Youtube Tutorial: Time-optimal river crossing. (Zermelo's navigation problem)

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(0) The Optimization Problem:

$$\begin{aligned}
&\min_{u(t)} \quad \int_0^{t_f} \underbrace{1}_{f_0} \mathrm{d}t. \\
&\text{s.t.} \quad \dot{x} = f(x_1, u) = \begin{bmatrix} v\ cos(u) \\ v\ sin(u) + d(x_1) \end{bmatrix}, \\
&\qquad d(x_1) = d_p\Big(\frac{4x_1}{x_{f,1}} - \frac{4x_1^2}{x_{f,1}^2}\Big), \\
&\qquad x(0) = [0, 0]^T, \\
&\qquad x(t_f) = [x_{f,1}, x_{f,2}]^T, \\
&\qquad t_f \text{ free}
\end{aligned} \tag{1}$$

(1) Optimality Conditions:

Hamiltonian:

$$H = f_0 + p^T f = 1 + p_1 v\ cos(u) + p_2\,(v\ sin(u) + d(x_1)) \tag{2}$$

System Dynamics:

$$\dot{x} = f(x_1, u) \tag{3}$$

Costate Eq:

$$\dot{p} = -H_x = \begin{bmatrix} p_2 d_p \left(\frac{8x_1}{x_{f,1}^2} - \frac{4}{x_{f,1}}\right) \\ 0 \end{bmatrix} \tag{4}$$

Stationary Cond.:

$$H_u = -p_1 v\ sin(u) + p_2 v\ cos(u) = 0 \quad \rightarrow \quad u = atan\left(\frac{p_2}{p_1}\right) \tag{5}$$

Legendre Cond.:

$$H_{uu} = -p_1 v\ cos(u) - p_2 v\ sin(u) \succeq 0 \tag{6}$$

Transversality Cond.:

$$H(x^*(t_f), u^*(t_f), p^*(t_f)) = 0 \tag{7}$$

(2) Numeric Solution of BVP:

Use a shooting method (fsolve, ode45) to numerically solve the BVP in Octave/Matlab. A tranformation to a fixed endtime t_f is helpful to get numeric stability.