Chapter 2
Heat Conduction in Cartesian Coordinates

1 Introduction

Here we investigate solutions to special cases of the following form of the heat equation

$$\rho C_p \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T)$$

where the ∇ operator in Cartesian coordinates (x, y, x) is given by

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

with $\mathbf{i}, \mathbf{j}, \mathbf{k}$ being the orthogonal unit vectors associated with the coordinate system. Solutions to the above equation must be obtained subject to properly stated initial and boundary conditions.

Selected steady state problems will also be discussed. Recall that under steady state conditions with constant thermal properties the energy balance equation is

$$\nabla^2 T = 0$$

Finally, at steady state but in the presence of distributed internal heat generation the energy equation is

$$\nabla \cdot (k \nabla T) = -g(\mathbf{r})$$

proper formulation requires the statement of boundary conditions.

2 Fundamental Solutions to Steady State Problems

Solutions to steady state problems in one dimensional systems exhibiting symmetry are easily obtained as solutions of ordinary differential equations by direct integration.

Consider a solid slab whose thickness L is much smaller than its width and its height. In Cartesian coordinates the steady state heat balance equation becomes

$$\frac{d^2T}{dx^2} = 0$$

the general solution of which is

$$T(x) = Ax + B$$

where the constants A and B must be determined from the specific boundary conditions involved. This represents the steady state loss of heat through a flat wall.

As an example of the implementation of boundary conditions consider the case where the heat flux q_1 is specified at $x = x_1$ (Neumann boundary condition) while the temperature T_2 is given at $x = x_2 = x_1 + L$.

Starting with the conditions at $x = x_1$ one has

$$q(x_1) = q_1 = -k\frac{dT}{dx}$$

so that

$$A = -\frac{q_1}{k}$$

The condition at $x = x_2$ yields

$$B = T_2 - Ax_2 = T_2 + \frac{q_1}{k}x_2$$

The desired solution is then

$$T(x) = \frac{q_1}{k}(x_2 - x) + T_2$$

And the unknown temperature at $x = x_1$ is

$$T_1 = \frac{q_1}{k}(x_2 - x_1) + T_2$$

Clearly, since $x_2 - x_1 > 0$, if $q_1 > 0$ (i.e. heat enters the slab at $x = x_1$) then $T_1 > T_2$.

3 Fundamental Solutions to Transient Problems

Transient problems resulting from the effect of instantaneous point, line and planar sources of heat lead to useful fundamental solutions of the heat equation. By considering media of infinite or semi-infinite extent one can conveniently ignore the effect of boundary conditions on the resulting solutions.

Let a fixed amount of energy $Q\rho C_p$ be released at time $t = 0$ at the origin of the three dimensional solid of infinite extent, initially at $T = 0$ everywhere. Assuming constant thermal properties the heat equation is

$$\frac{\partial T}{\partial t} = \alpha(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2})$$

and this must be solved subject to

$$T(x, y, z, 0) = 0$$

for all x, y, z and energy $Q\rho C_p$ released instantaneously at $t = 0$ at the origin.

The fundamental solution of this problem is

$$T(x, y, z, t) = \frac{Q}{(4\pi\alpha t)^{3/2}} e^{-\frac{x^2+y^2+z^2}{4\alpha t}}$$

This solution is useful in the study of thermal explosions where a buried explosive load located at $\mathbf{r} = 0$ is suddenly released at $t = 0$ and the subsequent distribution of temperature at various distances from the explosion is measured as a function of time. A slight modification of the solution produced by the method of reflexion constitutes an approximation to the problem of surface heating of bulk samples by short duration pulses of high energy beams.

Similarly, if the heat is released instantaneously at $t = 0$ but along the $z-$axis, the corresponding fundamental solution is

$$T(x, y, t) = \frac{Q}{4\pi\alpha t} e^{-\frac{x^2+y^2}{4\alpha t}}$$

where $Q\rho C_p$ is now the amount of heat released per unit length.

Finally, if the heat is instantaneously released at $t = 0$ but on the entire the $y - z$ plane at $x = 0$ the corresponding fundamental solution is

$$T(x, t) = \frac{Q}{(4\pi\alpha t)^{1/2}} e^{-\frac{x^2}{4\alpha t}}$$

where $Q\rho C_p$ is now the amount of heat released per unit area.

Another important solution is obtained for the case of a semi-infinite solid ($x \geq 0$) initially at $T = T_0$ everywhere and suddenly exposed to a fixed temperature $T = 0$ at $x = 0$. The statement of the problem is

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

subject to

$$T(x, 0) = T_0$$

and

$$T(0,t) = 0$$

The solution is easily obtained introducing the *Laplace transform*. Multiply the heat equation by $\exp(-st)$, where s is the parameter of the transform, and integrate with respect to t from 0 to ∞, i.e.

$$\frac{1}{\alpha}\int_0^\infty \exp(-st)\frac{\partial T}{\partial t}dt = \int_0^\infty \exp(-st)\frac{\partial^2 T}{\partial x^2}dt$$

introducing the notation $L[T] = T^* = \int_0^\infty \exp(-st)Tdt$ the transformed heat equation becomes

$$\frac{d^2T^*}{dx^2} = \frac{s}{\alpha}T^*$$

an ordinary differential equation which is readily solved for T^*. The desired result $T(x,t)$ is finally obtained from T by inverting the transform and is

$$T(x,t) = T_0 erf(\frac{x}{2\sqrt{\alpha t}})$$

where the *error function*, erf is defined as

$$erf(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp(-\xi^2)d\xi$$

The above solution is an appropriate mathematical approximation of the problem of quenching hot bulk metal samples.

4 The Method of Separation of Variables and Integral Transforms

Analytical solutions to boundary value problems in linear ordinary differential equations are usually obtained by first producing a *general solution*. The general solution is an expression obtained by integration and contains as many independent arbitrary constants as there are derivatives in the original differential equation. Incorporation of boundary conditions define the values of the integration constants.

In contrast, general solutions of linear partial differential equations involve arbitrary functions of specific functions. Incorporation of boundary conditions involves the determination of functional relationships and is rarely feasible or practical. An alternative approach to finding solutions is based instead on first determining a set of *particular* solutions directly and then combining these so as to satisfy the prescribed boundary conditions. A specific implementation of the above idea is known as the *method of separation of variables*. This section illustrates the application of the separation of variables method for the determination of analytical solutions of steady and transient one dimensional linear heat conduction problems.

In essence, the method is based on the assumption that if one is looking for a solution to a transient, one-dimensional heat conduction problem of the form $T(x,t)$ it is possible to express it as a product

$$T(x,t) = X(x)\Gamma(t)$$

where the functions $X(x)$ and $\Gamma(t)$ are each functions of a single independent variable satisfying specific ordinary differential equations. One proceeds by first solving the associated ODEs which are then combined in the product form given above.

The equation for $\Gamma(t)$ is always of first order and is readily solved by elementary methods.

The equation for $X(x)$ is always of second order and together with the boundary conditions leads to an eigenvalue problem (Sturm-Liouville system). The required solution for $X(x)$ can be formally and very generally expressed by the inversion formula

$$X(x) = \sum_{m=1}^{\infty} K(\beta_m, x)\bar{X}(\beta_m)$$

where the kernel functions $K(\beta_m, x)$ are the normalized eigenfunctions of the associated Sturm-Liouville system and the sum runs over all the eigenvalues of the system. Moreover, the integral transform $\bar{X}(\beta_m)$ is given by the formula

$$\bar{X}(\beta_m) = \int_{x'=0}^{L} K(\beta_m, x')X(x')dx'$$

Therefore, once the boundary conditions are translated from $T(x,t)$ to $X(x)$, the resulting Sturm-Liouville system is solved yielding the appropriate eigenfunctions and eigenvalues. From these one then determines the kernel $K(\beta_m, x)$ and the integral transform $\bar{X}(\beta_m)$. The inversion formula is next used to obtain the function $X(x)$. Finally, the desired solution $T(x,t)$ is obtained substituting in the assumed product form.

The transform method is also applicable to problems in semi-infinite domains. In this case the inversion formula is given instead by

$$X(x) = \int_{\beta=0}^{\infty} K(\beta_m, x)\bar{X}(\beta)d\beta$$

while the integral transform is

$$\bar{X}(\beta_m) = \int_{x'=0}^{\infty} K(\beta_m, x')X(x')dx'$$

Note that the sum over the discrete set of eigenvalues in the inversion formula has been replaced by an integral over the continuous spectrum of eigenvalues obtained in the semi-infinite case.

4.1 Solving Steady State Problems by Separation of Variables

At steady state $\partial T/\partial t = 0$ so that, in two dimensional systems, the temperature satisfies Laplace's equation

$$\nabla^2 T = 0$$

From Green's theorem, Laplace's equation requires that

$$\oint\oint \frac{\partial T}{\partial n} d\sigma = 0$$

which states that under steady state conditions the boundary heat flux cannot be chosen arbitrarily but must average zero.

Also, from Green's theorem, if T_1 and T_2 are two solutions of a steady state problem whose values coincide at the boundary

$$\int\int\int_V [\nabla(T_2 - T_1]^2 d\tau = 0$$

so that $T_2 - T_1$ = constant. The constant is zero when the problem involves only prescribed temperatures at the boundary (Dirichlet problem) and can be nonzero when normal derivatives of T at the boundary are specified (Neumann problem).

Consider steady state heat conduction in a thin rectangular plate of width l and height d. The edges $x = 0$, $x = l$ and $y = 0$ are maintained at $T = 0$ while at the edge $y = d$ $T(x, d) = f(x)$. No heat flow along the z direction perpendicular to the plate. The required temperature $T(x, y)$ satisfies

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

To find T by separation of variables we assume the a particular solution can be represented as a product of two functions each depending on a single coordinate, i.e.

$$T_p(x, y) = X(x)Y(y)$$

substituting into Laplace's equation gives

$$-\frac{1}{X}\frac{d^2 X}{dx^2} = \frac{1}{Y}\frac{d^2 Y}{dy^2} = k^2$$

where k^2 is a constant and this is true since the LHS is a function of x alone while the RHS a function of y alone. The constant is selected as k^2 in order to obtain a proper Sturm-Liouville problem for X (with real eigenvalues). With the above the original PDE problem has been transformed into a system of two ODE's, i.e.

$$X'' + k^2 X = 0$$

subject to $X(0) = X(l) = 0$ and

$$Y'' - k^2 Y = 0$$

subject to $Y(0) = 0$.

