Math 320 Midterm Study Guide Key

February 23, 2023

1)

1)

$$X \cap Y = \{0, 1, 2\} \cap \{1, 2, 3\} = \{1, 2\}$$

$$(X \cap Y) \cup Z = \{1, 2\} \cup \{0, 3\} = \{0, 1, 2, 3\}$$

2)

$$P(Y) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

$$P(Z) = \{\emptyset, \{0\}, \{3\}, \{0, 3\}\}$$

$$P(Y) \cap P(Z) = \{\emptyset, \{3\}\}$$

3)

$$X \times Y = \{0, 1, 2\} \times \{1, 2, 3\} = \{(0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}$$

2)

Recall that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. Also recall that $\cap$ is associative. Using these identities,

$$A \cup (B \cap C \cap D) = A \cup (B \cap (C \cap D)) = (A \cup B) \cap (A \cup (C \cap D))$$

$$= (A \cup B) \cap ((A \cup C) \cap (A \cup D)) = (A \cup B) \cap (A \cup C) \cap (A \cup D)$$

3)

To show this forms an equivalence relation, we need to show that congruence of triangles is **i)** Reflexive, **ii)** Symmetric, **iii)** Transitive.

i) Every triangle is congruent to itself, as every triangles will have the same side lengths and angles as itself.

ii) Suppose triangle A is congruent to triangle B. Then A has the same side lengths and angles as B, thus B has the same side lengths and angles as A. This implies B is also congruent to A.

iii) Suppose triangle A is congruent to triangle B and triangle B is congruent to triangle C. Then A, B share the same side lengths and angles, and B, C share the same side lengths and angles, thus A, C share the same side lengths and angles so A and C are congruent.

As all three properties are satisfied, then congruence of triangles is an equivalence relation.

4)

To show f is a bijection, we need to show f is onto and one-to-one.

onto: Let $(a, b, c) \in \mathbb{R}^3$ such that $f(x, y, z) = (a, b, c)$ for some x, y, z. This yields system of equations:

$$x + y = a$$

$$y + 3z = b$$

$$3z = c$$

If we divide the bottom equation by 3, we see

$$z = \frac{c}{3}$$

If we subtract equation 3 from equation 2, we see

$$y + 3z - 3z = b - c$$

$$y = b - c$$

If we substitute our result for y into equation 1, we get

$$x + (b - c) = a$$

$$x = a - b + c$$

Thus

$$(x, y, z) = \left(a - b + c, b - c, \frac{c}{3}\right)$$

and f is onto.

one-to-one: Now, suppose $\exists (x_1, y_1, z_1), (x_2, y_2, z_2)$ such that $f(x_1, y_1, z_1) = f(x_2, y_2, z_2)$. Then

$$f(x_1, y_1, z_1) = f(x_2, y_2, z_2)$$

$$(x_1 + y_1, y_1 + 3z_1, 3z_1) = (x_2 + y_2, y_2 + 3z_2, 3z_2)$$

Then, from the last coordinate:

$$3z_1 = 3z_2$$

$$z_1 = z_2$$

From the second coordinate:

$$y_1 + 3z_1 = y_2 + 3z_2$$

$$y_1 + 3z_1 = y_2 + 3z_1$$

$$y_1 = y_2$$

From the first coordinate:

$$x_1 + y_1 = x_2 + y_2$$

$$x_1 + y_1 = x_2 + y_1$$

$$x_1 = x_2$$

thus

$$(x_1, y_1, z_1) = (x_2, y_2, z_2)$$

and f is one-to-one. As f is both onto and one-to-one, then f is bijective.

5)

To show this forms an equivalence relation, we need to show that this relation is **i)** Reflexive, **ii)** Symmetric, **iii)** Transitive.

i) Every set A has a bijection $f : A \to A$ given by $f(a) = a$. This is the identity map that maps every element of A to itself, thus $|A| = |A|$

ii) Suppose A has the same cardinality as B. Then there exists a bijection $f : A \to B$. As f is a bijection, it is invertible and there exists a bijection $f^{-1} : B \to A$, thus $|A| = |B|$ implies $|B| = |A|$.

iii) Suppose $|A| = |B|$ and $|B| = |C|$. Then there exist bijections $f : A \to B$ and $g : B \to C$. The composition $g \circ f : A \to C$ will be a bijection as composition of bijections form a bijection, thus $|A| = |C|$.

