

# Solution of hydrodynamic equilibrium in black-oil reservoirs: the classical material balance equation

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## Abstract

It is generally known that the material balance equation can be derived from a 3D black-oil model typical of reservoir simulation. However, the existing literature is not comprehensive about what are the needed assumptions and how the pressure calculated from the material balance equation relates to the pressure field of a 3D black-oil model.

This work examines underlying details in obtaining the material balance equation from a 3D black-oil model. The black-oil equations are integrated in reservoir volume and quantities are averaged in space. The material balance equation is obtained for the averaged quantities and it is applied to calculate reservoir pressure. It is shown that this calculated pressure corresponds to the pressure at a chosen datum level in the 3D black-oil model (and not, in general, to the average pressure of the 3D model). The definition of a convenient datum level is also discussed. It is found that the physical correctness of PVT properties is a sufficient condition for solution uniqueness of the material balance equation when pressure is the unknown quantity. Finally, a numerical scheme for solving the material balance equation is proposed based on nonlinear root finding algorithms.

**Keywords:** material balance, black-oil equations, reservoir engineering.

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## 1. Introduction

The material balance equation is a classical tool in petroleum reservoir engineering. It consists of the algebraic equation originally presented in Schilthuis (1936) as a zero dimensional model for estimating reservoir pressure from mass balance. Schilthuis (1936) introduced the hypothesis of mechanical equilibrium attained at all times not necessarily in the entire reservoir volume, but in the regions of high permeability where most of the production would come from and where pressure would present a similar trend. Such regions were said to contain an active oil.

The equation developed by Schilthuis (1936) did not account for the non-existence of hydrodynamic, chemical or thermal equilibrium. An attempt to introduce non-equilibrium effects of hydrodynamic origin came with *black-oil* equations (Aziz and Settari, 1979; Trangenstein and Bell, 1989b). The novelty on hydrodynamics was the introduction of a macroscopic equation of motion using Darcy's law. Black-oil equations are usually solved for in three dimensional models that aim to predict the pressure field responsible for reservoir fluid flow.

Many works have reported applications of the material balance equation in the literature, but few have explored its connection with black-oil equations. Ertekin et al. (2001) briefly addressed this subject arguing both approaches to be equivalent if capillarity and gradients of flow potential ( $\nabla\Phi = \nabla p - \rho\vec{g}$ ) were neglected and if PVT properties were averaged in reservoir volume. This work develops on those concepts and proposes an averaging scheme for flow properties. Subsequently, it suggests

how the pressure computed from material balance equation can be compared to the pressure field in 3D black-oil models.

The term black-oil used in this work refers to specific hydrocarbon compositions to which phase equilibrium can be described by a two pseudo-component system. For such reservoir fluids, gas phase composition remains essentially constant in the range of pressure and temperature observed during production so that a pseudo-gas component can be defined. Similarly, a pseudo-oil component can be defined by subtraction of the pseudo-gas component from the oil phase composition. After defining the pseudo-components, the number of transport equations needed for hydrocarbon components is reduced from a usually large number to only two.

The same approach of varying bubble-point pressure widely used in reservoir simulation, which is found in Aziz and Settari (1979) and Ertekin et al. (2001), is also applied here. This approach extends the material balance equation to saturated reservoirs under pressure maintenance (e.g. fluid injection). The applicability, however, requires negligible non-equilibrium effects.

After presenting the material balance equation, sufficient conditions are shown for which the material balance equation has only one solution for pressure. These conditions require fluids to be compressible and gas vaporization to occur with increase of oil phase density. A simple scheme for numerical solution is also proposed based on classical root-finding algorithms for non-linear equations.

## 2. Derivation of equations

This section presents black-oil equations, performs the averaging of quantities in space and examines simplifying assumptions necessary to obtain the material balance equation.

### 2.1. Black-oil equations

We assume that the reservoir is a hydraulically connected volume where hydrocarbon saturation is nonzero. By a hydraulically connected volume, we mean that it is possible to observe a common pressure decline trend when any reservoir portion is under production (Dake, 2001). Mathematically, this is equivalent to having pressure diffusivity above certain minimal value everywhere. Reservoir fluids are composed of oil and gas pseudo-components and water component. There can be three fluid phases at most: oil, gas and liquid water. Oil phase is composed of oil and gas components, gas phase is composed uniquely by gas component and, water phase, uniquely by water component. The only allowed mass transfer is of gas component between oil and gas phases. The gas phase composition is always that of a dry gas.

