

# Solution of hydrodynamic equilibrium in black-oil reservoirs: the classical material balance equation

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## Abstract

This work examines the underlying details in obtaining the classical material balance equation from a three dimensional black-oil model typical of reservoir simulations. The three-dimensional model is integrated in reservoir volume and quantities such as fluid properties are spatially averaged. The material balance equation is obtained for the averaged quantities and it is applied to calculate reservoir pressure. It is shown that the calculated pressure corresponds to the local pressure at a defined depth and not to the average reservoir pressure in general. The definition of a convenient reference depth is discussed. Sufficient conditions for solution uniqueness of the material balance equation are provided for the case that pressure is the unknown quantity and one of such conditions is shown to check for physical correctness of PVT properties. Finally, a scheme for solving the material balance equation numerically is proposed.

*Keywords:* material balance, black-oil equations, reservoir engineering.

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## 1. Introduction

The material balance equation is a classical tool in petroleum reservoir engineering. It consists of the algebraic equation originally presented in Schilthuis (1936) and conceived as a zero dimensional model for estimating reservoir pressure from mass balance. Schilthuis (1936) introduced the

hypothesis of equilibrium attained at all times not necessarily in the entire reservoir volume, but in the regions of high permeability where most of the production would come from and where pressure would present a similar behavior. Such regions were said to contain an active oil.

The equation developed by Schilthuis (1936) did not account for the non-existence of mechanical, phase or thermal equilibrium. An attempt to introduce non-equilibrium effects of hydrodynamic origin came with *black-oil* equations (Aziz and Settari, 1979; Trangenstein and Bell, 1989b). The novelty on hydrodynamics was the introduction of a macroscopic equation of motion using Darcy’s law.

The term black-oil in this context refers to specific hydrocarbon compositions to which phase equilibrium for isothermal changes in pressure can be described by a two pseudo-component system. For such reservoir fluids, gas phase composition remains essentially constant in the range of pressures observed during production and a pseudo gas component can thus be defined. Similarly, a pseudo oil component can also be defined by subtracting all the pseudo gas component from oil phase composition. After defining these two pseudo-components, the number of transport equations needed for hydrocarbon components is reduced from a usually large number to only two.

A great number of applications of material balance equation have been reported in the literature, but few of them explore its connection with black-oil equations. Ertekin et al. (2001) briefly addressed this subject arguing both approaches to be equivalent if capillarity and gradients of flow potential ( $\nabla\Phi = \nabla p - \rho\vec{g}$ ) were neglected and if PVT properties were averaged in reservoir volume. This work develops this process examining underlying details.

The same approach of varying bubble-point pressure widely used in reservoir simulation, which is found in Aziz and Settari (1979) and Ertekin et al. (2001), is also applied here to material balance. This approach extends the material balance equation to saturated reservoirs under pressure maintenance (e.g. fluid injection). The applicability, however, depends on how important non-equilibrium effects are to phase equilibrium.

After presenting material balance equation, sufficient conditions are shown for which material balance equation has only one solution for pressure. These conditions require fluids to be compressible and that gas vaporization occurs with reduction of oil phase volume. A simple scheme for numerical solution is also proposed based on classical root-finding algorithms for non-linear equations.

## 2. Derivation of equations

This section presents black-oil equations, performs averaging of quantities in space and examines simplifying assumptions necessary to obtain the material balance equation.

### 2.1. Black-oil equations

We assume that the reservoir is a connected volume where hydrocarbon saturation is nonzero. Reservoir fluids are composed of oil and gas pseudo-components and water component. There can be three fluid phases: oil, gas and (liquid) water. Oil phase is composed of oil and gas components, gas phase is composed uniquely by gas component and, water phase, uniquely by water component. The only allowed mass transfer is of gas component between oil and gas phases. Gas phase composition is always that of a dry gas.

Mass balance equation for each component reads:

- water component:

$$\frac{\partial}{\partial t} (\rho_w S_w \phi) + \nabla \cdot (\rho_w \vec{u}_w) = 0; \quad (1)$$

- oil component:

$$\frac{\partial}{\partial t} (\rho_o Y_o S_o \phi) + \nabla \cdot (\rho_o Y_o \vec{u}_o) = 0; \quad (2)$$

- gas component:

$$\frac{\partial}{\partial t} (\rho_o Y_g S_o + \rho_g S_g) + \nabla \cdot (\rho_o Y_g \vec{u}_o + \rho_g \vec{u}_g) = 0. \quad (3)$$

Mass concentrations of oil and gas components in oil phase are denoted by  $Y_o$  and  $Y_g$ , respectively.

Initial conditions are of no fluid motion. Boundary conditions are of no normal flow at all boundaries except for interfaces with wells or aquifers.

A common simplification in black-oil formulation is to restrict mass transport to macroscopic advection. No diffusive or dispersive term is included in mass balance equations.

