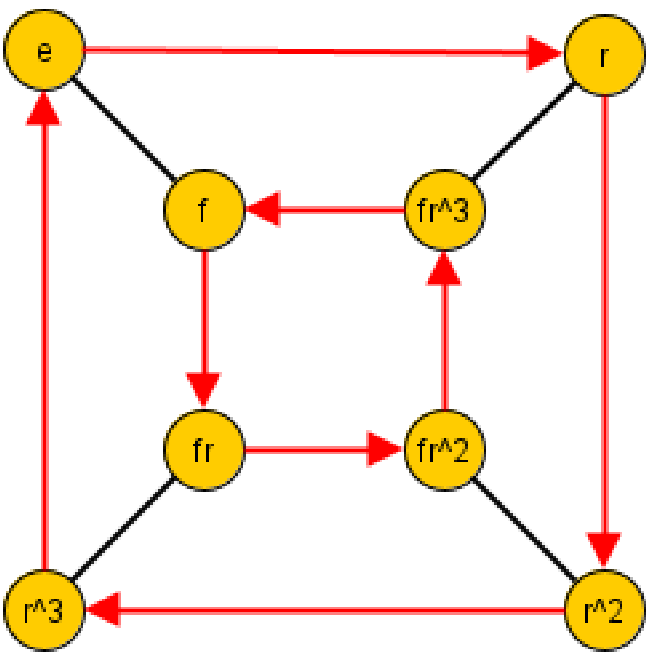
Math 320 Midterm Key

February 18, 2024

1.

There are multiple valid answers. Here is one:



2.

These answers are not unique. You might have a slightly different answer that is valid. Remember in cycle notation $(123) = (231) = (312)$ and that the product of disjoint cycles commutes.

a)

$$\alpha\beta = (143)(23)(34)(234)(12) = (1432)$$

b)

$$\beta\alpha = (34)(234)(12)(143)(23) = (3421)$$

3.

a)

U(5)				
*	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

b)

The subgroups are

$$\{1\}, \{1,4\}, \{1,2,3,4\}$$

4.

There are a few ways to show this.

Method 1:

We could show this function is onto and one-to-one directly. For onto, suppose $(a,b) \in \mathbb{R}^2$. Then if

$$(a,b) = (2x+y, y-3)$$

$$a = 2x+y, b = y-3$$

Thus

$$y = b+3$$

and

$$a = 2x+b+3$$

$$x = \frac{a-b-3}{2}$$

So for any $(a,b) \in \mathbb{R}^2$ there exists x, y such that $f(x,y) = (a,b)$. Now to show this function is one-to-one. Suppose $f(x,y) = f(z,w)$. Then

$$(2x+y, y-3) = (2z+w, w-3)$$

So

$$y-3 = w-3$$

and

$$y = w$$

and

$$2x+y = 2z+w = 2z+y$$

So

$$2x = 2z$$

and

$$x = z$$

Thus $(x,y) = (z,w)$ and f is one-to-one. As f is onto and one-to-one, then f is a bijection.

Method 2:

Since f is a affine transformation of the form $\vec{x} \to A\vec{x} + \vec{b}$, it is sufficient to show A is invertible to show f is invertible and thus a bijection.

$$f(x, y) = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -3 \end{pmatrix}$$

so

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$$

and as $\mathrm{Det}(A) = 2 * 1 = 2 \neq 0$, then A is invertible.

5.

a)

A set S with binary operation $\diamond$ forms an abelian group $G = (S, \diamond)$ if G satisfies the following:

1. $\exists e \in G$ such that $x \diamond e = e \diamond x = x$ for all $x \in G$ (Identity)
2. $(a \diamond b) \diamond c = a \diamond (b \diamond c)$ for all $a, b, c \in G$ (Associativity)
3. For each $a \in G$, there exists $a^{-1} \in G$ such that $a \diamond a^{-1} = a^{-1} \diamond a = e$ (Inverses)
4. For all $a, b \in G$, $a \diamond b = b \diamond a$ (Commutativity)

b)

