2.2. Signaleigenschaften
Signalbeziehungen

2.2.1 Symmetrieeigenschaften

$f(t) = f_g(t) + f_u(t)$ → reell!

$e^{-j\omega t} = \cos\omega t - j\sin\omega t$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\cdot e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} \left[f_g(t) + f_u(t)\right]\cdot\left[\cos\omega t - j\sin\omega t\right] dt$$

$$= \int_{-\infty}^{\infty} f_g(t)\cdot\cos\omega t\,dt + \underbrace{\int_{-\infty}^{\infty} f_u(t)\cdot\cos\omega t\,dt}_{=0}$$

$$- \underbrace{j\int_{-\infty}^{\infty} f_g(t)\cdot\sin\omega t\,dt}_{0} - j\int_{-\infty}^{\infty} f_u(t)\cdot\sin\omega t\,dt$$

$$= \int_{-\infty}^{\infty} f_g(t)\cdot\cos\omega t\,dt - j\int_{-\infty}^{\infty} f_u(t)\cdot\sin\omega t\,dt$$

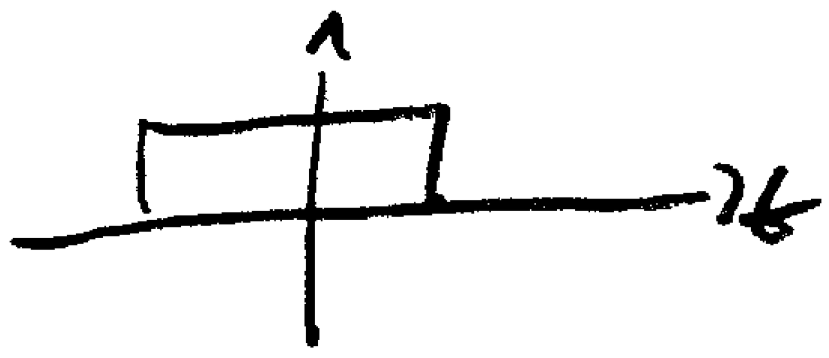
⇒ Bei reellen Zeitfunktionen gilt:

gerade Zeitfunktion ○—● reelles Spektrum

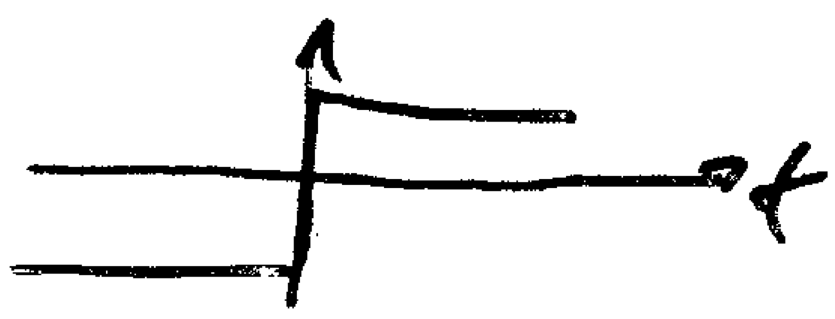
ungerade Zeitfunktion ○—● imaginäres Spektrum

Beispiele:

$\mathrm{rect}(t/T) \;\circ\!\!-\!\!\bullet\; T \cdot \mathrm{si}(\pi f T)$



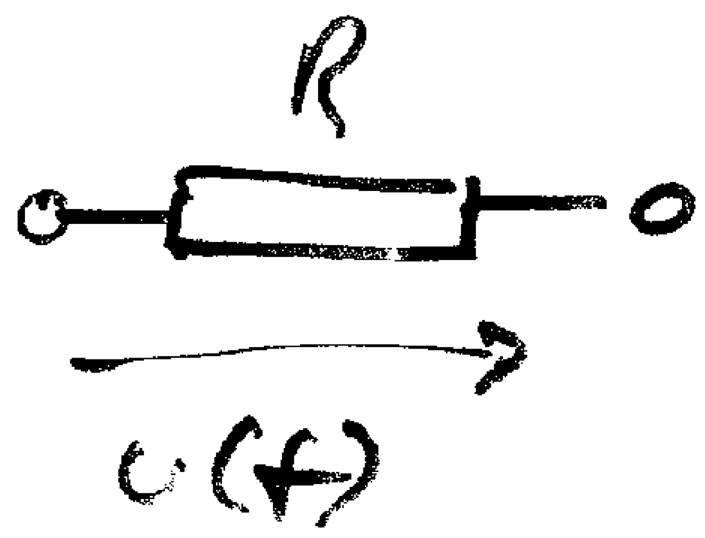
$\mathrm{sgn}(t) \;\circ\!\!-\!\!\bullet\; \frac{1}{j\pi f}$



$\cos(\omega_0 t) \;\circ\!\!-\!\!\bullet\; \frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)$

$\sin(\omega_0 t) \;\circ\!\!-\!\!\bullet\; \frac{1}{2j}\delta(f-f_0) - \frac{1}{2j}\delta(f+f_0)$

Leistung und Energie von Signalen



$$E_{el} = \frac{1}{R} \int_{t_1}^{t_2} u^2(t)\,dt$$

Elektr. Energie im Zeitabschnitt $t_1 \dots t_2$

2.2.2. Leistung und Energie von Signalen

$$E_{el} = \frac{1}{R} \int_{t_1}^{t_2} u^2(t)\,dt$$

Energie im Zeitabschnitt $t_1 \ldots t_2$ bei einer Spannung $u(t)$ an einem Widerstand R

Normierung:

$$E = \int_{t_1}^{t_2} a^2(t)\,dt$$

Normierung auf $R = 1\,\Omega$

Energiesignal: $E < \infty$

ja: ~~Impulse~~, zeitl. beschränkte Signale

nicht: periodische Signale

Leistungssignal: hier: $E_T \to \infty$

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} a^2(t)\,dt$$

$P < \infty$

~~2.2.3~~ 5.3 Parsevalsches Theorem

$$f_1(t)\cdot f_2(t) \;\circ\!\!-\!\!\!-\!\!\bullet\; F_1(f) * F_2(f)$$

$$F_1(f) * F_2(f) = \int_{-\infty}^{\infty} F_1(x)\cdot F_2(f-x)\,dx$$

$$= \int_{-\infty}^{\infty} f_1(t)\cdot f_2(t)\cdot e^{-j2\pi f t}\,dt$$

$f=0$: $$\int_{-\infty}^{\infty} F_1(x)\cdot F_2(-x)\,dx = \int_{-\infty}^{\infty} f_1(t)\cdot f_2(t)\,dt$$

$f_1(t)=f_2(t) \rightarrow F_1(x)=F_2(x)$:
$$\int_{-\infty}^{\infty} F_1(x)\cdot F_1(-x)\,dx = \int_{-\infty}^{\infty} f_1(t)\cdot f_1(t)\,dt$$
$$= \int_{-\infty}^{\infty} f_1^2(t)\,dt$$

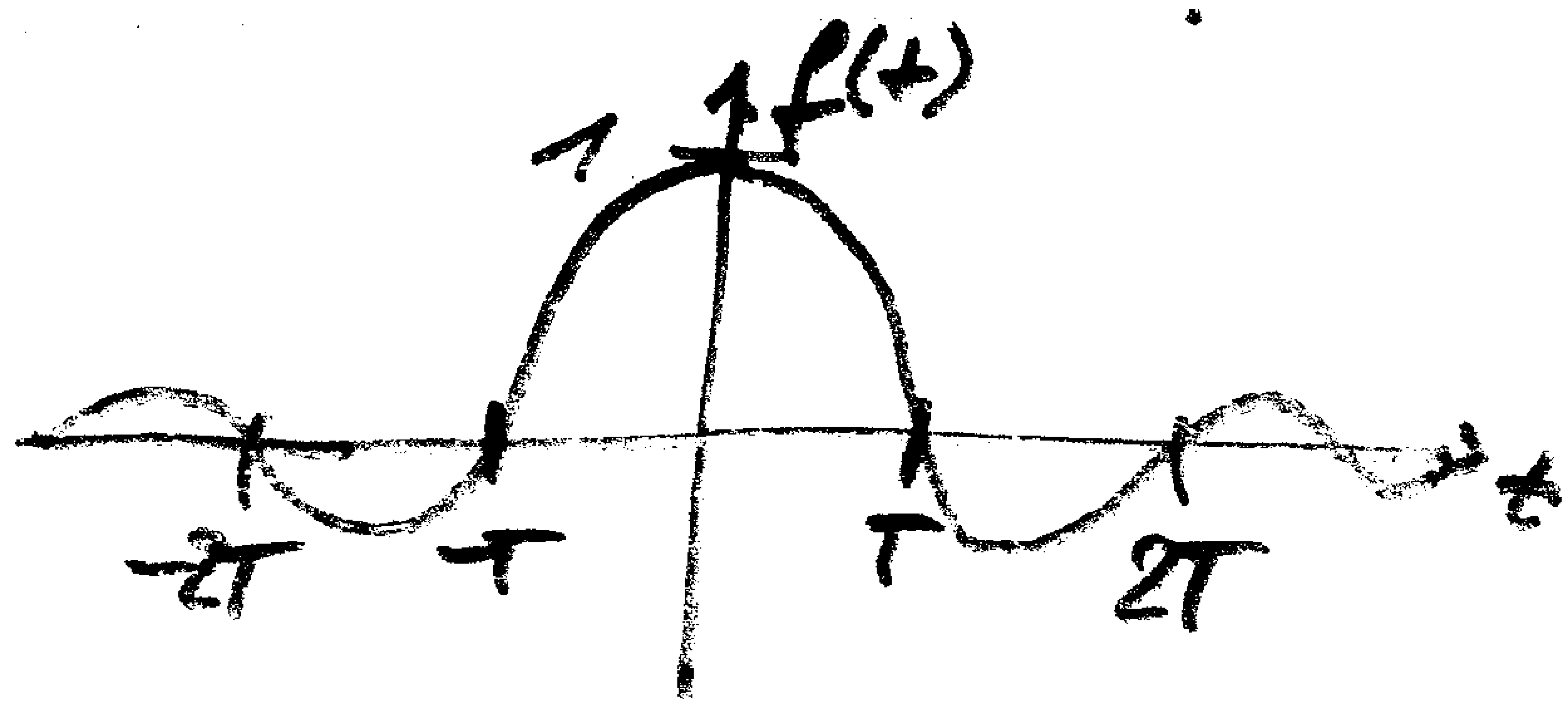
$F_1(-x) = F_1^*(x)$:
$$\int_{-\infty}^{\infty} F_1(x)\cdot F_1^*(x)\,dx$$
$$= \int_{-\infty}^{\infty} |F_1(x)|^2\,dx$$

$x=f$: $$\boxed{\int_{-\infty}^{\infty} |F_1(f)|^2\,df = \int_{-\infty}^{\infty} f_1^2(t)\,dt}$$