Mass balance equation for each component reads:

- water component:

$$\frac{\partial}{\partial t} (\rho_w S_w \phi) + \nabla \cdot (\rho_w \vec{u}_w) = 0; \quad (1)$$

- oil component:

$$\frac{\partial}{\partial t} (\rho_o Y_o S_o \phi) + \nabla \cdot (\rho_o Y_o \vec{u}_o) = 0; \quad (2)$$

- gas component:

$$\frac{\partial}{\partial t} [(\rho_o Y_g S_o + \rho_g S_g) \phi] + \nabla \cdot (\rho_o Y_g \vec{u}_o + \rho_g \vec{u}_g) = 0. \quad (3)$$

Mass concentrations of oil and gas components in oil phase are denoted by  $Y_o$  and  $Y_g$ , respectively.

Initial conditions are of no fluid motion. Boundary conditions are of no normal flow at all boundaries except for interfaces with wells or aquifers.

A common simplification in black-oil formulation is to restrict mass transport to macroscopic advection. No diffusive or dispersive term is included in mass balance equations.

Momentum balance for each phase is given by an extension of Darcy's law to multiphase flow. This extension consists in multiplying absolute (or single-phase) permeability ( $\vec{k}$ ) by a scalar function called phase relative permeability ( $k_{r,\alpha}$ ):

$$\vec{u}_\alpha = -\frac{k_{r,\alpha} \vec{k}}{\mu_\alpha} \cdot (\nabla p_\alpha - \rho_\alpha \vec{g}), \quad \alpha = o, w, g. \quad (4)$$

Another simplification is now made assuming thermal equilibrium in the reservoir volume, i.e.,

$$T_w = T_o = T_g = T_{rock} = T = \text{constant}. \quad (5)$$

Temperature is a constant prescribed value and no equation for energy balance is needed.

It follows from Duhem's theorem that the equilibrium of oil and gas phases is determined if, in addition to the mass balance equations, two other properties are prescribed. From Gibbs' phase rule, the oil and gas system has two degrees of freedom. It is then necessary to specify two intensive properties. The constant temperature is one and the local pressure is chosen as the other. Since temperature was assumed constant, oil phase density becomes a function of pressure alone. The same is valid for mass fractions of gas and oil components in the oil phase:

$$\rho_o = \rho_o(p_o), \quad Y_g = Y_g(p_o), \quad Y_o = Y_o(p_o). \quad (6)$$

When the oil phase is undersaturated, the system has one additional degree of freedom and both pressure and the amount of dissolved gas must be known to define its thermodynamic state, e.g.,  $\rho_o = \rho_o(p_o, Y_o)$ .

Porosity ( $\phi$ ) is assumed a function of pore pressure ( $p_P$ ). Because three phases are present in general, such quantity is defined by some relation of the type (Kim et al., 2011):

$$p_P = p_P(p_o, p_w, p_g, S_o, S_w, S_g). \quad (7)$$

To close the system of equations, a connection among different phase pressures is provided by capillary pressure relations:

$$p_{c,ow} = p_o - p_w, \quad (8)$$

$$p_{c,go} = p_g - p_o. \quad (9)$$

### 2.2. Material balance equation

Derivation of the material balance equation consists in integrating black-oil equations in reservoir volume ( $\Omega$ ).

Integrating Eqs. (1), (2) and (3) in volume and applying divergence theorem gives:

- water component:

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\Omega} \rho_w S_w \phi dV &= - \oint_{\partial\Omega} \rho_w (\vec{u}_w \cdot d\vec{S}) \\ &= \rho_w^{ST} (Q_{we}^{ST} + Q_{wi}^{ST} - Q_w^{ST}); \end{aligned} \quad (10)$$

- oil component:

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\Omega} \rho_o Y_o S_o \phi dV &= - \oint_{\partial\Omega} \rho_o Y_o (\vec{u}_o \cdot d\vec{S}) \\ &= -\rho_o^{ST} Q_o^{ST}; \end{aligned} \quad (11)$$

- gas component:

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\Omega} (\rho_o Y_g S_o + \rho_g S_g) \phi dV &= - \oint_{\partial\Omega} (\rho_o Y_g \vec{u}_o + \rho_g \vec{u}_g) \cdot d\vec{S} \\ &= \rho_g^{ST} (Q_{gi}^{ST} - Q_g^{ST}). \end{aligned} \quad (12)$$

### 2.2.1. Averaging quantities

Spatially averaged quantities are defined for porosity ( $\phi$ ), phase saturation ( $S_\alpha$ ), density ( $\rho_\alpha$ ) and mass concentration of a component  $k$  in phase  $\alpha$  ( $Y_{\alpha,k}$ ).