Momentum balance for each phase is given by an extension of Darcy's law to multiphase flow. This extension consists in multiplying absolute (or single-phase) permeability ( $\vec{k}$ ) by a scalar function called phase relative permeability ( $k_{r,\alpha}$ ):

$$\vec{u}_\alpha = - \frac{k_{r,\alpha} \vec{k}}{\mu_\alpha} \cdot (\nabla p_\alpha - \rho_\alpha \vec{g}), \quad \alpha = o, w, g. \quad (4)$$

Phase densities are functions of the local thermodynamic state:

$$\rho_w = \rho_w(p_w, T_w) \quad (5)$$

$$\rho_o = \rho_o(p_o, T_o, Y_g) \quad (6)$$

$$\rho_g = \rho_g(p_g, T_g) \quad (7)$$

Another simplification is now made assuming thermal equilibrium in the entire reservoir volume, i.e.,

$$T_w = T_o = T_g = T_{rock} = T = \text{constant} . \quad (8)$$

Temperature is a constant prescribed value and thus no equation for energy balance is needed. It follows from Gibbs' phase rule that the equilibrium of oil and gas phases has only two degrees of freedom if the component number is restricted to the two pseudo-components. As temperature is assumed constant, oil phase density is a function of pressure alone. The same is valid for mass fractions of gas and oil components in oil phase:

$$Y_g = Y_g(p_o) , \quad (9)$$

$$Y_o = Y_o(p_o) . \quad (10)$$

Porosity ( $\phi$ ) is assumed a function of pore pressure ( $p_P$ ). Because three phases are present in general, such quantity is defined by some relation of the type (Kim et al., 2011):

$$p_P = p_P(p_o, p_w, p_g, S_o, S_w, S_g) . \quad (11)$$

To close the system of equations, a connection among different phase pressures is provided by capillary pressure relations:

$$p_{c,ow} \left( \vec{k}, \phi, S_w \right) = p_o - p_w , \quad (12)$$

$$p_{c,go} \left( \vec{k}, \phi, S_g \right) = p_g - p_o . \quad (13)$$

## 2.2. Material balance equation

Derivation of the material balance equation consists in integrating black-oil equations in reservoir volume.

Integrating Eqs. (1), (2) and (3) in volume and applying divergence theorem gives:

- water component:

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\Omega} \rho_w S_w \phi dV &= - \oint_{\partial\Omega} \rho_w (\vec{u}_w \cdot d\vec{S}) \\ &= \rho_w^{ST} (Q_{we}^{ST} + Q_{wi}^{ST} - Q_w^{ST}) ; \end{aligned} \quad (14)$$

- oil component:

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\Omega} \rho_o Y_o S_o \phi dV &= - \oint_{\partial\Omega} \rho_o Y_o (\vec{u}_o \cdot d\vec{S}) \\ &= \rho_o^{ST} Q_o^{ST} ; \end{aligned} \quad (15)$$

- gas component:

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\Omega} (\rho_o Y_g S_o + \rho_g S_g) \phi dV &= - \oint_{\partial\Omega} (\rho_o Y_g \vec{u}_o + \rho_g \vec{u}_g) \cdot d\vec{S} \\ &= \rho_g^{ST} (Q_{gi}^{ST} - Q_g^{ST}) . \end{aligned} \quad (16)$$

### 2.2.1. Averaging quantities

Spatially averaged quantities are defined for porosity ( $\phi$ ), saturation ( $S_\alpha$ ), density ( $\rho_\alpha$ ) and mass concentration ( $Y_{\alpha,k}$ ) of a component  $k$  in phase  $\alpha$ .

All quantities are initially functions of space and time, but they become functions of time only after being averaged in space:

$$\bar{\phi}(t) = \frac{1}{\Omega} \int_{\Omega} \phi(\vec{x}, t) dV . \quad (17)$$

We proceed omitting function arguments:

$$\bar{S}_\alpha \bar{\phi} = \frac{1}{\Omega} \int_{\Omega} S_\alpha \phi dV, \quad (18)$$

$$\bar{\rho}_\alpha \bar{S}_\alpha \bar{\phi} = \frac{1}{\Omega} \int_{\Omega} \rho_\alpha S_\alpha \phi dV, \quad (19)$$

$$\bar{\rho}_\alpha \bar{Y}_{\alpha,k} \bar{S}_\alpha \bar{\phi} = \frac{1}{\Omega} \int_{\Omega} \rho_\alpha Y_{\alpha,k} S_\alpha \phi dV. \quad (20)$$

Eqs. (14), (15) and (16) are rewritten in terms of averaged quantities:

- water component:

$$\frac{\partial}{\partial t} (\bar{\rho}_w \bar{S}_w \bar{\phi} \Omega) = \rho_w^{ST} (Q_{we}^{ST} + Q_{wi}^{ST} - Q_w^{ST}) \quad (21)$$

- oil component:

$$\frac{\partial}{\partial t} (\bar{\rho}_o \bar{Y}_o \bar{S}_o \bar{\phi} \Omega) = \rho_o^{ST} Q_o^{ST} \quad (22)$$

- gas component:

$$\frac{\partial}{\partial t} [(\bar{\rho}_o \bar{Y}_g \bar{S}_o + \bar{\rho}_g \bar{S}_g) \bar{\phi} \Omega] = \rho_g^{ST} (Q_{gi}^{ST} - Q_g^{ST}). \quad (23)$$

We now introduce usual definitions from petroleum engineering for the description of volumetric and solubility changes in fluid phases. In what concerns volumetric changes,  $B_w$ ,  $B_o$  and  $B_g$ , the formation volume factors of water, oil and gas phases, respectively, express the ratio of a phase volume at reservoir condition to its volume at surface condition, i.e.:

$$B_w = \frac{\rho_w^{ST}}{\bar{\rho}_w}, \quad (24)$$

$$B_o = \frac{\rho_o^{ST}}{\bar{Y}_o \bar{\rho}_o}, \quad (25)$$

$$B_g = \frac{\rho_g^{ST}}{\bar{\rho}_g} . \quad (26)$$

In what concerns changes in gas solubility in oil phase, the solubility ratio ( $R_s$ ) gives the ratio between the volume of dissolved gas and the volume of oil both measured in surface condition:

$$R_s = \frac{\rho_o^{ST}/\bar{Y}_o}{\rho_g^{ST}/\bar{Y}_g} , \quad (27)$$

or using  $\bar{Y}_o + \bar{Y}_g = 1$ :

$$R_s = \frac{\rho_o^{ST}}{\rho_g^{ST}} \frac{\bar{Y}_g}{1 - \bar{Y}_g} . \quad (28)$$

