Dynamic Warhouse Location Problem

Feser Alexander
Hovorka Anja
Schaffner Patrick

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1 Model

1.1 Bellman Equation

In each period the entrepreneur observes the price p and chooses either to stay at his current location w or relocate to a more profitable location. If he chooses to relocate ($w' \neq w$), he has to pay the costs of relocation R but has an otherwise identical income as if he were already located at w'. Furthermore the price follows a random walk. The corresponding Bellman-Equation is then:

$$V(w,p) = \max_{w' \in W} \left[\sum_{m \in M} \pi_{w',m} - 1(w' \neq w) * R + \beta * \mathrm{E}[V(w', p') \mid p] \right]$$

$$\text{s.t.} \quad p' = p + \epsilon \quad , \quad \epsilon \sim N(0, \sigma^2)$$

$$\begin{aligned}
\pi_{w,m} &:= \text{Profits gained by serving market } m \text{ from warehouse } w \\
R &:= \text{Relocation cost} \\
\beta &:= \text{Discount factor} \\
\sigma &:= \text{Standard deviation of price changes}
\end{aligned}$$

1.2 Profits

We assume that the demand at a given market depends on the price and the time required for delivery. Furthermore we assume fixed variable cost and no fixed-costs. The profits generated by serving market m from location w is then given by

$$\pi_{w,m} = \max\left[D_m(p, t_{w,m}) * (p - c_{w,m}), 0\right]$$

$$\begin{aligned}
c_{w,m} &:= \text{Average cost per unit if delivered from warehouse } w \text{ to market } m \\
t_{w,m} &:= \text{Delivery time from warehouse } w \text{ to market } m \text{ (in hours)} \\
D_m(p,t) &:= \text{Demand-function of market } m
\end{aligned}$$

The demand-function $D_m(p,t)$ must be non-negative and decreasing in the price p and delivery-time t. We provide a linear demand-function that is exponentially discounted by the delivery time:

$$D_m(p,t) = \max\left[\left(\overline{q}_m - \frac{\overline{q}_m}{\overline{p}_m}p\right) \;*\; e^{-A_m * t}\;,\;0\right]$$

$$\begin{aligned}
\overline{q}_m &:= \text{Maximal demand of market } m \ (\overline{q}_m = D_m(0)).\\
\overline{p}_m &:= \text{Maximal valuation of market } m \ (D_m(\overline{p}_m) = 0).\\
A_m &:= \text{Competitiveness in delivery-time of market } m.
\end{aligned}$$

The user is free to use another specification as long as the conditions imposed on it are fulfilled.

The average cost per unit are calculated by adding transportation and production costs. Transportation costs depend on delivery-time and distance:

$$c_{w,m} = 2 * (t_{w,m} * L + d_{w,m} * T) + F$$

$$\begin{aligned}
d_{w,m} &:= \text{Distance from warehouse } w \text{ to market } m \text{ (in kilometers)}\\
L &:= \text{Hourly wage of delivery staff}\\
T &:= \text{Transportation cost per kilometer}\\
F &:= \text{Production cost}
\end{aligned}$$

1.3 Expectations

To calculate expectations for the different (discrete) price states, we must discretize the distribution of p'. We specify the grid by the tuple $\langle p_{min}, p_{max}, N\rangle$ where p_{min}, p_{max} specify the price-range we allow and N specifies the number of price states evenly spaced over $[p_{min}, p_{max}]$. Let G be the set of all prices $p_{1,2,\ldots,N}$, with $p_1 = p_{min}$ and $p_N = p_{max}$, then the distance between adjacent prices is given by η:

$$\begin{aligned}
G &= \{p_{min}, p_2, p_3, ..., p_{N-1}, p_{max}\}\\
p_i &= p_{min} + (i-1) * \eta\\
\eta &= \frac{p_{max} - p_{min}}{N-1}
\end{aligned}$$

To discretize the distribution we bunch together intervals of the outcome of p' to prices in G. The interval corresponding to p_j is denoted by I_j:

$$\begin{aligned}
I_G &= [p_{min}, p_{max}]\\
I_j &= \left[\max\left(p_j - \frac{\eta}{2}, p_{min}\right), \min\left(p_j + \frac{\eta}{2}, p_{max}\right)\right]
\end{aligned}$$

We then discretize then by the following rule:

$$p' \in I_j \;\rightarrow\; p' = p_j$$

To calculate the probability that p' takes one of the prices in G we must assume that p' always stays inside I_G. Otherwise the density function over G would not add up to one. By using Bayes' rule we get

$$\begin{aligned}
\mathrm{P}[p' = p_j \mid p] &= \mathrm{P}[p' \in I_j \mid p' \in I_G \ , \ p] \\
&= \frac{\overbrace{\mathrm{P}[p' \in I_G \mid p' \in I_j \ , \ p]}^{=1 \text{ since } p' \in I_j \Rightarrow p' \in I_G} * \mathrm{P}[p' \in I_j \mid p]}{\mathrm{P}[p' \in I_G \mid p]} \\
&= \frac{\mathrm{P}[p' \in I_j \mid p]}{\mathrm{P}[p' \in I_G \mid p]}
\end{aligned}$$

