2. Signale

- deterministisch
- stochastisch

2.1. Elementarsignale

2.1.1. komplexe Exponentialfunktion

$$x(t) = A \cdot e^{st} \qquad s = \sigma + j\omega$$
$$= A e^{\sigma t} e^{j\omega t}$$

$\sigma > 0$: aufklingend A.
$\sigma < 0$: abklingend A.
$\sigma = 0$: konstant Amplitude

$$x(t) = A \cdot e^{\sigma t} [\cos \omega t + j \sin \omega t]$$
$$\rightarrow \mathrm{Re}\{x(t)\} = A e^{\sigma t} \cdot \cos \omega t$$
$$\rightarrow \mathrm{Im}\{x(t)\} = \underline{A e^{\sigma t}} \cdot \sin \omega t$$

= Hüllkurve

~~2.1.2~~ Sinusförmige Funktionen

$$u(t) = U_0 \cdot \sin(\omega_0 t + \varphi_0)$$

$$\text{mit } \sin x = \frac{e^{jx} - e^{-jx}}{2j}$$

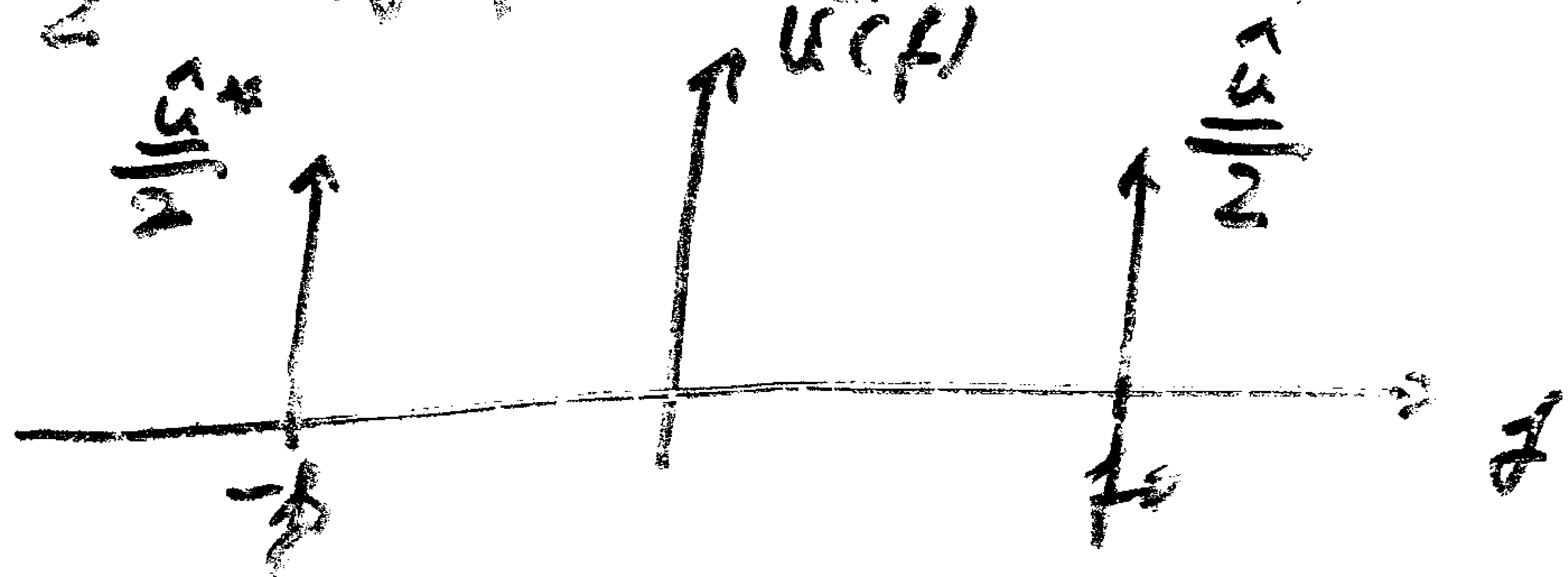
$$u(t) = \frac{U_0}{2j}\left(e^{j\varphi_0} e^{j\omega_0 t} - e^{-j\varphi_0} e^{-j\omega_0 t}\right)$$

$$\text{mit } \underline{\hat{u}} = \frac{U_0}{j} e^{j\varphi_0} \left(= U_0 \sin\varphi_0 - jU_0 \cos\varphi_0\right)$$

$$u(t) = \frac{\hat{u}}{2} e^{j\omega_0 t} + \frac{\hat{u}^*}{2} e^{-j\omega_0 t}$$

