Math 341 Homework 7 — Due 12/02/2020 before the start of class

Fall 2020, Dr. Honigs

Instructions: Write up all answers neatly and clearly labeled. Turn your homework in on canvas by uploading it as a **single** .pdf file.
Unless stated otherwise, show all your work and explain your reasoning. You are encouraged to work in groups, but your final written solutions must be in your own words.
This homework will be graded out of 50 points.

1. (10 points) The matrix A, on the left below, is row equivalent to the matrix B, on the right. Find a basis for $\text{Nul}(A)$ and a basis for $\text{Col}(A)$.

$$A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2. Consider the following linear transformation:

$$T : \mathbb{P}_2 \to \mathbb{R}^2$$
$$p(x) \mapsto \begin{bmatrix} p(1) \\ p(-1) \end{bmatrix}$$

   (a) (2 points) Describe in words, what is the kernel of this linear transformation?

   (b) (7 points) Let $\mathcal{B} = \{1, x, x^2\}$ be the standard basis for $\mathbb{P}_2$. Consider the linear transformation $S : \mathbb{P}_2 \to \mathbb{R}^3$ that sends each polynomial to its $\mathcal{B}$-coordinates. S is an isomorphism, so it has an inverse, which we will call $R : \mathbb{R}^3 \to \mathbb{P}_2$. Find the standard matrix for the composition $T \circ R : \mathbb{R}^3 \to \mathbb{R}^2$. We'll call this matrix A.

   A possibly helpful explanatory comment: The idea of the matrix A is that it does the same thing as T but instead acts on the $\mathcal{B}$-coordinates of $\mathbb{P}_2$, so we will get to learn about T by studying this matrix.

   Find a basis for $\text{Nul}(A)$ and then use it to find a basis for the kernel of T by converting it from $\mathcal{B}$-coordinates to vectors in $\mathbb{P}_2$.

   (c) (1 point) What is the dimension of the kernel?

3. Next consider the linear transformation

$$T : \mathbb{P}_3 \to \mathbb{R}^2$$
$$p(x) \mapsto \begin{bmatrix} p(1) \\ p(-1) \end{bmatrix}$$

We want to find a basis for the kernel and find out what its dimension is, but since this is a more difficult problem than the last one, we want to use coordinates so that our knowledge of linear algebra can help us out.

(a) (4 points) This part won't help us solve this overall problem, but will be a good warm-up. Consider the basis $\mathcal{C} = \{1 + 2x^2, 4 + x + 5x^2, x\}$ for $\mathbb{P}_2$. What are the $\mathcal{C}$-coordinates of $3 + 2x$?

(b) (4 points) Let $\mathcal{B} = \{1, x, x^2, x^3\}$ be the standard basis for $\mathbb{P}_3$. Consider the linear transformation $S : \mathbb{P}_3 \to \mathbb{R}^4$ that sends each polynomial to its $\mathcal{B}$-coordinates. S is an isomorphism, so it has an inverse, which we will call $R : \mathbb{R}^4 \to \mathbb{P}_3$. Find the standard matrix for the composition $T \circ R : \mathbb{R}^4 \to \mathbb{R}^2$. We'll call this matrix A.

A possibly helpful explanatory comment: The idea of the matrix A is that it does the same thing as T but instead acts on the $\mathcal{B}$-coordinates of $\mathbb{P}_3$, so we will get to learn about T by studying this matrix.

(c) (4 points) What is $\dim(\mathrm{Col}(A))$? Show your work.

(d) (2 points) The Rank-Nullity Theorem says that for any $m \times n$ matrix, $\dim(\mathrm{Col}(A)) + \dim(\mathrm{Nul}(A)) = n$. This theorem is in section 4.6 of the book. We'll probably be talking about why it's true after this is due, but we definitely know enough to apply it now!

Using this theorem and your answer to the previous question, calculate what the dimension of $\mathrm{Nul}(A)$ should be.

(e) (4 points) Find a basis for $\mathrm{Nul}(A)$. What is its dimension? Is this what you expected from the answer to the previous part?

(f) (2 points) Use your basis for $\mathrm{Nul}(A)$ to find a basis for the kernel of T by converting it from $\mathcal{B}$-coordinates to vectors in $\mathbb{P}_3$. This is telling you a cool fact about polynomials :)

4. Now, suppose we're doing a physics problem where we're solving a linear differential equation to find a function $f(t)$ describing the motion of an object. We haven't seen the definition of linear differential equations, but that's ok. The important thing for us to know about them here is that the set of all possible solutions to them is a vector space. More specifically, it's a subspace of the vector space of all functions $\mathbb{R} \to \mathbb{R}$. We're using the variable t here rather than x since these types of problems often come up in physical situations where the motion of an object is a function of time.

(a) (7 points) Suppose we found two solutions to our equation: $e^{2t}\sin(t)$ and $e^{2t}\cos(t)$. Show that these functions are linearly independent.

(b) (3 points) Now suppose the span of the two solutions in the previous part is all of the solutions to this differential equation. Now, suppose that we have the additional information that $f(0) = 3$ and $f(\frac{\pi}{2}) = 5$. Given these conditions, what is $f(t)$?

(In case you're curious to know, this problem would be describing something that is resonating, like when someone sings and it eventually breaks a glass because it's vibrating too much. If you graph the functions, they look like something that is oscillating more and more and more. These are the solutions to the differential equation

$$f''(t) - 4f'(t) + 5f(t) = 0$$

but you don't need to verify this.)