

# Solution of thermodynamic equilibrium in black-oil reservoirs: the classical material balance equation

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## Abstract

This work presents a derivation of the classical material balance equation starting from black-oil equations in integral form. The existing link between both approaches is examined in detail, including the necessary step of averaging PVT properties. An interpretation for pressure in material balance calculations is provided and the choice of a reference depth (datum) is discussed. The material balance equation is formulated for a varying bubble-point pressure and sufficient conditions for solution uniqueness are presented. A scheme for numerical solution is also proposed.

*Keywords:* material balance, black-oil equations, reservoir engineering.

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## 1. Introduction

The so called material balance equation is widely employed in reservoir engineering and has become a classical tool in oil reservoir management. It consists of an algebraic equation originally presented in Schilthuis (1936) and derived from a zero dimensional model for estimating reservoir pressure from mass balance. Schilthuis (1936) introduced the hypothesis of thermodynamic equilibrium occurring not in the entire reservoir volume, but in the regions of high permeability where most of the production would come from and where pressure would present a similar behavior. Such regions were said to contain

an *active oil*.

The equations developed by Schilthuis (1936) did not account for the non-existence of mechanical, phase or thermal equilibrium. Furthermore, the zero-dimensional approach excluded the possibility of mechanical equilibrium under gravity and capillary forces as they produce a variable fluid distribution in space. An attempt to introduce non-equilibrium effects of hydrodynamic origin came with *black-oil* equations (Aziz and Settari, 1979; Trangenstein and Bell, 1989b). The novelty on hydrodynamics was the introduction of a macroscopic equation of motion employing Darcy’s law.

Black-oil equations arise from the generalization of the zero dimensional concept to a three-dimensional model that incorporates Darcy’s law and entails the same approach for phase equilibrium (Aziz and Settari, 1979).

The term black-oil refers to specific hydrocarbon compositions whose phase equilibrium for isothermal changes in pressure may be described by a two pseudo-component system. For such reservoir fluids, gas phase composition remains essentially constant in the range of pressures observed during production and we may define this composition as a pseudo gas component. A pseudo oil component may then be defined by subtracting the gas component from oil phase composition. After defining the two pseudo components, the number of transport equations needed for hydrocarbon species is reduced from usually a large number to only two.

Although many applications of material balance equation have been reported in the literature, few explore its connection with black-oil equations. Ertekin et al. (2001) briefly addressed this issue arguing both approaches to be equivalent if capillarity and gradients of flow potential ( $\nabla\Phi = \nabla p - \rho\mathbf{g}$ )

were neglected and if PVT properties were averaged.

This work develops such assumptions and examines underlying details in obtaining material balance equation from the more general black-oil equations. We shall see that the process of integrating black-oil equations in the reservoir volume suggests the convenient averaging of PVT properties and provides a physical interpretation for pressure computed from material balance equation.

We employ the same approach of varying bubble-point pressure widely used in reservoir simulation that is found in Aziz and Settari (1979) and Ertekin et al. (2001). The employed approach extends material balance equation to saturated reservoirs under pressure maintenance (e.g. fluid injection). The applicability, however, depend on how important non-equilibrium effects are to the phase equilibrium.

After presenting material balance equation, sufficient conditions for solution uniqueness are imposed on PVT properties. These conditions require fluids to be compressible and that gas dissolution occurs with reduction of the overall system volume. A simple scheme for numerical solution is also proposed based on classical root-finding algorithms for non-linear equations.

## **2. Derivation of equations**

This section starts presenting black-oil equations and it proceeds averaging quantities in space and examining simplifying assumptions for which material balance equation are obtained.

### 2.1. Black-Oil equations

As already mentioned, black-oil is a designation to a simplified treatment of hydrocarbon composition and phase equilibrium. Hydrocarbon species are grouped into two pseudo-components called oil and gas, being a pseudo-component one that consists of more than one chemical molecule.

Oil component, also called stock tank oil, is defined as the hydrocarbon composition in liquid phase for standard surface pressure and temperature. Similarly, gas component is the hydrocarbon composition in gaseous phase for the same surface conditions. All species are generally present in both phases although the equilibrium composition is shifted to liquid phase (or gaseous phase) for components of greater (lower) molecular mass.

Reservoir fluids are then composed of the two pseudo-components (oil and gas) plus the water component. There can exist three fluid phases: oil, gas and liquid water. Oil phase may be composed of oil and gas components, gas phase may be composed uniquely by the gas component and, water phase, uniquely by water component. The only allowed mass transfer is of gas component between oil and gas phases.

In addition to the three fluid components and phases, there is the solid phase composed by the rock.

Mass balance equations for each component in fluid phase are:

- water component:

$$\frac{\partial}{\partial t} \int_{\Omega} \rho_w S_w \phi dV + \oint_{\partial\Omega} \rho_w (\mathbf{u}_w \cdot \mathbf{dS}) = \rho_w^{ST} Q_w^{ST}; \quad (1)$$

- oil component:

$$\frac{\partial}{\partial t} \int_{\Omega} \rho_o Y_o S_o \phi dV + \oint_{\partial\Omega} \rho_o Y_o (\mathbf{u}_o \cdot \mathbf{dS}) = \rho_o^{ST} Q_o^{ST}; \quad (2)$$

- gas component:

$$\frac{\partial}{\partial t} \int_{\Omega} (\rho_o Y_g S_o + \rho_g S_g) \phi dV + \oint_{\partial\Omega} (\rho_o Y_g \mathbf{u}_o + \rho_g \mathbf{u}_g) \cdot \mathbf{dS} = \rho_g^{ST} Q_g^{ST}. \quad (3)$$

Mass concentration of oil and gas components in oil phases are denoted as  $Y_o$  and  $Y_g$ , respectively.

