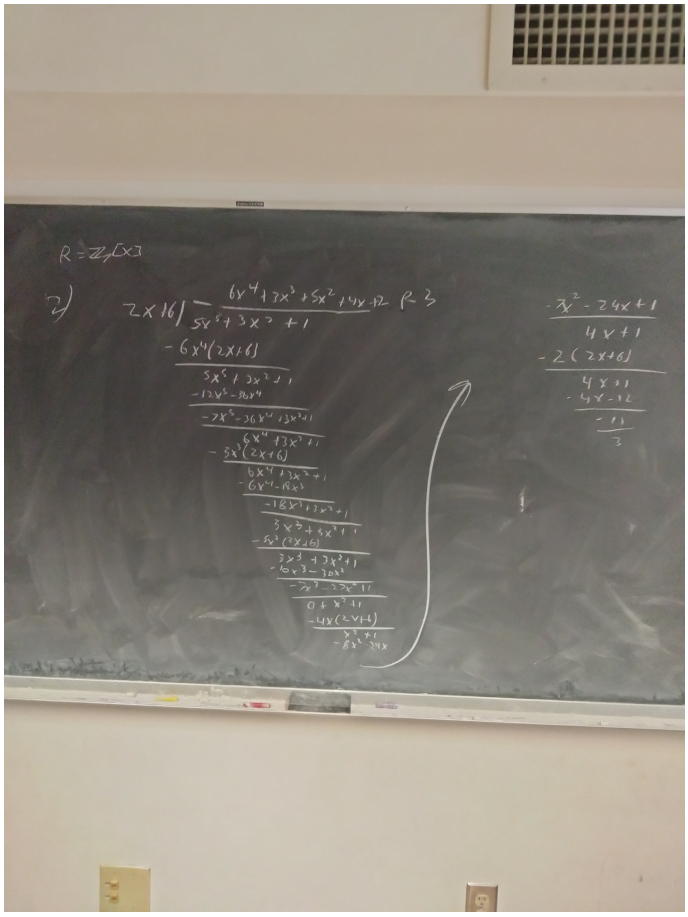

a)

b)



2.

It is important to note we are working in $\mathbb{Z}_7$. The long division is shown below.



This will yield a final answer of the form

$$5x^5 + 3x^2 + 1 = (2x + 6)\left(6x^4 + 3x^3 + 5x^2 + 4x + 2\right) + 3$$

3.

a)

$$\mathbb{Z}/7\mathbb{Z} = \{7\mathbb{Z}, 1 + 7\mathbb{Z}, 2 + 7\mathbb{Z}, 3 + 7\mathbb{Z}, 4 + 7\mathbb{Z}, 5 + 7\mathbb{Z}, 6 + 7\mathbb{Z}\}$$

b)

Let $\phi : \mathbb{Z}/7\mathbb{Z} \to \mathbb{Z}_7$ such that

$$\phi\left(x + 7\mathbb{Z}\right) = x \pmod 7$$

4.

a)

We need to show $f\left(\left(a + bi\right)\left(c + di\right)\right) = f\left(a + bi\right) f\left(c + di\right)$ for all $\left(a + bi\right), \left(c + di\right) \in \mathbb{C}^*$. Let $\left(a + bi\right), \left(c + di\right) \in \mathbb{C}^*$. Then

$$f\left(\left(a + bi\right)\left(c + di\right)\right) = f\left(ac + bci + bdi - bd\right)$$

$$= f\left(\left(ac - bd\right) + \left(bc + bd\right) i\right) = \left(ac - bd\right) - \left(bc + bd\right) i$$

and

$$f\left(a + bi\right) f\left(c + di\right) = \left(a - bi\right)\left(c - di\right) = ac - bci - adi - bd = \left(ac - bd\right) + \left(bc + bd\right) i$$

so

$$((a+bi)(c+di)) = f(a+bi)\, f(c+di)$$

b)

We need to find all of the the non-zero complex numbers of the form $a+bi$ such that $f(a+bi)=1$.

$$f(a+bi) = a - bi = 1 + 0i = 1$$

so

$$a = 1, -b = 0$$

Thus the kernel can only contain 1, yielding

$$\ker(f) = \{1\}$$

Proofs

5.

a)

There are a few possible definitions. These can be found in the lecture notes.

b)

Here, a is some element of R that is given. We know that $a^2 = 1$. We do not know anything else about a. Arbitrary elements of S are of the form $ara, asa \in S$ such that $s, r \in R$. We will use the subring test. We need to show closure under subtraction and multiplication. Let $ara, asa \in S$. First subtraction:

$$ara - asa = a(ra - sa) = a(r-s)\, a \in S$$

as $r - s \in R$. Now for multiplication:

$$(ara)(asa) = araasa = ar1sa = arsa = a(rs)\, a \in S$$

as $rs \in S$. As the subring test is satisfied, then S is a subring.

6.

We need to show that both statements imply each other as this is an if and only if statement. First, suppose

$$(a+b)^2 = a^2 + 2ab + b^2$$

Then we can expand the left hand side:

$$(a+b)^2 = (a+b)(a+b) = a^2 + ab + ba + b^2 = a^2 + 2ab + b^2$$

if we subtract a^2, ab, b^2 from both sides, we get the following:

$$a^2 + ab + ba + b^2 - a^2 - ab - b^2 = a^2 + 2ab + b^2 - a^2 - ab - b^2$$

$$ba = ab$$

so R must be commutative if the identity holds. Now, suppose R is commutative. That is $ab = ba$ for all $a, b \in R$. Then

$$(a+b)^2 = (a+b)(a+b) = a^2 + ab + ba + b^2$$

by commutativity:

$$= a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$$

as desired.

7.

a)

There are a few possible definitions. These can be found in the lecture notes.

b)

We will use the ideal test. We need to show that for $(2a+2bi), (2c+2di) \in S$ and for $(x+yi) \in R$ the following hold:

1. $(2a+2bi)-(2c+2di) \in S$
2. $(2a+2bi)(x+yi) \in S$
3. $(x+yi)(2a+2bi) \in S$

1. is closure under subtraction, **2.** and **3.** are the absorption laws. First, let's show closure under subtraction:

$$(2a+2bi)-(2c+2di) = 2a+2bi-2c-2di$$

$$= 2a-2c+2bi-2di = (2a-2c)+(2bi-2di)$$

$$= 2(a-c)+2(bi-di) = 2(a-c)+2(b-d)i$$

as $a, b, c, d \in \mathbb{Z}$ then $(a-c), (b-d) \in \mathbb{Z}$ thus $= 2(a-c), 2(b-d) \in 2\mathbb{Z}$ so $(2a+2bi)-(2c+2di) \in S$. Now, for the absorption laws. Here, we will note that $\mathbb{C}$ is a commutative ring, thus

$$(x+yi)(2a+2bi) = (2a+2bi)(x+yi)$$

As such, proving **2.** will also prove **3.**

$$(2a+2bi)(x+yi) = 2ax+2ayi+2bxi-2by$$

$$= 2ax-2by+2ayi+2bxi = (2ax-2by)+(2ayi+2bxi)$$

$$= 2(ax-by)+2(ay+bx)i$$

Thus the absorption laws hold. As we have satisfied all the requirements of the ideal test, then S is an ideal.