Klausur Mathe 3 Analysis ET/ TI - SS 2011

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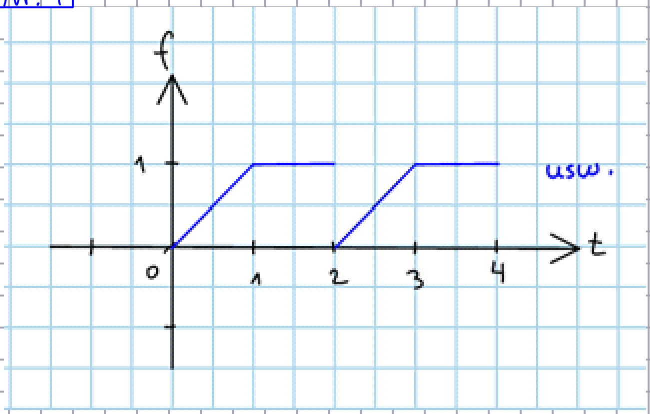
18:53

Nr. 1



$$y(t) = \begin{cases} t & \text{für } 0 \le t < 1 \\ 1 & \text{für } 1 \le t \le 2 \end{cases}$$ 1

$T = 2 \quad ; \quad \omega = \frac{2\pi}{T} = \pi$

$$c_0 = \frac{1}{T} \int_T y(t)\, dt = \frac{1}{2} \cdot \left(\int_0^1 t\, dt + \int_1^2 1\, dt \right)$$ 1 1

$$c_0 = \frac{1}{2} \cdot \left(\left[\frac{1}{2} t^2 \right]_0^1 + \left[t \right]_1^2 \right) = \frac{1}{2} \left(\frac{1}{2} + 2 - 1 \right) = \frac{3}{4}$$ 1

$$c_n = \frac{1}{T} \cdot \int_T y(t) \cdot e^{-jn\omega t}\, dt = \frac{1}{2} \cdot \left(\int_0^1 t \cdot e^{-jn\pi t}\, dt + \int_1^2 e^{-jn\pi t}\, dt \right)$$ 1

$$\left(\text{Formelsammlung:} \quad \int x e^{ax}\, dx = e^{ax} \cdot \left(\frac{x}{a} - \frac{1}{a^2} \right) \right)$$

$$c_n = \frac{1}{2} \cdot \left(\left[e^{-jn\pi t} \cdot \left(\frac{t}{-jn\pi} + \frac{1}{n^2\pi^2} \right) \right]_0^1 + \left[e^{-jn\pi t} \cdot \frac{1}{-jn\pi} \right]_1^2 \right) = \frac{1}{2} \cdot \left(\left[e^{-jn\pi t} \cdot \left(\frac{jt}{n\pi} + \frac{1}{n^2\pi^2} \right) \right]_0^1 + \left[e^{-jn\pi t} \cdot \frac{j}{n\pi} \right]_1^2 \right)$$ 1 1

$$c_n = \frac{1}{2} \cdot \left(e^{-jn\pi} \cdot \left(\frac{j}{n\pi} + \frac{1}{n^2\pi^2} \right) - \frac{1}{n^2\pi^2} + \frac{j}{n\pi} \cdot \left(e^{-2jn\pi} - e^{-jn\pi} \right) \right)$$ 1 1

$$c_n = e^{-jn\pi} \cdot \frac{1}{2n^2\pi^2} - \frac{1}{2n^2\pi^2} + \frac{j}{2n\pi} e^{-2jn\pi} = \left(\underbrace{\cos(n\pi)}_{=(-1)^n} - j \underbrace{\sin(n\pi)}_{=0} \right) \cdot \frac{1}{2n^2\pi^2} - \frac{1}{2n^2\pi^2} + \left(\underbrace{\cos(2n\pi)}_{=1} - j \underbrace{\sin(2n\pi)}_{=0} \right) \cdot \frac{j}{2n\pi}$$ 1 1

$$c_n = \frac{1}{2n^2\pi^2} \cdot \left((-1)^n - 1 \right) + j \cdot \frac{1}{2n\pi} = \begin{cases} j \cdot \frac{1}{2n\pi} & n \text{ gerade} \\ -\frac{1}{n^2\pi^2} + j \cdot \frac{1}{2n\pi} & n \text{ ungerade} \end{cases}$$ 1