As all three properties are satisfied, then equality of cardinality is an equivalence relation.

6)

Let $a, b \in G$. Then $ab = eabe = b^2aba^2 = bbabaa = b(ba)(ba)a = bea = ba$, thus G is abelian.

7)

To show this forms a group, we need to show that X is closed under composition of functions, X has an identity, each function as an inverse, and that composition of functions is associative.

Closure: We will prove this by exhaustion.

$$t \circ t = t,\ t \circ \frac{1}{t} = \frac{1}{t},\ t \circ -t = -t,\ t \circ -\frac{1}{t} = -\frac{1}{t}$$

$$\frac{1}{t} \circ t = \frac{1}{t},\ \frac{1}{t} \circ \frac{1}{t} = t,\ \frac{1}{t} \circ -t = -\frac{1}{t},\ \frac{1}{t} \circ -\frac{1}{t} = -t$$

$$-t \circ t = -t,\ -t \circ \frac{1}{t} = -\frac{1}{t},\ -t \circ -t = t,\ -t \circ -\frac{1}{t} = \frac{1}{t}$$

$$-\frac{1}{t} \circ t = -\frac{1}{t},\ -\frac{1}{t} \circ \frac{1}{t} = -t,\ -\frac{1}{t} \circ -t = \frac{1}{t},\ -\frac{1}{t} \circ -\frac{1}{t} = t$$

Identity: From the above compositions, we can see that t is an identity under function composition.

Inverses: From the above note that each function is its own inverse.

Associativity: Composition of functions is associative, as we previously showed in class.

As all conditions are satisfied, then X forms a group under function composition.

8)

Using subgroup test 2, we need to show $N_G(S)$ is closed and for each $a \in N_G(S) \implies a^{-1} \in N_G(S)$.

Closure: Suppose $a, b \in N_G(S), s \in S$. Then $bsb^{-1} = t$ for some $t \in S$, thus $(ab)s(ab)^{-1} = absb^{-1}a^{-1} = ata^{-1} \in S$, as $t \in S$, thus $N_G(S)$ is closed.

Inverses: Suppose $a \in N_G(S), s \in S$. Then $asa^{-1} = t$ for some $t \in S$. Let's show for a given t, there is exactly one s such that $asa^{-1} = t$. Suppose this was not the case. Then $as_1a^{-1} = as_2a^{-1} = t$, thus

$$a^{-1}as_1a^{-1}a = a^{-1}as_2a^{-1}a$$

$$s_1 = s_2$$

As each s maps to a unique t, and $asa^{-1} \in S$ for all $s \in S$, then every possible $t \in S$ should be mapped to by some $s \in S$ as $s = a^{-1}ta$. Thus $a^{-1} \in N_G(S)$.

9)

Since $\mathbb{Z}_8$ is cyclic, then we know each of its subgroups must also be cyclic. Lets find each of the subgroups generated by the elements of $\mathbb{Z}_8$:

$$\langle 0 \rangle = \{0\}$$

$$\langle 1 \rangle = \{0, 1, 2, 3, 4, 5, 6, 7\}$$

$$\langle 2 \rangle = \{0, 2, 4, 6\}$$

$$\langle 3 \rangle = \{0, 1, 2, 3, 4, 5, 6, 7\}$$

$$\langle 4 \rangle = \{0, 4\}$$

$$\langle 5 \rangle = \{0, 1, 2, 3, 4, 5, 6, 7\}$$

$$\langle 6 \rangle = \{0, 2, 4, 6\}$$

$$\langle 7 \rangle = \{0, 1, 2, 3, 4, 5, 6, 7\}$$

Thus all of the subgroups of $\mathbb{Z}_8$ are $\langle 0 \rangle, \langle 4 \rangle, \langle 2 \rangle, \langle 1 \rangle$.