Phase saturation is defined as its volumetric fraction of the pore space:

$$S_{\alpha} = \frac{\phi_{\alpha}}{\phi}, \quad (4)$$

where  $\phi_{\alpha}$  is the volumetric fraction of a fluid phase  $\alpha$  and  $\phi$  is the porosity, i.e., the volumetric fraction of the pore volume. Since we consider that pore volume is fully saturated with fluid,

$$S_o + S_g + S_w = 1. \quad (5)$$

A common simplification in black-oil formulations is to restrict mass transport to macroscopic advection. No diffusive or dispersive term is included in the mass balance equations. This simplification must be looked with care when gradients of species concentration are important. For instance, when gas is being dissolved in oil phase in a reservoir under repressurization, this condition is not generally true.

Momentum balance for each phase is given by an extension of Darcy's law to multiphase flow. This extension consists in multiplying absolute (or

single-phase) permeability ( $\mathbf{k}$ ) by a scalar function called phase relative permeability ( $k_{r,\alpha}$ ):

$$\mathbf{u}_\alpha = -\frac{k_{r,\alpha}\mathbf{k}}{\mu_\alpha}(\nabla p_\alpha - \rho_\alpha g \nabla z) , \quad \alpha = o, w, g . \quad (6)$$

Phase densities are functions of the local thermodynamic state:

$$\rho_w = \rho_w(p_w, T_w) \quad (7)$$

$$\rho_o = \rho_o(p_o, T_o, Y_g) \quad (8)$$

$$\rho_g = \rho_g(p_g, T_g) \quad (9)$$

Another simplification is now made assuming thermal equilibrium in the entire reservoir volume:

$$T_w = T_o = T_g = T_{rock} = T \quad (10)$$

and

$$\nabla T \equiv 0 . \quad (11)$$

Temperature is thus a constant prescribed value and no energy balance equation is needed. Also, it follows from Gibbs' phase rule that the equilibrium of oil and gas phases has only two degrees of freedom if we restrict component number to the two pseudo-components. As temperature is assumed constant, oil phase density in saturated states is a function of pressure or composition only. The same is valid for mass concentrations of gas and oil components in oil phase.

$$Y_g = Y_g(p) , \quad (12)$$

$$Y_o = Y_o(p) . \quad (13)$$

Porosity ( $\phi$ ) is assumed a function of pore pressure ( $p_P$ ). Since three phases are present in general, such quantity is defined by some relation of type (Kim et al., 2011):

$$p_P = p_P(p_o, p_w, p_g, S_o, S_w, S_g) . \quad (14)$$

In order to close the system of equations, a connection among different phase pressures must be provided. Such expressions are referred to as capillary pressures and are usually employed in the following forms (Aziz and Settari, 1979):

$$p_{c,ow}(\mathbf{k}, \phi, S_w) = p_o - p_w , \quad (15)$$

$$p_{c,go}(\mathbf{k}, \phi, S_g) = p_g - p_o . \quad (16)$$

## 2.2. Material balance equation

Derivation of the material balance equation consists in assuming no fluid motion and in integrating black-oil equations in reservoir volume. As it will be shown, the integration process requires all quantities to be either spatially averaged or assumed constant.

We start assuming mechanical equilibrium, i.e., no bulk motion:

$$\mathbf{u}_\alpha = -\frac{k_{r,\alpha}\mathbf{k}}{\mu_\alpha}(\nabla p_\alpha - \rho_\alpha g \nabla z) \equiv 0 . \quad (17)$$

This condition implies that pressure gradients must equal gravity force per unit volume:

$$\nabla p_\alpha = \rho_\alpha g \nabla z \quad (18)$$

and, consequently, pressure is a function of depth only:

$$\int_{p_\alpha^0}^{p_\alpha} \frac{dp'_\alpha}{\rho_\alpha(p'_\alpha, T)} = g(z - z_0) . \quad (19)$$

Black-oil equations then become:

- water component:

$$\frac{\partial}{\partial t} \int_{\Omega} \rho_w S_w \phi dV = \rho_w^{ST} Q_w^{ST} ; \quad (20)$$

- oil component:

$$\frac{\partial}{\partial t} \int_{\Omega} \rho_o Y_o S_o \phi dV = \rho_o^{ST} Q_o^{ST} ; \quad (21)$$

- gas component:

$$\frac{\partial}{\partial t} \int_{\Omega} (\rho_o Y_g S_o + \rho_g S_g) \phi dV = \rho_g^{ST} Q_g^{ST} . \quad (22)$$

### 2.2.1. Averaging quantities

We now define spatially averaged quantities for porosity ( $\phi$ ), saturation ( $S_\alpha$ ), density ( $\rho_\alpha$ ) and mass concentration ( $Y_{\alpha,k}$ ) of a component  $k$  in phase  $\alpha$ .

All quantities are initially functions of space and time that become functions of time alone after being averaged in space:

$$\bar{\phi}(t) = \frac{1}{\Omega} \int_{\Omega} \phi(\mathbf{x}, t) dV . \quad (23)$$

We proceed omitting function arguments:

$$\bar{S}_\alpha \bar{\phi} = \frac{1}{\Omega} \int_{\Omega} S_\alpha \phi dV , \quad (24)$$



$$\bar{\rho}_\alpha \bar{S}_\alpha \bar{\phi} = \frac{1}{\Omega} \int_\Omega \rho_\alpha S_\alpha \phi dV, \quad (25)$$

$$\bar{\rho}_\alpha \bar{Y}_{\alpha,k} \bar{S}_\alpha \bar{\phi} = \frac{1}{\Omega} \int_\Omega \rho_\alpha Y_{\alpha,k} S_\alpha \phi dV. \quad (26)$$

Eqs. (20), (21) and (22) rewritten in terms of averaged quantities read:

- water component:

$$\frac{\partial}{\partial t} (\bar{\rho}_w \bar{S}_w \bar{\phi} \Omega) = \rho_w^{ST} Q_w^{ST} \quad (27)$$

- oil component:

$$\frac{\partial}{\partial t} (\bar{\rho}_o \bar{Y}_o \bar{S}_o \bar{\phi} \Omega) = \rho_o^{ST} Q_o^{ST} \quad (28)$$

- gas component:

$$\frac{\partial}{\partial t} [(\bar{\rho}_o \bar{Y}_g \bar{S}_o + \bar{\rho}_g \bar{S}_g) \bar{\phi} \Omega] = \rho_g^{ST} Q_g^{ST}. \quad (29)$$

