

M2

ymonbru

April 9, 2024

Definition 0.1. *If $A_0^\bullet \rightarrow \dots \rightarrow A_n^\bullet$ is a sequence of maps of complex such that the composition of two consecutive maps is 0, then let's denote $[A_0^\bullet \rightarrow \dots \rightarrow A_n^{\bullet-n}]$ the total complex of this double complex, defined by: TODO*

Chapter 1

Presheaves and sheaves

Let X be a locally compact Hausdorff space.

1.1 Sheaves

Definition 1.1. A presheaf on X is a contravariant functor from the category of open sets of X to abelian groups.

Definition 1.2. If \mathcal{F} is a presheaf on X and $p \in X$ then the stalk of \mathcal{F} at p is the abelian group $\mathcal{F}_p := \varinjlim_{p \in U \text{ open}} \mathcal{F}(U)$.

Definition 1.3. If \mathcal{F} is a presheaf on X , it is said to be a sheaf if for any $U \subset X$ open and any covering family of U $(U_a)_{a \in A}$ one has the exact sequence:

$$0 \rightarrow \mathcal{F}(U) \rightarrow \prod_{a \in A} \mathcal{F}(U_a) \rightarrow \prod_{a, b \in A} \mathcal{F}(U_a \cap U_b) \quad (1.1)$$

1.2 \mathcal{K} -sheaves

Definition 1.4. A \mathcal{K} -presheaf on X is a contravariant functor from the category of compact sets of X to abelian groups.

Definition 1.5. If \mathcal{F} is a \mathcal{K} -presheaf on X and $p \in X$ then the stalk of \mathcal{F} at p is the abelian group $\mathcal{F}_p := \varinjlim_{p \in K \text{ compact}} \mathcal{F}(K) = \mathcal{F}(\{p\})$.

Definition 1.6. If \mathcal{F} is a \mathcal{K} -presheaf on X , it is said to be a \mathcal{K} -sheaf if the following conditions are satisfied:

•

$$\mathcal{F}(\emptyset) = 0 \quad (1.2)$$

• For K_1 and K_2 two compacts of X the following sequence is exact:

$$0 \rightarrow \mathcal{F}(K_1 \cup K_2) \rightarrow \mathcal{F}(K_1) \oplus \mathcal{F}(K_2) \rightarrow \mathcal{F}(K_1 \cap K_2) \quad (1.3)$$

- For any compact K of X , the following natural morphism is an isomorphism

$$\lim_{\substack{\longrightarrow \\ K \subset U \text{ open relatively compact}}} \mathcal{F}(\overline{U}) \rightarrow \mathcal{F}(K) \quad (1.4)$$

Remark 1.7. (1.4) is well defined because if K is a compact subset of X , then for $x \in K$ let U_x be an open neighborhood relatively compact (which exists by local compactness), the family $(U_x)_{x \in K}$ covers K then one can extract a finite cover of it : U_1, \dots, U_n and then $\cup_{i=1}^n U_i$ is an open neighborhood, and a finite union of relatively compact, then it's relatively compact. ($\cup_{i=1}^n U_i = \cup_{i=1}^n \overline{U_i}$)

1.3 Technical lemmas

Lemma 1.8. If K_1, \dots, K_n are compact of X then $\{U_1 \cap \dots \cap U_n\}_{U_i \supset K_i \text{ open in } X}$ is a cofinal system of neighborhoods of $K_1 \cap \dots \cap K_n$.

Proof. □

Lemma 1.9. If \mathcal{C} and \mathcal{D} are two categories, $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ two functors such that (F, G) is an adjoint pair. Then for (F, G) to be an equivalence of category, it's enough to have that these canonical natural transformations $\text{id}_{\mathcal{D}} \Rightarrow F \circ G$ and $G \circ F \Rightarrow \text{id}_{\mathcal{C}}$ are isomorphisms.

Proof. TODO □

Lemma 1.10. If $(K_a)_{a \in A}$ is a filtered directed system of compact subsets of X , and \mathcal{F} a \mathcal{K} -presheaf satisfying (1.4), then

$$\lim_{\substack{\longrightarrow \\ a \in A}} \mathcal{F}(K_a) \rightarrow \mathcal{F}\left(\bigcap_{a \in A} K_a\right)$$

is an isomorphism.

Proof. TODO □

1.4 Equivalence of category

Definition 1.11.

- If \mathcal{F} is a presheaf then let $\alpha^* \mathcal{F}$ be the \mathcal{K} -presheaf:

$$K \mapsto \lim_{\substack{\longrightarrow \\ K \subset U \text{ open}}} \mathcal{F}(U)$$

- If \mathcal{G} is a \mathcal{K} -presheaf then let $\alpha_* \mathcal{G}$ be the presheaf:

$$U \mapsto \lim_{\substack{\longleftarrow \\ U \supset K \text{ compact}}} \mathcal{G}(K)$$

Proposition 1.12. The pair (α^*, α_*) is an adjoint pair.

Proof. TODO □

Lemma 1.13.

- α^* send sheaves to \mathcal{K} -sheaves
- α^* send \mathcal{K} -sheaves to sheaves
- The restrictions obtained still form an adjoint pair.