Equations can now be rewritten using the above definitions and denoting porous volume by  $V_p = \bar{\phi}\Omega$ .

$$\frac{\partial}{\partial t} \left( \frac{\bar{S}_w V_p}{B_w} \right) = Q_{we}^{ST} + Q_{wi}^{ST} - Q_w^{ST} \quad (29)$$

$$\frac{\partial}{\partial t} \left( \frac{\bar{S}_o V_p}{B_o} \right) = Q_o^{ST} , \quad (30)$$

$$\frac{\partial}{\partial t} \left[ V_p \left( \frac{\bar{S}_g}{B_g} + \frac{\bar{S}_o R_s}{B_o} \right) \right] = (Q_{gi}^{ST} - Q_g^{ST}) , \quad (31)$$

$$\bar{S}_w + \bar{S}_o + \bar{S}_g = 1 . \quad (32)$$

Integrating Eqs. (29), (30) and (31) between times  $t_1$  and  $t_2$  gives:

$$\frac{V_{p2} \bar{S}_{w2}}{B_{w2}} = W + (W_i + W_e - W_p) , \quad (33)$$

$$\frac{\bar{S}_{o2} V_{p2}}{B_{o2}} = N - N_p , \quad (34)$$



$$V_{p2} \left( \frac{\bar{S}_{g2}}{B_{g2}} + \frac{\bar{S}_{o2} R_{s2}}{B_{o2}} \right) = N \left( m \frac{B_{o1}}{B_{g1}} + R_{s1} \right) + (G_i - G_p) , \quad (35)$$

$$\bar{S}_o + \bar{S}_w + \bar{S}_g = 1 . \quad (36)$$

Where  $N = V_{p1} \bar{S}_{o1} / B_{o1}$  and  $W = V_{p1} \bar{S}_{w1} / B_{w1}$  are the initial volumes of oil and water measured at surface condition and  $m = V_{p1} \bar{S}_{g1} / B_{o1} N$  is the ratio of the initial volumes of gas and oil phases measured in reservoir condition.

An equation for the bubble-point can be obtained by setting  $\bar{S}_{g2} = 0$  in Eq. (35) and solving it for the solubility ratio at bubble-point pressure, or  $R_s(p = P_b)$ :

$$R_{s2}(p = P_b) = \left[ N (m B_1 / B_{g1} + R_{s1}) + G_i - G_p \right] / (N - N_p) . \quad (37)$$

### 2.2.2. Material balance equation

It is now possible to obtain the so-called generalized material balance equation by substituting the equations for phase saturation, Eqs. (33), (34) and (35), into Eq. (36):

$$F_p - F_i = \underbrace{N E_o + m N E_g + W E_w}_{\text{fluid volume expansion (E)}} + \underbrace{V_{p1} - V_{p2}}_{\text{pore volume reduction (C)}} \quad (38)$$

or

$$F_p - F_i = E + C , \quad (39)$$

where new terms  $C$ ,  $E$ ,  $F_i$  and  $F_p$  are defined in Table 1.

Eq (38) involves fluid properties, original fluid volumes ( $N$ ,  $W$  and  $G$ ), cumulative production and injection and aquifer influx. It is an algebraic equation that can be solved for any one of its terms, provided that all remaining are known. This work focuses on its solution for pressure.

### 2.2.3. Relationship between the material balance equation and the reservoir pressure

The material balance equation is widely used to relate production, injection and reservoir pressure. Actually, all averaged fluid properties in Eq. (38) depend on the pressure field, and so does the pore volume. If the reservoir is under production, though, the pressure field is not known and for its prediction one would have to resort to more advanced techniques such as reservoir simulation.

To overcome this difficulty, a possible simplification to the pressure field is to neglect hydrodynamic gradients and assume hydrostatic equilibrium:  $\vec{u}_\alpha \equiv 0$ . This implies that  $\nabla p_\alpha \equiv \rho_\alpha \vec{g}$ , and, as a consequence, pressure becomes a function of time and depth only given by the following expression:

$$\int_{p_\alpha^{ref}(t)}^{p_\alpha(t,z)} \frac{dp'_\alpha}{\rho_\alpha(p'_\alpha)} = g(z - z_{ref}) , \quad (40)$$

where  $p_\alpha(t, z)$  is the pressure field of phase  $\alpha$  and  $p_\alpha^{ref}(t)$  is its pressure at some reference depth  $z_{ref}$ . It is now possible to compute the pressure field from Eq. (40) and average fluid properties accordingly.

Furthermore, it can be shown that neglecting the variation of fluid properties with depth does not introduce any important error. As pressure increases with depth according to the hydrostatic gradient, density also increases but by a negligible amount. For instance, the average oil density can be approximated to the local density at the reference level:  $\bar{\rho}_o(t) \approx \rho_o(t, z_{ref})$ ; the same applies to the formation volume factor:  $B_o \approx \rho_o(t, z_{ref})/\rho_o^{ST}$ .

