Thesis Draft: Algebrization

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Introduction

The P vs NP problem is perhaps the most important open problem in complexity theory.

Chapter 1

Preliminaries

1.1 Turing machines

Central to our definitions of complexity is that of a Turing machine. This is the most common mathematical model of a computer, and is the jumping-off point for mant variants. There are many ways to think of a Turing machine, but the most common is that of a small machine that can read and write to an arbitrarily-long "tape" according to some finite set of rules. We give a more formal definition below, and then we will attempt to take this definition into a more manageable form.

Definition 1.1.1 ([9, Def. 3.1])**.** A *Turing machine* is a 7-tuple $(Q, \Sigma, \Gamma, \delta, q_0, q_a, q_r)$ where Q, Σ, and Γ are all finite sets and

1. Q is the set of *states*,
2. Σ is the *input alphabet*,
3. Γ is the *tape alphabet*,
4. $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$ is the *transition function*,
5. $q_0 \in Q$ is the *start state*,
6. $q_a \in Q$ is the *accept state*,
7. $q_r \in Q$ is the *reject state*, with $q_a \neq q_r$.

While we have this formalism here as a useful reference, even here we will most frequently refer to Turing machines in a more intuitionisitc form. There are several ways we will think about Turing machines.

The first way to think about a Turing machine is as a little computing box with a tape. We let the box read and write to the tape, and each step it can move the tape one space in either direction. At some point, the machine can decide it is done, in which case we say it "halts"; however it does not necessarily need to halt. For this paper, we will only think about machines that *do* halt, and in particular we will care about how many it takes us to get there. Further, we will use this informalism as a base from which we can define our Turing machine variants intuitively, without needing to deal with the (potentially extremely convoluted) formalism.

Another way we think about a Turing machine is as an algorithm. Perhaps the foundational paper of modern computer science theory, the *Church-Turing thesis*, states that any actually-computable algorithm has an equivalent Turing machine, and vice versa. We will use this fact liberally; in many cases we will simply describe an algorithm and not deal with putting it into the context of a Turing machine. If we have explained the algorithm well enough that a reader can execute it (as we endeavor to do), then we know a Turing machine must exist.

1.2 Complexity classes

Complexity classes are the main way we think about the hardness of problems in computer science. A complexity class is a collection of languages that all share a common level of difficulty.

1.2.1 Time complexity

The most intuitive (and most important) notion of complexity is that of time complexity. Time complexity is the answer of the question of how long it takes to solve a problem. We begin with an abstract base for our time classes, and will then introduce some specific ones that we care about.

Definition 1.2.1 ([2, Def. 1.19])**.** Let $f : \mathbb{N} \to \mathbb{N}$ be a function. The class $\mathsf{DTIME}(f(n))$ is the class of all problems computable by a deterministic Turing machine in $O(f(n))$ steps for some constant $c > 0$.

While DTIME is a useful base to start from, it is rare that we deal with DTIME classes directly.

Definition 1.2.2 ([2, Def. 1.20])**.** The complexity class P is the class

$$\mathsf{P} = \bigcup_{c>0} \mathsf{DTIME}(n^c).$$

The class P is perhaps the most important complexity class. Mathematically, we care about P because it is closed under composition: a polynomial-time algorithm iterated a polynomial number of times is still in P. Further, P turns out to generally be invariant under change of (deterministic) computation model, which allows us to reason about P problems easily without needing to resort to the formal definition of a Turing machine. More philosophically, P generally represents the set of "efficient" algorithms in the real world.

As we have defined P in terms of DTIME, the question naturally arises of whether there is an equivalent in terms of NTIME. Naturally, there is, and we call it NP.

Definition 1.2.3 ([9, Cor. 7.22])**.** The complexity class NP is the class

$$\mathsf{NP} = \bigcup_{c>0} \mathsf{NTIME}(n^c).$$

While this definition demonstrates how NP is similar to P, there are other equivalent ones that we can use. In particular, we very often like to think of NP in terms of deterministic *verifiers*. Since nondeterministic machines do not exist in real life, this definition gives a practical meaning to NP.

Theorem 1.2.4 ([9, Def. 7.19])**.** NP *is exactly the class of all languages verifiable by a* P*-time Turing machine.*

1.2.2 Space complexity

In addition to time complexity, the an additional notion of complexity is that of space complexity. Space complexity is the question of how much space on its memory tape a machine needs in order to compute a problem. In many ways, our definitions of space complexity are analagous to those for time complexity that we have already defined. In particular, DSPACE will correspond nicely to DTIME, and NSPACE to NTIME.

Definition 1.2.5 ([2, Def. 4.1])**.** Let $f : \mathbb{N} \to \mathbb{N}$ be a function. A language L is in $\mathsf{DSPACE}(f(n))$ if there exists a deterministic Turing machine M such that the number of locations on the tape that are non-blank at some point during the execution of M is in $O(f(n))$.

In the same way as we have defined DSPACE for deterministic machines, we now need to define NSPACE for nondeterministic machines.

Definition 1.2.6 ([2, Def. 4.1])**.** Let $f : \mathbb{N} \to \mathbb{N}$ be a function. A language L is in $\mathsf{NSPACE}(f(n))$ if there exists a nondeterministic Turing machine M such that the number of locations on the tape that are non-blank at some point during the execution of M is in $O(f(n))$.

Analagously to P and NP, our two main classes of space complexity are PSPACE and NPSPACE.

