

Solution of hydrodynamic equilibrium in black-oil reservoirs: the classical material balance equation

Abstract

This work examines underlying details in obtaining the classical material balance equation from a three dimensional black-oil model typical of reservoir simulations. The three-dimensional model is integrated in reservoir volume and quantities such as fluid properties are spatially averaged. The material balance equation is obtained for averaged quantities and it is applied to calculate reservoir pressure. It is shown that the calculated pressure corresponds to the local pressure at an arbitrary reference depth and not to the average reservoir pressure in general. The definition of a convenient reference depth is discussed. Sufficient conditions for solution uniqueness of the material balance equation are provided for the case that pressure is the unknown quantity. One of such conditions provides a check for physical correctness of PVT properties obtained from laboratory experiments or empirical correlations. Finally, a numerical scheme for solving the material balance equation is proposed.

Keywords: material balance, black-oil equations, reservoir engineering.

1. Introduction

The material balance equation is a classical tool in petroleum reservoir engineering. It consists of the algebraic equation originally presented in Schilthuis (1936) as a zero dimensional model for estimating reservoir

pressure from mass balance. Schilthuis (1936) introduced the hypothesis of equilibrium attained at all times not necessarily in the entire reservoir volume, but in the regions of high permeability where most of the production would come from and where pressure would present a similar behavior. Such regions were said to contain an active oil.

The equation developed by Schilthuis (1936) did not account for the non-existence of mechanical, phase or thermal equilibrium. An attempt to introduce non-equilibrium effects of hydrodynamic origin came with *black-oil* equations (Aziz and Settari, 1979; Trangenstein and Bell, 1989b). The novelty on hydrodynamics was the introduction of a macroscopic equation of motion using Darcy’s law.

The term black-oil in this context refers to specific hydrocarbon compositions to which phase equilibrium for isothermal changes in pressure can be described by a two pseudo-component system. For such reservoir fluids, gas phase composition remains essentially constant in the range of pressures observed during production so that a pseudo-gas component can be defined. Similarly, a pseudo-oil component can be defined by subtracting the pseudo-gas component from oil phase composition. After defining the pseudo-components, the number of transport equations needed for hydrocarbon components is reduced from a usually large number to only two.

Many works have reported applications of the material balance equation in the literature, but few have explored its connection with black-oil equations. Ertekin et al. (2001) briefly addressed this subject arguing both approaches to be equivalent if capillarity and gradients of flow potential ($\nabla\Phi = \nabla p - \rho\vec{g}$) were neglected and if PVT properties were averaged in reservoir volume. This work develops this concept examining underlying details.

The same approach of varying bubble-point pressure widely used in reservoir simulation, which is found in Aziz and Settari (1979) and Ertekin et al. (2001), is also applied here to material balance. This approach extends the material balance equation to saturated reservoirs under pressure maintenance (e.g. fluid injection). The applicability, however, requires negligible non-equilibrium effects in phase equilibrium.

After presenting the material balance equation, sufficient conditions are shown for which material balance equation has only one solution for pressure. These conditions require fluids to be compressible and gas vaporization to occur with increase of oil phase density. A simple scheme for numerical solution is also proposed based on classical root-finding algorithms for non-linear equations.

2. Derivation of equations

This section presents black-oil equations, performs the averaging of quantities in space and examines simplifying assumptions necessary to obtain the material balance equation.

2.1. Black-oil equations

We assume that the reservoir is a hydraulically connected volume where hydrocarbon saturation is nonzero. By a hydraulically connected volume, we mean that it is possible to observe a common pressure decline trend when some region of the reservoir is under production, as discussed in Dake (2001). Reservoir fluids are composed of oil and gas pseudo-components and water component. There can be three fluid phases: oil, gas and (liquid) water. Oil phase is composed of oil and gas components, gas phase is composed uniquely by gas component and, water phase, uniquely by water component.

The only allowed mass transfer is of gas component between oil and gas phases, i.e., gas phase composition is always that of a dry gas.

Mass balance equation for each component reads:

- water component:

$$\frac{\partial}{\partial t} (\rho_w S_w \phi) + \nabla \cdot (\rho_w \vec{u}_w) = 0; \quad (1)$$

- oil component:

$$\frac{\partial}{\partial t} (\rho_o Y_o S_o \phi) + \nabla \cdot (\rho_o Y_o \vec{u}_o) = 0; \quad (2)$$

- gas component:

$$\frac{\partial}{\partial t} [(\rho_o Y_g S_o + \rho_g S_g) \phi] + \nabla \cdot (\rho_o Y_g \vec{u}_o + \rho_g \vec{u}_g) = 0. \quad (3)$$

Mass concentrations of oil and gas components in oil phase are denoted by Y_o and Y_g , respectively.

Initial conditions are of no fluid motion. Boundary conditions are of no normal flow at all boundaries except for interfaces with wells or aquifers.

A common simplification in black-oil formulation is to restrict mass transport to macroscopic advection. No diffusive or dispersive term is included in mass balance equations.

Momentum balance for each phase is given by an extension of Darcy's law to multiphase flow. This extension consists in multiplying absolute (or single-phase) permeability (\vec{k}) by a scalar function called phase relative permeability ($k_{r,\alpha}$):

$$\vec{u}_\alpha = -\frac{k_{r,\alpha} \vec{k}}{\mu_\alpha} \cdot (\nabla p_\alpha - \rho_\alpha \vec{g}), \quad \alpha = o, w, g. \quad (4)$$

Phase densities are functions of the local thermodynamic state:

$$\rho_w = \rho_w(p_w, T_w) \quad (5)$$

$$\rho_o = \rho_o(p_o, T_o, Y_g) \quad (6)$$

$$\rho_g = \rho_g(p_g, T_g) \quad (7)$$

Another simplification is now made assuming thermal equilibrium in the reservoir volume, i.e.,

$$T_w = T_o = T_g = T_{rock} = T = \text{constant} . \quad (8)$$

Temperature is a constant prescribed value and no equation for energy balance is needed.