This approximation can be quantified by putting together the expressions for compressibility and hydrostatic equilibrium. For this, we assume a

constant isothermal compressibility for oil phase,

$$\frac{1}{\rho_o} \left( \frac{\partial \rho_o}{\partial p_o} \right)_{T, Y_o} = c_o, \quad (41)$$

and we use hydrostatic equilibrium  $dp_o = \rho_o g dz$  to obtain the expression for the derivative of oil density with depth:

$$\frac{1}{\rho_o} \left( \frac{\partial \rho_o}{\partial z} \right)_{T, Y_o} = c_o \rho_o g. \quad (42)$$

Similarly for the gas phase, using the state equation  $pV = ZnRT$ , we obtain its isothermal compressibility:

$$\frac{1}{\rho_g} \left( \frac{\partial \rho_g}{\partial p_g} \right)_T = \frac{1}{p_g} - \frac{1}{Z} \frac{\partial Z}{\partial p_g}. \quad (43)$$

Neglecting the variation of the Z-factor and using hydrostatic equilibrium again, we obtain the derivative of gas density with depth as:

$$\frac{1}{\rho_g} \left( \frac{\partial \rho_g}{\partial z} \right)_T = \left( \frac{\mathcal{M}}{ZRT} \right) g. \quad (44)$$

For investigating the order of magnitude of Eqs. (42) and (44), we substitute in them the values of  $\rho_o = 10^3 \text{ kg/m}^3$  and  $c_o = 10^{-9} \text{ Pa}^{-1}$  for the oil phase, and  $\mathcal{M} = 19 \text{ kg/kmol}$  and  $T = 400 \text{ K}$  for the gas phase:

$$\frac{1}{\rho_o} \left( \frac{\partial \rho_o}{\partial z} \right)_{T, Y_o} \approx 1 \times 10^{-5} \text{ m}^{-1}, \quad (45)$$

$$\frac{1}{\rho_g} \left( \frac{\partial \rho_g}{\partial z} \right)_{T, Z} \approx 5 \times 10^{-5} \text{ m}^{-1}. \quad (46)$$

Oil and gas densities are thus approximately constant with depth. The reference depth for computing phase density (and also the formation volume factor) can be any arbitrary depth inside phase volume. Under the approximation of a constant density, it is straightforward to show that

$\bar{\rho}_\alpha = \rho_\alpha (\bar{p}_\alpha)$  and  $\bar{p}_\alpha = p_\alpha (z_\alpha^0)$  where  $z_\alpha^0$  is the depth of phase gravity center and  $\alpha = w, o, g$ . The pressure calculated in the material balance equation corresponds to the local pressure at  $z^{ref}$ , without fluid motion, and only equals the average reservoir pressure  $\bar{p}_\alpha$  if  $z^{ref} = z^0$ .

For the case of an oil reservoir with a gas cap, the reference depth can be defined, with advantage, at the gas-oil contact. The point here is that at phase contact both gas and oil densities are computed using the same pressure, i.e., the corresponding local value.

If the reference depth for oil phase is otherwise defined below gas-oil contact, gas density must be computed at some different depth still inside gas phase, i.e., at or above gas-oil contact. This is to avoid the effect of oil hydrostatic gradient in the computation of gas density,

$$\frac{1}{\rho_g} \left( \frac{\partial \rho_g}{\partial z} \right)_{T,Z} = \frac{\rho_o}{\rho_g} \left( \frac{\mathcal{M}}{ZRT} \right) g, \quad (47)$$

what is  $\rho_o/\rho_g$  times greater than the sole effect of gas hydrostatic pressure that was computed in Eq. (44).

It has been shown that pressure variation with depth is not important for the computation of gas, oil and water densities. Nevertheless, it was said that pressure also plays a second role in the material balance equation by defining phase equilibrium. For this purpose, pressure variation with depth may not be negligible in the evaluation of the solubility ratio:

$$\frac{1}{R_s} \frac{dR_s}{dz} = \frac{1}{R_s} \frac{dR_s}{dp_o} \frac{dp_o}{dz} = \frac{\rho_o g}{R_s} \frac{dR_s}{dp_o}. \quad (48)$$

For a non-volatile oil, we can approximate  $dR_s/dp_o$  to the ratio between the initial solubility ratio ( $R_{s,0}$ ) and saturation pressure ( $p_{sat}$ ), so that Eq. (48) becomes:

$$\frac{1}{R_s} \frac{dR_s}{dz} \approx \frac{\rho_o g}{p_{sat}} \approx 10^{-3} \text{ m}^{-1}, \quad (49)$$

where the following orders of magnitudes were assumed:  $p_{sat} \approx 10^7 \text{ Pa}$ ,  $\rho_o \approx 10^3 \text{ kg/m}^3$  and  $g \approx 10 \text{ m}^2/\text{s}$ .

From Eq. (49), we see that the variation of pressure with depth may not be negligible for the phase equilibrium, specially for thick reservoirs of light oil. In such cases, using an averaged value for the solubility ratio may be worth the accuracy gain in computing dissolved gas volume.

### 3. Solution of material balance equation

A numerical solution of Eq. (39) is presented here. We shall start defining a residual function ( $\varepsilon$ ) for which the pressure solving Eq. (39) is a root:

$$\varepsilon(p_2) = \frac{E + F_i + C - F_p}{V_{p1}}. \quad (50)$$

We proceed showing that  $\varepsilon$  is a monotonic function provided some conditions on fluid properties are observed and that, as a consequence, only one root of  $\varepsilon$  exists at most. Equivalently, we can say that only one solution exists for material balance equation.

#### 3.1. Mathematical aspects of the residual function

It is a known result of real analysis that a monotonic function  $f : D \subset \mathcal{R} \rightarrow \mathcal{R}$  can have at most one root in  $D$ . Further, if  $f$  is continuous in the closed interval  $I = [a, b] \subset D$  and  $f(a)f(b) < 0$ , then  $f$  has one (and only one) root in  $I$ .

We now state without proof that the following affirmatives are sufficient conditions to classify a function  $f$  as monotonic:

1.  $f$  is continuous;
2.  $f$  is differentiable everywhere except for a countable set of points;

3.  $f' < 0$  everywhere.

