M2

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June 26, 2024

Chapter 1

Introduction

1.1 Total Complexes

Definition 1.1. *If $A_0^\bullet \to ... \to A_n^\bullet$ is a sequence of maps (with $f_i : A_i \to A_{i+1}$)of complex such that the composition of two consecutive maps is 0, then let's denote $[A_0^\bullet \to ... \to A_n^{\bullet - n}]$ the total complex of this double complex, defined by the following data:*

- *The object in degré k is $\bigoplus_{i=0}^{n} A_i[-i]^k$*
- *The diferential is given by the matrix* $\begin{pmatrix} d_{A_0} & 0 & ... & ... & 0 \\ f_0 & -d_{A_1} & ... & ... & 0 \\ 0 & f_1 & ... & ... & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & (-1)^{n-1} d_{A_{n-1}} & 0 \\ 0 & 0 & ... & f_{n-1} & (-1)^n d_{A_n} \end{pmatrix}$

Proof. One needs to check that the matrix square is 0. Let M be this matrix and (i, j) be integers.

$$M^2[i, j] = \sum_{k=1}^{n} M[i, k]M[k, j] = M[i, i]M[i, j] + M[i, i-1]M[i-1, j]$$

One can distinguish four cases:

- If j is not in $\{i-2, i-1, i,\}$ then the two terms are 0.
- If $j = i$, then $M^2[i, j] = ((-1)^i d_{A_i})^2 + 0 = 0$.
- If $j = i-1$ then $M^2[i, j]0 + (-1)^i d_{A_i} \circ f_i + (-1)^{i+1} d_{A_{i+1}} \circ f_i = 0$ because f_i is a morphism of complex.
- If $j = i-2$ then $M^2[i, j] = f_i \circ f_{i-1} = 0$.

□

Remark 1.2. *In particular, if $f : A^\bullet \to B^\bullet$ is a morphism of complex then $[A^\bullet \to B^{\bullet -1}]$ is the cone of the morphism f.*

Lemma 1.3. *A morphism of complex $f : A^\bullet \to B^\bullet$ is a quasi isomorphism if and only if, its cone is acyclic.*

Proof. One get's a short exact sequence $0 \to B^\bullet[-1] \to [A^\bullet \to B^{\bullet-1}] \to A^\bullet \to 0$ by using the canonical inclusion and projection over the direct sum. The long exact sequence induced in cohomology is then:

$$... H^{k-1}A^\bullet \to H^kB^\bullet[-1] \to H^k[A^\bullet \to B^{\bullet-1}] \to H^kA^\bullet \to H^{k+1}B^\bullet[-1] ...$$

By using the fact that $H^kB^\bullet[-1] = H^{k-1}B^\bullet$ one gets:

$$... \to H^{k-1}A^\bullet \to H^{k-1}B^\bullet \to H^k[A^\bullet \to B^{\bullet-1}] \to H^kA^\bullet \to H^kB^\bullet ...$$

And then the statement is straightforward by reading the exact sequence. $\square$

Lemma 1.4. *If a complex $[A^\bullet \to B^{\bullet-1} \to C^{\bullet-2}]$ is acyclic then there is a long exact sequence*

$$... \to H^kA \to H^kB \to H^kC \to ...$$

Proof. One can see that by construction there is a canonical isomorphism of complexess: $[A^\bullet \to B^{\bullet-1} \to C^{\bullet-2}] = [A^\bullet \to [B^\bullet \to C^{\bullet-1}]^{\bullet-1}]$.

Then by the previous lemma: $A^\bullet \to [B^\bullet \to C^{\bullet-1}]$ is a quasi isomorphism. One can then rewrite the long exact sequence in cohomology givent by the short exact sequence $0 \to C^\bullet[-1] \to [B^\bullet \to C^{\bullet-1}] \to B^\bullet \to 0$ wich is (as in the previous lemma):

$$... \to H^{k-1}B^\bullet \to H^{k-1}C^\bullet \to H^k[B^\bullet \to C^{\bullet-1}] \to H^kB^\bullet \to H^kC^\bullet \to ...$$

.

The result is then a long exact sequence :

$$... \to H^{k-1}B^\bullet \to H^{k-1}C^\bullet \to H^kA^\bullet \to H^kB^\bullet \to H^kC^\bullet \to ...$$

$\square$

Chapter 2

Presheaves and sheaves

Let X be a locally compact Hausdorf space.

2.1 Sheaves

Definition 2.1. *A presheave on X is a contravariant functor from the category of open sets of X to abélian groups.*

Definition 2.2. *If $\mathcal{F}$ is a presheaf on X and $p \in X$ then the stalk of $\mathcal{F}$ at p is the abelian group $\mathcal{F}_p := \varinjlim_{p \in U \text{ open}} \mathcal{F}(U)$.*

Definition 2.3. *If $\mathcal{F}$ is a presheaf on X, it is said to be a sheaf if for any $U \subset X$ open and any covering family of U $(U_a)_{a \in A}$ one has the exact sequence:*

$$0 \to \mathcal{F}(U) \to \prod_{a \in A} \mathcal{F}(U_a) \to \prod_{a,b \in A} \mathcal{F}(U_a \cap U_b) \tag{2.1}$$

2.2 $\mathcal{K}$-sheaves

Definition 2.4. *A $\mathcal{K}$-presheave on X is a contravariant functor from the category of compact sets of X to abélian groups.*

Definition 2.5. *If $\mathcal{F}$ is a $\mathcal{K}$-presheaf on X and $p \in X$ then the stalk of $\mathcal{F}$ at p is the abelian group $\mathcal{F}_p := \varinjlim_{p \in K \text{ compact}} \mathcal{F}(K) = \mathcal{F}(\{p\})$.*

Definition 2.6. *If $\mathcal{F}$ is a $\mathcal{K}$-presheaf on X, it is said to be a $\mathcal{K}$-sheaf if the folowing conditions are satisfied:*

