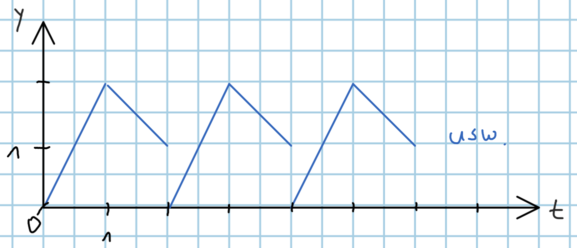
Klausur Mathe 3 ET/TI - WS 10/11 - Teil Analysis

Dienstag, 4. Januar 2011
12:47

Nr. 1



$$T = 2 \quad \Rightarrow \quad \omega = \frac{2\pi}{T} = \pi$$

$$y(t) = \begin{cases} 2t & 0 \le t < 1 \\ 3-t & 1 \le t < 2 \end{cases}$$

$$c_0 = \frac{a_0}{2} = \frac{1}{T}\int_0^T y(t)\,dt = \frac{1}{2}\left(\int_0^1 2t\,dt + \int_1^2 3-t\,dt\right) = \frac{1}{2}\left(\left[t^2\right]_0^1 + \left[3t - \frac{1}{2}t^2\right]_1^2\right)$$

$$c_0 = \frac{1}{2}\cdot\left(1 + \left(3\cdot 2 - \frac{1}{2}\cdot 2^2\right) - \left(3\cdot 1 - \frac{1}{2}\cdot 1^2\right)\right) = \frac{1}{2}\cdot\left(1 + 4 - \frac{5}{2}\right) = \frac{5}{4}$$

$$\left(\text{Aus Formelsammlung: } \int x e^{ax}\,dx = \frac{x}{a}e^{ax} - \frac{1}{a^2}e^{ax}\right)$$

$$c_n = \frac{1}{T}\int_0^T y(t)\cdot e^{-j\omega n t}\,dt = \frac{1}{2}\cdot\left(\int_0^1 2t\,e^{-j\pi n t}\,dt + \int_1^2 (3-t)\cdot e^{-j\pi n t}\,dt\right)$$

$$c_n = \frac{1}{2}\cdot\left(2\cdot\int_0^1 t\,e^{-j\pi n t}\,dt + 3\cdot\int_1^2 e^{-j\pi n t}\,dt - \int_1^2 t\cdot e^{-j\pi n t}\,dt\right)$$

$$c_n = \frac{1}{2}\cdot\left(2\cdot\left[\frac{t}{-j\pi n}e^{-j\pi n t} + \frac{1}{\pi^2 n^2}e^{-j\pi n t}\right]_0^1 + 3\cdot\left[\frac{1}{-j\pi n}e^{-j\pi n t}\right]_1^2 - \left[\frac{t}{-j\pi n}e^{-j\pi n t} + \frac{1}{\pi^2 n^2}e^{-j\pi n t}\right]_1^2\right)$$

$$c_n = \frac{1}{2}\cdot\left(2\cdot\left[\left(\frac{jt}{\pi n} + \frac{1}{\pi^2 n^2}\right)\cdot e^{-j\pi n t}\right]_0^1 + 3\cdot\left[\frac{j}{\pi n}e^{-j\pi n t}\right]_1^2 - \left[\left(\frac{jt}{\pi n} + \frac{1}{\pi^2 n^2}\right)e^{-j\pi n t}\right]_1^2\right)$$

$$c_n = \frac{1}{2}\cdot\left(2\cdot\left(\left(\frac{j}{\pi n} + \frac{1}{\pi^2 n^2}\right)e^{-j\pi n} - \frac{1}{\pi^2 n^2}\right) + 3\cdot\left(\frac{j}{\pi n}\cdot\left(e^{-2j\pi n} - e^{-j\pi n}\right)\right) - \left(\left(\frac{2j}{\pi n} + \frac{1}{\pi^2 n^2}\right)e^{-2j\pi n} - \left(\frac{j}{\pi n} + \frac{1}{\pi^2 n^2}\right)e^{-j\pi n}\right)\right)$$

$$c_n = -\frac{1}{\pi^2 n^2} + e^{-j\pi n}\cdot\left(\frac{j}{\pi n} + \frac{1}{\pi^2 n^2} - \frac{3j}{2\pi n} + \frac{j}{2\pi n} + \frac{1}{2\pi^2 n^2}\right) + e^{-2j\pi n}\cdot\left(\frac{3j}{2\pi n} - \frac{j}{\pi n} - \frac{1}{2\pi^2 n^2}\right)$$