It follows from Duhem's theorem that the equilibrium of oil and gas phases is determined if, in addition to the mass balance equations, two properties are prescribed. From Gibbs' phase rule, the system has two degrees of freedom if the component number is restricted to the two-pseudo components. It is then necessary to specify two intensive properties. The constant temperature is one and pressure is chosen as the other. Oil phase density now becomes a function of pressure alone. The same is valid for mass fractions of gas and oil components in oil phase:

$$Y_g = Y_g(p_o) , \quad (9)$$

$$Y_o = Y_o(p_o) . \quad (10)$$

Porosity (ϕ) is assumed a function of pore pressure (p_P). Because three phases are present in general, such quantity is defined by some relation of the type (Kim et al., 2011):

$$p_P = p_P(p_o, p_w, p_g, S_o, S_w, S_g) . \quad (11)$$

To close the system of equations, a connection among different phase pressures is provided by capillary pressure relations:

$$p_{c,ow} = p_o - p_w , \quad (12)$$

$$p_{c,go} = p_g - p_o . \quad (13)$$

2.2. Material balance equation

Derivation of the material balance equation consists in integrating black-oil equations in reservoir volume (Ω).

Integrating Eqs. (1), (2) and (3) in volume and applying divergence theorem gives:

- water component:

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\Omega} \rho_w S_w \phi dV &= - \oint_{\partial\Omega} \rho_w (\vec{u}_w \cdot d\vec{S}) \\ &= \rho_w^{ST} (Q_{we}^{ST} + Q_{wi}^{ST} - Q_w^{ST}) ; \end{aligned} \quad (14)$$

- oil component:

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\Omega} \rho_o Y_o S_o \phi dV &= - \oint_{\partial\Omega} \rho_o Y_o (\vec{u}_o \cdot d\vec{S}) \\ &= -\rho_o^{ST} Q_o^{ST} ; \end{aligned} \quad (15)$$

- gas component:

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\Omega} (\rho_o Y_g S_o + \rho_g S_g) \phi dV &= - \oint_{\partial\Omega} (\rho_o Y_g \vec{u}_o + \rho_g \vec{u}_g) \cdot d\vec{S} \\ &= \rho_g^{ST} (Q_{gi}^{ST} - Q_g^{ST}) . \end{aligned} \quad (16)$$

2.2.1. Averaging quantities

Spatially averaged quantities are defined for porosity (ϕ), saturation (S_α), density (ρ_α) and mass concentration ($Y_{\alpha,k}$) of a component k in phase α .

All quantities are initially functions of space and time, but they become functions of time alone after being averaged in space:

$$\bar{\phi}(t) = \frac{1}{\Omega} \int_{\Omega} \phi(\vec{x}, t) dV. \quad (17)$$

Omitting function arguments:

$$\bar{S}_\alpha \bar{\phi} = \frac{1}{\Omega} \int_{\Omega} S_\alpha \phi dV, \quad (18)$$

$$\bar{\rho}_\alpha \bar{S}_\alpha \bar{\phi} = \frac{1}{\Omega} \int_{\Omega} \rho_\alpha S_\alpha \phi dV, \quad (19)$$

$$\bar{\rho}_\alpha \bar{Y}_{\alpha,k} \bar{S}_\alpha \bar{\phi} = \frac{1}{\Omega} \int_{\Omega} \rho_\alpha Y_{\alpha,k} S_\alpha \phi dV. \quad (20)$$

Eqs. (14), (15) and (16) are rewritten in terms of averaged quantities:

- water component:

$$\frac{\partial}{\partial t} (\bar{\rho}_w \bar{S}_w \bar{\phi} \Omega) = \rho_w^{ST} (Q_{we}^{ST} + Q_{wi}^{ST} - Q_w^{ST}) \quad (21)$$

- oil component:

$$\frac{\partial}{\partial t} (\bar{\rho}_o \bar{Y}_o \bar{S}_o \bar{\phi} \Omega) = -\rho_o^{ST} Q_o^{ST} \quad (22)$$

- gas component:

$$\frac{\partial}{\partial t} [(\bar{\rho}_o \bar{Y}_g \bar{S}_o + \bar{\rho}_g \bar{S}_g) \bar{\phi} \Omega] = \rho_g^{ST} (Q_{gi}^{ST} - Q_g^{ST}). \quad (23)$$

We now introduce usual definitions from petroleum engineering for the description of volumetric and solubility changes. In what concerns volumetric changes, B_w , B_o and B_g , the formation volume factors of water, oil and gas phases, respectively, express the ratio of a phase volume at reservoir condition to its volume at surface condition, i.e.:

$$B_w = \frac{\rho_w^{ST}}{\bar{\rho}_w}, \quad (24)$$

$$B_o = \frac{\rho_o^{ST}}{\bar{Y}_o \bar{\rho}_o}, \quad (25)$$

$$B_g = \frac{\rho_g^{ST}}{\bar{\rho}_g}. \quad (26)$$

In what concerns changes in gas solubility in oil phase, the solubility ratio (R_s) gives the ratio between the volume of dissolved gas and the volume of oil both measured in surface condition:

$$R_s = \frac{\rho_o^{ST} / \bar{Y}_o}{\rho_g^{ST} / \bar{Y}_g}, \quad (27)$$

or using $\bar{Y}_o + \bar{Y}_g = 1$:

$$R_s = \frac{\rho_o^{ST}}{\rho_g^{ST}} \frac{\bar{Y}_g}{1 - \bar{Y}_g}. \quad (28)$$