- $$\mathcal{F}(\emptyset) = 0 \tag{2.2}$$
- *For K_1 and K_2 two comapcts of X the folowing sequence is exact:*

$$0 \to \mathcal{F}(K_1 \cup K_2) \to \mathcal{F}(K_1) \bigoplus \mathcal{F}(K_2) \to \mathcal{F}(K_1 \cap K_2) \tag{2.3}$$

- *For any compact K of X, the following natural morphism is an isomorphism*

$$\varinjlim_{K\subset U \text{ open relatively compact}} \mathcal{F}(\overline{U}) \to \mathcal{F}(K) \tag{2.4}$$

Remark 2.7. (2.4) *is well defined because if K is a compact subset of X, then for $x \in K$ let U_x be an open neighborhood relatively compact (wich exists by local compactness), the family $(U_x)_{x\in K}$ covers K then one can extract a finite cover of it : $U_1, ... U_n$ and then $\cup_{i=1}^n U_i$ is an open neighborhood, and a finite union of relatively comapct, then it's relatively compact. ($\overline{\cup_{i=1}^n U_i} = \cup_{i=1}^n \overline{U_i}$)*

2.3 Technical lemmas

Lemma 2.8. *If $K_1, ... K_n$ are comapcts of X then $\{U_1 \cap ... \cap U_n\}_{U_i \supset K_i \text{ open in } X}$ is a cofinal system of neighborhoods of $K_1 \cap ... \cap K_n$.*

Proof. It's the theorem IsCompact.nhdsSet-inter-eq in the File Mathlib/Topology/Separation.lean and the use of Filter.HasBasis.inf in the file Mathlib.Order.Filter.Bases

□

Lemma 2.9. *If $\mathcal{C}$ and $\mathcal{D}$ are two categories, $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ two functors such that (F, G) is an adjoint pair. Then for (F, G) to be an equivalence of category, it's enough to have that thes canonical naturals transformations $id_{\mathcal{D}} \Rightarrow F \circ G$ and $G \circ F \Rightarrow id_{\mathcal{D}}$ are isomorphisms.*

Proof. CategoryTheory.Adjunction.toEquivalence in mathlib □

2.4 Equivalence of category

Definition 2.10.

- *If $\mathcal{F}$ is a presheaf then let $\alpha^*\mathcal{F}$ be the $\mathcal{K}$-presheaf:*

$$K \mapsto \varinjlim_{K\subset U \text{ open}} \mathcal{F}(U)$$

- *If $\mathcal{G}$ is a $\mathcal{K}$-presheaf then let $\alpha_*\mathcal{G}$ be the presheaf:*

$$U \mapsto \varprojlim_{U\supset K \text{ compact}} \mathcal{G}(K)$$

Proposition 2.11. *The pair (α^*, α_*) is an adjonit pair.*

Proof.
- Let τ be an element of $\hom(\alpha^*\mathcal{F}, \mathcal{G})$. It's the data of morphism τ_K for K a compact of X such that for any K and K' compacts is a commutative square. Then for any U and V opens, by composing with the commutative square

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[d] & \varinjlim_{K\subset U} \mathcal{F}(U) \arrow[d] \\
\mathcal{F}(V) \arrow[r] & \varinjlim_{K'\subset U} \mathcal{F}(U)
\end{tikzcd}$$

$$\begin{tikzcd}
\varinjlim\limits_{K\subset U} \mathcal{F}(U) \arrow[r, "\tau_K"] \arrow[d] & \mathcal{G}(K) \arrow[d] \\
\varinjlim\limits_{K'\subset U} \mathcal{F}(U) \arrow[r, "\tau_{K'}"] & \mathcal{G}(K')
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{G}(K) \arrow[d] \\
\mathcal{F}(V) \arrow[r] & \mathcal{G}(K')
\end{tikzcd}$$

one get's a commutative square :. Conversely such data give rise (by taking the limit over U and V) to a commutative square such as in (2.4)

- On the other hand if one takes the limit over K and K' one get's a commutative square

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[d] & \varprojlim\limits_{K\subset U} \mathcal{G}(K) \arrow[d] \\
\mathcal{F}(V) \arrow[r] & \varprojlim\limits_{K\subset V} \mathcal{G}(K)
\end{tikzcd}$$

(that allow to recover the previous one in the same as before) wich is the data of an element of $\hom(\mathcal{F}, \alpha_*\mathcal{G})$.

Then the elements of $\hom(\alpha^*\mathcal{F}, \mathcal{G})$ and $\hom(\mathcal{F}, \alpha_*\mathcal{G})$ are both obtained by a natural construction (in $\mathcal{F}$ and $\mathcal{G}$) aplied to (2.4). $\square$

Lemma 2.12.

- α^* *send sheaves to* $\mathcal{K}$*-sheaves*
- α_* *send* $\mathcal{K}$*-sheaves to sheaves*
- *The reistrictions obtained still form an adjoint pair between shaeves and* $\mathcal{K}$*-sheaves.*

Proof.

- Let $\mathcal{F}$ be a sheaf. The condition $\emptyset \subset U$ is always satisfied and $\emptyset$ is minimal among open subset for the inclusion then $(\alpha^*)(\mathcal{F})(\emptyset) = \mathcal{F}(\emptyset)$. One can apply the sheaf condition to the empty family and obtain the exact sequence $0 \to \mathcal{F}(\emptyset) \to \Pi_\emptyset = 0$, and then (2.2).

  Let K_1, K_2 be two of compacts of X, let U_1, U_2 be a two opens such that $K_i \subset U_i$ for all i. Then the sheaf condition gives an exact sequence (because for abelian groups the product is the direct sum) $0 \to \mathcal{F}(U_1 \cup U_2) \to \mathcal{F}(U_1) \bigoplus \mathcal{F}(U_2) \to \mathcal{F}(U_1 \cap U_2)$. The injective limits are exacts then taking the limits over those opens gives an exact sequence:

$$0 \to \varinjlim_{K_i \subset U_i} \mathcal{F}(U_1 \cup U_2) \to \varinjlim_{K_i \subset U_i} \mathcal{F}(U_1) \times \mathcal{F}(U_2) \to \varinjlim_{K_i \subset U_i} \mathcal{F}(U_1 \cap U_2) \tag{2.5}$$

An open U contains $K_1 \cup K_2$ if and only if it's of the form $U_1 \cup U_2$ with $K_i \subset U_i$ (one can take $U_1 = U_2 = U$ for the direct implication), then by definition $\varinjlim_{K_i \subset U_i} \mathcal{F}(U_1 \cup U_2) = \alpha^* \mathcal{F}(K_1 \cup K_2)$.

