Math 528 - Catenary Project

Patrick McCaw

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1 Abstract

Catenary curves, defined by the hyperbolic cosine function, represent the equilibrium shape of a flexible chain suspended under gravity. Their discovery disproved the initial assumption of a parabolic form, and century revolutionized architecture and engineering. This paper details the historical evolution from Galileo's misconception to Hooke's arch construction application and Euler's investigation into their surface area-minimizing properties. The following derivation employs fundamental principles of physics, and the calculus of variations to show that the form taken is indeed a catenary. The resulting equation encapsulates a fundamental representation of nature's tendency to minimize potential energy, and is still used for many current practical applications.

2 Introduction

A catenary curve is a mathematical representation of the shape formed by a flexible, evenly dense chain or cable hanging freely under its own weight and supported at its ends. Its equation is usually found in the very generalized form:

$$y(x) = acosh\left(\frac{x}{a}\right)$$

The exact form of a catenary curve is significant because it is not a parabola. This discovery, along with revealing something fundamental about how the universe works, revolutionized architecture and engineering in the 17th century, and is still widely used today.

Figure 1: The flexing of the cables of the Golden Gate Bridge is modeled by a catenary curve

3 History

The term ”catenary” is derived from the Latin word ”catēna,” meaning ”chain.” Often attributed to Thomas Jefferson, the English word ”catenary” gained prominence in a letter to Thomas Paine discussing the equilibrium of arches. While Galileo initially believed a hanging chain formed a parabola, Joachim Jungius later demonstrated that it is not. Robert Hooke applied the catenary concept to arch construction during the rebuilding of St Paul's Cathedral, and in 1671, he claimed to solve the optimal arch shape problem. The solution, published cryptically, was decoded in 1705 as ”ut pendet continuum flexile, sic stabit contiguum rigidum inversum” (”As hangs a flexible cable so, inverted, stand the touching pieces of an arch”). In 1691, Leibniz, Huygens, and Johann Bernoulli derived the catenary equation, while Euler in 1744 showed it to be the curve minimizing surface area when rotated around the x-axis, forming a catenoid. Nicolas Fuss contributed equations describing a chain's equilibrium under any force in 1796.

4 Derivation

Catenary curves are unique because they are a fundamental part of nature. Due to this, the general equation of a catenary curve can be derived from simple physics. The derivation of a catenary curve relies on the calculus of variations and the fact that, in nature, systems equilibrate to their lowest potential energy state.

A diagram of a catenary curve is presented in **Figure 3**.

Here we see that we have a string hung between two points at a length of $2a$ apart. Let the length of the string be L ($L¿2a$), with a uniform density of ρ, and the force of gravity g. Also let the top of the poles from which the catenary hangs be $y = 0$ for simplicity in calculation. We will also denote potential energy as U.

Since the length of the string is always constant, that is our starting point. Recall from calculus that:

$$J = \int dS = \int_{-a}^{a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = L \tag{1}$$

Using this, we can break our string down into infinitesimally small pieces denoted as dS in the diagram, and using the standard formula for potential energy due to gravity, the following can be derived from $U = mgh$:

$$dU = gy\rho dS$$

This is the potential energy for each infinitesimally small piece of rope. To find the total potential energy of the rope, we integrate, recalling from (1) that we