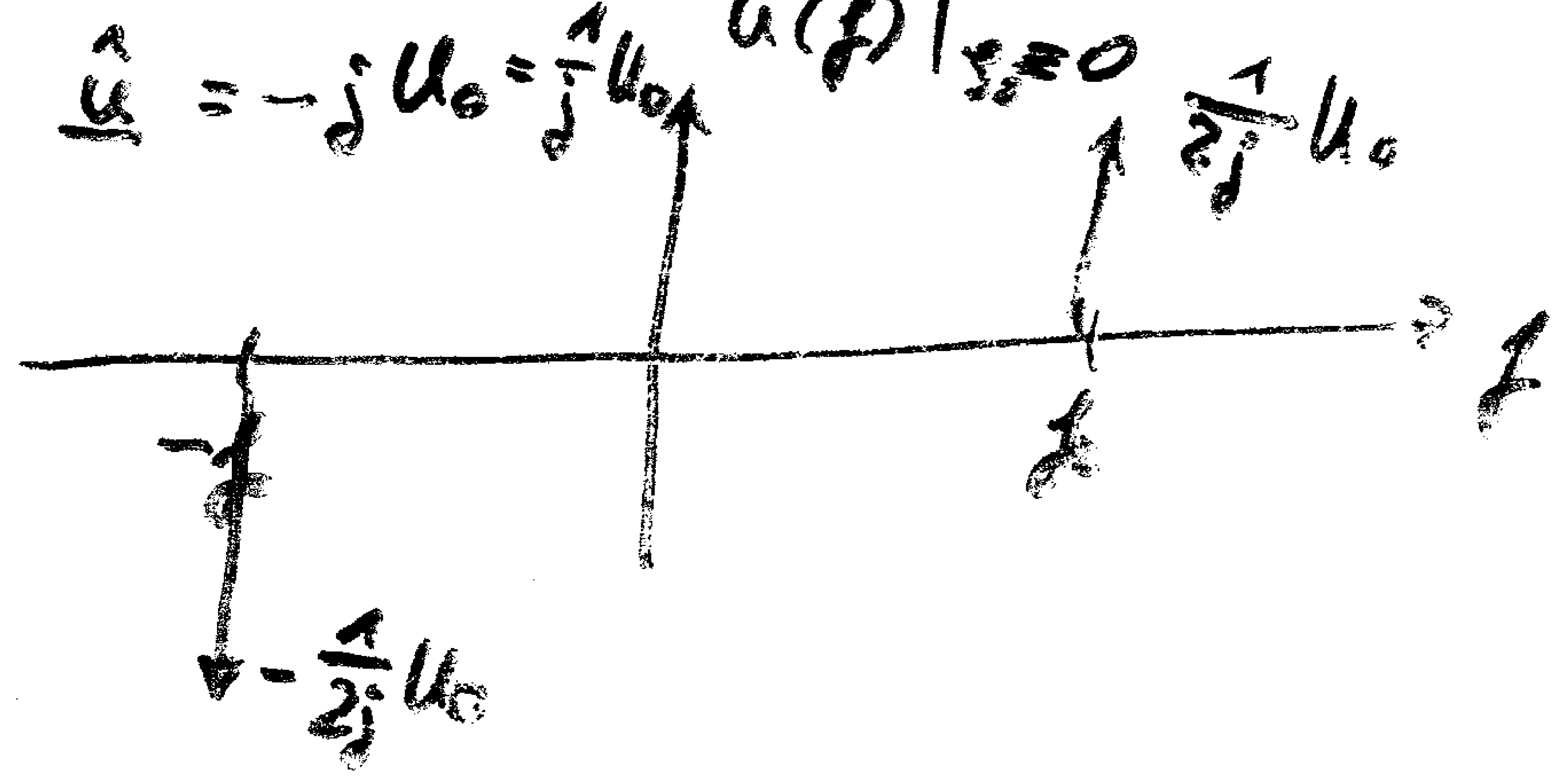
$$\circ\!-\!\bullet$$

$$U(f) = \frac{\hat{u}}{2}\,\delta(f - f_0) + \frac{\hat{u}^*}{2}\,\delta(f + f_0)$$



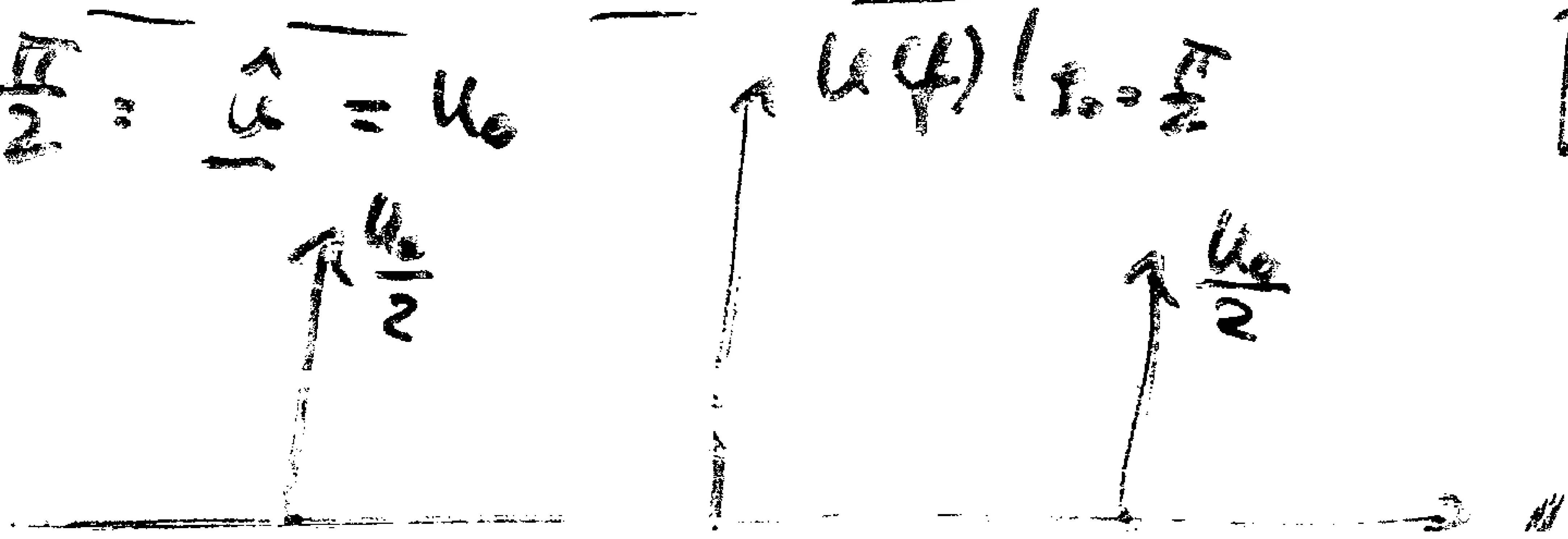
$\varphi_0 = 0$: $\underline{\hat{u}} = -jU_0 = \frac{1}{j}U_0$

$\boxed{U_0 \sin \omega_0 t}$



$\varphi_0 = \frac{\pi}{2}$: $\underline{\hat{u}} = U_0$

$\boxed{U_0 \cos \omega_0 t}$



$$\sigma(t) = \frac{1}{2} + \frac{1}{2}\,\mathrm{sgn}(t)$$

$$\Rightarrow \mathrm{sgn}(t) = 2\left(\sigma(t) - \frac{1}{2}\right)$$

$$= 2\sigma(t) - 1$$

$$\circ\!\!-\!\!\bullet$$

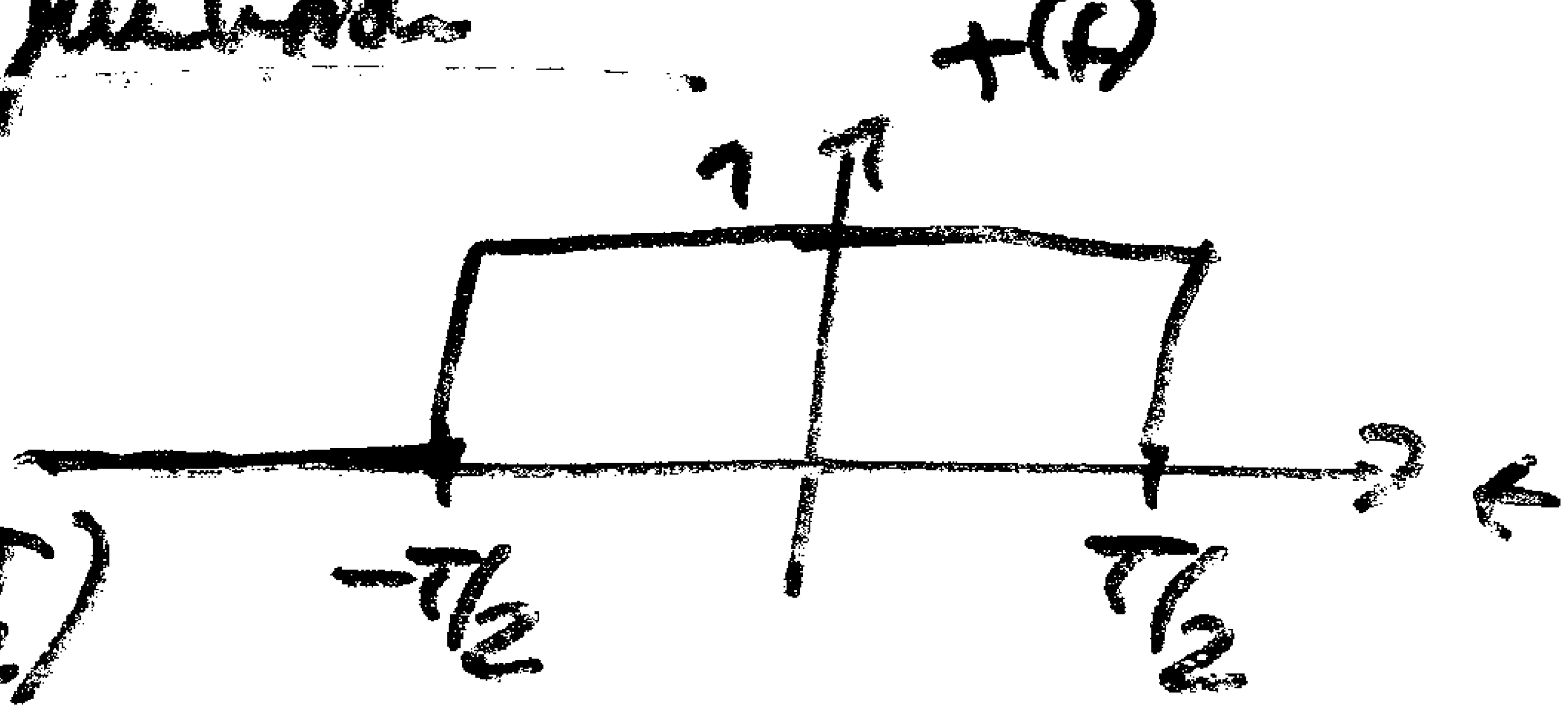
$$F\{\mathrm{sgn}(t)\} = 2\cdot\left(\frac{1}{2}\,\delta(f) + \frac{1}{2}\,\frac{1}{j\pi f}\right) - \delta(f)$$