We now introduce usual definitions from petroleum engineering for the description of volumetric and solubility changes in fluid phases. In what concerns volumetric changes,  $B_w$ ,  $B_o$  and  $B_g$ , the formation volume factors of water, oil and gas phases, respectively, express the ratio of a phase volume at some given pressure and temperature condition to the volume of the same phase at standard pressure and temperature, i.e.:

$$B_w = \frac{\rho_w^{ST}}{\bar{\rho}_w}, \quad (30)$$

$$B_o = \frac{\rho_o^{ST}}{\bar{Y}_o \bar{\rho}_o}, \quad (31)$$

$$B_g = \frac{\rho_g^{ST}}{\bar{\rho}_g} . \quad (32)$$

In what concerns changes in gas solubility in oil phase, the solubility ratio ( $R_s$ ) gives the ratio between volumes of dissolved gas and oil both measured in standard condition:

$$R_s = \frac{\rho_o^{ST}/\bar{Y}_o}{\rho_g^{ST}/\bar{Y}_g} , \quad (33)$$

or using  $\bar{Y}_o + \bar{Y}_g = 1$ :

$$R_s = \frac{\rho_o^{ST}}{\rho_g^{ST}} \frac{\bar{Y}_g}{1 - \bar{Y}_g} . \quad (34)$$

Equations may now be rewritten using the above definitions and denoting porous volume by  $V_p = \bar{\phi}\Omega$ .

$$\frac{\partial}{\partial t} \left( \frac{\bar{S}_w V_p}{B_w} \right) = Q_w^{ST} , \quad (35)$$

$$\frac{\partial}{\partial t} \left( \frac{\bar{S}_o V_p}{B_o} \right) = Q_o^{ST} , \quad (36)$$

$$\frac{\partial}{\partial t} \left[ V_p \left( \frac{\bar{S}_g}{B_g} + \frac{\bar{S}_o R_s}{B_o} \right) \right] = Q_g^{ST} , \quad (37)$$

$$\bar{S}_w + \bar{S}_o + \bar{S}_g = 1 . \quad (38)$$

Integrating Eqs. (35), (36) and (37) between times  $t_1$  and  $t_2$  gives:

$$\frac{V_{p2} \bar{S}_{w2}}{B_{w2}} = W + (W_i + W_e - W_p) , \quad (39)$$

$$\frac{\bar{S}_{o2} V_{p2}}{B_{o2}} = N - N_p , \quad (40)$$

$$V_{p2} \left( \frac{\bar{S}_{g2}}{B_{g2}} + \frac{\bar{S}_{o2} R_{s2}}{B_{o2}} \right) = N \left( m \frac{B_{o1}}{B_{g1}} + R_{s1} \right) + (G_i - G_p) , \quad (41)$$

$$\bar{S}_o + \bar{S}_w + \bar{S}_g = 1 . \quad (42)$$

Where  $N = V_{p1} \bar{S}_{o1} / B_{o1}$  and  $W = V_{p1} \bar{S}_{w1} / B_{w1}$  are the initial volumes of oil and water measured at standard conditions and  $m = V_{p1} \bar{S}_{g1} / B_{o1} N$  is the ratio of the initial volumes of gas and oil phases measured in reservoir conditions.

An equation for bubble-point may be obtained by setting  $S_g = 0$  in Eq. (41) and solving it for the solubility ratio at bubble-point pressure  $R_s(p = P_b)$ :

$$R_{s2}(p = P_b) = \left[ N (m B_1 / B_{g1} + R_{s1}) + G_i - G_p \right] / (N - N_p) . \quad (43)$$

### 2.2.2. Definition of a reference depth for pressure

Material balance equation is widely employed to establish a relationship between production (and injection) and reservoir pressure. It turns out, though, that pressure is different for different locations, mostly for different depths, and also for different phases. What is then the significance of having a single value of pressure for the entire reservoir volume?

For answering this question, we examine the two roles played by pressure in material balance equation: computation of averaged phase density and phase equilibrium.

We shall see that for the purpose of computing the averaged phase density, we may define some reference depth and use the corresponding local density

as an approximation to the averaged density without introducing any important error. Pressure increases with depth according to the hydrostatic gradient. Consequently, density also increases but by a negligible amount.

This approximation may be quantified by putting together the expressions for compressibility and hydrostatic equilibrium. For this, we assume a constant isothermal compressibility for oil phase

$$\left( \frac{1}{\rho_o} \frac{\partial \rho_o}{\partial p_o} \right)_{T, Y_o} = c_o \quad (44)$$

and we use hydrostatic equilibrium  $dp_o = \rho_o g dz$  to obtain the expression for the derivative of oil density with depth:

$$\left( \frac{1}{\rho_o} \frac{\partial \rho_o}{\partial z} \right)_{T, Y_o} = c_o \rho_o g. \quad (45)$$

Similarly for the gas phase, we obtain its isothermal compressibility from the gas state equation

$$\left( \frac{1}{\rho_g} \frac{\partial \rho_g}{\partial p_g} \right)_{T, Z} = \frac{1}{p_g} = \frac{\mathcal{M}}{\rho_g Z R T} \quad (46)$$

and again we use hydrostatic equilibrium to obtain the derivative of gas density with depth:

$$\left( \frac{1}{\rho_g} \frac{\partial \rho_g}{\partial z} \right)_{T, Z} = \left( \frac{\mathcal{M}}{Z R T} \right) g. \quad (47)$$

Substituting typical values in Eqs. (44) and (47), we see that oil and gas densities are rather constant with depth:

$$\left( \frac{1}{\rho_o} \frac{\partial \rho_o}{\partial z} \right)_{T, Y_o} \approx 10^{-6} \text{ m}^{-1}, \quad (48)$$

$$\left( \frac{1}{\rho_g} \frac{\partial \rho_g}{\partial z} \right)_{T, Z} \approx 10^{-5} \text{ m}^{-1}. \quad (49)$$

The reference depth for computing phase density (or formation volume factors) is then any arbitrary depth inside phase volume.

