

# Solution of hydrodynamic equilibrium in black-oil reservoirs: the classical material balance equation

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## Abstract

This work derives the classical material balance equation from a three dimensional black-oil model. Sufficient conditions for solution uniqueness of material balance equation are presented for the case that pressure is the unknown quantity. It is shown that spatially averaged PVT properties are required for reducing a black-oil model to the material balance approach. It is also shown that the averaging process may be neglected for many practical purposes. An interpretation for pressure in material balance equation is provided and its relation to a reference depth is examined. A scheme numerically solve material balance equation for pressure is also proposed.

*Keywords:* material balance, black-oil equations, reservoir engineering.

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## 1. Introduction

Material balance equation is a classical tool in petroleum reservoir engineering. It consists of an algebraic equation originally presented in Schilthuis (1936) and conceived as a zero dimensional model for estimating reservoir pressure from mass balance. Schilthuis (1936) introduced the hypothesis of equilibrium attained at all times not necessarily in the entire reservoir volume, but in the regions of high permeability where most of the production would come from and where pressure would present a similar behavior. Such regions were said to contain an active oil.

As mentioned, the equations developed by Schilthuis (1936) did not account for the non-existence of mechanical, phase or thermal equilibrium. An attempt to introduce non-equilibrium effects of hydrodynamic origin came with *black-oil*

equations (Aziz and Settari, 1979; Trangenstein and Bell, 1989b). The novelty on hydrodynamics was the introduction of a macroscopic equation of motion using Darcy’s law.

The term black-oil refers to specific hydrocarbon compositions to which phase equilibrium for isothermal changes in pressure can be described by a two pseudo-component system. For such reservoir fluids, gas phase composition remains essentially constant in the range of pressures observed during production and a pseudo gas component can be defined for a simplified description gas phase composition. A pseudo oil component can also be defined by subtracting the pseudo gas component from oil phase composition. After defining these two pseudo-components, the number of transport equations needed for hydrocarbon species is reduced from usually a large number to only two.

A great number of applications of material balance equation have been reported in the literature, but few of them explore its connection with black-oil equations. Ertekin et al. (2001) briefly addressed this subject arguing both approaches to be equivalent if capillarity and gradients of flow potential ( $\nabla\Phi = \nabla p - \rho\vec{g}$ ) were neglected and if PVT properties were averaged in reservoir volume. The intuitive notion is that material balance equation is obtained if fluid motion is neglected and black-oil equations are integrated in reservoir volume. This work develops this process examining underlying details.

The same approach of varying bubble-point pressure widely used in reservoir simulation, which is found in Aziz and Settari (1979) and Ertekin et al. (2001), is also applied here to material balance. This approach extends material balance equation to saturated reservoirs under pressure maintenance (e.g. fluid injection). The applicability, however, depends on how important non-equilibrium effects are to phase equilibrium.

After presenting material balance equation, sufficient conditions are shown for which material balance equation has only one solution for pressure. These conditions require fluids to be compressible and that gas dissolution occurs with reduction of the overall system volume. A simple scheme for numerical solution is also proposed based on classical root-finding algorithms for non-linear

equations.

## 2. Derivation of equations

This section starts presenting black-oil equations and it proceeds averaging quantities in space and examining simplifying assumptions for which material balance equation is obtained.

### 2.1. Black-oil equations

Reservoir fluids are composed of oil and gas pseudo-components and water component. There can be three fluid phases: oil, gas and (liquid) water. Oil phase may be composed of oil and gas components, gas phase may be composed uniquely by gas component and, water phase, uniquely by water component. The only allowed mass transfer is of gas component between oil and gas phases.

Mass balance equations for each component are:

- water component:

$$\frac{\partial}{\partial t} \int_{\Omega} \rho_w S_w \phi dV + \oint_{\partial\Omega} \rho_w (\vec{u}_w \cdot d\vec{S}) = \rho_w^{ST} Q_w^{ST}; \quad (1)$$

- oil component:

$$\frac{\partial}{\partial t} \int_{\Omega} \rho_o Y_o S_o \phi dV + \oint_{\partial\Omega} \rho_o Y_o (\vec{u}_o \cdot d\vec{S}) = \rho_o^{ST} Q_o^{ST}; \quad (2)$$

- gas component:

$$\frac{\partial}{\partial t} \int_{\Omega} (\rho_o Y_g S_o + \rho_g S_g) \phi dV + \oint_{\partial\Omega} (\rho_o Y_g \vec{u}_o + \rho_g \vec{u}_g) \cdot d\vec{S} = \rho_g^{ST} Q_g^{ST}. \quad (3)$$

Mass concentrations of oil and gas components in oil phases are denoted as  $Y_o$  and  $Y_g$ , respectively.

Phase saturation is defined as its volumetric fraction of the pore space:

$$S_{\alpha} = \frac{\phi_{\alpha}}{\phi}, \quad (4)$$

where  $\phi_\alpha$  is the volumetric fraction of a fluid phase  $\alpha$  and  $\phi$  is the porosity. Clearly, all saturations sum up to unity.

$$S_o + S_g + S_w = 1. \quad (5)$$

A common simplification in black-oil formulation is to restrict mass transport to macroscopic advection. No diffusive or dispersive term is included in mass balance equations.

