~~1.3.~~ 3. Systemreaktionen

Testsignal: Dirac-Impuls $\delta(t)$

→ enthält <u>alle</u> Frequenzen mit Amplitude $= 1$

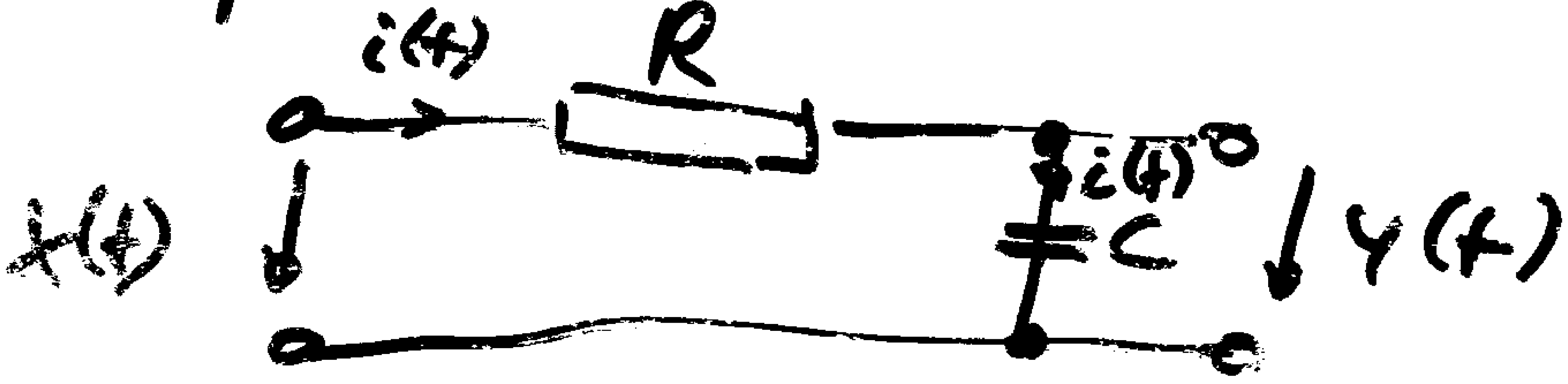
$$\delta(t) \circ\!\!-\!\!\bullet\; X(f) = 1$$

Für lineare Systeme gilt:

$$y(t) = \sum_{n=0}^{\infty} x_n(t) \quad , \quad x_n(t) = a_n \cdot \sin(\omega_n t)$$

3.1 ~~1.3.1.~~ Impulsantwort

Beispiel: RC-Glied



<u>DGL = ?</u>

Kirchhoff: $x(t) = i(t) \cdot R + y(t)$

$$i(t) = C \cdot \frac{dy}{dt}$$

$$\Rightarrow \quad x(t) = C \cdot \frac{dy}{dt} \cdot R + y(t)$$

$$= RC\,\dot{y}(t) + y(t)$$

$$\Rightarrow \quad \dot{y}(t) + \frac{1}{RC}\,y(t) = \frac{1}{RC}\,x(t)$$

$$\dot{y}(t) + \frac{1}{\tau}\,y(t) = \frac{1}{\tau}\,x(t) \quad ; \quad \tau = RC$$

Lösung: $y(t) = e^{-\int \frac{1}{\tau} dt} \left\{ \int \frac{1}{\tau} x(t)\, e^{\int \frac{1}{\tau} dt}\, dt + C \right\}$

$$= e^{-t/\tau} \left\{ \frac{1}{\tau} \cdot \int x(t)\, e^{t/\tau}\, dt + C \right\}$$

$Y(0) = 0$ kausal! (für $x(t) = 0$ für $t \leq 0$)

$\Rightarrow \underline{C = 0}$

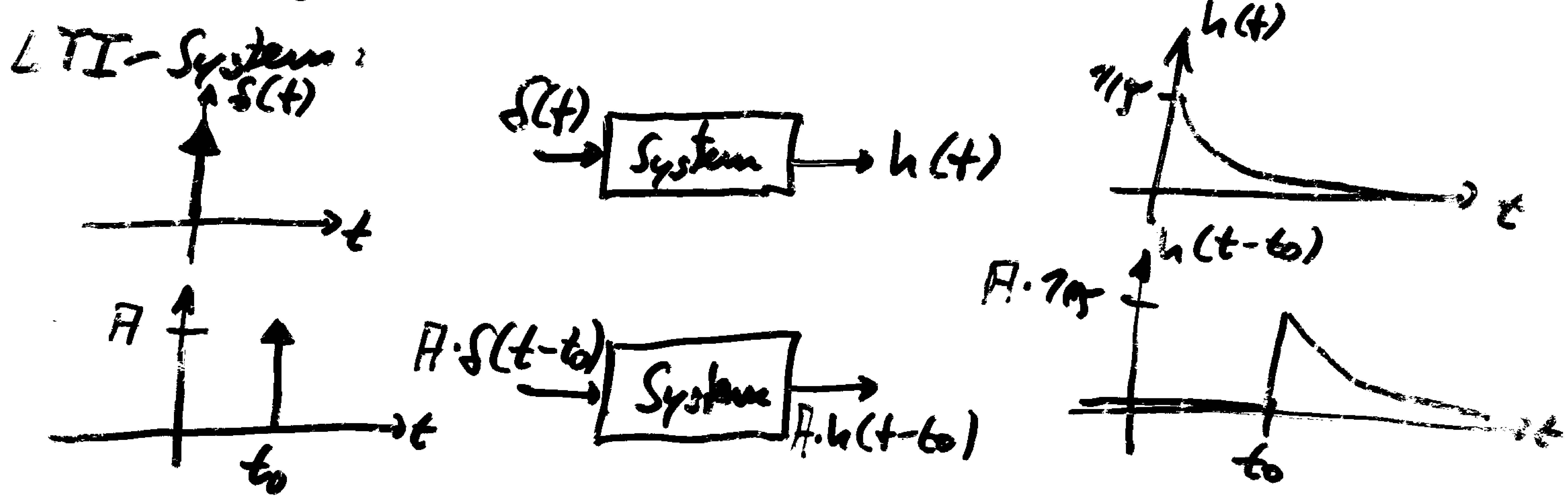
$$\Rightarrow y(t) = \frac{1}{\tau} e^{-t/\tau} \int x(t) e^{t/\tau} dt$$