Definition 1.2.7 ([2, Def. 4.5])**.** The complexity class PSPACE is the class

$$\mathsf{PSPACE} = \bigcup_{c>0} \mathsf{DSPACE}(n^c).$$

Definition 1.2.8 ([2, Def. 4.5])**.** The complexity class NPSPACE is the class

$$\mathsf{NPSPACE} = \bigcup_{c>0} \mathsf{NSPACE}(n^c).$$

Unlike with P and NP, the relationship between PSPACE and NPSPACE is well known. Due to the complexity of the proof of the theorem, we will not prove it here, as it is mostly not relevant to what we will be doing.

Theorem 1.2.9 (Savitch's theorem; [8])**.** PSPACE = NPSPACE.

Upon seeing this, one might ask why it is that we believe $\mathsf{P} \neq \mathsf{NP}$ if we know that PSPACE = NPSPACE, given they are defined analogously. The answer to this question boils down to the fact that we are able to reuse space, while we are not able to reuse time. Space on the tape that is no longer needed can be overwritten, while time that is no longer needed is gone forever.

Since PSPACE and NPSPACE are equal classes, it is relatively rare to see NPSPACE referred to. Here, we will only refer to it when it makes a class relationship clearer; most frequently when comparing NPSPACE to some other nondeterministic class.

1.2.3 Completeness

Even within a complexity class, not all problems are created equal. The notion of *completeness* gives us a mathematically-rigorous way to talk about which problems in a class are the hardest. Since putting upper bounds on hard problems naturally puts those same bounds on any easier problems, complete problems can be useful in reasoning about the relationship between complexity classes.

Definition 1.2.10 ([9, Def. 7.29])**.** A language A is *polynomial-time reducible* to a language B if a polynomial-time computable function $f : \Sigma^* \to \Sigma^*$ exists such that for all $w \in \Sigma^*$, $w \in A$ if and only if $f(w) \in B$.

Polynomial-time reductions are important because they give us a way to say that A is *no harder* than B. In particular, if we have an algorithm M that determines B, we can construct the following algorithm that determines A with only a polynomial amount of additional work:

Input: A string $w \in \Sigma^*$
Output: Whether $w \in A$
1 Compute $f(w)$;
2 Use M to check whether $f(w) \in B$;
3 **return** *the result of* M;

Algorithm 1: An algorithm to reduce A to B

Definition 1.2.11 ([9, Def. 7.34])**.** A language L is NP-complete if $L \in \mathsf{NP}$ and every $A \in \mathsf{NP}$ is polynomial-time reducible to L.

This is a practical use of our polynomial-time reductions: since an NP-complete language has a reduction from every other language in NP, it follows that it is *at least as hard* as any other language in NP. Of particular interest to complexity theorists is the fact that P = NP if and only if *any* NP-complete language is in P.

Just as we have NP-completeness for time complexity, we also have notions of completeness for space complexity. Since PSPACE = NPSPACE, instead of calling the class NPSPACE-complete, we call it PSPACE-complete.

Definition 1.2.12 ([9, Def. 8.8])**.** A language L is PSPACE-complete if $L \in \mathsf{PSPACE}$ and every $A \in \mathsf{PSPACE}$ is polynomial-time reducible to NP.

While this definition is mostly analagous to that of NP-completeness, one might wonder why we use a time complexity for our reduction when PSPACE is a space-complexity class. This is because if we were to use space complexity, we would want to use PSPACE-reductions, but that would make every language in PSPACE trivially PSPACE-complete. Since that is not a useful definition, we instead restrict ourselves to polynomial-time reductions.

1.3 Polynomials

Much of our work will deal with multivariate polynomials. For a given field $\mathbb{F}$, we will denote the set of m-variable polynomials over $\mathbb{F}$ with $\mathbb{F}[x_{1,\dots,m}]$.

Definition 1.3.1 ([1, p. 8])**.** The *multidegree* of a multivariate polynomial p, written $\mathrm{mdeg}(d)$, is the maximum degree of any variable x_i of p.

It is worth noting that for monovariate polynomials, multidegree and degree coincide. The difference between multidegree and degree is subtle, but important. We shall illustrate the difference with a simple example.

Example 1.3.1.1. Consider the polynomial $x_1^2 x_2 + x_2^2$. The multidegree of this polynomial is 2, while its degree is 3.

We denote by $\mathbb{F}[x_{1,\dots,m}^{\leq d}]$ the subset of $\mathbb{F}[x_{1,\dots,m}]$ of polynomials with multidegree at most d. We also need two special cases of these polynomials, which we will want to quickly be able to reference throughout the paper.

Definition 1.3.2 ([1, p. 8])**.** A polynomial is *multilinear* if it has multidegree at most 1. Similarly, a polynomial is *multiquadratic* if it has multidegree at most 2.

From here, we need to define the notion of an *extension polynomial*. This gives the ability to take an arbitrary multivariate function defined on a subset of a field and extend it to be a multivariate polynomial over the *whole* field.

Definition 1.3.3 ([1, p. 8])**.** Let $\mathbb{F}$ be a finite field, $H \subseteq \mathbb{F}$, $m \in \mathbb{N}$ a number, and $f : H^m \to \mathbb{F}$ be a function. An *extension polynomial* of f is any polynomial $f' \in \mathbb{F}[x_{1,\dots,m}]$ such that $f(h) = f'(h)$ for all $h \in H$.

It turns out that this polynomial needs only to be of a surprisingly low multidegree. Since polynomials of lower degree are generally easier to compute, we would like to have some measure of what a "small" polynomial actually is in this context.