The injective limit commute with the direct sum (as it is a coproduct), then:

$$\varinjlim_{K_i \subset U_i} \mathcal{F}(U_1) \times \mathcal{F}(U_2) = (\varinjlim_{K_i \subset U_i} \mathcal{F}(U_1)) \bigoplus (\varinjlim_{K_i \subset U_i} \mathcal{F}(U_2)) = \alpha^* \mathcal{F}(K_1) \bigoplus \alpha^* \mathcal{F}(K_2)$$

.

By the lemma 2.8 the limit $\varinjlim_{K_i \subset U_i} \mathcal{F}(U_1 \cap U_2)$ compute the same thing as $\varinjlim_{K_1 \cap K_2 \subset U} \mathcal{F}(U) = \alpha^* \mathcal{F}(K_1 \cap K_2)$.

Then the exact sequence (2.5) is in fact (2.3).

Let K be a compact, U a relatively comapct open such that $K \subset U$ and V an open suche that $\overline{U} \subset V$ then $K \subset V$. Conversely if V is an open containing K, then K is a comapct of V (locally compact as X is) and then admits an open neighborhood U relatively compact (in V).Thus (because the two limits are over the same set) one has the equality

$$\varinjlim_{K \subset U \text{open relatively compact}} \varinjlim_{\overline{U} \subset V \text{ open}} \mathcal{F}(V) = \varinjlim_{K \subset U \text{ open}} \mathcal{F}(V)$$

. Wich rewrite by definition as $\varinjlim_{K \subset U \text{ open relatively compact}} \alpha^* \mathcal{F}(\overline{U}) = \alpha^* \mathcal{F}(V)$ i.e. (2.4).

Then $\alpha^* \mathcal{F}$ is a $\mathcal{K}$-sheaf.

- Let $\mathcal{G}$ be a $\mathcal{K}$-sheaf. Let $K_1, \dots K_n$ be a family of compacts subsets let's show that the sequence $0 \to \mathcal{G}(\bigcup_{i=1}^{n} K_i) \to \prod_{i=1}^{n} \mathcal{G}(K_i) \to \prod_{i,j=1}^{n} \mathcal{G}(K_i \cap K_j)$.

  If the family is empty, then the sequence is $0 \to \mathcal{G} \to 0 \to 0$ wich is exact beause of (2.2).
  If $n = 1$ then the sequence is $0 \to \mathcal{G}(K_1) \to \mathcal{G}(K_2) \to 0$ wich is exact beause of (2.2).
  $\mathcal{G}$ is a $\mathcal{K}$-sheaf, then (by (2.3)) the map $\mathcal{G}(K \cup K') \to \mathcal{G}(K) \bigoplus \mathcal{G}(K')$ is injective, then (the direct products of two abélian groups is their product), then bi straightforward induction the map is injective $\mathcal{G}(\bigcup_{i=1}^{n} K_i) \to \prod_{i=1}^{n} \mathcal{G}(K_i)$.

Let's show the exactness of the other term in the sequence by induction, the base case of $n = 2$ is given by (2.3). Let's assume that it's exact for $n \in \mathbb{N}$ fixed. Let $K_1, \dots K_n, K_{n+1}$ be compacts subset and $(f_1, \dots, f_{n+1})$ be an element of the kernel of $\prod_{i=1}^{n+1} \mathcal{G}(K_i) \to \prod_{i,j=1}^{n+1} \mathcal{G}(K_i \cap K_j)$. Then $(f_1, \dots, f_n)$ is in the kernel of $\prod_{i=1}^{n} \mathcal{G}(K_i) \to \prod_{i,j=1}^{n} \mathcal{G}(K_i \cap K_j)$ so by induction hypothesis, it's of the form $(f|_{K_1}, \dots, f|_{K_n})$ for $f \in \mathcal{G}(K := \bigcup_{i=1}^{n} K_i)$. On the other hand, by the compatibility of reistriction $f|_{K \cap K_{n+1}} = f|_{K_1}|_{K \cap K_{n+1}} = f_1|_{K \cap K_{n+1}} = f_1|_{K_1 \cap K_{n+1}}|_{K \cap K_{n+1}}$ so $f|_{K \cap K_n+1} - f_{n+1}|_{K \cap K_n+1} = f_1|_{K_1 \cap K_{n+1}}|_{K \cap K_{n+1}} - f_{n+1}|_{K_1 \cap K_{n+1}}|_{K \cap K_{n+1}} = 0$ by hypothesis. Then by the exactness of (2.3) f and f_{n+1} are of the form $g|_K$ and $g|_{K_{n+1}}$, with $g \in \mathcal{G}(\bigcup_{i=1}^{n+1} K_i)$.

Then $(f_1, \dots, f_{n+1})$ is of the form $(g|_{K_1}, \dots g|_{K_{n+1}})$. Wicch conclude the exactness prof because the inclusion of the image into the kernel is starightforward by definition of the map.

Let $(U_a)_{a \in A}$ be a family of opens of X, one can consider the collections of family of compacts $(K_a)_{a \in A}$ such that $\forall a \in A/\ K_a \subset U_a$ and only a finite number of them are not empty (by the (2.2)adding enpty compacts in the family ad a product with zero in the exact sequence, wich does not changes the sequence) and take the projective limit of the previous exact sequence over it.