Für $x(t) = \delta(t)$ Impuls-Erregung

$y(t) = \underline{h(t)}$ Impulsantwort

$$\boxed{\Rightarrow h(t) = \frac{1}{\tau} e^{-t/\tau} \underbrace{\int \delta(t) e^{t/\tau} dt}_{=1}}$$

LTI-System:



$k_1 \delta(t-t_1) + k_2 \delta(t-t_2) \ \circ\!\!\longrightarrow \boxed{\quad\quad} \longrightarrow\!\!\circ \ k_1 h(t-t_1) + k_2 h(t-t_2)$

allg.: $\sum_{i=-\infty}^{\infty} k_i \delta(t-t_i) \ \circ\!\!\longrightarrow \boxed{\quad\quad} \longrightarrow\!\!\circ \ \sum_{i=-\infty}^{\infty} k_i h(t-t_i)$

$\downarrow$

Grenzübergang:

$x(t) \ \circ\!\!-\boxed{\text{System}}-\!\!\circ \ y(t)$

Faltung

$$x(t) = \int_{-\infty}^{\infty} \underbrace{k(\tau)}_{x(\tau)} \cdot \delta(t-\tau) d\tau$$

$$\int_{-\infty}^{\infty} \underbrace{k(\tau)}_{x(\tau)} \cdot h(t-\tau) d\tau = y(t)$$

Faltung: 3.2

$$y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau)\, d\tau$$

$$= x(t) * h(t)$$

Faltung(soperation)
(sintegral)

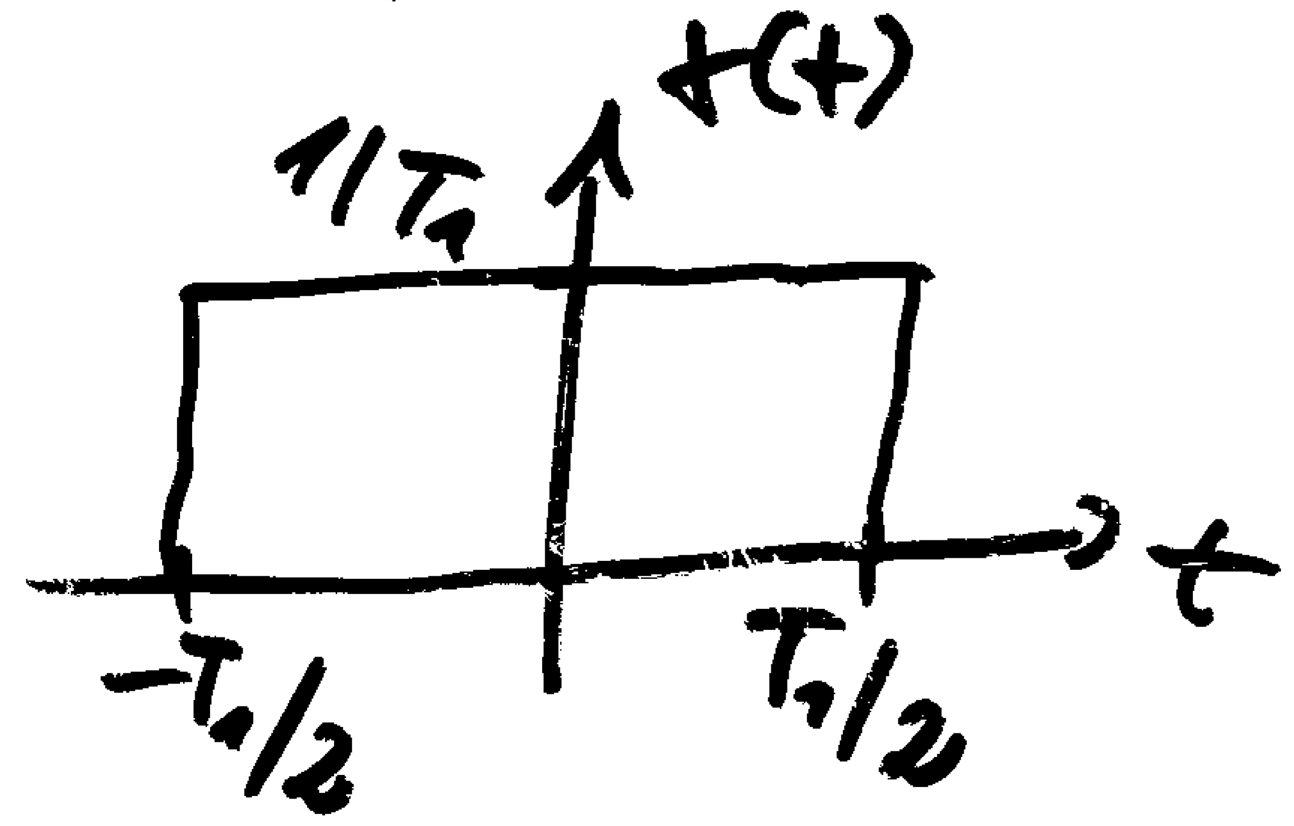
Für kausale Systeme:

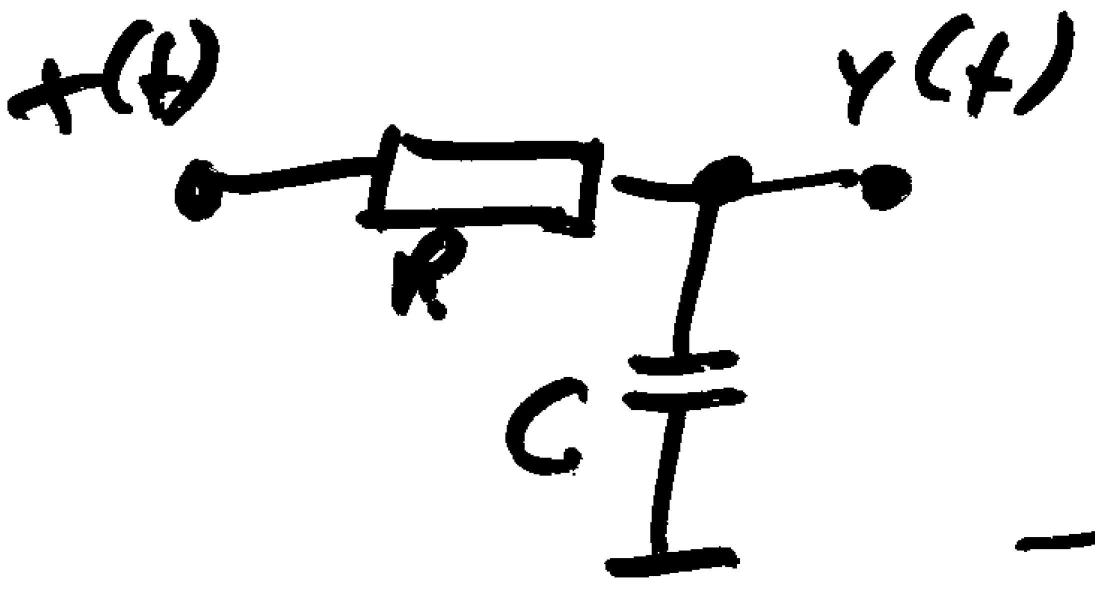
$$y(t) = \int_{0}^{\infty} x(\tau) \cdot h(t-\tau)\, d\tau$$