Equations can now be rewritten using the above definitions and denoting porous volume by $V_p = \bar{\phi}\Omega$.

$$\frac{\partial}{\partial t} \left(\frac{\bar{S}_w V_p}{B_w} \right) = Q_{we}^{ST} + Q_{wi}^{ST} - Q_w^{ST} \quad (29)$$

$$\frac{\partial}{\partial t} \left(\frac{\bar{S}_o V_p}{B_o} \right) = -Q_o^{ST}, \quad (30)$$

$$\frac{\partial}{\partial t} \left[V_p \left(\frac{\bar{S}_g}{B_g} + \frac{\bar{S}_o R_s}{B_o} \right) \right] = (Q_{gi}^{ST} - Q_g^{ST}) , \quad (31)$$

$$\bar{S}_w + \bar{S}_o + \bar{S}_g = 1 . \quad (32)$$

Integrating Eqs. (29), (30) and (31) between times t_1 and t_2 gives:

$$\frac{V_{p2} \bar{S}_{w2}}{B_{w2}} = W + (W_i + W_e - W_p) , \quad (33)$$

$$\frac{\bar{S}_{o2} V_{p2}}{B_{o2}} = N - N_p , \quad (34)$$

$$V_{p2} \left(\frac{\bar{S}_{g2}}{B_{g2}} + \frac{\bar{S}_{o2} R_{s2}}{B_{o2}} \right) = N \left(m \frac{B_{o1}}{B_{g1}} + R_{s1} \right) + (G_i - G_p) , \quad (35)$$

$$\bar{S}_o + \bar{S}_w + \bar{S}_g = 1 . \quad (36)$$

Where $N = V_{p1} \bar{S}_{o1} / B_{o1}$ and $W = V_{p1} \bar{S}_{w1} / B_{w1}$ are the initial volumes of oil and water measured at surface condition and $m = V_{p1} \bar{S}_{g1} / B_{o1} N$ is the ratio of the initial volumes of gas and oil phases measured in reservoir condition.

An equation for the bubble-point can be obtained by setting $\bar{S}_{g2} = 0$ in Eq. (35) and solving it for the solubility ratio at bubble-point pressure, or $R_s(p = P_b)$:

$$R_{s2}(p = P_b) = \left[N (m B_1 / B_{g1} + R_{s1}) + G_i - G_p \right] / (N - N_p) . \quad (37)$$

2.2.2. Material balance equation

It is now possible to obtain the so-called generalized material balance equation by substituting the equations for phase saturation, Eqs. (33), (34) and (35), into Eq. (36):

$$F_p - F_i = \underbrace{NE_o + mNE_g + WE_w}_{\text{fluid volume expansion } (E)} + \underbrace{V_{p1} - V_{p2}}_{\text{pore volume reduction } (C)} \quad (38)$$

or

$$F_p - F_i = E + C, \quad (39)$$

where new terms C , E , F_i and F_p are defined in Table 1.

Eq (38) involves fluid properties, original fluid volumes (N , W and G), cumulative production and injection and aquifer influx. It is an algebraic equation that can be solved for any one of its terms, provided that all remaining are known. This work focuses on its solution for pressure.

2.2.3. Relationship between the material balance equation and reservoir pressure

The material balance equation is widely used to relate production and injection to reservoir pressure. Actually, all averaged fluid properties in Eq. (38) depend on the pressure field, and so does the pore volume. If the reservoir is under production, though, the pressure field is not known and for its prediction one would have to resort to more advanced techniques such as reservoir simulation.

To overcome this difficulty, a possible simplification to the pressure field is to neglect hydrodynamic gradients and assume hydrostatic equilibrium: $\vec{u}_\alpha \equiv 0$. This implies that $\nabla p_\alpha \equiv \rho_\alpha \vec{g}$, and, as a consequence, pressure

becomes a function of time and depth given by the following expression:

$$\int_{p_{\alpha}^{ref}(t)}^{p_{\alpha}(t,z)} \frac{dp'_{\alpha}}{\rho_{\alpha}(p'_{\alpha})} = g(z - z_{ref}) , \quad (40)$$

where $p_{\alpha}(t, z)$ is the pressure field of phase α and $p_{\alpha}^{ref}(t)$ is the pressure at some reference depth z_{ref} . It is now possible to compute the pressure field from Eq. (40) and average fluid properties accordingly.

It can be shown that neglecting the variation of density with depth does not introduce any important error. As pressure increases with depth according to the hydrostatic gradient, density also increases but by a negligible amount. For instance, the average oil density can be approximated to the local density at the reference level: $\bar{\rho}_o(t) \approx \rho_o(t, z_{ref})$.