For the case of an oil reservoir covered by a gas cap, the reference depth may be defined, with advantage, at the gas-oil contact. The point here is that at phase contact both gas and oil densities may be computed using the same pressure, i.e., the corresponding local value.

If the reference depth for oil phase is otherwise defined below gas-oil contact, gas density must be computed at some different depth still inside gas phase, i.e., at or above gas-oil contact. This is to avoid the effect of oil hydrostatic gradient in the computation of gas density,

$$\left( \frac{1}{\rho_g} \frac{\partial \rho_g}{\partial z} \right)_{T,Z} = \frac{\rho_o}{\rho_g} \left( \frac{\mathcal{M}}{ZRT} \right) g, \quad (50)$$

what is  $\rho_o/\rho_g$  times greater than the sole effect of gas hydrostatic pressure on its density.

It has been shown that pressure variation with depth is not important for the computation of gas, oil and water densities. Nevertheless, it was said that pressure also plays a second role in material balance equation by defining phase equilibrium. For this purpose, pressure variation with depth may not be negligible in the evaluation of the solubility ratio:

$$\frac{1}{R_s} \frac{dR_s}{dz} = \frac{1}{R_s} \frac{dR_s}{dp_o} \frac{dp_o}{dz} = \frac{\rho_o g}{R_s} \frac{dR_s}{dp_o}. \quad (51)$$

For a non-volatile saturated oil, we may approximate  $dR_s/dp_o$  to the ratio between solubility ratio ( $R_s$ ) and saturation pressure ( $p_{sat}$ ), so that Eq. (51) becomes:

$$\frac{1}{R_s} \frac{dR_s}{dz} \approx \frac{\rho_o g}{p_{sat}} \approx 10^{-3} \text{ m}^{-1}, \quad (52)$$

where the following orders of magnitudes were assumed:  $p_{sat} \approx 10^7 \text{ Pa}$ ,  $\rho_o \approx 10^3 \text{ kg/m}^3$  and  $g \approx 10 \text{ m}^2/\text{s}$ .

From Eq. (52), we see that the variation of pressure with depth may not be negligible for the phase equilibrium, specially for reservoirs of light oils with large depth ranges. In such cases, using an averaged value for the solubility ratio may be worth the accuracy gain in dissolved gas volume.

### 2.2.3. Material balance as an equation for reservoir pressure

We now look to solve Eqs. (39), (40) and (41) for the averaged quantities.

A single equation for pressure may be obtained by substituting Eqs. (40), (41) and (39) into (42). The resulting equation allows pressure to be calculated from given production ( $F_p$ ) and injection ( $F_i$ ) volumes:

$$F_p - F_i = \underbrace{NE_o + mNE_g + WE_w}_{\text{fluid volume expansion (E)}} + \underbrace{V_{p1} - V_{p2}}_{\text{porous volume contraction (C)}} \quad (53)$$

or

$$F_p - F_i = E + C, \quad (54)$$

where newly introduced terms are defined in Table 1. Eq. (53) is commonly referred to as the Material Balance Equation (MBE).

## 3. Solution of Material balance equation

A numerical solution of Eq. (54) is presented here. We shall start defining a residual function ( $\varepsilon$ ) for which the pressure solving Eq. (54) is a root:

$$\varepsilon(p_2) = \frac{E + F_i + C - F_p}{V_{p1}}. \quad (55)$$

We proceed showing that  $\varepsilon$  is a monotonic function provided some conditions on fluid properties are observed and that, as a consequence, only one root of

$\varepsilon$  exists at most. Equivalently, we may say that only one solution exists for material balance equation.

### *3.1. Mathematical Aspects of the Residual Function*

It is a known result of real analysis that a monotonic function  $f : D \subset \mathcal{R} \rightarrow \mathcal{R}$  can have at most one root in  $D$ . Further, if  $f$  is continuous in the closed interval  $I = [a, b] \subset D$  and  $f(a)f(b) < 0$ , then  $f$  has one (and only one) root in  $I$ .

We now state without proof that the following affirmatives are sufficient conditions to classify a function  $f$  as monotonic:

- $f$  is continuous;
- $f$  is differentiable everywhere except for a countable set of points;
- $f' < 0$  everywhere.

Residual function clearly satisfies the first two affirmatives because fluid properties are continuous everywhere and are also differentiable everywhere except for the first order phase change at bubble-point pressure. The last affirmative is proved by rewriting  $d\epsilon/dp$  as a sum of four parcels corresponding each one to a different phase. Each phase parcel is composed of two factors:

current volume *in-situ* and volume derivative with respect to pressure:

$$\begin{aligned}
V_{p1} \frac{d\varepsilon}{dp} = & \underbrace{(N - N_p) \left( \frac{dB_o}{dp} - B_g \frac{dR_s}{dp} \right)}_{\text{oil phase}} \\
& + \underbrace{\left[ NR_{s1} + mN \frac{B_{o1}}{B_{g1}} + G_i - (N - N_p) R_{s2} - G_p \right] \frac{dB_g}{dp}}_{\text{gas phase}} \\
& + \underbrace{(W + W_i + W_e - W_p) \frac{dB_w}{dp}}_{\text{water phase}} \\
& - \underbrace{\frac{dV_p}{dp}}_{\text{porous volume}} .
\end{aligned} \tag{56}$$

*In-situ* volumes (e.g.  $N - N_p$  for oil phase) are non-negative values. Pressure derivatives of fluid volumes are all negative values (as fluid pressure increases, fluid volume contracts) and pressure derivative of rock porous volume is a positive value (as fluid pressure increases, porous volume expands).

The value of  $d\varepsilon/dp$  is thus negative because each term is a product of a negative and a positive value, being the sum of all the four terms also a negative value.

In summary, the following four relations constitute sufficient conditions for  $\varepsilon(p)$  be a strictly decreasing function and for material balance equation to have only one solution:

$$\left( \frac{dB_o}{dp} - B_g \frac{dR_s}{dp} \right) < 0, \tag{57}$$

$$\frac{dB_g}{dp} < 0, \tag{58}$$

$$\frac{dB_w}{dp} < 0, \tag{59}$$



$$-\frac{dV_p}{dp} < 0. \quad (60)$$

Switching inequalities to equalities in Eqs. (57), (58) and (59) is equivalent to enforcing system incompressibility. Eq. (57), for instance, would require a volume increase in oil phase to be equal to gas volume in gas phase for any additional gas dissolution.