$$= \cancel{\delta(f)} + \frac{1}{j\pi f} - \cancel{\delta(f)}$$

$$= \frac{1}{j\pi f}$$

~~2.1.5~~ Rechteckfunktion

$$x(t) = \mathrm{rect}\left(\frac{t}{T}\right)$$



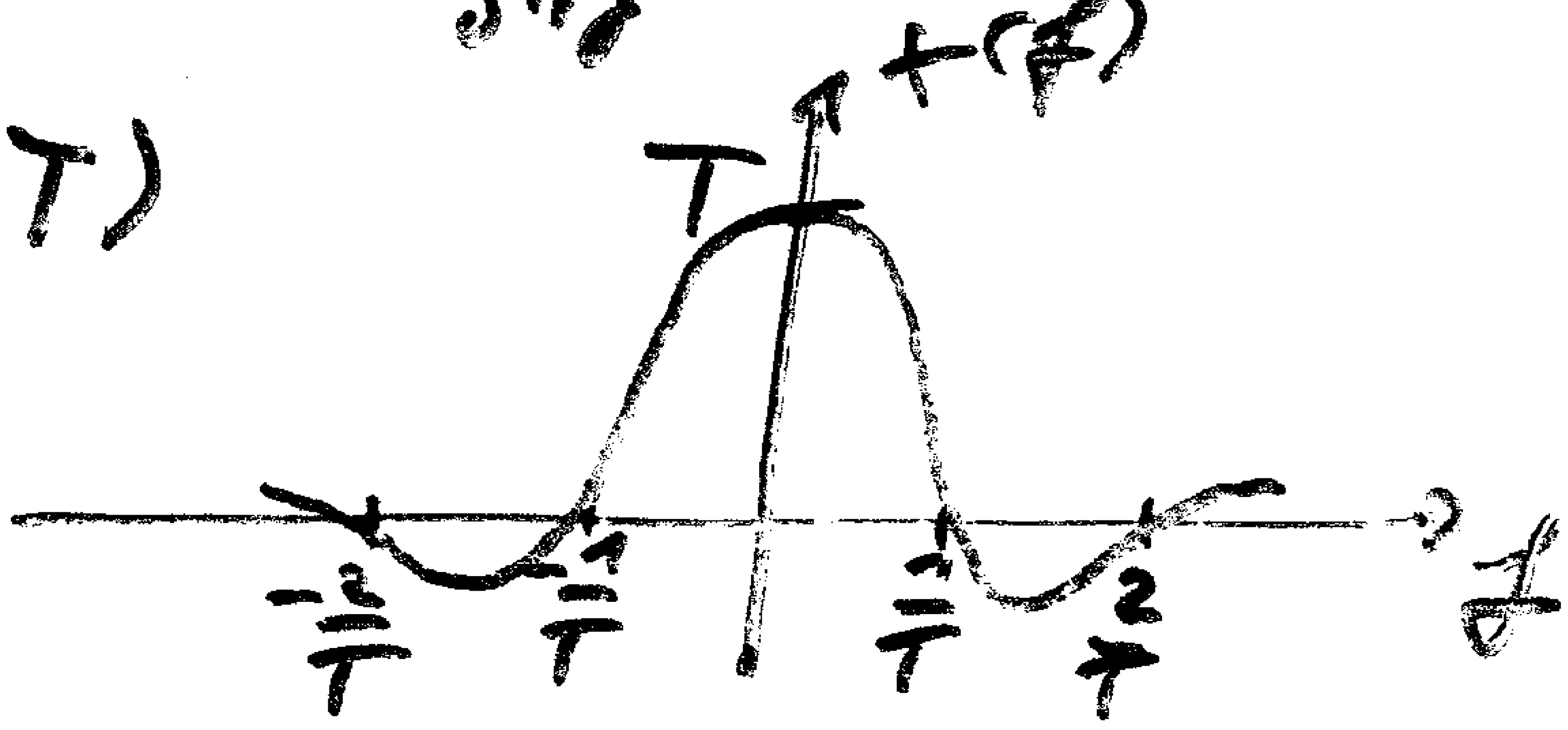
$$= \sigma\left(t+\frac{T}{2}\right) - \sigma\left(t-\frac{T}{2}\right)$$

$$\circ\!\!-\!\!\bullet$$

$$X(f) = \frac{1}{2}\delta(f)\cdot e^{j2\pi\frac{T}{2}f} + \frac{1}{2}\frac{1}{j\pi f}e^{j2\pi\frac{T}{2}f}$$