All quantities are initially functions of space and time, but they become functions of time alone after being averaged in space:

$$\bar{\phi}(t) = \frac{1}{\Omega} \int_{\Omega} \phi(\vec{x}, t) dV. \quad (13)$$

Omitting function arguments:

$$\bar{S}_\alpha \bar{\phi} = \frac{1}{\Omega} \int_{\Omega} S_\alpha \phi dV, \quad (14)$$

$$\bar{\rho}_\alpha \bar{S}_\alpha \bar{\phi} = \frac{1}{\Omega} \int_{\Omega} \rho_\alpha S_\alpha \phi dV, \quad (15)$$

$$\bar{\rho}_\alpha \bar{Y}_{\alpha,k} \bar{S}_\alpha \bar{\phi} = \frac{1}{\Omega} \int_{\Omega} \rho_\alpha Y_{\alpha,k} S_\alpha \phi dV. \quad (16)$$

Eqs. (10), (11) and (12) can now be rewritten in terms of averaged quantities:

- water component:

$$\frac{\partial}{\partial t} (\bar{\rho}_w \bar{S}_w \bar{\phi} \Omega) = \rho_w^{ST} (Q_{we}^{ST} + Q_{wi}^{ST} - Q_w^{ST}) \quad (17)$$

- oil component:

$$\frac{\partial}{\partial t} (\bar{\rho}_o \bar{Y}_o \bar{S}_o \bar{\phi} \Omega) = -\rho_o^{ST} Q_o^{ST} \quad (18)$$

- gas component:

$$\frac{\partial}{\partial t} [(\bar{\rho}_o \bar{Y}_g \bar{S}_o + \bar{\rho}_g \bar{S}_g) \bar{\phi} \Omega] = \rho_g^{ST} (Q_{gi}^{ST} - Q_g^{ST}). \quad (19)$$

Usual definitions from petroleum engineering can be introduced for the description of volumetric and solubility changes. For volumetric changes,  $B_w$ ,  $B_o$  and  $B_g$ , the formation volume factors of water, oil and gas phases, respectively, express the ratio of a phase volume at reservoir condition to its volume at standard surface condition, i.e.:

$$B_w = \frac{\rho_w^{ST}}{\bar{\rho}_w}, \quad (20)$$

$$B_o = \frac{\rho_o^{ST}}{\bar{Y}_o \bar{\rho}_o}, \quad (21)$$

$$B_g = \frac{\rho_g^{ST}}{\bar{\rho}_g}. \quad (22)$$

For changes in gas solubility in oil phase, the solubility ratio ( $R_s$ ) gives the ratio between the volume of dissolved gas and the volume of oil both measured at surface condition:

$$R_s = \frac{\rho_o^{ST} / \bar{Y}_o}{\rho_g^{ST} / \bar{Y}_g}, \quad (23)$$

or, using  $\bar{Y}_o + \bar{Y}_g = 1$ :

$$R_s = \frac{\rho_o^{ST}}{\rho_g^{ST}} \frac{\bar{Y}_g}{1 - \bar{Y}_g}. \quad (24)$$

Equations can now be rewritten using the above definitions and denoting porous volume by  $V_p = \bar{\phi} \Omega$ .

$$\frac{\partial}{\partial t} \left( \frac{\bar{S}_w V_p}{B_w} \right) = Q_{we}^{ST} + Q_{wi}^{ST} - Q_w^{ST} \quad (25)$$

$$\frac{\partial}{\partial t} \left( \frac{\bar{S}_o V_p}{B_o} \right) = -Q_o^{ST}, \quad (26)$$

$$\frac{\partial}{\partial t} \left[ V_p \left( \frac{\bar{S}_g}{B_g} + \frac{\bar{S}_o R_s}{B_o} \right) \right] = (Q_{gi}^{ST} - Q_g^{ST}), \quad (27)$$

$$\bar{S}_w + \bar{S}_o + \bar{S}_g = 1. \quad (28)$$

Integrating Eqs. (25), (26) and (27) between times  $t_1$  and  $t_2$  gives:

$$\frac{V_{p2} \bar{S}_{w2}}{B_{w2}} = W + (W_i + W_e - W_p), \quad (29)$$

$$\frac{\bar{S}_{o2} V_{p2}}{B_{o2}} = N - N_p, \quad (30)$$

$$V_{p2} \left( \frac{\bar{S}_{g2}}{B_{g2}} + \frac{\bar{S}_{o2} R_{s2}}{B_{o2}} \right) = N \left( m \frac{B_{o1}}{B_{g1}} + R_{s1} \right) + (G_i - G_p), \quad (31)$$

$$\bar{S}_o + \bar{S}_w + \bar{S}_g = 1. \quad (32)$$

Where  $N = V_{p1} \bar{S}_{o1} / B_{o1}$  and  $W = V_{p1} \bar{S}_{w1} / B_{w1}$  are the initial volumes of oil and water measured at surface condition and  $m = V_{p1} \bar{S}_{g1} / B_{o1} N$  is the ratio of the initial volumes of gas and oil phases measured at reservoir condition.

