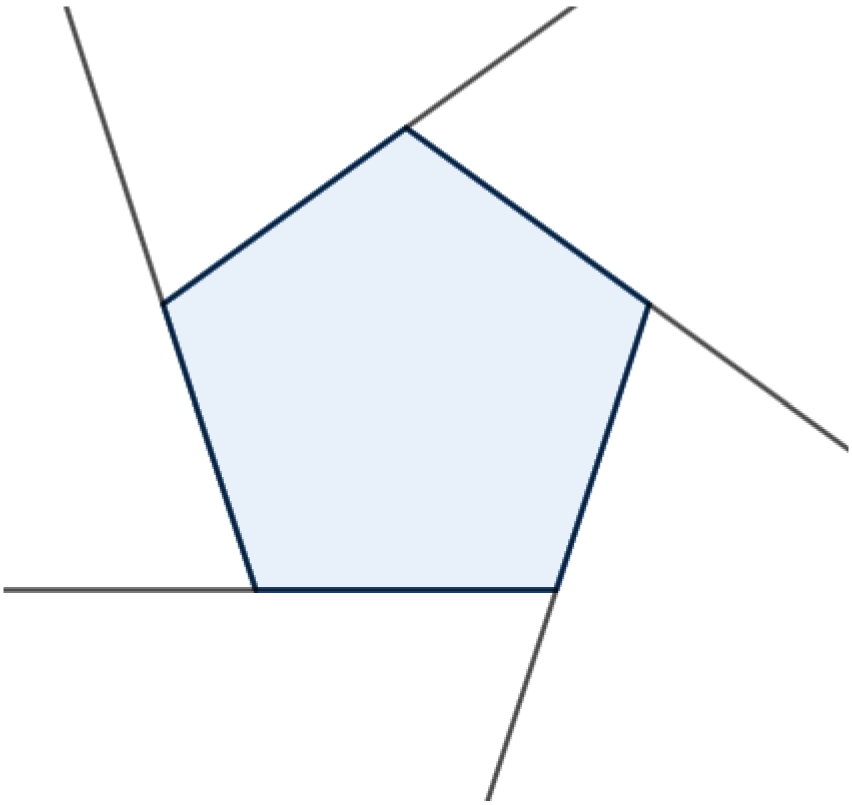
Math 320 Midterm Key

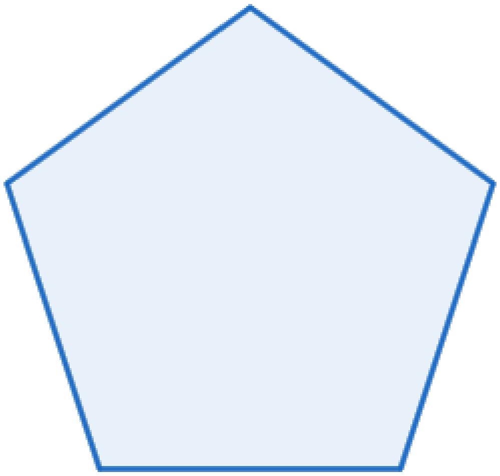
March 21, 2023

1.

a)



b)



2.

a)

$$\alpha\beta = (123)(578)(12)(53)(68) = (1)(257683)(4) = (257683)$$

b)

$$\beta\alpha = (12)(53)(68)(123)(578) = (137865)(2)(4) = (137865)$$

3.

As $U(7)$ is cyclic, we only need to consider the subgroups generated by each element of $U(7)$. These are

$$\langle 1 \rangle = \{1\}$$

$$\langle 2 \rangle = \{1, 2, 4\}$$

$$\langle 3 \rangle = \{1, 3, 2, 6, 4, 5\}$$

$$\langle 4 \rangle = \{1, 4, 2\}$$

$$\langle 5 \rangle = \{1, 5, 4, 6, 2, 3\}$$

$$\langle 6 \rangle = \{1, 6\}$$

Some of these are the same. As such, we get the following subgroups of $U(7)$:

$$\langle 1 \rangle, \langle 6 \rangle, \langle 2 \rangle, \langle 3 \rangle$$

4.

There are a few different ways this can be shown. If we show this directly, we need to show that

1. f is onto
2. f is one-to-one

1) Let $(x', y') \in \mathbb{R}^2$. Then we need to show $\exists\,(x, y) \in \mathbb{R}^2$ such that

$$f(x, y) = (x', y')$$

$$(x + y, x - y) = (x', y')$$

Thus

$$x + y = x', \; x - y = y'$$

Adding both equations:

$$2x = x' + y'$$

subtracting both equations:

$$2y = x' - y'$$

thus

$$x = \frac{x' + y'}{2}, \; y = \frac{x' - y'}{2}$$

and f is onto.
2) Now, let's show f is one-to-one. Suppose $f(x, y) = f(z, w)$. Then

$$f(x, y) = (x + y, x - y) = (z + w, z - w) = f(z, w)$$

Thus

$$x + y = z + w, \; x - y = z - w$$

Adding both equations:

$$2x = 2z$$

subtracting both equations:

$$2y = 2w$$

Thus

$$x = z, \; y = w$$

so f is one-to-one. As f is both onto and one-to-one, then f is bijective. For a second method, recall that a linear transformation is a bijection if the matrix representing the transformation has a non-zero determinant. As f is a linear transformation, we can equivalently define f as

$$f(x, y) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

as

$$\text{Det}\left(\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right) = 1 * -1 - 1 * 1 = -2 \neq 0$$

thus f is a bijection.

