

4. Distances

For $\delta(x, y)$ to be a metric, we have to show:

Non-negative $\delta(x, y) \geq 0$

Identity of indiscernibles $\delta(x, y) = 0$ iff $x = y$

Symmetry $\delta(x, y) = \delta(y, x)$

triangle inequality $\delta(x, y) \leq \delta(x, z) + \delta(z, y)$

(i) **Non-negative** because $|\cdot|$ is a norm.

Identity of indiscernibles Clearly the Hamming distance sums to zero iff x_r, x_s don't differ in more than one entry:

$$D_{\text{Hamming}} = 0 \Rightarrow x_r = x_s$$

Symmetry

$$\begin{aligned} D_{\text{Hamming}}(x_r, x_s) &= \sum_{j=1}^m |x_{rj} - x_{sj}| \\ &= \sum_{j=1}^m |x_{sj} - x_{rj}| = D_{\text{Hamming}}(x_s, x_r) \end{aligned}$$

triangle inequality

$$\forall x_r, x_s, x_t :$$

$$\begin{aligned} D_{\text{Hamming}}(x_r, x_t) &= \sum_{j=1}^m |x_{rj} - x_{tj}| \\ &= \sum_{j=1}^m |x_{rj} - x_{sj} + x_{sj} - x_{tj}| \\ &\leq \sum_{j=1}^m (|x_{rj} - x_{sj}| + |x_{sj} - x_{tj}|) \\ &= \sum_{j=1}^m |x_{rj} - x_{sj}| + \sum_{j=1}^m |x_{sj} - x_{tj}| \\ &= D_{\text{Hamming}}(x_r, x_s) + D_{\text{Hamming}}(x_s, x_t) \end{aligned}$$

(ii) **Non-negative** because $|\cdot|$ is a norm.

Identity of indiscernibles

$$D_{\Delta}(S, S) = |S \setminus S \cup S \setminus S| \quad (1)$$

$$= |\emptyset| = 0 \quad (2)$$

and Let $D_{\Delta}(A, B) = 0$ from that follows

$$0 = |A \setminus B \cup B \setminus A| \quad (3)$$

$$= |A \setminus B| + |B \setminus A| \quad \text{because } (A \setminus B) \cap (B \setminus A) = \emptyset \quad (4)$$

$$(5)$$

and because $|\cdot|$ is positive follows $|A \setminus B| \geq 0$ and $|B \setminus A| \geq 0$ which implies $|A \setminus B| = 0$ and $|B \setminus A| = 0$. From that follows $A \subset B$ and $B \subset A$ which implies $A = B$.

Symmetry $D_{\Delta}(A, B) = |A \setminus B \cup B \setminus A| = |B \setminus A \cup A \setminus B| = D_{\Delta}(B, A)$
because \cup is commutative.

triangle inequality Consider sets A, B and C . Then

$$A \setminus B = (A \setminus (B \cup C)) \cup ((A \cap C) \setminus B) \quad (6)$$

and

$$|(A \setminus (B \cup C)) \cup ((A \cap C) \setminus B)| = |(A \setminus (B \cup C))| + |((A \cap C) \setminus B)| \quad (7)$$

because all elements which get subtracted by C in the first term get added again in the second.

Then compute $D_{\Delta}(A, B) + D_{\Delta}(B, C)$

$$= |A \setminus B \cup B \setminus A| + |B \setminus C \cup C \setminus B| \quad (8)$$

$$= |A \setminus B| + |B \setminus A| + |B \setminus C| + |C \setminus B| \quad \text{same as in 4} \quad (9)$$

$$= |A \setminus (B \cup C) \cup (A \cap C) \setminus B| \quad (10)$$

$$+ |B \setminus (A \cup C) \cup (B \cap C) \setminus A| \quad (11)$$

$$+ |B \setminus (A \cup C) \cup (B \cup A) \setminus C| \quad (12)$$

$$+ |C \setminus (B \cup A) \cup (C \cap A) \setminus B| \quad \text{compare 6} \quad (13)$$

$$= |A \setminus (B \cup C)| + |(A \cap C) \setminus B| \quad (14)$$

$$+ |B \setminus (A \cup C)| + |(B \cap C) \setminus A| \quad (15)$$

$$+ |B \setminus (A \cup C)| + |(B \cup A) \setminus C| \quad (16)$$

$$+ |C \setminus (B \cup A)| + |(C \cap A) \setminus B| \quad \text{compare 7} \quad (17)$$

$$\geq |A \setminus (B \cup C)| \quad (18)$$

$$+ |(B \cap C) \setminus A| \quad (19)$$

$$+ |(B \cup A) \setminus C| \quad (20)$$

$$+ |C \setminus (B \cup A)| \quad (21)$$

$$= |A \setminus (B \cup C) \cup (B \cup A) \setminus C| \quad (22)$$

$$+ |C \setminus (B \cup A) \cup (B \cap C) \setminus A| \quad \text{compare 7} \quad (23)$$

$$= |A \setminus C| + |C \setminus A| \quad (24)$$

$$= |A \setminus C \cup C \setminus A| \quad (25)$$

$$= D_{\Delta}(A, C) \quad (26)$$

Consider sets A, B and C and distinguish two cases:

Let $x \in A \setminus C$. Then either

I did it
likes this:
wrong?

(a) $x \notin B \Rightarrow x \in A \setminus B$ or

(b) $x \in B \Rightarrow x \in B \setminus C$.

Similarly: Let $x \in C \setminus A$. Then either

(a) $x \notin B \Rightarrow x \in C \setminus B$ or

(b) $x \in B \Rightarrow x \in B \setminus A$.

$$\Rightarrow A \Delta C \subset (A \Delta B) \cup (B \Delta C)$$

$$\Rightarrow |A \Delta C| \leq |(A \Delta B)| + |(B \Delta C)|$$

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