Appendix A

maximization problem

The maximization problem arising in the Karhunen-Loéve formulation is stated as follows

$$\max_\sigma \frac{\langle(\sigma,u)^2\rangle}{(\sigma,\sigma)}. \tag{A.1}$$

where $u(x,t)$ is a velocity field in space direction x and at time t. The innerproducts are defined as $\langle\cdot\rangle = \lim_{T\to\infty}\int_0^T \cdot\, \mathrm{d}t$ and $(f,g) = \int_\Omega f\bar{g}\mathrm{d}\Omega$,
The maximization problem of (A.1) can also be written as a constrained maximization problem

$$\max_\sigma\langle(\sigma,u)^2\rangle, \mathrm{under}(\sigma,\sigma) = 1. \tag{A.2}$$

To apply the classical calculus of variations we first rewrite $\langle(\sigma,u)^2\rangle$

$$\langle(\sigma,u)^2\rangle = \lim_{T\to\infty}\frac{1}{T}\int_0^T\{\int_\Omega u(x,t)\sigma(x)\mathrm{d}x\int_\Omega u(x',t)\sigma(x')\mathrm{d}x'\}\mathrm{d}t \tag{A.3}$$

$$= \int_\Omega\int_\Omega\{\lim_{T\to\infty}\frac{1}{T}\int_0^T u(x,t)u(x',t)\mathrm{d}t\}\sigma(x)\sigma(x')\mathrm{d}x\mathrm{d}x' \tag{A.4}$$

$$= ((R(x,x'),\sigma(x)),\sigma(x')) \tag{A.5}$$

Where $R(x,x') = \lim_{T\to\infty}\frac{1}{T}\int_0^T u(x,t)u(x',t)\mathrm{d}t$ is the autocorrelation matrix.
If we define $K(\sigma)$ to be the right side of (A.5) and the constrained $(\sigma,\sigma)=1$ as $C(\sigma)=1$ we have the following Euler-Lagrange equations

$$\delta K = \lambda\delta C \tag{A.6}$$

with δK and δC the first variation of K and C.
The first variations of K and C are

$$\delta K = 2(R(x,x'),\sigma(x')), \delta C = 2\sigma(x) \tag{A.7}$$

This means that equation (A.6) is equal to

$$(R(x,x'),\sigma(x')) = \lambda\sigma(x) \tag{A.8}$$

The value of λ is maximum of (A.1). This follows directly if $\lambda\sigma(x)$ is substituted for $(R(x,x'),\sigma(x'))$ (eq. (A.8)) in (A.5).