$|F_1(f)|^2 \rightarrow$ Energiedichtespektrum

Übung 2.5. Parsevalsches Theorem

$$f(t) = si\left(\pi \frac{t}{T}\right)$$

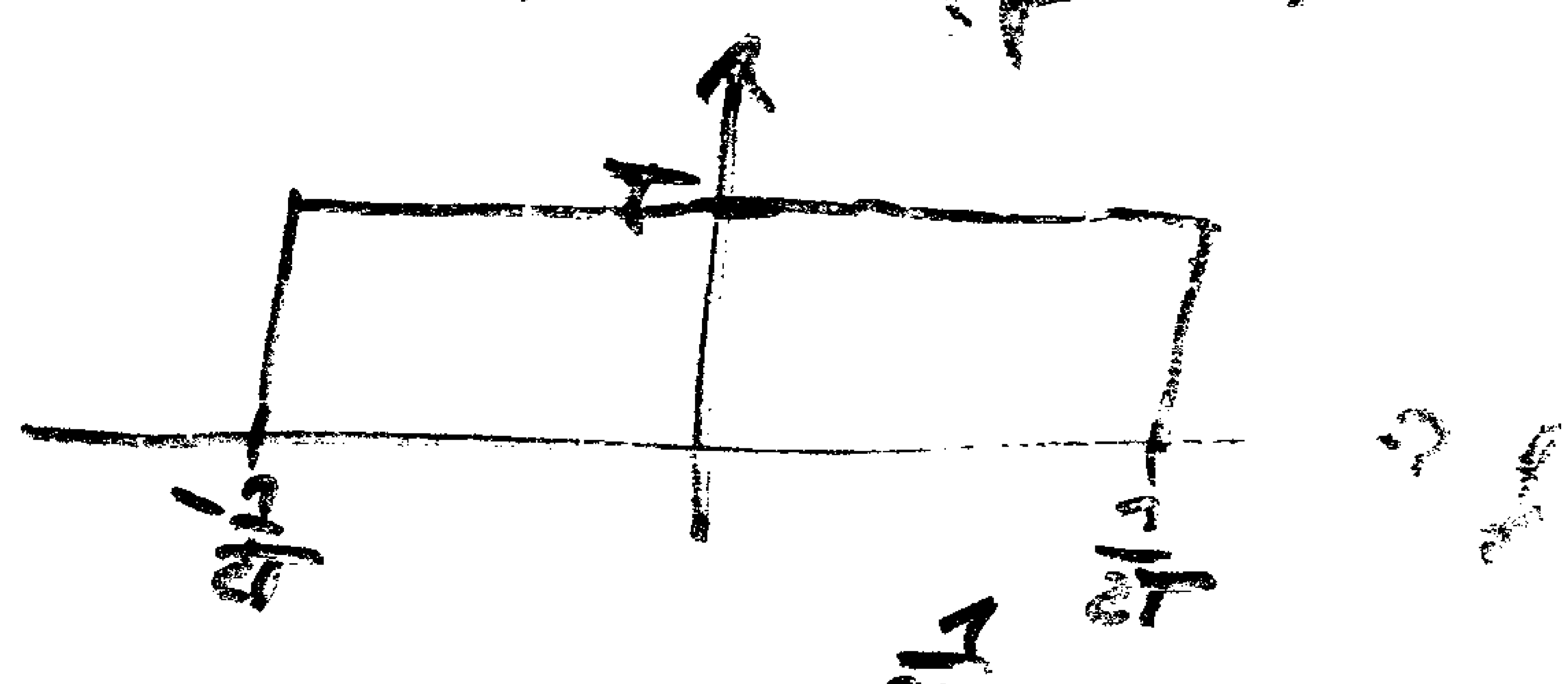


Energie E ?

$$E = \int_{-\infty}^{\infty} f^2(t)\, dt = \int_{-\infty}^{\infty} si^2\left(\pi \frac{t}{T}\right) dt = ???$$

$$\boxed{E = \int_{-\infty}^{\infty} |F(f)|^2\, df \qquad \text{Parsevalsches Theorem}}$$

F(f): $si\left(\pi \frac{t}{T}\right) \circ\!\!-\!\!\bullet\ T \cdot rect(f \cdot T)$



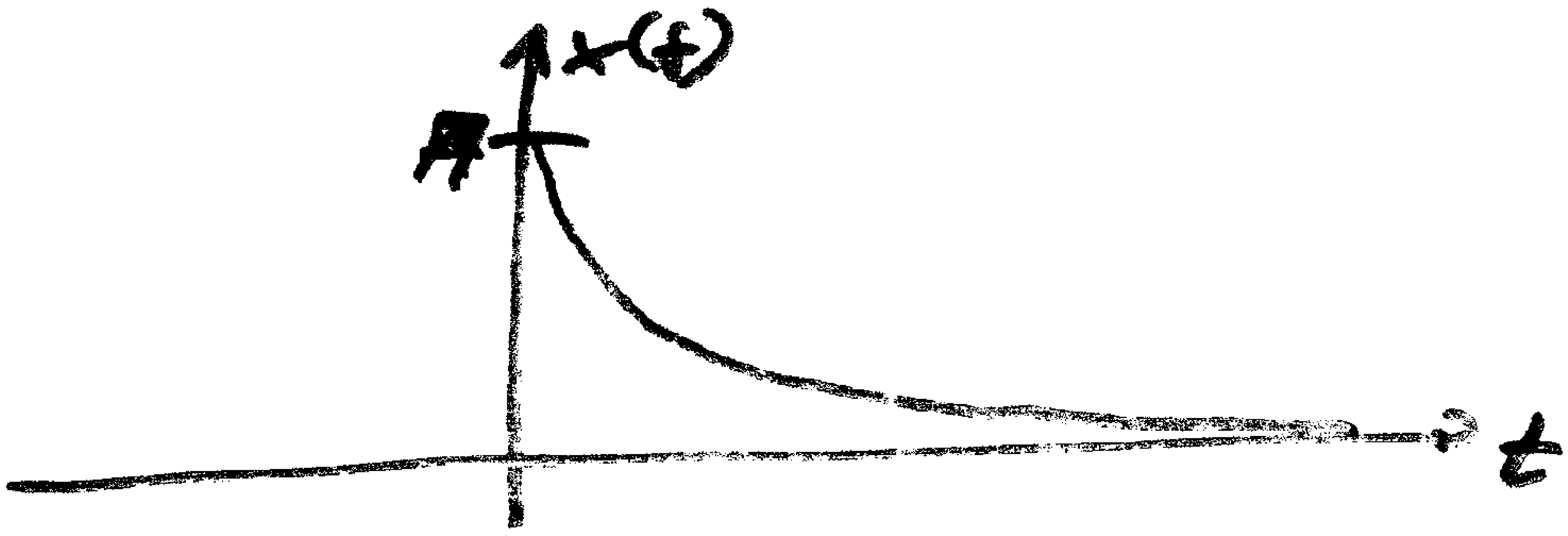
$$\Rightarrow E = \int_{-\infty}^{\infty} T^2 \cdot rect^2(fT)\, df = T^2 \int_{-\frac{1}{2T}}^{\frac{1}{2T}} 1 \cdot df$$

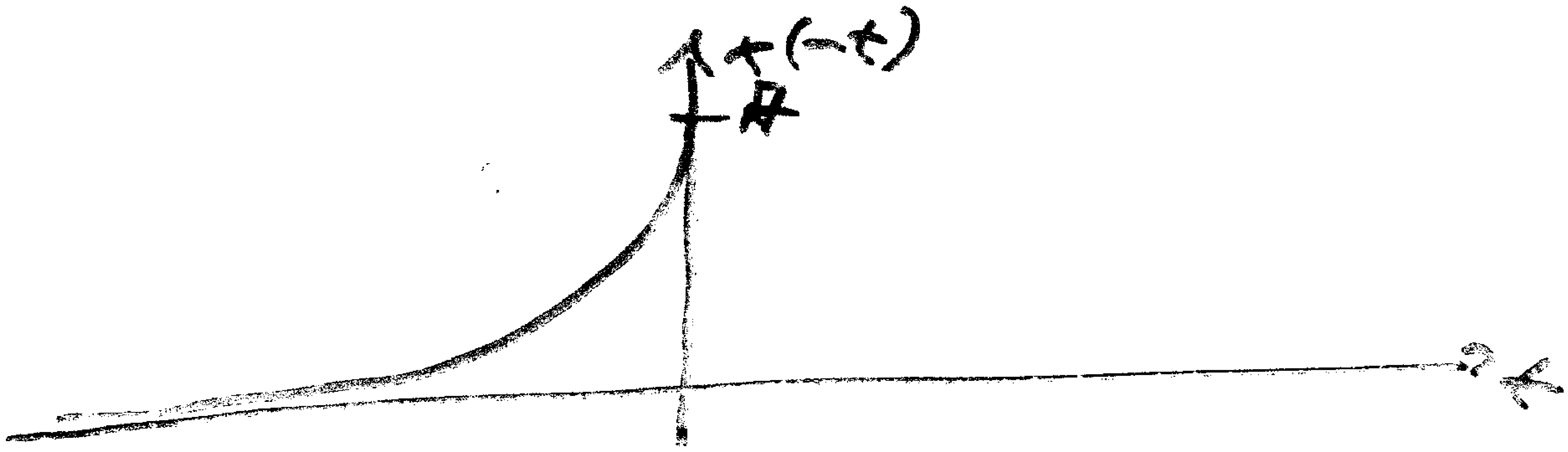
$$= T^2 \cdot f \Big|_{-\frac{1}{2T}}^{\frac{1}{2T}} = T^2 \left[\frac{1}{2T} - \left(-\frac{1}{2T}\right)\right]$$

$$= T^2 \cdot \frac{1}{T} = \underline{T}$$

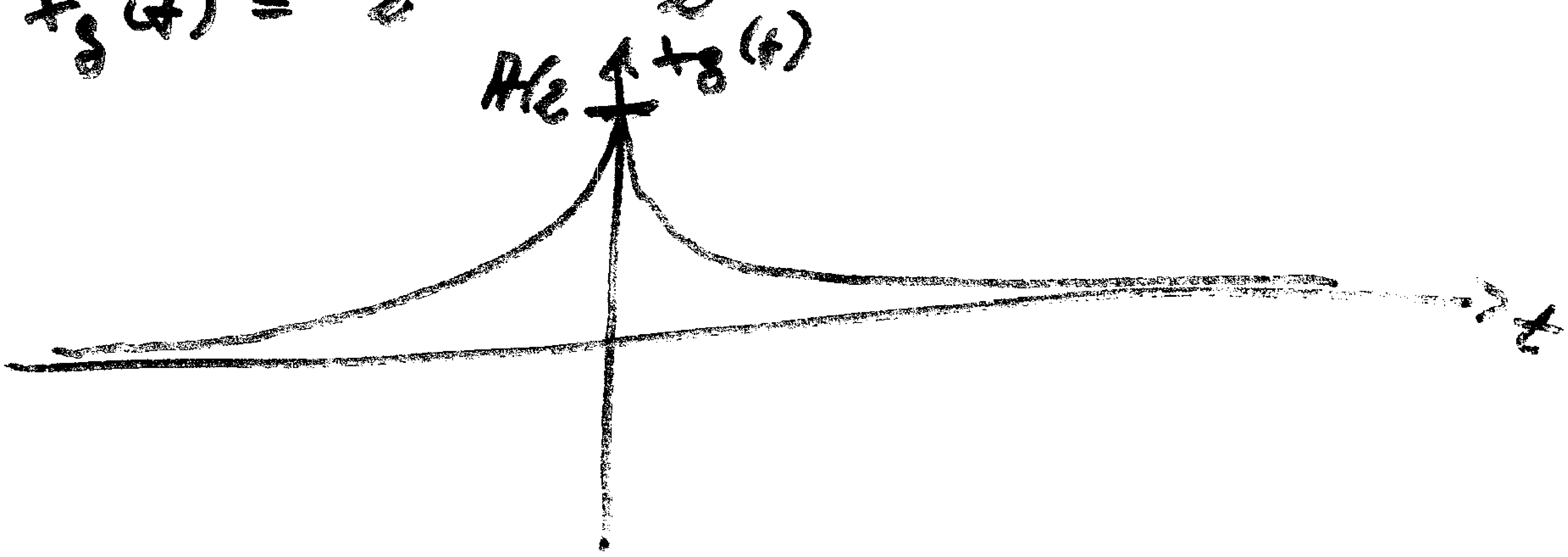
Beispiel:

$$x(t) = A \cdot e^{-t/T} \cdot \sigma(t)$$

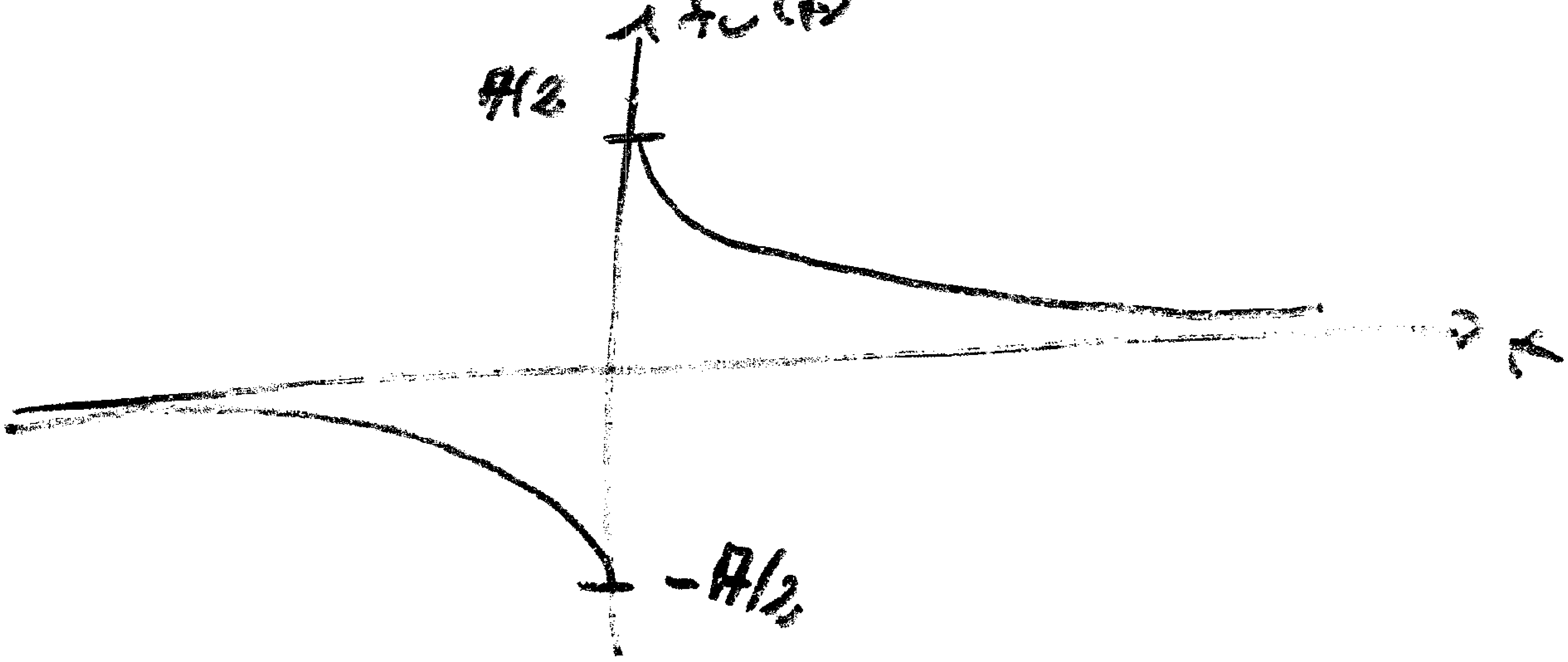




$$x_g(t) = \frac{x(t)}{2} + \frac{x(-t)}{2}$$



$$x_u(t) = \frac{x(t)}{2} - \frac{x(-t)}{2}$$

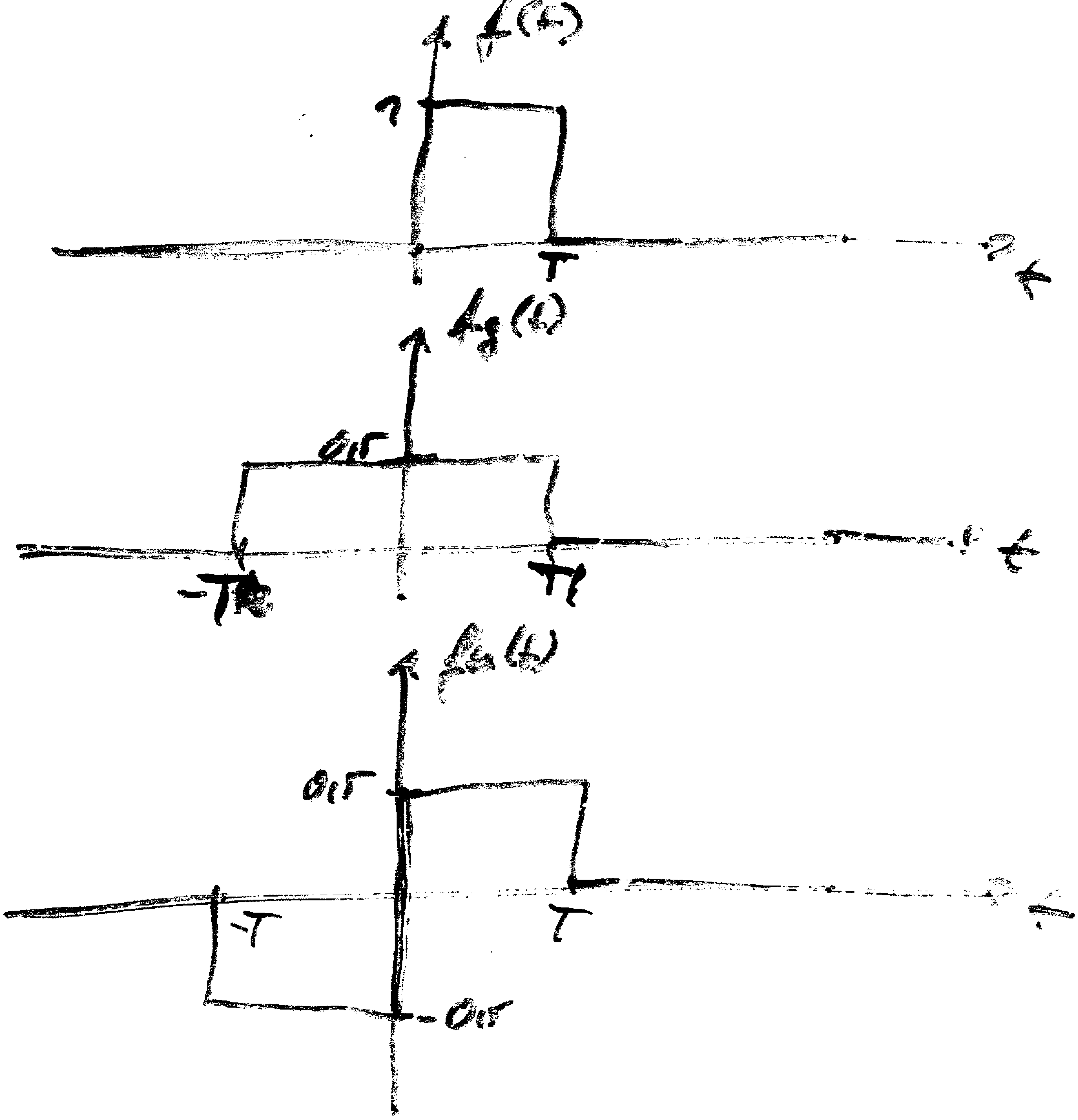


Übung 2.6. Hilbert-Transformator

a) kausale, reelle Zeitfunktion $f(t) = f_g(t) + f_u(t)$:

Zeige: $f_g(t) = f_u(t) \cdot \mathrm{sgn}(t)$

bzw. $f_u(t) = f_g(t) \cdot \mathrm{sgn}(t)$



b) Zusammenhang zwischen Real- und Imaginär-Teil des Spektrums

$$f_g(t) = f_u(t) \cdot \mathrm{sgn}(t) \qquad \Big| \; F(f) \bullet\!\!-\!\!\circ f(t)$$

$$j \cdot \mathrm{Im}\{F(f)\} * \frac{1}{j\pi f} = \mathrm{Im}\{F(f)\} * \frac{1}{\pi f}$$

$$\mathrm{Re}\{F(f)\} = \underline{\qquad\qquad\qquad}$$

$$f_u(t) = f_g(t) \cdot \mathrm{sgn}(t)$$

$$\mathrm{Re}\{F(f)\} * \frac{1}{j\pi f} = -j\,\mathrm{Re}\{F(f)\} * \frac{1}{\pi f}$$

$$\Rightarrow j\,\mathrm{Im}\{F(f)\} = \mathrm{Re}\{F(f)\} * \frac{1}{j\pi f}$$

$$\mathrm{Re}\{F(f)\} = \mathrm{Im}\{F(f)\} * \frac{1}{\pi f}$$

$$\mathrm{Im}\{F(f)\} = -\mathrm{Re}\{F(f)\} * \frac{1}{\pi f}$$

<u>Hilbert-Transformation</u>

2.2.4 Korrelationsfunktionen

$f(t)$, $g(t)$: Ähnlichkeit?

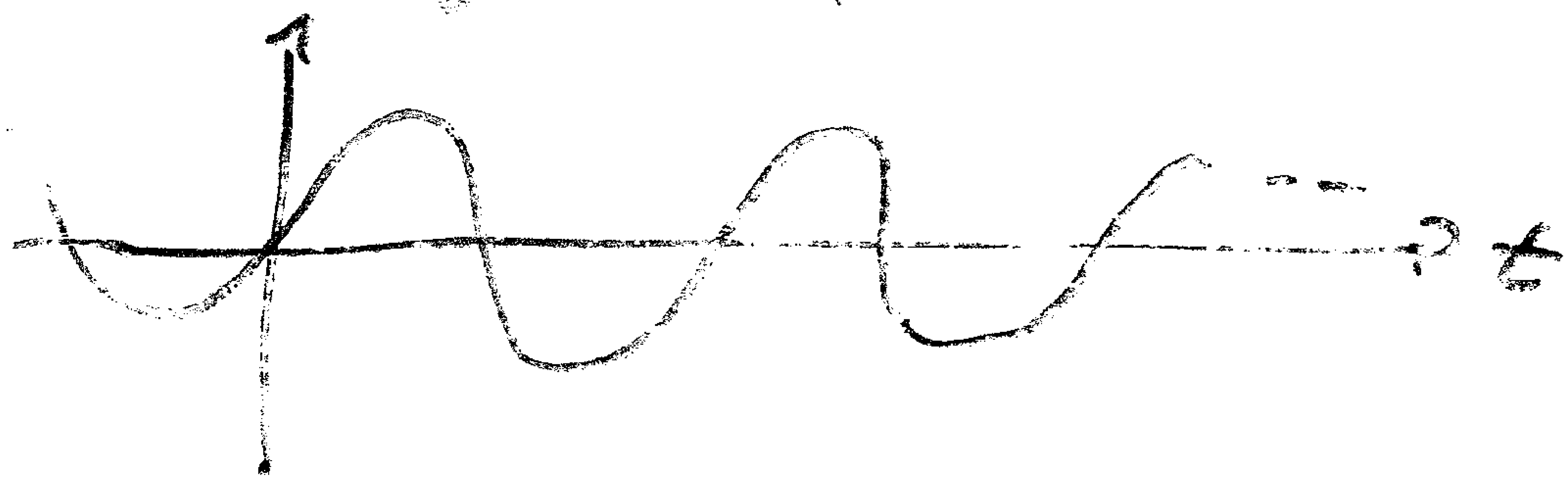
Ansatz: Differenzenergie: $E_\Delta = \int_{-\infty}^{\infty} [f(t) - g(t)]^2 \, dt$

$$\Rightarrow E_\Delta = \underbrace{\int_{-\infty}^{\infty} f^2(t)\, dt}_{E_f \,\hat{=}} + \underbrace{\int_{-\infty}^{\infty} g^2(t)\, dt}_{E_g \,\hat{=}} - 2\underbrace{\int_{-\infty}^{\infty} f(t) g(t)\, dt}_{f(f,g)}$$