Let's show each of the axioms. First, let's show the binary operation is well defined. As you are told this is a binary operation in the problem, you will not get penalized for not showing this step. Since, multiplication and addition of real numbers always results in a real number, we only need to make sure -1 is never a possible output from this binary operation. Suppose

$$-1 = a \diamond b = a + b + ab$$

for some a, b. Then

$$-1 = a(1 + b) + b$$

$$-1 - b = -1(1 + b) = a(1 + b)$$

As $b \neq -1$ (since -1 is not in the domain), then $(1 + b) \neq 0$. As such, we are able to cancel out the $(1 + b)$.

$$-1 = a$$

But as -1 is not in the domain, $a \neq -1$! Thus to avoid contradiction, we conclude that $a \diamond b \neq -1$ for any a, b in the domain. Now for the axioms.

Identity:

Suppose $b = 0$, then

$$a \diamond 0 = a + 0 + 0 = a$$

Similarly if $a = 0$, then

$$0 \diamond b = 0 + b + 0 = b$$

Associativity:

$$(a \diamond b) \diamond c = (a + b + ab) \diamond c$$

$$= a + b + ab + c + (a + b + ab)c$$

$$= a + b + ab + c + ac + bc + abc$$

$$= a + b + c + bc + a\left(b + c + bc\right)$$

$$= a \diamond (b + c + bc) = a \diamond (b \diamond c)$$

Inverses:

$$0 = a \diamond b = a + b + ab$$

$$0 = a + b\left(1 + a\right)$$

$$\frac{-a}{(1 + a)} = b = a^{-1}$$

To ensure this, consider

$$a \diamond a^{-1} = a + \frac{-a}{(1 + a)} + a\left(\frac{-a}{(1 + a)}\right)$$

$$= \frac{a\left(1 + a\right)}{(1 + a)} - \frac{a}{(1 + a)} - \frac{a^2}{(1 + a)} = 0$$

Or if we take the binary operation the other direction:

$$a^{-1} \diamond a = \frac{-a}{(1 + a)} + a + \frac{-a}{(1 + a)}a$$

$$= \frac{-a}{(1 + a)} + \frac{a\left(1 + a\right)}{(1 + a)} + \frac{-a^2}{(1 + a)} = 0$$

Commutativity:

$$a \diamond b = a + b + ab = b + a + ba = b \diamond a$$

6.

a)

A relation R on a set S is antisymmetric if for all a, b in $S:\ aRb, bRa$ implies $a = b$

b)

Suppose $a|b$ and $b|a$. Then $a = nb$ for some $n \in \mathbb{Z}^+$ and $b = ma$ for some $m \in \mathbb{Z}^+$, thus

$$a = n\left(ma\right) = \left(nm\right)a$$

for some $nm \in \mathbb{Z}^+$. Thus

$$mn = 1 \rightarrow m, n = 1$$

so

$$a = 1 * b = b$$

7.

a)

Let G, H be groups. We say $G \approx H$ if there exists a bijection $f : G \to H$ such that $f(ab) = f(a) f(b)$ for all $a, b \in G$.

b)

First note $\text{Fix}(\phi) \subseteq G$. As such, we will use a subgroup test to show this. You could also show this directly by proving all of the group axioms directly. First, let's show this is non-empty. Recall for all isomorphisms:

$$\phi(e_G) = e_H$$

As this is an automorphism, then

$$\phi(e) = e$$

so $e \in \text{Fix}(\phi)$. Now, suppose we use the first test: Let $a, b \in \text{Fix}(\phi)$

$$\phi\left(ab^{-1}\right) = \phi(a)\,\phi\left(b^{-1}\right) = \phi(a)\,\phi(b)^{-1} = ab^{-1}$$

So $ab^{-1} \in \text{Fix}(\phi)$. If we instead use the second test: Let $a, b \in \text{Fix}(\phi)$

$$\phi(ab) = \phi(a)\,\phi(b) = ab$$

so $ab \in \text{Fix}(\phi)$.

$$\phi\left(a^{-1}\right) = \phi(a)^{-1} = a^{-1}$$

so $a^{-1} \in \text{Fix}\left(a^{-1}\right)$.

8.

No comment