The solution for $X(x)$ is

$$X = X_n = A_n \sin(\frac{n\pi x}{l})$$

with eigenvalues

$$k_n = \frac{n\pi}{l}$$

for $n = 1, 2, 3, ...$

The solution of $Y(y)$ is

$$Y_n = B_n \sinh(\frac{n\pi y}{l})$$

The *principle of superposition* allows the creation of a more general solution from individual particular solutions by simple linear combination. Therefore the final form of the particular solution is

$$T_n = \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi x}{l}) \sinh(\frac{n\pi y}{l})$$

The a_n's are determined by making the above satisfy the non homogeneous condition at $y = d$, i.e.

$$f(x) = \sum_{n=1}^{\infty} [a_n \sinh(\frac{n\pi d}{l})] \sin(\frac{n\pi x}{l})$$

which is the *Fourier sine series representation* of $f(x)$ with coefficients

$$c_n = a_n \sinh(\frac{n\pi d}{l}) = \frac{2}{l} \int_0^l f(x) \sin(\frac{n\pi x}{l}) dx$$

so that the final solution is

$$T(x, y) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{l}) \frac{\sinh(\frac{n\pi y}{l})}{\sinh(\frac{n\pi d}{l})}$$

Therefore, as long as $f(x)$ is representable in terms of Fourier series, the obtained solution converges to the desired solution. Note also that the presence of homogeneous conditions at $x = 0$, $x = l$ made feasible the determination of the required eigenvalues.

4.2 Solving Transient Problems by Separation of Variables

4.2.1 Quenching of Slab with Fixed Temperature at its Boundary Surfaces

A simple but important conduction heat transfer problem consists of determining the temperature history inside a solid body which is quenched from a high temperature. More specifically, consider the homogeneous problem of finding the one-dimensional temperature distribution inside a slab of thickness L and thermal diffusivity α undergoing transient heat conduction. The initial temperature distribution of the slab is $T(x,0) = f(x)$. The slab is quenched by forcing the temperature at its two surfaces $x = 0$ and $x = L$ to become equal to zero (i.e. $(T(0,t) = T(L,t) = 0$; Dirichlet homogeneous conditions) for $t > 0$. For simplicity the thermal properties are assumed constant.

The mathematical statement of the heat equation for this problem is:

$$\frac{\partial T(x,t)}{\partial t} = \alpha \frac{\partial^2 T(x,t)}{\partial x^2}$$

subject to

$$T(0,t) = T(L,t) = 0$$

and

$$T(x,0) = f(x)$$

for all x when $t = 0$.

The method of separation of variables starts by assuming the solution to this problem has the following particular form

$$T(x,t) = X(x)\Gamma(t)$$

If the assumption is wrong, one discovers soon enough, but if it is correct then we may just find a solution to the problem! The latter turns out to be the case for this and many other similar problems.

Introducing the above assumption into the heat equation and rearranging yields

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{\alpha\Gamma}\frac{d\Gamma}{dt}$$

However since $X(x)$ and $\Gamma(t)$, the left hand side of this equation is *only* a function of x while the right hand side is a function only of t. For this to avoid being a contradiction (for arbitrary values of x and t) both sides must be equal to a constant. For physical reasons the required constant must be negative; let us call it $-\omega^2$.

Exercise: Show why the required constant must be negative.

Therefore, the original PDE is transformed into the following two ODE's

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\omega^2$$

and

$$\frac{1}{\alpha\Gamma}\frac{d\Gamma}{dt} = -\omega^2$$

General solutions to these equations are readily obtained by direct integration and are

$$X(x) = A' \cos(\omega x) + B' \sin(\omega x)$$

and

$$\Gamma(t) = C \exp(-\omega^2 \alpha t)$$

and substituting back into our original assumption yields

$$T(x,t) = X(x)\Gamma(t) = [A\cos(\omega x) + B\sin(\omega x)]\exp(-\omega^2\alpha t)$$

where the constant C has been combined with A' and B' to give A and B without loosing any generality.

Now we introduce the boundary conditions. Since $T(0,t) = 0$, necessarily $A = 0$. Furthermore, since also $T(L,t) = 0$, then $\sin(\omega L) = 0$ (since $B = 0$ is an uninteresting trivial solution.) There is an infinite number of values of ω which satisfy this conditions, i.e.

$$\omega_n = \frac{n\pi}{L}$$

with $n = 1, 2, 3, ...$ The ω_n's are the **eigenvalues** and the associated functions $\sin(\omega_n x)$ are the **eigenfunctions** of this quenching problem. These eigenvalues and eigenfunctions play in heat conduction a role analogous to that of the **deflection modes** in structural dynamics, the **vibration modes** in vibration theory and the **quantum states** in wave mechanics.

Note that each value of ω yields an independent solution satisfying the heat equation as well as the two boundary conditions. Therefore we have now an infinite number of independent solutions $T_n(x,t)$ for $n = 1, 2, 3, ...$ given by

$$T_n(x,t) = [B_n \sin(\omega_n x)]\exp(-\omega_n^2 \alpha t)$$

The **principle of superposition** allows the creation of a more general solution from the particular solutions above by simple linear combination to give

$$T(x,t) = \sum_{n=1}^{\infty} T_n(x,t) = \sum_{n=1}^{\infty}[B_n \sin(\omega_n x)]\exp(-\omega_n^2\alpha t) = \sum_{n=1}^{\infty}[B_n \sin(\frac{n\pi x}{L})]\exp(-(\frac{n\pi}{L})^2\alpha t)$$

The last step is to ensure the values of the constants B_n are chosen so as to satisfy the initial condition, i.e.

$$T(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi x}{L})$$

Note that this is the **Fourier sine series representation** of the function $f(x)$.

A key property of the eigenfunctions is the **orthonormality property** expressed in the case of the $\sin(\omega_n x)$ functions as

$$\int_0^L \sin(\frac{n\pi x}{L})\sin(\frac{m\pi x}{L})dx = \begin{cases} 0 & , n \neq m \\ L/2 & , n = m \end{cases}$$

Using the orthonormality property one can multiply the Fourier sine series representation of $f(x)$ by $\sin(\frac{m\pi x}{L})$ and integrate from $x = 0$ to $x = L$ to produce the result

$$B_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})dx$$

for $n = 1, 2, 3, ...$

Exercise: Prove the above result.

Explicit expressions for the B_n's can be obtained for simple $f(x)$'s, for instance if

$$f(x) = \begin{cases} x & , 0 \leq x \leq \frac{L}{2} \\ L - x & , \frac{L}{2} \leq x \leq L \end{cases}$$

then

$$B_n = \begin{cases} \frac{4L}{n^2\pi^2} & , n = 1, 5, 9, ... \\ -\frac{4L}{n^2\pi^2} & , n = 3, 7, 11, ... \\ 0 & , n = 2, 4, 6, ... \end{cases}$$

Finally, the resulting B_n's can be substituted into the general solution above to give

$$T(x,t) = \sum_{n=1}^{\infty} T_n(x,t) = \sum_{n=1}^{\infty}[\frac{2}{L}\int_0^L f(x')\sin(\frac{n\pi x'}{L})dx']\sin(\frac{n\pi x}{L})\exp(-\frac{n^2\pi^2\alpha t}{L^2})$$

and for the specific $f(x)$ given above

$$T(x,t) = \frac{4L}{\pi^2}[\exp(-\frac{\pi^2\alpha t}{L^2})\sin(\frac{\pi x}{L}) - \frac{1}{9}\exp(-\frac{9\pi^2\alpha t}{L^2})\sin(\frac{3\pi x}{L}) + ...]$$

Another important special case is when the initial temperature $f(x) = T_i =$ constant. The result in this case is

$$T(x,t) = \frac{4T_i}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)}\sin(\frac{(2n+1)\pi x}{L})\exp(-\frac{n^2\pi^2\alpha t}{L^2})$$

Exercise: Derive the above equation.

4.2.2 Quenching of a Slab with Convective Heat Losses at its Boundaries

The separation of variables method can also be used when the boundary conditions specify values of the normal derivative of the temperature (Newmann conditions) or when linear combinations of the normal derivative and the temperature itself are used (Convective conditions; Mixed conditions; Robin conditions). Consider the homogeneous problem of transient heat conduction in a slab initially at a temperature $T = f(x)$ and subject to convection losses into a medium at $T = 0$ at $x = 0$ and $x = L$. Convection heat transfer coefficients at $x = 0$ and $x = L$ are, respectively h_1 and h_2. Assume the thermal conductivity of the slab k is constant.

The mathematical formulation of the problem is to find $T(x,t)$ such that

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t}$$

$$-k\frac{\partial T}{\partial x} + h_1 T = 0$$

at $x = 0$ and

$$k\frac{\partial T}{\partial x} + h_2 T = 0$$

at $x = L$, with

$$T(x,0) = f(x)$$

for all x when $t = 0$.

Assume the solution is of the form $T(x,t) = X(x)\Gamma(t)$ and substitute to obtain

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{\alpha\Gamma}\frac{d\Gamma}{dt} = -\beta^2$$

The solution for $\Gamma(t)$ is

$$\Gamma(t) = \exp(-\alpha\beta^2 t)$$

while $X(x)$ is the solution of the following eigenvalue (**Sturm-Liouville**) problem

$$\frac{d^2X}{dx^2} + \beta^2 X = 0$$

with

$$-k\frac{dX}{dx} + h_1 X = 0$$

at $x = 0$ and

$$k\frac{dX}{dx} + h_2 X = 0$$

at $x = L$.

Let the eigenvalues of this problem be β_m and the eigenfunctions $X(\beta_m, x)$. Since the eigenfunctions are orthogonal

$$\int_0^L X(\beta_m, x)X(\beta_n, x)dx = \begin{cases} 0 & , n \neq m \\ N(\beta_m) & , n = m \end{cases}$$

where

$$N(\beta_m) = \int_0^L X(\beta_m, x)^2 dx$$

is the norm of the problem.