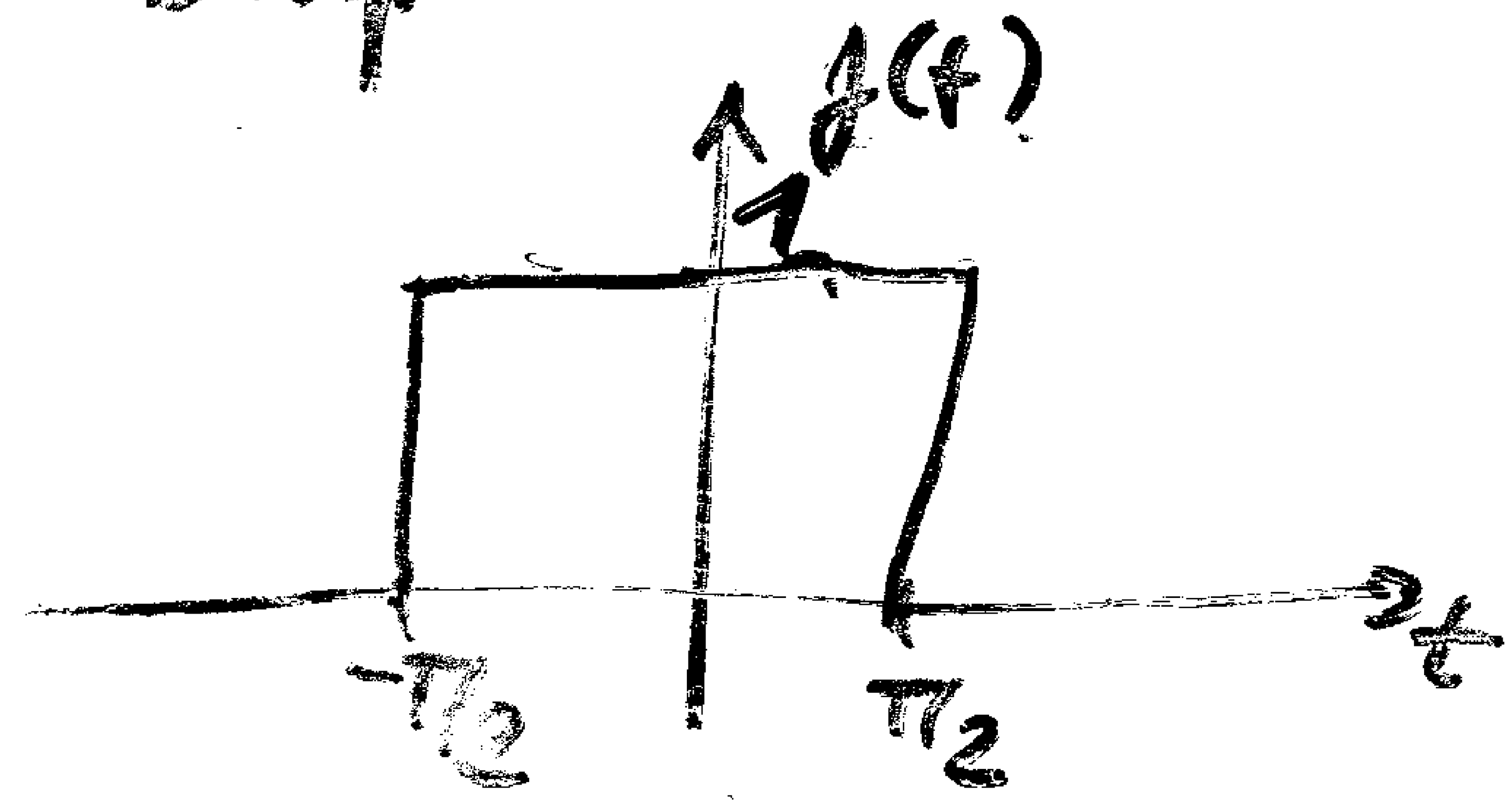
↳ Korrelationsprodukt:

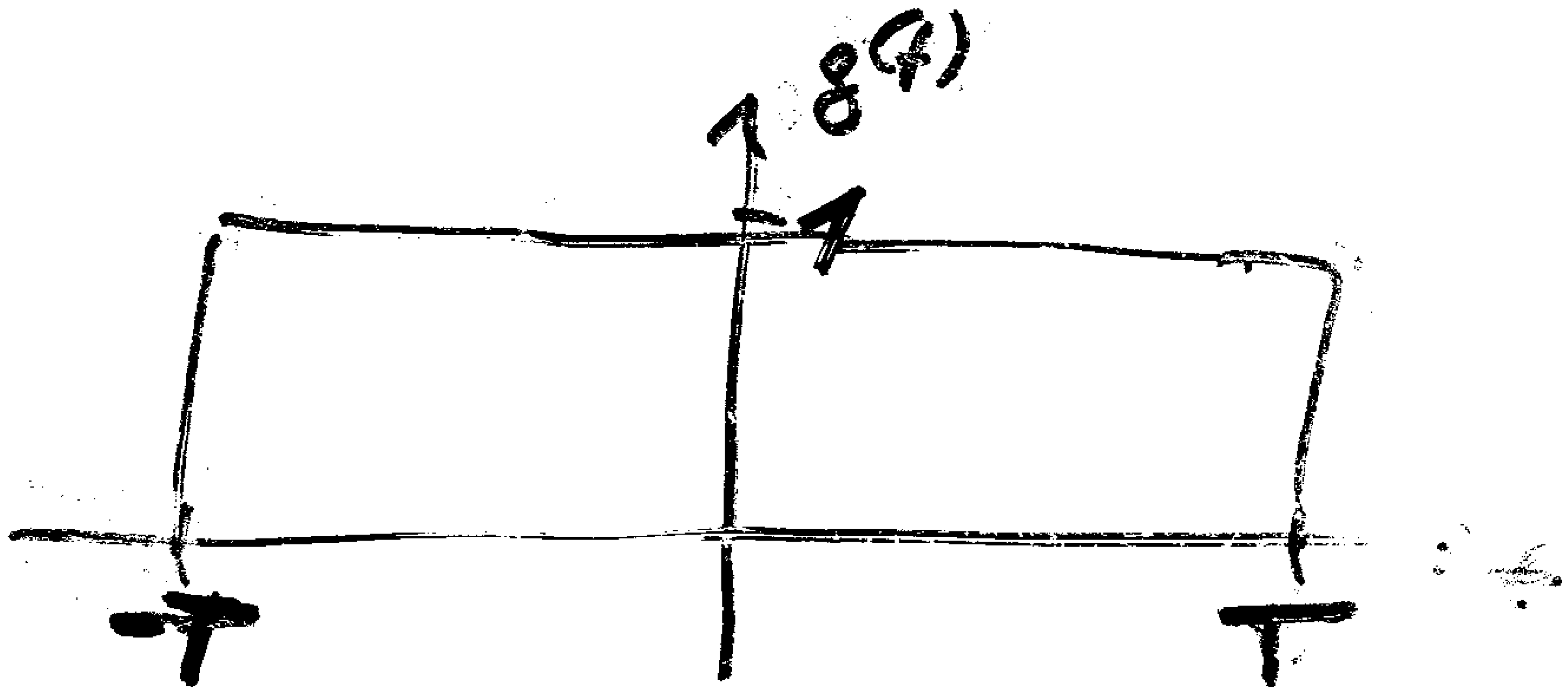
$$\varphi_{fg}(\tau) = \int_{-\infty}^{\infty} f(t) \cdot g(t+\tau)\, dt$$

z.B. Sinusfunktion $f(t) = g(t)$:



Beispiel:





$f(t)=\mathrm{rect}(t/T)$

$g(t)=\mathrm{rect}\left(\frac{t}{2T}\right)$

Fallunterscheidungen:

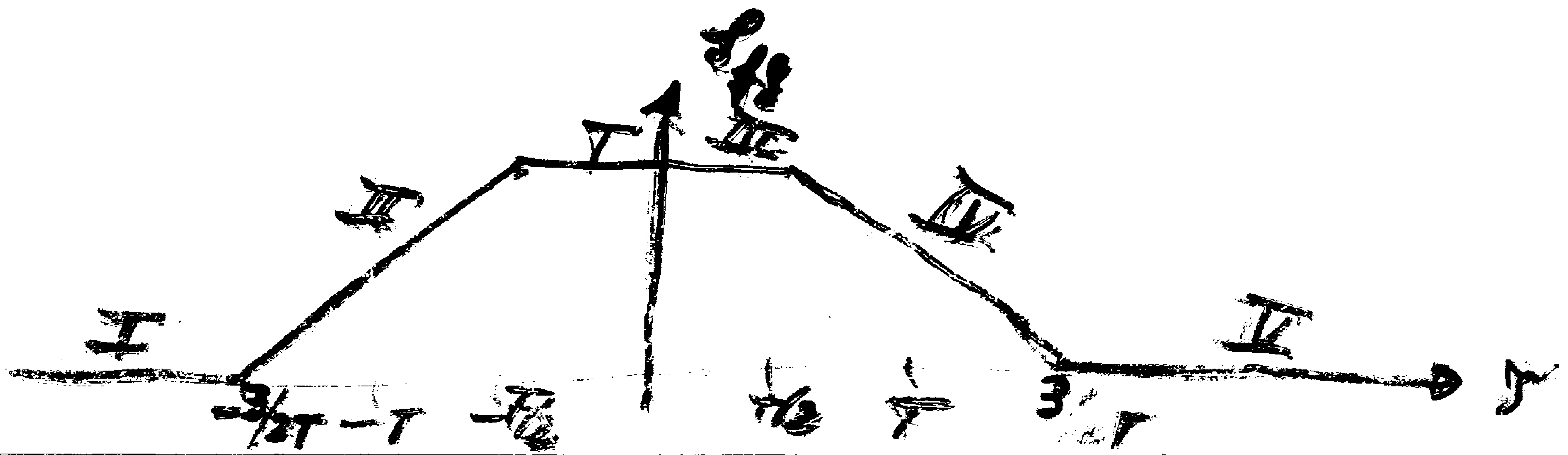
I) $-\infty < \tau < -\frac{3}{2}T$: $\varphi_{fg} = \underline{0}$

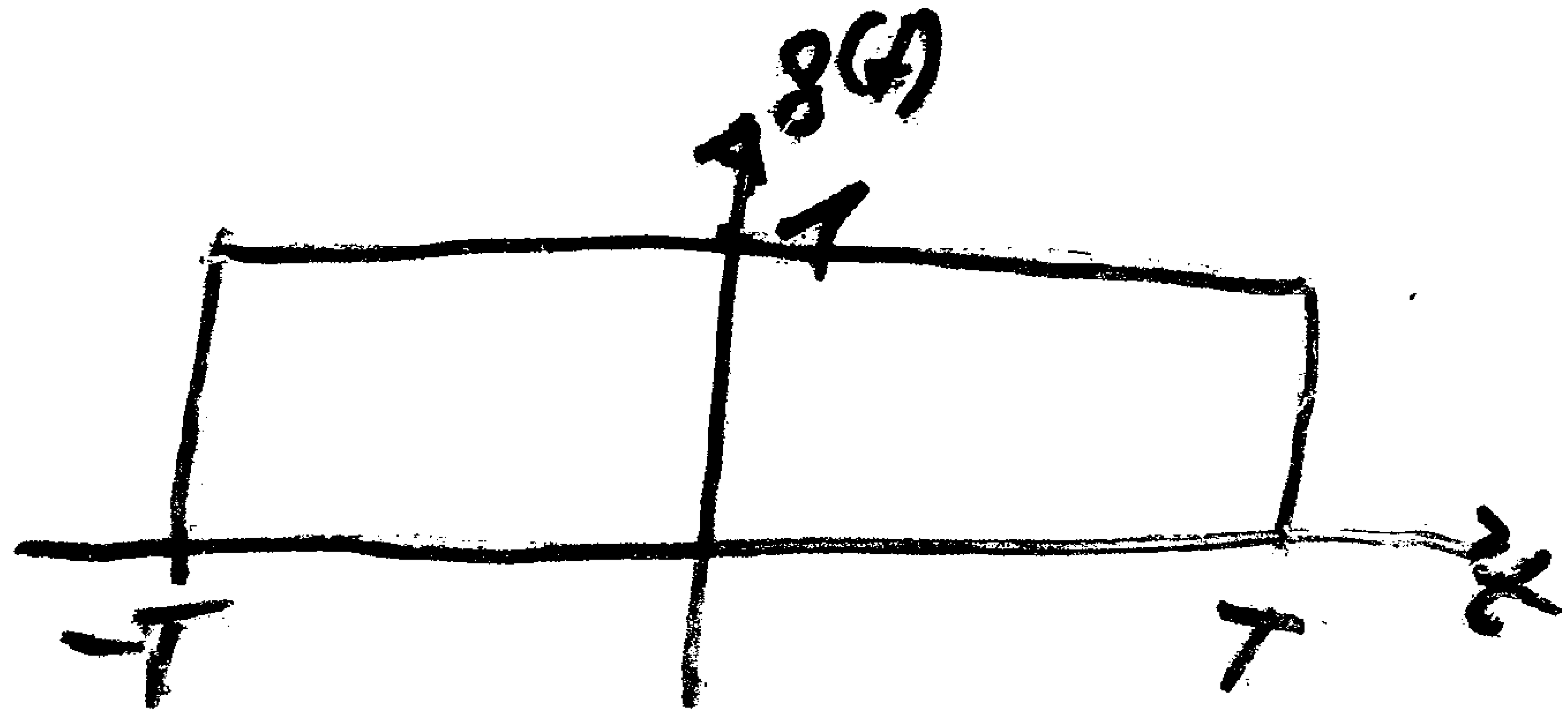
II) $-\frac{3}{2}T < \tau < -\frac{1}{2}T$ bzw. $\frac{1}{2}T < -\tau < \frac{3}{2}T$: $\varphi_{fg} = \int_{-T/2}^{-\tau+T} 1 \cdot dt = -\tau + T + \frac{T}{2} = \underline{-\tau + \frac{3}{2}T}$

