M2

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Definition 0.1. *If $A_0^\bullet \to ... \to A_n^\bullet$ is a sequence of maps (with $f_i : A_i \to A_{i+1}$)of complex such that the composition of two consecutive maps is 0, then let's denote $[A_0^\bullet \to ... \to A_n^{\bullet - n}]$ the total complex of this double complex, defined by the following data:*

- *The object in degré k is $\bigoplus_{i=0}^{n} A_i[-i]^n$*
- *The diferential is given by the matrix* $\begin{pmatrix} d_{A_0} & 0 & ... & ... & 0 \\ f_0 & -d_{A_1} & ... & ... & 0 \\ 0 & f_1 & ... & ... & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & (-1)^{n-1} d_{A_{n-1}} & 0 \\ 0 & 0 & ... & f_{n-1} & (-1)^n d_{A_n} \end{pmatrix}$

Proof. One needs to check that the matrix square is 0. Let M be this matrix and (i, j) be integers.

$$M^2[i,j] = \sum_{k=1}^{n} M[i,k]M[k,j] = M[i,i]M[i,j] + M[i,i-1]M[i-1,j]$$

One can distinguish four cases:

- If j is not in $\{i-2, i-1, i,\}$ then the two terms are 0.
- If $j = i$, then $M^2[i,j] = ((-1)^i d_{A_i})^2 + 0 = 0$.
- If $j = i-1$ then $M^2[i,j]0 + (-1)^i d_{A_i} \circ f_i + (-1)^{i+1} d_{A_{i+1}} \circ f_i = 0$ because f_i is a morphism of complex.
- If $j = i-2$ then $M^2[i,j] = f_i \circ f_{i-1} = 0$.

□

Remark 0.2. *In particular, if $f : A^\bullet \to B^\bullet$ is a morphism of complex then $[A^\bullet \to B^{\bullet -1}]$ is the cone of the morphism f.*

Lemma 0.3. *A morphism of complex $f : A^\bullet \to B^\bullet$ is a quasi isomorphism if and only if, its cone is acyclic.*

Proof. One get's a short exact sequence $0 \to B^\bullet[-1] \to [A^\bullet \to B^{\bullet -1}] \to A^\bullet \to 0$ by using the canonical inclusion and projection over the direct sum. The long exact sequence induced in cohomology is then:

$$... H^{k-1}A^\bullet \to H^k B^\bullet[-1] \to H^k[A^\bullet \to B^{\bullet -1}] \to H^k A^\bullet \to H^{k+1} B^\bullet[-1] ...$$

By using the fact that $H^k B^\bullet[-1] = H^{k-1} B^\bullet$ one gets:

$$... \to H^{k-1}A^\bullet \to H^{k-1}B^\bullet \to H^k[A^\bullet \to B^{\bullet -1}] \to H^k A^\bullet \to H^k B^\bullet ...$$

And then the statement is straightforward by reading the exact sequence. □

Lemma 0.4. *If a complex $[A^\bullet \to B^{\bullet -1} \to C^{\bullet -2}]$ is acyclic then there is a long exact sequence*

$$... \to H^k A \to H^k B \to H^k C \to ...$$

Proof. One can see that by construction there is a canonical isomorphism of complexess: $[A^\bullet \to B^{\bullet-1} \to C^{\bullet-2}] = [A^\bullet \to [B^\bullet \to C^{\bullet-1}]^{\bullet-1}]$.

Then by the previous lemma: $A^\bullet \to [B^\bullet \to C^{\bullet-1}]$ is a quasi isomorphism. One can then rewrite the long exact sequence in cohomology givent by the short exact sequence $0 \to C^\bullet[-1] \to [B^\bullet \to C^{\bullet-1}] \to B^\bullet \to 0$ wich is (as in the previous lemma):

$$... \to H^{k-1}B^\bullet \to H^{k-1}C^\bullet \to H^k[B^\bullet \to C^{\bullet-1}] \to H^kB^\bullet \to H^kC^\bullet \to ...$$

.

The result is then a long exact sequence :

$$... \to H^{k-1}B^\bullet \to H^{k-1}C^\bullet \to H^kA^\bullet \to H^kB^\bullet \to H^kC^\bullet \to ...$$

□

Chapter 1

Presheaves and sheaves

Let X be a locally compact Hausdorf space.

