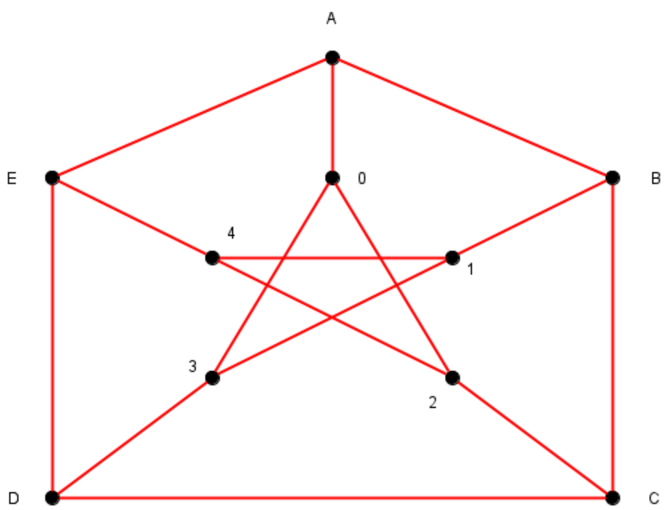
# Math 320 Project 2 Example

March 17, 2023

In graph theory, perhaps the most famous graph is the Petersen Graph The Petersen graph is important in graph theory, as it works as a non-trivial counterexample for many conjectures in graph theory. A picture of the Petersen graph is shown below:




The Petersen Graph

There are many different ways to draw this graph, but the above form is one of the more common drawings. This method of drawing has a couple of key features to note. First, note that the outside and inside sets of nodes both contain the same number of nodes (5). Second, note that we connect each node of the inside set of nodes with it's corresponding node in the outside set of nodes. Also note that the inside nodes are connecting by adding 2 mod 5 to each numbered node on the inside. Note, we made sure to label the nodes clockwise in the inside set of nodes. This leads to a more general notion of the Petersen graph.

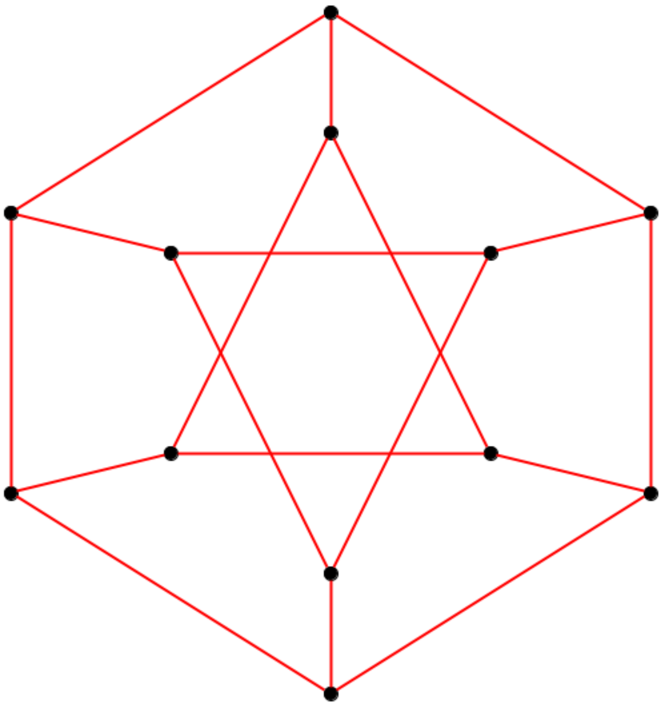
## $G(n,k)$

Informally, the generalized Petersen Graph $G(n,k)$ contains $n$ nodes in the inside set of nodes and the outside set of nodes. The outside nodes will all be connected in a single cycle, each inside node will be connected to exactly one inside node, and the inside nodes will be connected by adding $k$ mod $n$. (Provided we are careful to number our inside nodes in the same order as the outside nodes).

