

M2

ymonbru

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Definition 0.1. If $A_0^\bullet \rightarrow \dots \rightarrow A_n^\bullet$ is a sequence of maps (with $f_i : A_i \rightarrow A_{i+1}$) of complex such that the composition of two consecutive maps is 0, then let's denote $[A_0^\bullet \rightarrow \dots \rightarrow A_n^{\bullet-n}]$ the total complex of this double complex, defined by the following data:

- The object in degré k is $\bigoplus_{i=0}^n A_i[-i]^n$
- The differential is given by the matrix
$$\begin{pmatrix} d_{A_0} & 0 & \dots & \dots & 0 \\ f_0 & -d_{A_1} & \dots & \dots & 0 \\ 0 & f_1 & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & (-1)^{n-1}d_{A_{n-1}} & 0 \\ 0 & 0 & \dots & f_{n-1} & (-1)^n d_{A_n} \end{pmatrix}$$

Proof. One needs to check that the matrix square is 0. Let M be this matrix and (i, j) be integers.

$$M^2[i, j] = \sum_{k=1}^n M[i, k]M[k, j] = M[i, i]M[i, j] + M[i, i-1]M[i-1, j]$$

One can distinguish four cases:

- If j is not in $\{i-2, i-1, i, \}$ then the two terms are 0.
- If $j = i$, then $M^2[i, j] = ((-1)^i d_{A_i})^2 + 0 = 0$.
- If $j = i-1$ then $M^2[i, j] = 0 + (-1)^i d_{A_i} \circ f_i + (-1)^{i+1} d_{A_{i+1}} \circ f_i = 0$ because f_i is a morphism of complex.
- If $j = i-2$ then $M^2[i, j] = f_i \circ f_{i-1} = 0$.

□

Remark 0.2. In particular, if $f : A^\bullet \rightarrow B^\bullet$ is a morphism of complex then $[A^\bullet \rightarrow B^{\bullet-1}]$ is the cone of the morphism f .

Lemma 0.3. A morphism of complex $f : A^\bullet \rightarrow B^\bullet$ is a quasi isomorphism if and only if, its cone is acyclic.

Proof. One get's a short exact sequence $0 \rightarrow B^\bullet[-1] \rightarrow [A^\bullet \rightarrow B^{\bullet-1}] \rightarrow A^\bullet \rightarrow 0$ by using the canonical inclusion and projection over the direct sum. The long exact sequence induced in cohomology is then:

$$\dots H^{k-1}A^\bullet \rightarrow H^k B^\bullet[-1] \rightarrow H^k[A^\bullet \rightarrow B^{\bullet-1}] \rightarrow H^k A^\bullet \rightarrow H^{k+1} B^\bullet[-1] \dots$$

By using the fact that $H^k B^\bullet[-1] = H^{k-1} B^\bullet$ one gets:

$$\dots \rightarrow H^{k-1}A^\bullet \rightarrow H^{k-1}B^\bullet \rightarrow H^k[A^\bullet \rightarrow B^{\bullet-1}] \rightarrow H^k A^\bullet \rightarrow H^k B^\bullet \dots$$

And then the statement is straightforward by reading the exact sequence. □

Lemma 0.4. If a complex $[A^\bullet \rightarrow B^{\bullet-1} \rightarrow C^{\bullet-2}]$ is acyclic then there is a long exact sequence

$$\dots \rightarrow H^k A \rightarrow H^k B \rightarrow H^k C \rightarrow \dots$$

Proof. One can see that by construction there is a canonical isomorphism of complexess: $[A^\bullet \rightarrow B^{\bullet-1} \rightarrow C^{\bullet-2}] = [A^\bullet \rightarrow [B^\bullet \rightarrow C^{\bullet-1}]^{\bullet-1}]$.

Then by the previous lemma: $A^\bullet \rightarrow [B^\bullet \rightarrow C^{\bullet-1}]$ is a quasi isomorphism. One can then rewrite the long exact sequence in cohomology givent by the short exact sequence $0 \rightarrow C^\bullet[-1] \rightarrow [B^\bullet \rightarrow C^{\bullet-1}] \rightarrow B^\bullet \rightarrow 0$ wich is (as in the previous lemma):

$$\dots \rightarrow H^{k-1}B^\bullet \rightarrow H^{k-1}C^\bullet \rightarrow H^k[B^\bullet \rightarrow C^{\bullet-1}] \rightarrow H^k B^\bullet \rightarrow H^k C^\bullet \rightarrow \dots$$

.