1.1 Sheaves

Definition 1.1. *A presheave on X is a contravariant functor from the category of open sets of X to abélian groups.*

Definition 1.2. *If $\mathcal{F}$ is a presheaf on X and $p \in X$ then the stalk of $\mathcal{F}$ at p is the abelian group $\mathcal{F}_p := \varinjlim_{p\in U \ open} \mathcal{F}(U)$.*

Definition 1.3. *If $\mathcal{F}$ is a presheaf on X, it is said to be a sheaf if for any $U \subset X$ open and any covering family of U $(U_a)_{a\in A}$ one has the exact sequence:*

$$0 \to \mathcal{F}(U) \to \prod_{a\in A} \mathcal{F}(U_a) \to \prod_{a,b\in A} F(U_a \cap U_b) \tag{1.1}$$

1.2 $\mathcal{K}$-sheaves

Definition 1.4. *A $\mathcal{K}$-presheave on X is a contravariant functor from the category of compact sets of X to abélian groups.*

Definition 1.5. *If $\mathcal{F}$ is a $\mathcal{K}$-presheaf on X and $p \in X$ then the stalk of $\mathcal{F}$ at p is the abelian group $\mathcal{F}_p := \varinjlim_{p\in K \ compact} \mathcal{F}(K) = \mathcal{F}(\{p\})$.*

Definition 1.6. *If $\mathcal{F}$ is a $\mathcal{K}$-presheaf on X, it is said to be a $\mathcal{K}$-sheaf if the folowing conditions are satisfied:*

- $$\mathcal{F}(\emptyset) = 0 \tag{1.2}$$
- *For K_1 and K_2 two comapcts of X the folowing sequence is exact:*

$$0 \to \mathcal{F}(K_1 \cup K_2) \to \mathcal{F}(K_1) \bigoplus \mathcal{F}(K_2) \to \mathcal{F}(K_1 \cap K_2) \tag{1.3}$$

- *For any compact K of X, the following natural morphism is an isomorphism*

$$\varinjlim_{K \subset U \text{ open relatively compact}} \mathcal{F}(\overline{U}) \to \mathcal{F}(K) \tag{1.4}$$

Remark 1.7. (1.4) *is well defined because if K is a compact subset of X, then for $x \in K$ let U_x be an open neighborhood relatively compact (wich exists by local compactness), the family $(u_x)_{x \in K}$ covers K then one can extract a finite cover of it : $U_1, \dots U_n$ and then $\cup_{i=1}^n U_i$ is an open neighborhood, and a finite union of relatively comapct, then it's relatively compact. ($\overline{\cup_{i=1}^n U_i} = \cup_{i=1}^n \overline{U_i}$)*

1.3 Technical lemmas

Lemma 1.8. *If $K_1, \dots K_n$ are comapcts of X then $\{U_1 \cap \dots \cap U_n\}_{U_i \supset K_i \text{ open in } X}$ is a cofinal system of neighborhoods of $K_1 \cap \dots K_n$.*

Proof. □

Lemma 1.9. *If $\mathcal{C}$ and $\mathcal{D}$ are two categories, $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ two functors such that (F, G) is an adjoint pair. Then for (F, G) to be an equivalence of category, it's enough to have that thes canonical naturals transformations $id_{\mathcal{D}} \Rightarrow F \circ G$ and $G \circ F \Rightarrow id_{\mathcal{D}}$ are isomorphisms.*

Proof. CategoryTheory.Adjunction.toEquivalence dans mathlib □

1.4 Equivalence of category

Definition 1.10.

- *If $\mathcal{F}$ is a presheaf then let $\alpha^* \mathcal{F}$ ne the $\mathcal{K}$-presheaf:*

$$K \mapsto \varinjlim_{K \subset U \text{ open}} \mathcal{F}(U)$$

- *If $\mathcal{G}$ is a $\mathcal{K}$-presheaf then let $\alpha_* \mathcal{G}$ ne the presheaf:*

$$U \mapsto \varprojlim_{U \supset K \text{ compact}} \mathcal{F}(K)$$

Proposition 1.11. *The pair (α^*, α_*) is an adjonit pair.*

Proof. TODO □

Lemma 1.12.

- *α^* send sheaves to $\mathcal{K}$-sheaves*
- *α^* send $\mathcal{K}$-sheaves to sheaves*
- *The reistrictions obtained still form an adjoint pair.*

The previous adjoint pair give rise to an adjoint pair between shaeves and $\mathcal{K}$-sheaves

Proof. TODO □

Lemma 1.13. *The previous adjoint pair give rise to an equivalence of category between shaeves and $\mathcal{K}$-sheaves*

Proof. $\square$

Chapter 2

Homotopy sheaves

Definition 2.1. *Let $\mathcal{F}^\bullet$ be complex of $\mathcal{K}$-presheaves then taking the cohomology defines a $\mathcal{K}$-presheaf denoted $H^\bullet\mathcal{F}^\bullet$.*

