

M2

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# Chapter 1

## Introduction

### 1.1 Total Complexes

**Definition 1.1.** If  $A_0^\bullet \rightarrow \dots \rightarrow A_n^\bullet$  is a sequence of maps (with  $f_i : A_i \rightarrow A_{i+1}$ ) of complex such that the composition of two consecutive maps is 0, then let's denote  $[A_0^\bullet \rightarrow \dots \rightarrow A_n^{\bullet-n}]$  the total complex of this double complex, defined by the following data:

- The object in degré  $k$  is  $\bigoplus_{i=0}^n A_i[-i]^k$
- The differential is given by the matrix 
$$\begin{pmatrix} d_{A_0} & 0 & \dots & \dots & 0 \\ f_0 & -d_{A_1} & \dots & \dots & 0 \\ 0 & f_1 & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & (-1)^{n-1}d_{A_{n-1}} & 0 \\ 0 & 0 & \dots & f_{n-1} & (-1)^n d_{A_n} \end{pmatrix}$$

*Proof.* One needs to check that the matrix square is 0. Let  $M$  be this matrix and  $(i, j)$  be integers.

$$M^2[i, j] = \sum_{k=1}^n M[i, k]M[k, j] = M[i, i]M[i, j] + M[i, i-1]M[i-1, j]$$

One can distinguish four cases:

- If  $j$  is not in  $\{i-2, i-1, i\}$  then the two terms are 0.
- If  $j = i$ , then  $M^2[i, j] = ((-1)^i d_{A_i})^2 + 0 = 0$ .
- If  $j = i-1$  then  $M^2[i, j] = 0 + (-1)^i d_{A_i} \circ f_i + (-1)^{i+1} d_{A_{i+1}} \circ f_i = 0$  because  $f_i$  is a morphism of complex.
- If  $j = i-2$  then  $M^2[i, j] = f_i \circ f_{i-1} = 0$ .

□

**Remark 1.2.** In particular, if  $f : A^\bullet \rightarrow B^\bullet$  is a morphism of complex then  $[A^\bullet \rightarrow B^{\bullet-1}]$  is the cone of the morphism  $f$ .

**Lemma 1.3.** *A morphism of complex  $f : A^\bullet \rightarrow B^\bullet$  is a quasi isomorphism if and only if, its cone is acyclic.*

*Proof.* One get's a short exact sequence  $0 \rightarrow B^\bullet[-1] \rightarrow [A^\bullet \rightarrow B^{\bullet-1}] \rightarrow A^\bullet \rightarrow 0$  by using the canonical inclusion and projection over the direct sum. The long exact sequence induced in cohomology is then:

$$\dots H^{k-1}A^\bullet \rightarrow H^k B^\bullet[-1] \rightarrow H^k[A^\bullet \rightarrow B^{\bullet-1}] \rightarrow H^k A^\bullet \rightarrow H^{k+1} B^\bullet[-1] \dots$$

By using the fact that  $H^k B^\bullet[-1] = H^{k-1} B^\bullet$  one gets:

$$\dots \rightarrow H^{k-1}A^\bullet \rightarrow H^{k-1}B^\bullet \rightarrow H^k[A^\bullet \rightarrow B^{\bullet-1}] \rightarrow H^k A^\bullet \rightarrow H^k B^\bullet \dots$$

And then the statement is straightforward by reading the exact sequence. □

**Lemma 1.4.** *If a complex  $[A^\bullet \rightarrow B^{\bullet-1} \rightarrow C^{\bullet-2}]$  is acyclic then there is a long exact sequence*

$$\dots \rightarrow H^k A \rightarrow H^k B \rightarrow H^k C \rightarrow \dots$$

*Proof.* One can see that by construction there is a canonical isomorphism of complexess:  $[A^\bullet \rightarrow B^{\bullet-1} \rightarrow C^{\bullet-2}] = [A^\bullet \rightarrow [B^\bullet \rightarrow C^{\bullet-1}]^{\bullet-1}]$ .

