Math 341, Fall 2020, Dr. Honigs Final Exam

Name : ___________________________________

This exam has a total of 100 points.

You may use any resources you like to complete this exam *except* for consulting with another person. You also may not share or make the contents of this exam public in any way. You must also be present in the proctoring session on zoom with your camera on. By turning in your exam for grading, you are indicating that you complied with this condition.

To receive credit for an answer you MUST show your work unless the problem indicates otherwise. If you are unable to complete a problem, make sure to tell me what you do know about it for partial credit.

If you need clarification or are having trouble with the submission process, the instructor will be checking the zoom chat (you can message me privately), checking email regularly at the address honigs@uoregon.edu, and is on-call at the US phone number 510-684-1910.

To turn in your exam, upload one .pdf to canvas through the "Final" assignment.

1. Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & -1 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

(a) (8 points) Find a basis for $\text{Nul}(A)$.

(b) (3 points) What is the dimension of $\text{Nul}(A)$? If you weren't able to solve part (a), tell me how you would use the solution to part (a) to answer this problem.

(c) (3 points) Give the statement of the equation from the Rank-Nullity theorem and use it, with your answer to part (b), to solve for $\dim(\text{Col}(A))$.

(d) (6 points) Give a basis for $\text{Col}(A)$. Explain how you found it.

2. (a) (4 points) Suppose we have a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ that are in a vector space V. What properties must this set of vectors have to be a basis for V?

(b) (6 points) Is the following set of vectors a basis for $\mathbb{R}^3$? Explain your answer:

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 7 \\ -9 \end{bmatrix} \right\}$$

(c) (6 points) Suppose we have an invertible $n \times n$ matrix. Must its columns give a basis for $\mathbb{R}^n$? You may use facts from the Invertible Matrix Theorem in your answer.

For your convenience, the page of the textbook with the Invertible Matrix Theorem is attached at the end of this exam.

3. (15 points) Consider the following linear system in the variables x_1, x_2, x_3, x_4:

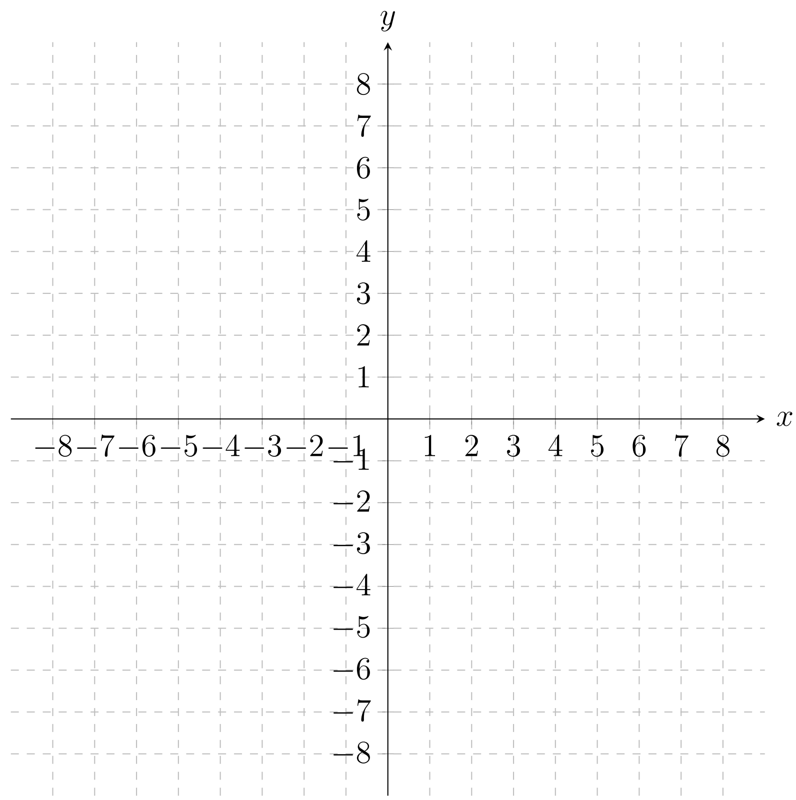
$$\begin{cases} 3x_1 - 7x_2 + 8x_3 - 5x_4 = 8 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 = 6 \\ 3x_2 - 6x_3 + 6x_4 = 3 \end{cases}$$

Put the linear system into an augmented matrix and perform row operations to get it into **reduced** echelon form.

Then, use the reduced echelon form matrix to find the set of all solutions to the linear system. If there aren't any solutions, explain how you know. If there are infinitely many solutions, express leading variables in terms of constants and free variable(s).

4. Consider the subset of $\mathbb{R}^2$ consisting of all vectors of the form $\begin{bmatrix} s+3 \\ s \end{bmatrix}$ where s can be any real number.