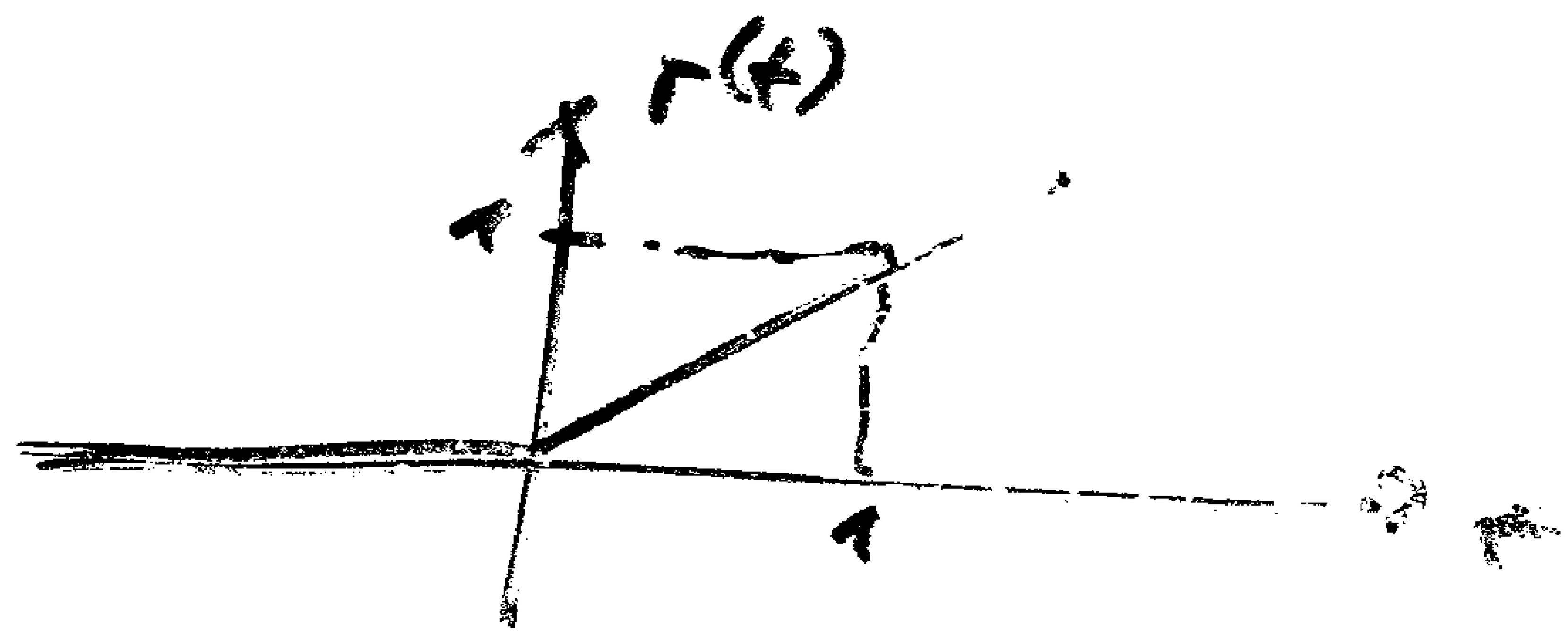
$$- \frac{1}{2}\delta(f)\cdot e^{-j2\pi\frac{T}{2}f} - \frac{1}{2}\frac{1}{j\pi f}e^{-j2\pi\frac{T}{2}f}$$

$$= T\cdot \mathrm{si}(\pi f T)$$



2.1.6 Rampe

$$r(t) = t\cdot\sigma(t)$$

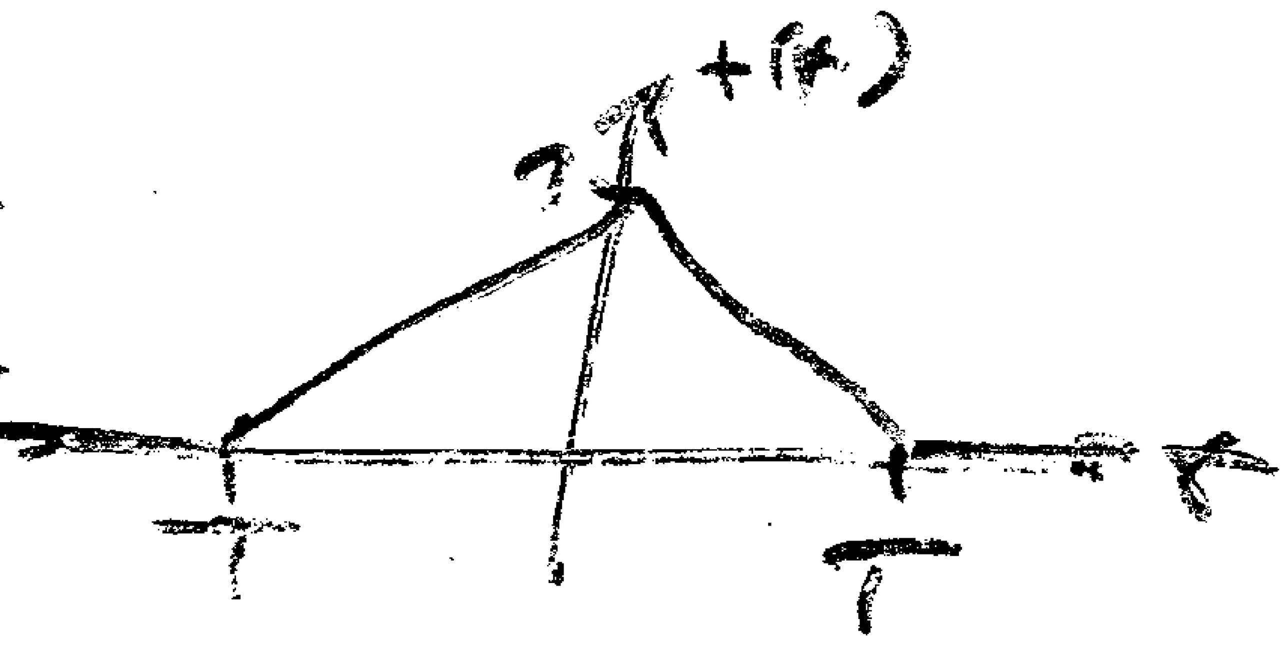


$$\circ\!\!-\!\!\bullet$$

$$R(f) = \frac{j}{2}\underbrace{\delta'(f)}_{\text{Doppel-Impuls}} - \frac{1}{4\pi^2 f^2}$$

2.1.7 Dreieck-Impuls

$$x(t) = \begin{cases} \frac{t+T}{T} & -T \le t < 0 \\ \frac{-t+T}{T} & 0 \le t \le T \\ 0 & \text{sonst} \end{cases}$$



$$= r(t+T) - 2r(t) + r(t-T)$$

$$X(f) = T\,\mathrm{si}^2(\pi f T)$$

$$X(f) = \frac{1}{2}\delta(f)\cdot e^{j2\pi\frac{T}{2}f} + \frac{1}{2}\frac{1}{j\pi f}e^{j2\pi T/2 f}$$
$$- \frac{1}{2}\delta(f)\cdot e^{-j2\pi\frac{T}{2}f} - \frac{1}{2}\frac{1}{j\pi f}e^{-j2\pi T/2 f}$$

$$= \frac{1}{2}\delta(f)\cdot\left[e^{j2\pi\frac{T}{2}f} - e^{-j2\pi\frac{T}{2}f}\right]$$
$$+ \frac{1}{2}\frac{1}{j\pi f}\left[e^{j2\pi T/2 f} - e^{-j2\pi T/2 f}\right]$$

$$= \left[\frac{1}{2}\delta(f) + \frac{1}{2}\frac{1}{j\pi f}\right]\left[e^{j2\pi\frac{T}{2}f} - e^{-j2\pi\frac{T}{2}f}\right]$$

$$= \left[\frac{1}{2}\delta(f) + \frac{1}{2}\frac{1}{j\pi f}\right]\cdot 2j\sin(\pi T f)$$