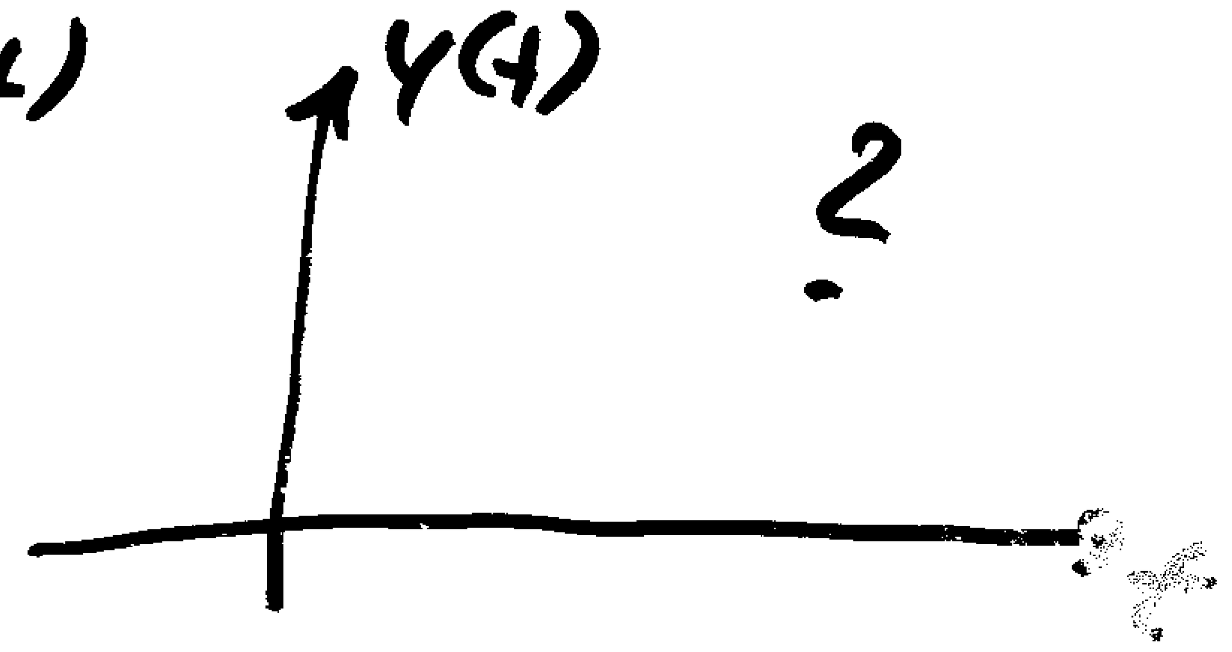
Kommutativität der Faltungsoperation:

$$x(t) * h(t) = h(t) * x(t)$$

Beispiel: Rechteck-Erregung auf RC-Glied



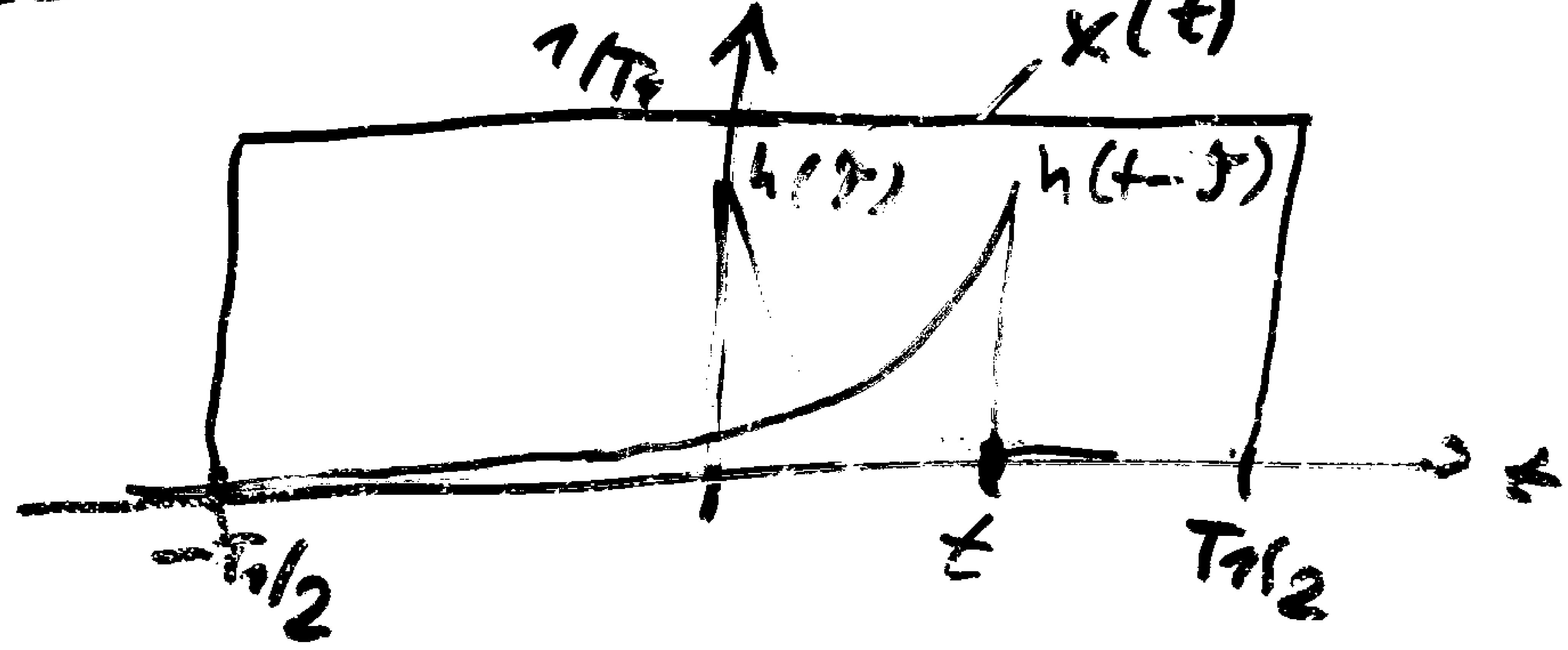




$x(t) = \frac{1}{T_1}\,\mathrm{rect}\left(\frac{t}{T_1}\right) \qquad h(t) = \frac{1}{T_{RC}}\, e^{-t/T_{RC}}$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

graphische Interpretation:



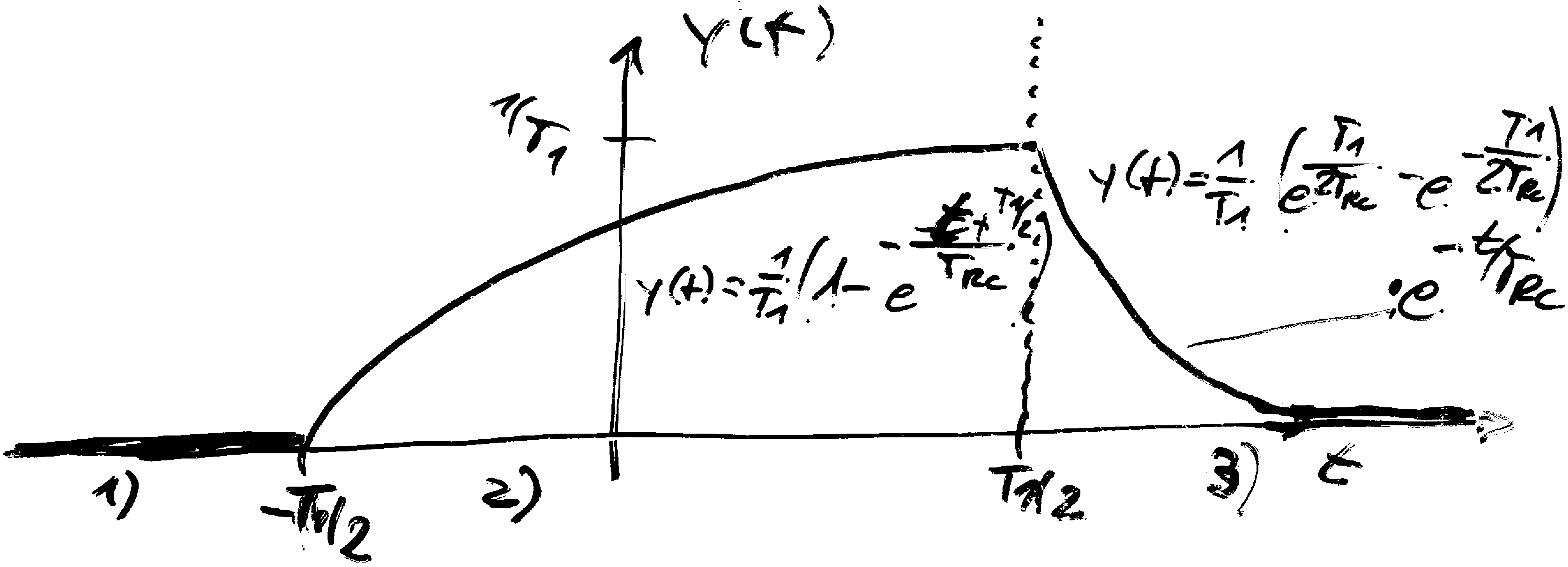
1) $t < -T_1/2$: $y(t) = 0$, da keine gemeinsame Fläche.

2) $-T_1/2 \le t \le T_1/2$: $y(t) = \int_{-T_1/2}^{t} \frac{1}{T_1}\cdot\frac{1}{T_{RC}}\, e^{-\frac{t-\tau}{T_{RC}}}\, d\tau = \frac{1}{T_1}\, e^{-\frac{t-\tau}{T_{RC}}}\Big|_{-T_1/2}^{t}$

$$= \frac{1}{T_1}\left(1 - e^{-\frac{t+T_1/2}{T_{RC}}}\right)$$

3) $t > \frac{T_1}{2}$: $y(t) = \int_{-T_1/2}^{T_1/2} \frac{1}{T_1}\cdot\frac{1}{T_{RC}}\, e^{-\frac{t-\tau}{T_{RC}}}\, d\tau = \frac{1}{T_1}\left(e^{-\frac{t-T_1/2}{T_{RC}}} - e^{-\frac{t+T_1/2}{T_{RC}}}\right)$

$$= \frac{1}{T_1}\left(e^{\frac{T_1}{2T_{RC}}} - e^{-\frac{T_1}{2T_{RC}}}\right) e^{-t/T_{RC}}$$

