Reading Questions Week #7

to: Prof. Andrew Cumming
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1) Before reading, give a definition of what you think chaos is, then update it as you go through the reading.

My definition of chaos is the behavior of a system whose evolution is highly sensitive to tiny fluctuations in its initial conditions (it notably makes me think of the weather and the butterfly effect).

Update: I was weirdly spot on... that's almost verbatim what the textbook said...

2) (a) How does the existence of the Jacobi constant reduce the dimensionality of phase space in the circular restricted three-body problem, and why is this reduction essential for constructing a Poincaré surface of section? (b) What's the physical meaning of a jacobi surface?

In phase space, there are 4 dimensions in which the particle can live. We showed previously that the Jacobi constant is a constant of motion for the restricted three-body problem, which relates the four quantities $(x, y, \dot{x}, \dot{y})$. Thus, the Jacobi constant makes it so that particles in the restricted three-body problem do not live anywhere in that 4 dimensional space; they live on a surface determined by the Jacobi constant C_J. The Jacobi surface describes the confined space where the particle evolves in all of phase space. It can be reduced and interpreted by looking at the intersection of this curve with different planes (e.g., $y = 0$).

3) What do the islands in Fig 9.5 symbolize physically and what happens when we place the starting condition further from the center of the island that straddles $\dot{x} = 0$?

The islands symbolize resonant motion, with a mean motion resonance of the form $p + q : p$, where p and q are integers, and q is the number of islands. If we place the starting condition further from the center of the island that straddles $\dot{x} = 0$, then the trajectory would three successive points at the center of each island.

4) What is the maximum Lyapounov characteristic exponent and what does it tell you?

The LCE is a measure of how do two nearby trajectories drift from each other over time. If two orbits have a separation d_0 in phase space at time t_0, and a separation d later at time t, then the LCE is $\gamma = \lim_{t\to\infty}(\ln(d/d_0)/(t - t_0))$.

5) What is problematic with modeling long term solar system dynamics with a perturbation expansion?

Traditionally, what is done is: write the Hamiltonian as the sum of a part that describes the independent Keplerian motion of the planets about the Sun plus a part called the disturbing function which accounts for pairwise interactions between the planets. One can generally expand the disturbing function in terms of the small parameters of the problem. But, for long-term behavior, one has to average the disturbing function over the mean motions of the planets (this results in the 'secular' part of the disturbing function). By including higher order terms though, the equations are no longer linear.

Another thing I found: If a Hamiltonian system has an integral of motion for each degree of freedom, then the system will be quasi-periodic: the orbits will then be confined to a multidimensional torus and the orbital elements should be describable by a sum of periodic terms. This would be true in the Solar System if the masses, eccentricities, and inclinations of its objects were sufficiently small. But, the real Solar System does not satisfy those requirements.

6) This review paper dates from 1993 and states that 'a detailed understanding of Pluto's behavior is likely to be o btained only with the next generation of simulations'. From you knowledge has this been done? (You can also quickly search this up).

From what I found: yes, this has been done, and it was showed that Pluto's orbit is chaotic but that there are no dramatic manifestations of that chaos on Gyr timescale. Source: CITA talk from 2022 https://www.cita.utoronto.ca/items/pluto-near-edge-chaos/

7) What is the general consensus of the numerical simulations mentioned in this chapter? Does the solar system exhibit long-term stability?

The latest simulations at the time integrated the evolution of the 5 outer planets for 100 Myr and the results suggested (but did not prove) that their orbits were chaotic. Round-off errors were also a problem at the time, which prevented robust predictions beyond 1 Gyr. Other studies also found that Pluto could be in a complicated system of three resonances, but that this outcome would be sensitive to the initial conditions. It sounds like they were on the verge (with future simulations) to be able to more robustly assess the chaotic nature of our Solar System, although many indicators seemed to show that it can be considered chaotic over $\sim 0.1 - 1$ Gyr time scale.