Momentum balance for each phase is given by an extension of Darcy's law to multiphase flow. This extension consists in multiplying absolute (or single-phase) permeability ( $\vec{k}$ ) by a scalar function called phase relative permeability ( $k_{r,\alpha}$ ):

$$\vec{u}_\alpha = -\frac{k_{r,\alpha}\vec{k}}{\mu_\alpha}(\nabla p_\alpha - \rho_\alpha g \nabla z), \quad \alpha = o, w, g. \quad (6)$$

Phase densities are functions of the local thermodynamic state:

$$\rho_w = \rho_w(p_w, T_w) \quad (7)$$

$$\rho_o = \rho_o(p_o, T_o, Y_g) \quad (8)$$

$$\rho_g = \rho_g(p_g, T_g) \quad (9)$$

Another simplification is now made assuming thermal equilibrium in the entire reservoir volume, i.e.,

$$T_w = T_o = T_g = T_{rock} = T = \text{constant}. \quad (10)$$

Temperature is a constant prescribed value and thus no equation for energy balance is needed. It follows from Gibbs' phase rule that the equilibrium of oil and gas phases has only two degrees of freedom if component number is restricted to the two pseudo-components. As temperature is assumed constant, oil phase density is a function of pressure alone. The same is valid for mass fractions of gas and oil components in oil phase:

$$Y_g = Y_g(p_o), \quad (11)$$

$$Y_o = Y_o(p_o) . \quad (12)$$

Porosity ( $\phi$ ) is assumed a function of pore pressure ( $p_P$ ). Because three phases are present in general, such quantity is defined by some relation of type (Kim et al., 2011):

$$p_P = p_P(p_o, p_w, p_g, S_o, S_w, S_g) . \quad (13)$$

To close the system of equations, a connection among different phase pressures is provided by capillary pressure relations:

$$p_{c,ow}(\vec{k}, \phi, S_w) = p_o - p_w , \quad (14)$$

$$p_{c,go}(\vec{k}, \phi, S_g) = p_g - p_o . \quad (15)$$

## 2.2. Material balance equation

Derivation of the material balance equation consists in assuming no fluid motion and in integrating black-oil equations in reservoir volume. As it will be shown, the integration process requires all quantities to be either spatially averaged or assumed constant.

We start assuming mechanical equilibrium, i.e., no bulk motion:

$$\vec{u}_\alpha = -\frac{k_{r,\alpha}\vec{k}}{\mu_\alpha}(\nabla p_\alpha - \rho_\alpha g \nabla z) \equiv 0 . \quad (16)$$

This condition implies that pressure gradients must equal gravity force per unit volume:

$$\nabla p_\alpha = \rho_\alpha g \nabla z \quad (17)$$

and, consequently, pressure is a function of depth only:

$$\int_{p_\alpha^0}^{p_\alpha} \frac{dp'_\alpha}{\rho_\alpha(p'_\alpha, T)} = g(z - z_0) . \quad (18)$$

Black-oil equations become:

- water component:

$$\frac{\partial}{\partial t} \int_{\Omega} \rho_w S_w \phi dV = \rho_w^{ST} Q_w^{ST} ; \quad (19)$$

- oil component:

$$\frac{\partial}{\partial t} \int_{\Omega} \rho_o Y_o S_o \phi dV = \rho_o^{ST} Q_o^{ST}; \quad (20)$$

- gas component:

$$\frac{\partial}{\partial t} \int_{\Omega} (\rho_o Y_g S_o + \rho_g S_g) \phi dV = \rho_g^{ST} Q_g^{ST}. \quad (21)$$

### 2.2.1. Averaging quantities

Spatially averaged quantities are defined for porosity ( $\phi$ ), saturation ( $S_\alpha$ ), density ( $\rho_\alpha$ ) and mass concentration ( $Y_{\alpha,k}$ ) of a component  $k$  in phase  $\alpha$ .

All quantities are initially functions of space and time and they become functions only of time after being averaged in space:

$$\bar{\phi}(t) = \frac{1}{\Omega} \int_{\Omega} \phi(\vec{x}, t) dV. \quad (22)$$

Omitting function arguments:

$$\bar{S}_\alpha \bar{\phi} = \frac{1}{\Omega} \int_{\Omega} S_\alpha \phi dV, \quad (23)$$

$$\bar{\rho}_\alpha \bar{S}_\alpha \bar{\phi} = \frac{1}{\Omega} \int_{\Omega} \rho_\alpha S_\alpha \phi dV, \quad (24)$$

$$\bar{\rho}_\alpha \bar{Y}_{\alpha,k} \bar{S}_\alpha \bar{\phi} = \frac{1}{\Omega} \int_{\Omega} \rho_\alpha Y_{\alpha,k} S_\alpha \phi dV. \quad (25)$$

Eqs. (19), (20) and (21) are rewritten in terms of averaged quantities:

- water component:

$$\frac{\partial}{\partial t} (\bar{\rho}_w \bar{S}_w \bar{\phi} \Omega) = \rho_w^{ST} Q_w^{ST} \quad (26)$$

- oil component:

$$\frac{\partial}{\partial t} (\bar{\rho}_o \bar{Y}_o \bar{S}_o \bar{\phi} \Omega) = \rho_o^{ST} Q_o^{ST} \quad (27)$$