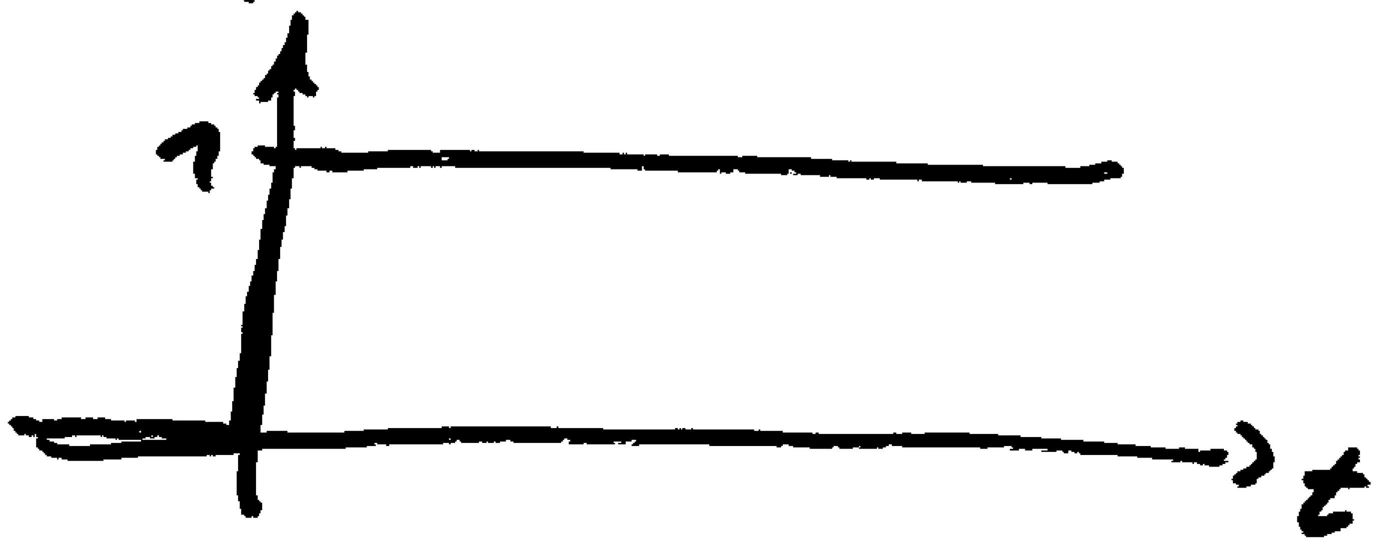


Y(t)
1/T1
Y(t) = 1/T1 (1 - e^(-(t+T1/2)/TRC))
Y(t) = 1/T1 (e^(T1/2TRC) - e^(-T1/2TRC)) · e^(-t/TRC)
1)
-T1/2
2)
T1/2
3)
t

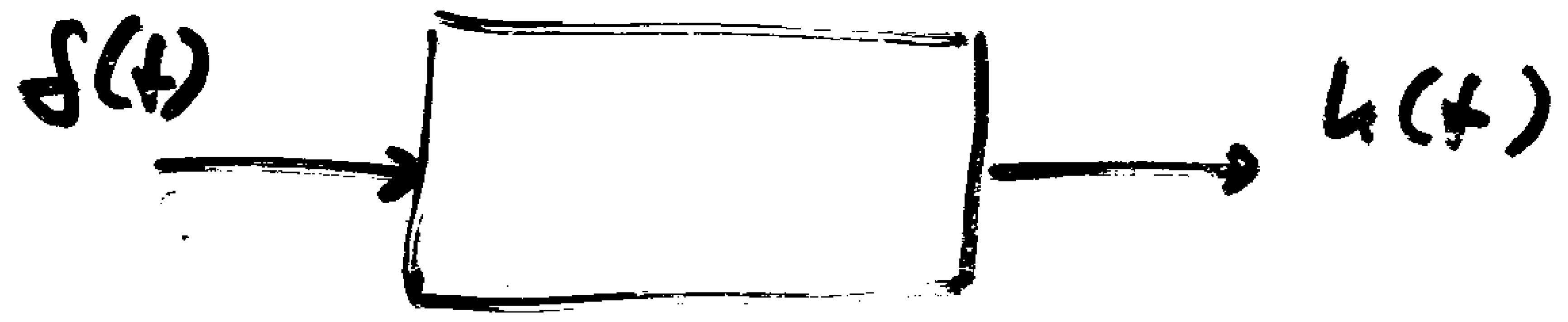
Beispiel 1.3.2 Sprungantwort

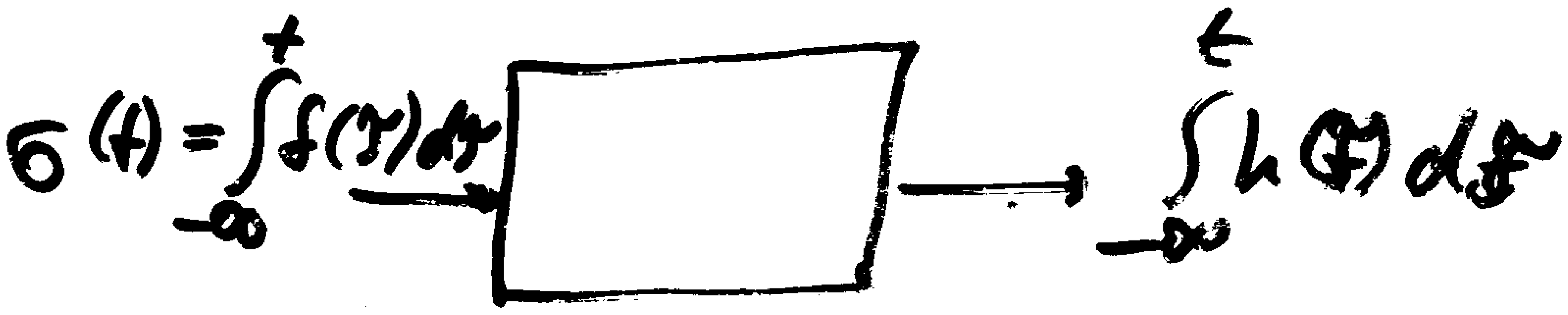
RC-Glied: $h(t) = \frac{1}{T_{RC}} e^{-t/T_{RC}}$

$x(t) = \sigma(t)$



$$\sigma(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$





$$y(t) = \int_{-\infty}^{t} h(\tau)\, d\tau = \frac{1}{T_{RC}} \int_{-\infty}^{t} e^{-\tau/T_{RC}}\, d\tau$$

$$= \frac{1}{T_{RC}} \cdot (-T_{RC})\, e^{-\tau/T_{RC}} \Big|_{\cancel{-\infty}\ 0}^{t}$$

$$= -\left[e^{-t/T_{RC}} - 1 \right] = \underbrace{1 - e^{-t/T_{RC}}}_{\text{Sprungantwort}}$$

