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MA-INF 4111 - Intelligent Learning and Analysis Systems: Machine Learning
Exercise Sheet 4

2. Backpropagation with the Hyperbolic Tangent Function

Since $(tanh(x))' = 1 - tanh(x)^2$, the weight update rules become:

"Do you want to know more?" - Starship Troopers ;-)

$$\delta_k \leftarrow \left(1 - o_k^2\right)(t_k - o_k)$$
$$\delta_h \leftarrow \left(1 - o_h^2\right) \sum_{k \in \text{outputs}} w_{h,k}\delta_k$$
$$w_{i,j} \leftarrow w_{i,j} + \Delta w_{i,j}, \text{ where } \delta w_{i,j} = \eta \delta_j x_{i,j}$$

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4. Distances

For $\delta(x, y)$ to be a metric, we have to show:

Non-negative $\delta(x, y) \geq 0$

Identity of indiscernibles $\delta(x, y) = 0$ iff $x = y$

Symmetry $\delta(x, y) = \delta(y, x)$

triangle inequality $\delta(x, y) \leq \delta(x, z) + \delta(z, y)$

(i) **Non-negative** because $|\cdot|$ is a norm.

Identity of indiscernibles Cleary the Hamming distance sums to zero iff x_r, x_s don't differ in more than one entry:
$D_{\text{Hamming}} = 0 \Rightarrow x_r = x_s$

Symmetry

$$\begin{aligned} D_{\text{Hamming}}(x_r, x_s) &= \sum_{j=1}^{m} |x_{rj} - x_{sj}| \\ &= \sum_{j=1}^{m} |x_{sj} - x_{rj}| = D_{\text{Hamming}}(x_s, x_r) \end{aligned}$$

triangle inequality

$$\begin{aligned}
&\forall x_r, x_s, x_t : \\
D_{\text{Hamming}}(x_r, x_t) &= \sum_{j=1}^{m} \left| x_{rj} - x_{tj} \right| \\
&= \sum_{j=1}^{m} \left| x_{rj} - x_{sj} + x_{sj} - x_{tj} \right| \\
&\leq \sum_{j=1}^{m} \left(\left| x_{rj} - x_{sj} \right| + \left| x_{sj} - x_{tj} \right| \right) \\
&= \sum_{j=1}^{m} \left| x_{rj} - x_{sj} \right| + \sum_{j=1}^{m} \left| x_{sj} - x_{tj} \right| \\
&= D_{\text{Hamming}}(x_r, x_s) + D_{\text{Hamming}}(x_s, x_t)
\end{aligned}$$

(ii) **Non-negative** because $|\cdot|$ is a norm.

Identity of indiscernibles

$$D_\Delta(S, S) = |S \setminus S \cup S \setminus S| \quad (1)$$
$$= |\emptyset| = 0 \quad (2)$$

and Let $D_\Delta(A, B) = 0$ from that follows

$$0 = |A \setminus B \cup B \setminus A| \quad (3)$$
$$= |A \setminus B| + |A \setminus B| \qquad \text{because } (A \setminus B) \cap (B \setminus A) = \emptyset \quad (4)$$
$$(5)$$

and because $|\cdot|$ is positive follows $|A \setminus B| \geq 0$ and $|B \setminus A| \geq 0$ which implies $|A \setminus B| = 0$ and $|B \setminus A| = 0$. From that follows $A \subset B$ and $B \subset A$ which implies $A = B$.

Symmetry $D_\Delta(A, B) = |A \setminus B \cup B \setminus A| = |B \setminus A \cup A \setminus B| = D_\Delta(B, A)$ because $\cup$ is commutative.

triangle inequality Consider sets A, B and C. Then

$$A \setminus B = (A \setminus (B \cup C)) \cup ((A \cap C) \setminus B) \quad (6)$$

and

$$|(A \setminus (B \cup C)) \cup ((A \cap C) \setminus B)| = |(A \setminus (B \cup C))| + |((A \cap C) \setminus B)| \quad (7)$$

because all elements which get subtracted by C in the first term get added again in the second.

Then compute $D_\Delta(A,B) + D_\Delta(B,C)$

$$
\begin{aligned}
&= |A \setminus B \cup B \setminus A| + |B \setminus C \cup C \setminus B| && \tag{8}\\
&= |A \setminus B| + |B \setminus A| + |B \setminus C| + |C \setminus B| && \text{same as in 4} \tag{9}\\
&= |A \setminus (B \cup C) \cup (A \cap C) \setminus B| && \tag{10}\\
&\quad + |B \setminus (A \cup C) \cup (B \cap C) \setminus A| && \tag{11}\\
&\quad + |B \setminus (A \cup C) \cup (B \cup A) \setminus C| && \tag{12}\\
&\quad + |C \setminus (B \cup A) \cup (C \cap A) \setminus B| && \text{compare 6} \tag{13}\\
&= |A \setminus (B \cup C)| + |(A \cap C) \setminus B| && \tag{14}\\
&\quad + |B \setminus (A \cup C)| + |(B \cap C) \setminus A| && \tag{15}\\
&\quad + |B \setminus (A \cup C)| + |(B \cup A) \setminus C| && \tag{16}\\
&\quad + |C \setminus (B \cup A)| + |(C \cap A) \setminus B| \text{ compare 7} && \tag{17}\\
&\geq |A \setminus (B \cup C)| && \tag{18}\\
&\quad + |(B \cap C) \setminus A| && \tag{19}\\
&\quad + |(B \cup A) \setminus C| && \tag{20}\\
&\quad + |C \setminus (B \cup A)| && \tag{21}\\
&= |A \setminus (B \cup C) \cup (B \cup A) \setminus C| && \tag{22}\\
&\quad + |C \setminus (B \cup A) \cup (B \cap C) \setminus A| && \text{compare 7} \tag{23}\\
&= |A \setminus C| + |C \setminus A| && \tag{24}\\
&= |A \setminus C \cup C \setminus A| && \tag{25}\\
&= D_\Delta(A,C) && \tag{26}
\end{aligned}
$$

Consider sets A, B and C and distinguish two cases:

I did it likes this: wrong?

Let $x \in A \setminus C$. Then either

(a) $x \notin B \Rightarrow x \in A \setminus B$ or

(b) $x \in B \Rightarrow x \in B \setminus C$.

Similarly: Let $x \in C \setminus A$. Then either

(a) $x \notin B \Rightarrow x \in C \setminus B$ or

(b) $x \in B \Rightarrow x \in B \setminus A$.

$$
\begin{aligned}
\Rightarrow (A \Delta B) \cup (B \Delta C) &= (A \setminus B) \cup (B \setminus A) \cup (B \setminus C) \cup (C \setminus B) \tag{27}\\
&\supset (A \setminus C) \cup (C \setminus A) \tag{28}\\
&= A \Delta C \tag{29}\\
& \tag{30}
\end{aligned}
$$

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\Rightarrow |A \Delta C| \leq |(A \Delta B)| + |(B \Delta C)| \tag{31}
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