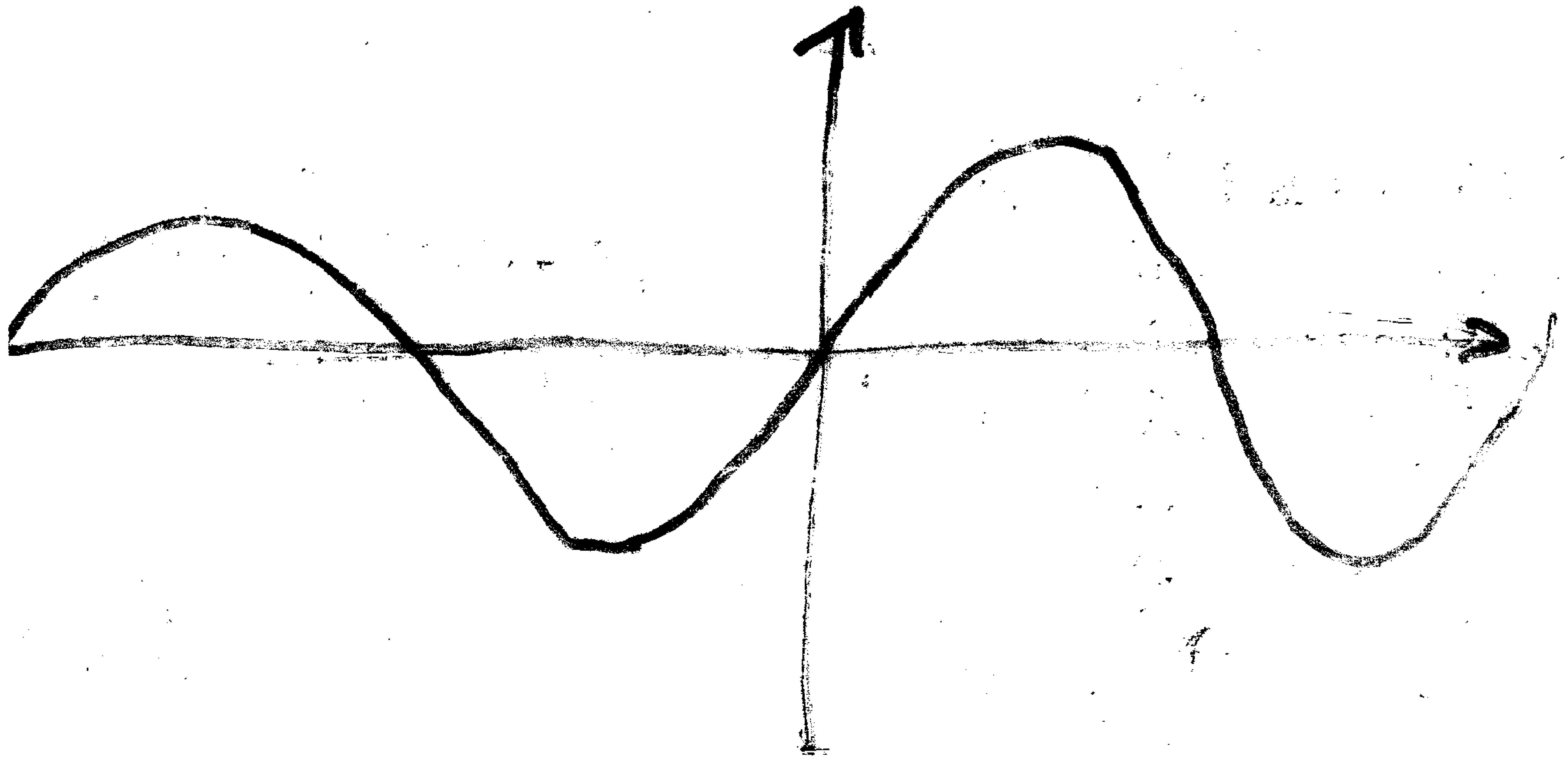
III) $-\frac{1}{2}T < -\tau < \frac{1}{2}T$: $\varphi_{fg} = \int_{-T/2}^{T/2} 1 \cdot dt = \underline{T}$

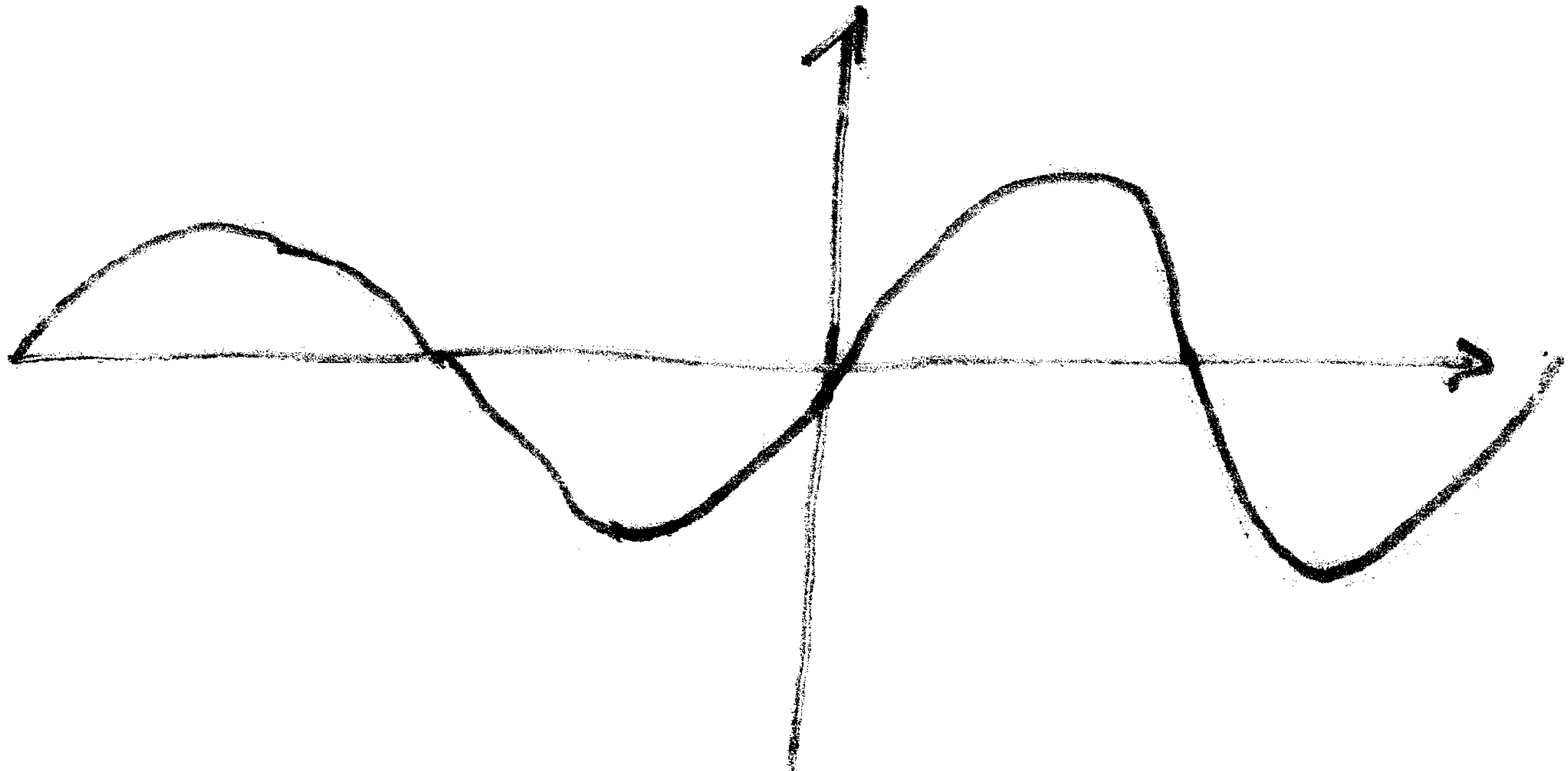
IV) $\frac{T}{2} < -\tau < \frac{3}{2}T$: $\varphi_{fg} = \int_{-\tau-T}^{T/2} 1 \cdot dt = \underline{\tau + \frac{3}{2}T}$

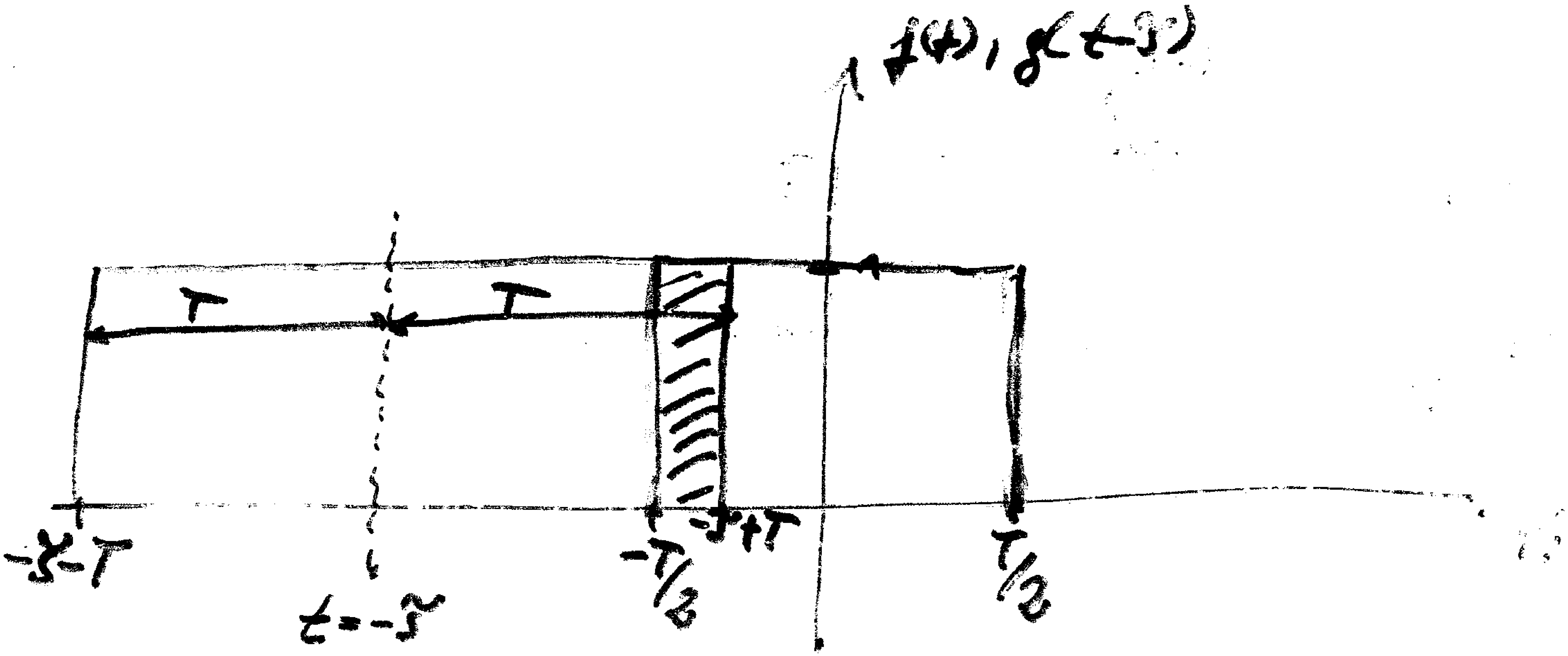
V) $\frac{3}{2}T < -\tau < \infty$: $\varphi_{fg} = \underline{0}$




g(t)
1
-T
T
t






f(t), g(t-ξ)
T
T
-ξ-T
t=-ξ
-T/2
-ξ+T
T/2

$$\varphi_{fg}(\tau) = \int_{-\infty}^{\infty} f(t)\cdot g(t+\tau)\,dt \qquad \text{mit } -\infty<\tau<\infty$$

Korrelations-funktion

$\vartheta = -t, \quad d\vartheta = -dt$

$$\varphi_{fg}(\tau) = \int_{\infty}^{-\infty} f(-\vartheta)\cdot g(-\vartheta+\tau)\,(-d\vartheta)$$

$$= \int_{-\infty}^{\infty} f(-\vartheta)\cdot g(\tau-\vartheta)\,d\vartheta$$

$$= f(-\tau) * g(\tau)$$

"Faltungs-integral"

für $f(t)=f(-t)$ gerade Fkt.!

$$\varphi_{fg}(\tau) = f(-\tau) * g(\tau) \overset{?}{\neq} g(\tau) * f(-\tau) = \varphi_{gf}(-\tau)$$

→ Korrelationsfunktion ist nicht kommutativ!