Definition 1.3.4 ([5, §5.1])**.** Let $\mathbb{F}$ be a finite field, $H \subseteq \mathbb{F}$, $m \in \mathbb{N}$ a number, and $f : H^m \to \mathbb{F}$ be a function. A *low-degree extension* $\hat{f}$ of f is an extension of f with multidegree at most $|H| - 1$.

It turns out that this is the minimum possible degree of any extension polynomial. Further, it turns out that for any f, there is a *unique* low-degree extension. Neither of these statements are particularly important for our further work, so we will not endeavor to prove them here. Something of practical use to us is an explicit formula for the low-degree extension, which we shall now calculate.

Theorem 1.3.5 ([5, §5.1])**.** *Let $\mathbb{F}$ be a finite field, $H \subseteq \mathbb{F}$, $m \in \mathbb{N}$ a number, and $f : H^m \to \mathbb{F}$. Then a low-degree extension $\hat{f}$ of f is the function*

$$\hat{f}(x) = \sum_{\beta \in H^m} \delta_\beta(x) f(\beta), \tag{1.3.1}$$

where δ is the polynomial

$$\delta_x(y) = \prod_{i=1}^{m} \left(\sum_{\omega \in H} \left(\prod_{\gamma \in H \setminus \{\omega\}} \frac{(x_i - \gamma)(y_i - \gamma)}{(\omega - \gamma)^2} \right) \right). \tag{1.3.2}$$

Proof. First, we must show $\hat{f}$ has multidegree $|H| - 1$. First, note that $\hat{f}$ is a linear combination of some $\delta_x$es; hence asking about the multidegree of $\hat{f}$ is really just asking about the multidegree of δ_x. Looking at δ_x, the innermost product has $|H| - 1$ terms, each with the same y_i; thus those terms have multidegree $|H| - 1$. Summing terms preserves their multidegree, and the outer product iterates over the variables, thus it preserves multidegree as well. Thus, δ_x has multidegree $|H| - 1$.

To understand why $\hat{f}(x)$ agrees with $f(x)$ on H, we first should look at $\delta_\beta(x)$. In particular, for all $x, y \in H^m$,

$$\delta_y(x) = \begin{cases} 1 & x = y \\ 0 & x \neq y. \end{cases}$$

This can be shown through some straightforward but tedious algebra which we have omitted here. The equivalence above is the reason we have chosen our notation here to be reminiscent of the Kronecker delta function.

Taking the above statement, we get that for all $x \in H^m$, the only nonzero term of $\hat{f}(x)$ is the term where $\beta = x$; thus $\hat{f}(x) = f(x)$. Hence, $\hat{f}$ is a low-degree extension of f. □

Of particular interest to us will be the case of low-degree extensions where $H = \{0, 1\}$. Since every field contains both 0 and 1, this will allow us to construct a set consisting of an extension for *every* field. Further, since $|H| = 2$ here, it means our low-degree extensions will be multilinear. Not only do we thus constrain our polynomial to have a very low multidegree, the δ function also dramatically simplifies in this case, which makes it much easier to reason about.

Corollary 1.3.6 ([1, §4.1])**.** *Let $\mathbb{F}$ be a finite field, $m \in \mathbb{N}$ a number, and $f : \{0,1\}^m \to \mathbb{F}$. Then*

$$\hat{f}(x) = \sum_{\beta \in \{0,1\}^m} \delta_\beta(x) f(\beta) \tag{1.3.3}$$

is a low-degree extension of f, where δ is the polynomial

$$\delta_x(y) = \left(\prod_{i : x_i = 1} y_i \right) \left(\prod_{i : x_i = 0} (1 - y_i) \right). \tag{1.3.4}$$

Note that in the product bound $i : x_i = 1$, we mean the product over all numbers i such that $x_i = 1$.

As we can see, the form of δ in Equation (1.3.4) is much more manageable than the form in Equation (1.3.2), and it is perhaps more immediately apparent here why δ has the property it does.

The form of δ_x defined in Equation (1.3.4) has further use to us than just being simpler.

Theorem 1.3.7 ([1, §4.1])**.** *For any field $\mathbb{F}$, the set $\{\delta_x \mid x \in \{0,1\}^n\}$ forms a basis for the vector space of multilinear polynomials $\mathbb{F}^n \to \mathbb{F}$.*

This is a particularly useful basis because it allows us to reason about multilinear polynomials in terms of their outcomes on the Boolean cube.

Theorem 1.3.8 ([1, Theorem 4.3])**.** *Let $\mathbb{F}$ be a field and $Y \subseteq \mathbb{F}^n$ be a set of t points $y_1, \ldots, y_t$. Then for at least $2^n - t$ Boolean points $w \in \{0,1\}^n$, there exists a multiquadratic extension polynomial $p : \mathbb{F}^n \to \mathbb{F}$ such that*

1. *$p(y_i) = 0$ for all $i \in [t]$,*
2. *$p(w) = 1$,*
3. *$p(z) = 0$ for all Boolean $z \neq w$.*

Proof. □

Lemma 1.3.9 ([1, Lemma 4.5])**.** *Let $\mathcal{F}$ be a collection of fields. Let $f : \{0,1\}^n \to \{0,1\}$ be a Boolean function, and for every $\mathbb{F} \in \mathcal{F}$, let $p_{\mathbb{F}} : \mathbb{F}^n \to \mathbb{F}$ be a multiquadratic polynomial over $\mathbb{F}$ extending f. Also let $\mathcal{Y}_{\mathbb{F}} \in \mathbb{F}^n$ for each $\mathbb{F} \in \mathcal{F}$, and define $t = \sum_{\mathbb{F}\in\mathcal{F}} |\mathcal{Y}_{\mathbb{F}}|$.*