### 3.2. Solution Algorithm

Several methods exist to find roots of nonlinear continuous functions (Hamming, 1987). One possible choice is Newton’s method, which linearizes the function using either its gradient or a secant line. The secant line approach introduces an artificial smoothness to the gradient direction near continuous but non-differentiable points, for what it is the approach employed here to deal with the bubble-point pressure.

An open source implementation of secant method is available in SciPy library (Jones et al., 2001–). This work used library version 0.12.

The steps to solve all material balance equation are summarized bellow:

**Step 1.** *Compute bubble-point pressure for instant  $t_2$ .*

Bubble-point pressure  $P_{b2}$  is only a function of the values in initial instant  $t_1$  and produced and injected volumes.

**Step 2.** *Compute fluid pressure for instant  $t_2$  finding the root of residual function  $(\varepsilon)$ .*

Pressure in Eq. (53) is only a function of fluid properties in instant  $t_1$ , produced and injected values and bubble-point pressure at instant  $t_2$ ,  $P_{b2}$ , which was already computed in step 1.

**Step 3.** *Compute fluid saturations  $t_2$ .*

Once  $S_{o2}$ ,  $S_{g2}$ ,  $S_{w2}$ ,  $p_2$  and  $P_{b2}$  are known, solution for instant  $t_2$  is complete.

#### 4. Results

Material balance equation were solved for a black-oil reservoir initially producing in depletion drive mechanism and later subjected to water injection. Fig. 1 shows in the solid line curve the severe depletion during initial phases of production. In the dashed curve, Fig. 1 shows the bubble-point pressure, indicating how far the saturated reservoir is from undersaturation and allowing for planning of pressure maintenance projects.

Water injection starts when oil recovery factor ( $RF$ ) is about  $RF = 0.08$  and it is maintained until the reservoir reaches a novel undersaturated state. Initially, the high compressibility of gas prevents pressure from raising and free gas phase is reduced due to redissolution. Finally, the reservoir reaches the undersaturated state and a steep pressure increase is seen in Fig. 1 for a recovery factor about  $RF = 0.15$ .

It must be observed that the result of gas reentering oil phase completely neglects non-equilibrium effects, e.g. compositional gradients, and must be used with care. For instance, it may be in strong disagreement with real reservoir behavior when gravity segregation is present.

Figure 4 shows the formation of a free gas phase with growing  $S_g$  and then its disappearance. Depending on the ratio of vertical and horizontal permeabilities or reservoir dip angle, it may also suggest the formation of a secondary gas cap.

Figure 5 shows the shape of residual function ( $\epsilon$ ) with respect to pressure, i.e., holding all produced or injected volumes constant and allowing only pressure to change. Each different solid line corresponds to a different volume of fluid production and injection. The function root is the solution of material balance equation, Eq. (54), and the non-differentiable point is the bubble-point pressure. Also, it is seen that function  $\epsilon$  monotonically decreases with pressure, as theoretically shown before.

Figure 6 shows the count of newtonian iterations required to compute pressure with an accuracy within 0.001 of initial bubble-point pressure. Typically, 12 iterations are required.

Figure 7 shows the residual of numerical solution as defined in Eq. (55). Material balance error is below  $10^{-7}$  units of porous volume for all time steps.

## 5. Conclusions

1. Material balance equation can be obtained from black-oil equations under the assumptions of:
  - (a) mechanical equilibrium;
  - (b) absence of gravity and capillary forces;
  - (c) spatially constant bubble-point pressure or, equivalently, fluid saturations.
2. One unique solution of pressure equation exists for a compressible system;
3. A residual function can be defined such that its root solves pressure equation. A numerical solution can be obtained with nonlinear root-finding algorithms.

## Nomenclature

$B_g$	gas formation volume factor
$B_o$	oil formation volume factor
$B_w$	water formation volume factor
$R_s$	solubility ratio of gas in oil phase
$S_g$	gas saturation
$S_o$	oil saturation
$S_w$	water saturation
$G_p$	produced volume of gas
$m$	ratio of original free gas volume to original oil phase volume in reservoir conditions
$N$	original volume of oil in place
$N_p$	produced volume of oil
$P_b$	bubble-point pressure
$V_p$	porous volume
$W_i$	injected volume of water
$W_p$	produced volume of water

## References

### References

- Allen III, M., Behie, A., Trangenstein, J., 1988. Multiphase flow in porous media .
- Aziz, K., Settari, A., 1979. Petroleum reservoir simulation. volume 476. Applied Science Publishers London.

- Bedrikovetsky, P., 1993. Mathematical theory of oil and gas recovery: with applications to ex-USSR oil and gas fields. volume 4. Springer.
- Danesh, A., 1998. PVT and phase behaviour of petroleum reservoir fluids. volume 47. Elsevier Science.
- Ertekin, T., Abou-Kassem, J., King, G., 2001. Basic applied reservoir simulation. Richardson, TX: Society of Petroleum Engineers.
- Firoozabadi, A., 1999. Thermodynamics of hydrocarbon reservoirs. volume 2. McGraw-Hill New York.
- Hamming, R., 1987. Numerical methods for scientists and engineers. Dover Publications.
- Herrera, I., Camacho, R., 1997. A consistent approach to variable bubble-point systems. Numerical Methods for Partial Differential Equations 13, 1–18.
- Hilfer, R., 1998. Macroscopic equations of motion for two-phase flow in porous media. Phys. Rev. E 58, 2090–2096. URL: <http://link.aps.org/doi/10.1103/PhysRevE.58.2090>, doi:10.1103/PhysRevE.58.2090.
- Ismael, H., 1996. Shocks and bifurcations in black-oil models. SPE Journal 1, 51–58.
- Jones, E., Oliphant, T., Peterson, P., et al., 2001–. SciPy: Open source scientific tools for Python. URL: <http://www.scipy.org/>.