It can be shown that for the above problem the eigenfunctions are

$$X(\beta_m, x) = \beta_m \cos \beta_m x + \frac{h_1}{k_1} \sin \beta_m x$$

while the eigenvalues are the roots of the transcendental equation

$$\tan \beta_m L = \frac{\beta_m(\frac{h_1}{k_1} + \frac{h_2}{k_2})}{\beta_m^2 - \frac{h_1}{k_1}\frac{h_2}{k_2}}$$

Therefore, the complete solution is of the form

$$T(x, t) = \sum_{m=1}^{\infty} c_m X(\beta_m, x) \exp(-\alpha \beta_m^2 t)$$

The specific eigenfunctions are obtained by incorporating the initial condition

$$f(x) = \sum_{m=1}^{\infty} c_m X(\beta_m, x)$$

which expresses the representation of $f(x)$ in terms of eigenfunctions and requires that

$$c_m = \frac{1}{N(\beta_m)} \int_0^L X(\beta_m, x) f(x) dx$$

Many problems are special cases of the above generalization. For example, the quenching problem under homogeneous Dirichlet conditions discussed before is a special case of the above in which the eigenfunctions are

$$X(\beta_m, x) = \sin(\beta_m x)$$

the eigenvalues are the roots of $\sin(\beta_m L)$, i.e.

$$\beta_m = \frac{m\pi}{L}$$

and the norm is simply

$$N(\beta_m) = \int_0^L X(\beta_m, x)^2 dx = \int_0^L \sin(\beta_m x)^2 = \frac{L}{2}$$

As another example, the case when $h_1 = 0$ and $h_2 = h$ yields the eigenfunctions

$$X(\beta_m, x) = \cos(\beta_m x)$$

and the eigenvalues are the roots of $\beta_m \tan(\beta_m L) = h/k$. Note that the eigenvalues in this case cannot be given explicitly but must be determined by numerical solution of the given transcendental equation. For this purpose, it is common to rewrite the transcendental equation as

$$\cot(\beta_m L) = \frac{\beta_m L}{Bi}$$

where $Bi = hL/k$. This can be easily solved either graphically or numerically by bisection, Newton's or secant methods. Finally, the norm in this case is

$$N(\beta_m) = \int_0^L X(\beta_m, x)^2 dx = \int_0^L \cos(\beta_m x)^2 dx =$$

$$= \frac{1}{2} \frac{\cos(\beta_m L)\sin(\beta_m L) + \beta_m L}{\beta_m} = \frac{L(\beta_m^2 + (h/k)^2) + (h/k)}{2(\beta_m^2 + (h/k)^2)}$$

Exercise: Derive the above expression for $N(\beta_m)$.

5 Non-homogeneous Problems

Consider the problem of finding the temperature field $T(x,t)$ resulting from transient heat conduction in a slab (1D) initially at $T(x,0) = f(x)$ whose surfaces at $x = 0$ and $x = l$ are maintained respectively at constant non-zero temperatures T_0 and T_l for $t > 0$. The mathematical formulation is

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

subject to

$$T(0,t) = T_0$$

$$T(l,t) = T_l$$

and

$$T(x,0) = f(x)$$

This problem can be transformed into two equivalent but simpler problems. A steady state non homogeneous problem and a transient homogeneous problem.

Introduce new functions $v(x)$ and $w(x,t)$ such that

$$T(x,t) = v(x) + w(x,t)$$

Substituting into the original PDE one gets

$$\frac{d^2 v}{dx^2} = 0$$

and

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{\alpha}\frac{\partial w}{\partial t}$$

subject to

$$w(0,t) = T_0 - v(0)$$

$$w(l,t) = T_l - v(l)$$

and

$$w(x,0) = f(x) - v(x)$$

Now we select $v(x)$ such that the boundary conditions for w become homogeneous, i.e.

$$v(x) = T_0 + \frac{x}{l}[T_l - T_0]$$

With this, the solution for w with $F(x) = f(x) - v(x)$ is readily obtained by separation of variables. Finally

$$T(x,t) = T_0 + (\frac{x}{l})(T_1 - T_0) + \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha t/l^2} \ \sin(\frac{n\pi x}{l})$$

with

$$c_n = \frac{2}{l}\int_0^l [f(x) - v(x)]\sin(\frac{n\pi x}{l})dx$$
$$= \frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi x}{l})dx + \frac{2}{n\pi}[(-1)^n T_1 - T_0]$$

Exercise: Derive the above results.

6 Separation of Variables for the Multidimensional Heat Equation

The corresponding problem in 3D is

$$\nabla^2 T(\mathbf{r}, t) = \frac{1}{\alpha} \frac{\partial T(\mathbf{r}, t)}{\partial t}$$

subject to

$$k_i \frac{\partial T}{\partial n_i} + h_i T = 0$$

on boundary S_i and

$$T(\mathbf{r}, 0) = f(\mathbf{r})$$

initially.

Separation of variables in this case requires a solution of the form

$$T(\mathbf{r}, t) = \psi(\mathbf{r})\Gamma(t)$$

which leads to

$$\frac{1}{\psi(\mathbf{r})} \nabla^2 \psi(\mathbf{r}) = \frac{1}{\alpha \Gamma(t)} \frac{d\Gamma}{dt} = -\lambda^2$$

The solution for Γ is the same as before while ψ must be found to satisfy

$$\nabla^2 \psi(\mathbf{r}) + \lambda^2 \psi(\mathbf{r}) = 0$$

subject to

$$k_i \frac{\partial \psi}{\partial n_i} + h_i \psi = 0.$$

This is called the Helmholtz equation and can in turn be solved by separating variables as well.

6.1 Separation of Variables for the Helmholtz Equation

Separation of variables of the Helmholtz equation requires assuming the solution has the form

$$\psi(x, y, z) = X(x)Y(y)Z(z)$$

substituting this produces

$$X'' + \beta^2 X = 0$$

$$Y'' + \gamma^2 Y = 0$$

$$Z'' + \eta^2 Z = 0$$

where $\beta^2 + \gamma^2 + \eta^2 = \lambda^2$. Further, the solution for the time dependent part is

$$\Gamma = \exp(-\alpha\lambda^2 t)$$

Finally, the complete solution is obtained by superposition of the individual solutions.

6.2 Multidimensional Homogeneous Problems

Consider the problem of transient 2D conduction in a plate of dimensions $a \times b$, initially at $T(x, y, 0) = f(x, y)$ and subject to an insulating condition at the boundary $x = 0$, a zero temperature condition at the boundary $y = 0$ and to convective heat exchange with a surrounding environment at zero temperature at boundaries $x = a$ and $y = b$ with heat transfer coefficients h_2 and h_4, respectively.

Separation of variables starts by assuming a solution of the form

$$T(x, y, t) = \Gamma(t)X(x)Y(y)$$

substituting one gets

$$X'' + \beta^2 X = 0$$

subject to

$$X' = 0$$

at $x = 0$ and

$$X' + \frac{h_2}{k} X = 0$$

at $x = a$, and

$$Y'' + \gamma^2 Y = 0$$

subject to

$$Y = 0$$

at $y = 0$ and

$$Y' + \frac{h_4}{k}Y = 0$$

at $y = b$. The complete solution is then

$$T(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn} e^{-\alpha(\beta_m^2+\gamma_n^2)t} \cos(\beta_m x) \sin(\gamma_n y)$$

where $\cos(\beta_m x)$ and $\sin(\gamma_n x)$ are the eigenfunctions of this problem and the Fourier coefficients c_{mn} must be obtained by substituting the initial condition $T(x, y, 0) = f(x, y)$ and taking advantage of the orthogonality of the eigenfunctions to yield

$$c_{mn} = \frac{1}{N(\beta_m)N(\gamma_n)} \int_0^a \int_0^b \cos(\beta_m x) \sin(\gamma_n x) f(x, y) dx dy$$

where the norms $N(\beta_m)$ and $N(\gamma_n)$ are

$$N(\beta_m) = \int_0^a \cos^2(\beta_m x) dx = \frac{1}{2} \frac{a(\beta_m^2 + (h_2/k)^2) + h_2/k}{\beta_m^2 + (h_2/k)^2}$$

and

$$N(\gamma_n) = \int_0^b \sin^2(\gamma_n y) dy = \frac{1}{2} \frac{b(\gamma_n^2 + (h_4/k)^2) + h_4/k}{\gamma_n^2 + (h_4/k)^2}$$

While the eigenvalues β_m are obtained as the roots of

$$\beta_m \tan \beta_m a = \frac{h_2}{k}$$

and the eigenvalues γ_n are the roots of

$$\gamma_n \cot \gamma_n b = -\frac{h_4}{k}$$

6.3 Quenching of a Billet

Consider a long billet of rectangular cross section $2a \times 2b$ initially at a uniform temperature T_i which is quenched by exposing it to convective exchange with an environment at T_∞ by means of a heat transfer coefficient h. Since the billet is long it is appropriate to neglect conduction along its axis. Because of symmetry it is enough to analyze one quarter of the cross section using a rectangular Cartesian system of coordinates with the origin at the center of the billet.

Define first the **shifted temperature** $\theta(x,y,t) = T(x,y,t) - T_\infty$. In terms of the shifted temperature the mathematical formulation of the problem is as follows.

$$\frac{\partial \theta}{\partial t} = \alpha(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2})$$

subject to

$$\theta(x,y,0) = T_i - T_\infty = \theta_i$$

$$\frac{\partial \theta}{\partial x} = 0$$

at $x = 0$,

$$\frac{\partial \theta}{\partial y} = 0$$

at $y = 0$,

$$\frac{\partial \theta}{\partial x} = -\frac{h}{k}\theta(a,y,t)$$

at $x = a$, and

$$\frac{\partial \theta}{\partial y} = -\frac{h}{k}\theta(x,b,t)$$

at $y = b$.