Then by the previous lemma:  $A^\bullet \rightarrow [B^\bullet \rightarrow C^{\bullet-1}]$  is a quasi isomorphism. One can then rewrite the long exact sequence in cohomology givent by the short exact sequence  $0 \rightarrow C^\bullet[-1] \rightarrow [B^\bullet \rightarrow C^{\bullet-1}] \rightarrow B^\bullet \rightarrow 0$  wich is (as in the previous lemma):

$$\dots \rightarrow H^{k-1}B^\bullet \rightarrow H^{k-1}C^\bullet \rightarrow H^k[B^\bullet \rightarrow C^{\bullet-1}] \rightarrow H^k B^\bullet \rightarrow H^k C^\bullet \rightarrow \dots$$

The result is then a long exact sequence :

$$\dots \rightarrow H^{k-1}B^\bullet \rightarrow H^{k-1}C^\bullet \rightarrow H^k A^\bullet \rightarrow H^k B^\bullet \rightarrow H^k C^\bullet \rightarrow \dots$$

□

## Chapter 2

# Presheaves and sheaves

Let  $X$  be a locally compact Hausdorff space.

### 2.1 Sheaves

**Definition 2.1.** A presheaf on  $X$  is a contravariant functor from the category of open sets of  $X$  to abelian groups.

**Definition 2.2.** If  $\mathcal{F}$  is a presheaf on  $X$  and  $p \in X$  then the stalk of  $\mathcal{F}$  at  $p$  is the abelian group  $\mathcal{F}_p := \varinjlim_{p \in U \text{ open}} \mathcal{F}(U)$ .

**Definition 2.3.** If  $\mathcal{F}$  is a presheaf on  $X$ , it is said to be a sheaf if for any  $U \subset X$  open and any covering family of  $U$   $(U_a)_{a \in A}$  one has the exact sequence:

$$0 \rightarrow \mathcal{F}(U) \rightarrow \prod_{a \in A} \mathcal{F}(U_a) \rightarrow \prod_{a, b \in A} \mathcal{F}(U_a \cap U_b) \quad (2.1)$$

### 2.2 $\mathcal{K}$ -sheaves

**Definition 2.4.** A  $\mathcal{K}$ -presheaf on  $X$  is a contravariant functor from the category of compact sets of  $X$  to abelian groups.

**Definition 2.5.** If  $\mathcal{F}$  is a  $\mathcal{K}$ -presheaf on  $X$  and  $p \in X$  then the stalk of  $\mathcal{F}$  at  $p$  is the abelian group  $\mathcal{F}_p := \varinjlim_{p \in K \text{ compact}} \mathcal{F}(K) = \mathcal{F}(\{p\})$ .

**Definition 2.6.** If  $\mathcal{F}$  is a  $\mathcal{K}$ -presheaf on  $X$ , it is said to be a  $\mathcal{K}$ -sheaf if the following conditions are satisfied:

•

$$\mathcal{F}(\emptyset) = 0 \quad (2.2)$$

• For  $K_1$  and  $K_2$  two compact sets of  $X$  the following sequence is exact:

$$0 \rightarrow \mathcal{F}(K_1 \cup K_2) \rightarrow \mathcal{F}(K_1) \oplus \mathcal{F}(K_2) \rightarrow \mathcal{F}(K_1 \cap K_2) \quad (2.3)$$

- For any compact  $K$  of  $X$ , the following natural morphism is an isomorphism

$$\lim_{\substack{\longrightarrow \\ K \subset U \text{ open relatively compact}}} \mathcal{F}(\overline{U}) \rightarrow \mathcal{F}(K) \quad (2.4)$$

**Remark 2.7.** (2.4) is well defined because if  $K$  is a compact subset of  $X$ , then for  $x \in K$  let  $U_x$  be an open neighborhood relatively compact (which exists by local compactness), the family  $(U_x)_{x \in K}$  covers  $K$  then one can extract a finite cover of it :  $U_1, \dots, U_n$  and then  $\cup_{i=1}^n U_i$  is an open neighborhood, and a finite union of relatively compact, then it's relatively compact.  $(\overline{\cup_{i=1}^n U_i} = \cup_{i=1}^n \overline{U_i})$

## 2.3 Technical lemmas

**Lemma 2.8.** If  $K_1, \dots, K_n$  are compact of  $X$  then  $\{U_1 \cap \dots \cap U_n\}_{U_i \supset K_i \text{ open in } X}$  is a cofinal system of neighborhoods of  $K_1 \cap \dots \cap K_n$ .