- gas component:

$$\frac{\partial}{\partial t} [(\bar{\rho}_o \bar{Y}_g \bar{S}_o + \bar{\rho}_g \bar{S}_g) \bar{\phi} \Omega] = \rho_g^{ST} Q_g^{ST}. \quad (28)$$

We now introduce usual definitions from petroleum engineering for the description of volumetric and solubility changes in fluid phases. In what concerns volumetric changes,  $B_w$ ,  $B_o$  and  $B_g$ , the formation volume factors of water, oil and gas phases, respectively, express the ratio of a phase volume at reservoir condition to the volume at surface condition, i.e.:

$$B_w = \frac{\rho_w^{ST}}{\bar{\rho}_w}, \quad (29)$$

$$B_o = \frac{\rho_o^{ST}}{\bar{Y}_o \bar{\rho}_o}, \quad (30)$$

$$B_g = \frac{\rho_g^{ST}}{\bar{\rho}_g}. \quad (31)$$

In what concerns changes in gas solubility in oil phase, the solubility ratio ( $R_s$ ) gives the ratio between volumes of dissolved gas and oil both measured in surface condition:

$$R_s = \frac{\rho_o^{ST} / \bar{Y}_o}{\rho_g^{ST} / \bar{Y}_g}, \quad (32)$$

or using  $\bar{Y}_o + \bar{Y}_g = 1$ :

$$R_s = \frac{\rho_o^{ST}}{\rho_g^{ST}} \frac{\bar{Y}_g}{1 - \bar{Y}_g}. \quad (33)$$

Equations can now be rewritten using the above definitions and denoting porous volume by  $V_p = \bar{\phi}\Omega$ .

$$\frac{\partial}{\partial t} \left( \frac{\bar{S}_w V_p}{B_w} \right) = Q_w^{ST}, \quad (34)$$

$$\frac{\partial}{\partial t} \left( \frac{\bar{S}_o V_p}{B_o} \right) = Q_o^{ST}, \quad (35)$$

$$\frac{\partial}{\partial t} \left[ V_p \left( \frac{\bar{S}_g}{B_g} + \frac{\bar{S}_o R_s}{B_o} \right) \right] = Q_g^{ST}, \quad (36)$$

$$\bar{S}_w + \bar{S}_o + \bar{S}_g = 1. \quad (37)$$

Integrating Eqs. (34), (35) and (36) between times  $t_1$  and  $t_2$  gives:

$$\frac{V_{p2}\bar{S}_{w2}}{B_{w2}} = W + (W_i + W_e - W_p) , \quad (38)$$

$$\frac{\bar{S}_{o2}V_{p2}}{B_{o2}} = N - N_p , \quad (39)$$

$$V_{p2} \left( \frac{\bar{S}_{g2}}{B_{g2}} + \frac{\bar{S}_{o2}R_{s2}}{B_{o2}} \right) = N \left( m \frac{B_{o1}}{B_{g1}} + R_{s1} \right) + (G_i - G_p) , \quad (40)$$

$$\bar{S}_o + \bar{S}_w + \bar{S}_g = 1 . \quad (41)$$

Where  $N = V_{p1}\bar{S}_{o1}/B_{o1}$  and  $W = V_{p1}\bar{S}_{w1}/B_{w1}$  are the initial volumes of oil and water measured at surface condition and  $m = V_{p1}\bar{S}_{g1}/B_{o1}N$  is the ratio of the initial volumes of gas and oil phases measured in reservoir condition.

An equation for bubble-point can be obtained by setting  $\bar{S}_{g2} = 0$  in Eq. (40) and solving it for the solubility ratio at bubble-point pressure  $R_s(p = P_b)$ :

$$R_{s2}(p = P_b) = \left[ N \left( mB_1/B_{g1} + R_{s1} \right) + G_i - G_p \right] / (N - N_p) . \quad (42)$$

### 2.2.2. Material balance equation

It is now possible to obtain the classical material balance equation substituting the equations for phase saturation, Eqs. (38), (39) and (40), into Eq. (41):

$$F_p - F_i = \underbrace{NE_o + mNE_g + WE_w}_{\text{fluid volume expansion (E)}} + \underbrace{V_{p1} - V_{p2}}_{\text{porous volume contraction (C)}} \quad (43)$$

or

$$F_p - F_i = E + C , \quad (44)$$

where new terms  $C$ ,  $E$ ,  $F_i$  and  $F_p$  are defined in Table 1.

Eq (43) involves fluid properties, original fluid volumes ( $N$ ,  $W$  and  $G$ ), cumulative production and injection and aquifer influx. It is an algebraic equation that can be solved to any of its terms provided all remaining are known. This work focuses on its solution for pressure.



### 2.2.3. An interpretation for calculated pressure and the definition of a reference depth

Material balance equation is widely used to establish a relationship between production (and injection) and reservoir pressure. In Eq. (43), however, formation volume factors were defined using averaged values of fluid properties, and they are function of pressure. It turns out, though, that pressure is a function of depth and phase.

What is then the meaning of pressure calculations in material balance equations?

For answering this question, we examine the two roles played by pressure in material balance equation: computation of phase density and phase equilibrium.