Beispiel: Frequenzgang RC-Glied

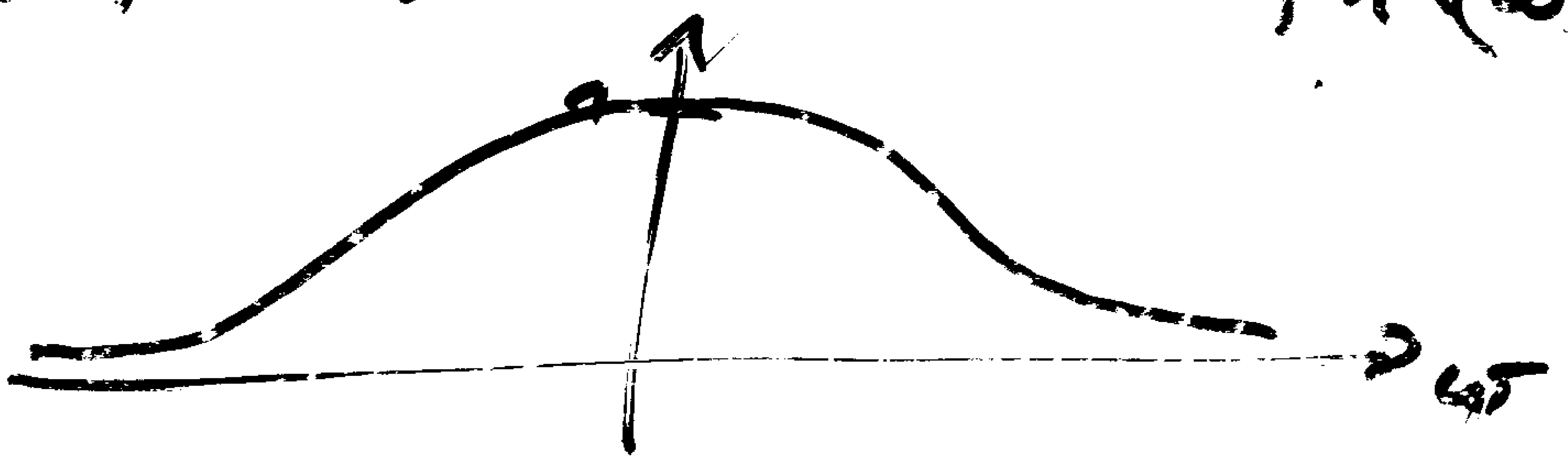
$$h(t) = \frac{1}{T_{RC}} e^{-t/T_{RC}} \Rightarrow H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$

$$\Rightarrow H(\omega) = \int_{0}^{\infty} \frac{1}{T_{RC}} e^{-t/T_{RC}} e^{-j\omega t}\, dt = \frac{1}{T_{RC}} \int_{0}^{\infty} e^{-\left(\frac{1}{T_{RC}} + j\omega\right)t}\, dt$$

$$= -\frac{\frac{1}{T_{RC}}}{\frac{1}{T_{RC}} + j\omega}\, e^{-\left(\frac{1}{T_{RC}} + j\omega\right)t}\Big|_{0}^{\infty} = 0 - \left(-\frac{1}{1 + j\omega T_{RC}}\right)$$

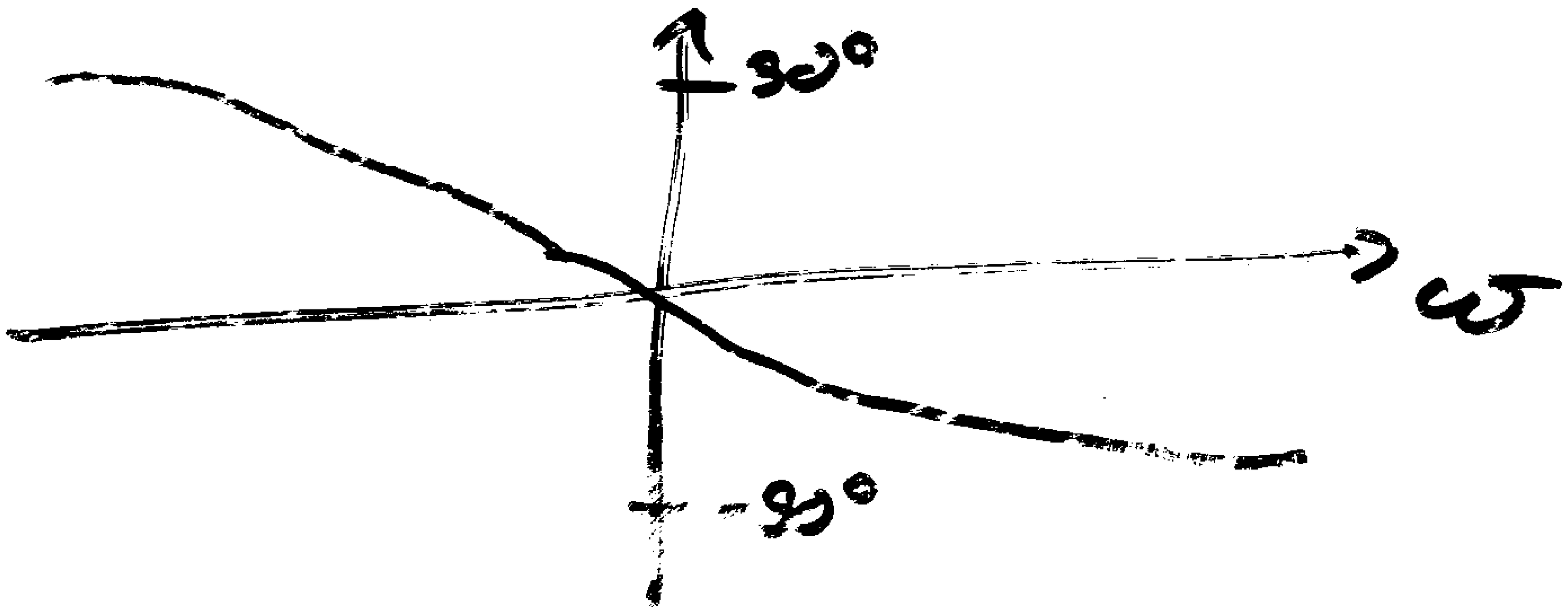
$$= \underline{\underline{\frac{1}{1 + j\omega T_{RC}}}}$$

Betragsfrequenzgang: $|H(\omega)| = \frac{1}{\sqrt{1 + (\omega T_{RC})^2}}$



TP-Charakter

Phasengang: $\varphi(\omega) = -\arctan \frac{\omega T_{RC}}{1} = -\arctan(\omega T_{RC})$



~~1.3.1.~~ Eigenfunktion und Frequenzgang 3.3.

