Math 320 Project 1 Example Write-Up

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The first thing we will prove is that congruence of triangles and similarity of triangles are both well defined equivalence relations.

Congruence

To show congruence forms an equivalence relation, we need to show that congruence of triangles is **i)** Reflexive, **ii)** Symmetric, **iii)** Transitive.

i) Every triangle is congruent to itself, as every triangles will have the same side lengths and angles as itself.

ii) Suppose triangle A is congruent to triangle B. Then A has the same side lengths and angles as B, thus B has the same side lengths and angles as A. This implies B is also congruent to A.

iii) Suppose triangle A is congruent to triangle B and triangle B is congruent to triangle C. Then A, B share the same side lengths and angles, and B, C share the same side lengths and angles, thus A, C share the same side lengths and angles so A and C are congruent.

As all three properties are satisfied, then congruence of triangles is an equivalence relation.

Similarity

To show similarity forms an equivalence relation, we need to show that similarity of triangles is **i)** Reflexive, **ii)** Symmetric, **iii)** Transitive.

i) Every triangle is similar to itself, as every triangles will have the same side lengths and angles as itself.

ii) Suppose triangle A is similar to triangle B. Then A has the same angles as B, thus B has the same side lengths and angles as A. This implies B is also similar to A by AA rule.

iii) Suppose triangle A is similar to triangle B and triangle B is similar to triangle C. Then A, B share the same angles, and B, C share the same side angles, thus A, C share the same side angles so A and C are similar by AA rule.

As all three properties are satisfied, then similarity of triangles is an equivalence relation.

Proofs of Congruence Rules

Here, we will give a sketch of a proof for why the SSS, SAS, ASA rules work, as well as reasoning why ASS will not work.

SSS

The law cosines states:

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

Where a, b, c are sides of the triangle, and C is the angle opposite side c. As this is a triangle, we know $0 < \gamma < \pi$, so we can uniquely determine C by solving the equation above for C.

$$C = \frac{\arccos\left(a^2 + b^2 - c^2\right)}{2ab}$$

As each angle can be found in the triangle by picking appropriate sides for a, b, c, then we can determine the angles of both triangles only with information of the side lengths, thus both triangles sharing the same side lengths is sufficient to show that the two triangles are congruent.

SAS

Once again, we can use the law of cosines. As the unknown side is the side opposite the angle C, then we simply solve for c to find the missing side:

$$c = \sqrt{a^2 + b^2 - 2ab\cos(C)}$$

As a triangle cannot have a negative side length, we can drop the $-$ sign when taking the square root. This means that c will have a unique value. Once we know all three side lengths we can rely on SSS to show both triangles are congruent.

ASA

This time around, we will use the law of sines. Recall the law of sines:

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

Here, each side is denoted by a lower case a,b, or c. The angle opposite of a given side is given by taking a capital letter. Before we use the law of sines, let's first note the sum of the angles in a triangle will always be π. This means that given the two known angles, we can find the third angle by taking $\pi - A - B$ (assuming A,B are the known angles). Once we know three angles, and have one known side (suppose WLOG a is the known side), then we can solve for the remaining unknown sides as

$$b = \frac{a\sin(B)}{\sin(A)}$$

and

$$c = \frac{a\sin(C)}{\sin(A)}$$

Why ASS fails

If we consider ASS, we do not have enough information to determine the missing side. Suppose WLOG the missing side is a. If we try using the law of cosines, we can rearrange the equation to get

$$0 = a^2 - 2ab\cos(C) + \left(b^2 - c^2\right)$$

Using quadratic formula:

$$a = \frac{2b\cos(C) \pm \sqrt{\left(-2b\cos(C)\right)^2 - 4\left(b^2 - c^2\right)}}{2}$$

