Class 10

February 8, 2024

Fixed Elements

Another way we might verify we have a subgroup is to look at fixed elements. Consider a permutation α. If element i moves to position j, we say $\alpha(i) = j$. If $\alpha(i) = i$, we say α fixes i.

Ex 1

Consider

$$\alpha = (234) \in S_5$$

Here,

$$\alpha(1) = 1, \alpha(2) = 3, \alpha(3) = 4, \alpha(4) = 2, \alpha(5) = 5$$

Thus α fixes 1 and 5. This can help us, as any set where each permutation fixes an element are quicker to check.

Ex 2

Let $H = \{\alpha \in S_4 | \alpha(4) = 4\} \leq S_4$. To show this is true, note that every element except 4 gets permuted. As 3 elements are being permuted, this implies $H = S_3$ (well, H and S_3 contain the same elements).

Ex 3

This can be generalized to show $S_1 \leq S_2 \leq \ldots \leq S_{n-1} \leq S_n$. To show $S_i \leq S_j$ for all $i \leq j$, permute the first i numbers, but leave the last $j - i$ numbers fixed. To do so, we will introduce the concept of a stabilizer.

Stabilizer

If we consider subsets of a permutation group G, (Not necessarily S_n), then the permutations that leave element a fixed is called the stabilizer of a. We call this group

$$\mathrm{Stab}_G(a) = \{\alpha \in G | \alpha(a) = a\}$$

Let's prove this forms a subgroup of G.

pf

Let $\alpha, \beta \in \mathrm{Stab}_G(a)$. This implies $\alpha(a), \beta(a) = a$, thus $\alpha(\beta(a)) = \alpha(a) = a$ (If both permutations fix a, so does their product). Now, consider $\alpha^{-1}(a)$. As α, both fix a, then $\alpha^{-1}(a) = \alpha^{-1}(\alpha(a)) = e(a) = a$, thus α^{-1} fixes a. Thus $\alpha^{-1} \in \mathrm{Stab}_G(a)$, thus $\mathrm{Stab}_G(a)$ forms a subgroup of G.

Ex 4

This idea of a stabilizer can be extended to symmetry groups of objects. Consider the symmetry group of a tetrahedron. The set of all rotations of the vertices that leave vertex 1 fixed form the $\mathrm{Stab}_{A_4}(1) = \{(1), (234), (243)\}$.

Orbits

The orbit of an element a in permutation group G is the set containing all of the locations element a visits. Formally,

$$\mathrm{Orb}_G(a) = \{\alpha(a) \in G\}$$

Ex 5

In the symmetry group of a tetrahedron, vertex 1 will visit vertices $1, 2, 3, 4$ in some rotation, thus $\mathrm{Orb}_{A_4}(1) = \{1, 2, 3, 4\}$.

Orbit-Stabilizer Theorem

When trying the size of a permutation group G, rather than counting each permutation explicitly, we can instead count the size of the orbit and stabilizer of any particular element, then for $a \in G$:

$$|G| = |\mathrm{Stab}_G(a)| * |\mathrm{Orb}_G(a)|$$

Here, note that $|G|$ is the number of elements of a group G or the number of elements in a set G. We call $|G|$ the order of G if G is a group.

Ex 6

We know that the symmetry group of a tetrahedron is A_4. Recall $|A_4| = \frac{4!}{2} = 12$. If we instead use orbit-stabilizer theorem, with G as the symmetry group of the tetrahedron, and using vertex 1 as the element, then

$$|G| = |\mathrm{Orb}_G(1)| * |\mathrm{Stab}_G(1)| = 4 * 3 = 12$$

Oftentimes it is hard to count the symmetries of a group by finding each symmetry explicitly. Orbit stabilizer can help make life easier. In class, we used orbit-stabilizer to count the rotational symmetries of each of the DnD dice (platonic solids and D10). This is good practice, and I suggest doing this for other solids. How might the orbit and stabilizer of these shapes change if we include the reflectional symmetries? What about other solids (such as the Archimedean solids)?

Subgroups of S_n

It isn't always clear when a set of permutations forms a subgroup of S_n. If you use the subgroup tests, you can verify when a given set is a subgroup. Usually, it is easiest to see if there is a commonality between the permutations in the set.

