Class 15

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Cosets

Cosets allow us to split the elements of a group into equal sized subsets. Suppose we have a group G with subset H. Then we define a left coset of H in G containing a as

$$aH = \{ah|h \in H\}$$

where $a \in G$. Similarly, we define the right coset of H in G containing a as

$$Ha = \{ha|h \in H\}$$

For coset aH or Ha, we call a the coset representative of H. We can also define the coset

$$aHa^{-1} = \left\{aha^{-1}|h \in H\right\}$$

We will also use the notation $|aH|$ to represent the number of elements in aH.

Ex 1

Let $H = \{e, r\}, G = D_3$. Then the possible left cosets are

$$eH, rH, r^2H, fH, frH, fr^2H$$

Let's compute each of these

$$eH = \{ee, er\} = \{e, r\}$$

$$rH = \{re, rr\} = \left\{r, r^2\right\}$$

$$r^2H = \left\{r^2e, r^2r\right\} = \left\{r^2, e\right\}$$

$$fH = \{fe, fr\} = \{f, fr\}$$

$$frH = \{fre, frr\} = \left\{fr, fr^2\right\}$$

$$fr^2H = \left\{fr^2e, fr^2r\right\} = \left\{fr^2, f\right\}$$

Similarly, for the right cosets, we have possible right cosets

$$He, Hr, Hr^2, Hf, Hfr, Hfr^2$$

Computing each

$$He = \{ee, re\} = \{e, r\}$$

$$Hr = \{er, rr\} = \{r, r^2\}$$

$$Hr^2 = \{er^2, rr^2\} = \{r^2, e\}$$

$$Hf = \{ef, rf\} = \{f, fr^2\}$$

$$Hfr = \{efr, rfr\} = \{fr, f\}$$

$$Hfr^2 = \{efr^2, rfr^2\} = \{fr^2, fr\}$$

Note that it is possible for $aH = Ha$ or for $aH \neq Ha$, depending on H and a.

Ex 2

This time, suppose we take $H = \{e, f\}$ and still keep $G = D_3$. Then we can compute the left and right cosets once again. First for the left cosets:

$$eH = \{ee, ef\} = \{e, f\}$$

$$rH = \{re, rf\} = \{r, fr^2\}$$

$$r^2H = \{r^2e, r^2f\} = \{r^2, fr\}$$

$$fH = \{fe, ff\} = \{f, e\}$$

$$frH = \{fre, frf\} = \{fr, r^2\}$$

$$fr^2H = \{fr^2e, fr^2f\} = \{fr^2, r\}$$

Notice that some of these cosets are the same. In fact,

$$eH = fH = \{e, f\}$$

$$rH = fr^2H = \{fr^2, r\}$$

$$r^2H = frH = \{r^2, fr\}$$

Similarly, if we compute the right cosets:

$$He = \{ee, fe\} = \{e, f\}$$

$$Hr = \{er, fr\} = \{r, fr\}$$

$$Hr^2 = \{er^2, fr^2\} = \{r^2, fr^2\}$$

$$Hf = \{ef, ff\} = \{f, e\}$$

$$Hfr = \{efr, ffr\} = \{fr, r\}$$

$$Hfr^2 = \{efr^2, ffr^2\} = \{fr^2, r^2\}$$

Once again, we see that some of the right cosets are the same as each other.

$$He = Hf = \{f, e\}$$

$$Hr = Hfr = \{r, fr\}$$

$$Hr^2 = Hfr^2 = \{fr^2, r^2\}$$

In general, we can can show this holds whenever H is a subgroup. In fact, we can make much stronger claims about cosets when H is a subgroup of G.

Thm

Let $H \leq G$, $a, b \in G$. Then

1. $a \in aH$
2. $aH = H$ iff $a \in H$
3. $(ab) H = a (bH)$
4. $aH = bH$ iff $a \in bH$
5. $aH = bH$ or $aH \cap bH = \emptyset$
6. $aH = bH$ iff $a^{-1}b \in H$
7. $|aH| = |bH|$
8. $aH = Ha$ iff $H = aHa^{-1}$
9. $aH \leq G$ iff $a \in H$

Equivalent theorems hold for the right cosets. In particular, note that the cosets of H will partition the elements of G if H is a subgroup.

Pf

1) As $H \leq G$, then $e \in H$ thus

$$a = a * e \in aH$$

2) First, suppose $a \in H$. Then $ah \in H$ as H is a subgroup and is closed, thus $aH \subseteq H$. Now, suppose $h \in H$. Then $ah \in H$, as H is closed, so $aH \subseteq H$, thus $aH = H$. Suppose $a \notin H$. Then $a \in aH$ by **1)**, but since $a \notin H$ then $aH \not\subseteq H$, thus $aH \neq H$.

3) Let $(ab) h \in (ab) H$. Then $(ab) h = a (bh) \in a (bH)$, thus $(ab) H \subseteq a (bH)$. Similarly, let $a (bh) \in a (bH)$. Then $a (bh) = (ab) h \in (ab) H$, thus $a (bH) \subseteq (ab) H$, thus $a (bH) = (ab) H$.

4) Let $aH = bH$. As $a \in aH$ by **1)**, then $a \in aH = bH$. Now, suppose $a \in bH$. Then $a = bh$ for some $h \in H$, thus $aH = (bh) H = b (hH) = bH$ (as **2)** implies $hH = H$ for $h \in H$).