Residual function clearly satisfies the first two affirmatives because fluid properties are continuous everywhere and are also differentiable everywhere except for the first order phase change at bubble-point pressure. The third affirmative is proved by rewriting  $d\epsilon/dp$  as a sum of four parcels corresponding each parcel to a different phase. Each parcel is composed of two factors: current volume *in-situ* and volume derivative with respect to pressure:

$$\begin{aligned}
V_{p1} \frac{d\epsilon}{dp} = & \underbrace{(N - N_p) \left( \frac{dB_o}{dp} - B_g \frac{dR_s}{dp} \right)}_{\text{oil phase}} \\
& + \underbrace{\left[ NR_{s1} + mN \frac{B_{o1}}{B_{g1}} + G_i - (N - N_p) R_{s2} - G_p \right] \frac{dB_g}{dp}}_{\text{gas phase}} \\
& + \underbrace{(W + W_i + W_e - W_p) \frac{dB_w}{dp}}_{\text{water phase}} \\
& + \underbrace{- \frac{dV_p}{dp}}_{\text{porous volume}} .
\end{aligned} \tag{51}$$

*In-situ* volumes (e.g.  $N - N_p$  for oil phase) are non-negative values. Pressure derivatives of fluid volumes are non-positive values (fluid volume cannot increase as pressure increases) and pressure derivative of porous volume is a positive value (as fluid pressure increases, porous volume expands owing to rock compression).

The value of  $d\epsilon/dp$  is thus negative if at least one phase is compressible because it would be a sum of products between negative and positive factors.

In summary, the following four relations constitute sufficient conditions for  $\epsilon(p)$  be a strictly decreasing function and, therefore, for material balance

equation to have only one solution:

$$\left( \frac{dB_o}{dp} - B_g \frac{dR_s}{dp} \right) \leq 0, \quad (52)$$

$$\frac{dB_g}{dp} \leq 0, \quad (53)$$

$$\frac{dB_w}{dp} \leq 0, \quad (54)$$

$$\frac{dV_p}{dp} \geq 0. \quad (55)$$

Eq. (52) also provides a condition to check whether PVT data obtained from laboratory or empirical correlations is physically correct. It can be alternatively formulated as shown bellow in Eq. (56). It simply indicates that any vaporization must be followed by an increase in oil phase density:

$$\frac{1}{\rho_o} \frac{d\rho_o}{dY_o} \geq Y_o \left( \frac{\rho_o}{\rho_g} - 1 \right). \quad (56)$$

### 3.2. Solution algorithm

Several methods exist to find roots of nonlinear continuous functions (Hamming, 1987). One possible choice is Newton's method, which linearizes the function using either its gradient or a secant line. The secant line approach introduces a numerical smoothness to the gradient near continuous but non-differentiable points, for what it is the approach used here to deal with the bubble-point pressure.

An open source implementation of secant method is available in SciPy library (Jones et al., 2001–). This work used library version 0.12.

The steps to solve all material balance equation are summarized bellow:

**Step 1.** *Compute bubble-point pressure for instant  $t_2$ .*

Bubble-point pressure  $P_{b2}$  is only a function of the values in initial instant  $t_1$  and produced and injected volumes.

**Step 2.** *Compute fluid pressure for instant  $t_2$  finding the root of residual function  $(\varepsilon)$ .*

Pressure in Eq. (38) is a function only of fluid properties in instant  $t_1$ , produced and injected values and bubble-point pressure at instant  $t_2$ ,  $P_{b2}$ , which was already computed in step 1.

**Step 3.** *Compute fluid saturations  $t_2$ .*

Once  $S_{o2}$ ,  $S_{g2}$ ,  $S_{w2}$ ,  $p_2$  and  $P_{b2}$  are known, solution for instant  $t_2$  is complete.

#### 4. Results

Material balance equation were solved for a black-oil reservoir initially producing in depletion drive mechanism and later subjected to water injection. Fig. 1 shows in the solid line curve the severe depletion during initial phases of production. In the dashed curve, Fig. 1 shows the bubble-point pressure, indicating how far the saturated reservoir is from undersaturation and allowing for planning of pressure maintenance projects.

Water injection starts when oil recovery factor ( $RF$ ) is about  $RF = 0.08$  and it is maintained until the reservoir reaches a novel undersaturated state. Initially, the high compressibility of gas prevents pressure from raising and free gas phase is reduced because of redissolution. Finally, the reservoir reaches the undersaturated state and a steep pressure increase is seen in Fig. 1 for a recovery factor about  $RF = 0.15$ .

It must be observed that the result of gas reentering oil phase completely neglects non-equilibrium effects, e.g. compositional gradients, and must be



used with care. For instance, it may be in strong disagreement with real reservoir behavior when gravity segregation is present.

Figure 4 shows the formation of a free gas phase with growing  $S_g$  and then its disappearance. Depending on the ratio of vertical and horizontal permeabilities or reservoir dip angle, it may also suggest the formation of a secondary gas cap.

Figure 5 shows the shape of residual function ( $\epsilon$ ) with respect to pressure, i.e., holding all produced or injected volumes constant and allowing only pressure to change. Each different solid line corresponds to a different volume of fluid production and injection. The function root is the solution of material balance equation, Eq. (39), and the non-differentiable point is the bubble-point pressure. Also, it is seen that function  $\epsilon$  monotonically decreases with pressure, as theoretically shown before.

Figure 6 shows the count of newtonian iterations required to compute pressure with an accuracy within 0.001 of initial bubble-point pressure. Typically, 12 iterations are required.

Figure 7 shows the residual of numerical solution as defined in Eq. (50). Material balance error is less than  $10^{-7}$  units of porous volume for all time steps.