*Proof.* □

**Lemma 2.9.** If  $\mathcal{C}$  and  $\mathcal{D}$  are two categories,  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $G : \mathcal{D} \rightarrow \mathcal{C}$  two functors such that  $(F, G)$  is an adjoint pair. Then for  $(F, G)$  to be an equivalence of category, it's enough to have that the canonical natural transformations  $\text{id}_{\mathcal{D}} \Rightarrow F \circ G$  and  $G \circ F \Rightarrow \text{id}_{\mathcal{C}}$  are isomorphisms.

*Proof.* CategoryTheory.Adjunction.toEquivalence in mathlib □

## 2.4 Equivalence of category

**Definition 2.10.**

- If  $\mathcal{F}$  is a presheaf then let  $\alpha^* \mathcal{F}$  be the  $\mathcal{K}$ -presheaf:

$$K \mapsto \lim_{\substack{\longrightarrow \\ K \subset U \text{ open}}} \mathcal{F}(U)$$

- If  $\mathcal{G}$  is a  $\mathcal{K}$ -presheaf then let  $\alpha_* \mathcal{G}$  be the presheaf:

$$U \mapsto \lim_{\substack{\longleftarrow \\ U \supset K \text{ compact}}} \mathcal{G}(K)$$

**Proposition 2.11.** The pair  $(\alpha^*, \alpha_*)$  is an adjoint pair.

*Proof.* • Let  $\tau$  be an element of  $\text{hom}(\alpha^* \mathcal{F}, \mathcal{G})$ . It's the data of morphism  $\tau_K$  for  $K$  a compact of  $X$  such that for any  $K$  and  $K'$  compacts

$$\begin{array}{ccc} \lim_{\substack{\longrightarrow \\ K \subset U}} \mathcal{F}(U) & \xrightarrow{\tau_K} & \mathcal{G}(K) \\ \downarrow & & \downarrow \\ \lim_{\substack{\longrightarrow \\ K' \subset U}} \mathcal{F}(U) & \xrightarrow{\tau_{K'}} & \mathcal{G}(K') \end{array} \quad (2.5)$$

is a commutative square. Then for any  $U$  and  $V$  opens, by composing with the commutative square

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \varinjlim_{K \subset U} \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \longrightarrow & \varinjlim_{K' \subset U} \mathcal{F}(U) \end{array}$$

one get's a commutative square :

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{G}(K) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \longrightarrow & \mathcal{G}(K') \end{array} \tag{2.6}$$

. Conversely such data give rise (by taking the limit over  $U$  and  $V$ ) to a commutative square such as in (2.5)

- On the other hand if one takes the limit over  $K$  and  $K'$  one get's a commutative square

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \varprojlim_{K \subset U} \mathcal{G}(K) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \longrightarrow & \varprojlim_{K \subset V} \mathcal{G}(K) \end{array}$$

(that allow to recover the previous one in the same as before) wich is the data of an element of  $\text{hom}(\mathcal{F}, \alpha_* \mathcal{G})$ .

Then the elements of  $\text{hom}(\alpha^* \mathcal{F}, \mathcal{G})$  and  $\text{hom}(\mathcal{F}, \alpha_* \mathcal{G})$  are both obtained by a natural construction (in  $\mathcal{F}$  and  $\mathcal{G}$ ) applied to (2.6).  $\square$

**Lemma 2.12.**

- $\alpha^*$  send sheaves to  $\mathcal{K}$ -sheaves
- $\alpha^*$  send  $\mathcal{K}$ -sheaves to sheaves
- The restrictions obtained still form an adjoint pair between sheaves and  $\mathcal{K}$ -sheaves.

*Proof.*

- Let  $\mathcal{F}$  be a sheaf. The condition  $\emptyset \subset U$  is always satisfied and  $\emptyset$  is minimal among open subset for the inclusion then  $(\alpha^*)(\mathcal{F})(\emptyset) = \mathcal{F}(\emptyset)$ . One can apply the sheaf condition to the empty family and obtain the exact sequence  $0 \rightarrow \mathcal{F}(\emptyset) \rightarrow \Pi_\emptyset = 0$ , and then (2.2).