5) Suppose $aH \neq bH$ and $aH \cap bH \neq \emptyset$. Then $\exists x \in aH, bH$ such that

$$x = ah_1 = bh_2$$

for some $h_1, h_2 \in H$. Then

$$a = ah_1h_1^{-1} = bh_2h_1^{-1} = b \left(h_2h_1^{-1}\right) \in bH$$

as H is a group so $h_2h_1^{-1} \in H$. But by **4)**, this means $aH = bH$, a contradiction! Thus to avoid contradiction, we must have $aH = bH$ or $aH \cap bH = \emptyset$. Further, as H is non-empty, this implies that $\{aH | a \in G\}$ will partition G.

6) Suppose $aH = bH$. Then

$$H = eH = a^{-1}aH = a^{-1}bH$$

But as

$$\left(a^{-1}b\right) H = H$$

then by **2)**, $a^{-1}b \in H$. Now, suppose $a^{-1}b \in H$. Then by **2)**,

$$a^{-1}bH = H$$

so

$$bH = aa^{-1}bH = aH$$

7) To show these sets have the same cardinality, we need to find a bijection between them. Let $f : aH \to bH$ given by

$$f (ah) = bh$$

for all $h \in H$. First, let's show f is onto. Let $bh \in bH$. Then

$$f(ah) = bh$$

Now, to show this function is one-to-one. Suppose

$$f(ah_1) = f(ah_2)$$

Then

$$bh_1 = bh_2$$

$$b^{-1}bh_1 = h_1 = h_2 = b^{-1}bh_2$$

Thus

$$ah_1 = ah_2$$

As f is a bijection, then $|aH| = |bH|$.
8) Suppose $aH = Ha$. Then

$$aHa^{-1} = Haa^{-1} = H$$

Now, suppose $H = aHa^{-1}$. Then

$$Ha = aHa^{-1}a = aH$$

9) Suppose $a \in H$. Then by **2)**, $aH = H$. But as we already know $H \leq G$, then $H = aH \leq G$. Now, suppose $a \notin H$. Then by **4)**, we know

$$eH = H \neq aH$$

as $a \notin H$. But by **4)**, this also implies $e \notin aH$. As aH does not contain an identity element, it cannot be a group (and therefore, $aH \nleq G$).

Normal Subgroups

A subgroup $H \leq G$ is a normal subgroup (denoted $H \trianglelefteq G$) if $aH = Ha$ for all $a \in G$. In particular, by **8)**, we also get $H \trianglelefteq G$ if $aHa^{-1} = H$ or $a^{-1}Ha = H$ for all $a \in G$). If $H < G$ and $H \trianglelefteq G$, then we say $H \triangleleft G$.

Ex 3

If we use the subgroup from **Ex 2**, we see that H is not a normal subgroup, as

$$rH \neq Hr$$

Ex 4

Let $H = \{e, r, r^2\}, G = D_3$. Then the possible left cosets are

$$eH, rH, r^2H, fH, frH, fr^2H$$

These will give

$$H = eH = rH = r^2H = \{e, r, r^2\}$$

$$fH = frH = fr^2H = \{f * e, f * r, f * r^2\} = \{f, fr, fr^2\}$$

Using **4)** allows us to only explicitly compute a given coset once. Once we know the elements of fH, we immediately know the elements of frH and fr^2H. Now for the right cosets:

$$He, Hr, Hr^2, Hf, Hfr, Hfr^2$$

$$H = He = Hr = Hr^2 = \{e, r, r^2\}$$

$$Hf = Hfr = Hfr^2 = \{ef, rf, r^2f\} = \{f, fr^2, fr\} = \{f, fr, fr^2\}$$

Since the left and right cosets are the same, then we can say $H \trianglelefteq G$.

Ex 5

Suppose $H = 3\mathbb{Z}, G = \mathbb{Z}$. Then

$$0 + 3\mathbb{Z} = \pm 3 + 3\mathbb{Z} = \pm 6 + 3\mathbb{Z} = \ldots = \{0, \pm 3, \pm 6, \ldots\} = \{3n | n \in \mathbb{Z}\}$$

$$1 + 3\mathbb{Z} = 4 + 3\mathbb{Z} = -2 + 3\mathbb{Z} = \ldots = \{1, 4, , -2, 7, -5, \ldots\} = \{3n + 1 | n \in \mathbb{Z}\}$$

$$2 + 3\mathbb{Z} = 5 + 3\mathbb{Z} = -1 + 3\mathbb{Z} = \ldots = \{2, 5, -1, 8, -4, \ldots\} = \{3n + 2 | n \in \mathbb{Z}\}$$

Or for the right cosets:

$$3\mathbb{Z} + 0 = 3\mathbb{Z} \pm 3 = 3\mathbb{Z} \pm 6 = \ldots = \{0, \pm 3, \pm 6, \ldots\} = \{3n | n \in \mathbb{Z}\}$$

$$3\mathbb{Z} + 1 = 3\mathbb{Z} + 4 = 3\mathbb{Z} - 2 = \ldots = \{1, 4, , -2, 7, -5, \ldots\} = \{3n + 1 | n \in \mathbb{Z}\}$$

$$3\mathbb{Z} + 2 = 3\mathbb{Z} + 5 = 3\mathbb{Z} - 1 = \ldots = \{2, 5, -1, 8, -4, \ldots\} = \{3n + 2 | n \in \mathbb{Z}\}$$

And as the left and right cosets are the same for H, then

$$H \trianglelefteq G$$