- Kim, J., Tchelepi, H., Juanes, R., 2011. Rigorous coupling of geomechanics and multiphase flow with strong capillarity, in: SPE Reservoir Simulation Symposium.
- Sammon, P., 1984. Gas phase appearance and disappearance in fully implicit black oil simulation. Old SPE Journal 24, 505–507.
- Schilthuis, R., 1936. Active oil and reservoir energy. Trans. AIME 118, 33.
- Steffensen, R., Sheffield, M., 1973. Reservoir simulation of a collapsing gas saturation requiring areal variation in bubble-point pressure, in: SPE Symposium on Numerical Simulation of Reservoir Performance.
- Thomas, L., Lumpkin, W., Reheis, G., 1976. Reservoir simulation of variable bubble-point problems. Old SPE Journal 16, 10–16.
- Trangenstein, J., Bell, J., 1989a. Mathematical structure of compositional reservoir simulation. SIAM journal on scientific and statistical computing 10, 817–845.
- Trangenstein, J., Bell, J., 1989b. Mathematical structure of the black-oil model for petroleum reservoir simulation. SIAM Journal on Applied Mathematics , 749–783.
- Udell, K., 1982. The thermodynamics of evaporation and condensation in porous media, in: SPE California Regional Meeting.
- Utseth, R., MacDonald, R., 1981. Numerical simulation of gas injection in oil reservoirs, in: SPE Annual Technical Conference and Exhibition.

## Highlights

1. The existing link between black-oil and material balance equation is examined in detail.
2. It is shown that gravity and capillary forces must be neglected and saturations must be constant in space so one can integrate black-oil equations in reservoir volume and obtain material balance equation.
3. Material balance equation are shown to have a unique solution for a compressible system.
4. Material balance equation are formulated for a variable bubble-point and they are employed to a reservoir under pressure maintenance.

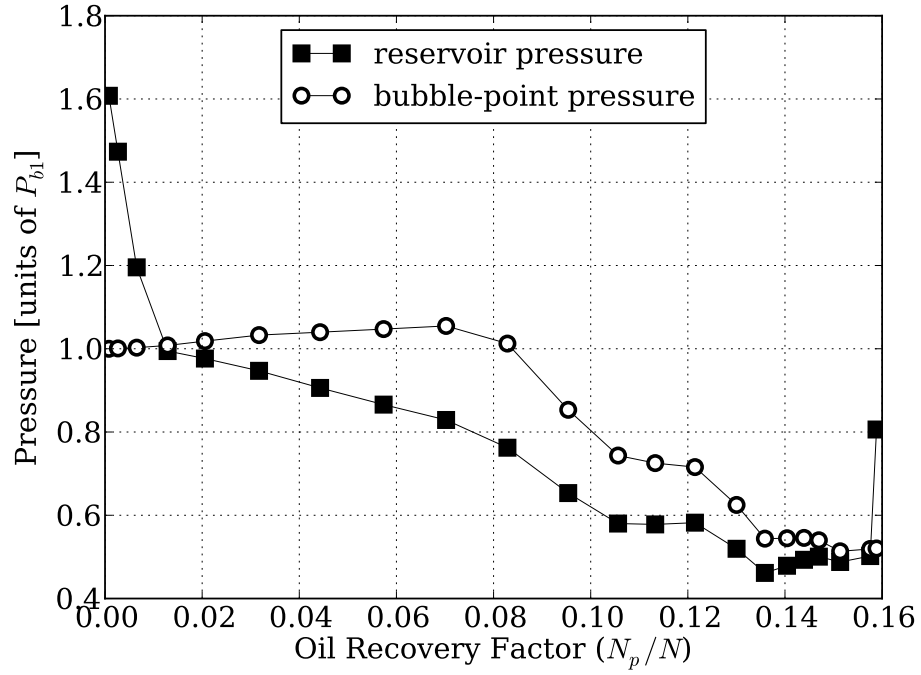


Figure 1: Reservoir pressure and bubble-point pressure as functions of recovery factor. Once depletion goes below bubble-point, pressure is hardly recovered for a saturated reservoir due to high system compressibility. Pressure is here given in units of the original bubble-point pressure.



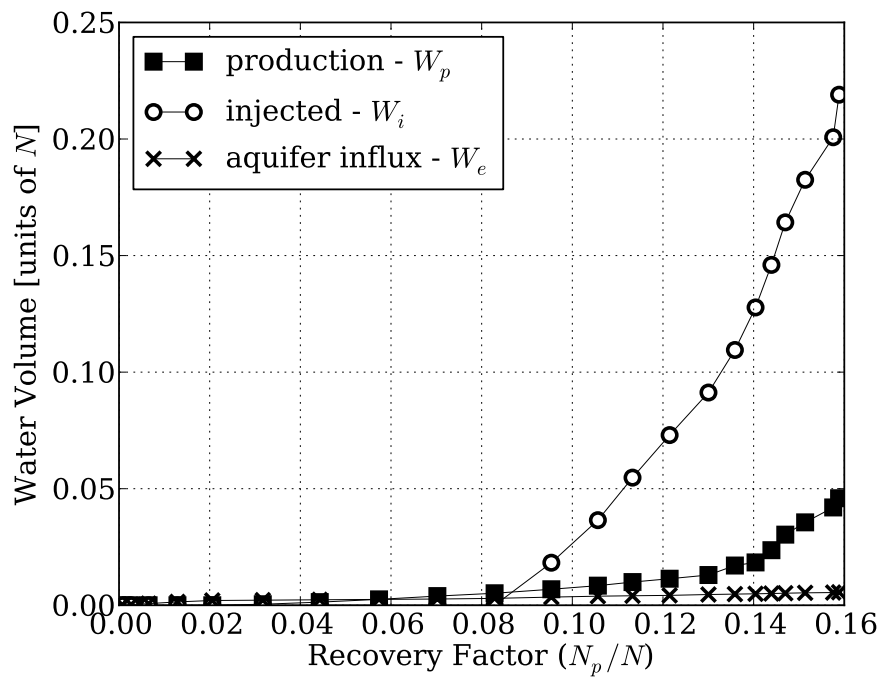


Figure 2: Volume of water production and injection and aquifer influx.