(a) (5 points) First, let's take a moment to understand what this set of vectors looks like. Pick three different vectors that are in this set and draw them on the following graph. (Hint: Pick three different values of s.) If you do this problem on separate paper, that's fine. Just make sure to draw your vectors neatly and label them so it's clear what they are.



(b) (5 points) What conditions does a subset of a vector space need to satisfy for it to be a subspace? (Hint: There are three of them.)

(c) (5 points) Is the subset of $\mathbb{R}^2$ given at the beginning of the problem a subspace? If yes, explain why it satisfies each of the three subspace criteria. If not, give an example showing that it doesn't satisfy at least one of the criteria.

5. (a) (5 points) Consider the following basis for $\mathbb{P}_2$: $\mathcal{B} = \{1 + x, 3 + x^2, -2x + x^2\}$. (Reminder: $\mathbb{P}_2$ is the vector space of all polynomials of degree at most 2)

If $p(x)$ is in $\mathbb{P}_2$ and we know $[p(x)]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$, what is $p(x)$?

(b) (5 points) Consider the following basis for $\mathbb{R}^2$: $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

If $\mathbf{v} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$, what is $[\mathbf{v}]_{\mathcal{B}}$?

6. (a) (8 points) The following matrix is invertible. Find its inverse:

$$A = \begin{bmatrix} 0 & -1 & 0 \\ -1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

(b) (6 points) Consider the following vector $\mathbf{b}$:

$$\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Use your solution to part (a) to solve the equation $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$.

7. (a) (8 points) Find the determinant of the following matrix. You may use any method you like. Show your work.

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 5 & 2 \\ 0 & 4 & -3 & 3 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

(b) (2 points) Use your answer to the previous part to decide whether the matrix from part (a) is invertible. If you didn't finish part (a), tell me how you would use your answer to decide.

THEOREM 8

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.
b. A is row equivalent to the $n \times n$ identity matrix.
c. A has n pivot positions.
d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e. The columns of A form a linearly independent set.
f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
h. The columns of A span $\mathbb{R}^n$.
i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
j. There is an $n \times n$ matrix C such that $CA = I$.
k. There is an $n \times n$ matrix D such that $AD = I$.
l. A^T is an invertible matrix.

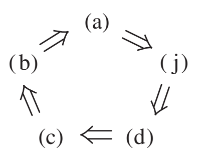
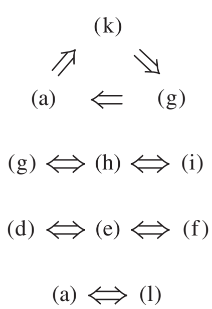


FIGURE 1

First, we need some notation. If the truth of statement (a) always implies that statement (j) is true, we say that (a) *implies* (j) and write (a) $\Rightarrow$ (j). The proof will establish the "circle" of implications shown in Figure 1. If any one of these five statements is true, then so are the others. Finally, the proof will link the remaining statements of the theorem to the statements in this circle.

PROOF If statement (a) is true, then A^{-1} works for C in (j), so (a) $\Rightarrow$ (j). Next, (j) $\Rightarrow$ (d) by Exercise 23 in Section 2.1. (Turn back and read the exercise.) Also, (d) $\Rightarrow$ (c) by Exercise 23 in Section 2.2. If A is square and has n pivot positions, then the pivots must lie on the main diagonal, in which case the reduced echelon form of A is I_n. Thus (c) $\Rightarrow$ (b). Also, (b) $\Rightarrow$ (a) by Theorem 7 in Section 2.2. This completes the circle in Figure 1.



Next, (a) $\Rightarrow$ (k) because A^{-1} works for D. Also, (k) $\Rightarrow$ (g) by Exercise 24 in Section 2.1, and (g) $\Rightarrow$ (a) by Exercise 24 in Section 2.2. So (k) and (g) are linked to the circle. Further, (g), (h), and (i) are equivalent for any matrix, by Theorem 4 in Section 1.4 and Theorem 12(a) in Section 1.9. Thus, (h) and (i) are linked through (g) to the circle.

Since (d) is linked to the circle, so are (e) and (f), because (d), (e), and (f) are all equivalent for *any* matrix A. (See Section 1.7 and Theorem 12(b) in Section 1.9.) Finally, (a) $\Rightarrow$ (l) by Theorem 6(c) in Section 2.2, and (l) $\Rightarrow$ (a) by the same theorem with A and A^T interchanged. This completes the proof. ■

Because of Theorem 5 in Section 2.2, statement (g) in Theorem 8 could also be written as "The equation $A\mathbf{x} = \mathbf{b}$ has a *unique* solution for each $\mathbf{b}$ in $\mathbb{R}^n$." This statement certainly implies (b) and hence implies that A is invertible.

The next fact follows from Theorem 8 and Exercise 8 in Section 2.2.

Let A and B be square matrices. If $AB = I$, then A and B are both invertible, with $B = A^{-1}$ and $A = B^{-1}$.