5.

a)

There are a few possible responses here that are valid. Here are some of them:

1. A subgroup is a group contained entirely within another group, under the same binary operation.
2. Test 1: $H \leq G$ if for all $a, b \in H \implies ab^{-1} \in H$
3. Test 2: $H \leq G$ if for all $a, b \in H \implies ab \in H$ and $a^{-1} \in H$

b)

Using test 1:
Let $x, y \in C(a)$. Then $xa = ax, ay = ya$ thus

$$y^{-1}ayy^{-1} = y^{-1}yay^{-1}$$

$$y^{-1}a = ay^{-1}$$

so

$$\left(xy^{-1}\right)a = xy^{-1}a = xay^{-1} = axy^{-1} = a\left(xy^{-1}\right)$$

Thus $xy^{-1} \in H$, so $H \leq G$.
Using test 2:
Let $x, y \in C(a)$. Then $xa = ax, ay = ya$ thus

$$(xy)a = xya = xay = axy = a(xy)$$

so $xy \in H$ and

$$xa = ax$$

$$x^{-1}xax^{-1} = x^{-1}axx^{-1}$$

$$ax^{-1} = x^{-1}a$$

thus $x^{-1} \in H$, so $H \leq G$.

6.

This is an iff statement. We need to show that both statements imply each other. Starting with G is abelian, let's show $(ab)^n = a^n b^n \ \forall n \geq 2$ by induction. Base case: let $n = 2$. Then

$$(ab)^2 = abab = a(ba)b = a(ab)b = aabb = a^2b^2$$

Now, for the induction step, suppose the claim $(ab)^k = a^k b^k$ holds for all k such that $2 \leq k \leq n$. Then

$$(ab)^n = a^n b^n$$

and

$$(ab)^{n+1} = (ab)^n (ab) = a^n b^n (ab) = a^n (b^n a) b = a^n (ab^n) b = a^n ab^n b = a^{n+1} b^{n+1}$$

as the base case and induction step hold, this completes our proof by induction. Now, let's show $(ab)^n = a^n b^n \ \forall n \geq 2$ implies G is abelian. Take $n = 2$. Then

$$(ab)^2 = a^2b^2$$

$$abab = a^2b^2$$

$$a^{-1}ababb^{-1} = a^{-1}a^2b^2b^{-1}$$

$$ba = ab$$

Thus G is abelian. As G is abelian implies $(ab)^n = a^nb^n \; \forall n \geq 2$ and $(ab)^n = a^nb^n \; \forall n \geq 2$ implies G is abelian, then we have shown G is abelian iff $(ab)^n = a^nb^n \; \forall n \geq 2$.

7.

a)

A relation R on a set S is an equivalence relation if it satisfies the following properties for all $x, y, z \in S$:

1. xRx (Reflexivity)
2. $xRy \implies yRx$ (Symmetry)
3. $xRy, yRz \implies xRz$ (Transitivity)

b)

This proof is copied from the class 11 notes.

To show that we have an equivalence relation, we need to show that for groups G, H, K

1. $G \approx G$ (Reflexivity)
2. $G \approx H \implies H \approx G$ (symmetry)
3. $G \approx H, H \approx K \implies G \approx K$ (transitivity)

1) Consider the identity function $\phi : G \to G$ such that $\phi(a) = a$. From a previous class, we know that the identity function is a bijection. As $\phi(ab) = ab = \phi(a)\phi(b)$, then ϕ is an isomorphism and $G \approx H$.

2) Suppose $G \approx H$. Then $\exists \phi : G \to H$ such that ϕ is an isomorphism. As ϕ is an isomorphism, then ϕ is bijective, thus ϕ^{-1} is bijective. As ϕ is an isomorphism, then $\phi(ab) = \phi(a)\phi(b)$. Applying ϕ^{-1} to both sides,

$$\phi^{-1}(\phi(a))\,\phi^{-1}(\phi(b)) = ab = \phi^{-1}(\phi(ab)) = \phi^{-1}(\phi(a)\phi(b))$$

As $\phi(a), \phi(b)$ are arbitrary elements of H, this combined with bijection implies ϕ^{-1} is an isomorphism from $H \to G$, thus $H \approx G$.

3) Suppose $G \approx H, H \approx K$. Then $\exists$ isomorphisms $\phi : G \to H, \theta : H \to K$. As ϕ, θ are isomorphisms, then both ϕ, θ are both bijections. Thus $\theta \circ \phi$ is a bijection. As ϕ, θ are isomorphisms, then

$$\theta(\phi(ab)) = \theta(\phi(a)\phi(b)) = \theta(\phi(a))\,\theta(\phi(b))$$

Thus $\theta \circ \phi$ is an isomorphism and $G \approx K$. As all three conditions are met, then isomorphism forms an equivalence relation.