The solution to this problem obtained by separation of variables is

$$\frac{\theta(x,y,t)}{\theta_i} = 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} e^{-\alpha(\lambda_n^2+\beta_m^2)t} \times \frac{\sin(\lambda_n a)\cos(\lambda_n x)\sin(\beta_m b)\cos(\beta_m y)}{[\lambda_n a + \sin(\lambda_n a)\cos(\lambda_n a)][\beta_m b + \sin(\beta_m b)\cos(\beta_m b)]}$$

where the eigenvalues λ_n and β_m are the roots of the following transcendental equations

$$\lambda_n \tan(\lambda_n a) = \frac{h}{k}$$

and

$$\beta_m \tan(\beta_m a) = \frac{h}{k}$$

6.4 Product Solutions

The solution above can be rewritten as

$$\frac{\theta(x,y,t)}{\theta_i} = 2\sum_{n=1}^{\infty} \frac{\sin(\lambda_n a)\cos(\lambda_n x)}{\lambda_n a + \sin(\lambda_n a)\cos(\lambda_n a)} e^{-\alpha\lambda_n^2 t}$$
$$\times 2\sum_{m=1}^{\infty} \frac{\sin(\beta_m b)\cos(\beta_m y)}{\beta_m b + \sin(\beta_m b)\cos(\beta_m b)} e^{-\alpha\beta_m^2 t}$$

Note that the first sum is precisely the solution to the 1D problem of quenching of a slab of thickness $2a$ while the second sum is the corresponding solution for a slab of thickness $2b$. Therefore

$$\frac{\theta(x,y,t)}{\theta_i}|_{billet} = \frac{\theta(x,t)}{\theta_i}|_{2a-slab} \times \frac{\theta(y,t)}{\theta_i}|_{2b-slab}$$

The method of obtaining solutions to multidimensional heat conduction problems by multiplication of the solutions of subsidiary problems of lower dimensionality is called the **method of product solutions**.

In general, if the initial temperature of the medium can be expressed as a product of single space variable functions one can produce solutions to multidimensional problems as products of single dimensional solutions.

Consider transient homogeneous 2D conduction in a plate with convective exchange on all four sides. Initially, $T(x,y,0) = F(x)F(y)$. This problem can be regarded as a composite of two one-dimensional ones with solutions $T_1(x,t)$ and $T_2(y,t)$. The product solution for the 2D problem is then

$$T(x,y,t) = T_1(x,t)T_2(y,t)$$

6.5 Non-homogeneous Problems

Consider the non homogeneous problem of transient multidimensional heat conduction with internal heat generation

$$\nabla^2 T(\mathbf{r},t) + \frac{1}{k}g(\mathbf{r}) = \frac{1}{\alpha}\frac{\partial T(\mathbf{r},t)}{\partial t}$$

subject to non homogeneous time independent conditions on at least part of the boundary.

This problem can be decomposed into a set of steady state non homogeneous problems in each of which a single non homogenous boundary condition occurs and a transient homogeneous problem. If the solutions of the steady state problems are $T_{0j}(\mathbf{r})$ and that for the transient problem is $T_h(\mathbf{r},t)$, the solution of the original problem is

$$T(\mathbf{r},t) = T_h(\mathbf{r},t) + \sum_{j=0}^{N} T_{0j}(\mathbf{r})$$

7 Transient Temperature Nomographs: Heisler Charts

The solutions obtained for 1D non homogeneous problems with Neumann boundary conditions in Cartesian coordinate systems using the method of separation of variables have been collected and assembled in the form of transient temperature nomographs by Heisler. The given charts are a very useful baseline against which to validate one's own analytical or numerical computations. The Heisler charts summarize the solutions to the following three important problems.

The first problem is the 1D transient homogeneous heat conduction in a plate of span L from an initial temperature T_i and with one boundary insulated and the other subjected to a convective heat flux condition into a surrounding environment at T_∞. (This problem is equivalent to the quenching of a slab of span $2L$ with identical heat convection at the external boundaries $x = -L$ and $x = L$).

Introduction of the following non dimensional parameters simplifies the mathematical formulation of the problem. First is the dimensionless distance

$$X = \frac{x}{L}$$

next, the dimensionless time

$$\tau = \frac{\alpha t}{L^2}$$

then the dimensionless temperature

$$\theta(X, \tau) = \frac{T(x,t) - T_\infty}{T_i - T_\infty}$$

and finally, the **Biot number**

$$Bi = \frac{hL}{k}$$

With the new variables, the mathematical formulation of the heat conduction problem becomes

$$\frac{\partial^2 \theta}{\partial X^2} = \frac{\partial \theta}{\partial \tau}$$

subject to

$$\frac{\partial \theta}{\partial X} = 0$$

at $X = 0$,

$$\frac{\partial \theta}{\partial X} + Bi\theta = 0$$

at $X = 1$, and

$$\theta = 1$$

in $0 \leq X \leq 1$ for $\tau = 0$.

8 Numerical Solution Methods: Finite Differences

Mathematical models of heat conduction problems must often be solved numerically. Three main approximation techniques are available: finite differences (FD), finite elements (FE) and finite volume (FV) methods. These methods are based on the idea of first discretizing the heat equation and then solving the resulting (algebraic) problem. Discretization is accomplished by regarding the medium as constituted by a collection of cells or volumes of finite size. Nodes are usually associated with each cell thus producing a **mesh** of points. The separation between any two nodes is the **mesh spacing**. Temperature at each cell is then represented by the temperature at the corresponding nodal location. A computer and a computer program are then used to solve the resulting algebraic problem. This section contains a survey of finite difference methods for the solution of one- and multidimensional transient heat conduction problems.

9 Finite Differences

The basic idea behind the **finite difference method** is to replace the various derivatives appearing in the mathematical formulation of the problem by suitable approximations on a finite difference mesh.

The simplest derivation of finite difference formulae makes use of Taylor series. The Taylor series expansions of a function $f(x)$ about a point x are:

$$f(x+\delta x) = f(x) + \delta x f'(x) + \frac{\delta x^2}{2!} f''(x) + \frac{\delta x^3}{3!} f'''(x) + \ldots$$

and

$$f(x-\delta x) = f(x) - \delta x f'(x) + \frac{\delta x^2}{2!} f''(x) - \frac{\delta x^3}{3!} f'''(x) + \ldots$$

where δx is the *mesh spacing.*

Solving the first equation above for $f'(x)$ gives

$$f'(x) = \frac{f(x+\delta x) - f(x)}{\delta x} - \frac{\delta x}{2} f''(x) - \frac{\delta x^2}{6} f'''(x) + \ldots$$

and solving the second one

$$f'(x) = \frac{f(x) - f(x-\delta x)}{\delta x} + \frac{\delta x}{2} f''(x) - \frac{\delta x^2}{6} f'''(x) + \ldots$$

Finally, from the first two equations

$$f'(x) = \frac{f(x+\delta x) - f(x-\delta x)}{2\delta x} - \frac{\delta x^2}{6} f'''(x) + \ldots$$

These are called respectively the forward, backward and central approximations to the derivative of $f(x)$. Note that the second term on the right hand side in the first two equations above is proportional to δx while the same second term in the third equation is proportional to δx^2. Therefore, the first two equations are regarded as leading to **first-order accurate** approximations to the derivative while the last formula leads to a **second-order accurate** approximation.

Note that the neglect of higher order terms in the above formulae for $f'(x)$ produces various approximation schemes for the derivative.

Second order derivative approximations can be similarly obtained. For example, expanding $f(x \pm \delta x)$ about x

$$f(x+2\delta x) = f(x) + 2\delta x f'(x) + 2\delta x^2 f''(x) + \frac{4}{3}\delta x^3 f'''(x) + ...$$

and

$$f(x-2\delta x) = f(x) - 2\delta x f'(x) + 2\delta x^2 f''(x) - \frac{4}{3}\delta x^3 f'''(x) + ...$$

Eliminating $f'(x)$ gives

$$f''(x) = \frac{f(x+\delta x) - 2f(x) + f(x-\delta x)}{\delta x^2} - \frac{1}{12}\delta x^2 f''''(x) + ...$$

Neglecting the higher order terms produces the central difference approximation to $f''(x)$. Note that this leads to a **second-order accurate** approximation of the second derivative.

Errors are always involved in performing any numerical computation. Round-off errors appear whenever computing takes place using a finite number of digits. This is the case when using modern computing machines. Truncation error is the error that exists even in the absence of round-off error and is the result of neglecting higher order terms in the finite difference approximations obtained from Taylor series expansions. Successful numerical work in conduction heat transfer requires attention to issues of accuracy and error control.

10 Solution of Conduction Problems by the Finite Difference Method

Consider the problem of transient 1D heat conduction inside a slab of span L with constant properties. The governing equation is

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

This equation must be solved subject to specific boundary and initial conditions in order to determine $T(x,t)$.

The formulae given above can now be used to produce finite difference equations approximating the heat conduction problem. Consider first a **mesh in space** formed by points separated by constant spacing Δx. The mesh points in space are $x_1, x_2, ..., x_{i-1}, x_i, x_{i+1}, ..., x_N$. Note that nodes x_1 and x_N are **boundary nodes** while all other nodes are **interior nodes**.

Consider also a **mesh in time** formed by instants of time separated by a constant amount of time Δt. The mesh points in time are $t_1, t_2, ..., t_j, t_{j+1}, ..., t_M$. Note that time level t_1 represents the initial condition of the system.

With the above notation, the temperature at a particular **node** (x_i, t_j) is given by $T(x_i, t_j) = T_{i,j}$.

10.1 The Explicit Scheme

We can now use the finite difference formulae given above to write approximations to the derivatives in the heat conduction equations. For instance, using a forward difference in time to represent the time rate of change of temperature at nodal location (x_i, t_j) yields

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} \approx \frac{1}{\alpha}\frac{T(x_i, t_j + \Delta t) - T(x_i, t_j)}{\Delta t} = \frac{1}{\alpha}\frac{T_{i,j+1} - T_{i,j}}{\Delta t}$$

A central difference approximation to the right hand side term around the same nodal location yields

$$\frac{\partial^2 T}{\partial x^2} \approx \frac{T(x_i + \Delta x, t_j) - 2T(x_i, t_j) + T(x_i - \Delta x, t_j)}{\Delta x^2} = \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2}$$

Substituting and rearranging gives an **explicit** formula for the calculation of $T_{i,j+1}$ for all interior nodes i, i.e.

$$T_{i,j+1} = T_{i,j} + \frac{\alpha \Delta t}{\Delta x^2}(T_{i+1,j} - 2T_{i,j} + T_{i-1,j})$$

Exercise: Derive the above result.