We shall see that to compute the averaged phase density, we may define some reference depth and use the corresponding local density as an approximation to the averaged density without introducing any important error.

Pressure increases with depth according to the hydrostatic gradient. Consequently, density also increases but by a negligible amount. This approximation may be quantified by putting together the expressions for compressibility and hydrostatic equilibrium. For this, we assume a constant isothermal compressibility for oil phase

$$\left( \frac{1}{\rho_o} \frac{\partial \rho_o}{\partial p_o} \right)_{T, Y_o} = c_o \quad (45)$$

and we use hydrostatic equilibrium  $dp_o = \rho_o g dz$  to obtain the expression for the derivative of oil density with depth:

$$\left( \frac{1}{\rho_o} \frac{\partial \rho_o}{\partial z} \right)_{T, Y_o} = c_o \rho_o g. \quad (46)$$

Similarly for the gas phase, we obtain its isothermal compressibility from the gas state equation

$$\left( \frac{1}{\rho_g} \frac{\partial \rho_g}{\partial p_g} \right)_{T, Z} = \frac{1}{p_g} = \frac{\mathcal{M}}{\rho_g ZRT} \quad (47)$$

and again we use hydrostatic equilibrium to obtain the derivative of gas density with depth:

$$\left( \frac{1}{\rho_g} \frac{\partial \rho_g}{\partial z} \right)_{T, Z} = \left( \frac{\mathcal{M}}{ZRT} \right) g. \quad (48)$$

Substituting typical values in Eqs. (45) and (48), we see that oil and gas densities are rather constant with depth:

$$\left( \frac{1}{\rho_o} \frac{\partial \rho_o}{\partial z} \right)_{T, Y_o} \approx 10^{-6} \text{ m}^{-1}, \quad (49)$$

$$\left( \frac{1}{\rho_g} \frac{\partial \rho_g}{\partial z} \right)_{T, Z} \approx 10^{-5} \text{ m}^{-1}. \quad (50)$$

Oil and gas densities are thus rather constant with depth and a zeroth order approximation of  $\bar{\rho}$  is accurate within 0.5% for material balance calculations.

The reference depth for computing phase density (or formation volume factors) is then any arbitrary depth inside phase volume. For a negligible variation of density with depth, it is easily seen that  $\bar{\rho} = \rho(\bar{p})$  and  $\bar{p} = p(z^0)$  where  $z^0$  is the depth of phase gravity center.

For the case of an oil reservoir covered by a gas cap, the reference depth may be defined, with advantage, at the gas-oil contact. The point here is that at phase contact both gas and oil densities can be computed using the same pressure, i.e., the corresponding local value (we are neglecting capillary pressure).

If the reference depth for oil phase is otherwise defined below gas-oil contact, gas density must be computed at some different depth still inside gas phase, i.e., at or above gas-oil contact. This is to avoid the effect of oil hydrostatic gradient in the computation of gas density,

$$\left( \frac{1}{\rho_g} \frac{\partial \rho_g}{\partial z} \right)_{T, Z} = \frac{\rho_o}{\rho_g} \left( \frac{\mathcal{M}}{ZRT} \right) g, \quad (51)$$

what is  $\rho_o/\rho_g$  times greater than the sole effect of gas hydrostatic pressure on its density.

It has been shown that pressure variation with depth is not important for the computation of gas, oil and water densities. Nevertheless, it was said that pressure also plays a second role in material balance equation by defining phase equilibrium. For this purpose, pressure variation with depth may not be negligible in the evaluation of the solubility ratio:

$$\frac{1}{R_s} \frac{dR_s}{dz} = \frac{1}{R_s} \frac{dR_s}{dp_o} \frac{dp_o}{dz} = \frac{\rho_o g}{R_s} \frac{dR_s}{dp_o}. \quad (52)$$

For a non-volatile saturated oil, we can approximate  $dR_s/dp_o$  to the ratio between solubility ratio ( $R_s$ ) and saturation pressure ( $p_{sat}$ ), so that Eq. (52) becomes:

$$\frac{1}{R_s} \frac{dR_s}{dz} \approx \frac{\rho_o g}{p_{sat}} \approx 10^{-3} \text{ m}^{-1}, \quad (53)$$

where the following orders of magnitudes were assumed:  $p_{sat} \approx 10^7 \text{ Pa}$ ,  $\rho_o \approx 10^3 \text{ kg/m}^3$  and  $g \approx 10 \text{ m}^2/\text{s}$ .

From Eq. (53), we see that the variation of pressure with depth may not be negligible for the phase equilibrium, specially for thick reservoirs of light oil. In such cases, using an averaged value for the solubility ratio may be worth the accuracy gain in computing dissolved gas volume.

### 3. Solution of material balance equation

A numerical solution of Eq. (44) is presented here. We shall start defining a residual function ( $\varepsilon$ ) for which the pressure solving Eq. (44) is a root:

$$\varepsilon(p_2) = \frac{E + F_i + C - F_p}{V_{p1}}. \quad (54)$$

We proceed showing that  $\varepsilon$  is a monotonic function provided some conditions on fluid properties are observed and that, as a consequence, only one root of  $\varepsilon$  exists at most. Equivalently, we may say that only one solution exists for material balance equation.

