Problem Set 6

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a) Show that $\sum_{x=0}^{N-1} \exp(-2\pi i kx/N) = \dfrac{1-\exp(-2\pi i k)}{1-\exp(-2\pi i k/N)}$

Recall geometric series: $\sum_{x=0}^{N} ar^x = a\dfrac{1-r^{N+1}}{1-r}$

Take $a = e^{-2\pi i k/N}$; then,

$$\sum_{x=0}^{N-1} e^{-2\pi i kx/N} = \sum_{x=0}^{N-1} a^x$$

$$= \frac{1-a^N}{1-a}$$

$$\boxed{\sum_{x=0}^{N-1} e^{-2\pi i k/N} = \frac{1-e^{-2\pi i k}}{1-e^{-2\pi i k/N}}}$$ ∎

b) Show this approaches N as $k \to 0$
& is equal to 0 for any integer k that is not a multiple of N

$\lim_{k\to 0}\left(\dfrac{1-e^{-2\pi i k}}{1-e^{-2\pi i k/N}}\right) \to$ goes towards $\dfrac{1-1}{1-1} = \dfrac{0}{0} \Rightarrow$ Use L'Hospital

$$= \lim_{k\to 0}\left(\frac{2\pi i e^{-2\pi i k}}{\frac{2\pi i}{N} e^{-2\pi i k/N}}\right)$$

$$= \frac{2\pi i}{\left(\frac{2\pi i}{N}\right)} \lim_{k\to 0}\left(\frac{e^{-2\pi i k}}{e^{-2\pi i k/N}}\right)^{1/1}$$

$$= N \cdot 1$$

$$\boxed{= N}$$ ∎

- For any integer k, $e^{-2\pi i k} = 1 \Rightarrow 1 - e^{-2\pi i k} = 1 - 1 = 0$

- For any integer k unequal to a multiple of N, $\frac{k}{N}$ is not an integer

  $\Rightarrow e^{-2\pi i k/N}$ is 1 only if $\frac{k}{N}$ is an integer,

  hence $e^{-2\pi i k/N}$ is never 1

  $\Rightarrow 1 - e^{-2\pi i k/N}$ is never zero

$$\therefore \frac{1 - e^{-2\pi i k}}{1 - e^{-2\pi i k/N}} = \frac{1-1}{\underbrace{A}_{\text{some constant}}} = \frac{0}{A} \boxed{= 0} \text{ as } A \text{ is never } 0$$

c)

Let's write down the DFT of a non-integer sine wave:

$$f(k') = \sum_{x=0}^{N-1} f(x)\, e^{-2\pi i k' x/N}$$

$$= \sum_{x=0}^{N-1} \sin(2\pi k x/N)\, e^{-2\pi i k' x/N}$$

$$= \sum_{x=0}^{N-1} \frac{e^{2\pi i k x/N} - e^{-2\pi i k x/N}}{2i}\, e^{-2\pi i k' x/N}$$

$$= \frac{1}{2i} \sum_{x=0}^{N-1} \left(e^{-2\pi i (k'-k)x/N} - e^{-2\pi i (k'+k)x/N}\right)$$

$$= \frac{1}{2i} \left(\frac{1 - e^{-2\pi i (k'-k)}}{1 - e^{-2\pi i (k'-k)/N}} - \frac{1 - e^{-2\pi i (k'+k)}}{1 - e^{-2\pi i (k'+k)/N}} \right)$$