PHYS 641
Fall 2022

Assignment 1

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[1]

Show that a Poisson distribution converges to a Gaussian in the limit of large λ

- Poisson distribution is $$P_P = \frac{e^{-\lambda}\lambda^k}{k!}$$

for λ = expected # of events in an interval
k = actual # of events in an interval

So effectively, we would like to show that for large λ:
$$P_p \rightarrow \frac{e^{-(k-\lambda)^2/2\lambda}}{\sqrt{2\pi\lambda}}$$

- For large λ, $k \sim$ continuous, so we can express it as $k = \lambda(1+\delta)$ with a deviation from the mean $\delta << \lambda$

- Stirling's approximation: $n! \approx n^n e^{-n}\sqrt{2\pi n}$

$$\Rightarrow P_p = \frac{e^{-\lambda}\lambda^k}{k!} = \frac{e^{-\lambda}\lambda^k}{\sqrt{2\pi k}\,e^{-k}k^k}$$

$$= \frac{e^{-\lambda}\lambda^{\lambda(1+\delta)}}{\sqrt{2\pi\lambda(1+\delta)}\;e^{-\lambda(1+\delta)}(\lambda(1+\delta))^{\lambda(1+\delta)}}$$

$$= \frac{\cancel{e^{-\lambda}}\,\cancel{\lambda^{\lambda(1+\delta)}}}{\sqrt{2\pi\lambda}\,\cancel{e^{-\lambda}}e^{-\lambda\delta}\cancel{\lambda^{\lambda(1+\delta)}}(1+\delta)^{\lambda(1+\delta)+\frac{1}{2}}}$$

$$= \frac{e^{\lambda\delta}(1+\delta)^{-\lambda(1+\delta)+\frac{1}{2}}}{\sqrt{2\pi\lambda}}$$

- Take log expansion:

$$\ln(P_p) = \ln\left(\frac{e^{\lambda\delta}}{\sqrt{2\pi\lambda}}\right) - \left(\lambda(1+\delta)+\frac{1}{2}\right)\ln(1+\delta)$$

- For small δ, $\ln(1+\delta) \simeq \delta - \frac{\delta^2}{2} + \underbrace{\theta(\delta^3)}_{\text{negligible}}$

$$\Rightarrow \ln(P_p) = \ln\left(\frac{e^{\lambda\delta}}{\sqrt{2\pi\lambda}}\right) - \left(\lambda(1+\delta)+\frac{1}{2}\right)\left(\delta - \frac{\delta^2}{2}\right)$$

$$= \ln\left(\frac{e^{\lambda\delta}}{\sqrt{2\pi\lambda}}\right) - \left(\lambda+\lambda\delta+\frac{1}{2}\right)\left(\delta - \frac{\delta^2}{2}\right)$$

$$= \ln\left(\frac{e^{\lambda\delta}}{\sqrt{2\pi\lambda}}\right) - \left(\lambda\delta - \frac{\lambda\delta^2}{2} + \lambda\delta^2 \underbrace{- \frac{\lambda\delta^3}{2} + \frac{\delta}{2} - \frac{\delta^2}{2}}_{\text{negligible}}\right)$$

$$= \ln\left(\frac{e^{\lambda\delta}}{\sqrt{2\pi\lambda}}\right) - \left(\lambda\delta - \frac{\lambda\delta^2}{2} + \lambda\delta^2\right)$$

$$= \ln\left(\frac{e^{\lambda\delta}}{\sqrt{2\pi\lambda}}\right) - \left(\lambda\delta + \frac{\lambda\delta^2}{2}\right)$$

$$k = \lambda(1+\delta) \Rightarrow \delta = \frac{k-\lambda}{\lambda}$$

- Take exponential:

$$P_p = e^{\ln\left(\frac{e^{\lambda\delta}}{\sqrt{2\pi\lambda}}\right) - \left(\lambda\delta + \frac{\lambda\delta^2}{2}\right)}$$

$$= \frac{e^{\lambda\delta}}{\sqrt{2\pi\lambda}} e^{-\left(\lambda\delta + \frac{\lambda\delta^2}{2}\right)}$$

$$= \frac{\cancel{e^{\lambda\delta}}}{\sqrt{2\pi\lambda}} \cancel{e^{-\lambda\delta}} e^{-\frac{\lambda\delta^2}{2}}$$

$$= \frac{e^{-\frac{\lambda\delta^2}{2}}}{\sqrt{2\pi\lambda}} \Rightarrow P_p = \frac{e^{-\lambda\left(\frac{k-\lambda}{\lambda}\right)^2/2}}{\sqrt{2\pi\lambda}}$$

mean = var ~ λ

$$\Rightarrow \boxed{P_p = e^{-(k-\lambda)^2/2\lambda} \Big/ \sqrt{2\pi\lambda}}$$

[2]

The gold standard for a believable result is usually 5σ. Let's define the Gaussian approximation as "good enough" if it agrees with the Poisson to within a factor of 2. How large does n need to be for the Gaussian to be good enough @ 5σ? 3σ?

- If λ is large enough, then the expected # of events ~ # of observed events

  i.e. $\lambda \sim n$

- We can set $P_P = 2 P_G$ with $k = \mu + \sigma^*$ & solve for λ

  with $\sigma^* = 3\sigma$ & 5σ

for which we would have $\mu \sim \lambda$

$\sigma^2 \sim \lambda \Rightarrow \sigma = \sqrt{\lambda}$

$$\Rightarrow \frac{e^{-\lambda}\lambda^{k}}{k!} = 2\,\frac{e^{-(k-\mu)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}$$

$$\Rightarrow \frac{e^{-\lambda}\lambda^{\lambda+\sigma^*}}{(\lambda+\sigma^*)!} = 2\,\frac{e^{-(\lambda+\sigma^*-\lambda)^2/2\lambda}}{\sqrt{2\pi\lambda}}$$

$$\Rightarrow \frac{e^{-\lambda}\lambda^{\lambda+\sigma^*}}{(\lambda+\sigma^*)!} = 2\,\frac{e^{-(\sigma^*)^2/2\lambda}}{\sqrt{2\pi\lambda}}$$

Take first $\sigma^* = 3\sigma = 3\sqrt{\lambda}$

$$\Rightarrow \frac{e^{-\lambda}\lambda^{\lambda+3\sqrt{\lambda}}}{(\lambda+3\sqrt{\lambda})!} = \frac{2\,e^{-(3\sqrt{\lambda})^2/2\lambda}}{\sqrt{2\pi\lambda}}$$

$$\Rightarrow \quad \frac{e^{-\lambda}\lambda^{\lambda+3\sqrt{\lambda}}}{(\lambda+3\sqrt{\lambda})!} = 2\,\frac{e^{-9/2}}{\sqrt{2\pi\lambda}}$$

We can plug e.g. in Wolfram & we find $\lambda = 8{,}217\ldots$

$\lambda \simeq 9 \quad = n$

$\boxed{\text{So for } 3\sigma, \text{ we need } n \simeq 9}$

- Now take $\sigma^* = 5\sigma = 5\sqrt{\lambda}$

$$\Rightarrow \quad \frac{e^{-\lambda}\lambda^{\lambda+5\sqrt{\lambda}}}{(\lambda+5\sqrt{\lambda})!} = 2\,\frac{e^{-25/2}}{\sqrt{2\pi\lambda}}$$

Again with Wolfram, we find $\lambda = \ldots$

Actually, Wolfram cannot solve it. Use Mathematica instead!

we find $\lambda \sim 575\ldots$

$\boxed{\text{So for } 5\sigma, \text{ we need } n = 576}$