Let  $K_1, K_2$  be two of compacts of  $X$ , let  $U_1, U_2$  be a two opens such that  $K_i \subset U_i$  for all  $i$ . Then the sheaf condition gives an exact sequence (because for abelian groups the product is the direct sum)  $0 \rightarrow \mathcal{F}(U_1 \cup U_2) \rightarrow \mathcal{F}(U_1) \oplus \mathcal{F}(U_2) \rightarrow \mathcal{F}(U_1 \cap U_2)$ . The injective limits are exact then taking the limits over those opens gives an exact sequence:

$$0 \rightarrow \lim_{\substack{\longrightarrow \\ K_i \subset U_i}} \mathcal{F}(U_1 \cup U_2) \rightarrow \lim_{\substack{\longrightarrow \\ K_i \subset U_i}} \mathcal{F}(U_1) \oplus \mathcal{F}(U_2) \rightarrow \lim_{\substack{\longrightarrow \\ K_i \subset U_i}} \mathcal{F}(U_1 \cap U_2) \quad (2.7)$$

An open  $U$  contains  $K_1 \cup K_2$  if and only if it's of the form  $U_1 \cup U_2$  with  $K_i \subset U_i$  (one can take  $U_1 = U_2 = U$  for the direct implication), then by definition  $\lim_{\substack{\longrightarrow \\ K_i \subset U_i}} \mathcal{F}(U_1 \cup U_2) = \alpha^* \mathcal{F}(K_1 \cup K_2)$ .

The injective limit commute with the direct product, then:

$$\lim_{\substack{\longrightarrow \\ K_i \subset U_i}} \mathcal{F}(U_1) \oplus \mathcal{F}(U_2) = (\lim_{\substack{\longrightarrow \\ K_i \subset U_i}} \mathcal{F}(U_1)) \oplus (\lim_{\substack{\longrightarrow \\ K_i \subset U_i}} \mathcal{F}(U_2)) = \alpha^* \mathcal{F}(K_1) \oplus \alpha^* \mathcal{F}(K_2)$$

.

By the lemma 2.8 the limit  $\lim_{\substack{\longrightarrow \\ K_i \subset U_i}} \mathcal{F}(U_1 \cap U_2)$  compute the same thing as  $\lim_{\substack{\longrightarrow \\ K_1 \cap K_2 \subset U}} \mathcal{F}(U) = \alpha^* \mathcal{F}(K_1 \cap K_2)$ .

Then the exact sequence (2.7) is in fact (2.3).

Let  $K$  be a compact,  $U$  a relatively comapct open such that  $K \subset U$  and  $V$  an open such that  $\bar{U} \subset V$  then  $K \subset V$ . Conversely if  $V$  is an open containing  $K$ , then  $K$  is a compact of  $V$  (locally compact as  $X$  is) and then admits an open neighborhood  $U$  relatively compact (in  $V$ ). Thus (because the two limits are over the same set) one has the equality

$$\lim_{\substack{\longrightarrow \\ K \subset U \text{ open relatively compact}}} \mathcal{F}(U) = \lim_{\substack{\longrightarrow \\ \bar{U} \subset V \text{ open}}} \mathcal{F}(U) = \lim_{\substack{\longrightarrow \\ K \subset U \text{ open}}} \mathcal{F}(U)$$

. Wich rewrite by definition as  $\lim_{\substack{\longrightarrow \\ K \subset U \text{ open relatively compact}}} \alpha^* \mathcal{F}(\bar{U}) = \alpha^* \mathcal{F}(V)$  i.e. (2.4)

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- A morphisme between two  $(\mathcal{K})$ -sheaves is by definition is by definition a morphisme between the two underling  $(\mathcal{K})$ -presheaves then, the natural equality  $\text{hom}_{\text{Sh}}(\alpha^* \mathcal{F}, \mathcal{G}) = \text{hom}_{\text{Sh}}(\mathcal{F}, \alpha_* \mathcal{G})$  is a consequence of 2.11

□

**Lemma 2.13.** *The previous adjoint pair give rise to an equivalence of category between shaeves and  $\mathcal{K}$ -sheaves*

*Proof.* By using 2.9, it's enough to show that for any sheaf  $\mathcal{F}$  and  $\mathcal{K}$ -sheaf  $\mathcal{G}$ , the natural maps  $\mathcal{F} \rightarrow \alpha_* \alpha^* \mathcal{F}$  and  $\alpha^* \alpha_* \mathcal{G} \rightarrow \mathcal{G}$  are isomorphism. The fact of being a natural isomorphism can be checked locally.