Then, there exists a subset $B \subseteq \{0,1\}^n$, with $|B| \leq t$, such that for all Boolean functions $f' : \{0,1\}^n \to \{0,1\}$ that agree with f on B, there exist multiquadratic polynomials $p'_{\mathbb{F}} : \mathbb{F}_n \to \mathbb{F}$ (one for each $\mathbb{F} \in \mathcal{F}$) such that

1. *$p'_{\mathbb{F}}$ extends f', and*
2. *$p'_{\mathbb{F}}(y) = p_{\mathbb{F}}(y)$ for all $y \in \mathcal{Y}_{\mathbb{F}}$.*

Proof. □

Lemma 1.3.10 ([6, Lemma 7])**.** *Let $m(x_1, \ldots, x_n)$ be a multilinear monomial. Over a field of characteristic other than 2, we have*

$$\sum_{b\in\{-1,1\}} m(b) = 0. \tag{1.3.5}$$

Proof. For some x_i, we can write $m = x_i \cdot m'$, where the degree of x_i in m' is 0. Then

$$\begin{aligned}
\sum_{b\in\{1,-1\}^n} m(b) &= \sum_{a\in\{-1,1\}} \sum_{b'\in\{1,-1\}^{n-1}} a \cdot m'(b') \\
&= \sum_{a\in\{-1,1\}} a \cdot \left(\sum_{b'\in\{1,-1\}^{n-1}} m'(b') \right) \\
&= \left(\sum_{b'\in\{1,-1\}^{n-1}} m'(b') \right) - \left(\sum_{b'\in\{1,-1\}^{n-1}} m'(b') \right) \\
&= 0.
\end{aligned}$$

□

Chapter 2

Relativization

An important prerequisite to understanding algebrization is the similar, but simpler, concept of *relativization*, also called *oracle separation*. To do this, we first must define an *oracle*.

Definition 2.0.1 ([1, Def. 2.1])**.** An *oracle* A is a collection of Boolean functions $A_m : \{0,1\}^m \to \{0,1\}$, one for each natural number m.

There are several ways to think of an oracle; this will extend the most naturally when it comes time to define an extension oracle in Definition 3.0.1. Another way to think of an oracle is as a subset $A \subseteq \{0,1\}^*$. This allows us to think of A as a language. Since we can do this, it gives us the ability to think of the complexity of the oracle. If we want to think about the subset in terms of our functions, we can write A as

$$A = \bigcup_{m \in \mathbb{N}} \{x \in \{0,1\}^m \mid A_m(x) = 1\}. \tag{2.0.1}$$

An oracle is not particularly interesting mathematical object on its own; its utility comes from when it interacts with a Turing machine.

Definition 2.0.2. A *Turing machine with an oracle* is

Of course, the question now becomes how we can effectively use an oracle in an algorithm. The previously-mentioned conception of an oracle as a set of strings is useful here. If we consider the set of strings as being a *language* in its own right, then querying the oracle is the same as determining whether a string is in the langauge, just in one step. If the language is computationally hard, this means our machine can get a significant power boost from the right oracle.

Definition 2.0.3 ([1, Def. 2.1])**.** For any complexity class $\mathcal{C}$, the complexity class $\mathcal{C}^A$ is the class of all languages determinable by a Turing machine with access to A in the number of steps defined for $\mathcal{C}$.

We will be using this definition in many places, so we should take a moment to look at it in more depth. First, it is important to realize that $\mathcal{C}^A$ is a set of *languages*, not *machines*: despite the notation, augmenting $\mathcal{C}$ with an oracle does not modify any languages, it just adds new ones that are computable. Second, since a machine can always ignore its oracle, it follows that adding an oracle can only increase the number of languages in the class, never decrease it.

Lemma 2.0.4. *For any complexity class $\mathcal{C}$ and oracle A, $\mathcal{C} \subseteq \mathcal{C}^A$.*

Proof. □

2.1 Defining relativization

We are now ready to define what relativization is. First, note that relativization is a statement about a *result*: we talk about inclusions algebrizing, not sets themselves.

Definition 2.1.1. Let $\mathcal{C}$ and $\mathcal{D}$ be complexity classes such that $\mathcal{C} \subseteq \mathcal{D}$. We say the result $\mathcal{C} \subseteq \mathcal{D}$ *relativizes* if $\mathcal{C}^A \subseteq \mathcal{D}^A$ for all oracles A. Conversely, if there exists A such that $\mathcal{C} \nsubseteq \mathcal{D}$, we say that the result $\mathcal{C} \subseteq \mathcal{D}$ *does not algebrize*.

Definition 2.1.2. Let $\mathcal{C}$ and $\mathcal{D}$ be complexity classes such that $\mathcal{C} \nsubseteq \mathcal{D}$. We say the result $\mathcal{C} \nsubseteq \mathcal{D}$ *relativizes* if $\mathcal{C}^A \nsubseteq \mathcal{D}^A$ for all oracles A. Conversely, if there exists A such that $\mathcal{C} \subseteq \mathcal{D}$, we say that the result $\mathcal{C} \nsubseteq \mathcal{D}$ *does not algebrize*.

We start with a very straightforward example of a relativizing result.