An implicit equation for the bubble-point can be obtained by setting  $\bar{S}_{g2} = 0$  in Eq. (31) and solving it for the solubility ratio at bubble-point pressure,  $R_s(p = P_b)$ :

$$R_{s2}(p = P_b) = \left[ N (m B_{11} / B_{g1} + R_{s1}) + G_i - G_p \right] / (N - N_p). \quad (33)$$

### 2.2.2. Material balance equation

It is now possible to obtain the generalized material balance equation by substituting the equations for phase saturation, Eqs. (29), (30) and (31), into Eq. (32):

$$F_p - F_i = \underbrace{N E_o + m N E_g + W E_w}_{\text{fluid volume expansion (E)}} + \underbrace{V_{p1} - V_{p2}}_{\text{pore volume compaction (C)}} \quad (34)$$

or

$$F_p - F_i = E + C, \quad (35)$$

where new terms  $C$ ,  $E$ ,  $F_i$  and  $F_p$  are defined in Table 1.

Eq (34) involves fluid properties, original phase volumes ( $N$ ,  $W$  and  $G$ ), cumulative production and injection and aquifer influx. It is an algebraic equation that can be solved for any one of its terms, provided that all remaining are known. This work focuses on its solution for pressure.

### 2.2.3. Relationship between the material balance equation and reservoir pressure

The material balance equation is mostly used to relate production and injection to reservoir pressure. In fact, all averaged phase properties in Eq. (34) depend on the pressure field. If the reservoir is under production, though, the pressure field is not known and for its prediction one would have to resort to more advanced techniques such as reservoir simulation. Further, even if the pressure field were known, one would still need to average fluid properties in space in order to apply Eq. (34). In practice, the reservoir engineer is unlikely to perform any averaging of properties since the main reason for using the material balance equation is to avoid the complexities of a 3D reservoir model. It can be shown that the averaging of fluid properties can be neglected under the following assumptions:

- Negligible hydrodynamic gradients and capillarity. This implies that pressure at any time is only a function of depth:  $\nabla p = \partial p / \partial z = \rho_o \vec{g}$ .
- Phase densities and gas solubility ratio are approximately constant with depth.

To show that phase densities can be considered approximately constant with depth, a first order Taylor expansion

$$\rho_o(z) \approx \rho_o(z_{\text{datum}}) + \left( \frac{\partial \rho_o}{\partial Y_o} \frac{\partial Y_o}{\partial z} + \frac{\partial \rho_o}{\partial p_o} \frac{\partial p_o}{\partial z} \right) dz \quad (36)$$

It will be shown that the pressure field can be further simplified by neglecting the variation of density with depth, what allows us to express the pressure field in the following simple expression:

$$p_\alpha(t, z) = p_\alpha(t, z_{\text{datum}}) + \rho_\alpha(t, z_{\text{datum}})g(z - z_{\text{datum}}). \quad (37)$$

$$\rho_o = \rho_o(p_o, Y_o) \quad (38)$$

For an undersaturated oil,  $\rho_o$  is a function of pressure and the amount of dissolved gas:

$$\frac{1}{\rho_o} \left( \frac{\partial \rho_o}{\partial z} \right)_T = \frac{1}{\rho_o} \left( \frac{\partial \rho_o}{\partial p_o} \frac{\partial p_o}{\partial z} + \frac{\partial \rho_o}{\partial Y_o} \frac{\partial Y_o}{\partial z} \right). \quad (39)$$

Neglecting any compositional gradient,  $\partial Y_o / \partial z = 0$ , and using the hydrostatic gradient:

$$\frac{1}{\rho_o} \left( \frac{\partial \rho_o}{\partial z} \right)_T = c_o \rho_o g, \quad (40)$$

where  $c_o$  is the isothermal compressibility.

For a saturated oil,

$$\frac{1}{\rho_o} \left( \frac{\partial \rho_o}{\partial z} \right)_T = g \left( \frac{\partial \rho_o}{\partial p_o} \right)_T = g \frac{d}{dp_o} \left( \frac{\rho_o^{ST} + R_s \rho_g^{ST}}{B_o} \right) \quad (41)$$

Similarly for the gas phase, using the state equation  $pV = ZnRT$ , we obtain its isothermal compressibility:

$$\frac{1}{\rho_g} \left( \frac{\partial \rho_g}{\partial p_g} \right)_T = \frac{1}{p_g} - \frac{1}{Z} \frac{\partial Z}{\partial p_g}. \quad (42)$$

Assuming reservoir temperature to be sufficiently above gas phase critical temperature, we have a negligible variation of the Z-factor with pressure,  $\partial Z / \partial p_g \approx 0$ . We use hydrostatic equilibrium again and obtain the derivative of gas density with depth:

$$\frac{1}{\rho_g} \left( \frac{\partial \rho_g}{\partial z} \right)_T = \frac{\rho_g \vec{g}}{p_g} = \left( \frac{\mathcal{M}}{ZRT} \right) g. \quad (43)$$