$$c_n = -\frac{1}{\pi^2 n^2} + e^{-j\pi n}\cdot\frac{3}{2\pi^2 n^2} + e^{-2j\pi n}\cdot\left(\frac{j}{2\pi n} - \frac{1}{2\pi^2 n^2}\right)$$

$$e^{-2j\pi n} = \underbrace{\cos(-2n\pi)}_{=1} + j\underbrace{\sin(-2n\pi)}_{=0} = 1$$

$$e^{-j\pi n} = \cos(-\pi n) + j\cdot\underbrace{\sin(-\pi n)}_{=0} = (-1)^n$$

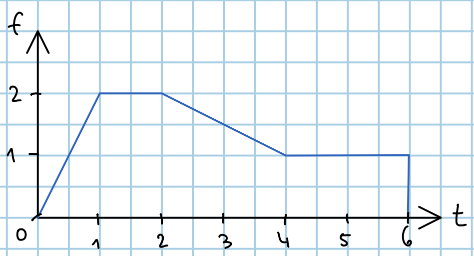
$$c_n = -\frac{1}{\pi^2 n^2} + (-1)^n\cdot\frac{3}{2\pi^2 n^2} + \frac{j}{2\pi n} - \frac{1}{2\pi^2 n^2} = \left((-1)^n - 1\right)\cdot\frac{3}{2\pi^2 n^2} + \frac{j}{2\pi n}$$

$$\Rightarrow \quad a_n = 2\cdot \operatorname{Re}(c_n) = \frac{3}{\pi^2 n^2}\cdot\left((-1)^n - 1\right) \qquad a_0 = 2\cdot c_0 = \frac{5}{2}$$

$$b_n = -2\cdot \operatorname{Im}(c_n) = -\frac{1}{\pi n}$$

15

Nr. 2



$$f(t) = \begin{cases} 2t & 0 \le t < 1 \\ 2 & 1 \le t < 2 \\ 3 - \frac{1}{2}t & 2 \le t \le 4 \\ 1 & 4 \le t \le 6 \end{cases} \quad 1$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t}\, dt = 2\int_0^1 t \cdot e^{-j\omega t}\, dt + 2 \cdot \int_1^2 e^{-j\omega t}\, dt + \int_2^4 (3 - \tfrac{1}{2}t)\, e^{-j\omega t}\, dt + \int_4^6 e^{-j\omega t}\, dt$$

(Aus Formelsammlung: $\int x e^{ax}\, dx = \frac{x}{a} e^{ax} - \frac{1}{a^2} e^{ax} = \left(\frac{x}{a} - \frac{1}{a^2}\right) \cdot e^{ax}$)

$$F(\omega) = 2 \cdot \left[\left(\frac{t}{-j\omega} + \frac{1}{\omega^2}\right) \cdot e^{-j\omega t}\right]_0^1 + 2 \cdot \left[\frac{1}{-j\omega} e^{-j\omega t}\right]_1^2 + 3 \cdot \left[\frac{1}{-j\omega} e^{-j\omega t}\right]_2^4 - \frac{1}{2} \cdot \left[\left(\frac{t}{-j\omega} + \frac{1}{\omega^2}\right) e^{-j\omega t}\right]_2^4 + \left[\frac{1}{-j\omega} \cdot e^{-j\omega t}\right]_4^6$$

$$F(\omega) = 2 \cdot \left(\left(\frac{j}{\omega} + \frac{1}{\omega^2}\right) \cdot e^{-j\omega} - \frac{1}{\omega^2}\right) + \frac{2j}{\omega}\left(e^{-2j\omega} - e^{-j\omega}\right) + \frac{3j}{\omega} \cdot \left(e^{-4j\omega} - e^{-2j\omega}\right) - \frac{1}{2}\left(\left(\frac{4j}{\omega} + \frac{1}{\omega^2}\right) e^{-4j\omega} - \left(\frac{2j}{\omega} + \frac{1}{\omega^2}\right) e^{-2j\omega}\right) + \frac{j}{\omega}\left(e^{-6j\omega} - e^{-4j\omega}\right)$$

$$F(\omega) = -\frac{2}{\omega^2} + e^{-j\omega} \cdot \left(\frac{2j}{\omega} + \frac{2}{\omega^2} - \frac{2j}{\omega}\right) + e^{-2j\omega}\left(\frac{2j}{\omega} - \frac{3j}{\omega} + \frac{j}{\omega} + \frac{1}{2\omega^2}\right) + e^{-4j\omega}\left(\frac{3j}{\omega} - \frac{2j}{\omega} - \frac{1}{2\omega^2} - \frac{j}{\omega}\right) + e^{-6j\omega} \cdot \frac{j}{\omega}$$

