1. Notes on `valuations.lean`

We define `linear_ordered_comm_group` and `linear_ordered_comm_monoid` to be a commutative group / monoid with a linear order satisfying $a \leq b \implies ca \leq cb$. Within the namespace `linear_ordered_comm_group` we prove some basic lemmas.

We then define `is_convex`, a property of subgroups of a linearly ordered commutative group. A subgroup H is convex if whenever $x \leq z \leq y$ and $x, y \in H$ we have $z \in H$. An example to keep in mind is that if $\mathbb{R}^2$ has the lexicographic order $(a, b) < (c, d) \iff a < c \vee (a = c \wedge b < d)$ then the subgroup

$$\{0\} \times \mathbb{R}$$

is convex. The kernel of an order-preserving group homomorphism is shown to be convex (an example would be the projection onto the first factor in the $\mathbb{R}^2$ example above).

The height of a linearly ordered commutative group is the number of proper convex subgroups. For example one can check that $\mathbb{R}^2$ above has height 2.

The construction of a totally ordered monoid $\Gamma \cup \{0\}$ from a totally ordered group Γ is done using the `option` function – the "extra" element `none` is identified with zero. It is proved that this extension of a linearly ordered group is a linearly ordered monoid. All of this happens within the `linear_ordered_comm_group` namespace.

We then define a valuation on a commutative ring, taking values in the monoid associated to a linearly ordered commutative group. Within the `is_valuation` namespace we prove some basic results about valuations. For some reason `is_valuation` is a typeclass. Some basic lemmas about valuations are proved, including the fact that the support of a valuation is a prime ideal.