To investigate the order of magnitude of Eqs. (??) and (43), we substitute into them the values of  $\rho_o = 10^3 \text{ kg/m}^3$  and  $c_o = 10^{-9} \text{ Pa}^{-1}$  for the oil phase, and  $\mathcal{M} = 19 \text{ kg/kmol}$  and  $T = 400 \text{ K}$  for the gas phase:

$$\frac{1}{\rho_o} \left( \frac{\partial \rho_o}{\partial z} \right)_{T, Y_o} \approx 1 \times 10^{-5} \text{ m}^{-1}, \quad (44)$$

$$\frac{1}{\rho_g} \left( \frac{\partial \rho_g}{\partial z} \right)_{T, Z} \approx 5 \times 10^{-5} \text{ m}^{-1}. \quad (45)$$

Oil and gas densities are thus approximately constant with depth and their average values can be approximated to the local values at the datum level. The pressure calculated in the material balance equation thus corresponds to the local pressure at  $z_{\text{datum}}$ .

For the case of an oil reservoir with a gas cap, it is more accurate to define the datum level at the gas-oil contact. The point is that a higher gas density will be computed for a datum level bellow gas-oil contact because of the higher hydrostatic gradient of oil phase,

$$\frac{1}{\rho_g} \left( \frac{\partial \rho_g}{\partial z} \right)_{T, Z} = \frac{\rho_o}{\rho_g} \left( \frac{\mathcal{M}}{ZRT} \right) g, \quad (46)$$

what is  $\rho_o / \rho_g$  times greater than the gas hydrostatic gradient used in Eq. (43).

It was shown that the pressure variation with depth can be neglected for the computation of gas, oil and water densities. This is true provided that phase changes do not occur.

for what the pressure variation with depth may not be negligible in the solubility ratio:

$$\frac{1}{R_s} \frac{dR_s}{dz} = \frac{1}{R_s} \frac{dR_s}{dp_o} \frac{dp_o}{dz} = \frac{\rho_o g}{R_s} \frac{dR_s}{dp_o}. \quad (47)$$

For a non-volatile oil, we can approximate  $dR_s/dp_o$  to the ratio between the initial solubility ratio ( $R_{s,0}$ ) and saturation pressure ( $p_{sat}$ ), so that Eq. (47) becomes:

$$\frac{1}{R_s} \frac{dR_s}{dz} \approx \frac{\rho_o g}{p_{sat}} \approx 10^{-3} \text{ m}^{-1}, \quad (48)$$

where the following orders of magnitudes were assumed:  $p_{sat} \approx 10^7 \text{ Pa}$ ,  $\rho_o \approx 10^3 \text{ kg/m}^3$  and  $g \approx 10 \text{ m}^2/\text{s}$ .

From Eq. (48), we see that the variation of pressure with depth may not be negligible for the phase equilibrium, specially for thick reservoirs of low saturation pressure. In such cases, using an averaged value for the solubility ratio may be worth the accuracy gain in computing dissolved gas volume.

### 3. Solution of material balance equation

A numerical solution of Eq. (35) is presented here. We start by defining a residual function ( $\varepsilon$ ) for which the pressure solving Eq. (35) is a root:

$$\varepsilon(p_2) = \frac{E + F_i + C - F_p}{V_{p1}}. \quad (49)$$

We proceed showing that  $\varepsilon$  is a monotonic function provided some conditions on fluid properties are observed and that only one root of  $\varepsilon$  exists at most. Equivalently, we can say that only one solution exists for the material balance equation.

#### 3.1. Mathematical aspects of the residual function

It is a known result of real analysis that a monotonic function  $f : D \subset \mathcal{R} \rightarrow \mathcal{R}$  can have at most one root in  $D$ . Further, if  $f$  is continuous in the closed interval  $I = [a, b] \subset D$  and  $f(a)f(b) < 0$ , then  $f$  has one (and only one) root in  $I$ .

We now state without proof that the following affirmatives are sufficient conditions to classify a function  $f$  as monotonic:

1.  $f$  is continuous;
2.  $f$  is differentiable everywhere except for a countable set of points;
3.  $f' < 0$  everywhere.

The residual function clearly satisfies the first two affirmatives because phase properties are continuous everywhere and are also differentiable everywhere except at bubble-point pressure.

The third affirmative is proved by rewriting  $d\varepsilon/dp$  as a

sum of four negative terms:

$$\begin{aligned} V_{p1} \frac{d\varepsilon}{dp} = & \underbrace{(N - N_p) \left( \frac{dB_o}{dp} - B_g \frac{dR_s}{dp} \right)}_{\text{oil phase}} \\ & + \underbrace{\left[ NR_{s1} + mN \frac{B_{o1}}{B_{g1}} + G_i - (N - N_p) R_{s2} - G_p \right] \frac{dB_g}{dp}}_{\text{gas phase}} \\ & + \underbrace{(W + W_i + W_e - W_p) \frac{dB_w}{dp}}_{\text{water phase}} \\ & + \underbrace{\left( -\frac{dV_p}{dp} \right)}_{\text{porous volume}}. \end{aligned} \quad (50)$$

Each term in Eq. (50) corresponds to a different phase (three fluid phases and the solid rock). For the fluid phases, each term is the product of the current volume *in-situ* (a positive quantity) and a volume derivative with respect to pressure (a negative quantity). Since pore volume expands with an increase of fluid pressure,  $dV_p/dp$  is positive. The value of  $d\varepsilon/dp$  is thus necessarily negative.

