1a) Let x be arbitrary
Assume $7 | (x-1)$, $\exists k \in \mathbb{Z}$ $7k = x-1$
Construct candidate q
Show that $7q = x^2 + x - 2$

b) Proof:

Let x be arbitrary. Suppose $7 | (x-1)$, that is, $\exists k \in \mathbb{Z}$, $7k = x-1$. Algebraically, this is equivalent to $x = 7k+1$. We will define a new integer $q = k(7k+3)$. We will now verify that $7q = x^2+x-2$

$$\begin{aligned} 7q &= 7(k(7k+3)) \\ &= (7k)(7k+3) \\ &= ((7k+1)-1)\cdot((7k+1)+2) \\ &= (x-1)(x+2) \\ 7q &= x^2+x-2 \quad // \end{aligned}$$

Since there is some integer q such that $7q = x^2+x-2$, by the definition of divides, $7 | x^2+x-2$ ■

2a) Let x be arbitrary
Assume x is even, $\exists k \in \mathbb{Z}$ $x = 2k$
Construct candidate q
Show that $x^2+3x-3 = 2q+1$

b) Proof:

Let x be arbitrary. Suppose x is even, that is, $\exists k \in \mathbb{Z}$, $x = 2k$. We will define a new integer variable $q = 2k^2+3k-2$. We will now verify that $x^2+3x-3 = 2q+1$

$$\begin{aligned} 2q+1 &= 2(2k^2+3k-2)+1 \\ &= 4k^2+6k-4+1 \\ &= 4k^2+6k-3 \\ &= (2k)^2+3(2k)-3 \\ &= x^2+3x-3 \quad // \end{aligned}$$

Since there is some integer q such that $x^2+3x-3 = 2q+1$, by the definition of odd, x^2+3x-3 is odd. ■

3a) Let x be arbitrary
Assume $x \bmod 8 = 5$, or, $\exists k \in \mathbb{Z}, 8k+5 = x$
Construct Candidate q
Show $4q+3 = 3x$

b) Proof

Let x be arbitrary. Suppose $x \bmod 8 = 5$, that is, $\exists k \in \mathbb{Z}, 8k+5 = x$. We will define a new integer variable $q = 6k+3$. We will now verify that $4q+3 = 3x$

$$\begin{aligned} 4q+3 &= 4(6k+3)+3 \\ &= 24k+12+3 \\ &= 24k+15 \\ &= 3(8k+5) \\ &= 3x \,_{//} \end{aligned}$$

Since there exists some integer q such that $4q+3 = 3x$, by the definition of the Division Algorithm, $3x \bmod 4 = 3$. ■

4) Let $a \in \mathbb{R}^+$ be arbitrary
[Construct candidate $x \in \mathbb{R}$
[[Let $N \in \mathbb{Z}^+$ be arbitrary
[[[Construct candidate $n \in \mathbb{Z}^+$
[[[[Suppose $n > N$
[[[[[Show that $((x - \frac{1}{n})^2 - x^2 \geq a) \vee ((x - \frac{1}{n}) - x^2 \leq -a)$]]]]]

5) The given proof does <u>not</u> prove the given statement. We must construct the candidate u first, before allowing w to be arbitrary. As written, the proof allows u to change in value in order to allow the predicate to always produce a True statement. However, the value for u must be fixed before we can consider its interactions with w. Generally, this statement is False.

6) This is a valid syllogism, by Modus Ponens.

$P(x)$ = X is a flying animal

$Q(x)$ = X has feathers,

$$\forall x, \quad P(x) \rightarrow Q(x)$$

$$\frac{P(\text{Bat})}{\therefore\ Q(\text{Bat})}$$

Whether this syllogism is True or False is a different question entirely.

E.C.) Why are proofs important?

As an exercise for a student, proof writing forces us to think deeply about rules and concepts that we tend to take for granted. It gives us the opportunity to more fully understand the exact nature of mathematics; how rules work and fail, how they can be connected, and how they can be extrapolated to other problems.

As a tool for the Philisophical/Mathematical community, proofs give us the tools to be confident in our correctness. They allow us to create a logical foundation, on which new ideas can be built. And they give us the tools necessary to support those new ideas. They help ensure that ideas are consistent and complete, or at least find the ways that they aren't.