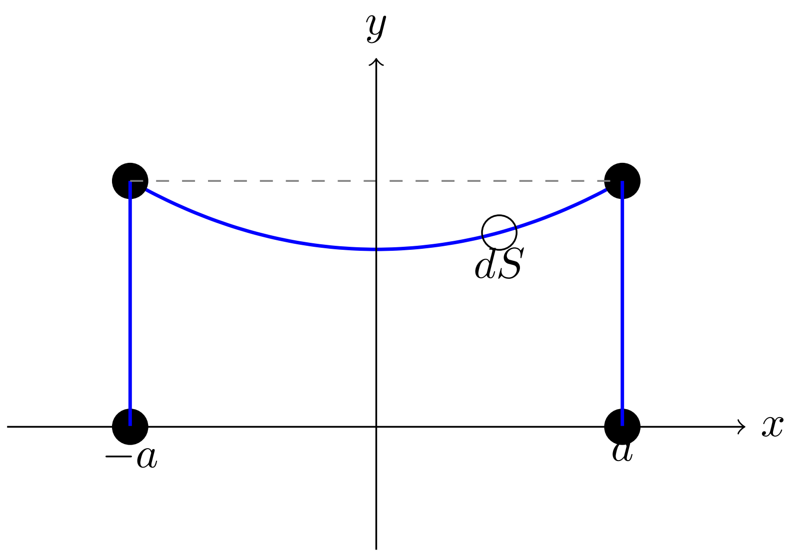

Figure 2: **Diagram of a catenary curve.**

can put L in terms of dx instead of dS:

$$U = \int dU = \int_{-a}^{a} \rho g y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Remembering that we are trying to minimize potential energy given a fixed length we can see that this is a constrained variation problem, so we construct a functional K:

$$K = U + \lambda J = \int_{-a}^{a} \rho g y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} + \lambda \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{-a}^{a} F(x, y, y')dx$$

Usually, in order to solve such an equation, the Euler-Lagrange equations would need to be used. In this case, we can use a simplified version, known as the Beltrami Identity since x is not used explicitly. The Beltrami Identity is noted below:

$$F - y' \frac{\partial F}{\partial y'} = c_1$$

After taking the derivatives, plugging into the Beltrami Identity, and canceling, we are left with:

$$y' = \sqrt{\frac{(\rho g y + \lambda)^2}{c_1^2} - 1}$$

Using separation of variables, and the proper substitution in terms of the dummy variable, u, we see that:

$$\cosh(u) = \frac{\rho g y + \lambda}{c_1}$$

$$\int \frac{dy}{\sqrt{\cosh^2(u) - 1}} = x + c_2$$

$$dy = \frac{c_1 \sinh(u)}{\rho g} du$$

Putting together and realizing hyperbolic trig identities:

$$\int \frac{c_1 \sinh(u)}{\rho g \sqrt{\cosh^2(u) - 1}} du = x + c_2$$

$$\frac{c_1}{\rho g} u = c_1 \cosh^{-1}\left(\frac{\rho g y + \lambda}{c_1}\right) = x + c_2$$

Finally, isolating y:

$$y = \frac{c_1}{\rho g} \cosh\left(\frac{\rho g x}{c_1}\right) - \frac{\lambda}{\rho g}$$

This is the final form of a catenary, however, there are still three unknown constants. Two of these can be obtained from boundary values, while the third can be found using the constraint equation for L. To find the values using boundary values, simply notice that for $x = -a$ and $x = a$, $y = 0$. From this, we find that:

$$\frac{c_1}{\rho g} \cosh\left(\frac{\rho g}{c_1}(-a + c_2)\right) = \frac{\lambda}{\rho g}$$

and:

$$\frac{c_1}{\rho g} \cosh\left(\frac{\rho g}{c_1}(a + c_2)\right) = \frac{\lambda}{\rho g}$$

Given that $\cosh(-q) =$

cosh(q), it is evident that $c_2 = 0$ and that:

$$\lambda = c_1 \cosh\left(\frac{\rho g a}{c_1}\right)$$

$$y = \frac{c_1}{\rho g}\left[\cosh\left(\frac{\rho g x}{c_1}\right) - \cosh\left(\frac{\rho g a}{c_1}\right)\right]$$

So all that remains to be found is c_1, which needs to be found using the constraint equation for length. By plugging in what we have found so far to (1), we find that c_1 should be obtained from the equation:

$$\frac{2c_1}{\rho g} \sinh\left(\frac{\rho g a}{c_1}\right) = L$$

Thus, we have found from basic physics that the equation for any catenary of this type takes the form:

$$y = \frac{c_1}{\rho g}\left[\cosh\left(\frac{\rho g x}{c_1}\right) - \cosh\left(\frac{\rho g a}{c_1}\right)\right]$$

where c_1 is found from this nonlinear equation:

$$\frac{2c_1}{\rho g}\sinh\left(\frac{\rho g a}{c_1}\right) = L$$

5 Plots

Now, let's create plots of various catenary curves with their a adjusted based on the equation $y(x) = acosh(\frac{x}{a})$ to see how they differ.