$$a = \frac{2b\cos(C) \pm \sqrt{4b^2\cos^2(C) - 4\left(b^2 - c^2\right)}}{2}$$

$$a = b\cos(C) \pm \sqrt{b^2\cos^2(C) - b^2 + c^2}$$

$$a = b\cos(C) \pm \sqrt{b^2\left(\cos^2(C) - 1\right) + c^2}$$

$$a = b\cos(C) \pm \sqrt{b^2\left(-\sin^2(C)\right) + c^2}$$

As we have a valid triangle, we will assume that $b^2\left(-\sin^2(C)\right) + c^2 \geq 0$. If it were not, we could not have a side length of a, as it would be imaginary. Further, we can assume that from the $\pm$ we will only take the $+$ if the $-$ forces $a \leq 0$. If we have the case where $b\cos(C) > \sqrt{b^2\left(-\sin^2(C)\right) + c^2}$, then we may have two possible answers. Suppose we instead try using the law of sines:

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

WLOG, we can try solving for an unknown angle. Suppose this angle is B and we know the angle C. Then

$$B = \arcsin\left(\frac{b\sin(C)}{c}\right)$$

Here, we run into an issue. If we take arcsin, we will get a value for B such that $-\frac{\pi}{2} < B < \frac{\pi}{2}$, as the range of arcsin is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The valid angles for a triangle lie in $(0, \pi)$ however. arcsin will not be able to tell us if we have an angle of B or $\pi - B$, as these will yield the same result from arcsin. As the law of cosines and law of sines both may yield two correct answers, ASS does not prove congruence of triangles in general.

Proofs of Similarity Rules

These similarity rules rely on the law of sines. Recall

$$\frac{\sin(A)}{a}=\frac{\sin(B)}{b}=\frac{\sin(C)}{c}$$

If we rearrange the law of sines, we can show the following ratios hold:

$$\frac{a}{b}=\frac{\sin(A)}{\sin(B)},\frac{a}{c}=\frac{\sin(A)}{\sin(C)},\frac{b}{c}=\frac{\sin(B)}{\sin(C)}$$

With this in mind, we can show why similarity holds for the following rules:

AA

Suppose we know two of the angles. Then we can find the third angle by subtracting the two known angles from π. Next, suppose the first triangle have side lengths a, b, c and the second triangle has side lengths xa, xb, xc where x is some positive scaling factor. Then

$$\frac{xa}{xb}=\frac{a}{b}=\frac{\sin(A)}{\sin(B)},\frac{xa}{xc}=\frac{a}{c}=\frac{\sin(A)}{\sin(C)},\frac{xb}{xc}=\frac{b}{c}=\frac{\sin(B)}{\sin(C)}$$

The only way for all of the angles to be the same in each triangle is for the corresponding sides to be proportional.

SAS

If we know two of the sides are proportional, and we know that the angle between the two sides, then we can show that the unknown side scales appropriately using the law of cosines. Recall

$$c=\sqrt{a^2+b^2-2ab\cos(C)}$$

As c is opposite angle C, we will assume c is the unknown side. Suppose the first triangle have side lengths a, b and angle C and the second triangle has side lengths xa, xb and angle C, where x is some positive scaling factor. Then

$$\sqrt{(xa)^2+(xb)^2-2(xa)(xb)\cos(C)}=\sqrt{x^2a^2+x^2b^2-2x^2ab\cos(C)}$$

As $x>0$

$$=x\sqrt{a^2+b^2-2ab\cos(C)}=xc$$

thus all of the sides of triangle 2 are proportional to those of triangle 1. As such, we can use the law of sines to solve for the remaining two angles for both triangles. These will both yield the same result as

$$\frac{xa}{xc}\sin(C)=\frac{a}{c}\sin(C)=\sin(A)$$

$$\frac{xb}{xc}\sin(C)=\frac{b}{c}\sin(C)=\sin(B)$$

and as we will know the ratios of the side lengths, we can be certain the law of sines is well defined, the only angle that can possibly be obtuse is an angle opposite of the longest side of the triangle. If we wait to solve for this angle until we know the other two angles, then we can take the difference of π and the two acute angles.