## 5. Conclusions

1. Material balance equation can be obtained from black-oil equations under the assumptions of:
  - (a) mechanical equilibrium;
  - (b) absence of gravity and capillary forces;
  - (c) spatially constant bubble-point pressure or, equivalently, fluid saturations.

2. One unique solution of pressure equation exists for a compressible system;
3. A residual function can be defined such that its root solves pressure equation. A numerical solution can be obtained with nonlinear root-finding algorithms.

### Nomenclature

$B_g$	gas formation volume factor
$B_o$	oil formation volume factor
$B_w$	water formation volume factor
$R_s$	solubility ratio of gas in oil phase
$S_g$	gas saturation
$S_o$	oil saturation
$S_w$	water saturation
$G_p$	produced volume of gas
$m$	ratio of original free gas volume to original oil volume in reservoir condition
$N$	original volume of oil in place
$N_p$	produced volume of oil
$P_b$	bubble-point pressure
$V_p$	porous volume
$W_i$	injected volume of water
$W_p$	produced volume of water

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## Highlights

1. The existing link between black-oil and material balance equation is examined in detail.
2. It is shown that gravity and capillary forces must be neglected and saturations must be constant in space so one can integrate black-oil equations in reservoir volume and obtain material balance equation.
3. Material balance equation are shown to have a unique solution for a compressible system.
4. Material balance equation are formulated for a variable bubble-point and they are used to a reservoir under pressure maintenance.

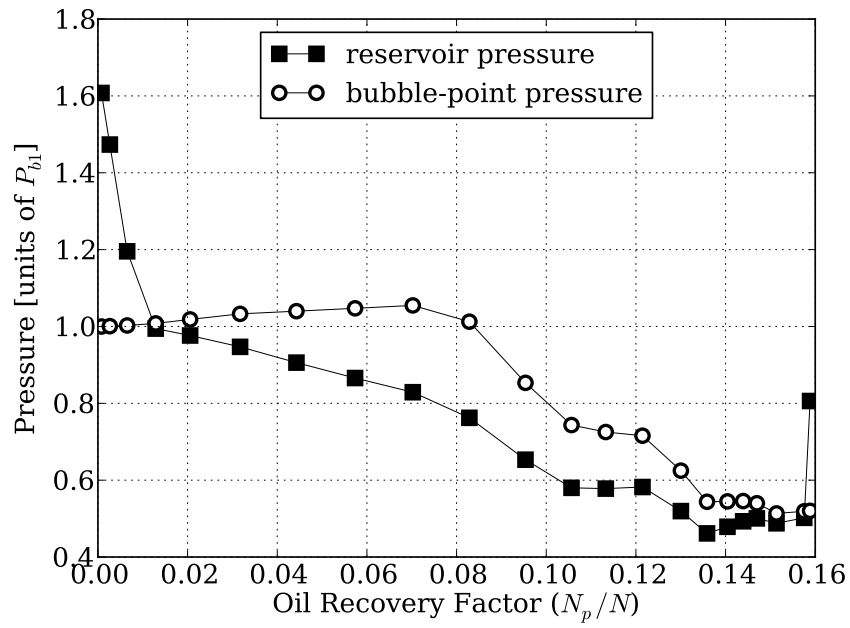


Figure 1: Reservoir pressure and bubble-point pressure as functions of recovery factor. Once depletion goes below bubble-point, pressure is hardly recovered for a saturated reservoir because of high system compressibility. Pressure is here given in units of the original bubble-point pressure.

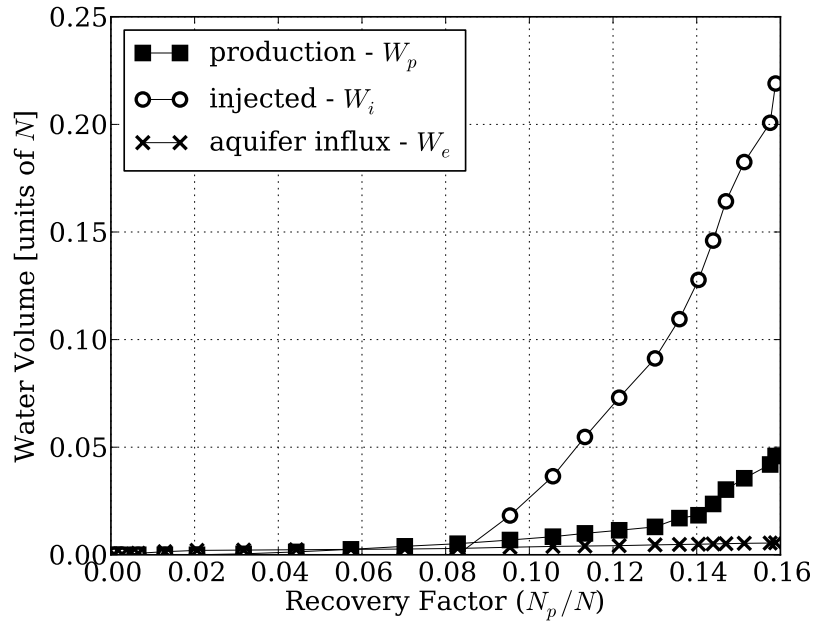


Figure 2: Volume of water production and injection and aquifer influx.