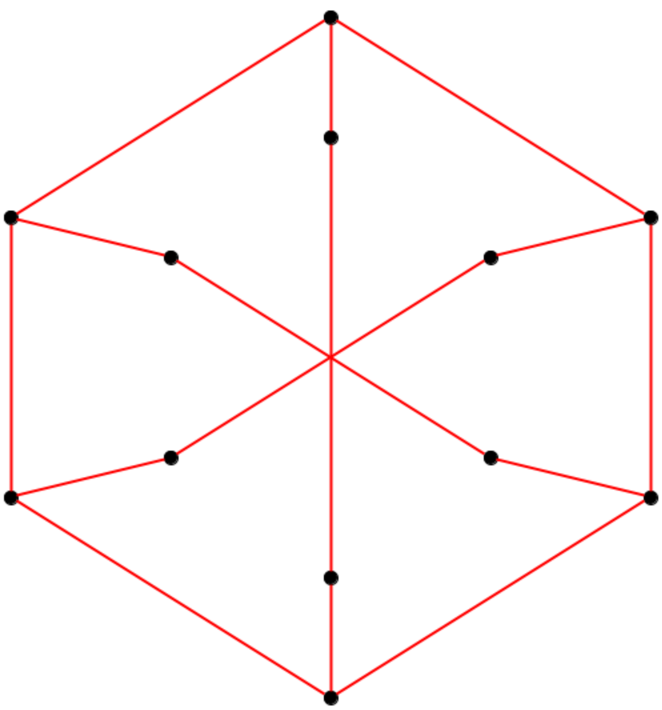
### Examples

**1)** The ordinary Petersen Graph is $G(5,2)$ as there are 5 nodes in the inside and outside set of nodes, and we are adding 2 mod 5.

**2)** $G(6,2)$ and $G(6,3)$ are shown below. Notice how the two graphs have different structure with the same $n$ value but different $k$ values.

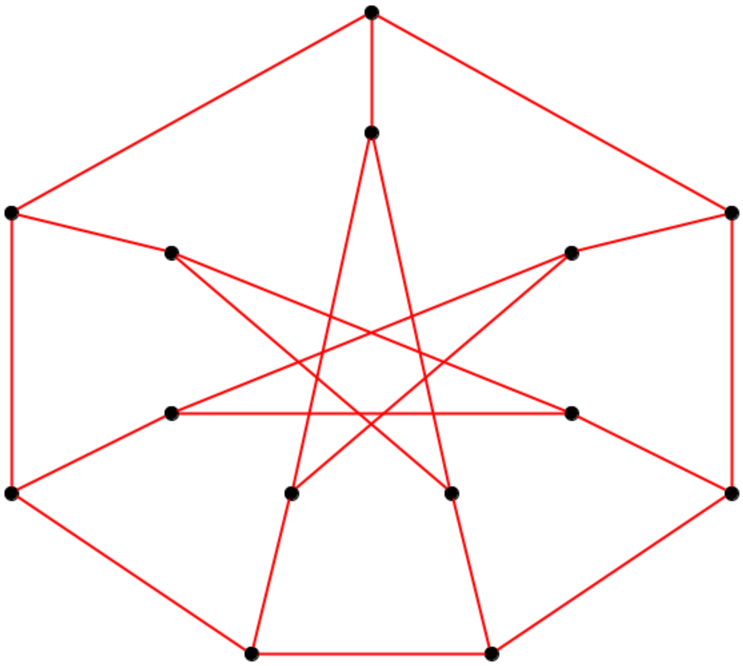


G(6,2)



G(6,3)

**3)** If we instead take $G\left(7,3\right)$, notice how the inside structure changes.



G(7,3)

## Properties

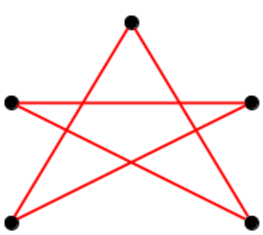
After drawing a few of these types of graphs, we may ask questions about when certain properties of the graph apply. Let's try to prove a couple!

### 1)

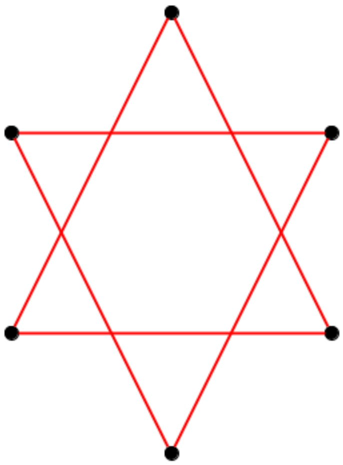
The inside subgraph will be connected if and only if $n, k$ are relatively prime.