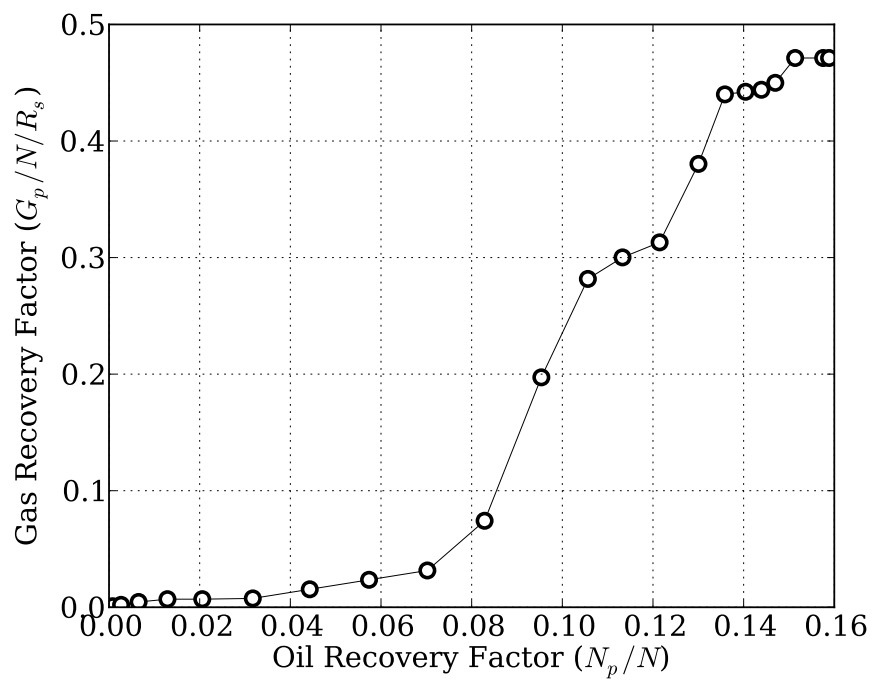


Figure 3: Volume of gas production.

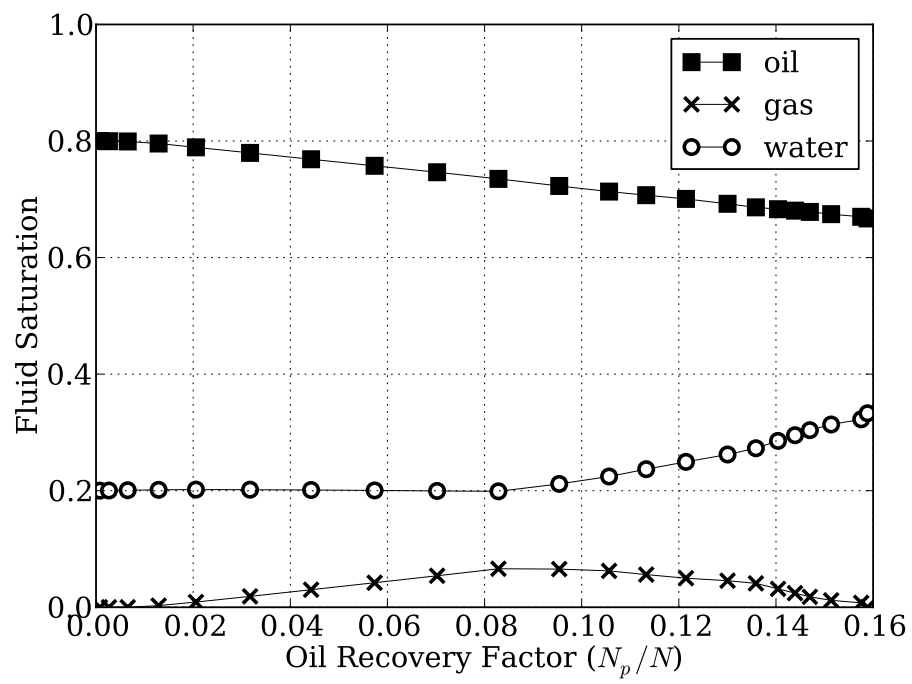


Figure 4: Phase saturations. Gas saturation is reduced as water injection progresses.

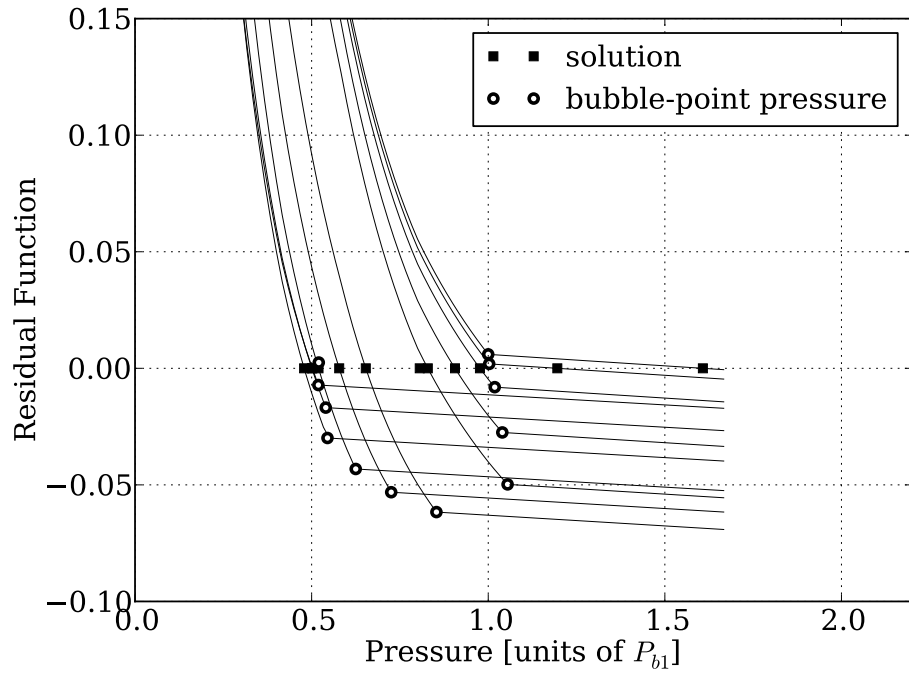


Figure 5: Shape of residual function ( $\varepsilon$ ). Each curve corresponds to different volumes of production and injection. Residual function monotonically decreases with pressure and it has no derivative at the bubble-point.