The result is then a long exact sequence :

$$\dots \rightarrow H^{k-1}B^\bullet \rightarrow H^{k-1}C^\bullet \rightarrow H^k A^\bullet \rightarrow H^k B^\bullet \rightarrow H^k C^\bullet \rightarrow \dots$$

□

Chapter 1

Presheaves and sheaves

Let X be a locally compact Hausdorff space.

1.1 Sheaves

Definition 1.1. A presheaf on X is a contravariant functor from the category of open sets of X to abelian groups.

Definition 1.2. If \mathcal{F} is a presheaf on X and $p \in X$ then the stalk of \mathcal{F} at p is the abelian group $\mathcal{F}_p := \varinjlim_{p \in U \text{ open}} \mathcal{F}(U)$.

Definition 1.3. If \mathcal{F} is a presheaf on X , it is said to be a sheaf if for any $U \subset X$ open and any covering family of U $(U_a)_{a \in A}$ one has the exact sequence:

$$0 \rightarrow \mathcal{F}(U) \rightarrow \prod_{a \in A} \mathcal{F}(U_a) \rightarrow \prod_{a, b \in A} \mathcal{F}(U_a \cap U_b) \quad (1.1)$$

1.2 \mathcal{K} -sheaves

Definition 1.4. A \mathcal{K} -presheaf on X is a contravariant functor from the category of compact sets of X to abelian groups.

Definition 1.5. If \mathcal{F} is a \mathcal{K} -presheaf on X and $p \in X$ then the stalk of \mathcal{F} at p is the abelian group $\mathcal{F}_p := \varinjlim_{p \in K \text{ compact}} \mathcal{F}(K) = \mathcal{F}(\{p\})$.

Definition 1.6. If \mathcal{F} is a \mathcal{K} -presheaf on X , it is said to be a \mathcal{K} -sheaf if the following conditions are satisfied:

- $$\mathcal{F}(\emptyset) = 0 \quad (1.2)$$

- For K_1 and K_2 two compacts of X the following sequence is exact:

$$0 \rightarrow \mathcal{F}(K_1 \cup K_2) \rightarrow \mathcal{F}(K_1) \oplus \mathcal{F}(K_2) \rightarrow \mathcal{F}(K_1 \cap K_2) \quad (1.3)$$

- For any compact K of X , the following natural morphism is an isomorphism

$$\lim_{\substack{K \subset U \\ U \text{ open relatively compact}}} \mathcal{F}(\overline{U}) \rightarrow \mathcal{F}(K) \quad (1.4)$$

Remark 1.7. (1.4) is well defined because if K is a compact subset of X , then for $x \in K$ let U_x be an open neighborhood relatively compact (which exists by local compactness), the family $(U_x)_{x \in K}$ covers K then one can extract a finite cover of it : U_1, \dots, U_n and then $\bigcup_{i=1}^n U_i$ is an open neighborhood, and a finite union of relatively compact, then it's relatively compact. $(\bigcup_{i=1}^n U_i)^\circ = \bigcup_{i=1}^n U_i^\circ$

1.3 Technical lemmas

Lemma 1.8. If K_1, \dots, K_n are compact of X then $\{U_1 \cap \dots \cap U_n\}_{U_i \supset K_i \text{ open in } X}$ is a cofinal system of neighborhoods of $K_1 \cap \dots \cap K_n$.

Proof. □

Lemma 1.9. If \mathcal{C} and \mathcal{D} are two categories, $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ two functors such that (F, G) is an adjoint pair. Then for (F, G) to be an equivalence of category, it's enough to have that the canonical natural transformations $\text{id}_{\mathcal{D}} \Rightarrow F \circ G$ and $G \circ F \Rightarrow \text{id}_{\mathcal{C}}$ are isomorphisms.

Proof. TODO □

1.4 Equivalence of category

Definition 1.10.