The sequence remains exact because projective limits are left exacts:

$$0 \to \varprojlim \mathcal{G}(\bigcup_{a \in A} K_a) \to \varprojlim \prod_{a \in A} \mathcal{G}(K_a) \to \varprojlim \prod_{a,b \in A} \mathcal{G}(K_a \cap K_b)$$

. The projective limits commmute with products, then the sequence is

$$0 \to \varprojlim \mathcal{G}(\bigcup_{a \in A} K_a) \to \prod_{a \in A} \varprojlim \mathcal{G}(K_a) \to \prod_{a,b \in A} \varprojlim \mathcal{G}(K_a \cap K_b)$$

By definition (because it's for a fixed a and does not depend of the family for other indexes) $\varprojlim \mathcal{G}(K_a) = \alpha_* \mathcal{G}(U_a)$ and $\varprojlim \mathcal{G}(K_a \cap K_b) = \alpha_* \mathcal{G}(U_a \cap U_b)$. Any compact included in $\bigcup_{a \in A} U_a$ is included in a finite number of the opens then $\varprojlim \mathcal{G}(\bigcup_{a \in A} K_a)$ compute $\alpha_* \mathcal{G}(\bigcup_{a \in A} U_a)$, then one get's the sheaf condition for $\alpha_* \mathcal{G}$.

- A morphisme between two ($\mathcal{K}$-)sheaves is by definition is by definition a morphisme between the two underling ($\mathcal{K}$-)presheaves then, the natural equality $\hom_{\mathrm{Sh}}(\alpha^* \mathcal{F}, \mathcal{G}) = \hom_{\mathrm{Sh}}(\mathcal{F}, \alpha_* \mathcal{G})$ is a consequence of 2.11

□

Lemma 2.13. *The previous adjoint pair give rise to an equivalence of category between shaeves and $\mathcal{K}$-sheaves*

Proof. By using 2.9, it's enough to show that for any sheaf $\mathcal{F}$ and $\mathcal{K}$-sheaf $\mathcal{G}$, the natural maps $\mathcal{F} \to \alpha_* \alpha^* \mathcal{U}$ and $\alpha^* \alpha_* \mathcal{G} \to \mathcal{G}$ are isomorphism. The fact of being a natural isomorphism can be checked locally.

- Let K be a comapct of X. One has to check that $\varinjlim_{K\subset U \text{ open}} \varprojlim_{U\supset K' \text{ compact}} \mathcal{G}(K') \to \mathcal{G}(K)$ is an isomorphism.

  Let U be an open relatively compact that contain K, for any $K' \subset U$ comapct,$\mathcal{G}$ define compatible maps $\mathcal{G}(\overline{U}) \to \mathcal{G}(K')$, then by the universal proprety of the projective limit one get's a map $\mathcal{G}(\overline{U}) \to \varprojlim_{U\supset K'} \mathcal{G}(K')$ such that the map $\mathcal{G}(\overline{U}) \to \varprojlim_{U\supset K'} \mathcal{G}(K') \to \mathcal{G}(K)$ is $\mathcal{G}(\overline{U}) \to \mathcal{G}(K)$. Then by taking the inductive limit over U, one get's ($\mathcal{G}(K)$ does not depend on U) that the canonical morphism $\varinjlim_{K\subset U} \mathcal{G}(\overline{U}) \to \mathcal{G}(K)$ factors that way: $\varinjlim_{K\subset U} \mathcal{G}(\overline{U}) \to \varinjlim_{K\subset U} \varprojlim_{U\supset K'} \mathcal{G}(K') \to \mathcal{G}(K)$. Then by (2.4) it's enough to show that the map

$$\varinjlim_{K\subset U \text{open relatively compact}} \mathcal{G}(\overline{U}) \to \varinjlim_{K\subset U \text{open relatively compact}} \varprojlim_{U\supset K' \text{compact}} \mathcal{G}(K')$$

  is an isomorphism.

  Let's build the map in the other direcction. By using the universal property of the inductive limit, one needs for any open relatively comapct U that contains K to build maps (compatibles with inclusion of opens)

$$\varprojlim_{U\supset K' \text{ compact}} \mathcal{G}(K') \to \varinjlim_{K\subset U \text{ open relatively comapct}} \mathcal{G}(\overline{U})$$

  K is a compact of U, then let V be an open subset (of U then of X) such that $\overline{V} \subset U$. $\overline{V}$ is a compact in U then there is a canonical projection $\varprojlim_{U\supset K' \text{ compact}} \mathcal{G}(K') \to \mathcal{G}(\overline{V})$ and V is an open relatively comapct that contains K then there is a canonical inclusion $\mathcal{G}(\overline{V}) \to \varinjlim_{K\subset U \text{ open relatively comapct}} \mathcal{G}(\overline{U})$, the composition of the two maps gives the desired morphism.

  One has to chack that it does not depend of the choice of V. If V' is an other choice, then $V \cup V'$ is an open subset of U that contains K and $\overline{V \cup V'} = \overline{V} \cup \overline{V'} \subset U$. $V \cap V'$ is an open subset of U that contains K and $\overline{V \cap V'} \subset \overline{V} \cap \overline{V'} \subset U$, then all the triangle of the following diagram are commutative (and thus the two morphism are equal):

$$\begin{tikzcd}
 & & \mathcal{G}(\overline{V'}) \arrow[dr] \arrow[drr] & & \\
\varprojlim_{U\supset K'} \mathcal{G}(K') \arrow[urr] \arrow[r] \arrow[drr] & \mathcal{G}(\overline{V \cup V'}) \arrow[ur] \arrow[dr] & & \mathcal{G}(\overline{V \cap V'}) \arrow[r] & \varinjlim_{K\subset U} \mathcal{G}(\overline{U}) \\
 & & \mathcal{G}(\overline{V'}) \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

  If W is an open relatively compact that contains U then (by universal property of the projective limit) there is a commutative triangle :

$$\begin{tikzcd}
\varprojlim\limits_{U\supset K'} \mathcal{G}(K') \arrow[r] & \mathcal{G}(\overline{V}) \\
\varprojlim\limits_{W\supset K'} \mathcal{G}(K') \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Then the triangle is also commutative:

$$\begin{tikzcd}
\varprojlim\limits_{U\supset K'} \mathcal{G}(K') \arrow[r] & \varinjlim\limits_{K\subset U} \mathcal{G}(\overline{U}) \\
\varprojlim\limits_{W\supset K'} \mathcal{G}(K') \arrow[u] \arrow[ur] &
\end{tikzcd}$$

That concludes the compatibility condition.