This approximation can be quantified by putting together the expressions for compressibility and hydrostatic equilibrium. For this, we assume a constant isothermal compressibility for oil phase,

$$\frac{1}{\rho_o} \left(\frac{\partial \rho_o}{\partial p_o} \right)_{T, Y_o} = c_o , \quad (41)$$

and we use hydrostatic equilibrium $dp_o = \rho_o g dz$ to obtain the expression for the derivative of oil density with depth:

$$\frac{1}{\rho_o} \left(\frac{\partial \rho_o}{\partial z} \right)_{T, Y_o} = c_o \rho_o g . \quad (42)$$

Similarly for the gas phase, using the state equation $pV = ZnRT$, we obtain its isothermal compressibility:

$$\frac{1}{\rho_g} \left(\frac{\partial \rho_g}{\partial p_g} \right)_T = \frac{1}{p_g} - \frac{1}{Z} \frac{\partial Z}{\partial p_g} . \quad (43)$$

Assuming reservoir temperature to be sufficiently above gas critical temperature for negligible variation of the Z-factor with pressure, we use hydrostatic equilibrium again and obtain the derivative of gas density with depth:

$$\frac{1}{\rho_g} \left(\frac{\partial \rho_g}{\partial z} \right)_T = \frac{\rho_g \vec{g}}{p_g} = \left(\frac{\mathcal{M}}{ZRT} \right) g . \quad (44)$$

For investigating the order of magnitude of Eqs. (42) and (44), we substitute into them the values of $\rho_o = 10^3 \text{ kg/m}^3$ and $c_o = 10^{-9} \text{ Pa}^{-1}$ for the oil phase, and $\mathcal{M} = 19 \text{ kg/kmol}$ and $T = 400 \text{ K}$ for the gas phase:

$$\frac{1}{\rho_o} \left(\frac{\partial \rho_o}{\partial z} \right)_{T, Y_o} \approx 1 \times 10^{-5} \text{ m}^{-1}, \quad (45)$$

$$\frac{1}{\rho_g} \left(\frac{\partial \rho_g}{\partial z} \right)_{T, Z} \approx 5 \times 10^{-5} \text{ m}^{-1}. \quad (46)$$

Oil and gas densities are thus approximately constant with depth. The reference depth for computing phase density (and also the formation volume factor) can be any arbitrary depth inside phase volume. Under the approximation of a constant density, it is straightforward to show that $\bar{\rho}_\alpha = \rho_\alpha(\bar{p}_\alpha)$ and $\bar{p}_\alpha = p_\alpha(z_\alpha^0)$, where z_α^0 is the depth of phase gravity center and $\alpha = w, o, g$. The pressure calculated in the material balance equation corresponds to the local pressure at z^{ref} , without fluid motion, and only equals the average reservoir pressure \bar{p}_α if the reference depth is that of the gravity center, $z^{ref} = z^0$.

For the case of an oil reservoir with a gas cap, the reference depth should be defined at the gas-oil contact. The point here is that at gas-oil contact both gas and oil densities are computed using the same pressure, its local value. If the reference depth, common to all phases, is otherwise defined below gas-oil contact, gas density will increase with depth at a higher rate because of the oil hydrostatic gradient,

$$\frac{1}{\rho_g} \left(\frac{\partial \rho_g}{\partial z} \right)_{T, Z} = \frac{\rho_o}{\rho_g} \left(\frac{\mathcal{M}}{ZRT} \right) g, \quad (47)$$

what is ρ_o/ρ_g times greater than the gas hydrostatic gradient used in Eq. (44).

It has been shown that the pressure variation with depth is not important for the computation of gas, oil and water densities. Nevertheless, pressure

also plays a second role in the material balance equation by defining phase equilibrium, for what the pressure variation with depth may not be negligible in the solubility ratio:

$$\frac{1}{R_s} \frac{dR_s}{dz} = \frac{1}{R_s} \frac{dR_s}{dp_o} \frac{dp_o}{dz} = \frac{\rho_o g}{R_s} \frac{dR_s}{dp_o}. \quad (48)$$

For a non-volatile oil, we can approximate dR_s/dp_o to the ratio between the initial solubility ratio ($R_{s,0}$) and saturation pressure (p_{sat}), so that Eq. (48) becomes:

$$\frac{1}{R_s} \frac{dR_s}{dz} \approx \frac{\rho_o g}{p_{sat}} \approx 10^{-3} \text{ m}^{-1}, \quad (49)$$

where the following orders of magnitudes were assumed: $p_{sat} \approx 10^7 \text{ Pa}$, $\rho_o \approx 10^3 \text{ kg/m}^3$ and $g \approx 10 \text{ m}^2/\text{s}$.

From Eq. (49), we see that the variation of pressure with depth may not be negligible for the phase equilibrium, specially for thick reservoirs of light oil. In such cases, using an averaged value for the solubility ratio may be worth the accuracy gain in computing dissolved gas volume.

3. Solution of material balance equation

A numerical solution of Eq. (39) is presented here. We shall start defining a residual function (ε) for which the pressure solving Eq. (39) is a root:

$$\varepsilon(p_2) = \frac{E + F_i + C - F_p}{V_{p1}}. \quad (50)$$

We proceed showing that ε is a monotonic function provided some conditions on fluid properties are observed and that, as a consequence, only one root of ε exists at most. Equivalently, we can say that only one solution exists for the material balance equation.

3.1. Mathematical aspects of the residual function

It is a known result of real analysis that a monotonic function $f : D \subset \mathcal{R} \rightarrow \mathcal{R}$ can have at most one root in D . Further, if f is continuous in the closed interval $I = [a, b] \subset D$ and $f(a)f(b) < 0$, then f has one (and only one) root in I .

We now state without proof that the following affirmatives are sufficient conditions to classify a function f as monotonic:

1. f is continuous;
2. f is differentiable everywhere except for a countable set of points;
3. $f' < 0$ everywhere.