Lemma 2.1.3. *For any oracle A, $\mathsf{P}^A \subseteq \mathsf{NP}^A$. Equivalently, the result $\mathsf{P} \subseteq \mathsf{NP}$ relativizes.*

Proof. Since any deterministic Turing machine is also a nondeterministic machine, it follows that a machine that solves a P^A problem is also an NP^A machine. Hence, $\mathsf{P}^A \subseteq \mathsf{NP}^A$. □

This result tells us that not *everything* is weird in the world of relativization: if we have a machine that can do more operations without an oracle, it can still do so with an oracle. Further, for the question of P vs. NP that we will discuss in Section 2.3, this means that the question we care about is whether $\mathsf{NP} \subseteq^? \mathsf{P}$ relativizes. As such, the question we are asking simplifies to determining where $\mathsf{P}^A = \mathsf{NP}^A$ and where $\mathsf{P}^A \subsetneq \mathsf{NP}^A$.

2.2 Query complexity

The goal of query complexity is to ask questions about some Boolean function $A : \{0,1\}^n \to \{0,1\}$ by querying A. For this, we will interchangeably think of A as a *function* as well as a bit string of length $N = 2^n$, where each string element is A applied to the ith string of length n, arranged in some lexicographical order. We can further think of the property itself as being a Boolean function; a function that takes as input the bit-string representation of A and outputs whether or not A has the given property. We will call the function representing the property f. When viewed like this, f is a function from $\{0,1\}^N$ to $\{0,1\}$. We define three types of query complexity for three of the most common types of computing paradigms: deterministic, randomized, and quantum. Nondeterministic query complexity is interesting, but it is outside the scope of this paper.

Definition 2.2.1 ([1, p. 17])**.** Let $f : \{0,1\}^N \to \{0,1\}$ be a Boolean function. Then the *deterministic query complexity* of f, which we write $D(f)$, is the minimum number of queries made by any deterministic algorithm with access to an oracle A that determines the value of $f(A)$.

To make this more clear, let us give an example problem.

Definition 2.2.2. The OR problem is the following oracle problem:

> Let $A : \{0,1\}^n \to \{0,1\}$ be an oracle. The function $\mathsf{OR}(A)$ returns 1 if there exists a string on which A returns 1, and 0 otherwise.

The question is then what the deterministic query complexity of the OR function is.

Theorem 2.2.3. *The OR problem has a deterministic query complexity of 2^n.*

Proof. □

For the next two definitions, since their Turing machines include some element of randomness, we only require that they succeed with a 2/3 probability. This is in line with most definitions of complexity classes involving random computers.

Definition 2.2.4 ([1, p. 17])**.** Let $f : \{0,1\}^N \to \{0,1\}$ be a Boolean function. Then the *randomized query complexity* of f, which we write $D(f)$, is the minimum number of queries made by any randomized algorithm with access to an oracle A that evaluates $f(A)$ with probability at least 2/3.

Definition 2.2.5 ([1, p. 17])**.** Let $f : \{0,1\}^N \to \{0,1\}$ be a Boolean function. Then the *quantum query complexity* of f, which we write $D(f)$, is the minimum number of queries made by any quantum algorithm with access to an oracle A that evaluates $f(A)$ with probability at least 2/3.

2.3 Relativization of P vs. NP

An important example of relativization is that of P and NP. While the question of if $\mathsf{P} = \mathsf{NP}$ is still open, we aim to show that *regardless of the answer*, the result does not algebrize. To do this, we show that there are some oracles A where $\mathsf{P}^A = \mathsf{NP}^A$, and some where $\mathsf{P}^A \neq \mathsf{NP}^A$.

Additionally, it should be noted that the similarity of relativization to algebrization means that the structure of these proofs will return in Section 3.2 when we show the algebrization of P and NP.

2.3.1 Equality

The more straightforward of the two proofs is the oracle where $\mathsf{P}^A = \mathsf{NP}^A$, so we shall begin with that.

Theorem 2.3.1 ([4, Theorem 2])**.** *There exists an oracle A such that $\mathsf{P}^A = \mathsf{NP}^A$.*

Proof. For this, we can let A be any PSPACE-complete language. By letting our machine in P be the reducer from A to any other language in PSPACE, we therefore get that $\mathsf{PSPACE} \subseteq \mathsf{P}^A$. Similarly, if we have a problem in NP^A, we can verify it in polynomial space without talking to A at all (by having our machine include a determiner for A). Hence, we have that $\mathsf{NP}^A \subseteq \mathsf{NPSPACE}$. Further, a celebrated result of Savitch [8] is that $\mathsf{PSPACE} = \mathsf{NPSPACE}$. Combining all these results, we get the chain

$$\mathsf{NP}^A \subseteq \mathsf{NPSPACE} = \mathsf{PSPACE} \subseteq \mathsf{P}^A \subseteq \mathsf{NP}^A. \tag{2.3.1}$$

This is a circular chain of subset relations, which means everything in the chain must be equal. Hence, $\mathsf{P}^A = \mathsf{NP}^A = \mathsf{PSPACE}$. □

For a slightly more intuitive view of what this proof is doing, what we have done is found an oracle that is so powerful that it dwarfs any amount of computation our actual Turing machine can do. Hence, the power of our machine is really just the same as the power of our oracle, and since we have given both the P and NP machine the same oracle, they have the same power.

2.3.2 Inequality

Having shown that an oracle exists where $\mathsf{P}^A = \mathsf{NP}^A$, we now endeavor to find one where $\mathsf{P}^A \neq \mathsf{NP}^A$. This piece of the proof is less simple than the previous section, and it uses a diagonalization argument to construct the oracle. Before we dive in to the main proof, however, we need to define a few preliminaries.