Let's now check that the two maps are the inverse of one another. If U and V are opens relatively compacts of X such that $K \subset V \subset U$, and W is an open such that $K \subset W \subset \overline{W} \subset V$, then y the previous constructions, the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{G}(\overline{U}) \arrow[rr] \arrow[dr] \arrow[dd] & & \varprojlim\limits_{U\supset K'} \mathcal{G}(K') \arrow[d] \arrow[dr] & & \\
 & \varinjlim\limits_{K\subset U} \mathcal{G}(\overline{U}) \arrow[r] & \varinjlim\limits_{K\subset U} \varprojlim\limits_{U\supset K'} \mathcal{G}(K') \arrow[r] & \mathcal{G}(\overline{W}) \arrow[r] & \varinjlim\limits_{K\subset U} \mathcal{G}(\overline{U}) \\
\mathcal{G}(\overline{V}) \arrow[ur] \arrow[rr] & & \varprojlim\limits_{V\supset K'} \mathcal{G}(K') \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

In particular to conclude that the following square is commutative:

$$\begin{tikzcd}
\mathcal{G}(\overline{U}) \arrow[r] \arrow[d] & \varinjlim\limits_{K\subset U} \mathcal{G}(\overline{U}) \arrow[d] \\
\mathcal{G}(\overline{V}) \arrow[r] & \varinjlim\limits_{K\subset U} \mathcal{G}(\overline{U})
\end{tikzcd}$$

One has to check that the morphism $\mathcal{G}(\overline{V}) \to \varprojlim\limits_{V\supset K'} \mathcal{G}(K') \to \mathcal{G}(\overline{W}) \to \varinjlim\limits_{K\subset U} \mathcal{G}(\overline{U})$ is the canonical inclusion. By the compatibility condition of the universal property of the projective limit $\mathcal{G}(\overline{V}) \to \varprojlim_{V \supset K'} \mathcal{G}(K') \to \mathcal{G}(\overline{W})$ is the map $\mathcal{G}(\overline{V} \supset \overline{W})$. And the maps $\mathcal{G}(\overline{V}) \to \mathcal{G}(\overline{W}) \to \varinjlim_{K \subset U} \mathcal{G}(\overline{U})$ is the canonical inclusion by the compatibility condition of the universal property of the injective limit.

However by the universal property of the inductive limit the only morphism that make this diagram commute is the identity.

For the other direction, one keeps the same notations. By the previous constructions the following diagram is commutative:

$$\begin{tikzcd}
\varprojlim\limits_{U \supset K'} \mathcal{G}(K') \arrow[d] \arrow[dr] & & \\
\varinjlim\limits_{K \subset U} \varprojlim\limits_{U \supset K'} \mathcal{G}(K') \arrow[r] & \mathcal{G}(\overline{W}) \arrow[r] \arrow[d] & \varinjlim\limits_{K \subset U} \mathcal{G}(\overline{U}) \arrow[d] \\
 & \varprojlim\limits_{W \supset K'} \mathcal{G}(K') \arrow[r] & \varinjlim\limits_{K \subset U} \varprojlim\limits_{U \supset K'} \mathcal{G}(K') \\
\varprojlim\limits_{V \supset K'} \mathcal{G}(K') \arrow[uu] \arrow[uur] & &
\end{tikzcd}$$

One can remark that the map $\varprojlim_{V \supset K'} \mathcal{G}(K') \to \mathcal{G}(\overline{W}) \to \varprojlim_{W \supset K'} \mathcal{G}(K')$ is the canonical map. By the universal property of the projective limit, it's enough to show that it's true when postcomposed by the maps (for K' a compact of W) $\varprojlim_{W \supset K'} \mathcal{G}(K') \to \mathcal{G}(K')$. By construction $\mathcal{G}(\overline{W}) \to \varprojlim_{W \supset K'} \mathcal{G}(K') \to \mathcal{G}(K') = \mathcal{G}(\overline{W} \subset K')$, so the statment is a compatibility condition in the universal property of the projective limit.

Then by the compatibility condition of the universal property of the inductive limit the map $\varprojlim_{W \supset K'} \mathcal{G}(K') \to \varprojlim_{W \supset K'} \mathcal{G}(K') \to \varinjlim_{K \subset U} \varprojlim_{U \supset K'} \mathcal{G}(K')$ is also the canonical map. Then the following diagram is commutative:

$$\begin{tikzcd}
\varprojlim\limits_{U \supset K'} \mathcal{G}(K') \arrow[r] \arrow[d] & \varinjlim\limits_{K \subset U} \varprojlim\limits_{U \supset K'} \mathcal{G}(K') \arrow[d] \\
\varprojlim\limits_{V \supset K'} \mathcal{G}(K') \arrow[r] & \varinjlim\limits_{K \subset U} \varprojlim\limits_{U \supset K'} \mathcal{G}(K')
\end{tikzcd}$$

An again by uniqueness it must be the identity, that concludes the proof.

- Let U be an open of X. One has to check that $\mathcal{F}(U) \to \varprojlim_{U \supset K \text{ compact}} \varinjlim_{K \subset U' \text{ open}} \mathcal{F}(U')$ is an isomorphism.

  One can apply the previous item with the compact $K = \{p\}$ (for $p \in X$) and then get that α_* preserve the stlaks. The fact that α^* preserves the stalks is straightforward by definition. Then the map $\mathcal{F}(U) \to \varprojlim_{U \supset K \text{ compact}} \varinjlim_{K \subset U' \text{ open}} \mathcal{F}(U')$ is an isomorphism once restricted to stalks, and because the two are sheaves on X they are then isomorphics.