The residual function clearly satisfies the first two affirmatives because fluid properties are continuous everywhere and are also differentiable everywhere except for the first order phase change at bubble-point pressure. The third affirmative is proved by rewriting $d\epsilon/dp$ as a sum of four parcels, each corresponding to a different phase (three fluid phases and the solid rock). For the fluid phases, each parcel is composed of two factors: current volume *in-situ* and volume derivative with respect to pressure:

$$\begin{aligned}
 V_{p1} \frac{d\epsilon}{dp} = & \underbrace{(N - N_p) \left(\frac{dB_o}{dp} - B_g \frac{dR_s}{dp} \right)}_{\text{oil phase}} \\
 & + \underbrace{\left[NR_{s1} + mN \frac{B_{o1}}{B_{g1}} + G_i - (N - N_p) R_{s2} - G_p \right] \frac{dB_g}{dp}}_{\text{gas phase}} \\
 & + \underbrace{(W + W_i + W_e - W_p) \frac{dB_w}{dp}}_{\text{water phase}} \\
 & - \underbrace{\frac{dV_p}{dp}}_{\text{porous volume}} .
 \end{aligned} \tag{51}$$

In-situ volumes (e.g. $N - N_p$ for oil phase) are non-negative values. Pressure derivatives of fluid volumes are non-positive values (fluid volume cannot increase as pressure increases) and pressure derivative of porous volume is a positive value (as fluid pressure increases, porous volume expands owing to rock compression).

The value of $d\varepsilon/dp$ is thus negative if at least one phase is compressible because it would be a sum of products between negative and positive factors.

In summary, the following four relations constitute sufficient conditions for $\varepsilon(p)$ be a strictly decreasing function and, therefore, for material balance equation to have only one solution:

$$\left(\frac{dB_o}{dp} - B_g \frac{dR_s}{dp} \right) \leq 0, \quad (52)$$

$$\frac{dB_g}{dp} \leq 0, \quad (53)$$

$$\frac{dB_w}{dp} \leq 0, \quad (54)$$

$$\frac{dV_p}{dp} \geq 0. \quad (55)$$

Eq. (52) also provides a condition to check whether PVT data obtained from laboratory or empirical correlations is physically correct. It can be alternatively formulated in terms of densities as shown bellow in Eq. (56). It simply indicates that any vaporization must be followed by an increase in oil phase density:

$$\frac{1}{\rho_o} \frac{d\rho_o}{dY_o} \geq Y_o \left(\frac{\rho_o}{\rho_g} - 1 \right). \quad (56)$$

3.2. Solution algorithm

Several methods exist to find roots of nonlinear continuous functions (Hamming, 1987). One possible choice is Newton's method, which linearizes

the function using either its gradient or a secant line. The secant line approach introduces a numerical smoothness to the gradient near continuous but non-differentiable points, for what it is the approach used here to deal with the bubble-point pressure.

An open source implementation of the secant method is available in SciPy library (Jones et al., 2001–). This work used library version 0.12.

The steps to solve the material balance equation are summarized bellow:

Step 1. *Compute bubble-point pressure for instant t_2 using Eq. (37).*

Bubble-point pressure P_{b2} is only a function of values at instant t_1 and produced and injected volumes.

Step 2. *Compute pressure for instant t_2 by finding the root of residual function $\varepsilon(p)$.*

Pressure in Eq. (38) is only a function of fluid properties in instant t_1 , produced and injected values and bubble-point pressure at instant t_2 , P_{b2} , which was already computed in step 1.

Step 3. *Compute fluid saturations t_2 .*

Once S_{o2} , S_{g2} , S_{w2} , p_2 and P_{b2} are known, solution for instant t_2 is complete.

4. Results

Material balance equation were solved for a black-oil reservoir initially producing in depletion drive mechanism and later subjected to water injection. Fig. 1 shows in the solid line curve the severe depletion during initial

phases of production. In the dashed curve, Fig. 1 shows the bubble-point pressure, indicating how far the saturated reservoir is from undersaturation and allowing for planning of pressure maintenance projects.

Water injection starts when oil recovery factor (RF) is about $RF = 0.08$ and it is maintained until the reservoir reaches a novel undersaturated state. Initially, the high compressibility of gas prevents pressure from raising and free gas phase is reduced because of redissolution. Finally, the reservoir reaches the undersaturated state and a steep pressure increase is seen in Fig. 1 for a recovery factor about $RF = 0.15$.

It must be observed that the result of gas reentering oil phase completely neglects non-equilibrium effects, e.g. compositional gradients, and must be used with care. For instance, it may be in strong disagreement with real reservoir behavior when gravity segregation is present.

Figure 4 shows the formation of a free gas phase with growing S_g and then its disappearance. Depending on the ratio of vertical and horizontal permeabilities or reservoir dip angle, it may also suggest the formation of a secondary gas cap.

Figure 5 shows the shape of residual function (ϵ) when pressure is varying and all remaining terms are held constant. Each different solid line corresponds to a different instant of time and therefore different volumes of cumulative production and injection. The function root is the solution of material balance equation, Eq. (39), and the non-differentiable point is the bubble-point pressure. Also, it is seen that function ϵ monotonically decreases with pressure, as theoretically shown before.

Figure 6 shows the count of newtonian iterations required to compute pressure with an accuracy within 0.1% of the initial bubble-point pressure. Typically, 12 iterations are required.

Figure 7 shows the residual of numerical solution as defined in Eq. (50). Material balance error is less than 10^{-7} units of porous volume for all time steps.

5. Conclusions

1. Material balance equation can be obtained from black-oil equations under the assumptions of:
 - (a) mechanical equilibrium;
 - (b) absence of gravity and capillary forces;
 - (c) spatially constant bubble-point pressure or, equivalently, fluid saturations.
2. One unique solution of pressure equation exists for a compressible system;
3. A residual function can be defined such that its root solves pressure equation. A numerical solution can be obtained with nonlinear root-finding algorithms.