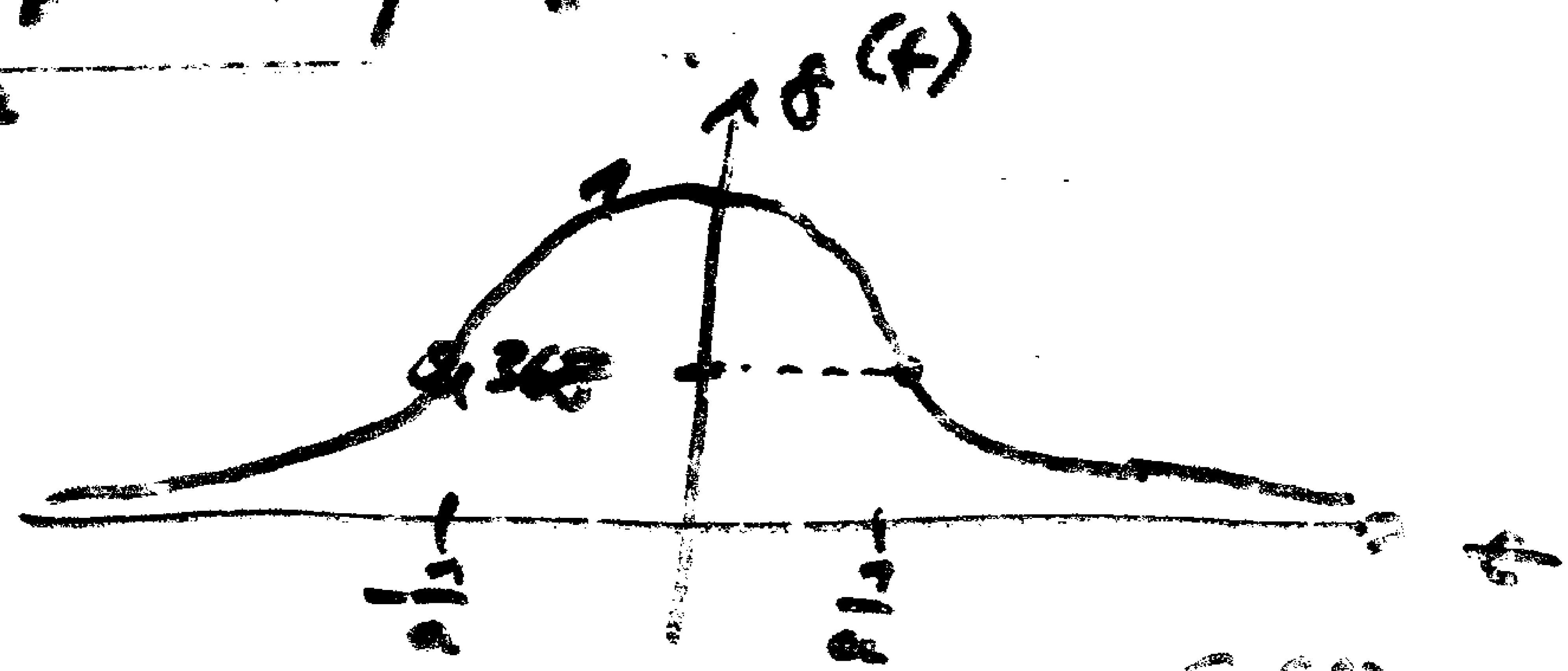
$$= j\delta(f)\sin(\pi T f) + \frac{1}{\pi f}\sin(\pi T f)$$

$$= \quad 0 \quad + \frac{T}{\pi T f}\sin(\pi T f)$$

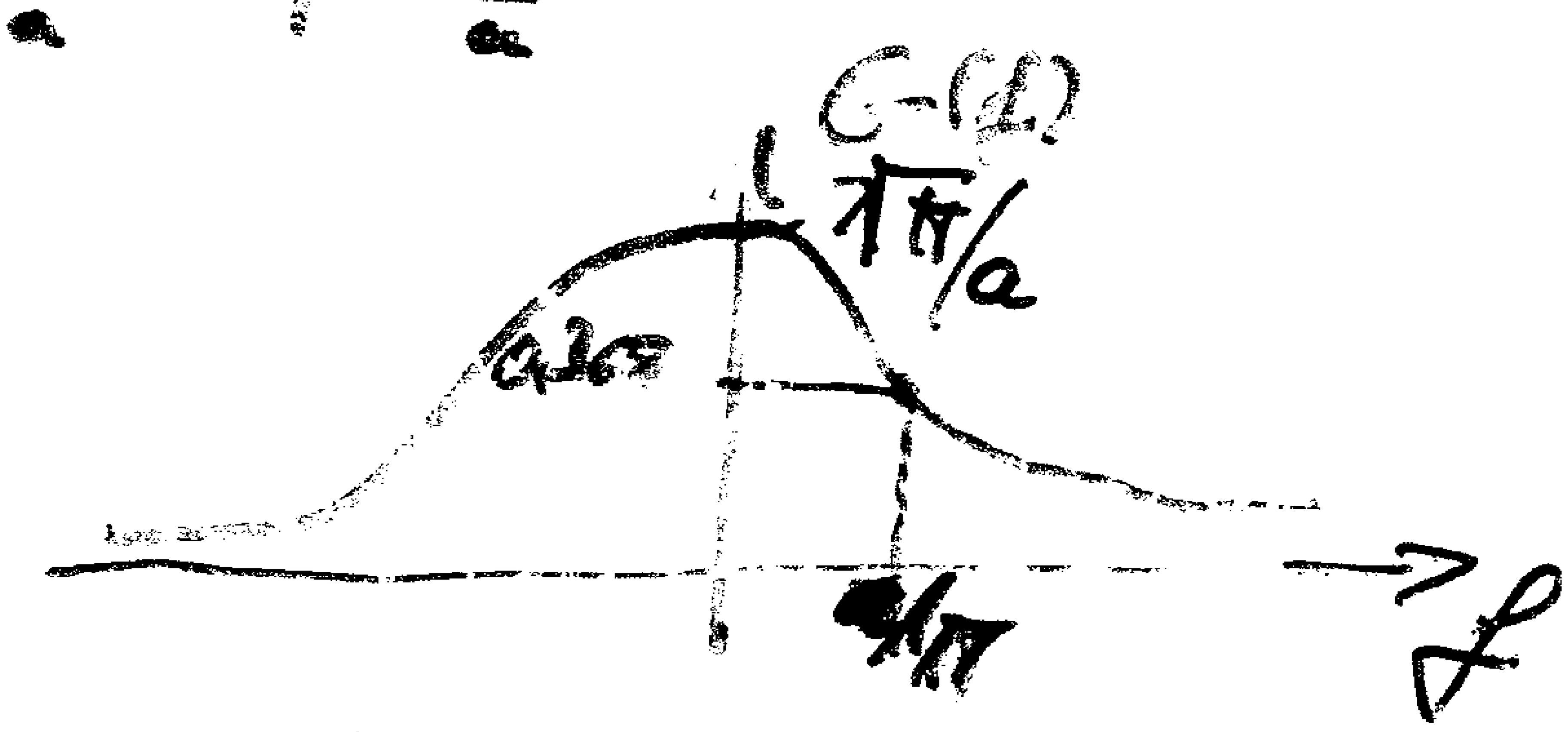
$$= T\cdot\frac{\sin \pi T f}{\pi T f} = T\cdot si(\pi T f)$$

~~2.1.8~~ Gauß-Impuls

$g(t) = e^{-a^2 t^2}$



$$G(f) = \frac{\sqrt{\pi}}{a} e^{-\pi^2 f^2 / a^2}$$



2.1.9 Kausale Signale

$\rightarrow x(t) = 0$ für $t < 0$!