$\square$

Chapter 3

Homotopy sheaves

Definition 3.1. *Let $\mathcal{F}^\bullet$ be complex of $\mathcal{K}$-presheaves then taking the cohomology defines a $\mathcal{K}$-presheaf denoted $H^\bullet\mathcal{F}^\bullet$.*

Definition 3.2. *A morphisme of complex of $\mathcal{K}$-presheave $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ is called quasi-isomorphism if it induces isomorphisms $H^i\mathcal{F}^\bullet \to H^i\mathcal{G}^\bullet$ for all i.*

Definition 3.3. *A complex of $\mathcal{K}$-presheaves $\mathcal{F}^\bullet$ is said to be a Homotopy-$\mathcal{K}$-sheave if the folowing conditions are satisfied:*

- $$\mathcal{F}^\bullet(\emptyset) \text{ is an acyclic complex} \tag{3.1}$$
- *For K_1 and K_2 two comapcts of X the folowing complex is acyclic:*
$$[\mathcal{F}^\bullet(K_1 \cup K_2) \to \mathcal{F}^{\bullet-1}(K_1) \bigoplus \mathcal{F}^{\bullet-1}(K_2) \to \mathcal{F}^{\bullet-2}(K_1 \cap K_2)] \tag{3.2}$$
- *For any compact K of X, the following natural morphism is a quasi-isomorphism*
$$\varinjlim_{K \subset U \text{ open relatively compact}} \mathcal{F}^\bullet(\overline{U}) \to \mathcal{F}^\bullet(K) \tag{3.3}$$

Lemma 3.4. *By using 1.4, (3.3) give rise to a "Mayer-Vietoris" long exact sequence:*

$$... \to H^k\mathcal{F}^\bullet(K_1 \cup K_2) \to H^k\mathcal{F}^\bullet(K_1) \bigoplus H^k\mathcal{F}^\bullet(K_2) \to H^k\mathcal{F}^\bullet(K_1 \cap K_2) \to ...$$

Lemma 3.5. *Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{K}$-presheaves. If $\mathcal{F}^\bullet$ has a finite filtration whose associated graded is a Homotopy-$\mathcal{K}$-sheaf, then $\mathcal{F}^\bullet$ is a Homotopy-$\mathcal{K}$-sheaf.*

Proof. TODO □

Lemma 3.6. *If $\mathcal{F}^\bullet$ is a homotopy-$\mathcal{K}$-sheaf, and $H^{-1}\mathcal{F}^\bullet = 0$ then $H^0\mathcal{F}^\bullet$ is a $\mathcal{K}$-sheaf*

Proof.

- $\mathcal{F}^\bullet(\emptyset)$ is acyclic then in particular it's cohomology in degree 0 is 0, then one gets (2.2)
- $H^{-1}\mathcal{F}^\bullet(K_1 \cap K_2) = 0$ then the first terms 3.4 gives the exact sequence of (2.3)

- Let K be a compact of X, the quasi-isomorphism of (2.4) gives (in particular) that $H^0\mathcal{F}^\bullet(K) = H^0(\varinjlim \mathcal{F}^\bullet(\bar{U}))$. To conclude one has to apply that the cohomology commute with inductive limit of a complex (Bouraki algèbre prop 1 X.28) and that in the category of presheaves of abélian groups, the limits are computed objectvise.

$\square$

Chapter 4

Pushforward, exceptional pushforward, and pullback

Let X and Y be two loccaly compacts hausdorf spaces and $f : X \to Y$ be a continuous map.

4.1 For Sheaves

Definition 4.1. *If $\mathcal{F}$ is a pre-sheaf over X, then the rule $U \mapsto \mathcal{F}(f^{-1}(U))$ defines a pre-shaef over Y.*

The functor obtained is denoted f_* and named the pushforward by f.

f_* send sheaves over X into sheaves over Y.

Proof. Let $(U_a)_{a\in A}$ be a family of opens of Y. Then one ca apply the sheaf condition of $\mathcal{F}$ with the family of opens of X: $(f^{-1}(U_a))_{a\in A}$. The result is the exact sequence:

$$0 \to \mathcal{F}(\bigcup_{a\in A} f^{-1}(U_a)) \to \prod_{a\in A} \mathcal{F}(f^{-1}(U_a)) \to \prod_{a,b\in A} \mathcal{F}(f^{-1}(U_a) \cap f^{-1}(U_b))$$

. On the other hand, the inverse image commute with union and intersections, then the previous exact sequence rewrites to

$$0 \to f_*\mathcal{F}(\bigcup_{a\in A} U_a) \to \prod_{a\in A} f_*\mathcal{F}(U_a) \to \prod_{a,b\in A} f_*\mathcal{F}(U_a \cap U_b)$$

. In other words, $f_*\mathcal{F}$ is a sheaf. □

Definition 4.2. *If $\mathcal{F}$ is a pre-sheaf over Y, then the rule $U \mapsto \varinjlim_{f(U)\subset V} \mathcal{F}(V)$ defines a pre-sheaf over Y.*

If $\mathcal{F}$ is a sheaf, the sheafification of the previous pre-sheaf is a denoted $f^*\mathcal{F}$ and called the pullback by f.

Definition 4.3. *If $f : X \to Y$ is the inclusion of an open subset, the exceptional pushforward by f: $f_!$ is defined by $f_!\mathcal{F}(U)$ being the subset of $f_*\mathcal{F}(U)$ of sections that vanish over a neighborhood of $Y - X$.*

It send the sheaves over X into the sheaves over Y

Proof. Let $U \supset V$ be two opens of Y and h be an element of $f_!\mathcal{F}(U)$, then h is an element of $\mathcal{F}(U \cap X)$ such that there is a W open that contains $Y \backslash X$ and such that $h|_{U \cap W \cap X} = 0$. Then $0 = h|_V|_{U \cap W \cap X} = h|_{V \cap U \cap W \cap X} = h|_{V \cap W \cap X}$ so $h|_V$ is in $f_!\mathcal{F}(V)$. So $f_!\mathcal{F}$ is well defined.