Definition 2.2. *A morphisme of complex of $\mathcal{K}$-presheave $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ if it induces isomorphisms $H^i\mathcal{F}^\bullet \to H^i\mathcal{G}^\bullet$ for all i.*

Definition 2.3. *A complex of $\mathcal{K}$-presheaves $\mathcal{F}^\bullet$ is said to be a Homotopy-$\mathcal{K}$-sheave if the folowing conditions are satisfied:*

- $$\mathcal{F}^\bullet(\emptyset) \text{ is an acyclic complex} \tag{2.1}$$
- *For K_1 and K_2 two comapcts of X the folowing complex is acyclic:*
$$[\mathcal{F}^\bullet(K_1 \cup K_2) \to \mathcal{F}^{\bullet-1}(K_1) \bigoplus \mathcal{F}^{\bullet-1}(K_2) \to \mathcal{F}^{\bullet-2}(K_1 \cap K_2)] \tag{2.2}$$
- *For any compact K of X, the following natural morphism is a quasi-isomorphism*
$$\varinjlim_{K \subset U \text{ open relatively compact}} \mathcal{F}^\bullet(\overline{U}) \to \mathcal{F}^\bullet(K) \tag{2.3}$$

Lemma 2.4. *By using 0.4, (2.3) give rise to a "Mayer-Vietoris" long exact sequence:*

$$\ldots \to H^k\mathcal{F}^\bullet(K_1 \cup K_2) \to H^k\mathcal{F}^\bullet(K_1) \bigoplus H^k\mathcal{F}^\bullet(K_2) \to H^k\mathcal{F}^\bullet(K_1 \cap K_2) \to \ldots$$

Lemma 2.5. *Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{K}$-presheaves. If $\mathcal{F}^\bullet$ has a finite filtration whose associated graded is a Homotopy-$\mathcal{K}$-sheaf, then $\mathcal{F}^\bullet$ is a Homotopy-$\mathcal{K}$-sheaf.*

Proof. TODO □

Lemma 2.6. *If $\mathcal{F}^\bullet$ is a homotopy-$\mathcal{K}$-sheaf, and $H^{-1}\mathcal{F}^\bullet = 0$ then $H^0\mathcal{F}^\bullet$ is a $\mathcal{K}$-sheaf*

Proof.

- $\mathcal{F}^\bullet(\emptyset)$ is acyclic then in particular it's cohomology in degree 0 is 0, then one gets (1.2)
- $H^{-1}\mathcal{F}^\bullet(K_1 \cap K_2) = 0$ then the first terms 2.4 gives the exact sequence of (1.3)
- TODO

□

Chapter 3

Pushforward, exceptional pushforward, and pullback

Let X and Y be two loccaly compacts hausdorf spaces and $f : X \to Y$ be a continuous map.

3.1 For Sheaves

Definition 3.1. *If $\mathcal{F}$ is a pre-sheaf over X, then the rule $U \mapsto \mathcal{F}(f^{-1}(U))$ defines a pre-shaef over Y.*

The functor obtained is denoted f_* and named the pushforward by f.

f_* send sheaves over X into sheaves over Y.

Proof. TODO ☐

Definition 3.2. *If $\mathcal{F}$ is a pre-sheaf over Y, then the rule $U \mapsto \varinjlim_{f(U) \subset V} \mathcal{F}(V)$ defines a pre-sheaf over Y.*

If $\mathcal{F}$ is a sheaf, the sheafification of the previous pre-sheaf is a denoted $f^*\mathcal{F}$ and called the pullback by f.

Proof. TODO ☐

Definition 3.3. *If $f : X \to Y$ is the inclusion of an open subset, the exceptional pushforward by f: $f_!$ is defined by $f_!\mathcal{F}(U)$ being the subset of $f_*\mathcal{F}(U)$ of sections that vanish over a neighborhood of $Y - X$.*

It send the sheaves over X into the sheaves over Y

Proof. TODO ☐

3.2 For $\mathcal{K}$-sheaves

Let's assume that f is proper.

Definition 3.4. *If $\mathcal{F}$ is a pre-$\mathcal{K}$-sheaf over X, then the rule $K \mapsto \mathcal{F}(f^{-1}(K))$ defines a pre-$\mathcal{K}$-sheaf over Y.*

The functor obtained is denoted f_* and named the pushforward by f.

f_* send $\mathcal{K}$-sheaves over X into $\mathcal{K}$-sheaves over Y.