Nomenclature

B_g	gas formation volume factor
B_o	oil formation volume factor
B_w	water formation volume factor
R_s	solubility ratio of gas in oil phase
S_g	gas saturation
S_o	oil saturation
S_w	water saturation
G_p	produced volume of gas
m	ratio of original free gas volume to original oil volume in reservoir condition
N	original volume of oil in place
N_p	produced volume of oil
P_b	bubble-point pressure
V_p	porous volume
W_i	injected volume of water
W_p	produced volume of water

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Highlights

1. The existing link between black-oil and material balance equation is examined in detail.
2. It is shown that gravity and capillary forces must be neglected and saturations must be constant in space so one can integrate black-oil equations in reservoir volume and obtain material balance equation.
3. Material balance equation are shown to have a unique solution for a compressible system.
4. Material balance equation are formulated for a variable bubble-point and they are used to a reservoir under pressure maintenance.

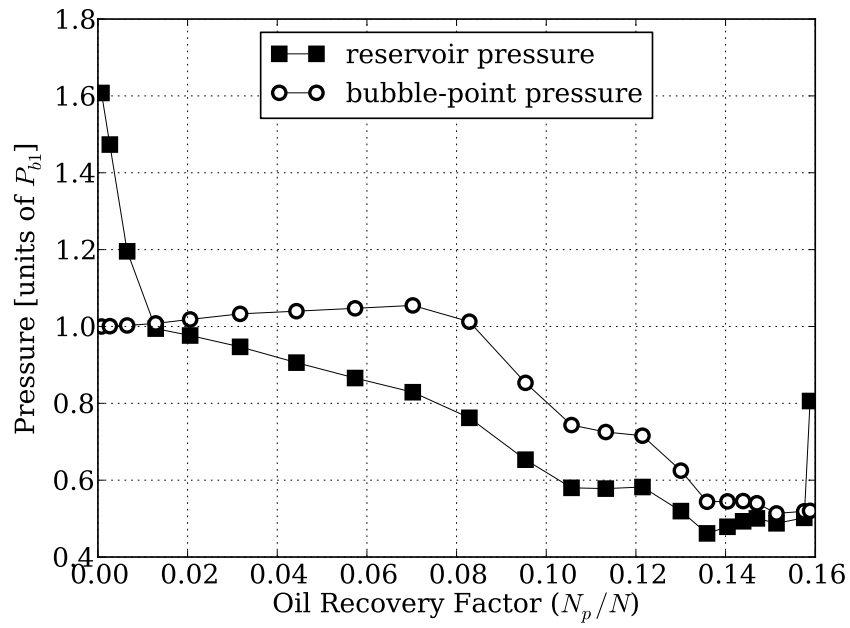


Figure 1: Reservoir pressure and bubble-point pressure as functions of recovery factor. Once depletion goes below bubble-point, pressure is hardly recovered for a saturated reservoir because of high system compressibility. Pressure is here given in units of the original bubble-point pressure.

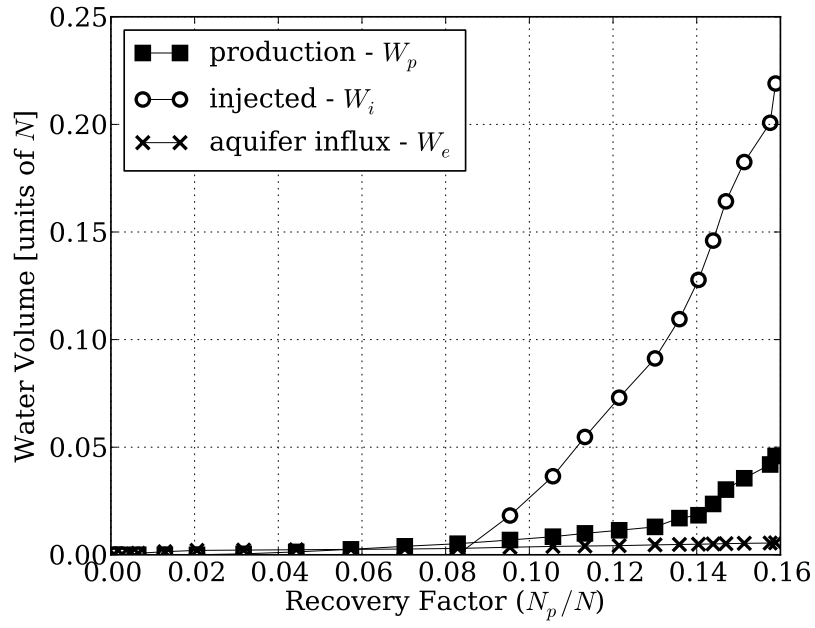


Figure 2: Volume of water production and injection and aquifer influx.