If the problem to be solved involves Dirichlet boundary conditions, for instance

$$T(0, t) = T(x_1, t_j) = T_{1,j} = T_1$$

and

$$T(L, t) = T(x_N, t_j) = T_{N,j} = T_N$$

The above equations completely define the temperatures at all the nodal locations. Since $T_{1,j}$ and $T_{N,j}$ are given one computes only $T_{i,j}$ for $i = 2, 3, ..., N-1$ and for all time levels $j + 1$.

An important limitation of the explicit scheme is that it is **conditionally stable**. It can be shown that the calculation of $T_{i,j}$ with the above method produces **stable** and physically meaningful results only as long as the **Courant-Friedrichs-Lewy (CFL)** condition is fulfilled, i.e. as long as

$$\frac{\alpha \Delta t}{\Delta x^2} \leq \frac{1}{2}$$

10.2 The Implicit Scheme

An alternative scheme is obtained by using instead a central difference approximation to the right hand side term around the nodal location (x_i, t_{j+1}). This gives

$$\frac{\partial^2 T}{\partial x^2} \approx \frac{T(x_i + \Delta x, t_{j+1}) - 2T(x_i, t_{j+1}) + T(x_i - \Delta x, t_{j+1})}{\Delta x^2} =$$

$$= \frac{T_{i+1,j+1} - 2T_{i,j+1} + T_{i-1,j+1}}{\Delta x^2}$$

Substituting and rearranging gives an **implicit** formula for the calculation of $T_{i,j+1}$ for all spatial nodes i, i.e.

$$-\frac{\alpha \Delta t}{\Delta x^2} T_{i-1,j+1} + (2\frac{\alpha \Delta t}{\Delta x^2} + 1)T_{i,j+1} - \frac{\alpha \Delta t}{\Delta x^2} T_{i+1,j+1} = T_{i,j}$$

Exercise: Derive the above result.

For given temperatures at the boundaries, the above formula constitutes a **system of simultaneous algebraic equations** the solution of which yields the desired values of $T_{i,j+1}$ for all $i = 2, 3, ..., N-1$ and for all time levels $j+1$.

Introducing the expressions

$$a_i = c_i = \frac{\alpha \Delta t}{\Delta x^2}$$

$$b_i = 2\frac{\alpha \Delta t}{\Delta x^2} + 1$$

and

$$d_i = T_{i,j}$$

the system of equations becomes

$$-a_i T_{i-1,j+1} + b_i T_{i,j+1} - c_i T_{i+1,j+1} = d_i$$

for $i = 2, 3, ..., N-1$ for each and every time level $j+1$ such that $j = 1, 2, ..., M$. Recall that the values of $T_{i,1}$ are given for all i by the initial condition of the problem .

The resulting system is **tridiagonal**, the associated matrix is **diagonally dominant** and it is easily and efficiently solved by the **Thomas algorithm**.

The main idea in the Thomas algorithm is to first reduce the original tridiagonal system of equations to a simple upper triangular form and then back substitute to determine the values of all the unknowns. Specifically, new variables β_i, S_i and D_i for $i = 1, 2, 3, ..., N$ are introduced and computed as follows, first

$$\beta_1 = \frac{d_1}{b_1}$$

next, recursively for $i = 2, 3, ..., N$

$$S_i = \frac{a_i}{b_{i-1}}$$

$$D_i = b_i - S_i c_{i-1}$$

$$\beta_i = d_i - S_i \beta_{i-1}$$

Then, back substitution is performed to obtain the solution, first for $i = N$

$$T_{N,j+1} = \frac{\beta_N}{D_N}$$

and finally for $i = N-1, N-2, ..., 2, 1$ as follows

$$T_{i,j+1} = \frac{(\beta_i - c_i T_{i+1,j+1})}{D_i}$$

An advantage of the implicit scheme is that stability is unconditionally stable. However, accuracy considerations preclude the use of large values of Δt.

10.3 The Semi-implicit Scheme

Still another possibility is to use weight averaging for the finite difference approximation of the right hand side term. Specifically, consider the following approximation

$$\frac{\partial^2 T}{\partial x^2} \approx \theta \frac{T(x_i + \Delta x, t_{j+1}) - 2T(x_i, t_{j+1}) + T(x_i - \Delta x, t_{j+1})}{\Delta x^2} +$$
$$+(1-\theta)\frac{T(x_i + \Delta x, t_j) - 2T(x_i, t_j) + T(x_i - \Delta x, t_j)}{\Delta x^2} =$$
$$= \theta \frac{T_{i+1,j+1} - 2T_{i,j+1} + T_{i+1,j+1}}{\Delta x^2} + (1-\theta)\frac{T_{i+1,j} - 2T_{i,j} + T_{i+1,j}}{\Delta x^2}$$

where $0 \leq \theta \leq 1$.

Substituting and rearranging gives a **semi-implicit** formula for the calculation of $T_{i,j+1}$ for all spatial nodes i, i.e.

$$-\frac{\theta \alpha \Delta t}{\Delta x^2} T_{i-1,j+1} + (2\theta \frac{\alpha \Delta t}{\Delta x^2} + 1) T_{i,j+1} - \frac{\theta \alpha \Delta t}{\Delta x^2} T_{i+1,j+1} =$$
$$= T_{i,j} + (1-\theta)\frac{\alpha \Delta t}{\Delta x^2}(T_{i+1,j} - 2T_{i,j} + T_{i+1,j})$$

The resulting system is also **tridiagonal** and it is also easily and efficiently solved by the **Thomas algorithm**. The scheme is also unconditionally stable. If $\theta = \frac{1}{2}$ one obtains the **Crank-Nicolson scheme**.

Note that the above formula reduces to the explicit scheme when $\theta = 0$ and to the implicit scheme when $\theta = 1$.

Exercise: Obtain the expressions for a_i, b_i, c_i and d_i for the C-N scheme.

10.4 More General Boundary Conditions for a Slab

Consider the same problem as above except that the slab is assumed insulated at $x = 0$ and that exchanges heat with an environment at T_∞ by means of a heat transfer coefficient h at $x = L$. The boundary conditions are thus

$$k\frac{\partial T}{\partial x} = 0$$

at $x = 0$, and and

$$-\frac{\partial T}{\partial x} = h(T - T_\infty)$$

at $x = L$ a convenient strategy is to introduce **auxiliary nodes** outside the slab (symmetrically located with respect to x_1 and x_N, respectively. Therefore, the auxiliary node to the left of node 1 will be labeled node -2 and the one to the right of node N will be node $N+1$.

Next, a central difference formula is used to approximate the derivatives at the boundaries and the values of temperature at the auxiliary nodes are eliminated by combining the result with the difference approximation of the heat equation.

A second order accurate finite difference approximation at $x = 0$ is

$$k\frac{T_{2,j} - T_{-2,j}}{2\Delta x} = 0$$

or $T_{2,j} = T_{-2,j}$. A similar approximation at $x = L$ yields

$$-k\frac{T_{N+1,j} - T_{N-1,j}}{2\Delta x} = h(T_N - T_\infty)$$

Combining with the finite difference approximation according to the explicit scheme yields

$$T_{1,j+1} = T_{1,j} + 2\alpha\frac{\Delta t}{\Delta x^2}(T_{2,j} - T_{1,j})$$

and

$$T_{N,j+1} = T_{N,j} + 2\alpha\frac{\Delta t}{\Delta x^2}[T_{N-1,j} - T_{N,j}(1 + \Delta x\frac{h}{k}) + T_\infty(\Delta x\frac{h}{k})]$$

The above expressions provide explicit formulae for the calculation of the temperatures of the boundary nodes.

While combining with the finite difference approximation according to the implicit scheme yields

$$(1 + 2\alpha\frac{\Delta t}{\Delta x^2})T_{1,j+1} - (2\alpha\frac{\Delta t}{\Delta x^2})T_{2,j+1} = T_{1,j}$$

and

$$-(2\alpha\frac{\Delta t}{\Delta x^2})T_{N-1,j+1} + (1 + 2\alpha\frac{\Delta t}{\Delta x^2}[1 + \Delta x\frac{h}{k}])T_{N,j+1} = T_{N,j} + T_\infty(2\alpha\frac{\Delta t}{\Delta x^2})(\Delta x\frac{h}{k})$$

The above expressions provide the necessary additional equations for the calculation of the temperatures of the boundary nodes.

10.5 Steady State Conduction in Two Dimensions

The 2D steady state heat equation with internal heat generation in rectangular Cartesian coordinates is

$$\frac{\partial}{\partial x}(k\frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(k\frac{\partial T}{\partial y}) + g(x,y) = 0$$

If k and g are assumed constant this becomes

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{g}{k} = 0$$

This is called the **Poisson's equation**. Furthermore, if $g = 0$ this reduces to

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

and this is **Laplace's equation**. In any case, the resulting equation must be solved subject to suitable boundary conditions and the result is the function $T(x, y)$.

Finite difference methods easily and readily yield approximate solutions to steady state heat conduction problems. Let's start with Laplace's equation in a rectangular plate of width X and height Y subject to Dirichlet conditions as follows

$$T(x,0) = 0$$

$$T(0,y) = 0$$

$$T(L_x,y) = 0$$

and

$$T(x,L_y) = f(x)$$

First we create a **rectangular mesh** of points $x_i, i = 1,2,3...,N$ and $y_j, j = 1,2,3,...,M$. Nodes x_1, x_N, y_1 and y_M are **boundary nodes** while the rest are **interior nodes**. The temperature at a specific nodal location (x_i, y_j), will be denoted by $T(x_i, y_j) = T_{i,j}$.

For the interior nodes a second order accurate finite difference approximation of Laplace's equation is

$$\frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{\Delta x^2} + \frac{T_{i,j-1} - 2T_{i,j} + T_{i,j+1}}{\Delta y^2} = 0$$

Equations for the boundary nodes are simply

$$T(x_i, 0) = T_{i,1} = 0$$

for $i = 1, 2, ..., N$

$$T(0, y_j) = T_{1,j} = T(x_N, y_j) = T_{N,j} = 0$$

and

$$T(x_i, y_M) = T_{i,M} = f(x_i)$$

The above is a system of $N \times M$ simultaneous linear algebraic equations which must be solved to obtain the approximate values of the temperatures at the interior nodes. The solution method is described next.