- Let  $U$  be an open of  $X$ . One has to check that  $\mathcal{F}(U) \rightarrow \varprojlim_{U \supset K \text{ compact}} \varinjlim_{K \subset U' \text{ open}} \mathcal{F}(U')$  is an isomorphism.

•

□



## Chapter 3

# Homotopy sheaves

**Definition 3.1.** Let  $\mathcal{F}^\bullet$  be complex of  $\mathcal{K}$ -presheaves then taking the cohomology defines a  $\mathcal{K}$ -presheaf denoted  $H^\bullet \mathcal{F}^\bullet$ .

**Definition 3.2.** A morphism of complex of  $\mathcal{K}$ -presheaf  $\mathcal{F}^\bullet \rightarrow \mathcal{G}^\bullet$  is called quasi-isomorphism if it induces isomorphisms  $H^i \mathcal{F}^\bullet \rightarrow H^i \mathcal{G}^\bullet$  for all  $i$ .

**Definition 3.3.** A complex of  $\mathcal{K}$ -presheaves  $\mathcal{F}^\bullet$  is said to be a Homotopy- $\mathcal{K}$ -sheaf if the following conditions are satisfied:

- 

$$\mathcal{F}^\bullet(\emptyset) \text{ is an acyclic complex} \quad (3.1)$$

- For  $K_1$  and  $K_2$  two compact of  $X$  the following complex is acyclic:

$$[\mathcal{F}^\bullet(K_1 \cup K_2) \rightarrow \mathcal{F}^{\bullet-1}(K_1) \bigoplus \mathcal{F}^{\bullet-1}(K_2) \rightarrow \mathcal{F}^{\bullet-2}(K_1 \cap K_2)] \quad (3.2)$$

- For any compact  $K$  of  $X$ , the following natural morphism is a quasi-isomorphism

$$\lim_{\substack{\longrightarrow \\ K \subset U \text{ open relatively compact}}} \mathcal{F}^\bullet(\overline{U}) \rightarrow \mathcal{F}^\bullet(K) \quad (3.3)$$

**Lemma 3.4.** By using 1.4, (3.3) give rise to a "Mayer-Vietoris" long exact sequence:

$$\dots \rightarrow H^k \mathcal{F}^\bullet(K_1 \cup K_2) \rightarrow H^k \mathcal{F}^\bullet(K_1) \bigoplus H^k \mathcal{F}^\bullet(K_2) \rightarrow H^k \mathcal{F}^\bullet(K_1 \cap K_2) \rightarrow \dots$$

**Lemma 3.5.** Let  $\mathcal{F}^\bullet$  be a complex of  $\mathcal{K}$ -presheaves. If  $\mathcal{F}^\bullet$  has a finite filtration whose associated graded is a Homotopy- $\mathcal{K}$ -sheaf, then  $\mathcal{F}^\bullet$  is a Homotopy- $\mathcal{K}$ -sheaf.

*Proof.* TODO □

**Lemma 3.6.** If  $\mathcal{F}^\bullet$  is a homotopy- $\mathcal{K}$ -sheaf, and  $H^{-1} \mathcal{F}^\bullet = 0$  then  $H^0 \mathcal{F}^\bullet$  is a  $\mathcal{K}$ -sheaf

*Proof.*

- $\mathcal{F}^\bullet(\emptyset)$  is acyclic then in particular it's cohomology in degree 0 is 0, then one gets (2.2)
- $H^{-1} \mathcal{F}^\bullet(K_1 \cap K_2) = 0$  then the first terms 3.4 gives the exact sequence of (2.3)
- TODO

□

## Chapter 4

# Pushforward, exceptional pushforward, and pullback

Let  $X$  and  $Y$  be two locally compact hausdorff spaces and  $f : X \rightarrow Y$  be a continuous map.

### 4.1 For Sheaves

**Definition 4.1.** If  $\mathcal{F}$  is a pre-sheaf over  $X$ , then the rule  $U \mapsto \mathcal{F}(f^{-1}(U))$  defines a pre-sheaf over  $Y$ .