Proof. TODO □

Chapter 4

Čech cohomology

4.1 Čech cohomology of sheaves

Definition 4.1. *If X is a topological space, and $\mathcal{F}$ a sheaf over X, then let $\check{H}^\bullet(X;\mathcal{F}))$ be the Čech cohomology of X with coeficient in $\mathcal{F}$*

Definition 4.2. *If X is a topological space, K a comapct subset of X and $\mathcal{F}$ a sheaf over X, then let $\check{H}_K^\bullet(X;\mathcal{F}))$ be the Čech cohomology of X with support in K with coeficient in $\mathcal{F}$*

Definition 4.3. *Let $f : X \to Y$ be a continuous map between topological spaces, and $\mathcal{F}$ a sheaf over X. f induces a natural map $\check{H}^\bullet(Y; f_*\mathcal{F}) \to \check{H}^\bullet(X;\mathcal{F})$.*

Moreover if f is proper, one gets a natural map $\check{H}_c^\bullet(Y; f_*\mathcal{F}) \to \check{H}_c^\bullet(X;\mathcal{F})$

Lemma 4.4. *Let $f : X \to Y$ be an inclusion of open subset, then there is a natural isomorphism $f_! : \check{H}_c^\bullet(X;\mathcal{F}) \to \check{H}^\bullet(Y; f_!\mathcal{F})$*

Proof. □

4.2 Čech cohomology of complex of $\mathcal{K}$-sheaves

Definition 4.5. *Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{K}$-presheaves on a compact space X then we define the Čech cohomology $\check{H}(X;\mathcal{F}^\bullet)$ by TODO*

Remark 4.6. *By using the inclusion of $\mathcal{K}$-presheaves into complexes of $\mathcal{K}$-presheave, one get's a definition of Čech cohomology for $\mathcal{K}$-presheave.*

Lemma 4.7. *Let $mathcalF^\bullet$ be an acyclic complex of $\mathcal{K}$-presheaves, then $\check{H}^k(X; mathcalF^\bullet) = 0$*

Proof. TODO □

Lemma 4.8. *Let $0 \to \mathcal{F}^\bullet \to \mathcal{G}^\bullet \to \mathcal{H}^\bullet \to 0$ be a short exact sequence of complex of $\mathcal{K}$-presheaves. Then there is a long exact seuquence in čech cohomology:*

$$... \to \check{H}^k(X;\mathcal{F}^\bullet) \to \check{H}^k(X;\mathcal{G}^\bullet) \to \check{H}^k(X;\mathcal{H}^\bullet) \to 0$$

Proof. TODO □

Lemma 4.9. *If $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism then the induced maps $\check{H}^i\mathcal{F}^\bullet \to \check{H}^i\mathcal{G}^\bullet$ are isomorphims.*

Proof. By 0.3, the complex $[\mathcal{F}^\bullet \to \mathcal{G}^{\bullet-1}]$ is acyclic then by 4.7, it's čech cohomology is zero.

But there is a short exact sequence $0 \to \mathcal{G}^\bullet[-1] \to [\mathcal{F}^\bullet \to \mathcal{G}^{\bullet-1}] \to \mathcal{F}^\bullet \to 0$, then the long exact sequence induced by 4.8 gives the claimed isomorphisms. □

Proposition 4.10. *Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{K}$-presheaves that verify (2.1) and (2.2) then the canonical map $H^\bullet\mathcal{F}^\bullet \to \check{H}^\bullet(X;\mathcal{F}^\bullet)$ is an isomorphism.*

Proof. TODO

□

4.3 Čech cohomology is determined by stalks

Lemma 4.11. *Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{K}$-presheaves that verify (1.4) and such that all the stalks are 0 then $\check{H}^\bullet(X;\mathcal{F}) = 0$*

Proof. □

Lemma 4.12. *Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{K}$-presheaves that verify (2.3) and $H^i\mathcal{F}^\bullet = 0$ for $i << 0$.*

Then if the stalks of $\mathcal{F}^\bullet$ are acyclics, $\check{H}^\bullet(X;\mathcal{F}^\bullet) = 0$

Proof. TODO □

Proposition 4.13. *Let $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ be complexes of $\mathcal{K}$-presheaves that verify (2.3) and $H^i\mathcal{F}^\bullet = H^i\mathcal{G}^\bullet = 0$ for i small enough.*

Then if a morphism $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ induces a quasi-isomorphism on stalks, $\check{H}^\bullet(X;\mathcal{F}^\bullet) = \check{H}^\bullet(X;\mathcal{G}^\bullet)$

Proof. □

Chapter 5

Purehomotopy $\mathcal{K}$-sheaves

Definition 5.1. *A homotopy $\mathcal{K}$-sheaf $\mathcal{F}^\bullet$ is said to be pure on X if:*

