

1. Voronoi diagrams

(a) Proof by induction over the number of points on a circle with radius r in \mathbb{R}^2 :

Induction start: $n = 3$ Three point lying on a circle with radius r are non-collinear and the resulting cells are unbounded (figure 1a).

Induction step: $n \rightarrow n + 1$ Let the induction assumption be true for n points. Let p, q be two arbitrary neighboring points on the circle. We insert a new point o exactly at the intersection of the bisector of p, q and the circles radius r . The resulting cells are unbounded (figure 1b).

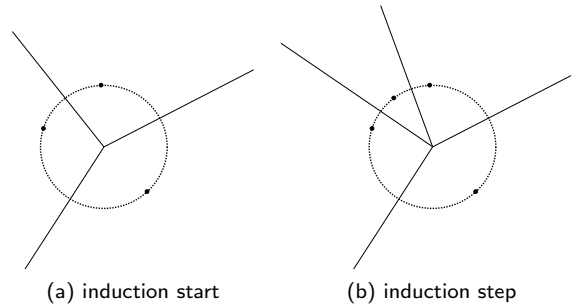


Figure 1: proof by induction over points on a circle

(b) Let $\text{Vor}(P)$ denote the Voronoi diagram of points P . Euler's formula for planer graphs states that

$$v - e + f = 2,$$

where v, e, f denote the number of vertices, edges and faces of the graph.

The number of faces is $f = |P| = n = \# \text{cells}(\text{Vor}(P))$. To make use of Euler's formula, we have to construct a planar Graph from $\text{Vor}(P)$, by adding an additional vertex to the diagram, situated at infinity, and connecting every infinite edge (from the unbounded cells) to this vertex (figure 2).

We note that each edge in $\text{Vor}(P)$ has exactly two incident vertices and each vertex of $\text{Vor}(P) + \infty$ has at least degree (deg) 3. Therefor we conclude

$$\sum_{\text{vertex} \in \text{Vor}(P) + \infty} \text{deg}(\text{vertex}) = 2e \geq 3(v + 1) \quad (1)$$

Since we inserted our additional point ∞ , Euler's Formula for our constructed graph is:

$$(v + 1) - e + f = (v + 1) - e + n = 2$$

Inserting this into (1) gives us:

$$\begin{aligned} 2e &\geq 3(2 + e - n) = 6 + 3e - 3n \Rightarrow \\ e &\leq 3n - 6 \end{aligned}$$

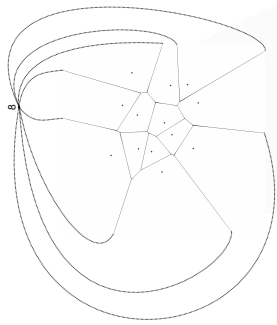


Figure 2: augmenting Voronoi diagram to planar graph

Inserting this result into (1):

$$v \leq 2n - 5$$

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