M2

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April 1, 2024

Chapter 1

Presheaves and sheaves

Let X be a locally compact Hausdorf space.

1.1 Sheaves

Definition 1.1. *A presheave on X is a contravariant functor from the category of open sets of X to abélian groups.*

Definition 1.2. *If $\mathcal{F}$ is a presheaf on X and $p \in X$ then the stalk of $\mathcal{F}$ at p is the abelian group $\mathcal{F}_p := \varinjlim_{p \in U \ open} \mathcal{F}(U)$.*

Definition 1.3. *If $\mathcal{F}$ is a presheaf on X, it is said to be a sheaf if for any $U \subset X$ open and any covering family of U $(U_a)_{a \in A}$ one has the exact sequence:*

$$0 \to \mathcal{F}(U) \to \prod_{a \in A} \mathcal{F}(U_a) \to \prod_{a,b \in A} F(U_a \cap U_b) \tag{1.1}$$

1.2 $\mathcal{K}$-sheaves

Definition 1.4. *A $\mathcal{K}$-presheave on X is a contravariant functor from the category of compact sets of X to abélian groups.*

Definition 1.5. *If $\mathcal{F}$ is a $\mathcal{K}$-presheaf on X and $p \in X$ then the stalk of $\mathcal{F}$ at p is the abelian group $\mathcal{F}_p := \varinjlim_{p \in K \ compact} \mathcal{F}(K) = \mathcal{F}(\{p\})$.*

Definition 1.6. *If $\mathcal{F}$ is a $\mathcal{K}$-presheaf on X, it is said to be a $\mathcal{K}$-sheaf if the folowing conditions are satisfied:*

- $$\mathcal{F}(\emptyset) = 0 \tag{1.2}$$

- *For K_1 and K_2 two comapcts of X the folowing sequence is exact:*
$$0 \to \mathcal{F}(K_1 \cup K_2) \to \mathcal{F}(K_1) \bigoplus \mathcal{F}(K_2) \to \mathcal{F}(K_1 \cap K_2) \tag{1.3}$$

- *Pour tout compact K de X, le morphisme naturel suivant est un isomorphisme*
$$\varinjlim_{K \subset U \ open \ relatively \ compact} \mathcal{F}(\overline{U}) \to \mathcal{F}(K) \tag{1.4}$$

1.3 Technical lemmas

Lemma 1.7. *If $K_1, \ldots K_n$ are comapcts of X then $\{U_1 \cap \ldots \cap U_n\}_{U_i \supset K_i \ open \ in \ X}$ is a cofinale systeme of neighborhoods of $K_1 \cap \ldots K_n$.*

Proof. TODO $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Lemma 1.8. *If $\mathcal{C}$ and $\mathcal{D}$ are two categories, $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ two functors such that (F, G) is an adjoint pair. Then for (F, G) to be an equivalence of category, it's enough to have that thes canonical naturals transformations $id_{\mathcal{D}} \Rightarrow F \circ G$ and $G \circ F \Rightarrow id_{\mathcal{D}}$ are isomorphisms.*

Proof. TODO $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Lemma 1.9. *If $(K_a)_{a \in A}$ is a filtered directed system of comapcts substes of X, and $\mathcal{F}$ a $\mathcal{K}$-presheaf satisfying , then*
$$\varinjlim_{a \in A} \mathcal{F}(K_a) \to \mathcal{F}(\bigcap_{a \in A} K_a)$$

is an isomorphism.

1.4 Equivalence of category

Definition 1.10. *adjoint pair between K-presheaves and presheaves*

Lemma 1.11. *The previous adjoint pair give rise to an equivalence of category between shaeves and $\mathcal{K}$-sheaves*

Proof. TODO

$\square$

Chapter 2

Homotopy sheaves

Chapter 3

Pushforward, exceptional pushforward, and pullback

Chapter 4

Čech cohomology

Chapter 5

Purehomotopy $\mathcal{K}$-sheaves

Chapter 6

Poincaré–Lefschetz duality

Chapter 7

Homotopy colimits

Chapter 8

Homotopy colimits of pure homotopy $\mathscr{K}$-sheaves

Chapter 9

Steenrod homology