#### 3.1. Mathematical aspects of the residual function

It is a known result of real analysis that a monotonic function  $f : D \subset \mathcal{R} \rightarrow \mathcal{R}$  can have at most one root in  $D$ . Further, if  $f$  is continuous in the closed interval  $I = [a, b] \subset D$  and  $f(a)f(b) < 0$ , then  $f$  has one (and only one) root in  $I$ .

We now state without proof that the following affirmatives are sufficient conditions to classify a function  $f$  as monotonic:

1.  $f$  is continuous;
2.  $f$  is differentiable everywhere except for a countable set of points;

3.  $f' < 0$  everywhere.

Residual function clearly satisfies the first two affirmatives because fluid properties are continuous everywhere and are also differentiable everywhere except for the first order phase change at bubble-point pressure. The third affirmative is proved by rewriting  $d\epsilon/dp$  as a sum of four parcels corresponding each one to a different phase. Each parcel is composed of two factors: current volume *in-situ* and volume derivative with respect to pressure:

$$\begin{aligned}
V_{p1} \frac{d\epsilon}{dp} = & \underbrace{(N - N_p) \left( \frac{dB_o}{dp} - B_g \frac{dR_s}{dp} \right)}_{\text{oil phase}} \\
& + \underbrace{\left[ NR_{s1} + mN \frac{B_{o1}}{B_{g1}} + G_i - (N - N_p) R_{s2} - G_p \right] \frac{dB_g}{dp}}_{\text{gas phase}} \\
& + \underbrace{(W + W_i + W_e - W_p) \frac{dB_w}{dp}}_{\text{water phase}} \\
& + \underbrace{\frac{dV_p}{dp}}_{\text{porous volume}} .
\end{aligned} \tag{55}$$

*In-situ* volumes (e.g.  $N - N_p$  for oil phase) are non-negative values. Pressure derivatives of fluid volumes are non-positive values (fluid volume cannot increase as pressure increases) and pressure derivative of porous volume is a positive value (as fluid pressure increases, porous volume expands owing to rock compression).

The value of  $d\epsilon/dp$  is thus negative if at least one phase is compressible because it would be a sum of products between negative and positive factors.

In summary, the following four relations constitute sufficient conditions for  $\epsilon(p)$  be a strictly decreasing function and, therefore, for material balance equation to have only one solution:

$$\left( \frac{dB_o}{dp} - B_g \frac{dR_s}{dp} \right) \leq 0, \tag{56}$$

$$\frac{dB_g}{dp} \leq 0, \tag{57}$$

$$\frac{dB_w}{dp} \leq 0, \tag{58}$$

$$\frac{dV_p}{dp} \geq 0. \quad (59)$$

Above relations, specially Eq. (56), provide a test to check whether PVT data obtained from laboratory or empirical correlations is physically correct.

### 3.2. Solution algorithm

Several methods exist to find roots of nonlinear continuous functions (Hamming, 1987). One possible choice is Newton's method, which linearizes the function using either its gradient or a secant line. The secant line approach introduces an artificial smoothness to the gradient direction near continuous but non-differentiable points, for what it is the approach used here to deal with the bubble-point pressure.

An open source implementation of secant method is available in SciPy library (Jones et al., 2001–). This work used library version 0.12.

The steps to solve all material balance equation are summarized bellow:

**Step 1.** *Compute bubble-point pressure for instant  $t_2$ .*

Bubble-point pressure  $P_{b2}$  is only a function of the values in initial instant  $t_1$  and produced and injected volumes.

**Step 2.** *Compute fluid pressure for instant  $t_2$  finding the root of residual function ( $\varepsilon$ ).*

Pressure in Eq. (43) is a function only of fluid properties in instant  $t_1$ , produced and injected values and bubble-point pressure at instant  $t_2$ ,  $P_{b2}$ , which was already computed in step 1.

**Step 3.** *Compute fluid saturations  $t_2$ .*

Once  $S_{o2}$ ,  $S_{g2}$ ,  $S_{w2}$ ,  $p_2$  and  $P_{b2}$  are known, solution for instant  $t_2$  is complete.

#### 4. Results

Material balance equation were solved for a black-oil reservoir initially producing in depletion drive mechanism and later subjected to water injection. Fig. 1 shows in the solid line curve the severe depletion during initial phases of production. In the dashed curve, Fig. 1 shows the bubble-point pressure, indicating how far the saturated reservoir is from undersaturation and allowing for planning of pressure maintenance projects.

Water injection starts when oil recovery factor ( $RF$ ) is about  $RF = 0.08$  and it is maintained until the reservoir reaches a novel undersaturated state. Initially, the high compressibility of gas prevents pressure from raising and free gas phase is reduced because of redissolution. Finally, the reservoir reaches the undersaturated state and a steep pressure increase is seen in Fig. 1 for a recovery factor about  $RF = 0.15$ .

It must be observed that the result of gas reentering oil phase completely neglects non-equilibrium effects, e.g. compositional gradients, and must be used with care. For instance, it may be in strong disagreement with real reservoir behavior when gravity segregation is present.

Figure 4 shows the formation of a free gas phase with growing  $S_g$  and then its disappearance. Depending on the ratio of vertical and horizontal permeabilities or reservoir dip angle, it may also suggest the formation of a secondary gas cap.