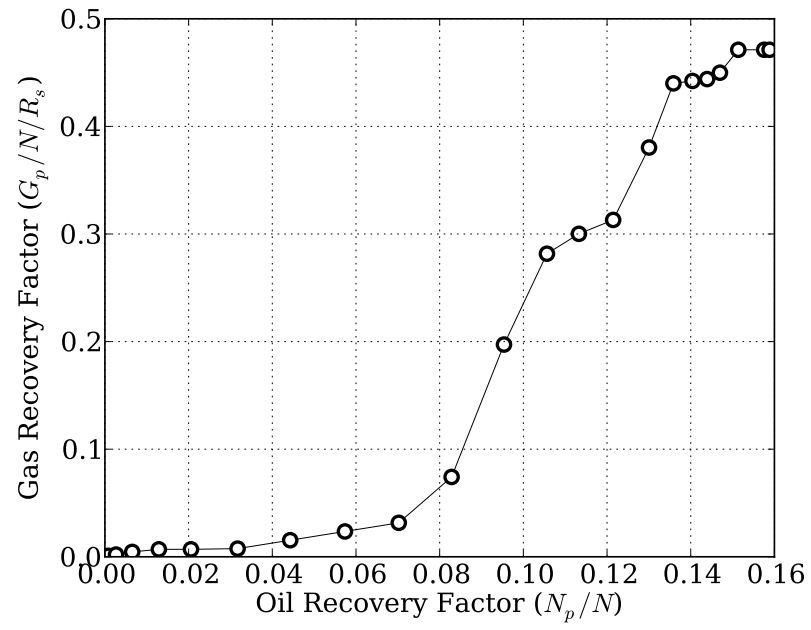


Figure 3: Volume of gas production.



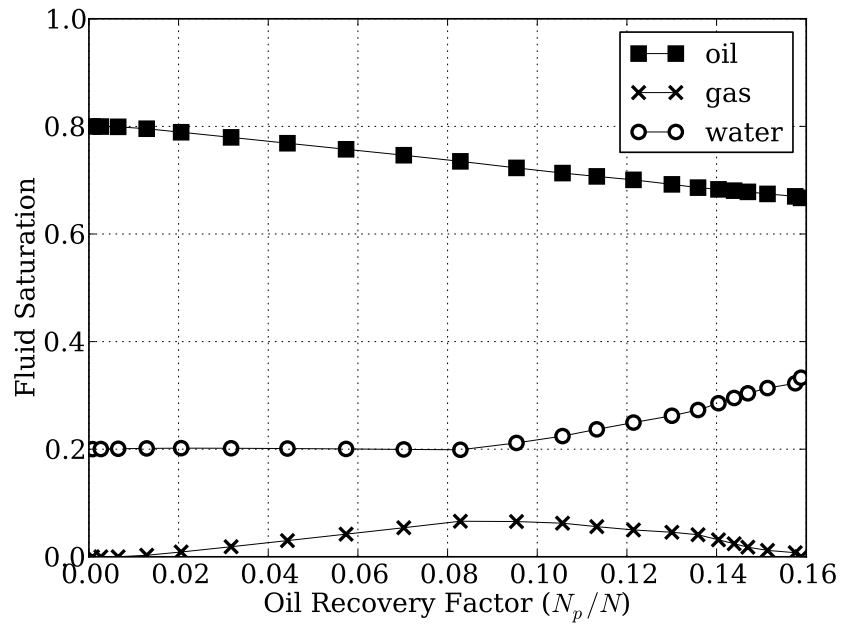


Figure 4: Phase saturations. Gas saturation is reduced as water injection progresses.

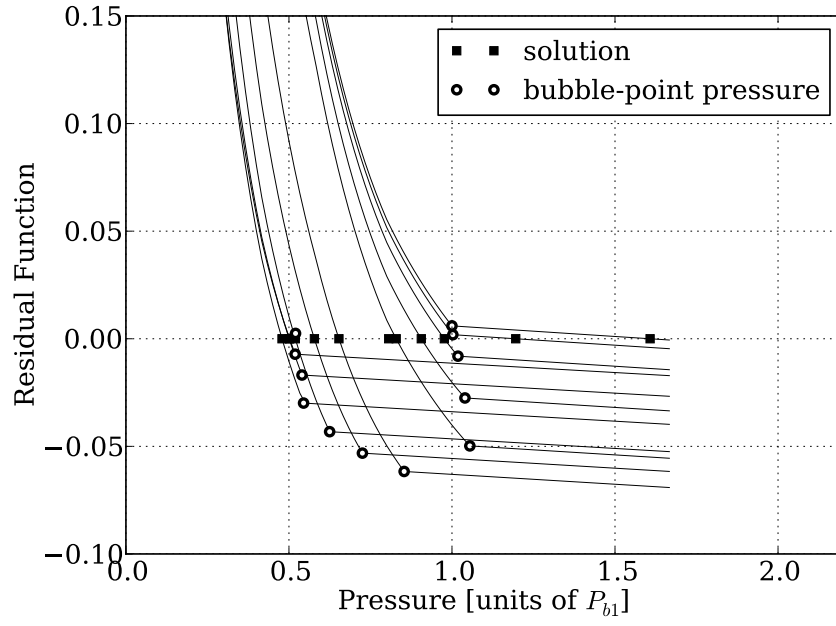


Figure 5: Shape of residual function ( $\varepsilon$ ). Each curve corresponds to different volumes of production and injection. Residual function monotonically decreases with pressure and it has no derivative at the bubble-point.