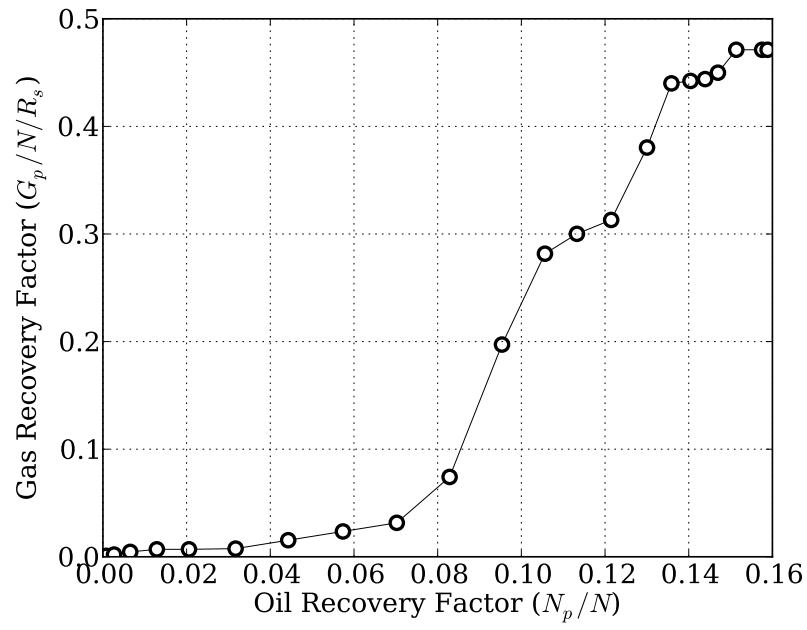


Figure 3: Volume of gas production.

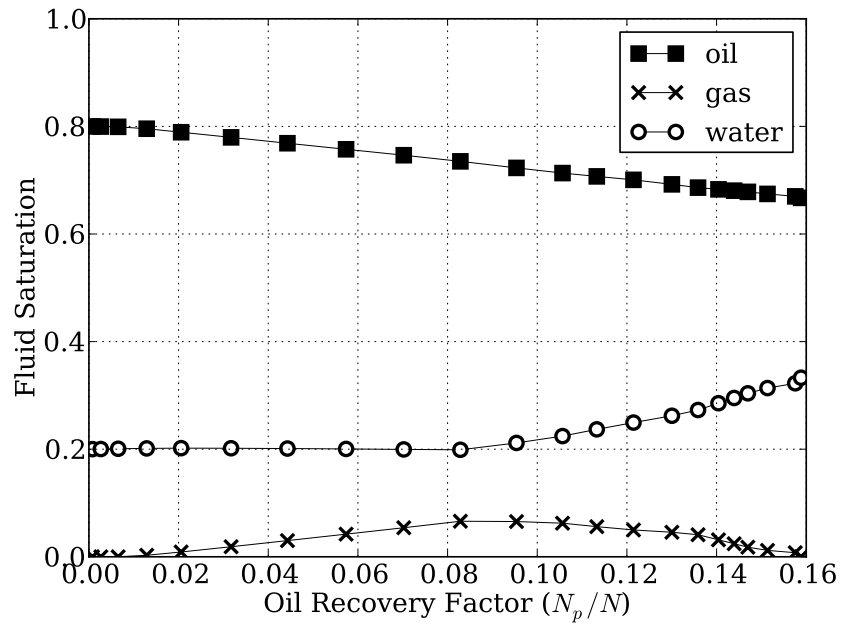


Figure 4: Phase saturations. Gas saturation is reduced as water injection progresses.

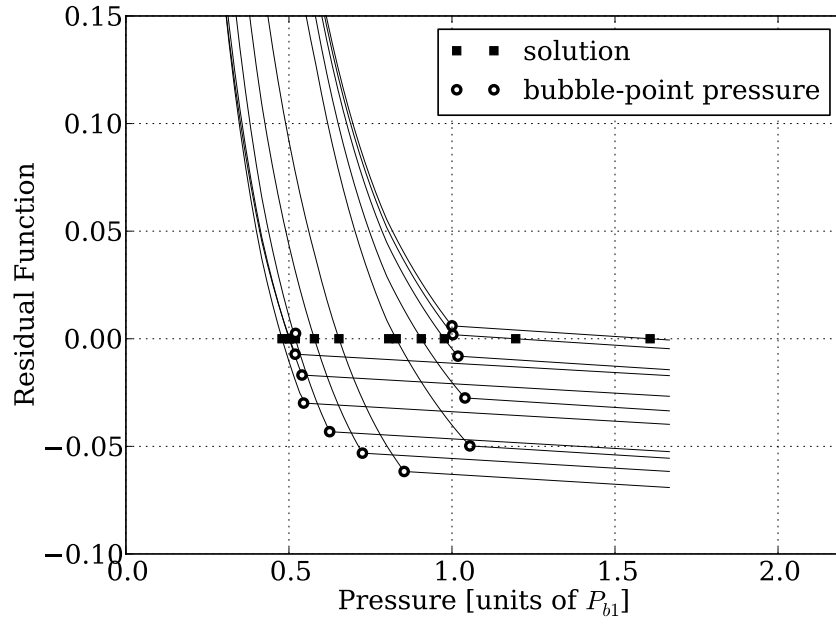


Figure 5: Shape of residual function (ε). Each curve corresponds to different volumes of production and injection. Residual function monotonically decreases with pressure and it has no derivative at the bubble-point.