We normalize p'

$$p' \sim N(p, \sigma^2) \quad \Leftrightarrow \quad \frac{p' - p}{\sigma} \sim N(0,1)$$

The fraction we got for $\mathrm{P}[p' = p_j \mid p]$ consists of probabilty measures that we can now calculate with the cumulative distribution function Φ of the standard normal distribution:

$$\mathrm{P}[p' \in [a,b] \mid p] \ = \ \mathrm{P}[p' \leq b \mid p] - \mathrm{P}[p' < a \mid p] \ = \ \Phi\left(\frac{b-p}{\sigma}\right) - \Phi\left(\frac{a-p}{\sigma}\right)$$

More specifically, the probability from transitioning from p_i to p_j is

$$\mathrm{P}[p' = p_j \mid p = p_i] \ = \ \frac{\Phi\left(\frac{\min\left(p_j + \frac{\eta}{2}, p_{max}\right) - p_i}{\sigma}\right) - \Phi\left(\frac{max\left(p_j - \frac{\eta}{2}, p_{min}\right) - p_i}{\sigma}\right)}{\Phi\left(\frac{p_{max} - p_i}{\sigma}\right) - \Phi\left(\frac{p_{min} - p_i}{\sigma}\right)}$$

The state transition matrix is then given by

$$\Sigma_{ij} = \mathrm{P}[p' = p_j \mid p = p_i]$$

Expectations are then simply

$$\mathrm{E}[V(w', p') \mid p] = V * \Sigma'$$

1.4 Required input parameters

Let us shortly summarize the parameters of the model:

$$\begin{aligned}
d_{w,m} &:= \text{Distance from warehouse } w \text{ to market } m \text{ (in kilometers)} \\
t_{w,m} &:= \text{Delivery time from warehouse } w \text{ to market } m \text{ (in hours)} \\
\langle \overline{q}_m, \overline{p}_m, A_m \rangle &:= \text{Demand specification of market } m \\
\langle p_{min}, p_{max}, N \rangle &:= \text{Grid specification} \\
L &:= \text{Hourly wage of delivery staff} \\
T &:= \text{Transportation cost per kilometer} \\
R &:= \text{Relocation cost} \\
F &:= \text{Production cost per unit} \\
\beta &:= \text{Discount factor} \\
\sigma &:= \text{Standard deviation of price changes}
\end{aligned}$$

Note you don't actually have to specify $d_{w,m}$ and $t_{w,m}$. These values will be fetched from the *Google Distance Matrix API*[1]. You just have to specify market and warehouse locations by addresses.

[1] https://developers.google.com/maps/documentation/distancematrix/

2 Value-function iteration

The problem contains some structure that can be exploited for fast value-function iteration. Let us conceptually split up the value-function:

$$\begin{aligned} V^+(w,p) &= \sum_{m\in M} \pi_{w,m} + \beta * \mathrm{E}[V(w,p') \mid p] \\ V^-(w,p) &= V^+(w,p) - R \end{aligned}$$

An entrepreneur gets V^+ if he chooses to stay at w. If he chooses to relocate to w he will get V^-. Since V^- is a positive monotone transformation of V^+ they share the same maxima given the same price:

$$w_p^* = \arg\max_{w\in W} V^+(w,p) = \arg\max_{w\in W} V^-(w,p)$$

The Bellman-equation may now be reformulated in terms of a binary choice between staying and relocating:

$$V(w,p) = \max\bigl[V^+(w,p)\ ,\ V^-(w_p^*,p)\bigr] = \max\bigl[V^+(w,p)\ ,\ V^+(w_p^*,p) - R\bigr]$$

The policy-function for w' in the original formulation is then simply

$$w'(w,p) = \begin{cases} w & \text{if} \quad V^+(w,p) \geq V^+(w_p^*,p) - R \\ w_p^* & \text{otherwise} \end{cases}$$

This leads to the following algorithm for value-function iteration:

1. Set $V = \Pi + \beta * V^{prime} * \Sigma'$ where V^{prime} is the value-function of the previous iteration and Π is the matrix of per-period profits (i.e. initialize V with V^+).

2. For every price p (for every column of V) do the following steps:

   (a) Find $w_p^* = \arg\max_{w\in W} V_{w,p}$.

   (b) Calculate $v_p^* = V^-(w_p^*,p) = V_{w_p^*,p} - R$.

   (c) For every $w \in W$ for which $V_{w,p} < v_p^*$, modify V by setting $V_{w,p} = v_p^*$ (i.e. relocate to w_p^*).

3. Repeat until $V - V^{prime}$ becomes sufficiently small.