- If \mathcal{F} is a presheaf then let $\alpha^* \mathcal{F}$ be the \mathcal{K} -presheaf:

$$K \mapsto \lim_{\substack{K \subset U \\ U \text{ open}}} \mathcal{F}(U)$$

- If \mathcal{G} is a \mathcal{K} -presheaf then let $\alpha_* \mathcal{G}$ be the presheaf:

$$U \mapsto \lim_{\substack{U \supset K \\ K \text{ compact}}} \mathcal{G}(K)$$

Proposition 1.11. The pair (α^*, α_*) is an adjoint pair.

Proof. TODO □

Lemma 1.12.

- α^* send sheaves to \mathcal{K} -sheaves
- α_* send \mathcal{K} -sheaves to sheaves
- The restrictions obtained still form an adjoint pair.

The previous adjoint pair give rise to an adjoint pair between sheaves and \mathcal{K} -sheaves

Proof. TODO □

Lemma 1.13. *The previous adjoint pair give rise to an equivalence of category between shaeves and \mathcal{K} -sheaves*

Proof.

□

Chapter 2

Homotopy sheaves

Definition 2.1. Let \mathcal{F}^\bullet be complex of \mathcal{K} -presheaves then taking the cohomology defines a \mathcal{K} -presheaf denoted $H^\bullet \mathcal{F}^\bullet$.

Definition 2.2. A morphism of complex of \mathcal{K} -presheaf $\mathcal{F}^\bullet \rightarrow \mathcal{G}^\bullet$ if it induces isomorphisms $H^i \mathcal{F}^\bullet \rightarrow H^i \mathcal{G}^\bullet$ for all i .

Definition 2.3. A complex of \mathcal{K} -presheaves \mathcal{F}^\bullet is said to be a Homotopy- \mathcal{K} -sheaf if the following conditions are satisfied:

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$$\mathcal{F}^\bullet(\emptyset) \text{ is an acyclic complex} \quad (2.1)$$

- For K_1 and K_2 two compact of X the following complex is acyclic:

$$[\mathcal{F}^\bullet(K_1 \cup K_2) \rightarrow \mathcal{F}^{\bullet-1}(K_1) \bigoplus \mathcal{F}^{\bullet-1}(K_2) \rightarrow \mathcal{F}^{\bullet-2}(K_1 \cap K_2)] \quad (2.2)$$

- For any compact K of X , the following natural morphism is a quasi-isomorphism

$$\lim_{\substack{\longrightarrow \\ K \subset U \text{ open relatively compact}}} \mathcal{F}^\bullet(\overline{U}) \rightarrow \mathcal{F}^\bullet(K) \quad (2.3)$$

Remark 2.4. By using 0.4, (2.3) give rise to a "Mayer-Vietoris" long exact sequence:

$$\dots \rightarrow H^k \mathcal{F}^\bullet(K_1 \cup K_2) \rightarrow H^k \mathcal{F}^\bullet(K_1) \bigoplus H^k \mathcal{F}^\bullet(K_2) \rightarrow H^k \mathcal{F}^\bullet(K_1 \cap K_2) \rightarrow \dots$$

Lemma 2.5. Let \mathcal{F}^\bullet be a complex of \mathcal{K} -presheaves. If \mathcal{F}^\bullet has a finite filtration whose associated graded is a Homotopy- \mathcal{K} -sheaf, then \mathcal{F}^\bullet is a Homotopy- \mathcal{K} -sheaf.

Proof. TODO □

Lemma 2.6. If \mathcal{F}^\bullet is a homotopy- \mathcal{K} -sheaf, and $H^{-1} \mathcal{F}^\bullet = 0$ then $H^0 \mathcal{F}^\bullet$ is a \mathcal{K} -sheaf

Proof.

- $\mathcal{F}^\bullet(\emptyset)$ is acyclic then in particular it's cohomology in degree 0 is 0, then one gets (1.2)
- $H^{-1} \mathcal{F}^\bullet(K_1 \cap K_2) = 0$ then the first terms 2.4 gives the exact sequence of (1.3)
- TODO

□

Chapter 3

Pushforward, exceptional pushforward, and pullback

3.1 For Sheaves

Definition 3.1.

Definition 3.2.

Definition 3.3.

3.2 For \mathcal{K} -sheaves

Definition 3.4.

Definition 3.5.

Definition 3.6.

