Math 320 Take Home Final

April 29, 2024

You will have from the start of class until midnight on May 1st to work on and submit the take home final. Please clearly number the start of a problem for your scratch work and for your solution. You are welcome to use any class notes and previous homework assignments when working on this exam. If you use any other resources (which is discouraged, but still allowed), please cite your sources.

Computation

Each of these questions will be worth 10 points. Show your work to receive full credit.

1.

Write the elements of $\mathbb{Z}_3 \times D_3$.

2.

Let $G = \mathbb{Z}, H = 4\mathbb{Z}$. Find all of the cosets in G/H.

3.

Let $G = (\mathbb{R}^+, *)$ be the group of positive real numbers under multiplication. Show $\phi : G \to G$ given by $\phi(x) = \sqrt{x}$ is an automorphism.

4.

For $R = \mathbb{C}[x]$. Find $q(x), r(x)$ such that $2x^2 + ix + 1 - i = (x+1)q(x) + r(x)$ and such that $\deg(r(x)) = 0$.

5.

Let $x^3 + 4x + 3, 4x^3 + 2x^2 + 1 \in \mathbb{Z}_7[x]$. Compute the product

$$\left(x^3 + 4x + 3\right)\left(4x^3 + 2x^2 + 1\right)$$

Proofs

Your lowest scoring response to the next 6 questions will be dropped. Each question is worth 10 points.

6.

Consider the set of even real-valued functions $E = \{f : \mathbb{R} \to \mathbb{R} | f(-x) = f(x)\}$. Let $f, g \in E$ and define addition

$$(f+g)(x) = f(x) + g(x)$$

and multiplication

$$(f * g)(x) = f(x)\, g(x)$$

Show $(E, +, *)$ forms a ring.

7.

a) Consider the integral domain $\mathbb{Q}[x]$. Show

$$\left\langle x^2 - n \right\rangle = \left\{ r(x) \left(x^2 - n \right) | r(x) \in \mathbb{Q}[x], n \in \mathbb{Z}^+ \right\}$$

is an ideal of $\mathbb{Q}[x]$.

b) Show $\mathbb{Q}[x] / \left\langle x^2 - n \right\rangle \approx \mathbb{Q}[\sqrt{n}]$ when n is not a perfect square.

8.

Consider $\mathbb{Q}[x]$ and the ideal $\left\langle x^2 - n \right\rangle$ from problem **7**a). If $n \in \mathbb{Z}^+$, then for what values of n will $\left\langle x^2 - n \right\rangle$ be

a) A prime ideal

b) A maximal ideal

(Note: n may be a perfect square in this problem.)

9.

Suppose $H = \{z \in \mathbb{C} \setminus \{0\} \,|\, |z| = 1\}$.

a) Show H is a normal subgroup of $G = (\mathbb{C} \setminus \{0\}, *)$ the group of non-zero complex numbers under multiplication (Recall $|ab| = |a|\,|b|$).

b) Show $G/H \approx (\mathbb{R}^+, *)$, the group of positive real numbers under multiplication.

10.

Let R be a commutative ring. An $x \in R$ is said to be nilpotent if $\exists n \in \mathbb{Z}^+$ such that $x^n = 0$. Show the set of all nilpotent elements of R form a subring of R.

11.

Let R be a ring. An element $a \in R$ is said to be idempotent if $a^2 = a$. Let $\phi : R \to S$ be a ring homomorphism. Show that $\phi(a)$ is idempotent in S if a is idempotent in R.

Bonus Question

If you answer this question, you will get 3 bonus points. If you do not wish to answer this question, write "no response" and you will still get full credit for this question.

12.

What is your favorite topic we covered in this semester?