Let $(U_a)_{a \in A}$ be a family of opens of Y. The map $f_!\mathcal{F}(\bigcup_{a \in A} U_a \cap X) \to \prod_{a \in A} f_!\mathcal{F}(U_a)$ is a reistriction of an injective map (because of the sheaf condition of $f_*\mathcal{F}$), then it's also injective.

Let (h_a) be an element of the kernel of $\prod_{a \in A} f_!\mathcal{F}(U_a) \to \prod_{a,b \in A} f_!\mathcal{F}(U_a \cap U_b)$. By the sheaf condition of $f_*\mathcal{F}$, it's of the form $(h|_{U_a})$ with $h \in f_*\mathcal{F}(\bigcup_{a \in A} U_a)$. To conclude the sheaf condition for $f_!\mathcal{F}$ one has to check that h is $f_!\mathcal{F}(\bigcup_{a \in A} U_a)$.

By definition for any $a \in A$ there is an open V_a of Y that contains $Y \backslash X$ and such that $h_a|_{U_a \cap V_a \cap X} = 0$. So for all $a \in A$ $h_{U_a \cap V_a \cap X} = 0$. Let V be the union of the V_a, it contains $Y \backslash X$. the reistriction of $h|_V$ to all $V_a \cap X$ are 0, then by the first part of the sheaf condition, $h|_{V \cap X}$ is also 0, then h is in $f_!\mathcal{F}(\bigcup_{a \in A} U_a)$. □

4.2 For $\mathcal{K}$-sheaves

Let's assume that f is proper.

Lemma 4.4. *If K is a compact of Y, then $\{f^{-1}(U)\}_{K \subset U}$ is a basis of open neighborhoods of $f^{-1}(K)$.*

Proof. □

Definition 4.5. *If $\mathcal{F}$ is a pre-$\mathcal{K}$-sheaf over X, then the rule $K \mapsto \mathcal{F}(f^{-1}(K))$ defines a pre-$\mathcal{K}$-sheaf over Y.*

The functor obtained is denoted f_* and named the pushforward by f.

f_* send $\mathcal{K}$-sheaves over X into $\mathcal{K}$-sheaves over Y.

Proof. By the lemma 4.4, for K a compact of Y, $\varinjlim_{K \subset U \text{ open in Y}} \mathcal{F}(f^{-1}(U))$ computes $\varinjlim_{f^{-1}(K) \subset U \text{ open in X}} \mathcal{F}(U)$. In other words $f_*(\alpha^*\mathcal{F}) = \alpha^* f_*(\mathcal{F})$.

Then if $\mathcal{G}$ is a $\mathcal{K}$-sheaf, it's of the form $\alpha^*\mathcal{F}$ for $\mathcal{F}$ some sheaf. Then $f_*\mathcal{G}$ is isomorphic to $\alpha^* f_*(\mathcal{F})$ wich is a sheaf because of 4.1 and 2.12 □

Chapter 5

Čech cohomology

5.1 Čech cohomology of sheaves

Definition 5.1. *If X is a topological space, and $\mathcal{F}$ a sheaf over X, then let $\check{H}^\bullet(X;\mathcal{F}))$ be the Čech cohomology of X with coeficient in $\mathcal{F}$*

Definition 5.2. *If X is a topological space, K a comapct subset of X and $\mathcal{F}$ a sheaf over X, then let $\check{H}^\bullet_K(X;\mathcal{F}))$ be the Čech cohomology of X with support in K with coeficient in $\mathcal{F}$*

Definition 5.3. *Let $f : X \to Y$ be a continuous map between topological spaces, and $\mathcal{F}$ a sheaf over X. f induces a natural map $\check{H}^\bullet(Y;f_*\mathcal{F}) \to \check{H}^\bullet(X;\mathcal{F})$.*

Moreover if f is proper, one gets a natural map $\check{H}^\bullet_c(Y;f_*\mathcal{F}) \to \check{H}^\bullet_c(X;\mathcal{F})$

Lemma 5.4. *Let $f : X \to Y$ be an inclusion of open subset, then there is a natural isomorphism $f_! : \check{H}^\bullet_c(X;\mathcal{F}) \to \check{H}^\bullet(Y;f_!\mathcal{F})$*

Proof. □

5.2 Čech cohomology of complex of $\mathcal{K}$-sheaves

Definition 5.5. *Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{K}$-presheaves on a compact space X then we define the Čech cohomology $\check{H}(X;\mathcal{F}^\bullet)$ by TODO*

Remark 5.6. *By using the inclusion of $\mathcal{K}$-presheaves into complexes of $\mathcal{K}$-presheave, one get's a definition of Čech cohomology for $\mathcal{K}$-presheave.*

Lemma 5.7. *Let $mathcalF^\bullet$ be an acyclic complex of $\mathcal{K}$-presheaves, then $\check{H}^k(X;mathcalF^\bullet) = 0$*

Proof. TODO □

Lemma 5.8. *Let $0 \to \mathcal{F}^\bullet \to \mathcal{G}^\bullet \to \mathcal{H}^\bullet \to 0$ be a short exact sequence of complex of $\mathcal{K}$-presheaves. Then there is a long exact seuquence in čech cohomology:*

$$... \to \check{H}^k(X;\mathcal{F}^\bullet) \to \check{H}^k(X;\mathcal{G}^\bullet) \to \check{H}^k(X;\mathcal{H}^\bullet) \to ...$$

Proof. TODO □

Lemma 5.9. *If $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism then the induced maps $\check{H}^i\mathcal{F}^\bullet \to \check{H}^i\mathcal{G}^\bullet$ are isomorphims.*

Proof. By 1.3, the complex $[\mathcal{F}^\bullet \to \mathcal{G}^{\bullet-1}]$ is acyclic then by 5.7, it's čech cohomology is zero.