Reell: $a_n = 2 \cdot \mathrm{Re}(c_n) = \frac{1}{n^2\pi^2} \cdot \left((-1)^n - 1 \right)$ 1

$b_n = -2 \cdot \mathrm{Im}(c_n) = -\frac{1}{n\pi}$ 1

$a_0 = 2 \cdot c_0 = \frac{3}{2}$ 1

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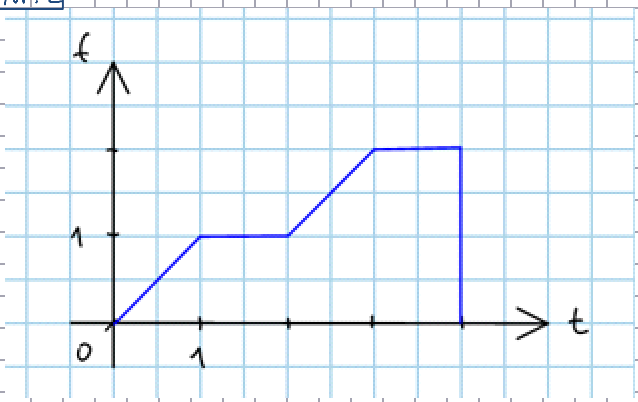
Nr. 2



$$f(t) = \begin{cases} t & 0 \le t < 1 \\ 1 & 1 \le t < 2 \\ t-1 & 2 \le t < 3 \\ 2 & 3 \le t \le 4 \end{cases}$$

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) \cdot e^{-j\omega t} \, dt$$

$$F(\omega) = \int_0^1 t \cdot e^{-j\omega t} \, dt + \int_1^2 e^{-j\omega t} \, dt + \int_2^3 (t-1) \cdot e^{-j\omega t} \, dt + \int_3^4 2 \cdot e^{-j\omega t} \, dt$$

$$F(\omega) = \left[e^{-j\omega t} \cdot \left(\frac{t}{-j\omega} + \frac{1}{\omega^2} \right) \right]_0^1 + \left[e^{-j\omega t} \cdot \frac{1}{-j\omega} \right]_1^2 + \left[e^{-j\omega t} \cdot \left(\frac{t}{-j\omega} + \frac{1}{\omega^2} + \frac{1}{j\omega} \right) \right]_2^3 + 2 \cdot \left[e^{-j\omega t} \cdot \frac{1}{-j\omega} \right]_3^4$$

$$F(\omega) = e^{-j\omega} \left(\frac{j}{\omega} + \frac{1}{\omega^2} \right) - \frac{1}{\omega^2} + \frac{j}{\omega} \cdot \left(e^{-2j\omega} - e^{-j\omega} \right) + e^{-3j\omega} \cdot \left(\frac{3j}{\omega} + \frac{1}{\omega^2} - \frac{j}{\omega} \right) - e^{-2j\omega} \left(\frac{2j}{\omega} + \frac{1}{\omega^2} - \frac{j}{\omega} \right) + \frac{2j}{\omega} \left(e^{-4j\omega} - e^{-3j\omega} \right)$$

$$F(\omega) = -\frac{1}{\omega^2} + e^{-j\omega} \cdot \frac{1}{\omega^2} - e^{-2j\omega} \cdot \frac{1}{\omega^2} + e^{-3j\omega} \cdot \frac{1}{\omega^2} + e^{-4j\omega} \cdot \frac{2j}{\omega}$$

$$F(\omega) = \frac{1}{\omega^2} \cdot \left(-1 + e^{-j\omega} - e^{-2j\omega} + e^{-3j\omega} + 2j\omega \cdot e^{-4j\omega} \right)$$

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Nr 3. Gegeben ist die Laplacetransformierte

$$F(p) = \frac{3 \cdot p}{2 \cdot p^2 + 8} - \frac{4}{p \cdot (p-1)}$$

a) Berechnen Sie unter Verwendung von $F(p)$ die Grenzwerte $\lim_{t \to 0} f(t)$ und $\lim_{t \to \infty} f(t)$

b) Berechnen Sie die Rücktransformierte $f(t)$ von $F(p)$ mithilfe der Partialbruchzerlegung und den Laplace-Korrespondenzen.