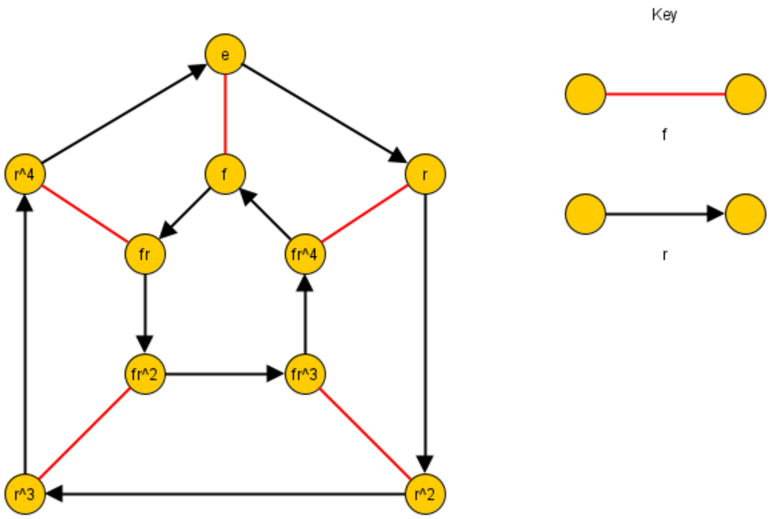
10)

U(11)										
*	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	1	3	5	7	9
3	3	6	9	1	4	7	10	2	5	8
4	4	8	1	5	9	2	6	10	3	7
5	5	10	4	9	3	8	2	7	1	6
6	6	1	7	2	8	3	9	4	10	5
7	7	3	10	6	2	9	5	1	8	4
8	8	5	2	10	7	4	1	9	6	3
9	9	7	5	3	1	10	8	6	4	2
10	10	9	8	7	6	5	4	3	2	1

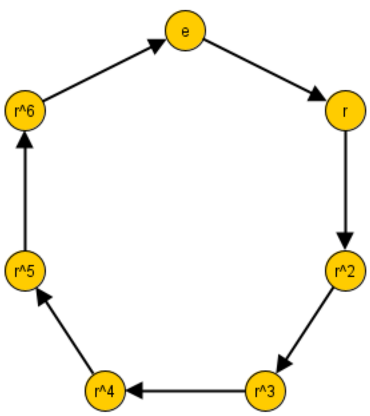
11)

Note that these digraphs need not be unique. I also chose to label the nodes, but the nodes do not need to be labeled for a Cayley Digraph to be valid.

a)



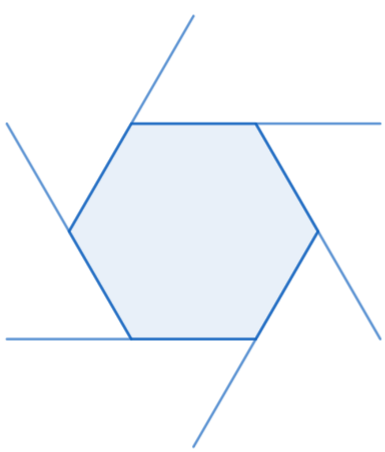
b)



12)

Note that these answers are not unique:

a)



b)

13)

a)

$$\alpha\beta = (145)(723)(8147)(23)(34)(651)$$

$$= (174)(2)(3865) = (174)(3865)$$

b)

$$\beta\gamma = (23)(34)(651)(34567)(12)$$

$$= (173)(25)(4)(6)(8) = (173)(25)$$

c)

$$\alpha\gamma = (145)(723)(8147)(34567)(12)$$

$$= (1382467)(5) = (1382467)$$

14)

Suppose α, β are odd cycles. This means that $\alpha = C_{\alpha 1}...C_{\alpha n}$, where n is odd, and each $C_{\alpha i}$ is a 2-cycle. Similarly, $\beta = C_{\beta 1}...C_{\beta m}$, where m is odd, and each $C_{\alpha i}$ is a 2-cycle. Thus

$$\alpha\beta = C_{\alpha 1}...C_{\alpha n}C_{\beta 1}...C_{\beta m}$$

As n, m are both odd, $n + m$ is even and $\alpha\beta$ is a product of an even number of 2-cycles, thus $\alpha\beta$ is even.