### Terminology)

Before we prove this, let's make sure we are clear on some terminology. The inside subgraph is the graph that is formed by removing every node in the outside set of nodes as well as all edges containing these nodes. The inside subgraph of $G(5,2)$ and $G(6,2)$ are pictured below:



Inside Subgraph of G(5,2)



Inside Subgraph of G(6,2)

Informally, we say a graph is connected if it is in one piece. We say a graph is disconnected if it is in more than one piece. Formally, we say a graph is connected if you can find a path from any node to any other node in the graph, and a graph is disconnected otherwise. Note that $G(5,2)$ is connected, but $G(6,2)$ is not. Now, we have the tools we need to prove the property!

**pf)**

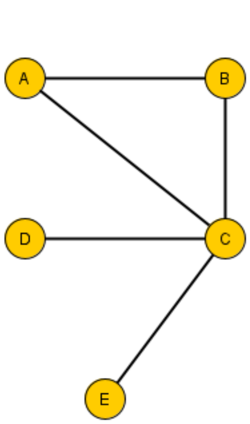
Recall the for the group $\mathbb{Z}_n$ under addition, that $x \in \mathbb{Z}_n$ is a generator of $\mathbb{Z}_n$ iff $x$ is relatively prime to $n$. That is each element in $\mathbb{Z}_n$ can be visited by repeatedly adding $x$ to itself. This allows us to visit each node in the inside subgraph by repeatedly connected every $k$ nodes if $k$ is relatively prime to $n$. As this property of $\mathbb{Z}_n$ is an iff statement, then it is also true that $G(n,k)$ will be disconnected if $n, k$ are not relatively prime. This is because $k$ is not a generator of $\mathbb{Z}_n$, so it cannot be possible to visit every element of $\mathbb{Z}_n$ by repeatedly adding $k$ (and thus it is not possible to visit every vertex by connecting every $k$ vertices).

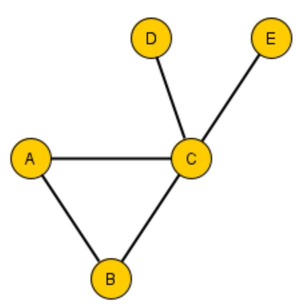
**2)**

Graphs $G(n,k)$ and $G(n,n-k)$ are isomorphic.

**Terminology)**

Before we prove this, let's make sure we are clear on terminology. When we say two graphs are isomorphic, we mean that the two graphs have the same structure. Informally, we can pair each node in the first graph with exactly one other node in the second graph such that if two nodes are connected in the first graph, then the corresponding nodes in the second graph must be connected, and vice versa. An example of two isomorphic graphs is shown below:




Notice that in both graphs, each labeled node in the left graph is connected to nodes with the exact same labels in the right graph and vice versa. Also note that each node contains exactly one label and that each label appears exactly once in each graph. With this idea of isomorphism in mind, let's prove the theorem!

**pf)**

Suppose we use the same labeling scheme for both $G(n,k)$ and $G(n,n-k)$. The outside nodes will contain the same connections in both graphs, as both sets of outside nodes will be a cycle labeled clockwise. Likewise, the connections between the outside and inside nodes will be the same as these graphs will share the same labeling scheme, and thus the same connection rule between the inside and outside sets of nodes. What remains is to show that the inside set of nodes shares the exact same connections in both graphs. That is inside nodes $u, v$ will be connected in the first graph iff corresponding nodes $u', v'$ are connected in the second graph. We know if two inside nodes in the first graph are connected, then

$$u - v \equiv_n k$$

or

$$v - u \equiv_n k$$

Suppose WLOG that

$$u - v \equiv_n k$$

then

$$-(u-v) \equiv_n -k$$

$$v - u \equiv_n n - k$$

Thus nodes $u', v'$ must be connected in the second graph. Now, suppose nodes $u', v'$ are connected in the second graph. That is

$$u' - v' \equiv_n n - k$$

or

$$v' - u' \equiv_n n - k$$

Suppose WLOG that

$$u' - v' \equiv_n n - k$$

then

$$-(u' - v') \equiv_n -(n-k)$$

$$v' - u' \equiv_n k - n \equiv_n k$$

Thus nodes $u, v$ must be connected in the first graph. As we have shown a connection between two the inside nodes in $G(n,k)$ implies connection of the corresponding nodes in $G(n,n-k)$ and vice versa, then we have proven the theorem.

## Conclusion

This is just scratching the surface of the properties of these types of graphs, but hopefully this provides an introduction to this family of graphs. For more information about these graphs, a good place to start is on Wikipedia. These proofs are my own, but more formal definitions also came from the same Wikipedia page.

https://en.wikipedia.org/wiki/Generalized_Petersen_graph