To simplify, consider the case of the square plate $L_x = L_y$ with $N = M$ and a uniform mesh with $\Delta x = \Delta y = \Delta$. With this the finite difference equation for the interior node (i, j) becomes

$$T_{i-1,j} + T_{i,j-1} - 4T_{i,j} + T_{i+1,j} + T_{i,j+1} = 0$$

This is sometimes called the **five point formula**.

To obtain a **banded** matrix the mesh points must be relabeled sequentially from left to right and from top to bottom. The resulting system can be solved by Gaussian elimination if N and M are small and by SOR iteration when they are large.

Iteration methods can be used to solve this system of equations. These methods require an initial guess for the temperature field but the final result must be independent of the guess. The iteration **converges** when the calculated temperature values at one iteration differ little from those obtained at the subsequent iteration.

In the **Jacobi iteration method** the interior nodes are visited sequentially and an improved guess of the value of $T_{i,j}$ is calculated using the previous guess values of all neighboring nodes. If $T_{i,j}$ represents the new value and $T^o_{i,j}$ the old one the algorithm is

$$T_{i,j} = \frac{T^o_{i-1,j} + T^o_{i,j-1} + T^o_{i+1,j} + T^o_{i,j+1}}{4}$$

Note that in the Jacobi method, the temperature field is only updated once all the nodes have been visited.

Typically, nodes are visited from left to right and from bottom to top. I.e. $(2, 2), (3, 2), ..., (N-1, 2), (2, 3), (3, 3), ..., (N-1, M-1)$. Note that in this scheme, when visiting node (i, j) nodes $(i-1, j)$ and $(i, j-1)$ would have already been visited (and improved guesses $T_{i-1,j}$ and $T_{i,j-1}$ would be available. Therefore, the **Gauss-Seidel iteration** uses calculated values as soon as they are available and the algorithm is

$$T_{i,j} = \frac{T^o_{i-1,j} + T^o_{i,j-1} + T_{i+1,j} + T_{i,j+1}}{4}$$

As a result of updating as the calculation proceeds, the Gauss-Seidel method converges at a faster rate than the Jacobi method.

It is possible to increase even more the speed of convergence by using the **Successive Over relaxation (SOR) method**. First, note that the Gauss-Seidel algorithm can be rewritten as

$$T_{i,j} = T^o_{i,j} + \frac{T_{i-1,j} + T_{i,j-1} - 4T^o_{i,j} + T^o_{i+1,j} + T^o_{i,j+1}}{4}$$

The second term on the right hand side is the **correction** or **displacement** in the value of the temperature at node (i, j) from $T^o_{i,j}$ to $T_{i,j}$.

The SOR method converges at a faster rate because larger corrections are done at each iteration, i.e.

$$T_{i,j} = T^o_{i,j} + \omega\frac{T_{i-1,j} + T_{i,j-1} - 4T^o_{i,j} + T^o_{i+1,j} + T^o_{i,j+1}}{4}$$

where the **over relaxation parameter** ω is $1 \leq \omega \leq 2$.

Finally consider now the problem of steady state conduction with internal heat generation described by Poisson's equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = -\frac{g(x, y)}{k}$$

on (x, y) in the **interior** $R = \{(x, y) : a < x < b, c < y < d\}$ of a planar region and subject to

$$T(x, y) = T_S(x, y)$$

along the **boundary** S of the region. As long as g and T_S are continuous a unique solution exists.

As before, a simple approach to the numerical solution of this problem consists in partitioning the intervals $[a, b]$ and $[c, d]$ respectively into $N-1$ and $M-1$ subintervals with step sizes $\Delta x = (b-a)/(N-1)$ and $\Delta y = (d-c)/(M-1)$ so that the *node* or *mesh point* (x_i, y_j) is at $x_i = a + (i-1)\Delta x$ for $i = 1, .., N$ and $y_j = c + (j-1)\Delta y$ for $j = 1, .., M$. Using now second order accurate centered finite difference approximations for the partial derivatives the following five point formula for the temperature at node (x_i, y_j) is obtained.

$$-\frac{1}{\Delta x^2}T_{i-1,j} - \frac{1}{\Delta x^2}T_{i+1,j} - \frac{1}{\Delta y^2}T_{i,j-1} - \frac{1}{\Delta y^2}T_{i,j+1} + (\frac{2}{\Delta x^2} + \frac{2}{\Delta y^2})T_{i,j} = \frac{g(x_i, y_j)}{k}$$

The boundary conditions imposed along the *boundary nodes* become

$$\begin{aligned} T_{0,j} &= T_S(x_1, y_j) \quad j = 1, 2, .., M \\ T_{N,j} &= T_S(x_N, y_j) \quad j = 1, 2, .., M \\ T_{i,0} &= T_S(x_i, y_1) \quad i = 1, 2, .., N \\ T_{i,M} &= T_S(x_i, y_M) \quad i = 1, 2, .., N \end{aligned}$$

This is a system of simultaneous algebraic equations. To obtain a **banded** matrix the mesh points must be relabeled sequentially from left to right and from top to bottom. The resulting system can be solved by Gaussian elimination if N and M are small and by SOR iteration when they are large.

11 Numerical Solution Methods: Finite Volume and Finite Elements

This section contains a survey of finite volume and finite element methods for the solution of one- and multidimensional dimensional transient heat conduction problems.

12 Solution of Conduction Problems by the Finite Volume Method

An alternative discretization method is based on the idea of regarding the computation domain as subdivided into a collection of *finite volumes*. In this view, each finite volume is represented by a line in 1D, an area in 2D and a volume in 3D. *Nodes*, located inside each finite volume, become the locus of computational values. In rectangular Cartesian coordinates in 2D the simplest finite volumes are rectangles. For each node, the rectangle faces are formed by drawing perpendiculars through the midpoints between contiguous nodes. Discretization equations are obtained by integrating the original partial differential equation over the span of each finite volume. The method is easily extended to nonlinear problems.

12.1 Steady State Conduction in a Slab with Internal Heat Generation

The problem of steady state heat transport through the thickness L of a large slab with constant internal heat generation g is described by the following form of the heat or diffusion equation

$$\frac{d}{dx}(k\frac{dT}{dx}) + g = 0$$

To implement the finite volume method first subdivide the thickness of the slab into a collection of N adjoining segments of thickness not necessarily of uniform size (finite volumes) and place a node inside each volume. Thus, an arbitrary node will be called P, its size is Δx and the nodes to its left and right, respectively W and E.

Note that two different types of nodes result. While *interior* nodes are surrounded by finite volume on both sides, *boundary* nodes contain material only on one side. Here we shall concentrate on the derivation of discrete equations for the interior nodes. Those for the boundary nodes will be discussed later.

The distance between nodes W and P is δx_w and that between P and E, δx_e. The locations of the finite volume boundaries corresponding to node P will be denoted by w and e. Finally, the distance between P and e is called δx_{e-} and that between e and E is δx_{e+}. If P is located in the center of the finite volume then $\delta x_{e-} = \delta x_{e+} = \frac{1}{2}\delta x_e$.

To implement the finite volume method we now integrate the above equation over the span of the finite volume associated with node P, i.e. from x_w to x_e

$$\int_w^e d(k\frac{dT}{dx}) + \int_w^e g dx = (k\frac{dT}{dx})|_e - (k\frac{dT}{dx})|_w + g\Delta x = 0$$

Next, approximate the derivatives by piecewise linear profiles to give

$$k_e \frac{T_E - T_P}{\delta x_e} - k_w \frac{T_P - T_W}{\delta x_w} + g\Delta x = 0$$

where the conductivities at the finite volume faces are calculated as the harmonic means of the values at the neighboring nodes, i.e.

$$k_e = [\frac{1 - \delta x_{e+}/\delta x_e}{k_P} + \frac{\delta x_{e+}/\delta x_e}{k_E}]^{-1}$$

and

$$k_w = [\frac{1 - \delta x_{w+}/\delta x_w}{k_W} + \frac{\delta x_{w+}/\delta x_w}{k_P}]^{-1}$$

Rearranging one obtains the algebraic equation

$$a_P T_P = a_E T_E + a_W T_W + b$$

where the coefficients are given by

$$a_E = \frac{k_e}{\delta x_e}$$

$$a_W = \frac{k_w}{\delta x_w}$$

$$a_P = a_E + a_W$$

and

$$b = g\Delta x$$

One algebraic equation like the above, relating the values of T at three contiguous nodes, is obtained for each of the N nodes. Adding the this set the discrete equations associated with the boundary nodes one obtains a consistent set of interlinked simultaneous algebraic equations which must be solved to give the values of T for all nodal locations..

For the special case of constant thermal properties and finite volumes of uniform size ($\delta x_e = \delta x_w = \Delta x$, the above is easily rearranged as

$$\frac{T_E - 2T_P + T_W}{\Delta x^2} + \frac{g}{k} = 0$$

which coincides with the FD formula obtained before.

12.2 Transient Conduction in a Slab

Now we apply the finite volume method to the discretization of a transient problem. Consider the problem of determining $T(x,t)$ for the slab $x \in [0, L]$ and $t > 0$ such that

$$\rho C_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}(k\frac{\partial T}{\partial x})$$

subject to

$$T(x,0) = T_0(x)$$

and

$$T(0,t) = T(L,t) = 0$$

The above is an approximate representation of the problem of quenching a slab.

Subdivide the slab into a collection of N adjacent segments of thickness (finite volumes) and introduce a set of $N+1$ nodes (one inside each volume and one on each boundary). The positions of nodes are then labeled from left to right in the form of a sequence $x_1, x_2, x_3, ..., x_i, ...x_{N+1}$. To discretize time simply select time intervals of duration Δt at which the calculations will be performed. Uniform time intervals will be assumed here. As before, we focus on the derivation of the discretized heat equation for all the interior volumes.