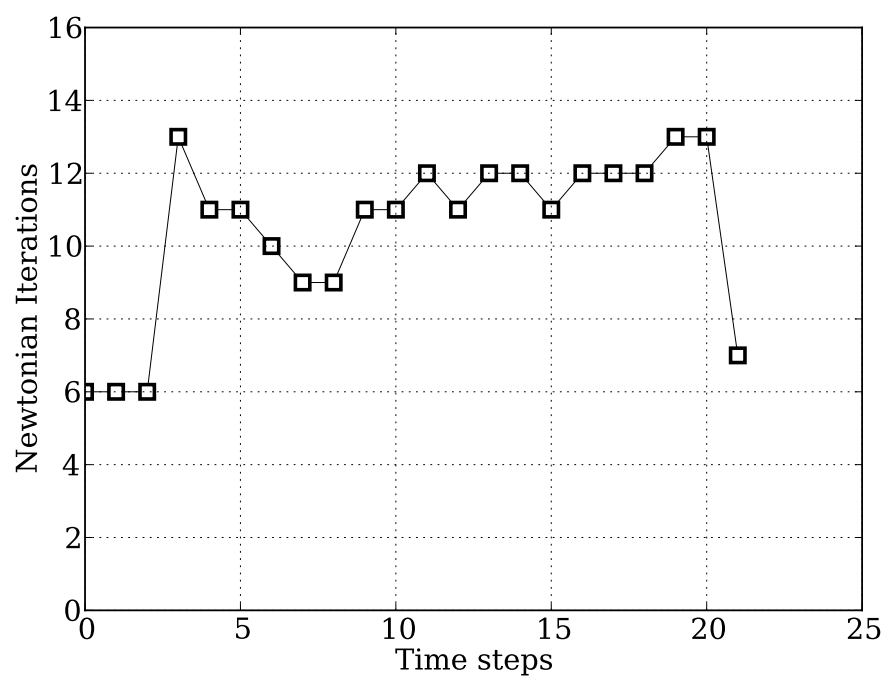


Figure 6: Count of newtonian iterations required to compute pressure with accuracy within 0.001 units of initial bubble-point pressure.

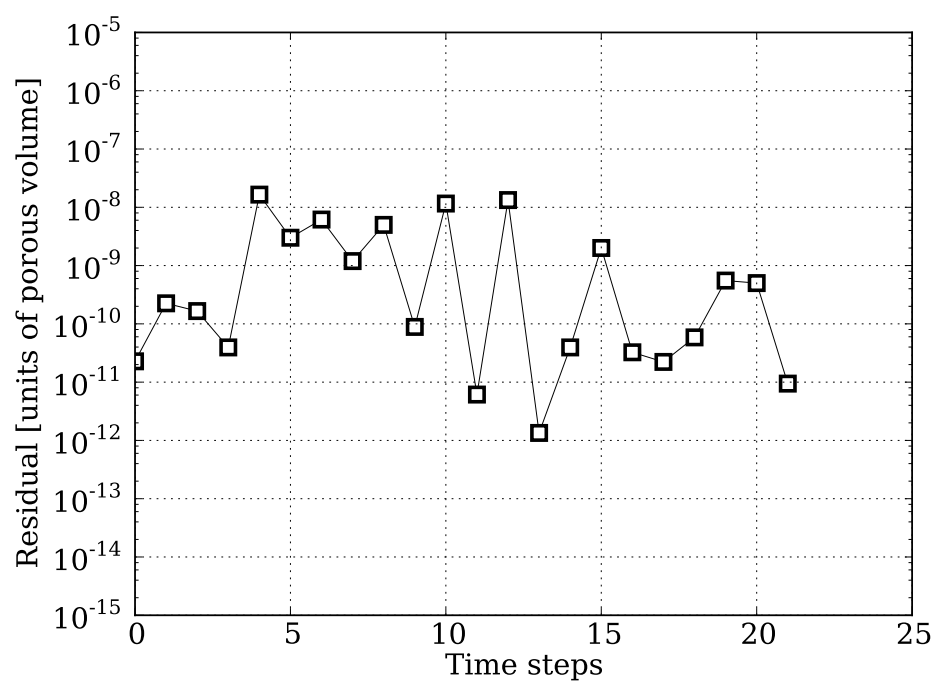


Figure 7: Residual of numerical solution in units of porous volume.

<i>Material balance equation</i>	
oil	$S_{o2} = (N - N_p) B_{o2}/V_{p2}$
water	$S_{w2} = (W + W_i + W_e - W_p) B_{w2}/V_{p2}$
gas	$S_{g2} = \{N [(R_{s1} - R_{s2}) + mB_{o1}/B_{g1}] + N_p R_{s2} + G_i - G_p\} B_{g2}/V_{p2}$
bubble-point pressure	$R_{s2}(P_b) = [N (mB_1/B_{g1} + R_{s1}) + G_i - G_p] / (N - N_p)$
pressure	$F_p - F_i = NE_o + mNE_g + WE_w + C$
<i>Definition of terms in pressure equation</i>	
fluid production	$F_p = N_p [B_{o2} + (G_p/N_p - R_{s2}) B_{g2}] + W_p B_{w2}$
fluid injection and influx	$F_i = (W_i + W_e) B_{w2} + G_i B_{g2}$
oil phase expansion	$E_o = (B_{o2} - B_{o1}) - (R_{so2} - R_{so1}) B_{g2}$
gas phase expansion	$E_g = B_{o1} (B_{g2}/B_{g1} - 1)$
water volume expansion	$E_w = B_{w2} - B_{w1}$
porous volume contraction	$C = V_{p2} - V_{p1}$

Table 1: Complete set of material balance equations.