Definition 2.3.2 ([4, p. 436])**.** Let X be an oracle. The language $L(X)$ is the set

$$L(X) = \{x \mid \text{there is } y \in X \text{ such that } |y| = |x|\}.$$

Our eventual goal will be to construct a language X such that $L(X) \in \mathsf{NP}^X \setminus \mathsf{P}^X$. Of particular note is that we can rather nicely put a upper bound on the complexity of $L(X)$ when given X as an oracle, regardless of the value of X. This fact is what gives us the freedom to construct X in such a way that $L(X)$ will not be in P^X.

Lemma 2.3.3 ([4, p. 436])**.** *For any oracle X, $L(X) \in \mathsf{NP}^X$.*

Proof. Let S be a string of length n. If $S \in L(X)$, then a witness for S is any string S' such that $|S| = |S'|$ and $S' \in X$. Since a machine with query access to X can query whether S' is in X in one step, it follows that we can verify that $S \in L(X)$ in polynomial time. □

With this lemma as a base, we can now move on to our main theorem.

Theorem 2.3.4 ([4, Theorem 3])**.** *There exists an oracle A such that $\mathsf{P}^A \neq \mathsf{NP}^A$.*

Proof. Our goal is to construct a set B such that $L(B) \notin \mathsf{P}^B$. We shall construct B in an interative manner. We do this by taking a sequence $\{P_i\}$ of all machines that recongize some language in P^A, and then constructing B such that for each machine in the sequence, there is some part of $L(B)$ it cannot recognize. This technique is called *diagonalization*, and it is used in many places in computer science theory.[1] Additionally, we define $p_i(n)$ to be the maximum running time of P_i on an input of length n. We give the following algorithm to construct B:

Input: A sequence of P oracle machines $\{P_i\}_{i=1}^{\infty}$
Output: A set B such that $L(B) \notin \mathsf{P}^B$

```
1  B(0) ← ∅;
2  n_0 ← 0;
3  for i starting at 1 do
4  |  Let n > n_i be large enough that p_i(n) < 2^n;
5  |  Run P_i^{B(i-1)} on input 0^n;
6  |  if P_i^{B(i-1)} rejects 0^n then
7  |  |  Let x be a string of length n not queried during the above computation;
8  |  |  B(i) ← B(i-1) ⊔ {x};
9  |  end
10 |  n_{i+1} ← 2^n;
11 end
12 B ← ⋃_i B(i);
```

Algorithm 2: An algorithm for constructing B

Now that we have presented the algorithm, let us demonstrate its soundness. First, note that since P_i runs in polynomial time, $p_i(n)$ is bounded above by a polynomial, and hence there will always exist an n as defined in line 4. Next, since there are 2^n strings of length n and since $p_i(n) < 2^n$, we know that there must be some x to make line 7 well-defined. While our algorithm allows x to be any string, if it is necessary to be explicit in which we choose, then picking x to be the smallest string in lexicographic order is a standard choice.

We should also briefly mention that this algorithm does not terminate. This is okay because we are only using it to construct the set B, which does not need to be bounded. If this were to be made practical, since the sequence of n_is is monotonically increasing, the set could be constructed "lazily" on each query by only running the algorithm until n_i is greater than the length of the query.

Next, we demonstrate that $L(B) \notin \mathsf{P}^B$. The end goal of our instruction is a set B such that if P_i^B accepts 0^n then there are no strings of length n in B, and if P_i^B rejects, then there is a string of length n in B. This means that no P_i accepts $L(B)$, and hence $L(B) \notin \mathsf{NP}^B$.

The central idea behind the proper functioning of our algorithm is that adding strings to our oracle *cannot change the output if they are not queried*. This is what we do in line 4: we need our input length to be long enough to guarantee that a non-queried string exists. Since the number of queried strings is no greater than $p_i(n)$, and there are 2^n strings of length n, there must be some string not queried.

Next, we run $P_i^{B(i-1)}$ on all the strings we have already added. If it accepts, then we want to make sure that no string of length n is in B; that is, 0^n is not in $L(B)$. Hence, in this particular loop we add nothing to $B(i)$. If $P_i^{B(i-1)}$ rejects, we then need to make sure that $0^n \in L(B)$ but in a way that does not affect the output of $P_i^{B(i-1)}$. Hence, we find a string that $P_i^{B(i-1)}$ did not query (and thus will not affect the result) and add it to $B(i)$.

Having done this, we then set n_{i+1} to be 2^n. Since $p_i(n) < 2^n$, it follows that no previous machine could have queried any strings of length n_{i+1}.[2] This way, we ensure our previous machines do not accidentally have their output change due to us adding a string they queried.

Having run this over all polynomial-time Turing machines, we have a set $L(B)$ such that no machine in P^B accepts it, which tells us $L(B) \notin \mathsf{P}^B$. But, Lemma 2.3.3 already told us $L(B) \in \mathsf{NP}^B$.

[1] This argument style is named after *Cantor's diagonal argument*, which was originally used to prove that the real numbers are uncountable [7, Thm. 2.14].

[2] A word of caution: we only care about what P_i does on input n_i, *not any other input*. This is because we only need each machine to be incorrect for some i, not all i.

Hence, $\mathsf{P}^B \neq \mathsf{NP}^B$. □

2.4 Diagonalization relativizes

Of course, determining that P vs NP does not relativize is only important if the proof techniques used in practice *do* in fact relativize. Rather unfortunately, it turns out that simple diagonalization is a relativizing result.