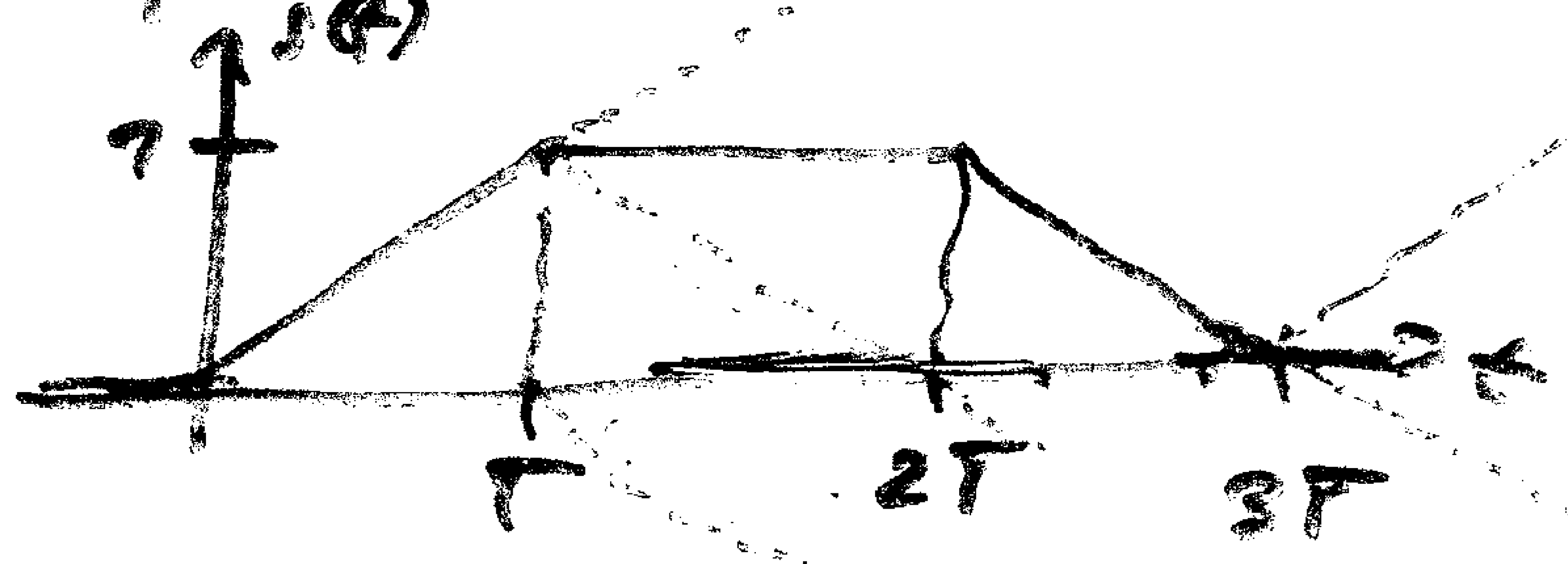
Kausalität wird erzwungen durch:

$$\underline{x_k(t) = x(t) \cdot \sigma(t)}$$

20.1.10 (4.3) Signal-Bausteine

Zusammensetzen beliebiger zeitverschobener Elementarsignale → einfache Fouriertransformation!

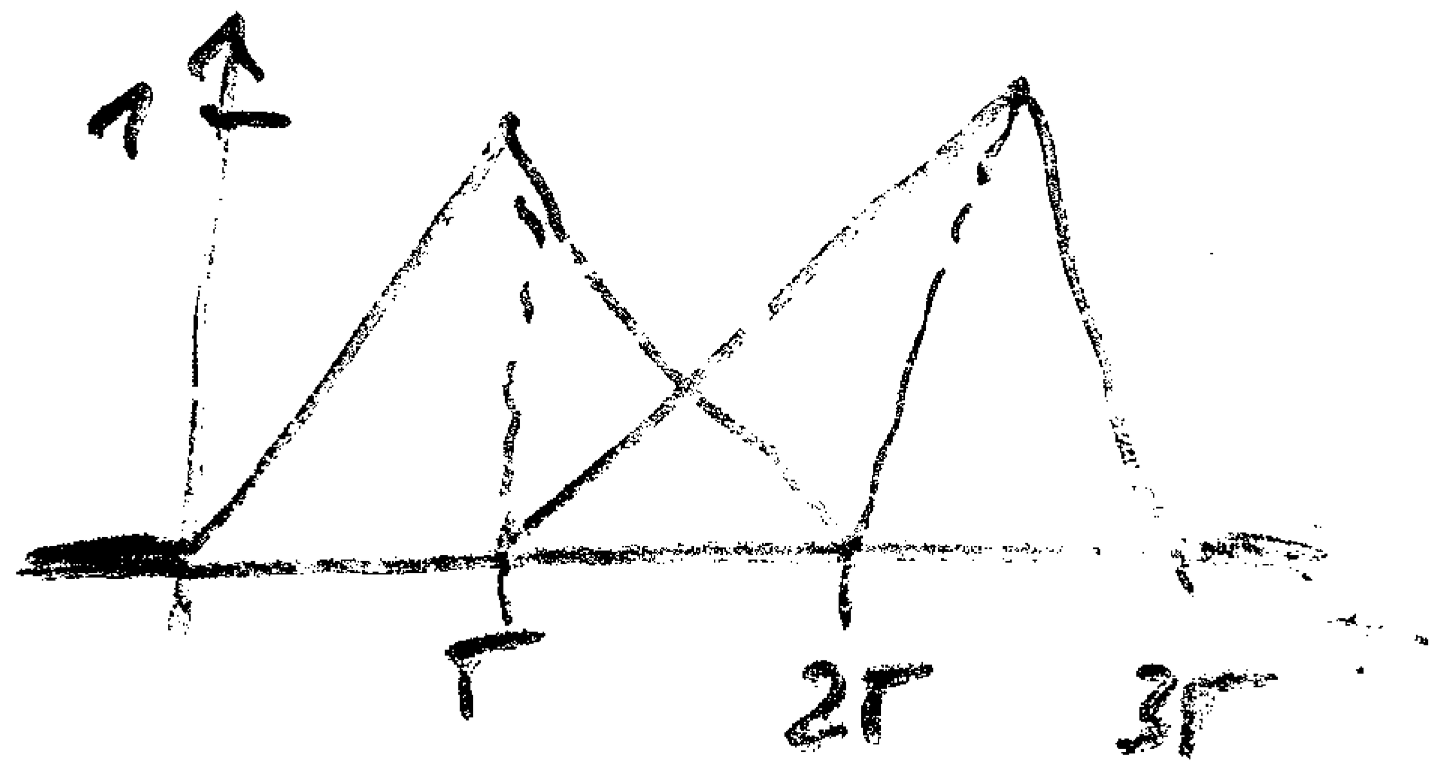
Beispiel 1:



$$s(t) = r(t/T) - r\left(\frac{t-T}{T}\right) - r\left(\frac{t-2T}{T}\right) + r\left(\frac{t-3T}{T}\right)$$

bzw. mit 2 Dreieckfunktionen:

$\Rightarrow$



$$S(f) = T\,si^2(\pi f T)\cdot e^{-j2\pi f T}$$

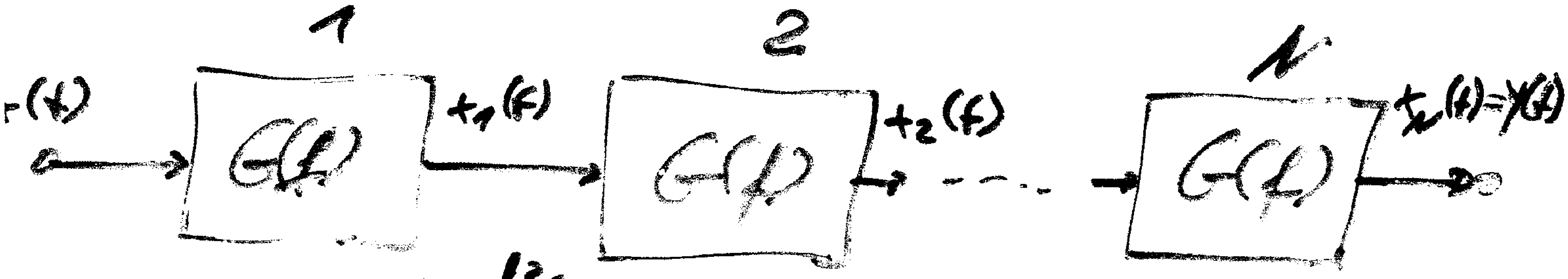
$$+ T\,si^2(\pi f T)\cdot e^{-j2\pi f 2T}$$

$$= T\,si^2(\pi f T)\left[e^{-j2\pi f T} + e^{-j2\pi f 2T}\right]$$

$$= T\,si^2(\pi f T)\,e^{-j3\pi f T}\left[e^{j\pi f T} + e^{-j\pi f T}\right]$$

$$= 2e^{-j3\pi f T}\cdot T\,si^2(\pi f T)\cdot \cos(\pi f T)$$

Übung 2.1. Gauß-Tiefpaß



$$G(f) = e^{-\pi^2 f^2/a^2}$$

Gesamtübertragungsfunktion $H(f) = ?$

$$X_1(f) = X(f)\cdot G(f)$$

$$X_2(f) = X_1(f)\cdot G(f) = X(f)\cdot [G(f)]^2$$

$$\vdots$$

$$X_N(f) = X(f)\cdot [G(f)]^N = X(f)\cdot H(f)$$

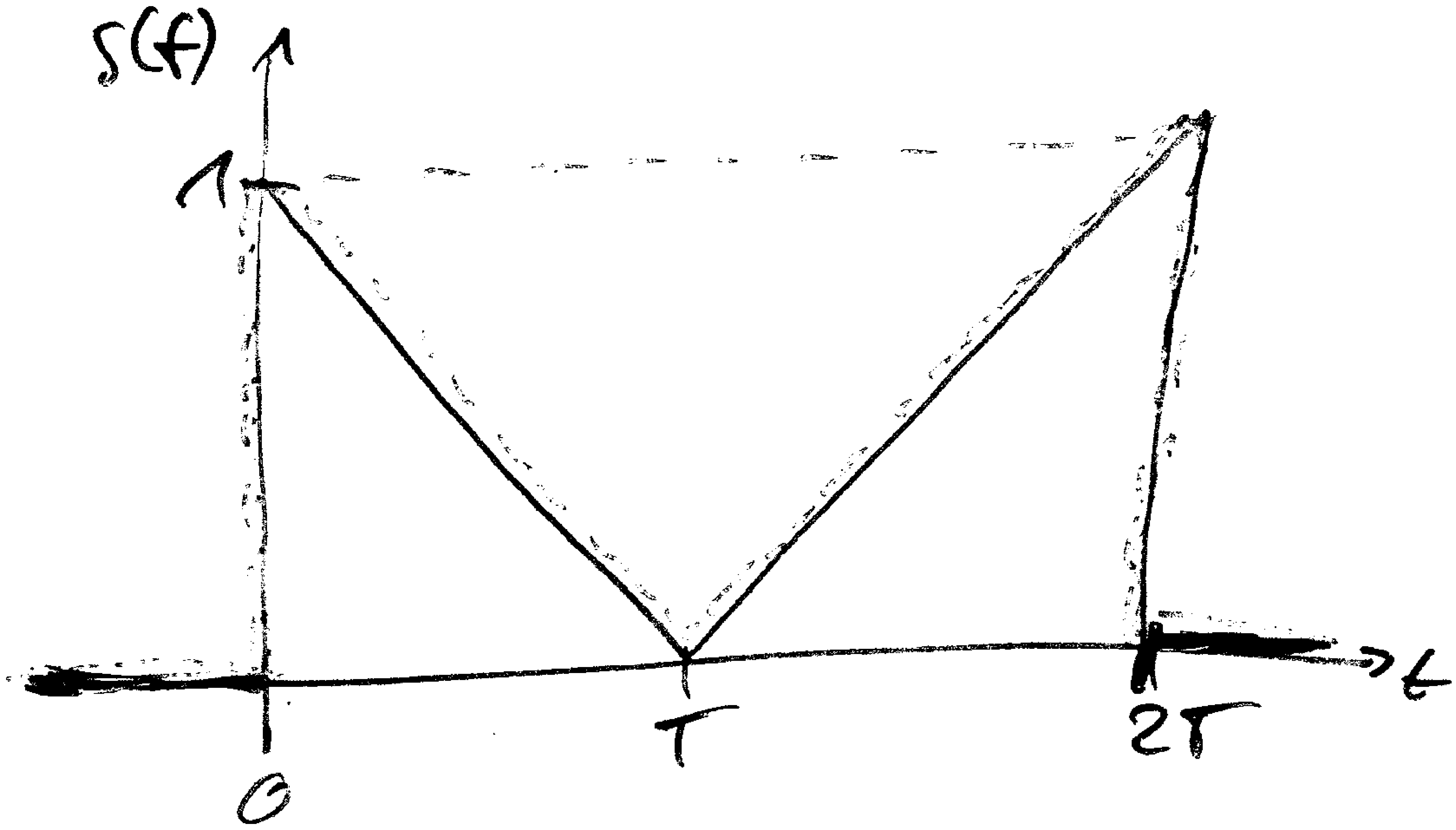
$$\Rightarrow \underline{\underline{H(f)}} = [G(f)]^N = e^{-\pi^2 f^2 \frac{N}{a^2}} = \underline{\underline{e^{-\pi^2 f^2 \left(\frac{\sqrt{N}}{a}\right)^2}}}$$

$$\circ\!\!-\!\!\bullet$$

$$h(t) = \frac{a/\sqrt{N}}{\sqrt{\pi}}\, e^{-\left(\frac{a}{\sqrt{N}}\right)^2 t^2}$$

→ wieder einen Gauß-TP, aber mit $\sqrt{N}$-fach niedrigerer Grenzfrequenz

Übung 2.2.