The functor obtained is denoted  $f_*$  and named the pushforward by  $f$ .  
 $f_*$  send sheaves over  $X$  into sheaves over  $Y$ .

*Proof.* TODO □

**Definition 4.2.** If  $\mathcal{F}$  is a pre-sheaf over  $Y$ , then the rule  $U \mapsto \varinjlim_{f(U) \subset V} \mathcal{F}(V)$  defines a pre-sheaf over  $Y$ .

If  $\mathcal{F}$  is a sheaf, the sheafification of the previous pre-sheaf is denoted  $f^*\mathcal{F}$  and called the pullback by  $f$ .

*Proof.* TODO □

**Definition 4.3.** If  $f : X \rightarrow Y$  is the inclusion of an open subset, the exceptional pushforward by  $f$ :  $f_!$  is defined by  $f_!\mathcal{F}(U)$  being the subset of  $f_*\mathcal{F}(U)$  of sections that vanish over a neighborhood of  $Y - X$ .

It send the sheaves over  $X$  into the sheaves over  $Y$

*Proof.* TODO □

### 4.2 For $\mathcal{K}$ -sheaves

Let's assume that  $f$  is proper.

**Definition 4.4.** If  $\mathcal{F}$  is a pre- $\mathcal{K}$ -sheaf over  $X$ , then the rule  $K \mapsto \mathcal{F}(f^{-1}(K))$  defines a pre- $\mathcal{K}$ -sheaf over  $Y$ .

The functor obtained is denoted  $f_*$  and named the pushforward by  $f$ .  
 $f_*$  send  $\mathcal{K}$ -sheaves over  $X$  into  $\mathcal{K}$ -sheaves over  $Y$ .

*Proof.* TODO

□

## Chapter 5

# Čech cohomology

### 5.1 Čech cohomology of sheaves

**Definition 5.1.** If  $X$  is a topological space, and  $\mathcal{F}$  a sheaf over  $X$ , then let  $\check{H}^\bullet(X; \mathcal{F})$  be the Čech cohomology of  $X$  with coefficient in  $\mathcal{F}$

**Definition 5.2.** If  $X$  is a topological space,  $K$  a compact subset of  $X$  and  $\mathcal{F}$  a sheaf over  $X$ , then let  $\check{H}_K^\bullet(X; \mathcal{F})$  be the Čech cohomology of  $X$  with support in  $K$  with coefficient in  $\mathcal{F}$

**Definition 5.3.** Let  $f : X \rightarrow Y$  be a continuous map between topological spaces, and  $\mathcal{F}$  a sheaf over  $X$ .  $f$  induces a natural map  $\check{H}^\bullet(Y; f_*\mathcal{F}) \rightarrow \check{H}^\bullet(X; \mathcal{F})$ .

Moreover if  $f$  is proper, one gets a natural map  $\check{H}_c^\bullet(Y; f_*\mathcal{F}) \rightarrow \check{H}_c^\bullet(X; \mathcal{F})$

**Lemma 5.4.** Let  $f : X \rightarrow Y$  be an inclusion of open subset, then there is a natural isomorphism  $f_! : \check{H}_c^\bullet(X; \mathcal{F}) \rightarrow \check{H}^\bullet(Y; f_!\mathcal{F})$

*Proof.* □

### 5.2 Čech cohomology of complex of $\mathcal{K}$ -sheaves

**Definition 5.5.** Let  $\mathcal{F}^\bullet$  be a complex of  $\mathcal{K}$ -presheaves on a compact space  $X$  then we define the Čech cohomology  $\check{H}(X; \mathcal{F}^\bullet)$  by `TODO`

**Remark 5.6.** By using the inclusion of  $\mathcal{K}$ -presheaves into complexes of  $\mathcal{K}$ -presheaves, one gets a definition of Čech cohomology for  $\mathcal{K}$ -presheaves.