While diagonalization itself does not have a formal definition, we can still think about it informally. Looking at our construction of B, which we did using diagonalization, notice that our definition never really cared about how the P_i worked, just about the results it produced. Hence, if it were to be possible to modify Algorithm 2 to construct $B \in \mathsf{NP} \setminus \mathsf{P}$, the proof would remain the same if we were to replace our sequence $\{P_i\}$ with a sequence of machines in P^A for some PSPACE-complete A. However, this would lead to a contradiction, as we showed in Theorem 2.3.1 that in that case, $\mathsf{P}^A = \mathsf{NP}^A$! This tells us that a simple diagonalization argument would not suffice to determine separation between P and NP.

Chapter 3

Algebrization

Algebrization, originally described by Aaronson and Wigderson [1], is an extension of relativization. While relativization deals with oracles that are Boolean functions, algebrization extends oracles to be a collection of polynomials over finite fields.

Definition 3.0.1 ([1, Def. 2.2])**.** Let $A_m : \{0,1\}^m \to \{0,1\}$ be a Boolean function and let $\mathbb{F}$ be a finite field. Then an *extension* of A_m over $\mathbb{F}$ is a polynomial $\tilde{A}_{m,\mathbb{F}} : \mathbb{F}^m \to \mathbb{F}$ such that $\tilde{A}_{m,\mathbb{F}}(x) = A_m(x)$ whevever $x \in \{0,1\}^m$. Also, given an oracle $A = (A_m)$, an extension $\tilde{A}$ of A is a collection of polynomials $\tilde{A}_{m,\mathbb{F}} : \mathbb{F}^m \to \mathbb{F}$, one for each positive integer m and finite field $\mathbb{F}$, such that

1. $\tilde{A}_{m,\mathbb{F}}$ is an extension of A_m for all $m, \mathbb{F}$, and
2. there exists a constant c such that $\mathrm{mdeg}(\tilde{A}_{m,\mathbb{F}}) \leq c$ for all $m, \mathbb{F}$.

Definition 3.0.2 ([1, Def. 2.2])**.** For any complexity class $\mathcal{C}$ and extension oracle $\tilde{A}$, the complexity class $\mathcal{C}^{\tilde{A}}$ is the class of all languages determinable by a Turing machine with access to $\tilde{A}$ with the requirements for $\mathcal{C}$.

Next, we need to formally define what algebrization is.

Definition 3.0.3 ([1, Def. 2.3])**.** Let $\mathcal{C}$ and $\mathcal{D}$ be complexity classes such that $\mathcal{C} \subseteq \mathcal{D}$. We say the result $\mathcal{C} \subseteq \mathcal{D}$ *algebrizes* if $\mathcal{C}^A \subseteq \mathcal{D}^{\tilde{A}}$ for all oracles A and finite field extensions $\tilde{A}$ of A. Conversely, if there exists A and $\tilde{A}$ such that $\mathcal{C} \not\subseteq \mathcal{D}$, we say that the result $\mathcal{C} \subseteq \mathcal{D}$ *does not algebrize*.

Definition 3.0.4 ([1, Def. 2.3])**.** Let $\mathcal{C}$ and $\mathcal{D}$ be complexity classes such that $\mathcal{C} \not\subseteq \mathcal{D}$. We say the result $\mathcal{C} \not\subseteq \mathcal{D}$ *algebrizes* if $\mathcal{C}^A \not\subseteq \mathcal{D}^{\tilde{A}}$ for all oracles A and finite field extensions $\tilde{A}$ of A. Conversely, if there exists A and $\tilde{A}$ such that $\mathcal{C} \subseteq \mathcal{D}$, we say that the result $\mathcal{C} \not\subseteq \mathcal{D}$ *does not algebrize*.

3.1 Algebraic query complexity

Similarly to how we defined query complexity in Section 2.2, our notion of algebrization requires a definition of *algebraic* query complexity.

Definition 3.1.1 ([1, Def. 4.1])**.** Let $f : \{0,1\}^N \to \{0,1\}$ be a Boolean function, $\mathbb{F}$ be a field, and c be a positive integer. Also, let $\mathbb{M}$ be the set of deterministic algorithms M such that $M^{\tilde{A}}$ outputs $f(A)$ for every oracle $A : \{0,1\}^n \to \{0,1\}$ and every finite field extension $\tilde{A} : \mathbb{F}^n \to \mathbb{F}$ of A with $\mathrm{mdeg}(\tilde{A}) \leq c$. Then, the deterministic algebraic query complexity of f over $\mathbb{F}$ is defined as

$$\tilde{D}_{\mathbb{F},c}(f) = \min_{M \in \mathcal{M}} \left(\max_{A, \tilde{A}: \mathrm{mdeg}(\tilde{A}) \leq c} T_M(\tilde{A}) \right), \tag{3.1.1}$$

where $T_M(\tilde{A})$ is the number of queries to $\tilde{A}$ made by $M^{\tilde{A}}$.

Our goal here is to find the *worst*-case scenario for the *best* algorithm that calculates the property f. The difference between this and Definition 2.2.1 is twofold: first, our algorithm M has access to an extension oracle of A, and second, that we can limit our $\tilde{A}$ in its maximum multidegree. For the most part, we will focus on equations with multidegree 2, which is enough to get the results we want.

As an example, let us look at the same OR problem we defined in Definition 2.2.2.

Theorem 3.1.2 ([1, Thm. 4.4])**.** *$\tilde{D}_{\mathbb{F},2}(\mathsf{OR}) = 2^n$ for every field $\mathbb{F}$.*

Proof. □

This gives us a potentially counterintuitive property of algebraic query complexity: while it would seem that giving our machine a polynomial (and a polynomial of multidegree only 2, at that) would give us the ability to solve the hardest problems more quickly, that turns out not to be the case.