Figure 5 shows the shape of residual function ( $\epsilon$ ) with respect to pressure, i.e., holding all produced or injected volumes constant and allowing only pressure to change. Each different solid line corresponds to a different volume of fluid production and injection. The function root is the solution of material balance equation, Eq. (44), and the non-differentiable point is the bubble-point pressure. Also, it is seen that function  $\epsilon$  monotonically decreases with pressure, as theoretically shown before.

Figure 6 shows the count of newtonian iterations required to compute pressure with an accuracy within 0.001 of initial bubble-point pressure. Typically, 12 iterations are required.

Figure 7 shows the residual of numerical solution as defined in Eq. (54). Material balance error is less than  $10^{-7}$  units of porous volume for all time steps.

## 5. Conclusions

1. Material balance equation can be obtained from black-oil equations under the assumptions of:
    - (a) mechanical equilibrium;
    - (b) absence of gravity and capillary forces;
    - (c) spatially constant bubble-point pressure or, equivalently, fluid saturations.
  2. One unique solution of pressure equation exists for a compressible system;
  3. A residual function can be defined such that its root solves pressure equation.
- A numerical solution can be obtained with nonlinear root-finding algorithms.

## Nomenclature

$B_g$	gas formation volume factor
$B_o$	oil formation volume factor
$B_w$	water formation volume factor
$R_s$	solubility ratio of gas in oil phase
$S_g$	gas saturation
$S_o$	oil saturation
$S_w$	water saturation
$G_p$	produced volume of gas
$m$	ratio of original free gas volume to original oil phase volume in reservoir condition
$N$	original volume of oil in place
$N_p$	produced volume of oil
$P_b$	bubble-point pressure
$V_p$	porous volume
$W_i$	injected volume of water
$W_p$	produced volume of water

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## Highlights

1. The existing link between black-oil and material balance equation is examined in detail.

2. It is shown that gravity and capillary forces must be neglected and saturations must be constant in space so one can integrate black-oil equations in reservoir volume and obtain material balance equation.
3. Material balance equation are shown to have a unique solution for a compressible system.
4. Material balance equation are formulated for a variable bubble-point and they are used to a reservoir under pressure maintenance.

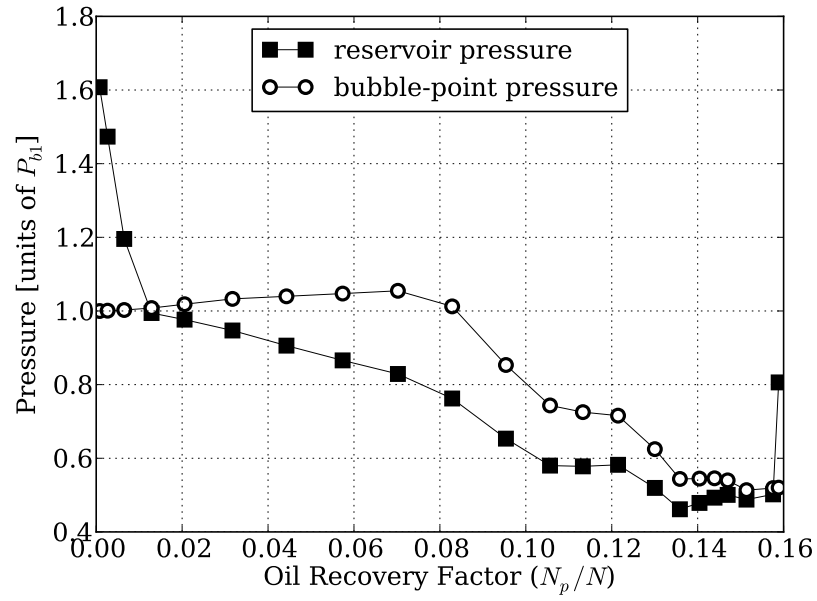


Figure 1: Reservoir pressure and bubble-point pressure as functions of recovery factor. Once depletion goes below bubble-point, pressure is hardly recovered for a saturated reservoir because of high system compressibility. Pressure is here given in units of the original bubble-point pressure.

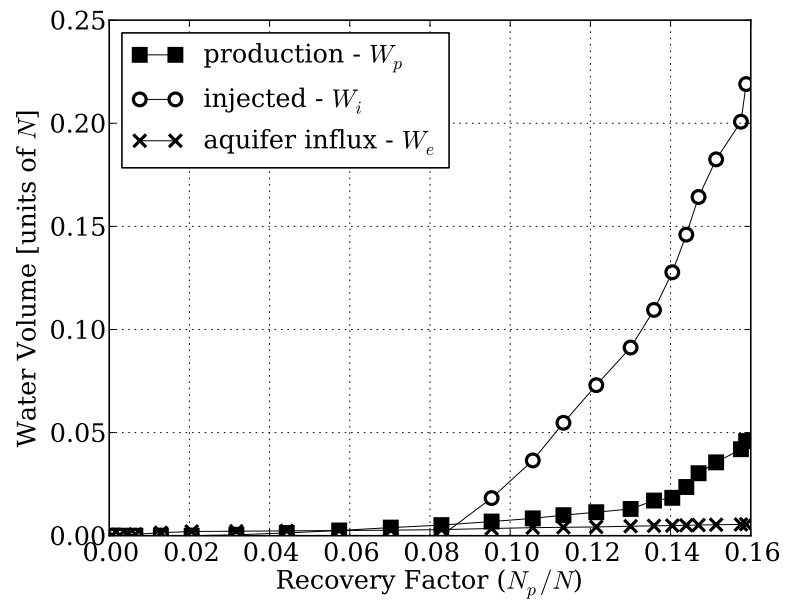


Figure 2: Volume of water production and injection and aquifer influx.

