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MA-INF 4111 - Intelligent Learning and Analysis Systems: Machine Learning
Exercise Sheet 4

4. Distances

done. will write up in the evening

For $\delta(x,y)$ to be a metric, we have to show:

1. $\delta(x,y) \geq 0, \delta(x,y) = 0$ iff $x = y$
2. $\delta(x,y) = \delta(y,x)$
3. $\delta(x,y) \leq \delta(x,z) + \delta(z,y)$

(i) done

(ii) **Non-negative** because $|\cdot|$ is a norm.

Identity of indiscernibles

$$\begin{aligned} D_\Delta(S,S) &= |S \setminus S \cup S \setminus S| && (1) \\ &= |\emptyset| = 0 && (2) \end{aligned}$$

and Let $D_\Delta(A,B) = 0$ from that follows

$$\begin{aligned} 0 &= |A \setminus B \cup B \setminus A| && (3) \\ &= |A \setminus B| + |A \setminus B| \qquad \text{because } (A \setminus B) \cap (B \setminus A) = \emptyset && (4) \\ & && (5) \end{aligned}$$

and because $|\cdot|$ is positive follows $|A \setminus B| \geq 0$ and $|B \setminus A| \geq 0$ which implies $|A \setminus B| = 0$ and $|B \setminus A| = 0$. From that follows $A \subset B$ and $B \subset A$ which implies $A = B$.

Symmetry $D_\Delta(A,B) = |A \setminus B \cup B \setminus A| = |B \setminus A \cup A \setminus B| = D_\Delta(B,A)$ because $\cup$ is commutative.

triangle inequality Consider sets A, B and C. Then

$$A \setminus B = (A \setminus (B \cup C)) \cup ((A \cap C) \setminus B) \tag{6}$$

and

$$|(A \setminus (B \cup C)) \cup ((A \cap C) \setminus B)| = |(A \setminus (B \cup C))| + |((A \cap C) \setminus B)| \tag{7}$$

because all elements which get subtracted by C in the first term get added again in the second.

Then compute $D_\Delta(A,B) + D_\Delta(B,C)$

$$
\begin{aligned}
&= |A \setminus B \cup B \setminus A| + |B \setminus C \cup C \setminus B| && && (8)\\
&= |A \setminus B| + |B \setminus A| + |B \setminus C| + |C \setminus B| && \text{same as in 4} && (9)\\
&= |A \setminus (B \cup C) \cup (A \cap C) \setminus B| && && (10)\\
&\quad + |B \setminus (A \cup C) \cup (B \cap C) \setminus A| && && (11)\\
&\quad + |B \setminus (A \cup C) \cup (B \cup A) \setminus C| && && (12)\\
&\quad + |C \setminus (B \cup A) \cup (C \cap A) \setminus B| && \text{compare 6} && (13)\\
&= |A \setminus (B \cup C)| + |(A \cap C) \setminus B| && && (14)\\
&\quad + |B \setminus (A \cup C)| + |(B \cap C) \setminus A| && && (15)\\
&\quad + |B \setminus (A \cup C)| + |(B \cup A) \setminus C| && && (16)\\
&\quad + |C \setminus (B \cup A)| + |(C \cap A) \setminus B| \text{ compare 7} && && (17)\\
&\geq |A \setminus (B \cup C)| && && (18)\\
&\quad + |(B \cap C) \setminus A| && && (19)\\
&\quad + |(B \cup A) \setminus C| && && (20)\\
&\quad + |C \setminus (B \cup A)| && && (21)\\
&= |A \setminus (B \cup C) \cup (B \cup A) \setminus C| && && (22)\\
&\quad + |C \setminus (B \cup A) \cup (B \cap C) \setminus A| && \text{compare 7} && (23)\\
&= |A \setminus C| + |C \setminus A| && && (24)\\
&= |A \setminus C \cup C \setminus A| && && (25)\\
&= D_\Delta(A,C) && && (26)
\end{aligned}
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