Chapter 4

Čech cohomology

4.1 Čech cohomology of sheaves

Definition 4.1. If X is a topological space, and \mathcal{F} a sheaf over X , then let $\check{H}^\bullet(X; \mathcal{F})$ be the Čech cohomology of X with coefficient in \mathcal{F}

Definition 4.2. If X is a topological space, K a compact subset of X and \mathcal{F} a sheaf over X , then let $\check{H}_K^\bullet(X; \mathcal{F})$ be the Čech cohomology of X with support in K with coefficient in \mathcal{F}

Definition 4.3. Let $f : X \rightarrow Y$ be a continuous map between topological spaces, and \mathcal{F} a sheaf over X . f induces a natural map $\check{H}^\bullet(Y; f_*\mathcal{F}) \rightarrow \check{H}^\bullet(X; \mathcal{F})$.

Moreover if f is proper, one gets a natural map $\check{H}_c^\bullet(Y; f_*\mathcal{F}) \rightarrow \check{H}_c^\bullet(X; \mathcal{F})$

Lemma 4.4. Let $f : X \rightarrow Y$ be an inclusion of open subset, then there is a natural isomorphism $f_! : \check{H}_c^\bullet(X; \mathcal{F}) \rightarrow \check{H}^\bullet(Y; f_!\mathcal{F})$

Proof. □

4.2 Čech cohomology of complex of \mathcal{K} -sheaves

Definition 4.5. Let \mathcal{F}^\bullet be a complex of \mathcal{K} -presheaves on a compact space X then we define the Čech cohomology $\check{H}(X; \mathcal{F}^\bullet)$ by `TODO`

Remark 4.6. By using the inclusion of \mathcal{K} -presheaves into complexes of \mathcal{K} -presheaves, one gets a definition of Čech cohomology for \mathcal{K} -presheaves.

Lemma 4.7. If $\mathcal{F}^\bullet \rightarrow \mathcal{G}^\bullet$ is a quasi-isomorphism then the induced maps $\check{H}^i \mathcal{F}^\bullet \rightarrow \check{H}^i \mathcal{G}^\bullet$ are isomorphisms.

Proof. `TODO` □

Proposition 4.8. Let \mathcal{F}^\bullet be a complex of \mathcal{K} -presheaves that verify (2.1) and (2.2) then the canonical map $H^\bullet \mathcal{F}^\bullet \rightarrow \check{H}^\bullet(X; \mathcal{F}^\bullet)$ is an isomorphism.

Proof. □

4.3 Čech cohomology is determined by stalks

Lemma 4.9. *Let \mathcal{F}^\bullet be a complex of \mathcal{K} -presheaves that verify (1.4) and such that all the stalks are 0 then $\check{H}^\bullet(X; \mathcal{F}) = 0$*

Proof.

□

Lemma 4.10. *Let \mathcal{F}^\bullet be a complex of \mathcal{K} -presheaves that verify (2.3) and $H^i \mathcal{F}^\bullet = 0$ for $i \ll 0$. Then if the stalks of \mathcal{F}^\bullet are acyclics, $\check{H}^\bullet(X; \mathcal{F}^\bullet) = 0$*

Proof. TODO

□

Proposition 4.11. *Let \mathcal{F}^\bullet and \mathcal{G}^\bullet be complexes of \mathcal{K} -presheaves that verify (2.3) and $H^i \mathcal{F}^\bullet = H^i \mathcal{G}^\bullet = 0$ for i small enough.*

Then if a morphism $\mathcal{F}^\bullet \rightarrow \mathcal{G}^\bullet$ induces a quasi-isomorphism on stalks, $\check{H}^\bullet(X; \mathcal{F}^\bullet) = \check{H}^\bullet(X; \mathcal{G}^\bullet)$

Proof.

□

Chapter 5

Purehomotopy \mathcal{K} -sheaves

Definition 5.1. A homotopy \mathcal{K} -sheaf \mathcal{F}^\bullet is said to be pure on X if:

- For $p \in X$ and $i \neq 0$, $(H^i \mathcal{F}^\bullet)_p = 0$
- $H^i \mathcal{F}^\bullet = 0$ for $i < 0$ locally on X : ie for all $p \in X$ there is an open neighbourhood U of p and an integer N such that for $i \leq N$ and $K \subset U$: $H^i \mathcal{F}^\bullet(K) = 0$

Lemma 5.2. Let \mathcal{F}^\bullet be a pure-homotopy \mathcal{K} -sheaf. Then:

- For $i < 0$ $H^i \mathcal{F}^\bullet = 0$
- $H^0 \mathcal{F}^\bullet$ is a \mathcal{K} -sheaf.