(12 Punkte)

a) $\lim\limits_{t \to 0} f(t) = \lim\limits_{p \to \infty} \left(p \cdot F(p)\right) = \lim\limits_{p \to \infty} \left(\frac{3p^2}{2p^2+8} - \frac{4}{p-1} \right) = \frac{3}{2}$ 1 1 1

$\lim\limits_{t \to \infty} f(t) = \lim\limits_{p \to 0} \left(p \cdot F(p)\right) = \lim\limits_{p \to 0} \left(\frac{3p^2}{2p^2+8} - \frac{4}{p-1} \right) = 4$ 1 1

b) $F(p) = \frac{3p}{2p^2+8} - \frac{4}{p(p-1)} = \frac{3}{2} \cdot \frac{p}{p^2+4} - 4 \cdot \frac{1}{p \cdot (p-1)}$

und $\frac{1}{p \cdot (p-1)} = \frac{A}{p} + \frac{B}{p-1}$ 1 $\Leftrightarrow$ $1 = A \cdot (p-1) + B \cdot p$

$p = 1$: $1 = B$ 1

$p = 0$: $-1 = A$

$\Rightarrow \frac{1}{p \cdot (p-1)} = -\frac{1}{p} + \frac{1}{p-1}$ 1

$\Rightarrow F(p) = \frac{3}{2} \cdot \frac{p}{p^2+4} - 4 \cdot \frac{1}{p \cdot (p-1)}$ 1

$F(p) = \frac{3}{2} \cdot \frac{p}{p^2+4} + 4 \cdot \frac{1}{p} - 4 \cdot \frac{1}{p-1}$ 1

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$f(t) = \frac{3}{2} \cdot \cos(2t) + 4 - 4 \cdot e^t$ 1

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Nr 4. Ermitteln Sie die Glieder der zur gegebenen z-Transformierten $F_z(z) = \frac{6\cdot z^3+2}{2\cdot z^3+5\cdot z^2-3\cdot z}$ gehörenden Zahlenfolge f_n.

(6 Punkte)

$$F_z(z) = \frac{6z^3+2}{2z^3+5z^2-3z} = \frac{6z^3+15z^2-9z-15z^2+9z+2}{2z^3+5z^2-3z} = 3 + \frac{-15z^2+9z+2}{2z\left(z^2+\frac{5}{2}z-\frac{3}{2}\right)}$$ 1

$$F_z(z) = 3 + \frac{-\frac{15}{2}z^2+\frac{9}{2}z+1}{z\left(z-\frac{1}{2}\right)(z+3)} = 3 + \frac{A}{z} + \frac{B}{z-\frac{1}{2}} + \frac{C}{z+3}$$ 1

$$-\frac{15}{2}z^2+\frac{9}{2}z+1 = A\cdot\left(z-\frac{1}{2}\right)\cdot(z+3) + B\cdot z\cdot(z+3) + C\cdot z\cdot\left(z-\frac{1}{2}\right)$$ 1

$z=0$: $\quad 1 = -\frac{3}{2}A \iff A = -\frac{2}{3}$

$z=\frac{1}{2}$: $\quad -\frac{15}{8}+\frac{9}{4}+1 = \frac{7}{4}B \iff B = \frac{11}{14}$

$z=-3$: $\quad -\frac{15}{2}\cdot 9 - \frac{27}{2} + 1 = \frac{21}{2}C \iff C = -\frac{160}{21}$ 1

$$\Rightarrow F_z(z) = 3 - \frac{2}{3}\cdot\frac{1}{z} + \frac{11}{14}\cdot\frac{1}{z-\frac{1}{2}} - \frac{160}{21}\cdot\frac{1}{z+3}$$ 1

$$f_n = 3\cdot\delta_{n,0} - \frac{2}{3}\cdot\delta_{n,1} + \frac{11}{14}\cdot\sigma(n-1)\cdot\left(\frac{1}{2}\right)^{n-1} - \frac{160}{21}\cdot\sigma(n-1)\cdot(-3)^{n-1}$$

$$f_n = \begin{cases} 3 & n=0 \\ -\frac{2}{3}+\frac{11}{14}-\frac{160}{21} = -\frac{15}{2} & n=1 \\ -\frac{2}{3}\cdot\frac{1}{2^{n-1}} - \frac{160}{21}\cdot(-3)^{n-1} & n\geq 2 \end{cases}$$ 1

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