**Lemma 5.7.** Let  $\mathcal{F}^\bullet$  be an acyclic complex of  $\mathcal{K}$ -presheaves, then  $\check{H}^k(X; \mathcal{F}^\bullet) = 0$

*Proof.* `TODO` □

**Lemma 5.8.** Let  $0 \rightarrow \mathcal{F}^\bullet \rightarrow \mathcal{G}^\bullet \rightarrow \mathcal{H}^\bullet \rightarrow 0$  be a short exact sequence of complex of  $\mathcal{K}$ -presheaves. Then there is a long exact sequence in Čech cohomology:

$$\dots \rightarrow \check{H}^k(X; \mathcal{F}^\bullet) \rightarrow \check{H}^k(X; \mathcal{G}^\bullet) \rightarrow \check{H}^k(X; \mathcal{H}^\bullet) \rightarrow \dots$$

*Proof.* `TODO` □

**Lemma 5.9.** *If  $\mathcal{F}^\bullet \rightarrow \mathcal{G}^\bullet$  is a quasi-isomorphism then the induced maps  $\check{H}^i \mathcal{F}^\bullet \rightarrow \check{H}^i \mathcal{G}^\bullet$  are isomorphisms.*

*Proof.* By 1.3, the complex  $[\mathcal{F}^\bullet \rightarrow \mathcal{G}^{\bullet-1}]$  is acyclic then by 5.7, it's Čech cohomology is zero.

But there is a short exact sequence  $0 \rightarrow \mathcal{G}^\bullet[-1] \rightarrow [\mathcal{F}^\bullet \rightarrow \mathcal{G}^{\bullet-1}] \rightarrow \mathcal{F}^\bullet \rightarrow 0$ , then the long exact sequence induced by 5.8 gives the claimed isomorphisms.  $\square$

**Proposition 5.10.** *Let  $\mathcal{F}^\bullet$  be a complex of  $\mathcal{K}$ -presheaves that verify (3.1) and (3.2) then the canonical map  $H^\bullet \mathcal{F}^\bullet \rightarrow \check{H}^\bullet(X; \mathcal{F}^\bullet)$  is an isomorphism.*

*Proof.* TODO  $\square$

### 5.3 Čech cohomology is determined by stalks

**Lemma 5.11.** *Let  $\mathcal{F}^\bullet$  be a complex of  $\mathcal{K}$ -presheaves that verify (2.4) and such that all the stalks are 0 then  $\check{H}^\bullet(X; \mathcal{F}) = 0$*

*Proof.*  $\square$

**Lemma 5.12.** *Let  $\mathcal{F}^\bullet$  be a complex of  $\mathcal{K}$ -presheaves that verify (3.3) and  $H^i \mathcal{F}^\bullet = 0$  for  $i \ll 0$ . Then if the stalks of  $\mathcal{F}^\bullet$  are acyclics,  $\check{H}^\bullet(X; \mathcal{F}^\bullet) = 0$*

*Proof.* TODO  $\square$

**Proposition 5.13.** *Let  $\mathcal{F}^\bullet$  and  $\mathcal{G}^\bullet$  be complexes of  $\mathcal{K}$ -presheaves that verify (3.3) and  $H^i \mathcal{F}^\bullet = H^i \mathcal{G}^\bullet = 0$  for  $i$  small enough.*

*Then if a morphism  $\mathcal{F}^\bullet \rightarrow \mathcal{G}^\bullet$  induces a quasi-isomorphism on stalks,  $\check{H}^\bullet(X; \mathcal{F}^\bullet) = \check{H}^\bullet(X; \mathcal{G}^\bullet)$*

*Proof.*  $\square$

## Chapter 6

# Purehomotopy $\mathcal{K}$ -sheaves

**Definition 6.1.** A homotopy  $\mathcal{K}$ -sheaf  $\mathcal{F}^\bullet$  is said to be pure on  $X$  if:

- For  $p \in X$  and  $i \neq 0$ ,  $(H^i \mathcal{F}^\bullet)_p = 0$
- $H^i \mathcal{F}^\bullet = 0$  for  $i \ll 0$  locally on  $X$ : ie for all  $p \in X$  there is an open neighbourhood  $U$  of  $p$  and an integer  $N$  such that for  $i \leq N$  and  $K \subset U$ :  $H^i \mathcal{F}^\bullet(K) = 0$

**Lemma 6.2.** Let  $\mathcal{F}^\bullet$  be a pure-homotopy  $\mathcal{K}$ -sheaf. Then:

- For  $i < 0$   $H^i \mathcal{F}^\bullet = 0$
- $H^0 \mathcal{F}^\bullet$  is a  $\mathcal{K}$ -sheaf.