$f(t), g(t)$: Kreuzkorrelationsfunktion KKF

$f(t), f(t)$: Autokorrelationsfunktion AKF
($\varphi_{ff}(\tau)$)

Eigenschaften der AKF:

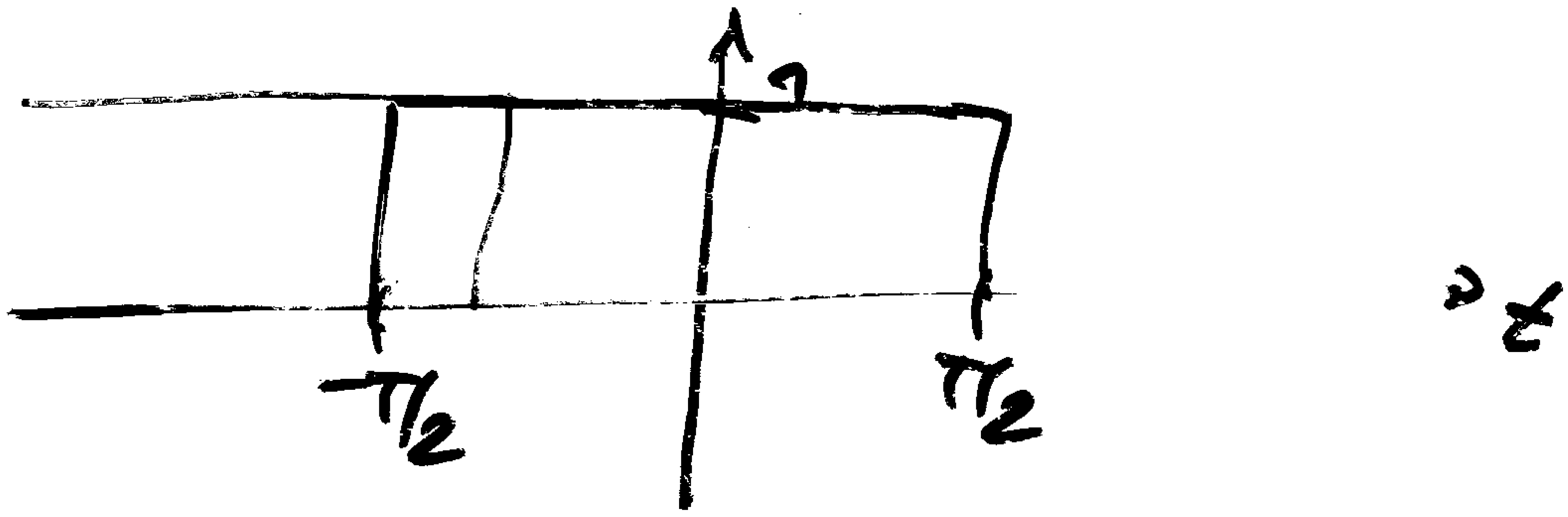
1) immer symmetrisch $\varphi_{ff}(\tau) = \varphi_{ff}(-\tau)$
2) Maximum immer bei $\tau=0$: $\varphi_{ff}(0) = E = \int_{-\infty}^{\infty} f^2(t)\,dt$
3) immer doppelte Breite von $f(t)$ (bei zeitlich begrenzten Signalen)

Beispiel

$$\varphi_{gg}(\tau) = g(-\tau) * g(\tau) = \int_{-\infty}^{\infty} g(t)\cdot g(t+\tau)\,dt$$

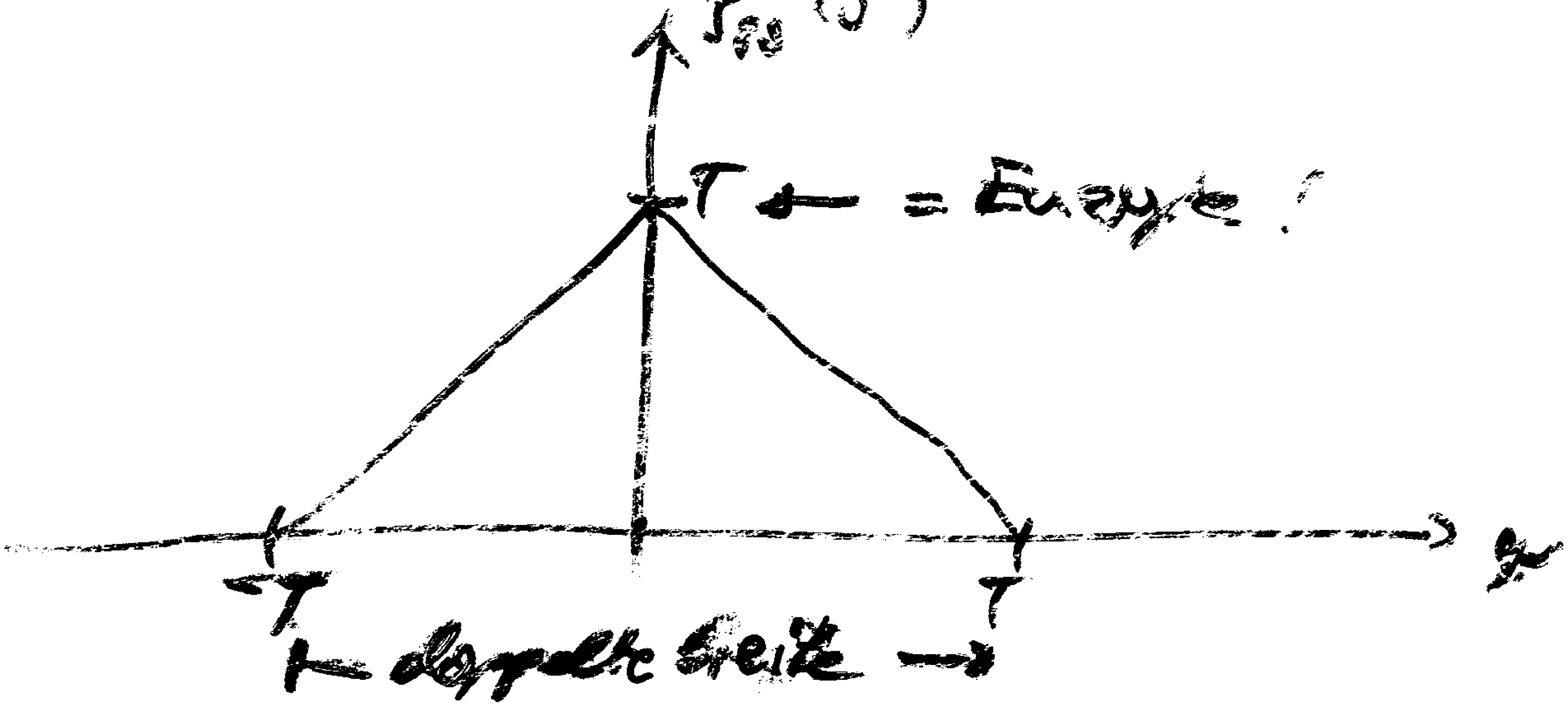
AKF des Rechteck-Impulses

$$g(t) = \mathrm{rect}(t/T)$$



I): $-\tau < -T$: $\varphi_{gg} = 0$

II): $-\tau : 0 \ldots T$: $\varphi_{gg} = \int_{-\tau-\frac{T}{2}}^{-T/2} 1\cdot dt$



2.2.5 Wiener-Khintchine-Theorem

$$\varphi_{gg}(\tau) = g(-\tau) * g(\tau)$$

$$\begin{array}{ccc} \circ & \circ & \circ \\ | & | & | \\ \bullet & \bullet & \bullet \end{array}$$

$$G^*(f) \cdot G(f) = |G(f)|^2$$

Energiedichtespektrum

↳ $E = \int_{-\infty}^{\infty} F\{\varphi_{gg}(\tau)\}\, df$ mit $\underbrace{\varphi_{gg}(\tau) \circ\!\!-\!\!\bullet\, |G(f)|^2}_{\text{Wiener-Khintchine-Theorem}}$

Sonderfall $\tau = 0$:

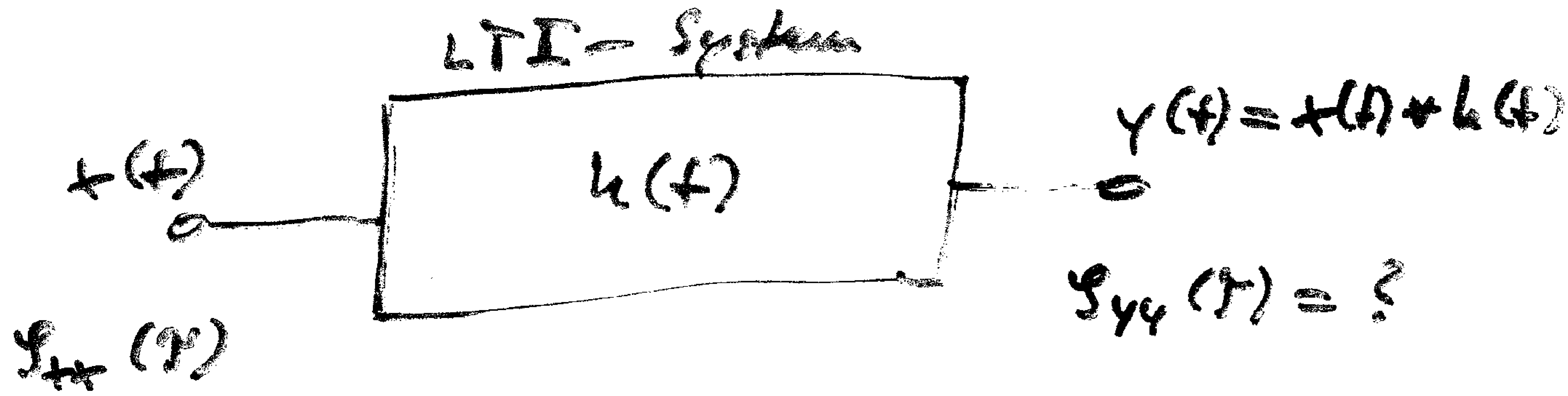
$$\varphi_{gg}(0) = E = \int_{-\infty}^{\infty} |G(f)|^2\, df$$

