Class 2

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Sets

Informally, a set is a non-contradictory, unordered collection of unique objects. When working with sets, we only care about whether or not an object is inside of a set. Note that sets can be finite or infinite.

Ex

$$A = \{a, b, c\}$$

$$B = \mathbb{R}$$

$$C = \{x \in \mathbb{Z} | x \text{ is even}\}$$

$$D = \{\square, \triangle, \blacklozenge, \smiley\}$$

If an element a is in set A, we denote this as $a \in A$. The set with no elements is called the empty set. We denote this as $\emptyset$.

Subsets

We say a set A is a subset of set B if $\forall a \in A$, $a \in B$. We denote this as $A \subseteq B$.

Ex

$$A = \{1, 2\}$$

$$B = \{1, 2, 3\}$$

$$A \subseteq B$$

Note that the empty set is a subset of all sets as it contains no elements. As the empty set contains no elements, it allows us to make vacuous claims such as

$$\forall x \in \emptyset,\ x \text{ is even, odd, a square, a cat, tired, a number, and not a number}$$

While the above is true, any claims about existence such as

$$\exists x \in \emptyset \text{ such that } x \text{ is even}$$

are false.

Equality of Sets

We say two sets A, B are equal if

$$A \subseteq B \text{ and } B \subseteq A$$

This is why we only care about unique elements when describing sets.

Ex

Show $A = B$ if $A = \{a, b, b, c, c\}$, $B = \{a, b, c\}$

pf

WTS $A \subseteq B$ and $B \subseteq A$

First, let's show $A \subseteq B$

$$a \in B$$

$$b \in B$$

$$b \in B$$

$$c \in B$$

$$c \in B$$

Thus all elements of A are in B, thus $A \subseteq B$. Now to show $B \subseteq A$:

$$a \in A$$

$$b \in A$$

$$c \in A$$

Thus all elements of B are in A, thus $B \subseteq A$. As $A \subseteq B$ and $B \subseteq A$, then $A = B$.

Strict Subset

If $A \subseteq B$ and $A \neq B$, then we say A is a strict subset of B. We denote this as $A \subset B$ or $A \subsetneq B$. As a word of warning, some texts use $A \subset B$ to mean A is a subset of B rather than A is a strict subset of B.

Ex

$$A = \{1, 2\}$$

$$B = \{1, 2, 3\}$$

$$A \subseteq B$$

but

$$3 \in B, 3 \notin A$$

thus

$$A \nsubseteq B$$

and thus $A \neq B$. This tells us $A \subsetneq B$.

Combining Sets

There are a few ways that we can combine sets. Here are the methods we may find useful for this class.

Union

The union of two sets A, B is the set containing all of the elements of A and B. This is denoted as $A \cup B = \{x | x \in A \text{ or } x \in B\}$. When we wish to take the union of many sets, we use the shorthand $\cup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$. This is analogous to $\sum$ notation for sums.

Ex

$$A = \{1, 2\}$$

$$B = \{2, 3\}$$

$$A \cup B = \{1, 2, 3\}$$

Note that

$$A \cup \emptyset = A, \ \forall A$$

Intersection

The intersection of two sets A, B is the set containing all of the elements that are in A and B. This is denoted as $A \cap B = \{x | x \in A \text{ and } x \in B\}$. When we wish to take the intersection of many sets, we use the shorthand $\cap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$. This is analogous to $\sum$ notation for sums.

Ex

$$A = \{1, 2\}$$

$$B = \{2, 3\}$$

$$A \cap B = \{2\}$$

Note that

$$A \cap \emptyset = \emptyset, \ \forall A$$

We say sets A, B are disjoint if they share no elements, formally, A, B are disjoint if $A \cap B = \emptyset$.

Difference

The difference between sets A, B is the set containing all of the elements in A that are not in B. This is denoted as $A - B = \{x | x \in A, x \notin B\} =$. It is also common to see this written as $A \backslash B$.

Ex

$$A = \{1, 2\}$$

$$B = \{2, 3\}$$

$$A - B = \{1\}$$

Note that

$$A - \emptyset = A, \ \forall A$$

Complement

Suppose $A \subseteq X$. We say the complement of A in X is the set containing all elements of X that are not in A. Formally, $A^c = X - A$. If it is cleat by context what X is, we will not explicitly state what X is.

Ex

$$A = \{1, 2\}$$

$$X = \{1, 2, 3, 4, 5\}$$

$$A^c = \{3, 4, 5\}$$

Cartesian Product

Another way to combine two sets is to take the Cartesian product of two sets. Even though it is called a product, it is best to think of this as pairing object, similar to how we have ordered pairs when plotting functions on the Cartesian plane. Formally, the Cartesian product is given as $A \times B = \{(a, b) \,|a \in A, b \in B\}$. The pair (a, b) is an ordered pair, so in general $(a, b) \neq (b, a)$.

Ex

$$A = \{1, 2\}$$

$$B = \{2, 3\}$$

$$A \times B = \{(1, 2), (1, 3), (2, 2), (2, 3)\}$$

Note that

$$A \times \emptyset = \emptyset, \ \forall A$$

Power set

We can generate a new set from set A called the power set of A, by taking all possible subsets of A. This is denoted as $P(A) = \{A' | A' \subseteq A\}$. This is also denoted as 2^A (the reason why will be clear later).