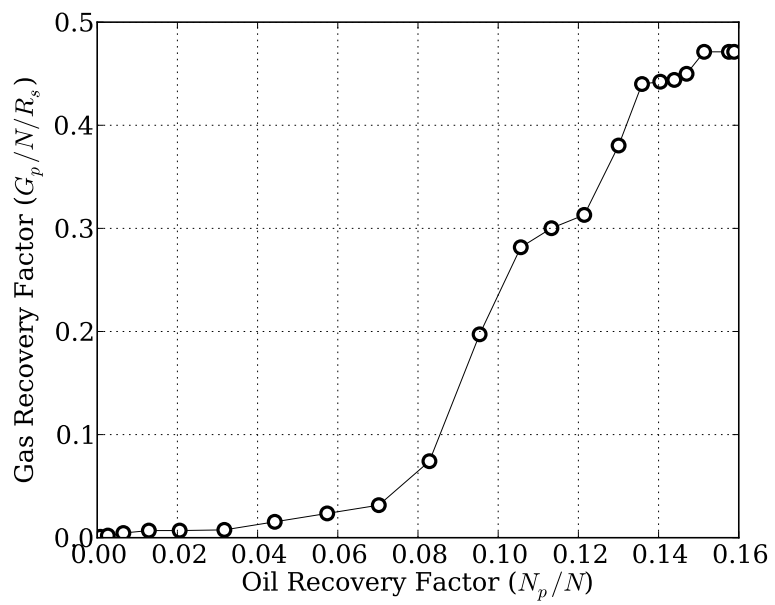


Figure 3: Volume of gas production.

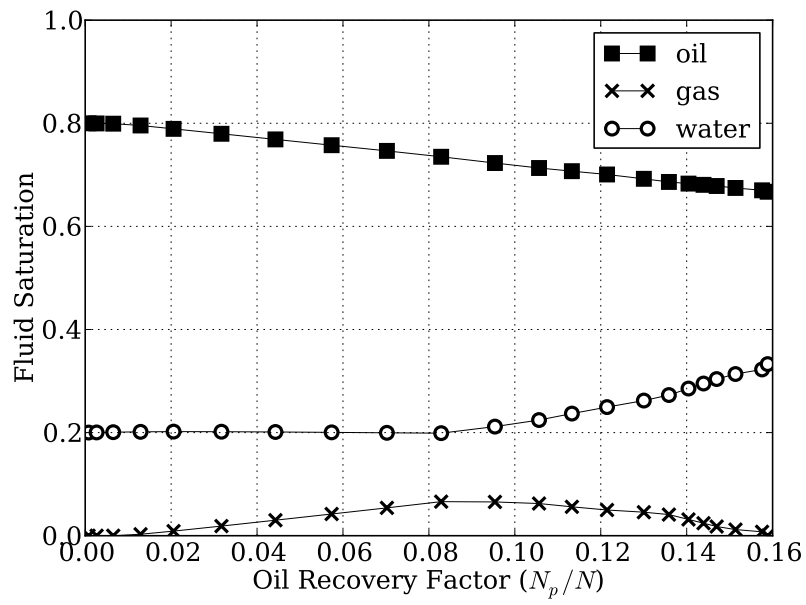


Figure 4: Phase saturations. Gas saturation is reduced as water injection progresses.

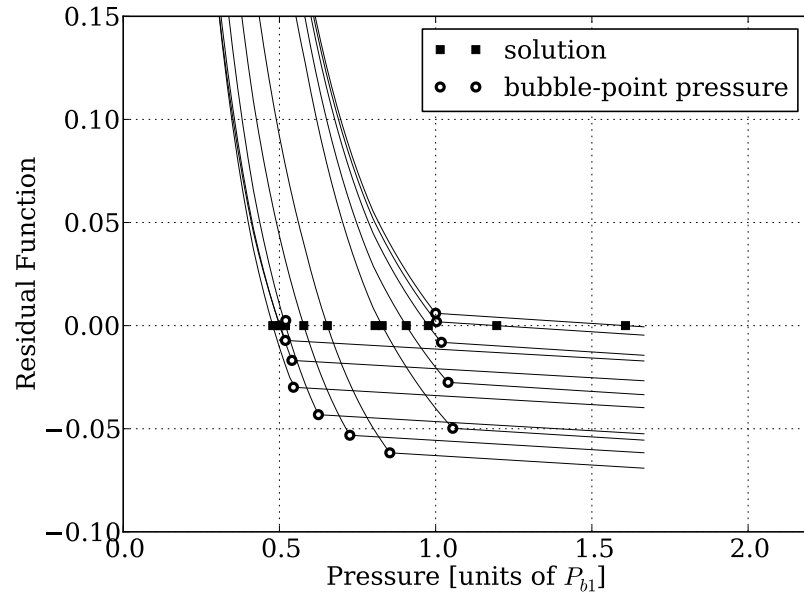


Figure 5: Shape of residual function ( $\varepsilon$ ). Each curve corresponds to different volumes of production and injection. Residual function monotonically decreases with pressure and it has no derivative at the bubble-point.