In summary, the following four relations constitute sufficient conditions for  $\varepsilon(p)$  be a strictly decreasing function and, therefore, for material balance equation to have only one solution:

$$\left( \frac{dB_o}{dp} - B_g \frac{dR_s}{dp} \right) < 0, \quad (51)$$

$$\frac{dB_g}{dp} < 0, \quad (52)$$

$$\frac{dB_w}{dp} < 0, \quad (53)$$

$$\frac{dV_p}{dp} > 0. \quad (54)$$

Eq. (51) also provides a condition to check whether PVT data obtained from laboratory or empirical correlations is physically correct. It can be alternatively formulated in terms of densities as shown bellow in Eq. (55). It simply indicates that any vaporization must be followed by an increase in oil phase density:

$$\frac{1}{\rho_o} \frac{d\rho_o}{dY_o} > Y_o \left( \frac{\rho_o}{\rho_g} - 1 \right). \quad (55)$$

#### 3.2. Solution algorithm

Several methods exist to find roots of nonlinear continuous functions (Hamming, 1987). One possible choice is Newton's method with a switch from the tangent to the secant line near the non-differentiable bubble-point pressure. Both methods are available in the open source SciPy library (Jones et al., 2001–). This work used the library version 0.12.

The steps to solve the material balance equation are summarized bellow:

**Step 1.** Compute bubble-point pressure for instant  $t_2$  using Eq. (33).

Bubble-point pressure  $P_{b2}$  is only a function of values at instant  $t_1$  and produced and injected volumes.

**Step 2.** Compute pressure for instant  $t_2$  by finding the root of residual function  $\varepsilon(p)$ .

Pressure in Eq. (34) is only a function of fluid properties in instant  $t_1$ , produced and injected values and bubble-point pressure at instant  $t_2$ ,  $P_{b2}$ , which was already computed in step 1.

**Step 3.** Compute fluid saturations for instant  $t_2$ .

Once  $S_{o2}$ ,  $S_{g2}$ ,  $S_{w2}$ ,  $p_2$  and  $P_{b2}$  are known, solution for instant  $t_2$  is complete.

## 4. Results

To illustrate the numerical procedure, the material balance equation was solved for a fictitious black-oil reservoir initially producing in a pure depletion drive mechanism and later subjected to water injection. Fig. 1 shows in the solid line curve the severe depletion during initial phases of production. In the dashed curve, Fig. 1 shows the bubble-point pressure, indicating how far the saturated reservoir is from undersaturation and allowing for planning of pressure maintenance projects.

Water injection starts when oil recovery factor ( $RF$ ) is  $RF = 0.08$  and it is maintained until the reservoir reaches a novel undersaturated state. Initially, the high compressibility of gas prevents pressure from raising and free gas phase is reduced because of redissolution. Finally, the reservoir reaches the undersaturated state and a steep pressure increase is seen in Fig. 1 for a recovery factor about  $RF = 0.15$ .

It must be observed that the result of gas reentering oil phase completely neglects non-equilibrium effects, e.g. compositional gradients and diffusion of matter, and must be used with care.

Figure 4 shows the formation of a free gas phase with growing  $S_g$  and then its disappearance. Depending on the ratio of vertical and horizontal permeabilities or reservoir dip angle, it may also suggest the formation of a secondary gas cap.

Figure 5 shows the shape of residual function ( $\epsilon$ ) when pressure is varying and all remaining terms are held constant. Each different solid line corresponds to a different instant of time and, therefore, to different volumes of cumulative production and injection. The function root is the solution of material balance equation, Eq. (35), and the non-differentiable point is the bubble-point pressure. Also, it is seen that function  $\epsilon$  monotonically decreases with pressure, as theoretically shown before.

Figure 6 shows the count of newtonian iterations required to compute pressure with an accuracy within 0.1%

of the initial bubble-point pressure. Typically, 12 iterations are required.

Figure 7 shows the residual of numerical solution as defined in Eq. (49). Material balance error is less than  $10^{-7}$  units of porous volume for all time steps.