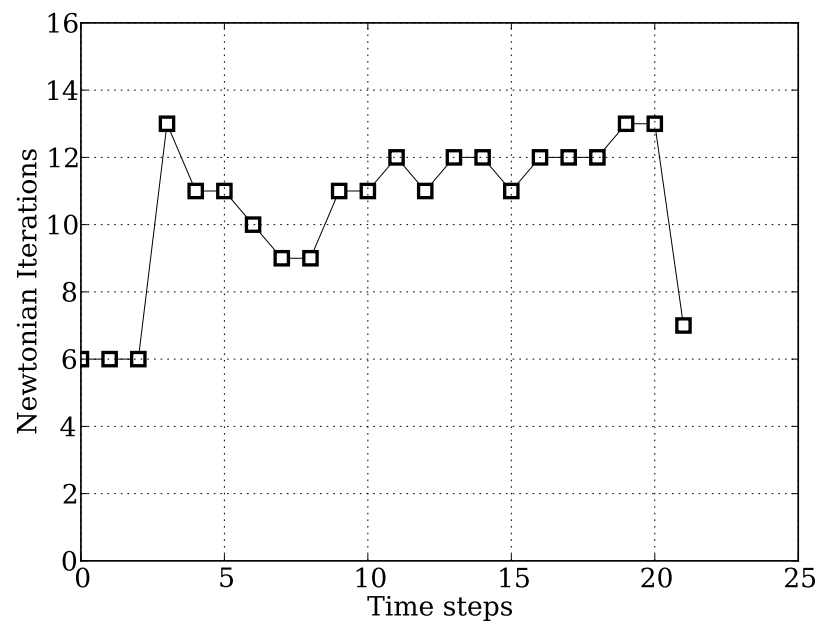


Figure 6: Count of newtonian iterations required to compute pressure with accuracy within 0.001 units of initial bubble-point pressure.

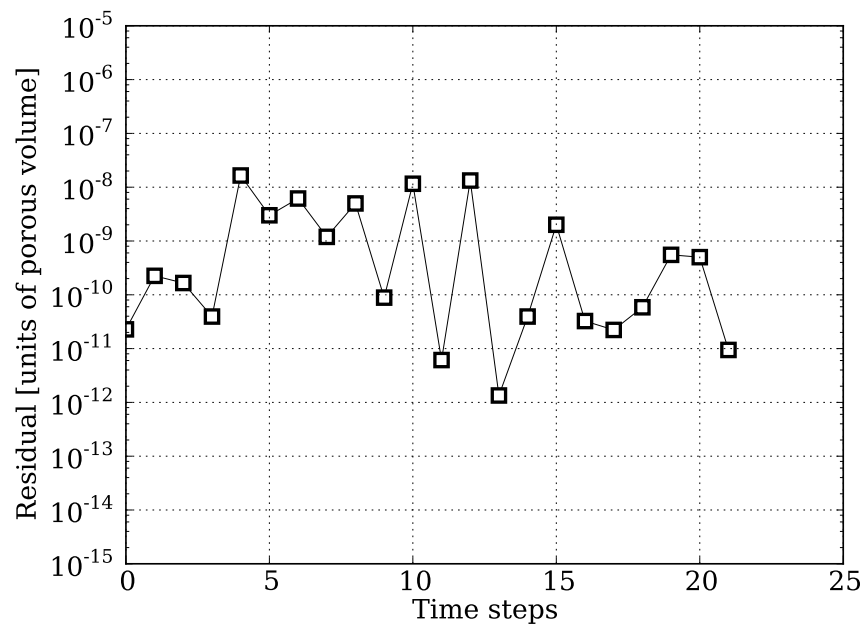


Figure 7: Residual of numerical solution in units of porous volume.

<i>Material balance equation</i>	
pressure	$F_p - F_i = NE_o + mNE_g + WE_w + C$
<i>Phase saturations</i>	
oil	$S_{o2} = (N - N_p) B_{o2}/V_{p2}$
water	$S_{w2} = (W + W_i + W_e - W_p) B_{w2}/V_{p2}$
gas	$S_{g2} = \{N [(R_{s1} - R_{s2}) + mB_{o1}/B_{g1}] + N_p R_{s2} + G_i - G_p\} B_{g2}/V_{p2}$
<i>List of terms</i>	
fluid production	$F_p = N_p [B_{o2} + (G_p/N_p - R_{s2}) B_{g2}] + W_p B_{w2}$
fluid injection and influx	$F_i = (W_i + W_e) B_{w2} + G_i B_{g2}$
fluid expansion	$E = NE_o + mNE_g + WE_w$
oil phase expansion	$E_o = (B_{o2} - B_{o1}) - (R_{so2} - R_{so1}) B_{g2}$
gas phase expansion	$E_g = B_{o1} (B_{g2}/B_{g1} - 1)$
water volume expansion	$E_w = B_{w2} - B_{w1}$
porous volume contraction	$C = V_{p2} - V_{p1}$
bubble-point pressure	$R_{s2} (P_b) = [N (mB_1/B_{g1} + R_{s1}) + G_i - G_p] / (N - N_p)$

Table 1: Complete set of material balance equations.

Appendix A. Compressibility

Appendix A.1. Oil compressibility

Zeroth order approximation of density:

$$p_o - p_o^{ref} = \rho_o^{ref} g(z - z^{ref}) \quad (\text{A.1})$$

First order approximation of density:

$$\frac{1}{\rho_o} \left(\frac{\partial \rho_o}{\partial p_o} \right)_{T, Y_o} = c_o \quad (\text{A.2})$$

Thus,

$$p_o - p_o^{ref} = \frac{1}{c_o} \left\{ \exp \left[\rho_o^{ref} c_o g(z - z^{ref}) \right] - 1 \right\}. \quad (\text{A.3})$$

Appendix A.2. Gas compressibility

Zeroth order approximation of density:

$$p_g - p_g^{ref} = \rho_g^{ref} g(z - z^{ref}). \quad (\text{A.4})$$

Using gas equation of state for constant temperature and Z-factor:

$$p_g - p_g^{ref} = p^{ref} \left\{ \exp \left[\frac{\rho_g^{ref} g(z - z^{ref})}{p^{ref}} \right] - 1 \right\}. \quad (\text{A.5})$$