Consider now an arbitrary finite volume of size Δx. Its representative node is locate at x_P and its boundaries are located at x_w and x_e. . The two contiguous nodes to the left and right of node P are W and E and their locations are x_W and x_E. Introduce again the various mesh spacings as in the steady case $\delta x_e, \delta x_{e+}, \delta x_{e-}, \delta x_w, \delta x_{w+}, \delta x_{w-}$ and Δx

Integrate now the heat equation over the span of the finite volume (i.e. from x to $x+\Delta x$) and also over the time interval from t to $t+\Delta t$, i.e.

$$\int_t^{t+\Delta t}\int_{x_w}^{x_e} \rho C_p \frac{\partial T}{\partial t} dx dt = \int_t^{t+\Delta t}\int_{x_w}^{x_e} \frac{\partial}{\partial x}(k\frac{\partial T}{\partial x}) dx dt = \int_t^{t+\Delta t}\int_{x_w}^{x_e} \partial(k\frac{\partial T}{\partial x}) dt$$

The double integration on the left hand side is straightforward and yields

$$\int_t^{t+\Delta t}\int_{x_w}^{x_e} \rho C_p \frac{\partial T}{\partial t} dx dt = \rho C_p \Delta x (T_P - T_P^o)$$

where T_P is the value of T at P at time $t+\Delta t$ and T_P^o is the value of T at P at time t.

On the right hand side, the space integration is performed first and the resulting derivatives are approximated from a piecewise linear profile to yield

$$\int_t^{t+\Delta t}\int_{x_w}^{x_e} \partial(k\frac{\partial T}{\partial x}) dt = \int_t^{t+\Delta t} [(k\frac{\partial T}{\partial x})|_e - (k\frac{\partial T}{\partial x})|_w] dt =$$
$$= \int_t^{t+\Delta t} [k_e \frac{T_E - T_P}{\delta x_e} - k_w \frac{T_P - T_W}{\delta x_w}] dt$$

To complete the process an important decision must be made when carrying out the time integration. A very general proposition is to assume that the integrals are given by time-weighed averages as follows

$$\int_t^{t+\Delta t} T_P dt = [\theta T_P + (1-\theta) T_P^o]\Delta t$$

$$\int_t^{t+\Delta t} T_E dt = [\theta T_E + (1-\theta) T_E^o]\Delta t$$

and

$$\int_t^{t+\Delta t} T_W dt = [\theta T_W + (1-\theta) T_W^o]\Delta t$$

where the weighing factor θ is a pure number with $0 \leq \theta \leq 1$.

Introducing the above, the integrated heat equation then finally becomes

$$\frac{\Delta x}{\Delta t}(T_P - T_P^o) = \theta[\frac{k_e(T_E - T_P)}{\delta x_e} - \frac{k_w(T_P - T_W)}{\delta x_w}] +$$
$$+(1-\theta)[\frac{k_e(T_E^o - T_P^o)}{\delta x_e} - \frac{k_w(T_P^o - T_W^o)}{\delta x_w}]$$

Rearrangement gives

$$a_P T_P = a_E[\theta T_E + (1-\theta)T_E^o] + a_W[\theta T_W + (1-\theta)T_W^o] +$$
$$+[a_P^o - (1-\theta)a_E - (1-\theta)a_W]T_P^o$$

where

$$a_E = \frac{k_e}{\delta x_e}$$

$$a_W = \frac{k_w}{\delta x_w}$$

$$a_P^o = \rho C_p \frac{\Delta x}{\Delta t}$$

and

$$a_P = \theta a_E + \theta a_W + a_P^o$$

Note that if finite volumes of uniform size are used with nodes at their midpoints, transport properties are assumed constant and $\theta = 0$ is assumed, the above reduces, after some rearrangement to

$$\frac{T_P - T_P^o}{\Delta t} = \frac{k}{\rho C_p} \frac{T_W^o - 2T_P^o + T_E^o}{\Delta x^2}$$

which is identical to the result obtained earlier using the finite difference method with forward differencing in time.

12.3 Transient Conduction in Multidimensional Systems

Consider the problem of estimating the temperature $T(x, y, z, t)$ in a three-dimensional brick $x \in [0, X], y \in [0, Y], z \in [0, Z]$ undergoing transient heat conduction with constant internal heat generation. The heat equation for this case is

$$\rho C_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}(k\frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(k\frac{\partial T}{\partial y}) + \frac{\partial}{\partial z}(k\frac{\partial T}{\partial z}) + g$$

subject to specified conditions at the boundaries of the domain.

The finite volume formulation regards the brick as composed of a set of adjoining volumes each containing a node. The dimensions of the typical volume are $\Delta x \times \Delta y \times \Delta z$ and the typical node P has six neighbors E, W, N, S, T and B. Proceeding exactly as above, the finite volume method produces the following result (with $\theta = 1$)

$$a_P T_P = a_E T_E + a_W T_W + a_N T_N + a_S T_S + a_T T_T + a_B T_B + b$$

where

$$a_E = \frac{k_e \Delta y \Delta z}{\delta x_e}$$

$$a_W = \frac{k_w \Delta y \Delta z}{\delta x_w}$$

$$a_N = \frac{k_n \Delta z \Delta x}{\delta y_n}$$

$$a_S = \frac{k_w \Delta z \Delta x}{\delta y_s}$$

$$a_T = \frac{k_t \Delta x \Delta y}{\delta z_t}$$

$$a_B = \frac{k_b \Delta x \Delta y}{\delta z_b}$$

$$a_P^o = \rho C_p \frac{\Delta x \Delta y \Delta z}{\Delta t}$$

$$b = g \Delta x \Delta y \Delta z + a_P^o T_P^o$$

and

$$a_P = a_E + a_W + a_N + a_S + a_T + a_B + a_P^o$$

Note that all the above expressions can all be generically written as

$$a_p T_P = \sum a_{nb} T_{nb} + b$$

where the summation contains just T_E and T_W in the case of uni-dimensional systems, contains T_E, T_W, T_N and T_S in the case of 2D systems and all T_E, T_W, T_N, T_S, T_T and T_B in the case of three dimensional systems.

The discrete equation derived using the FV method can be rearranged for $\theta = 1$ to produce a very appealing and physically meaningful expression. Consider two dimensional systems. The generic discrete form of the conservation equation is

$$\lambda_P (\phi_P - \phi_P^o) \frac{\Delta V}{\Delta t} = J_w A_w - J_e A_e + J_s A_s - J_n A_n + g \Delta V$$

where the dependent variable ϕ_i is the temperature T, for heat transfer and the concentration c, for mass transfer. Further, the J_i's are the fluxes of the transported quantity and the A_i's are the cross-sectional areas of the finite volume faces through which the transported quantity enters/leaves the finite volume. Therefore the products $J_i A_i$ are the rates of transport of energy or mass through the various control volume faces. Moreover, $\lambda = \rho C_p$ for heat transfer and $= 1$ for mass transfer. The transport rates are given as

$$J_e A_e = [(\frac{\delta x_{e-}}{\Gamma_P}) + (\frac{\delta x_{e+}}{\Gamma_E})]^{-1} (\phi_P - \phi_E) A_e$$

$$J_w A_w = [(\frac{\delta x_{w-}}{\Gamma_W}) + (\frac{\delta x_{w+}}{\Gamma_P})]^{-1}(\phi_W - \phi_P)A_w$$

$$J_n A_n = [(\frac{\delta y_{n-}}{\Gamma_P}) + (\frac{\delta y_{n+}}{\Gamma_N})]^{-1}(\phi_P - \phi_N)A_n$$

$$J_s A_s = [(\frac{\delta y_{s-}}{\Gamma_S}) + (\frac{\delta y_{s+}}{\Gamma_P})]^{-1}(\phi_S - \phi_P)$$

where $\Gamma_i = k_i$ for heat transfer and $= D$ for mass transfer.

Many special forms can be obtained by simplification of the above. For instance, for a rectangle with constant thermal properties, without internal heat generation at steady state and using $\Delta x = \Delta y$ one obtains

$$T_P = \frac{1}{4}[T_E + T_W + T_N + T_S]$$

which is identical to the expression obtained before using the method of finite differences.

13 Solution of Conduction Problems by the Finite Element Method

Consider the problem of multi-dimensional steady state heat conduction with internal heat generation and constant thermal properties

$$\nabla^2 T = -\frac{g}{k}$$

inside the domain Ω, and subject to $T = 0$ at the boundary. Intimately associated with this boundary value problem is the following *energy functional* I

$$I(v) = \int_\Omega [\frac{1}{2}|\nabla v|^2 - \frac{g}{k}v]dxdy$$

where v are functions among which the solution of the original heat conduction problem is included. It can be shown that the solution of the boundary value problem above is the function $T(\mathbf{r}) = v(\mathbf{r})$ that produces the minimum value of the energy functional I, i.e.

$$\delta I(T) = 0$$

and this function must clearly come from the set of functions which satisfy the stated boundary conditions.

Discretization in the finite element method is obtained by subdiving the computational domain into subdomains (*finite elements*), commonly triangles, rectangles, tetrahedra or rectangular parallelepipeds (bricks). The vertices in each of these geometries are called *nodes*. The methods seeks an approximation inside each element of the form

$$T(\mathbf{r}) = \sum_{i=1}^{m} c_i \phi_i(\mathbf{r})$$

where the $\phi_i(\mathbf{r})$ are known linearly independent piecewise polynomials called the *shape functions* and the c_i are unknown coefficients actually representing the values of T at the given nodal locations. The finite element method determines the specific values of these coefficients which minimize the functional I. (i.e. $\frac{\partial I}{\partial c_i} = 0$).

Minimization of I with respect to the c_i's produces a set of simultaneous linear algebraic equations of the form

$$A\mathbf{c} = \mathbf{b}$$

where $\mathbf{c} = (c_1, c_2, ..., c_m)^T$ is the solution vector and $\mathbf{b} = (b_1, b_2, ..., b_m)^T$ is the forcing vector. Solution of the above system using standard techniques yields the desired approximation.