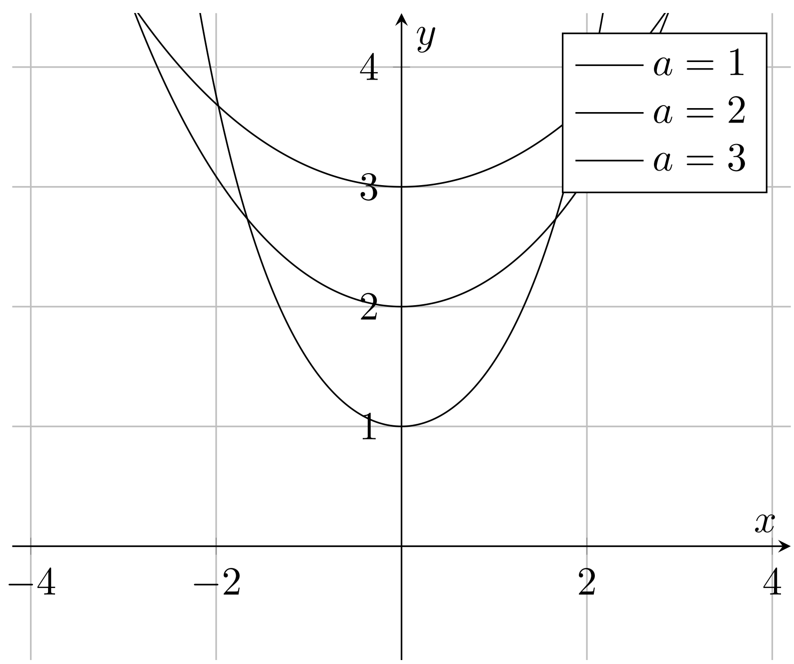

Figure 3: **Plot of Catenary Curves, varying a**

6 Applications

Catenary curves are special in two ways. They are interesting a theoretical sense because they reveal something very fundamental about nature. Things always try to minimize potential energy, and the form of catenary curves appear in all sorts of physical phenomenon from hanging ropes under different gravitational forces, to the path of a moving charge in an electromagnetic following a catenary, to the solutions of Maxwell's Equations. However, knowing this fundamental nature of catenary curves also has a myriad of practical applications as well. Catenary curves are using in architecture, because a catenary curve inverted is the strongest type of arch. Catenary curves are also used in by civil engineers to model how suspension bridges will perform and stress forces on beams. Electrical engineers use catenary curves to model transmission efficiency of hanging power lines. The applications of catenary curves are virtually endless, and most fields dealing with some object spanning a space make use of the equation governing catenary curves.

7 Weird and Wonderful

1. For an object undergoing rolling motion due to gravity on tracks of the same absolute length (the horizontal distance between the starting and ending point), the object will reach the bottom of a catenary curved track before a straight track even though the catenary track is technically longer.

2. While catenary curves and parabolas are two very distinct mathematical entities, they are very closely bonded. The focus of a parabola rolling on a straight line traces a catenary. In more formal mathematical language, the envelope of the directrix of a parabola is a catenary.

3. The strongest anchor rode, the line between a boat or other marine object and the mooring position of the anchor, is a catenary curve.

8 Conclusion

In summary, catenary curves, derived from the equilibrium-seeking principle in physics, offer both mathematical elegance and practical utility. Historically, they played a pivotal role in understanding arches and optimal shapes. The catenary equation, rooted in minimizing potential energy, finds applications in architecture and civil engineering, especially in modeling the behavior of hanging structures and suspension bridges. In addition to their practical applications, catenary curves exhibit intriguing mathematical connections, such as being the envelope of the directrix of a parabola. These curves serve as efficient models in various scenarios, from electromagnetic phenomena to optimal anchor design. In conclusion, the study of catenary curves reveals a balance between theoretical elegance and real-world functionality, making them a valuable tool in diverse fields of science and engineering.

9 References

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2. Khan. *The Catenary Problem and Solution.* YouTube, 22 April 2018, https://www.youtube.com/watch?v=npt6IkyL_f4. Accessed 26 November 2023.