Math 320 Midterm Study Guide

February 8, 2023

The problems on this study guide will be similar to the problems on the exam. This study guide has 3 times as many problems as the exam, so there will be plenty of practice problems. Attempt a problem on your own before checking the answer key.

Set Theory

1)

Compute the following operations for sets $X = \{0, 1, 2\}, Y = \{1, 2, 3\},$ and $Z = \{0, 3\}$

1. $(X \cap Y) \cup Z$
2. $P(Y) \cap P(Z)$
3. $X \times Y$

2)

Show that for sets A, B, C, D the following holds:

$$A \cup (B \cap C \cap D) = (A \cup B) \cap (A \cup C) \cap (A \cup D)$$

You are welcome to use any previous result from homework or lecture.

Functions and Relations

3)

Show that congruence of triangles forms an equivalence relation.

4)

Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a function such that $f(x, y, z) = (x + y, y + 3z, 3z)$. Show that f is a bijection.

5)

Let X, Y be sets. We say X, Y are of the same cardinality if there exists a bijection $f : X \to Y$. We denote this as $|X| = |Y|$. Show that this notion of equal cardinality forms an equivalence relation.

Groups

6)

Suppose G is a group such that for all $a \in G$, $a^2 = e$. Show that G must be abelian.

7)

Show that the set of functions $X = \left\{t, \frac{1}{t}, -t, \frac{-1}{t}\right\}$ form a group under composition of functions.

Subgroups

8)

Suppose G is a group. The normalizer $N_G(S)$ of $S \subseteq G$ is defined as

$$N_G(S) = \{a \in G | asa^{-1} \in S \ \forall s \in S\}$$

Show that H is a subgroup of G.

9)

Find all of the subgroups of $\mathbb{Z}_8$, the group of integers mod 8 under addition.

10)

Draw a multiplication table for $U(11)$, the group of integers from 1 to 10 under multiplication mod 11.

11)

Draw a Cayley Digraph for the following groups:

a) D_5
b) C_7

12)

Draw a shape with the following symmetry group:

a) C_6
b) D_4

Permutations

13)

Compute the following permutations for $\alpha = (145)(723)(8147)$, $\beta = (23)(34)(651)$, $\gamma = (34567)(12)$. Be sure to write your final answer as the product of disjoint cycles.

a) $\alpha\beta$
b) $\beta\gamma$
c) $\alpha\gamma$

14)

Show that the product of two odd cycles is an even cycle.

Orbits and Stabilizers

15)

Compute the orbits and stabilizers of $1, 2, 3, 4$ in A_4.

16)

Count the number of symmetries of a dodecahedron using orbit-stabilizer theorem. Recall a dodecahedron has 12 regular pentagonal faces.

Isomorphism

17)

Let G, H be isomorphic groups, and let $f : G \to H$ be an isomorphism from G to H. Show the following:

a) f maps the identity of G to the identity of H
b) $(f(a))^{-1} = f\left(a^{-1}\right)$ for all $a \in G$
c) If a, b commute in G, then $f(a), f(b)$ commute in H
d) $f(a^n) = (f(a))^n$

18)

Show that if $G \approx H$, and G is cyclic, then H is cyclic.

19)

Show that the set of clockwise rotation matrices of the form $R_{12} = \left\{ \begin{pmatrix} \cos\left(\frac{n\pi}{6}\right) & \sin\left(\frac{n\pi}{6}\right) \\ -\sin\left(\frac{n\pi}{6}\right) & \cos\left(\frac{n\pi}{6}\right) \end{pmatrix} | n \in \{0, 1, 2, 3, \ldots, 11\} \right\}$ is isomorphic to $\mathbb{Z}_{12}$