*Proof.* TODO □

**Proposition 6.3.** Let  $\mathcal{F}^\bullet$  be a pure-homotopy  $\mathcal{K}$ -sheaf. Then there is a canonical isomorphism:

$$H^\bullet \mathcal{F}^\bullet(X) = \check{H}^\bullet(X; H^0 \mathcal{F}^\bullet)$$

More generally: Let  $[\mathcal{F}_0^\bullet \rightarrow \dots \mathcal{F}_n^{\bullet-n}]$  be a complex of pure-homotopy  $\mathcal{K}$ -sheaves, then there is a canonical isomorphism:

$$H^\bullet[\mathcal{F}_0^\bullet(X) \rightarrow \dots \mathcal{F}_n^{\bullet-n}(X)] = \check{H}^\bullet(X; [H^0 \mathcal{F}_0^\bullet \rightarrow \dots \rightarrow (H^0 \mathcal{F}_n^\bullet)[n]])$$

*Proof.* TODO □

## Chapter 7

# Poincaré–Lefschetz duality

Uses check cohomology with compact supports for sheaves

**Definition 7.1.** Let  $M$  be a topological manifold, the rule  $\mathfrak{o}_M : K \mapsto H_{\dim M}(M, M \setminus K)$  defines a  $\mathcal{K}$ –sheaf, called the orientation  $\mathcal{K}$ –sheaf of  $M$ .

If  $M$  is a manifold with boundary, let  $j : M \setminus \partial M \rightarrow M$  denote the canonical inclusion. The the orientation shaeves of  $M$  are defined as follows:

- $\mathfrak{o}_M := j_* \mathfrak{o}_{M \setminus \partial M}$
- $\mathfrak{o}_{M \text{ rel } \partial} := j_! \mathfrak{o}_{M \setminus \partial M}$

**Definition 7.2.** Singular chains

**Lemma 7.3.** Let  $X$  be a topological manifold, then we have:

- $C_\bullet(X, X)$  is an acyclic complex
- For  $A$  and  $B$  two closed subsets of  $X$  the folowing complex is acyclic:

$$[C_\bullet(X, X \setminus (A \cup B)) \rightarrow C_{\bullet+1}(X, X \setminus A) \bigoplus C_{\bullet+1}(X, X \setminus B) \rightarrow C_{\bullet+2}(X, X \setminus (A \cap B))]$$

- For a family  $(K_a)_{a \in A}$  of closed subsets of  $X$  wich is filtered ( for any  $a, b \in A$  there is  $c \in A$  such that  $K_c \subset K_a \cap K_b$ ) any compact  $K$  of  $X$ , the following natural morphism is a quasi-isomorphism

$$\lim_{\substack{\longrightarrow \\ a \in A}} C_\bullet(X, X \setminus K_a) \rightarrow C_\bullet(X, X \setminus (\bigcap_{a \in A} K_a))$$

*Proof.* TODO □

**Lemma 7.4.** Let  $M$  be a topological manifold of dimension  $n$  with boundary,  $i : X \rightarrow M$  a closed subset, and  $N \subset \partial M$  a closed subset that locally looks like  $\emptyset \subset \mathbb{R}^{n-1}, \mathbb{R}_{>0} \times \mathbb{R}^{n-2} \subset \mathbb{R}^{n-1}$  or  $\mathbb{R}^{n-1}$ . Let  $j : \overset{\circ}{M} \cup \overset{\circ}{N} \rightarrow M$  be the canonical inclusion. Then there is a canonical isomorphism:

$$H^\bullet[C_{n-1-\bullet}(N, N \setminus X) \rightarrow C_{n-1-\bullet}(M, M \setminus X)] = \check{H}_c^\bullet(X; i^* j_! j^* \mathfrak{o}_M)$$

*Proof.* uses tout les poussé en avatn, tiré en arrière et tout ça, la cohomologie à support compact □

## Chapter 8

# Homotopy colimits

### 8.1 Homotopy colimits

Definition 8.1.

Definition 8.2.

### 8.2 Homotopy colimits of pure homotopy $\mathcal{K}$ -sheaves

Lemma 8.3.

*Proof.*

□

Lemma 8.4.



## Chapter 9

# Steenrod homology