↳ Energie-Methode 1: $E = \int_{-\infty}^{\infty} g^2(t)\, dt$ Zeitbereich

Energie-Methode 2: $E = \int_{-\infty}^{\infty} |G(f)|^2\, df$ Frequenzbereich

Energie-Methode 3: $E = \varphi_{gg}(0)$ AKF

~~2.2.6~~ Wiener-Lee-Beziehung



$$\left.\begin{aligned} \varphi_{xx}(\tau) &= x(-\tau) * x(\tau) \\ \varphi_{yy}(\tau) &= y(-\tau) * y(\tau) \end{aligned}\right\} \text{AKF} \begin{array}{l} \text{Eingang} \\ \text{Ausgang} \end{array}$$

$$\hookrightarrow \varphi_{yy}(\tau) = [x(-\tau) * x(\tau)] * h(t)$$

$$= [\underbrace{x(-\tau) * h(-\tau)}_{y(-\tau)}] * [\underbrace{x(\tau) * h(\tau)}_{y(\tau)}]$$

$$= x(-\tau) * x(\tau) * h(-\tau) * h(\tau)$$

(Faltung ist kommutativ, d.h. $x(\tau) * h(\tau) = h(\tau) * x(\tau)$)

$$\boxed{\varphi_{yy}(\tau) = \varphi_{xx}(\tau) * \varphi_{hh}(\tau)}$$

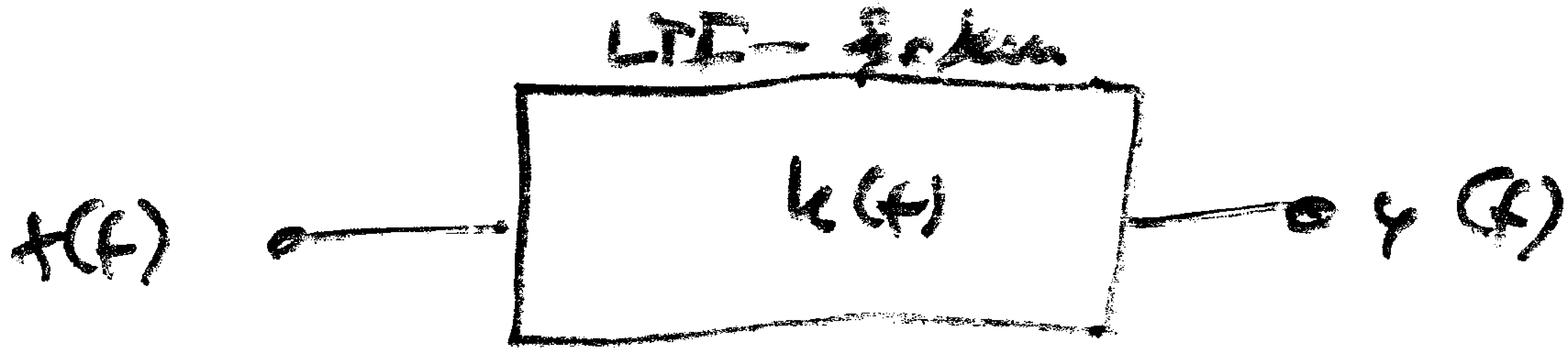
Wiener-Lee-Beziehung

$\varphi_{hh}(\tau) \rightarrow$ System-autokorrelierte

im Frequenzbereich:

$$|Y(f)|^2 = |X(f)|^2 \cdot |H(f)|^2$$

$|H(f)|^2 \rightarrow$ Leistungsübertragungsfunktion



Zeitbereich:	$x(t)$	$h(t)$	$y(t) = x(t) * h(t)$
Frequenzbereich:	$X(f)$	$H(f)$	$Y(f) = X(f) \cdot H(f)$
AKF	$\varphi_{xx}(\tau)$	$\varphi_{hh}(\tau)$	$\varphi_{yy}(\tau) = \varphi_{xx}(\tau) * \varphi_{hh}(\tau)$
Energie (Energiedichtespektrum)	$\lvert X(f)\rvert^2$	$\lvert H(f)\rvert^2$	$\lvert Y(f)\rvert^2 = \lvert X(f)\rvert^2 \cdot \lvert H(f)\rvert^2$

Beispiel:

$$g(t) = e^{-\pi t^2} \qquad \text{Gauß-Impuls}$$

AKF: $\varphi_{gg}(\tau) = g(-\tau) * g(\tau)$

$$= \int_{-\infty}^{\infty} g(t) \cdot g(t+\tau)\, dt$$

$$= \int_{-\infty}^{\infty} e^{-\pi t^2} \cdot e^{-\pi (t+\tau)^2}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-\pi(2t^2 + 2t\tau + \tau^2)}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-\pi\left(\sqrt{2}t + \frac{1}{\sqrt{2}}\tau\right)^2} \cdot e^{-\pi \frac{1}{2}\tau^2}\, dt$$

mit $\left(\sqrt{2}t + \frac{1}{\sqrt{2}}\tau\right)^2 = 2t^2 + 2\tau t + \frac{1}{2}\tau^2$

Subst.: $u = \sqrt{2}t + \frac{1}{\sqrt{2}}\tau$, $\quad du = \sqrt{2}\, dt$

$$\Rightarrow \varphi_{gg}(\tau) = \frac{1}{\sqrt{2}} e^{-\pi\frac{1}{2}\tau^2} \underbrace{\int_{-\infty}^{\infty} e^{-\pi u^2}\, du}_{=1} \qquad \text{mit } \int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

$$= \boxed{\frac{1}{\sqrt{2}} e^{-\pi\frac{1}{2}\tau^2}}$$

Energie: $E = \varphi_{gg}(0) = \underline{\underline{\frac{1}{\sqrt{2}}}}$

Energiedichtespektrum:

$$|G(f)|^2 = F\{\varphi_{gg}(\tau)\} = F\left\{\frac{1}{\sqrt{2}} e^{-\frac{\pi}{2}\tau^2}\right\}$$

Korrespondenz: $e^{-a^2 t^2} \circ\!\!-\!\!\bullet \frac{\sqrt{\pi}}{a} e^{-\frac{\pi^2}{a^2} f^2}$

1) mit $\frac{\sqrt{\pi}}{a} = \frac{1}{\sqrt{2}} \Rightarrow a = \sqrt{2\pi}$

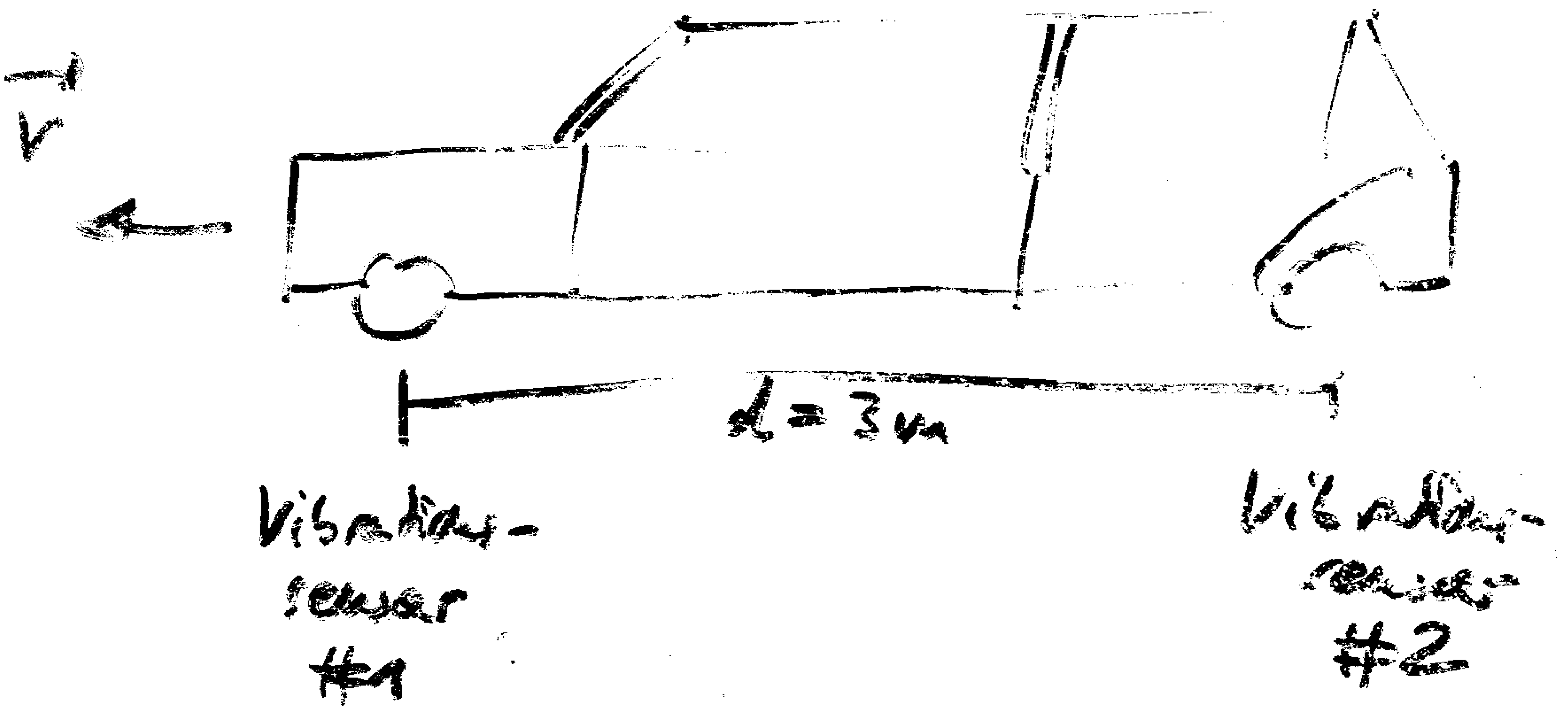
$$\Rightarrow \underline{|G(f)|^2 = e^{-2\pi f^2}}$$ (t, f vertauscht)

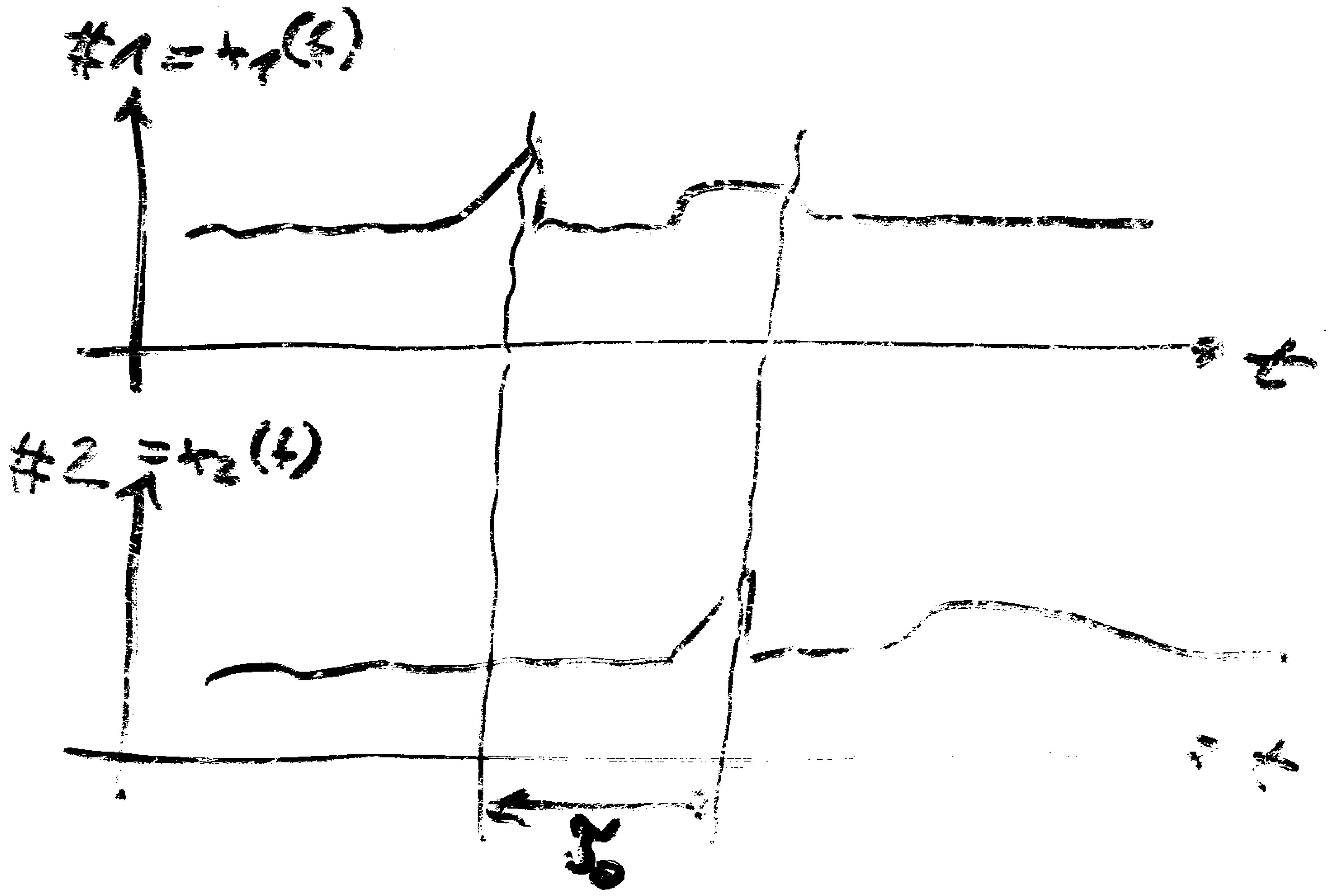
2) mit $a = \sqrt{\frac{\pi}{2}}$

$$\Rightarrow |G(f)|^2 = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{\pi}}{\sqrt{\frac{\pi}{2}}} e^{-\frac{\pi^2}{\pi/2} f^2}$$