Proof. TODO □

Proposition 5.3. Let \mathcal{F}^\bullet be a pure-homotopy \mathcal{K} -sheaf. Then there is a canonical isomorphism:

$$H^\bullet \mathcal{F}^\bullet(X) = \check{H}^\bullet(X; H^0 \mathcal{F}^\bullet)$$

More generally: Let $[\mathcal{F}_0^\bullet \rightarrow \dots \mathcal{F}_n^{\bullet-n}]$ be a complex of pure-homotopy \mathcal{K} -sheaves, then there is a canonical isomorphism:

$$H^\bullet[\mathcal{F}_0^\bullet(X) \rightarrow \dots \mathcal{F}_n^{\bullet-n}(X)] = \check{H}^\bullet(X; [H^0 \mathcal{F}_0^\bullet \rightarrow \dots \rightarrow (H^0 \mathcal{F}_n^\bullet)[n]])$$

Proof. TODO □

Chapter 6

Poincaré–Lefschetz duality

Uses chech cohomology with compact supports for sheaves

Definition 6.1. Let M be a topological manifold, the rule $\mathfrak{o}_M : K \mapsto H_{\dim M}(M, M \setminus K)$ defines a \mathcal{K} –sheaf, called the orientation \mathcal{K} –sheaf of M .

If M is a manifold with boundary, let $j : M \setminus \partial M \rightarrow M$ denote the canonical inclusion. The the orientation shaeves of M are defined as follows:

- $\mathfrak{o}_M := j_* \mathfrak{o}_{M \setminus \partial M}$
- $\mathfrak{o}_{M \text{ rel } \partial} := j_! \mathfrak{o}_{M \setminus \partial M}$

Definition 6.2. Singular chains

Lemma 6.3. Let X be a topological manifold, then we have:

- $C_\bullet(X, X)$ is an acyclic complex
- For A and B two closed subsets of X the folowing complex is acyclic:

$$[C_\bullet(X, X \setminus (A \cup B)) \rightarrow C_{\bullet+1}(X, X \setminus A) \bigoplus C_{\bullet+1}(X, X \setminus B) \rightarrow C_{\bullet+2}(X, X \setminus (A \cap B))]$$

- For a family $(K_a)_{a \in A}$ of closed subsets of X wich is filtered (for any $a, b \in A$ there is $c \in A$ such that $K_c \subset K_a \cap K_b$) any compact K of X , the following natural morphism is a quasi-isomorphism

$$\lim_{\substack{\longrightarrow \\ a \in A}} C_\bullet(X, X \setminus K_a) \rightarrow C_\bullet(X, X \setminus (\bigcap_{a \in A} K_a))$$

Proof. TODO □

Lemma 6.4. Let M be a topological manifold of dimension n with boundary, $i : X \rightarrow M$ a closed subset, and $N \subset \partial M$ a closed subset that locally looks like $\emptyset \subset \mathbb{R}^{n-1}, \mathbb{R}_{>0} \times \mathbb{R}^{n-2} \subset \mathbb{R}^{n-1}$ or \mathbb{R}^{n-1} . Let $j : \overset{\circ}{M} \cup \overset{\circ}{N} \rightarrow M$ be the canonical inclusion. Then there is a canonical isomorphism:

$$H^\bullet[C_{n-1-\bullet}(N, N \setminus X) \rightarrow C_{n-1-\bullet}(M, M \setminus X)] = \check{H}_c^\bullet(X; i^* j_! j^* \mathfrak{o}_M)$$

Proof. uses tout les poussé en avatn, tiré en arrière et tout ça, la cohomologie à support compact □

Chapter 7

Homotopy colimits

7.1 Homotopy colimits

Definition 7.1.

Definition 7.2.

7.2 Homotopy colimits of pure homotopy \mathcal{K} -sheaves

Lemma 7.3.

Proof.

□

Lemma 7.4.

Chapter 8

Steenrod homology