M2

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Chapter 1

Presheaves and sheaves

Let X be a locally compact Hausdorf space.

1.1 Sheaves

Definition 1.1. *A presheave on X is a contravariant functor from the category of open sets of X to abélian groups.*

Definition 1.2. *If $\mathcal{F}$ is a presheaf on X and $p \in X$ then the stalk of $\mathcal{F}$ at p is the abelian group $\mathcal{F}_p := \varinjlim_{p \in U \ open} \mathcal{F}(U)$.*

Definition 1.3. *If $\mathcal{F}$ is a presheaf on X, it is said to be a sheaf if for any $U \subset X$ open and any covering family of U $(U_a)_{a \in A}$ one has the exact sequence:*

$$0 \to \mathcal{F}(U) \to \prod_{a \in A} \mathcal{F}(U_a) \to \prod_{a,b \in A} F(U_a \cap U_b) \tag{1.1}$$

1.2 $\mathcal{K}$-sheaves

Definition 1.4. *A $\mathcal{K}$-presheave on X is a contravariant functor from the category of compact sets of X to abélian groups.*

Definition 1.5. *If $\mathcal{F}$ is a $\mathcal{K}$-presheaf on X and $p \in X$ then the stalk of $\mathcal{F}$ at p is the abelian group $\mathcal{F}_p := \varinjlim_{p \in K \ compact} \mathcal{F}(K) = \mathcal{F}(\{p\})$.*

Definition 1.6. *If $\mathcal{F}$ is a $\mathcal{K}$-presheaf on X, it is said to be a $\mathcal{K}$-sheaf if the folowing conditions are satisfied:*

- $$\mathcal{F}(\emptyset) = 0 \tag{1.2}$$
- *For K_1 and K_2 two comapcts of X the folowing sequence is exact:*

$$0 \to \mathcal{F}(K_1 \cup K_2) \to \mathcal{F}(K_1) \bigoplus \mathcal{F}(K_2) \to \mathcal{F}(K_1 \cap K_2) \tag{1.3}$$

- *Pour tout compact K de X, le morphisme naturel suivant est un isomorphisme*

$$\varinjlim_{K\subset U \text{ open relatively compact}} \mathcal{F}(\overline{U}) \to \mathcal{F}(K) \tag{1.4}$$

Remark 1.7. (1.4) *is well defined because if K is a compact subset of X, then for $x \in K$ let U_x be an open neighborhood relatively compact (wich exists by local compactness), the family $(u_x)_{x\in K}$ covers K then one can extract a finite cover of it : $U_1, \dots U_n$ and then $\cup_{i=1}^n U_i$ is an open neighborhood, and a finite union of relatively comapct, then it's relatively compact. ($\overline{\cup_{i=1}^n U_i} = \cup_{i=1}^n \overline{U_i}$)*

1.3 Technical lemmas

Lemma 1.8. *If $K_1, \dots K_n$ are comapcts of X then $\{U_1 \cap \dots \cap U_n\}_{U_i \supset K_i \text{ open in } X}$ is a cofinal system of neighborhoods of $K_1 \cap \dots K_n$.*

Proof. Let U_i be a relatively comapct open neighborhood of K_i and $U = \cup_{i=1}^n U_i$. Then $\overline{U}$ is compact

If $n = 2$, let V be a neighborhood of $K_1 \cap K_2$
If the result is true for $n - 1$, □

Lemma 1.9. *If $\mathcal{C}$ and $\mathcal{D}$ are two categories, $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ two functors such that (F, G) is an adjoint pair. Then for (F, G) to be an equivalence of category, it's enough to have that thes canonical naturals transformations $id_{\mathcal{D}} \Rightarrow F \circ G$ and $G \circ F \Rightarrow id_{\mathcal{D}}$ are isomorphisms.*

Proof. TODO □

Lemma 1.10. *If $(K_a)_{a\in A}$ is a filtered directed system of comapcts substes of X, and $\mathcal{F}$ a $\mathcal{K}$-presheaf satisfying (1.4), then*

$$\varinjlim_{a\in A} \mathcal{F}(K_a) \to \mathcal{F}(\bigcap_{a\in A} K_a)$$

is an isomorphism.

Proof. TODO □

1.4 Equivalence of category

Definition 1.11.

- *If $\mathcal{F}$ is a presheaf then let $\alpha^*\mathcal{F}$ ne the $\mathcal{K}$-presheaf :*

$$K \mapsto \varinjlim_{K\subset U \text{ open}} \mathcal{F}(U)$$

- *If $\mathcal{G}$ is a $\mathcal{K}$-presheaf then let $\alpha_*\mathcal{G}$ ne the presheaf :*

$$U \mapsto \varprojlim_{U\supset K \text{ compact}} \mathcal{F}(K)$$

Proposition 1.12. *The pair (α^*, α_*) is an adjonit pair.*

Proof. TODO □

Lemma 1.13.

- *α^* send sheaves to $\mathcal{K}$-sheaves*
- *α^* send $\mathcal{K}$-sheaves to sheaves*
- *The reistrictions obtained still form an adjoint pair.*

The previous adjoint pair give rise to an adjoint pair between shaeves and $\mathcal{K}$-sheaves

Proof. TODO □

Lemma 1.14. *The previous adjoint pair give rise to an equivalence of category between shaeves and $\mathcal{K}$-sheaves*

Proof.

□

Chapter 2

Homotopy sheaves

Chapter 3

Pushforward, exceptional pushforward, and pullback

Chapter 4

Čech cohomology

Chapter 5

Purehomotopy $\mathcal{K}$-sheaves

Chapter 6

Poincaré–Lefschetz duality

Chapter 7

Homotopy colimits

Chapter 8

Homotopy colimits of pure homotopy $\mathscr{K}$-sheaves

Chapter 9

Steenrod homology