But there is a short exact sequence $0 \to \mathcal{G}^\bullet[-1] \to [\mathcal{F}^\bullet \to \mathcal{G}^{\bullet-1}] \to \mathcal{F}^\bullet \to 0$, then the long exact sequence induced by 5.8 gives the claimed isomorphisms. □

Proposition 5.10. *Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{K}$-presheaves that verify (3.1) and (3.2) then the canonical map $H^\bullet\mathcal{F}^\bullet \to \check{H}^\bullet(X;\mathcal{F}^\bullet)$ is an isomorphism.*

Proof. TODO

□

5.3 Čech cohomology is determined by stalks

Lemma 5.11. *Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{K}$-presheaves that verify (2.4) and such that all the stalks are 0 then $\check{H}^\bullet(X;\mathcal{F}) = 0$*

Proof. □

Lemma 5.12. *Let $\mathcal{F}^\bullet$ be a complex of $\mathcal{K}$-presheaves that verify (3.3) and $H^i\mathcal{F}^\bullet = 0$ for $i << 0$.*

Then if the stalks of $\mathcal{F}^\bullet$ are acyclics, $\check{H}^\bullet(X;\mathcal{F}^\bullet) = 0$

Proof. TODO □

Proposition 5.13. *Let $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ be complexes of $\mathcal{K}$-presheaves that verify (3.3) and $H^i\mathcal{F}^\bullet = H^i\mathcal{G}^\bullet = 0$ for i small enough.*

Then if a morphism $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ induces a quasi-isomorphism on stalks, $\check{H}^\bullet(X;\mathcal{F}^\bullet) = \check{H}^\bullet(X;\mathcal{G}^\bullet)$

Proof. □

Chapter 6

Purehomotopy $\mathcal{K}$-sheaves

Definition 6.1. *A homotopy $\mathcal{K}$-sheaf $\mathcal{F}^\bullet$ is said to be pure on X if:*

- *For $p \in X$ and $i \neq 0$, $(H^i\mathcal{F}^\bullet)_p = 0$*
- *$H^i\mathcal{F}^\bullet = 0$ for $i << 0$ locally on X: ie for all $p \in X$ there is an open neighbourhoud U of p and an integer N such that for $i \leq N$ and $K \subset U$: $H^i\mathcal{F}^\bullet(K) = 0$*

Lemma 6.2. *Let $\mathcal{F}^\bullet$ be a pure-homotopy $\mathcal{K}$-sheaf. Then:*

- *For $i < 0$ $H^i\mathcal{F}^\bullet = 0$*
- *$H^0\mathcal{F}^\bullet$ is a $\mathcal{K}$-sheaf.*

Proof. TODO □

Proposition 6.3. *Let $\mathcal{F}^\bullet$ be a pure-homotopy $\mathcal{K}$-sheaf. Then there is a canonical isomorphism:*

$$H^\bullet\mathcal{F}^\bullet(X) = \check{H}^\bullet(X; H^0\mathcal{F}^\bullet)$$

.

More generaly: Let $[\mathcal{F}^\bullet_0 \to \dots \mathcal{F}^{\bullet-n}_n]$ be a complex of pure-homotopy $\mathcal{K}$-sheaves, then there is a canonical isomorphism:

$$H^\bullet[\mathcal{F}^\bullet_0(X) \to \dots \mathcal{F}^{\bullet-n}_n(X)] = \check{H}^\bullet(X; [H^0\mathcal{F}^\bullet_0 \to \dots \to (H^0\mathcal{F}^\bullet_n)[n]])$$

Proof. TODO □

Chapter 7

Poincaré–Lefschetz duality

Uses chech cohomology with compact supports for sheaves

Definition 7.1. *Let M be a topological manifold, the rule $\mathfrak{o}_M : K \mapsto H_{\dim M}(M, M\backslash K)$ defines a $\mathcal{K}-$sheaf, called the orientation $\mathcal{K}-$sheaf of M.*

If M is a manifold with boundary, let $j : M\backslash\partial M \to M$ denote the canonical inclusion. The the orientation shaeves of M are defined as follows:

- $\mathfrak{o}_M := j_*\mathfrak{o}_{M\backslash\partial M}$
- $\mathfrak{o}_{M\quad rel\partial} := j_!\mathfrak{o}_{M\backslash\partial M}$

Definition 7.2. *Singular chains*

Lemma 7.3. *Let X be a topological manifold, then we have:*

- *$C_\bullet(X, X)$ is an acyclic complex*
- *For A and B two closed subsets of X the folowing complex is acyclic:*

$$[C_\bullet(X, X\backslash(A\cup B)) \to C_{\bullet+1}(X, X\backslash A) \bigoplus C_{\bullet+1}(X, X\backslash B) \to C_{\bullet+2}(X, X\backslash(A\cap B))]$$

- *For a family $(K_a)_{a\in A}$ of closed subsets of X wich is filtered (for any $a, b \in A$ there is $c \in A$ such that $K_c \subset K_a \cap K_b$)any compact K of X, the following natural morphism is a quasi-isomorphism*

$$\varinjlim_{a\in A} C_\bullet(X, X\backslash K_a) \to C_\bullet(X, X\backslash(\bigcap_{a\in A} K_a))$$

Proof. TODO □

Lemma 7.4. *Let M be a topological manifold of dimension n with boundary, $i : X \to M$ a closed subset, and $N \subset \partial M$ a closed subset that locally looks like $\emptyset \subset \mathbb{R}^{n-1}, \mathbb{R}_{>0} \times R^{n-2} \subset \mathbb{R}^{n-1}$ or $\mathbb{R}^{n-1}$. Let $j : \mathring{M} \cup \mathring{N} \to M$ be the canonical inclusion. Then there is a canonical isomorphism:*

$$H^\bullet[C_{n-1-\bullet}(N, N\backslash X) \to C_{n-1-\bullet}(M, M\backslash X)] = \check{H}^\bullet_c(X; i^*j_!j^*\mathfrak{o}_M)$$

Proof. uses tout les poussé en avatn, tiré en arière et tout ça, la cohomologie à support compact □

Chapter 8

Homotopy colimits

8.1 Homotopy colimits

Definition 8.1.

Definition 8.2.

8.2 Homotopy colimits of pure homotopy $\mathscr{K}$-sheaves

Lemma 8.3.

Proof. □

Lemma 8.4.

Chapter 9

Steenrod homology