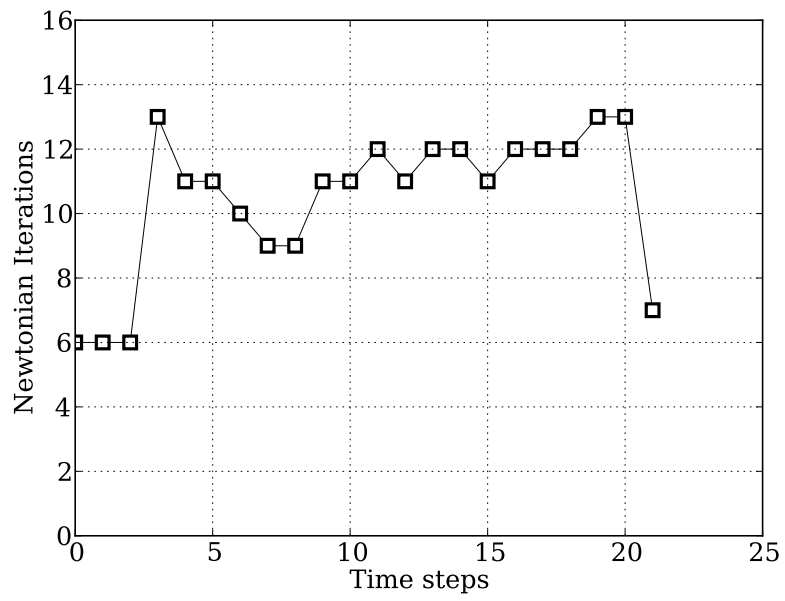


Figure 6: Count of newtonian iterations required to compute pressure with accuracy within 0.001 units of initial bubble-point pressure.



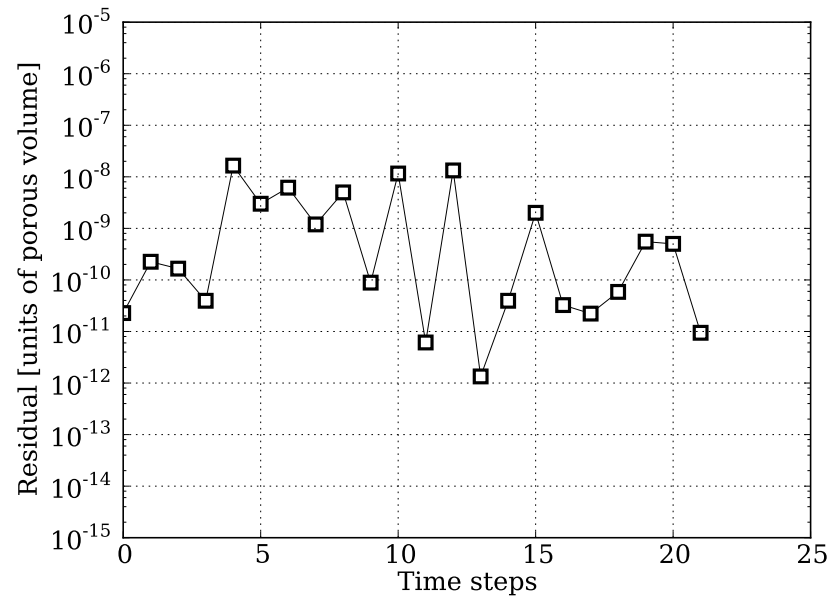


Figure 7: Residual of numerical solution in units of porous volume.

<i>Material balance equation</i>	
pressure	$F_p - F_i = NE_o + mNE_g + WE_w + C$
<i>Phase saturations</i>	
oil	$S_{o2} = (N - N_p) B_{o2}/V_{p2}$
water	$S_{w2} = (W + W_i + W_e - W_p) B_{w2}/V_{p2}$
gas	$S_{g2} = \{N [(R_{s1} - R_{s2}) + mB_{o1}/B_{g1}] + N_p R_{s2} + G_i - G_p\} B_{g2}/V_{p2}$
<i>List of terms</i>	
fluid production	$F_p = N_p [B_{o2} + (G_p/N_p - R_{s2}) B_{g2}] + W_p B_{w2}$
fluid injection and influx	$F_i = (W_i + W_e) B_{w2} + G_i B_{g2}$
fluid expansion	$E = NE_o + mNE_g + WE_w$
oil phase expansion	$E_o = (B_{o2} - B_{o1}) - (R_{so2} - R_{so1}) B_{g2}$
gas phase expansion	$E_g = B_{o1} (B_{g2}/B_{g1} - 1)$
water volume expansion	$E_w = B_{w2} - B_{w1}$
porous volume contraction	$C = V_{p2} - V_{p1}$
bubble-point pressure	$R_{s2} (P_b) = [N (mB_1/B_{g1} + R_{s1}) + G_i - G_p] / (N - N_p)$

Table 1: Complete set of material balance equations.