- *For $p \in X$ and $i \neq 0$, $(H^i\mathcal{F}^\bullet)_p = 0$*
- *$H^i\mathcal{F}^\bullet = 0$ for $i << 0$blocally on X: ie for all $p \in X$ there is an open neighbourhoud U of p and an integer N such that for $i \leq N$ and $K \subset U$: $H^i\mathcal{F}^\bullet(K) = 0$*

Lemma 5.2. *Let $\mathcal{F}^\bullet$ be a pure-homotopy $\mathcal{K}$-sheaf. Then:*

- *For $i < 0$ $H^i\mathcal{F}^\bullet = 0$*
- *$H^0\mathcal{F}^\bullet$ is a $\mathcal{K}$-sheaf.*

Proof. TODO □

Proposition 5.3. *Let $\mathcal{F}^\bullet$ be a pure-homotopy $\mathcal{K}$-sheaf. Then there is a canonical isomorphism:*

$$H^\bullet\mathcal{F}^\bullet(X) = \check{H}^\bullet(X; H^0\mathcal{F}^\bullet)$$

.

More generaly: Let $[\mathcal{F}_0^\bullet \to \dots \mathcal{F}_n^{\bullet - n}]$ be a complex of pure-homotopy $\mathcal{K}$-sheaves, then there is a canonical isomorphism:

$$H^\bullet[\mathcal{F}_0^\bullet(X) \to \dots \mathcal{F}_n^{\bullet - n}(X)] = \check{H}^\bullet(X; [H^0\mathcal{F}_0^\bullet \to \dots \to (H^0\mathcal{F}_n^\bullet)[n]])$$

Proof. TODO □

Chapter 6

Poincaré–Lefschetz duality

Uses chech cohomology with compact supports for sheaves

Definition 6.1. *Let M be a topological manifold, the rule $\mathfrak{o}_M : K \mapsto H_{\dim M}(M, M\backslash K)$ defines a $\mathcal{K}-$sheaf, called the orientation $\mathcal{K}-$sheaf of M.*

If M is a manifold with boundary, let $j : M\backslash\partial M \to M$ denote the canonical inclusion. The the orientation shaeves of M are defined as follows:

- $\mathfrak{o}_M := j_* \mathfrak{o}_{M\backslash\partial M}$
- $\mathfrak{o}_{M \quad rel\partial} := j_! \mathfrak{o}_{M\backslash\partial M}$

Definition 6.2. *Singular chains*

Lemma 6.3. *Let X be a topological manifold, then we have:*

- *$C_\bullet(X, X)$ is an acyclic complex*
- *For A and B two closed subsets of X the folowing complex is acyclic:*

$$[C_\bullet(X, X\backslash(A \cup B)) \to C_{\bullet+1}(X, X\backslash A) \bigoplus C_{\bullet+1}(X, X\backslash B) \to C_{\bullet+2}(X, X\backslash(A \cap B))]$$

- *For a family $(K_a)_{a\in A}$ of closed subsets of X wich is filtered (for any $a, b \in A$ there is $c \in A$ such that $K_c \subset K_a \cap K_b$)any compact K of X, the following natural morphism is a quasi-isomorphism*

$$\varinjlim_{a\in A} C_\bullet(X, X\backslash K_a) \to C_\bullet(X, X\backslash(\bigcap_{a\in A} K_a))$$

Proof. TODO □

Lemma 6.4. *Let M be a topological manifold of dimension n with boundary, $i : X \to M$ a closed subset, and $N \subset \partial M$ a closed subset that locally looks like $\emptyset \subset \mathbb{R}^{n-1}, \mathbb{R}_{>0} \times R^{n-2} \subset \mathbb{R}^{n-1}$ or $\mathbb{R}^{n-1}$. Let $j : \mathring{M} \cup \mathring{N} \to M$ be the canonical inclusion. Then there is a canonical isomorphism:*

$$H^\bullet[C_{n-1-\bullet}(N, N\backslash X) \to C_{n-1-\bullet}(M, M\backslash X)] = \check{H}^\bullet_c(X; i^* j_! j^* \mathfrak{o}_M)$$

Proof. uses tout les poussé en avatn, tiré en arière et tout ça, la cohomologie à support compact □

Chapter 7

Homotopy colimits

7.1 Homotopy colimits

Definition 7.1.

Definition 7.2.

7.2 Homotopy colimits of pure homotopy $\mathscr{K}$-sheaves

Lemma 7.3.

Proof. $\square$

Lemma 7.4.

Chapter 8

Steenrod homology