komplexe Exp.-fkt.: $x(t) = \underline{U} \cdot e^{j\omega t}$

= Eigenfunktionen von LTI-Systemen

$$y(t) = x(t) * h(t) = h(t) * x(t)$$

$$= \int_{-\infty}^{\infty} h(\tau) \cdot x(t-\tau)\, d\tau$$

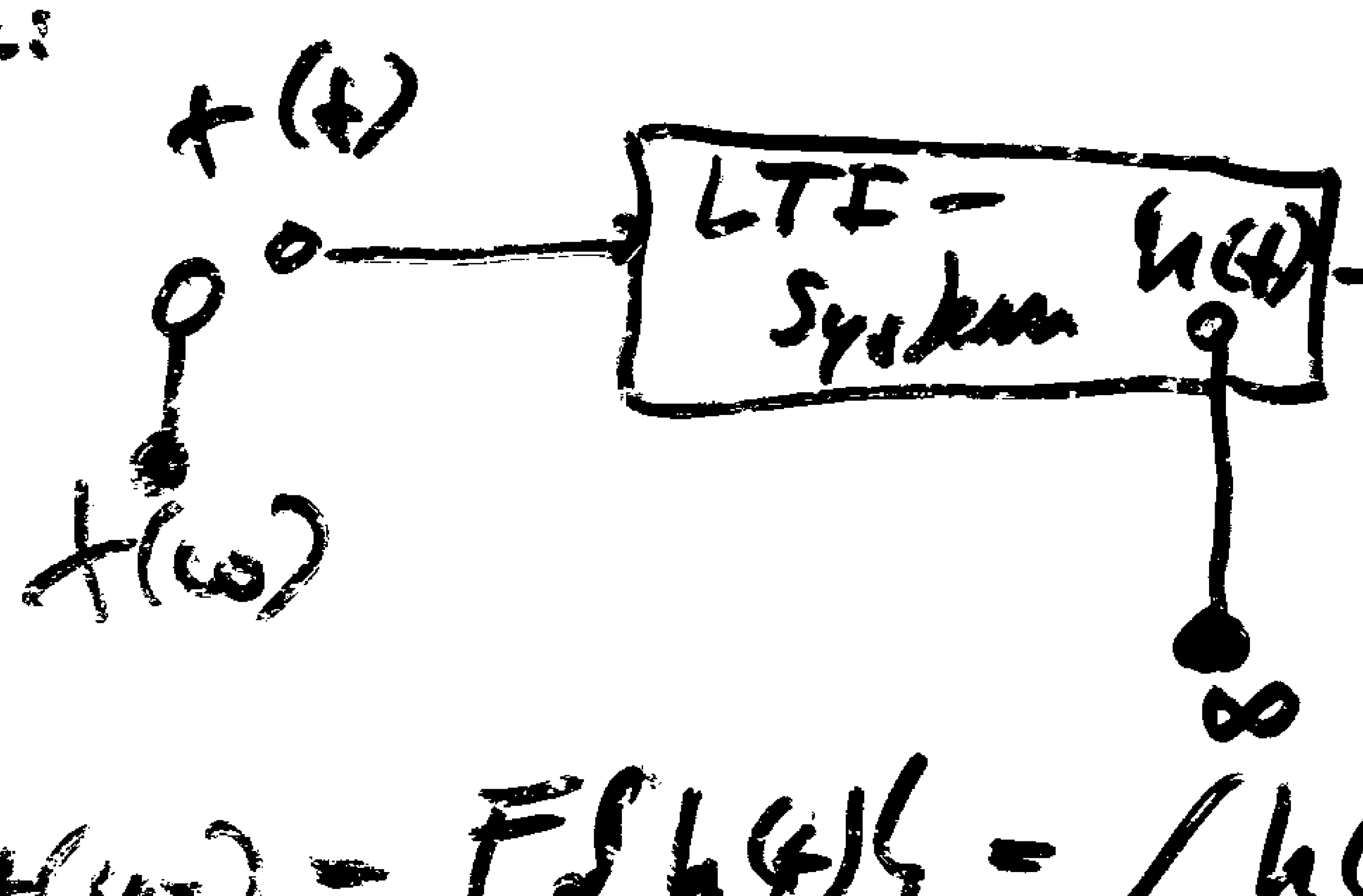
$$= \int_{-\infty}^{\infty} h(\tau) \cdot \underline{U} \cdot e^{j\omega(t-\tau)}\, d\tau$$

$$= \underbrace{\left[\int_{-\infty}^{\infty} h(\tau) \cdot e^{-j\omega\tau}\, d\tau\right]}_{F\{h(t)\} = H(\omega)} \cdot \underbrace{\underline{U} \cdot e^{j\omega t}}_{x(t)}$$

Frequenzgang

Übertragungsfunktion

Allgemein:



$y(t) = x(t) * h(t)$

$Y(\omega) = X(\omega) \cdot H(\omega)$

$$H(\omega) = F\{h(t)\} = \int_{-\infty}^{\infty} h(\tau) \cdot e^{-j\omega\tau}\, d\tau$$

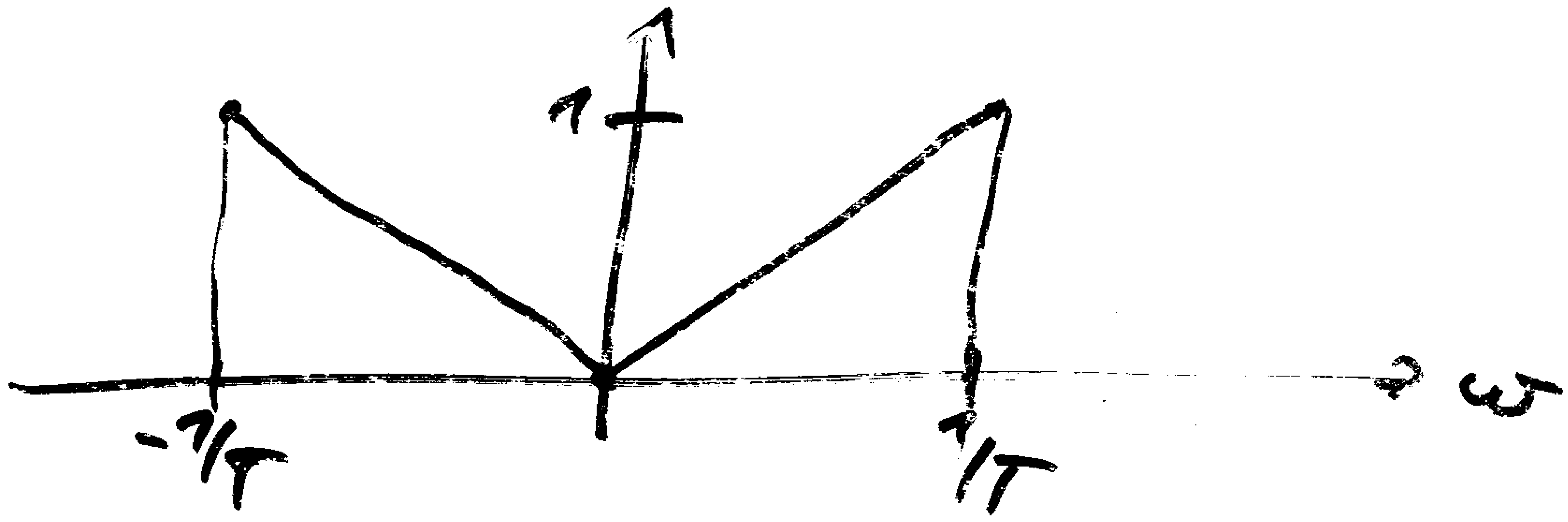
$|H(\omega)|$: Betragsfrequenzgang | $\varphi(\omega) = \arg\{H(\omega)\}$ Phasengang