$$\underline{= e^{-2\pi f^2}}$$

Übung: Korrelationsmeßtechnik





$$x_2(t) \approx x_1(t - \tau_0)$$

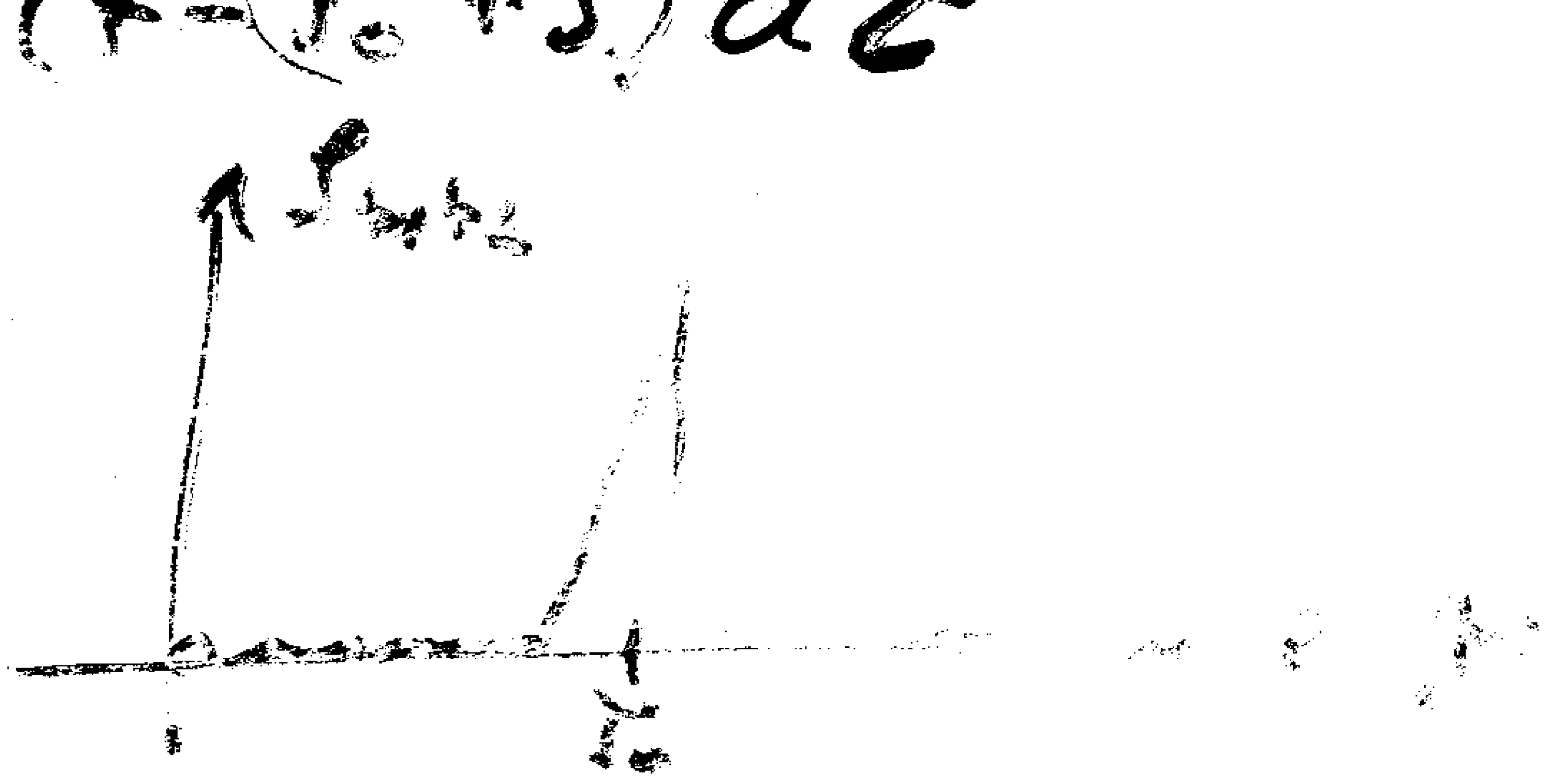
$$\varphi_{x_1 x_2}(\tau) = \int_{-\infty}^{\infty} x_1(t) \cdot x_2(t+\tau)\, dt$$

$$= \int_{-\infty}^{\infty} x_1(t) \cdot x_1(t - \tau_0 + \tau)\, dt$$

→ Maximum bei $\tau = \tau_0$

→ $\varphi_{x_1 x_2}(\tau_0) = \int_{-\infty}^{\infty} x_1^2(t)\, dt = E_{x_1}$

→ $v = \dfrac{d}{\tau_0}$



Übung: Korrelation von Leistungssignalen

Energie $\to \infty$

$$\varphi^2_{s_1 s_2}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} s_1(t)\cdot s_2(t+\tau)\, dt$$

KKF²

$s_1(t) = \cos\omega t$, $s_2(t) = \sin\omega t$:

$$\varphi^2_{s_1 s_2}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \cos\omega t \cdot \sin(\omega(t+\tau))\, dt \overset{?}{=}$$

AKF²

$$\varphi^2_{s_1 s_1}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \cos \underbrace{\omega t}_{\alpha} \cdot \cos \underbrace{(\omega(t+\tau))}_{\beta}\, dt$$

$$= \frac{1}{2}\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left[\cos\omega\tau + \cos\omega(2t+\tau)\right] dt$$

mit: $\cos\alpha\cdot\cos\beta = \frac{1}{2}\left(\cos(\alpha-\beta) + \cos(\alpha+\beta)\right)$

$$= \frac{1}{2}\lim_{T\to\infty} \frac{1}{2T}\left[t\cdot\cos\omega\tau + \frac{1}{2\omega}\sin\omega(2t+\tau)\right]_{-T}^{T}$$

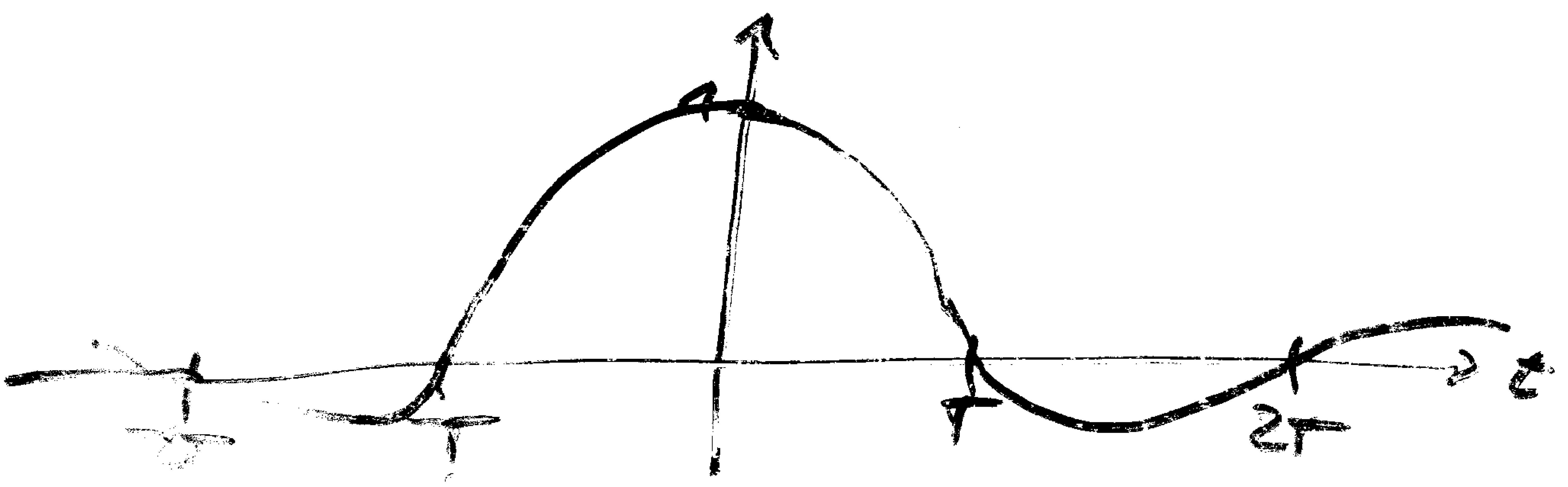
$$= \frac{1}{2}\lim_{T\to\infty} \frac{1}{2T}\left[2T\cos\omega\tau + \cancel{\frac{1}{2\omega}\sin\omega(2T+\tau)} - \cancel{\frac{1}{2\omega}\sin\omega(-2T+\tau)}\right]$$

$$= \frac{1}{2}\lim_{T\to\infty}\cos\omega\tau = \underline{\underline{\frac{1}{2}\cos\omega\tau}}$$

$P_{s_1} = \varphi^2_{s_1 s_1}(0) = \frac{1}{2}$ | $P_{s_2} = \varphi^2_{s_2 s_2}(0) = \frac{1}{2}$

Si-Funktion

$g(t) = si(\pi\, t/T)$

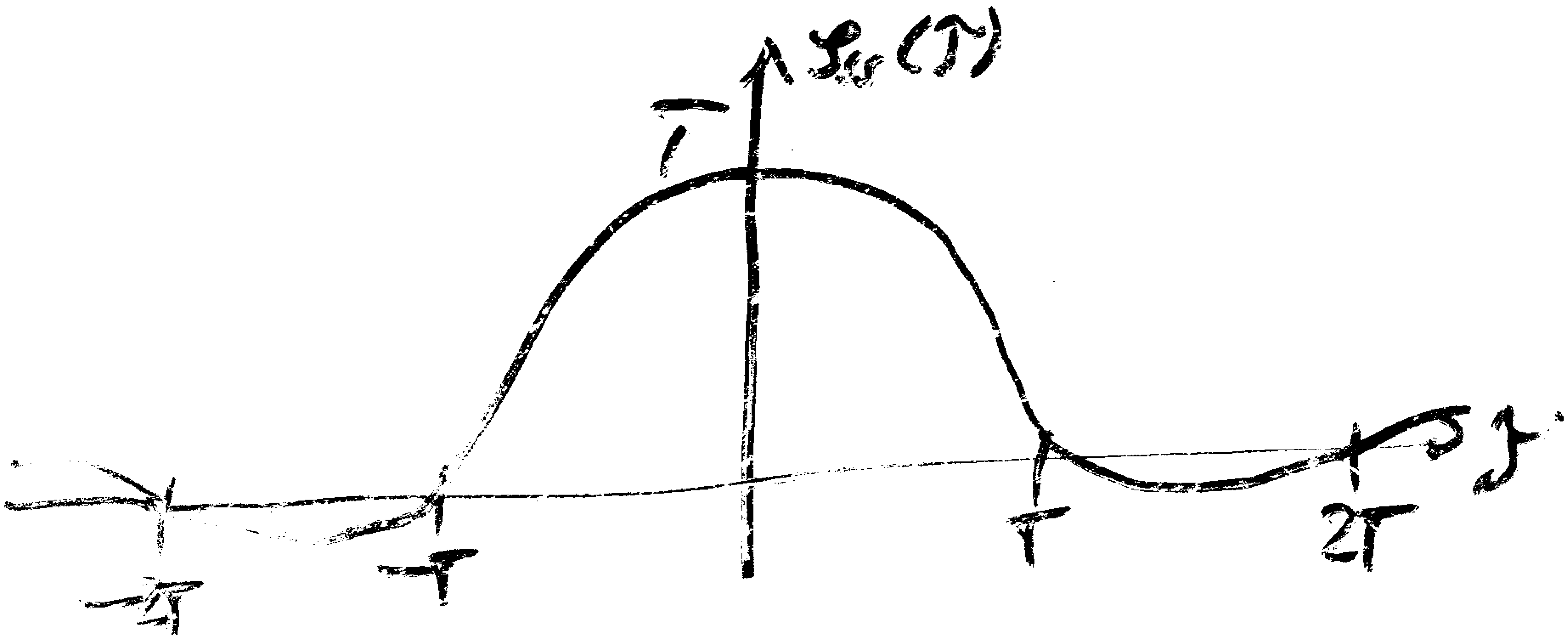


$\to \varphi_{gg}(\tau) \circ\!\!-\!\!\bullet\ |G(f)|^2$

1) $G(f) = T\,\mathrm{rect}(fT)$

2) $|G(f)|^2 = T^2\,\mathrm{rect}(fT)$ Energiedichtespektrum

3) $\varphi_{gg}(\tau) = T \cdot si(\pi\, \tau/T)$ AKF



$E = \varphi_{gg}(0) = \underline{\underline{T}}$ Energie