15)

Let $X = \{1, 2, 3, 4\}$. Recall A_4 will only contain even cycles. For 4 elements, this corresponds to any permutations with 0 disjoint 2-cycles or 2 disjoint 2-cycles. This gives us

$$A_4 = \{(1), (12)(34), (13)(24), (14)(23), (123), (132), (124), (142), (134), (143), (234), (243)\}$$

First, for the orbits:

$$\text{Orb}(1) = \{1, 2, 3, 4\}$$

$$\text{Orb}(2) = \{1, 2, 3, 4\}$$

$$\text{Orb}(3) = \{1, 2, 3, 4\}$$

$$\text{Orb}(4) = \{1, 2, 3, 4\}$$

These can be found by looking at the first 4 permutations. Note that each element of X will be in a 2-cycle with every other element of X, and every element must be in its own orbit, as the identity permutation is in A_4. To find the stabilizers, we look for the permutations that fix a given element. These are the permutations that do not contain the element in any cycles of length 2 or more.

$$\text{Stab}(1) = \{(1), (234), (243)\}$$

$$\text{Stab}(2) = \{(1), (134), (143)\}$$

$$\text{Stab}(3) = \{(1), (124), (142)\}$$

$$\text{Stab}(4) = \{(1), (123), (132)\}$$

16)

Let G be the group of symmetries of the dodecahedron. First, let's pick an arbitrary face of the dodecahedron. Let's call this face 1. There are 12 faces on the dodecahedron, so the dodecahedron could be rotated to put 1 into any of the locations of the other faces, thus

$$|\text{Orb}(1)| = 12$$

Each face of the dodecahedron is a pentagon, so this pentagon can be rotated 5 times about 1, yielding 5 permutations of the faces that leave 1 fixed. This implies

$$|\text{Stab}(1)| = 5$$

Thus by orbit-stabilizer theorem,

$$|G| = |\text{Orb}(1)|\,|\text{Stab}(1)| = 12 * 5 = 60$$

17)

a)

$$f(e) = f(e * e) = f(e) f(e)$$

As the identity is unique, this implies $f(e)$ must be the identity in H.
b)

$$f(a) f\left(a^{-1}\right) = f\left(aa^{-1}\right) = f(e)$$

By **a)**, $f(e)$ is the identity in H, thus $f\left(a^{-1}\right) = (f(a))^{-1}$
c)

$$f(a) f(b) = f(ab) = f(ba) = f(b) f(a)$$

d)
$n = 1$ is trivial. We will prove the rest using induction. If $n = 2$, then

$$f\left(a^2\right) = f(aa) = f(a) f(a) = f(a)^2$$

Now, for the inductive step. Suppose $f(a^n) = f(a)^n$ for all n such that $2 \leq n \leq k$. Then

$$f\left(a^{k+1}\right) = f\left(a^k\right) f(a) = f(a)^k f(a) = f(a)^k$$

18)

By **d)**, we know $f(a^n) = f(a)^n$. Suppose $f(b) \in H$. Then $b = a^n$ for some $n \in \mathbb{Z}$. Thus

$$f(b) = f(a^n) = f(a)^n$$

19)

Let $A^n = \begin{pmatrix} \cos\left(\frac{n\pi}{6}\right) & \sin\left(\frac{n\pi}{6}\right) \\ -\sin\left(\frac{n\pi}{6}\right) & \cos\left(\frac{n\pi}{6}\right) \end{pmatrix}$. Then let $f(A^n) = n$ be an isomorphism from $R_{12} \to \mathbb{Z}_{12}$. We will show that f is an isomorphism, thus $R_{12} \approx \mathbb{Z}_{12}$. To show this, we need to show $f(AB) = f(A)f(B)$, and that f is bijective. Let $A^n, A^m \in R_{12}$. Then

$$f(A^n A^m) = f\left(A^{n+m}\right) = f\left(A^{n+m_{mod\ 12}}\right) = n + m_{mod\ 12} = n + m = f(A^n) + f(A^m)$$

Note that we used a $+$ as our binary operation in $\mathbb{Z}_{12}$. Now, to show f is bijective. First, let's show f is one-to-one. Suppose $f(A^n) = f(A^m)$. Then

$$f(A^n) = n = m = f(A^m)$$

Thus

$$A^n = A^m$$

Now, to show f in onto. Let $n \in \mathbb{Z}_{12}$. Then

$$f(A^n) = n$$

and as $A^n \in R_{12}$ for all $n \in \mathbb{Z}_{12}$, then f is onto. As f is both one-to-one and onto, f is bijective. As all of the conditions are satisfied, then f is an isomorphism and $R_{12} \approx \mathbb{Z}_{12}$.