Fourier Transformierte $S(f)$?

1) $\mathrm{rect}\left(\frac{t-T}{2T}\right) \circ\!\!-\!\!\bullet\; 2T\,\mathrm{si}(2\pi fT)\,e^{-j2\pi fT}$

2) $\Lambda\left(\frac{t-T}{T}\right) \circ\!\!-\!\!\bullet\; T\,\mathrm{si}^2(\pi fT)\cdot e^{-j2\pi fT}$

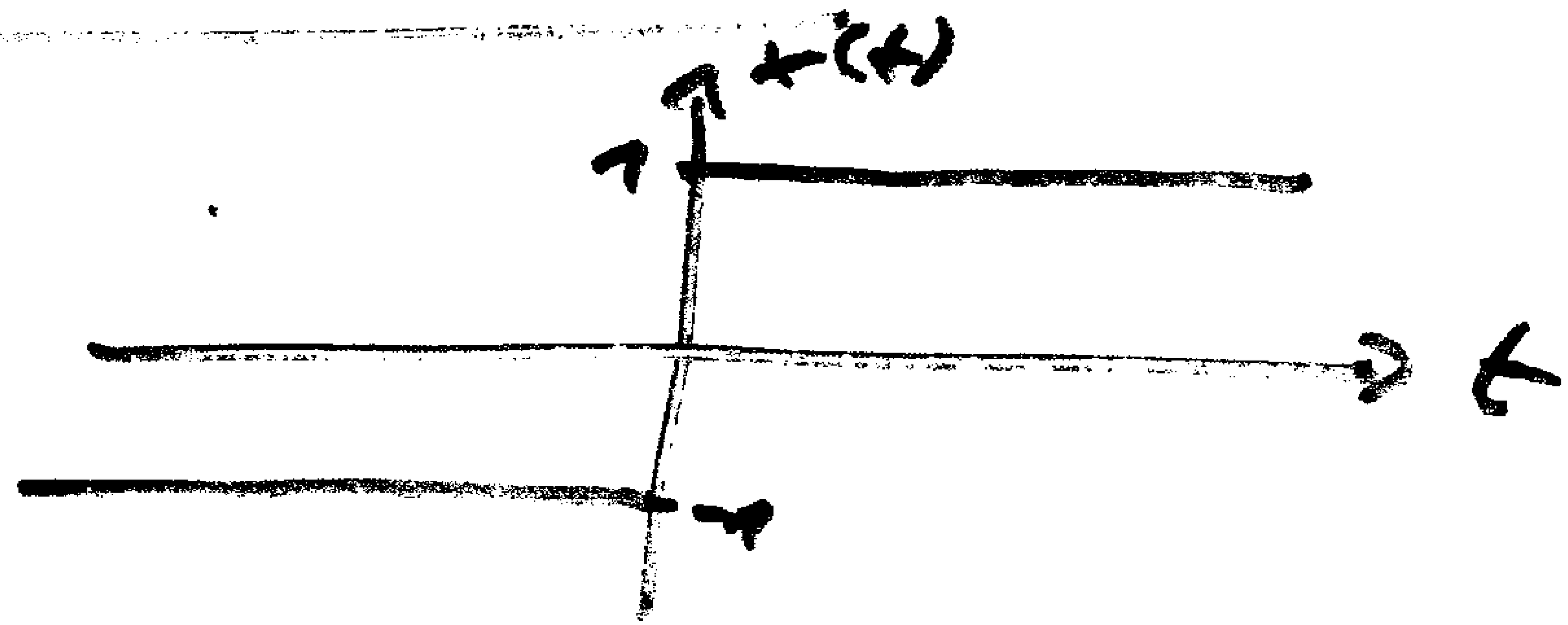
$\Rightarrow$ $S(f) = 1) - 2)$

$$= \left[2T\,\mathrm{si}(2\pi fT) - T\,\mathrm{si}^2(\pi fT)\right] e^{-j2\pi fT}$$

$$= \left[2\,\mathrm{si}(2\pi fT) - \mathrm{si}^2(\pi fT)\right]\cdot T\cdot e^{-j2\pi fT}$$

~~[illegible]~~ Signum – Funktion

$x(t) = \mathrm{sgn}(t)$



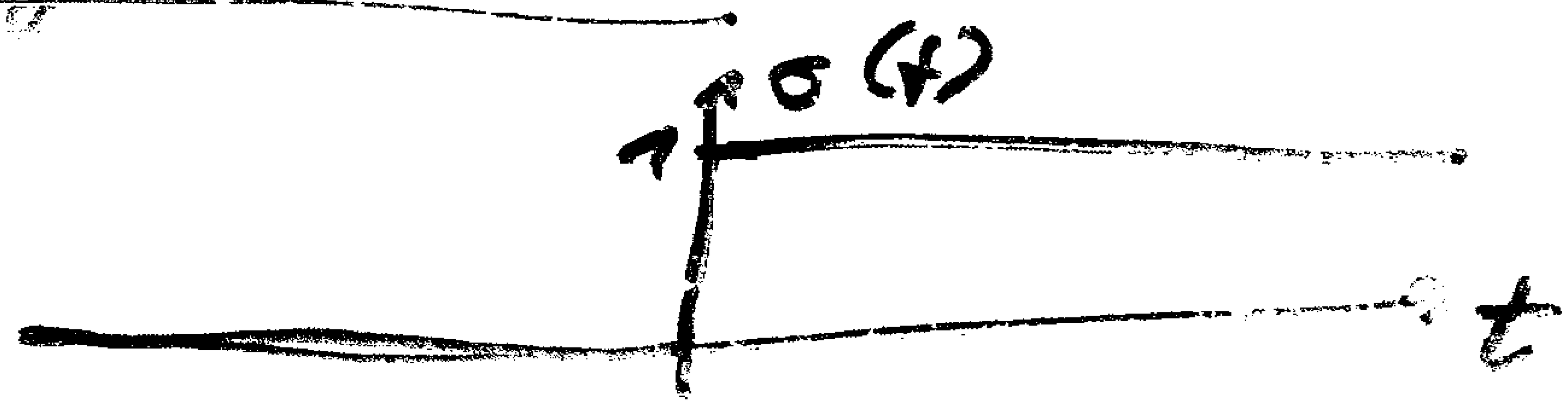
$X(f) = \frac{1}{j\pi f}$ über: $\frac{d}{dt}\mathrm{sgn}(t) = 2\,\delta(t)$ ○—● 2

$\to \int \frac{d}{dt}\mathrm{sgn}(t)\, dt$

$\frac{1}{j2\pi f}\cdot 2 = \underline{\underline{\frac{1}{j\pi f}}}$

~~[illegible]~~ Sprung – Funktion

$x(t) = \sigma(t)$



$= \frac{1}{2} + \frac{1}{2}\,\mathrm{sgn}\, t$

○—●

$X(f) = \frac{1}{2}\delta(f) + \frac{1}{2}\,\frac{1}{j\pi f}$