Ex 7

Consider $H = \{(1), (12), (34), (12)(34)\} \leq S_4$. This is a subgroup. This can be verified by explicitly checking each product of permutations in H. Notice H is closed.

$$(1)(1) = (1),\ (12)(1) = (12),\ (34)(1) = (34).\ (12)(34)(1) = (12)(34)$$

$$(1)(12) = (12),\ (12)(12) = (1),\ (34)(12) = (12)(34),\ (12)(34)(12) = (34)$$

$$(1)(34) = (34),\ (12)(34) = (12)(34),\ (34)(34) = (1),\ (12)(34)(34) = (12)$$

$$(1)(12)(34) = (12)(34),\ (12)(12)(34) = (34),\ (34)(12)(34) = (12),\ (12)(34)(12)(34) = (1)$$

Notice in each row there is the identity permutation (1), and that each permutation is its own inverse. As such, H is a subgroup. With larger H, checking every case gets unwieldy very quickly. One way we might speed process up is to note that the product of disjoint cycles commutes. Furthermore, we might note that each 2-cycle is it's own inverse. Note that $(12), (34)$ 2-cycles, and are disjoint. This implies that $(12)(34)$ will be in H, and will also be its own inverse. We can extend this idea further, for larger n.

Ex 8

Consider $K = \{(12), (34), (56), (78)\} \subseteq S_8$. Notice that K is not a subgroup as $(1) \notin K$. If we use $H = \langle k | k \in K \rangle$, then H is a subgroup of K. As each of the cycles in K is disjoint, we can see if every possible product of some number of permutations is in a subset. If we take all possible products, we get

$$(1), (12), (34), (56), (78), (12)(34), (12)(56), (12)(78), (34)(56), (34)(78), (56)(78)$$

$$(12)(34)(56), (12)(34)(78), (12)(56)(78), (34)(56)(78), (12)(34)(56)(78)$$

If instead we were given each of these permutations in a set, we could work backwards and see we have all the permutations generated by elements in K.

Cayley's Theorem

It turns out that for any group G with n elements, $G \leq S_n$. This allows us to represent any group as a set of permutations on n elements. Informally this can be shown by considering each row of a multiplication table as a permutation on n elements. As each group admits a multiplication table, then each group is a subgroup of S_n. This is known as Cayley's theorem.

Ex 9

Consider the group $G = D_3$. The multiplication table for C_5 looks like the table below:

D3

*	e	r	r2	f	fr	fr2
e	e	r	r2	f	fr	fr2
r	r	r2	e	fr2	f	fr
r2	r2	e	r	fr	fr2	f
f	f	fr	fr2	e	r	r2
fr	fr	fr2	f	r2	e	r
fr2	fr2	f	fr	r	r2	e

From this table, we can define a permutation corresponding to each element, by taking the permutation that appears in the row corresponding to that element.

$$\begin{array}{ccc} e & \rightarrow & (1) \\ r & \rightarrow & (132)(456) \\ r^2 & \rightarrow & (123)(465) \\ f & \rightarrow & (14)(25)(36) \\ fr & \rightarrow & (15)(26)(34) \\ fr^2 & \rightarrow & (16)(24)(35) \end{array}$$

If you combine all of these elements above, I claim that this product will result in one of the above elements (after appropriate simplification). If you wish to show this explicitly, it is good practice, but we will not do so here. e, is always the permutation corresponding to the first row. The permutations described in r, r^2 are inverses as

$$rr^2 \rightarrow (132)(456)(123)(465)$$

$$= (132)(456)(465)(123) = (132)\,e\,(123) = (132)(123) = e$$

Here, we used the property that the product of disjoint cycles commutes. Similarly, we can show f, fr, fr^2's permutations are their own inverses.

$$ff \rightarrow (14)(25)(36)(14)(25)(36)$$

$$= (14)(25)(36)(36)(25)(14) = (14)(25)\,e\,(25)(14)$$

$$= (14)(25)(25)(14) = (14)\,e\,(14) = (14)(14) = e$$

And similarly for $frfr$ and fr^2fr^2.