The previous adjoint pair give rise to an adjoint pair between sheaves and \mathcal{K} -sheaves

Proof. TODO

□

Lemma 1.14. *The previous adjoint pair give rise to an equivalence of category between sheaves and \mathcal{K} -sheaves*

Proof.

□

Chapter 2

Homotopy sheaves

Definition 2.1. Let \mathcal{F}^\bullet be complex of \mathcal{K} -presheaves then taking the cohomology defines a \mathcal{K} -presheaf denoted $H^\bullet \mathcal{F}^\bullet$.

Definition 2.2. A morphisme of complex of \mathcal{K} -presheave $\mathcal{F}^\bullet \rightarrow \mathcal{G}^\bullet$ if it induces isomorphisms $H^i \mathcal{F}^\bullet \rightarrow H^i \mathcal{G}^\bullet$ for all i .

Definition 2.3. A complex of \mathcal{K} -presheaves \mathcal{F}^\bullet is said to be a Homotopy- \mathcal{K} -sheaf if the folowing conditions are satisfied:

•

$$\mathcal{F}^\bullet(\emptyset) \text{ is an acyclic complex} \quad (2.1)$$

• For K_1 and K_2 two comapcts of X the folowing complex is acyclic:

$$[\mathcal{F}^\bullet(K_1 \cup K_2) \rightarrow \mathcal{F}^{\bullet-1}(K_1) \bigoplus \mathcal{F}^{\bullet-1}(K_2) \rightarrow \mathcal{F}^{\bullet-2}(K_1 \cap K_2)] \quad (2.2)$$

• For any compact K of X , the following natural morphism is a quasi-isomorphism

$$\lim_{\substack{\longrightarrow \\ K \subset U \text{ open relatively compact}}} \mathcal{F}^\bullet(\overline{U}) \rightarrow \mathcal{F}^\bullet(K) \quad (2.3)$$

Lemma 2.4. Let \mathcal{F}^\bullet be a complex of \mathcal{K} -presheaves. If \mathcal{F}^\bullet has a finite filtration whose associated graded is a Homotopy- \mathcal{K} -sheaf, then \mathcal{F}^\bullet is a Homotopy- \mathcal{K} -sheaf.

Proof. TODO

□

Lemma 2.5. If \mathcal{F}^\bullet is a homotopy- \mathcal{K} -sheaf, and $H^{-1} \mathcal{F}^\bullet = 0$ then $H^0 \mathcal{F}^\bullet$ is a \mathcal{K} -sheaf

Proof. TODO

□

Chapter 3

Pushforward, exceptional pushforward, and pullback

Chapter 4

Čech cohomology

4.1

Definition 4.1. Let \mathcal{F}^\bullet be a complex of \mathcal{K} -presheaves on a compact space X then we define the Čech cohomology $\check{H}(X; \mathcal{F}^\bullet)$ by *TODO*

Remark 4.2. By using the inclusion of \mathcal{K} -presheaves into complexes of \mathcal{K} -presheave, one get's a definition of Čech cohomology for \mathcal{K} -presheave.

Lemma 4.3. If $\mathcal{F}^\bullet \rightarrow \mathcal{G}^\bullet$ is a quasi-isomorphism then the induced maps $\check{H}^i \mathcal{F}^\bullet \rightarrow \check{H}^i \mathcal{G}^\bullet$ are isomorphisms.

Proof. *TODO*

□

Proposition 4.4. If \mathcal{F}^\bullet be a complex of \mathcal{K} -presheaves that verify ?? and ?? then the canonical map $H^\bullet \mathcal{F}^\bullet \rightarrow \check{H}^\bullet(X; \mathcal{F}^\bullet)$ is an isomorphism.

Proof.

□

4.2 Čech cohomology is determined by stalks

Lemma 4.5. *TODO*

Proof.

□

Lemma 4.6. *TODO*

Proof.

□

Proposition 4.7. *TODO*

Proof.

□

Chapter 5

Purehomotopy \mathcal{K} -sheaves

Definition 5.1. A homotopy \mathcal{K} -sheaf \mathcal{F}^\bullet is said to be pure on X if:

-
-

Lemma 5.2. Let \mathcal{F}^\bullet be a pure-homotopy \mathcal{K} -sheaf. Then:

- For $i < 0$ $H^i \mathcal{F}^\bullet = 0$
- $H^0 \mathcal{F}^\bullet$ is a \mathcal{K} -sheaf.

Proof. TODO

□

Proposition 5.3. Let \mathcal{F}^\bullet be a pure-homotopy \mathcal{K} -sheaf. Then there is a canonical isomorphism:

$$H^\bullet \mathcal{F}^\bullet(X) = \check{H}^\bullet(X; H^0 \mathcal{F}^\bullet)$$

.

More generally: TODO

Proof. TODO

□

Chapter 6

Poincaré–Lefschetz duality

Definition 6.1.

Lemma 6.2.

Proof.

□

Lemma 6.3.

Proof. uses tout les poussé en avatn, tiré en arière et tout ça, la cohomologie à support compact

□

Chapter 7

Homotopy colimits

7.1 Homotopy colimits

Definition 7.1.

Definition 7.2.

7.2 Homotopy colimits of pure homotopy \mathcal{K} -sheaves

Lemma 7.3.

Proof.

□

Lemma 7.4.

Chapter 8

Steenrod homology