

3. Properties of the Entropy

(i)

$$\begin{aligned}
H(X) &= H(p_1 \dots p_n) = \sum_{i=1}^n -p_i \log_2 p_i \\
&= \sum_{i=1}^n p_i \log_2 \frac{1}{p_i} \\
&= \mathbb{E}[\log_2(X)] \leq \log_2(\mathbb{E}[X]) \quad (1) \\
&= \log_2\left(\sum_{i=1}^n p_i \frac{1}{p_i}\right) \\
&= \log_2 n
\end{aligned}$$

We used the well known concavity of the logarithm in applying Jensen's inequality in (1).

(ii)

$$\begin{aligned}
\text{Gain}(X|Y) &= H(X) - H(X|Y) \\
&= \sum_X -P(X) \log_2 P(X) - \sum_{v \in \text{value}(Y)} P(Y=v) H(X|Y=v) \\
&= \sum_X -P(X) \log_2 P(X) \\
&\quad - \sum_{v \in \text{value}(Y)} P(Y=v) \sum_X (-P(X|Y=v) \log_2 P(X|Y=v))
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow -\text{Gain}(X|Y) &= \sum_X P(X) \log_2 P(X) \\
&\quad - \sum_{v \in \text{value}(Y)} P(Y=v) \sum_X P(X|Y=v) \log_2 P(X|Y=v) \\
&= \sum_X \sum_{v \in \text{value}(Y)} P(X, Y=v) \log_2 P(X) \\
&\quad - \sum_{v \in \text{value}(Y)} \sum_X P(Y=v) P(X|Y=v) \log_2 P(X|Y=v) \\
&= \sum_X \sum_{v \in \text{value}(Y)} P(X, Y=v) \log_2 P(X) \\
&\quad - \sum_{v \in \text{value}(Y)} \sum_X P(X, Y=v) \log_2 P(X|Y=v)
\end{aligned}$$

$$\begin{aligned}
&= \sum_X \sum_{v \in \text{value}(Y)} P(X, Y = v) (\log_2 P(X) - \log_2 P(X|Y = v)) \\
&= \sum_X \sum_{v \in \text{value}(Y)} P(X, Y = v) \left(\log_2 \left(\frac{P(X)}{P(X|Y = v)} \right) \right) \\
&= \sum_X \sum_{v \in \text{value}(Y)} P(Y = v) P(X|Y = v) \left(\log_2 \left(\frac{P(X)}{P(X|Y = v)} \right) \right) \\
&= \sum_{v \in \text{value}(Y)} P(Y = v) \sum_X P(X|Y = v) \left(\log_2 \left(\frac{P(X)}{P(X|Y = v)} \right) \right) \quad (2) \\
&\leq \sum_{v \in \text{value}(Y)} P(Y = v) \left(\log_2 \left(\sum_X \frac{P(X) P(X|Y = v)}{P(X|Y = v)} \right) \right) \quad (3) \\
&\leq \log_2 \left(\sum_{v \in \text{value}(Y)} \sum_X \frac{P(X) P(X|Y = v) P(Y = v)}{P(X|Y = v)} \right) \quad (4) \\
&= \log_2 \left(\sum_{v \in \text{value}(Y)} P(Y = v) \sum_X P(X) \right) \\
&\leq \log_2 \left(\sum_{v \in \text{value}(Y)} P(Y = v) \right) \\
&= \log_2(1) = 0
\end{aligned}$$

So we conclude:

$$-\text{Gain}(X|Y) \leq \log_2(1) = 0 \Leftrightarrow \text{Gain}(X|Y) \geq 0$$

We used Jensen's inequality from task (i) from equation (2) to (3) and again from (3) to (4). ■