Ex

$$A = \{1, 2, 3\}$$

$$P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Note that

$$P(\emptyset) = \{\emptyset\} \neq \emptyset$$

Other Useful Terms

Partition

A partition P of a set A is a collection of subsets of A such that each pair of distinct subsets is disjoint and the union over all subsets is A. Formally, we might say $P = \{P_1, P_2, \ldots, P_n\}$ where $P_i \cap P_j = \emptyset$ whenever $i \neq j$, $P_i \subseteq A \ \forall i$, $\cup_{i=1}^{n} P_i = A$

Ex

$$A = \{1, 2, 3, 4, 5\}$$

$$P = \{\{1, 2\}, \{3\}, \{4, 5\}, \emptyset\}$$

Notice that

$$\{1, 2\} \cup \{3\} \cup \{4, 5\} \cup \{\emptyset\} = A$$

And any two distinct sets $P_i \in P$ are disjoint.

$$\{1, 2\} \cap \{3\} = \emptyset$$

$$\{1, 2\} \cap \{4, 5\} = \emptyset$$

$$\{1, 2\} \cap \emptyset = \emptyset$$

$$\{3\} \cap \{4, 5\} = \emptyset$$

$$\{3\} \cap \emptyset = \emptyset$$

$$\{4, 5\} \cap \emptyset = \emptyset$$

Cardinality

Informally, the cardinality of a set is the number of distinct elements in a set. If a set has infinitely many elements, we say that a set has infinite cardinality. For the purposes of this course, we will mainly focus finite sets. We denote the cardinality of a set A as $|A|$.

Ex

$$A = \{1, 2, 3\}$$

$$|A| = 3$$

Note that

$$|\emptyset| = 0$$

Also note that $|P(A)| = 2^{|A|}$ for finite sets (hence the notation $P(A) = 2^A$). An example and proof of this claim are below. Using the A from above

$$P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

$$|P(A)| = 8$$

Now for a proof of the claim.

pf

The easiest way to show this is to think of each element of set A either being included or not included in a particular subset of A. With each element of A, we make the decision to include the element or to not include the element. This yields 2 decisions per element, thus 2^n total decisions for a set with n elements. As A has n elements, then $|A| = n$ and we make $2^{|A|}$ possible decisions when making all of the possible subsets of A, hence $|P(A)| = 2^{|A|}$. ■

Practice Problems

1) Determine which of the sets below are subsets of each other:

$$A = \{x \in \mathbb{Z} \text{ s.t. } 3|x\}$$

$$B = \{x \in \mathbb{Z} \text{ s.t. } 2|x\}$$

$$C = \{x \in \mathbb{Z} \text{ s.t. } 12|x\}$$

$$D = \{x \in \mathbb{Z} \text{ s.t. } x|120\}$$

2) Find $P(A \times B)$ for

$$A = \{a, b\}$$

$$B = \{c, d\}$$

3) Show

$$P = \{\{(a, b) | b \in \mathbb{R}\} | a \in \mathbb{R}\}$$

is a partition of $\mathbb{R}^2$.

Solutions

1)

$$C \subseteq A, C \subseteq B$$

Otherwise, none of the other sets are subsets of each other. Let's prove this! If $12|x$, then $2|x$ and $3|x$, so if $x \in C$, then $x \in A, B$. Thus $C \subseteq A, C \subseteq B$. Now, we will show that the other sets are not subsets of each other. Consider $123 \in A$. $123 \notin B, C, D$ so $A \nsubseteq B, C, D$. Consider $122 \in B$. $122 \notin A, C, D$ so $B \nsubseteq A, C, D$. Consider $144 \in C$. $144 \notin D$ so $C \nsubseteq D$. Consider $2 \in D$. $2 \notin A, C$ so $D \nsubseteq A, C$. Consider $3 \in D$. $3 \notin B$ so $D \nsubseteq B$.

2)

$$A \times B = \{(a, c), (a, d), (b, c), (b, d)\}$$

so

$$P(A \times B) = \{\emptyset, \{(a, c)\}, \{(a, d)\}, \{(b, c)\}, \{(b, d)\},$$

$$\{(a, c), (a, d)\}, \{(a, c), (b, c)\}, \{(a, c), (b, d)\}, \{(a, d), (b, c)\}, \{(a, d), (b, d)\}, \{(b, c), (b, d)\},$$

$$\{(a, c), (a, d), (b, c)\}, \{(a, c), (a, d), (b, d)\}, \{(a, c), (b, c), (b, d)\}, \{(a, d), (b, c), (b, d)\}, \{(a, c), (a, d), (b, c), (b, d)\}\}$$

3)

For convenience, let

$$A_a = \{(a, b) | b \in \mathbb{R}\}$$

First, let's show $A_a \subseteq \mathbb{R}^2$. Let $(a, b) \in A_a$. Then as $a, b \in \mathbb{R}$, then $(a, b) \in \mathbb{R}^2$ so $A_a \subseteq \mathbb{R}^2$. Suppose $i \neq j$. Then

$$A_i = \{(i, b) | b \in \mathbb{R}\}, \ A_j = \{(j, b) | b \in \mathbb{R}\}$$

and

$$A_i \cap A_j = \emptyset$$

as none of the points in A_i have the same x coordinate as any of the points in A_j. Finally, consider

$$\cup_{a \in \mathbb{R}} A_a = S$$

To show $S = \mathbb{R}^2$, we need to show $S \subseteq \mathbb{R}^2$ and $\mathbb{R}^2 \subseteq S$. Suppose $(a, b) \in S$. Then $\exists A_a$ such that $(a, b) \in A_a$. As $A_a \subseteq \mathbb{R}^2$ for all a, then $(a, b) \in \mathbb{R}^2$, thus $S \subseteq \mathbb{R}^2$. Now, suppose $(a, b) \in \mathbb{R}^2$. Then $(a, b) \in A_a$, thus $(a, b) \in \cup_{a \in \mathbb{R}} A_a = S$ so $\mathbb{R}^2 \subseteq S$ and $S = \mathbb{R}^2$.