## 5. Conclusions

1. Material balance equation can be obtained from black-oil equations under the assumptions of:
  - (a) mechanical equilibrium;
  - (b) absence of gravity and capillary forces;
  - (c) spatially constant bubble-point pressure or, equivalently, fluid saturations.
2. One unique solution of pressure equation exists for a compressible system;
3. A residual function can be defined such that its root solves pressure equation. A numerical solution can be obtained with nonlinear root-finding algorithms.

## Nomenclature

|       |   |
|-------|---|
| $B_g$ | gas formation volume factor   |
| $B_o$ | oil formation volume factor   |
| $B_w$ | water formation volume factor   |
| $R_s$ | solubility ratio of gas in oil phase  |
| $S_g$ | gas saturation  |
| $S_o$ | oil saturation  |
| $S_w$ | water saturation  |
| $G_p$ | produced volume of gas  |
| $m$   | ratio of original free gas volume to original oil volume at reservoir condition |
| $N$   | original volume of oil in place   |
| $N_p$ | produced volume of oil  |
| $P_b$ | bubble-point pressure   |
| $V_p$ | porous volume   |
| $W_i$ | injected volume of water  |
| $W_p$ | produced volume of water  |

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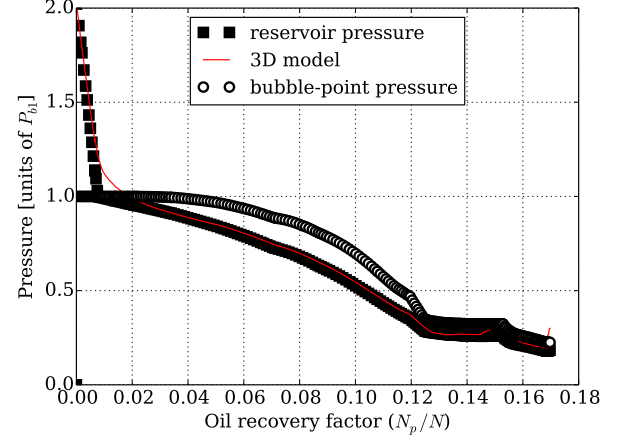


Figure 1: Reservoir pressure and bubble-point pressure as functions of recovery factor. Once depletion goes below bubble-point, pressure is hardly recovered for a saturated reservoir because of high system compressibility. Pressure is here given in units of the original bubble-point pressure.

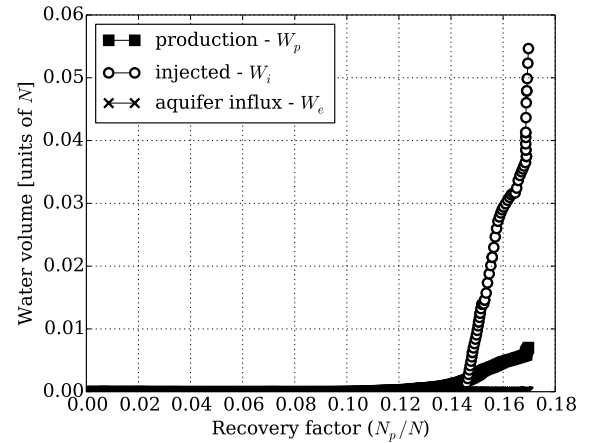


Figure 2: Volume of water production and injection and aquifer influx.

|                                  |  |
|----------------------------------|--|
| <i>Material balance equation</i> |  |
| pressure                         | $F_p - F_i = NE_o + mNE_g + WE_w + C$  |
| <i>Phase saturations</i>         |  |
| oil                              | $S_{o2} = (N - N_p) B_{o2}/V_{p2}$   |
| water                            | $S_{w2} = (W + W_i + W_e - W_p) B_{w2}/V_{p2}$   |
| gas                              | $S_{g2} = \{N [(R_{s1} - R_{s2}) + mB_{o1}/B_{g1}] + N_p R_{s2} + G_i - G_p\} B_{g2}/V_{p2}$ |
| <i>List of terms</i>             |  |
| fluid production                 | $F_p = N_p [B_{o2} + (G_p/N_p - R_{s2}) B_{g2}] + W_p B_{w2}$                                |
| fluid injection and influx       | $F_i = (W_i + W_e) B_{w2} + G_i B_{g2}$  |
| fluid expansion                  | $E = NE_o + mNE_g + WE_w$  |
| oil phase expansion              | $E_o = (B_{o2} - B_{o1}) - (R_{so2} - R_{so1}) B_{g2}$                                       |
| gas phase expansion              | $E_g = B_{o1} (B_{g2}/B_{g1} - 1)$   |
| water volume expansion           | $E_w = B_{w2} - B_{w1}$  |
| porous volume compaction         | $C = V_{p1} - V_{p2}$  |
| bubble-point pressure            | $R_{s2}(P_b) = [N(mB_1/B_{g1} + R_{s1}) + G_i - G_p] / (N - N_p)$                            |

Table 1: Complete set of material balance equations.