Now, while this is true for polynomials of multidegree 2, it turns out that if we restrict our oracles to being simply *multilinear* polynomials, we do get a speedup.

Theorem 3.1.3 ([6, Thm. 3])**.** *$\tilde{D}_{\mathbb{F},1}(\mathsf{OR}) = 1$ for every field $\mathbb{F}$ with characteristic not equal to 2.*

Proof. Let $A : \{0,1\}^n \to \{0,1\}$ and $\tilde{A}$ be our extension polynomial. Consider the value of $p(1/2, \dots, 1/2)$. We aim to show that this value is equal to 0 if and only if A is the zero oracle.

Consider the function

$$p'(x_1, \dots, x_n) = p(1 - 2x_1, \dots, 1 - 2x_n). \tag{3.1.2}$$

Since $1 - 2x$ is a linear polynomial, it follows that p' is itself a multilinear polynomial. Further, since the sum over $\{1, -1\}^n$ of a non-constant multilinear monomial is 0 as per Lemma 1.3.10, it follows that

$$\sum_{b \in \{-1,1\}^n} p'(b) = p'(0, \dots, 0), \tag{3.1.3}$$

i.e., the constant term of p'. Further, from our definition of p', we have that $p'(0, \dots, 0) = p(1/2, \dots, 1/2)$. Hence, we have

$$\sum_{b \in \{0,1\}^n} p(b) = p(1/2, \dots, 1/2). \tag{3.1.4}$$

Since $p(b) \geq 0$ for all $b \in \{0,1\}^n$, it follows that $p(1/2, \dots, 1/2)$ is 0 if and only if $p(b) = 0$ for all $b \in \{0,1\}^n$, i.e. exactly when A is the zero function. □

3.2 Algebrization of P vs. NP

As with relativization, an important application of algebrization is in regards to the P vs. NP problem.

Lemma 3.2.1. *The multilinear extension of any* PSPACE*-complete language is also in* PSPACE*.*

Proof. □

This result is incredibly important in Theorem 3.2.2: this tells us that PSPACE-complete languages are not made any more powerful when extended to finite fields. This allows to reuse the same analysis we made earlier in Theorem 2.3.1 mostly as-is.

Theorem 3.2.2 ([1, Theorem 5.1])**.** *There exist A, $\tilde{A}$ such that $\mathsf{NP}^A = \mathsf{P}^{\tilde{A}}$.*

Proof. For this theorem, we use the same technique we did in our proof of Theorem 2.3.1: find a PSPACE-complete language A and work from there. If we let $\tilde{A}$ be the unique multilinear extension of A, Babai, Fortnow, and Lund [3] have observed that the multilinear extension of any PSPACE language is also in PSPACE. Hence, reusing our argument from Theorem 2.3.1, we have

$$\mathsf{NP}^{\tilde{A}} = \mathsf{NP}^{\mathsf{PSPACE}} = \mathsf{PSPACE} = \mathsf{P}^A. \tag{3.2.1}$$

□

Theorem 3.2.3 ([1, Theorem 5.3]). *There exist A, $\tilde{A}$ such that $\mathsf{NP}^A \neq \mathsf{P}^{\tilde{A}}$.*

Proof. Like in Theorem 2.3.4, we aim to "diagonalize": iterate over all $\mathsf{P}^{\tilde{A}}$ machines to construct a language that none of them can recognize. Also like before, we will do this by constructing an oracle extension $\tilde{A}$ such that $L(A) \notin \mathsf{P}^{\tilde{A}}$. Since we only give an algebraic extension to P and not NP, we can resuse the result from Lemma 2.3.3 that $L(B) \in \mathsf{NP}^A$. We shall construct $\tilde{A}$ using the following algorithm:

Input: A sequence of P oracle machines $\{P_i\}_{i=1}^{\infty}$
Output: An extension oracle $\tilde{A}$ such that $L(A) \notin \mathsf{P}^{\tilde{A}}$

$\tilde{A} \leftarrow \varnothing$;
$n_0 \leftarrow 0$;
for *i starting at* 1 **do**
  Let $n > n_i$ be large enough that $p_i(n) < 2^n$;
  $T_j \leftarrow \bigcup_{j<i} S_j$;
  Run $P_i^{\tilde{A}}$ on input 0^n;
  if $P_i^{B(i-1)}$ *rejects* 0^n **then**
    Let $\mathcal{Y}_{\mathbb{F}}$ be the set of all $y \in \mathbb{F}^{n_i}$ queried during the above computation;
    `// See Lemma 1.3.9 for why we can do this`
    Let $w \in \{0,1\}^n$ such that the following works;
    for *all* $\mathbb{F}$ **do**
      Set $\tilde{A}_{n_i,\mathbb{F}}$ to be a multiquadratic polynomial such that $\tilde{A}_{n_i,\mathbb{F}}(w) = 1$ and $\tilde{A}_{n_i,\mathbb{F}}(y) = 0$ for all $y \in \mathcal{Y}_{\mathbb{F}} \cup (\{0,1\}^{n_i} \setminus \{w\})$;
    end
  else
    Set $\tilde{A}_{n_i,\mathbb{F}} = 0$ for all $\mathbb{F}$;
  end
  $n_{i+1} \leftarrow 2^n$;
end
$B \leftarrow \bigcup_i B(i)$;

Algorithm 3: An algorithm for constructing $\tilde{A}$

As before, we will start by demonstrating soundness and then move on to why the constructed oracle provides the separation we seek. □

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