To illustrate the practical implementation of the finite element methodology we now consider the case of one-dimensional steady heat conduction with internal heat generation inside a slab, i.e.

$$\frac{d}{dx}(k\frac{dT}{dx}) + g = 0 \tag{1}$$

in $0 \leq x \leq 1$ subject to the boundary conditions

$$T(0) = 0 \tag{2}$$

and

$$T(1) = 0 \tag{3}$$

In the Galerkin formulation of the finite element method, the domain $0 \leq x \leq 1$ is first subdivided into a set of n intervals (finite elements) of size Δx connected at their ends (nodes). For each node i, introduce the following *shape function* of position

$$\phi_i(x) = \begin{cases} 0 & 0 \leq x \leq x_{i-1} \\ N_1(x) & x_{i-1} \leq x \leq x_i \\ N_2(x) & x_i \leq x \leq x_{i+1} \\ 0 & x_{i+1} \leq x \leq 1 \end{cases} \tag{4}$$

where

$$N_1(x) = \frac{x - x_{i-1}}{\Delta x} \tag{5}$$

and

$$N_2(x) = \frac{x_{i+1} - x}{\Delta x} \tag{6}$$

The function ϕ_i is continuous and zero everywhere except for the interval $x_{i-1} \leq x \leq x_{i+1}$, on which it consists of two linear segments with a maximum value of 1 attained at $x = x_i$. As a set, these functions possess the very interesting property that any continuous piecewise-linear function of position, such as $T(x)$, can be represented by the linear combination

$$T(x) \approx \sum_{j=1}^{n} c_j \phi_j(x) \tag{7}$$

where the coefficients c_j are the values of T at the nodes $j = 1, 2, ..., n$, i.e. $c_j = T_j$.

Furthermore, the set of functions ϕ_i possess the fundamental property of *orthogonality* i.e.

$$\int_0^1 \phi_i \phi_j dx = \begin{cases} 0 & j \leq i-2 \\ \Delta x/6 & j = i-1 \\ \Delta x/3 & j = i \\ \Delta x/6 & j = i+1 \\ 0 & j \geq i+2 \end{cases} \tag{8}$$

The Galerkin method requires consideration of the *scalar product* of the (approximated) energy equation with the shape functions, i.e.

$$\int_0^1 (\frac{d}{dx}(k\frac{dT}{dx}) + g)\phi_i(x) dx = 0 \tag{9}$$

Integrating the first term by parts and noting the behavior of the shape functions at the boundary nodes, one obtains,

$$\int_0^1 (-k\frac{dT}{dx}\frac{d\phi_i}{dx} + g\phi_i) dx = 0 \tag{10}$$

Since the same value of the integral is obtained if one integrates first over the interval $[x_{i-1}, x_{i+1}]$ for all i and then adds up all the resulting integrals one can write

$$\sum_{i=1}^{n} \int_{x_{i-1}}^{x_{i+1}} (-k\frac{dT}{dx}\frac{d\phi_i}{dx} + g\phi_i) dx = 0 \tag{11}$$

Performing the integration over $[x_{i-1}, x_{i+1}]$ with the given ϕ_i's (and the approximation $T(x) = \sum_{j=1}^{n} c_j \phi_j(x)$) one obtains,

$$\int_{x_{i-1}}^{x_i} -k\frac{dT}{dx}\frac{d\phi_i}{dx} dx = \begin{cases} 0 & j \neq i \\ -k\frac{T_i - T_{i-1}}{\Delta x} & j = i \end{cases} \tag{12}$$

and

$$\int_{x_i}^{x_{i+1}} -k\frac{dT}{dx}\frac{d\phi_i}{dx} dx = \begin{cases} 0 & j \neq i \\ k\frac{T_{i+1} - T_i}{\Delta x} & j = i \end{cases} \tag{13}$$

and

$$\int_{x_{i-1}}^{x_{i+1}} g\phi_i dx = \int_{x_{i-1}}^{x_i} gN_1 dx + \int_{x_i}^{x_{i+1}} gN_2 dx = g\Delta x \tag{14}$$

Therefore, the integrated energy equation is

$$\frac{k}{(\Delta x)^2}(-T_{i-1} + 2T_i - T_{i+1}) = g \tag{15}$$

for $i = 1, 2, 3, ..., n$. Note this is exactly the same result obtained using the finite difference or the finite volume methods.

This result can be written using matrix notation as

$$A\mathbf{c} = \mathbf{b} \tag{16}$$

which is simply a tridiagonal system of linear algebraic equations where A is the *global stiffness matrix*,

$$A = \begin{pmatrix} K_{11}^1 & 0 & 0 & 0 & . & . \\ K_{21}^1 & (K_{22}^1 + K_{11}^2) & K_{12}^2 & 0 & . & . \\ 0 & K_{21}^2 & (K_{22}^2 + K_{11}^3) & K_{12}^3 & 0 & . \\ 0 & 0 & K_{21}^3 & (K_{22}^3 + K_{11}^4) & K_{12}^4 & 0 \\ 0 & . & . & . & . & . \end{pmatrix} \tag{17}$$

The entries K_{11}^i, K_{12}^i, K_{21}^i, and K_{22}^i are the components of the *element stiffness matrices* for the i-th element

$$\mathbf{K}^i = \begin{pmatrix} K_{11}^i & K_{12}^i \\ K_{21}^i & K_{22}^i \end{pmatrix} = \frac{k}{(\Delta x)^2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{18}$$

Further, $\mathbf{c} = \mathbf{T}$ is the *vector of nodal temperatures*,

$$\mathbf{c} = \begin{pmatrix} T_1 \\ T_2 \\ . \\ . \\ . \\ T_n \end{pmatrix} \tag{19}$$

and $\mathbf{b}$ is the *force vector*.

$$\mathbf{b} = \begin{pmatrix} 0 \\ g \\ . \\ . \\ g \\ 0 \end{pmatrix} \tag{20}$$

Note that the simplest way to handle the boundary conditions in this problem consists of resetting the values of all the entries in the first and last rows of the stiffness matrix to zero (except for the first entry in the first row and last entry in the last row, which should be set to 1), together with resetting the first and last elements of the force vector to zero (the specified temperatures at the boundaries).

As a specific illustration of the development of the finite element formulation for transient problems consider the problem of one-dimensional, transient heat conduction without internal heat generation through a plane wall of thickness $L = 1$ with constant thermal properties. The heat equation is

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{21}$$

The problem is to be solved subject to the *initial condition*

$$T(x, 0) = T_o \tag{22}$$

and the *boundary conditions*

$$T(0, t) = T(1, t) = 0 \tag{23}$$

The Galerkin finite element method subdivides the domain $0 \leq x \leq 1$ into a set of n intervals (finite elements) of size Δx connected at their ends (nodes). For each node i, introduce the following *shape function* of position

$$\phi_i(x) = \begin{cases} 0 & 0 \leq x \leq x_{i-1} \\ N_1(x) & x_{i-1} \leq x \leq x_i \\ N_2(x) & x_i \leq x \leq x_{i+1} \\ 0 & x_{i+1} \leq x \leq 1 \end{cases} \tag{24}$$

where

$$N_1(x) = \frac{x - x_{i-1}}{\Delta x} \tag{25}$$

and

$$N_2(x) = \frac{x_{i+1} - x}{\Delta x} \tag{26}$$

The function ϕ_i is continuous and zero everywhere except for the interval $x_{i-1} \leq x \leq x_{i+1}$, on which it consists of two linear segments with a maximum value of 1 attained at $x = x_i$. The set of N_i's is used to construct a linear combination to approximate $T(x, t)$ as follows

$$T(x, t) = \sum_{j=1}^{n} c_j(t)\phi_j(x) = \sum_{j=1}^{n} T_j(t) N_j(x) \tag{27}$$

where the Fourier coefficients T_j are the values of T at the nodes $j = 1, 2, ..., n$. Note that the Galerkin formulation separates the dependency of the problem on t from that of x through the use of the shape functions.

Furthermore, the set of functions N_i possess an orthogonality property such that

$$\int_0^1 N_i N_j dx = \begin{cases} 0 & j \le i-2 \\ \Delta x/6 & j = i-1 \\ \Delta x/3 & j = i \\ \Delta x/6 & j = i+1 \\ 0 & j \ge i+2 \end{cases} \tag{28}$$

The Galerkin method is based in the weighed minimization of the residual $R = T_t - \alpha T_{xx}$ by constructing the *scalar product* (R, N_i) of the (approximated) energy equation with the shape functions, i.e.

$$(R, N_i) = \int_0^1 \int_0^t [\frac{\partial T}{\partial t} - \alpha \frac{\partial^2 T}{\partial x^2}] N_i(x) dx dt = 0 \tag{29}$$

Note that the scalar product now involves integration over time as well as over space.

The same value of the integral is obtained if one first integrates over the interval $[x_{i-1}, x_{i+1}]$, for all i and then collects all the resulting integrals, i.e.

$$\sum_{i=1}^{n} \int_{x_{i-1}}^{x_{i+1}} \int_0^t [\frac{\partial T}{\partial t} - \alpha \frac{\partial^2 T}{\partial x^2}] N_i(x) dx dt = 0 \tag{30}$$

Given the dependencies on x and t, this can be written as

$$\sum_{i=1}^{n} \int_{x_{i-1}}^{x_{i+1}} \int_0^t [\frac{dT_j(t)}{dt} N_j(x) N_i(x) - T_j(t) \alpha \frac{d^2 N_j(x)}{dx^2}) N_i(x)] dx dt = 0 \tag{31}$$

This can now be written using matrix notation as a set of ordinary differential equations

$$\mathbf{A} \frac{d\vec{T}}{dt} + \mathbf{B} \vec{T} = 0 \tag{32}$$

where $\vec{T}$ is the vector of nodal temperatures

$$\vec{T} = (T_1, T_2, ..., T_n)^{\mathbf{T}} \tag{33}$$

and $\mathbf{A}$ and $\mathbf{B}$ are matrices defined by the scalar products

$$\mathbf{A} = a_{ij} = (N_j, N_i) \tag{34}$$

and

$$\mathbf{B} = b_{ij} = (-\alpha \frac{d^2 N_j(x)}{dx^2}, N_i) \tag{35}$$

In order to solve the resulting system of ordinary differential equations, consider the following generic implicit approximation with parameter θ such that $0 \le \theta \le 1$,

$$\mathbf{A}(\frac{\vec{T}^{t+\Delta t} - \vec{T}^t}{\Delta t}) + \mathbf{B}(\theta \vec{T}^{t+\Delta t} + (1-\theta)\vec{T}^t) = 0 \tag{36}$$

For any value of θ, a system of algebraic equations has to be solved. As in the steady case, introduction of the boundary conditions requires modification of the finite element equations for the boundary nodes.