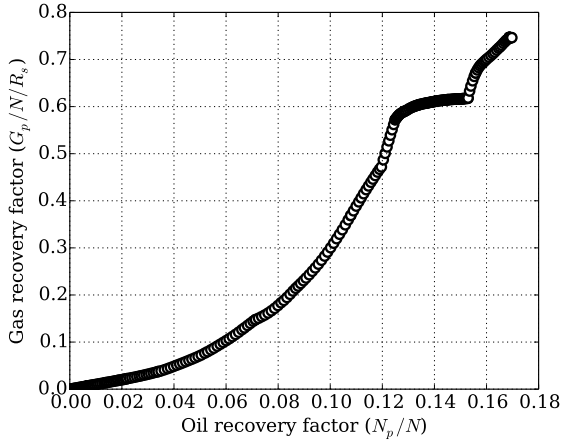


Figure 3: Volume of gas production.

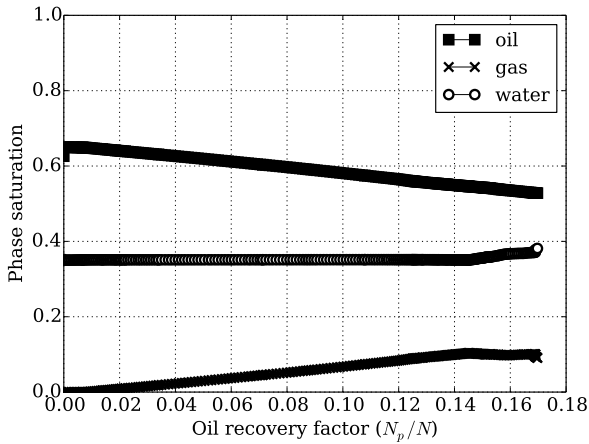


Figure 4: Phase saturations. Gas saturation is reduced as water injection progresses.

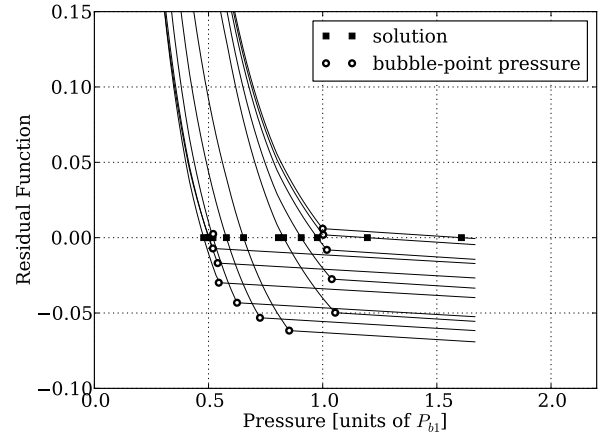


Figure 5: Shape of residual function ( $\epsilon$ ). Each curve corresponds to different volumes of production and injection. Residual function monotonically decreases with pressure and it has no derivative at the bubble-point.



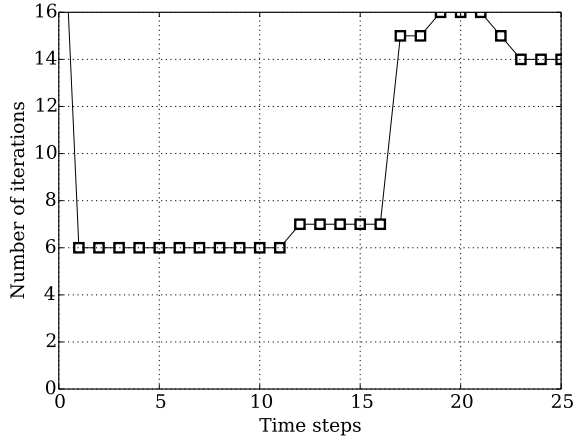


Figure 6: Count of newtonian iterations required to compute pressure with accuracy within 0.001 units of initial bubble-point presure.

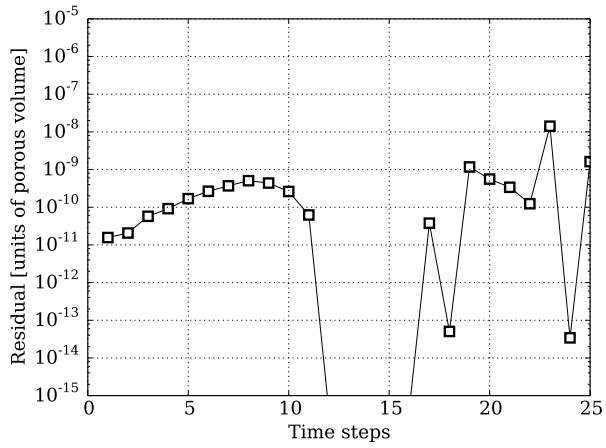


Figure 7: Residual of numerical solution in units of porous volume.