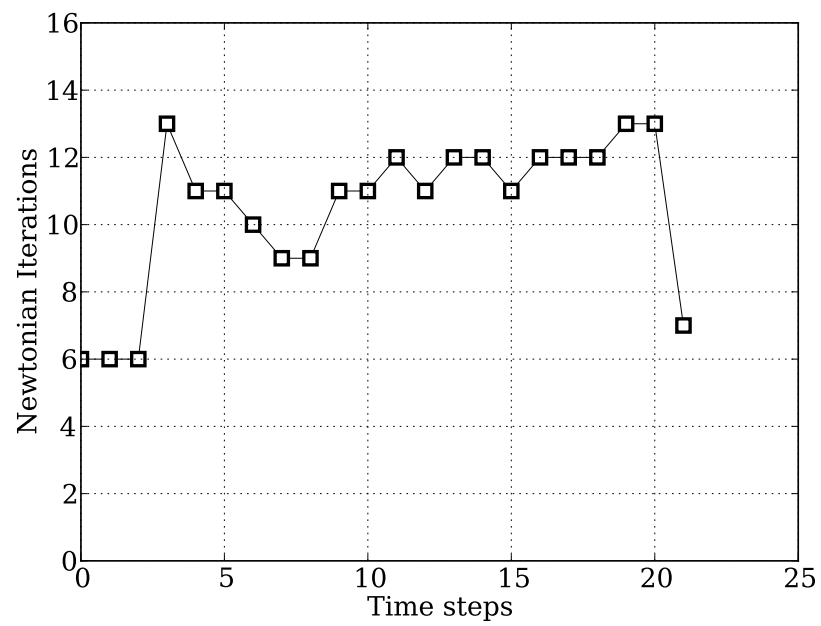


Figure 6: Count of newtonian iterations required to compute pressure with accuracy within 0.001 units of initial bubble-point pressure.

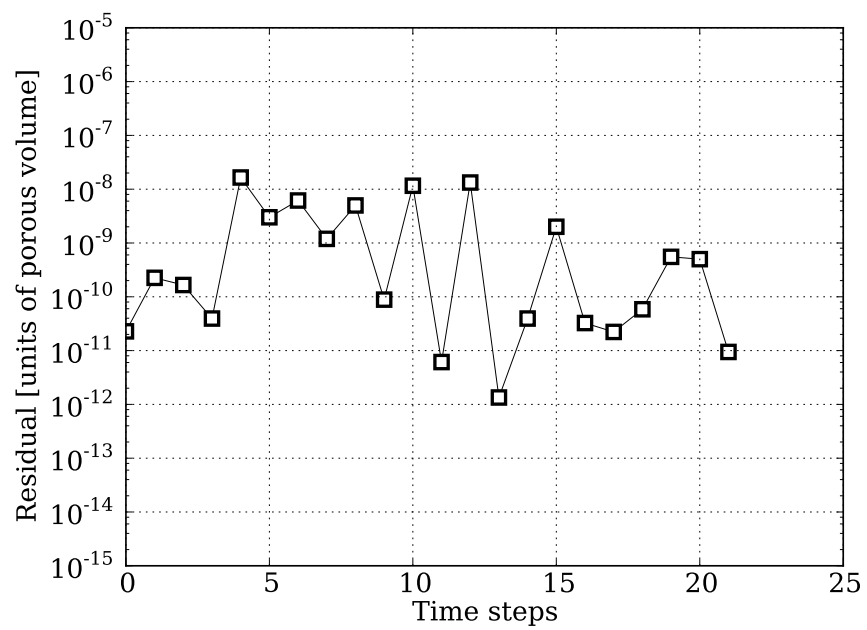


Figure 7: Residual of numerical solution in units of porous volume.

<i>Material balance equation</i>	
pressure	$F_p - F_i = NE_o + mNE_g + WE_w + C$
<i>Phase saturations</i>	
oil	$S_{o2} = (N - N_p) B_{o2}/V_{p2}$
water	$S_{w2} = (W + W_i + W_e - W_p) B_{w2}/V_{p2}$
gas	$S_{g2} = \{N [(R_{s1} - R_{s2}) + mB_{o1}/B_{g1}] + N_p R_{s2} + G_i - G_p\} B_{g2}/V_{p2}$
<i>List of terms</i>	
fluid production	$F_p = N_p [B_{o2} + (G_p/N_p - R_{s2}) B_{g2}] + W_p B_{w2}$
fluid injection and influx	$F_i = (W_i + W_e) B_{w2} + G_i B_{g2}$
fluid expansion	$E = NE_o + mNE_g + WE_w$
oil phase expansion	$E_o = (B_{o2} - B_{o1}) - (R_{so2} - R_{so1}) B_{g2}$
gas phase expansion	$E_g = B_{o1} (B_{g2}/B_{g1} - 1)$
water volume expansion	$E_w = B_{w2} - B_{w1}$
porous volume contraction	$C = V_{p2} - V_{p1}$
bubble-point pressure	$R_{s2}(P_b) = [N (mB_1/B_{g1} + R_{s1}) + G_i - G_p] / (N - N_p)$

Table 1: Complete set of material balance equations.

## Appendix A. Compressibility

### Appendix A.1. Oil compressibility

Zeroth order approximation of density:

$$p_o - p_o^{ref} = \rho_o^{ref} g(z - z^{ref}) \quad (\text{A.1})$$

First order approximation of density:

$$\frac{1}{\rho_o} \left( \frac{\partial \rho_o}{\partial p_o} \right)_{T, Y_o} = c_o \quad (\text{A.2})$$

Thus,

$$p_o - p_o^{ref} = \frac{1}{c_o} \left\{ \exp \left[ \rho_o^{ref} c_o g(z - z^{ref}) \right] - 1 \right\} . \quad (\text{A.3})$$

### Appendix A.2. Gas compressibility

Zeroth order approximation of density:

$$p_g - p_g^{ref} = \rho_g^{ref} g(z - z^{ref}) . \quad (\text{A.4})$$

Using gas equation of state for constant temperature and Z-factor:

$$p_g - p_g^{ref} = p^{ref} \left\{ \exp \left[ \frac{\rho_g^{ref} g(z - z^{ref})}{p^{ref}} \right] - 1 \right\} . \quad (\text{A.5})$$