Beispiel: Differenzierer

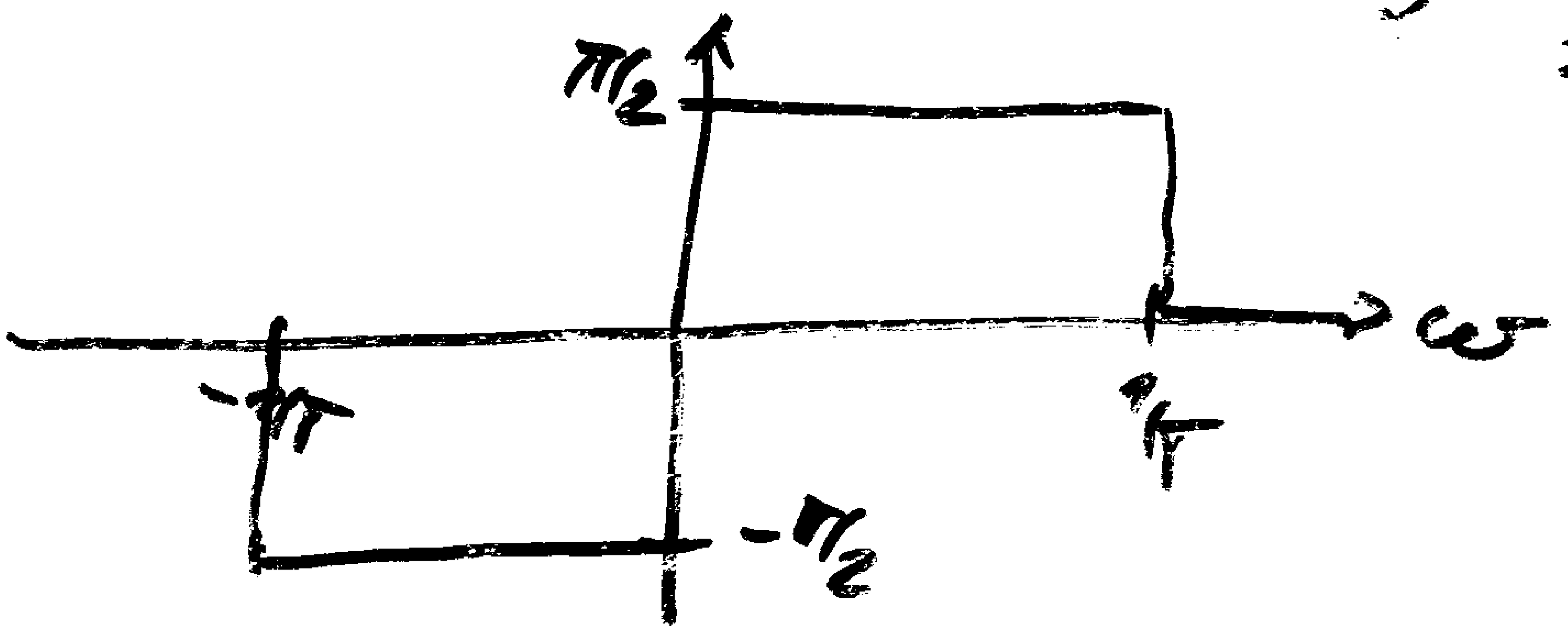
$$H(\omega) = \begin{cases} j\omega T & \text{f. } -\frac{1}{T} \le f \le \frac{1}{T} \\ 0 & \text{sonst} \end{cases}$$

$$x(t) = \cos(2\pi f_0 t + \vartheta) \quad , \quad f_0 = \frac{1}{2T}$$

a) Betrag: $|H(\omega)| = |j\omega T|$



b) Phase: $\varphi(\omega) = \arctan\left\{\frac{\omega T}{0}\right\} = \arctan(\pm\infty)$
$= \pm \pi/2$



→ 90°- Phasenschieber

→ Hilberttransformator

$$Y(\omega) = X(\omega) \cdot H(\omega)$$

$$X(\omega) = F\left\{ \underbrace{\tfrac{1}{2} e^{j\omega t} e^{j\vartheta} + \tfrac{1}{2} e^{-j\omega t}}_{\cos(\omega t + \vartheta)} e^{-j\vartheta} \right\}$$

$$= \tfrac{1}{2} e^{j\vartheta} \delta(f - f_0) + \tfrac{1}{2} e^{-j\vartheta} \delta(f + f_0)$$

$$Y(\omega) = j\omega T \cdot \tfrac{1}{2} \cdot \left\{ e^{j\vartheta} \delta(f - f_0) + e^{-j\vartheta} \delta(f + f_0) \right\}$$

Ausblendeigenschaft $\delta(f)$

$$\bullet\!\!-\!\!\circ$$

$$y(t) = j \underbrace{2\pi f_0}_{\omega_0} T \cdot \tfrac{1}{2} \cdot \left\{ e^{j\vartheta} e^{j2\pi f_0 t} - e^{-j\vartheta} e^{-j2\pi f_0 t} \right\}$$

$$= \underbrace{2\pi f_0}_{\omega_0} T \cdot \sin(\underbrace{2\pi f_0}_{\omega_0} t + \vartheta)$$