$$F(\omega) = -\frac{2}{\omega^2} + e^{-j\omega} \cdot \frac{2}{\omega^2} + e^{-2j\omega} \cdot \frac{1}{2\omega^2} - e^{-4j\omega} \cdot \frac{1}{2\omega^2} + e^{-6j\omega} \cdot \frac{j}{\omega} \quad 1$$

$$F(\omega) = \frac{1}{\omega^2} \cdot \left(-2 + 2e^{-j\omega} + \frac{1}{2} e^{-2j\omega} - \frac{1}{2} e^{-4j\omega} + j\omega\, e^{-6j\omega}\right) \quad 1$$

12

Nr. 4

$$F_z(z) = \frac{z^2 + 6z - 8}{z^2 + z - 2} = \frac{z^2 + z - 2 + 5z - 6}{z^2 + z - 2} = 1 + \frac{5z - 6}{z^2 + z - 2} \quad 1$$

$$F_z(z) = 1 + \frac{5z - 6}{(z+2)(z-1)} = 1 + \frac{A}{(z-1)} + \frac{B}{(z+2)} = 1 + \frac{\left(-\frac{1}{3}\right)}{z-1} + \frac{\frac{16}{3}}{z+2}$$

$$5z - 6 = A \cdot (z+2) + B \cdot (z-1)$$

$z = 1$: $\quad -1 = 3A \iff A = -\frac{1}{3}$ 1

$z = -2$: $\quad -16 = -3B \iff B = \frac{16}{3}$

$$f_n = \delta_{n,0} + \left(-\tfrac{1}{3}\right) \cdot \sigma(n-1) \cdot 1^{n-1} + \tfrac{16}{3} \cdot \sigma(n-1) \cdot (-2)^{n-1} \quad 1$$

$$f_n = \begin{cases} 1 & \text{für } n = 0 \\ -\frac{1}{3} + \frac{16}{3} \cdot (-2)^{n-1} & \text{für } n \ge 1 \end{cases} \quad 1$$

6

$\boxed{\text{Nr. 3}}$

Gegeben ist die Laplacetransformierte

$$F(p) = \frac{p}{p^2+1} + \frac{1}{p^2(p^2+1)} \quad ; \quad F(p) \bullet\!\!-\!\!\circ\, f(t)$$

a) Berechnen Sie unter Verwendung von $F(p)$ die Grenzwerte

$$\lim_{t\to 0}\big(f(t)\big) \quad \text{und} \quad \lim_{t\to\infty}\big(f(t)\big)$$

b) Berechnen Sie die Rücktransformierte $f(t)$ von $F(p)$ mithilfe der Partialbruchzerlegung und den Laplace-Korrespondenzen.

Lsg.: a) $p\cdot F(p) = \frac{p^2}{p^2+1} + \frac{1}{p(p^2+1)}$ 1

und $\lim_{t\to 0} f(t) = \lim_{p\to\infty}\big(p\cdot F(p)\big) = 1$ 1 1

$\lim_{t\to\infty} f(t) = \lim_{p\to 0}\big(p\cdot F(p)\big) = \infty$ 1 1

b) $F(p) = \frac{p}{p^2+1} + \frac{1}{p^2(p^2+1)} = \frac{p}{p^2+1} + \frac{A}{p} + \frac{B}{p^2} + \frac{Cp+D}{p^2+1}$ 1

$\Rightarrow \quad 1 = A\,p(p^2+1) + B(p^2+1) + (Cp+D)\cdot p^2$ 1

$p=0: \quad \boxed{1 = B}$ 1

$p=1: \quad \text{I)}\ 1 = 2A + 2 + C + D$

$p=-1: \quad \text{II)}\ 1 = -2A + 2 - C + D$

$\oplus: \quad 2 = 4 + 2D \quad \Rightarrow \quad \boxed{D = -1}$ 1

$p=2: \quad \text{III)}\ 1 = 10A + 5 + 8C - 4$

$\text{I)}\quad 0 = 2A + C$

$\text{III)}\quad 0 = 10A + 8C$

$\text{III)} - 5\cdot\text{I)}: \quad 0 = 3C \quad \Rightarrow \quad \boxed{C = A = 0}$ 1

$\Rightarrow \quad F(p) = \frac{p}{p^2+1} + \frac{1}{p^2} - \frac{1}{p^2+1}$ 1

$\circ\!\!-\!\!\bullet$

$f(t) = \cos t + t - \sin t$ 1

12

45