CSC165H1 Problem Set 2

Yulin WANG, Qidi Zhou, Dana Zhao

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1. AND vs. IMPLIES

a)

Proof:
Header:
Let n$\in \mathbb{N}$
Assume that n > 15
We want to prove that n^3 - 10n^2 + 3 $\geq$ 165.

Body:

$$\begin{aligned} n^3 - 10n^2 + 3 &\geq n^3 - 10n^2 \quad (since\ 3 \geq 0) \\ &= n^2(n-10) \\ &\geq n^2 \times 1 = n^2 = 15^2 \quad (since\ n > 15 \Rightarrow n - 10 \geq 1) \\ &\geq 165 \end{aligned}$$

Therefore, we have proven that $\forall$n $\in \mathbb{N}$, n > 15 $\Rightarrow$ n^3 -10n^2 + 3 $\geq$ 165.

□

b)

Since this statement is False, so we want to disprove it.
We can prove that its negation is True.
Negation of the original statement: $\exists$n $\in \mathbb{N}$, n $\leq$ 15 $\vee$ n^3 - 10n^2 + 3 < 165.
Proof:
Header:
Let n = 1, then n $\in \mathbb{N}$
We want to prove n $\leq$ 15 $\vee$ n^3 - 10n^2 + 3 < 165.
Body:
Since n = 1, so n $\leq$ 15.
Thus, n $\leq$ 15 $\vee$ n^3 - 10n^2 + 3 < 165 is True.

Therefore, negation of the original statement is True, then the original statement is False.

$\square$

2. Ceiling function

a)

Translation: $\forall$n,m $\in \mathbb{N}$, n < m $\Rightarrow \lceil \frac{m-1}{m} \cdot$ n$\rceil$ = n
Proof:
Header:
Let n, m $\in \mathbb{N}$
Assume that n < m
We want to prove that $\lceil \frac{(m-1)}{m} \cdot$ n$\rceil$ = n.
Body:

$$\lceil \frac{(m-1)}{m} \cdot n \rceil = \lceil (1 - \frac{1}{m}) \cdot n \rceil$$
$$= \lceil -\frac{n}{m} + n \rceil$$

By Fact 2, since -$\frac{n}{m} \in \mathbb{R}$ and n$\in \mathbb{Z}$, so we have:
$\lceil$-$\frac{n}{m}$ + n$\rceil$ = $\lceil$-$\frac{n}{m}\rceil$ + n
Since n, m $\in \mathbb{N}$ and n < m, we know that 0 $\leq \frac{n}{m}$ < 1
Then, -1 < - $\frac{n}{m} \leq$ 0
So, $\lceil$-$\frac{n}{m}\rceil$ = 0
Thus, $\lceil$ $\frac{(m-1)}{m}$ $\cdot$ n $\rceil$ = $\lceil$-$\frac{n}{m}\rceil$ + n = 0 + n = n

Therefore, we have proven that $\forall$ n,m $\in \mathbb{N}$, n < m $\Rightarrow \lceil \frac{(m-1)}{m} \cdot$n$\rceil$ = n.

$\square$

b)

Translation:
$\forall n \in \mathbb{N}$, $(50 | \text{nextFifty}(n) \land \text{nextFifty}(n) \geq n) \land (\forall k \in \mathbb{N},\ 50|k \land k \geq n \Rightarrow k \geq \text{nextFifty}(n))$

$\Leftrightarrow \forall n \in \mathbb{N}$, $(50|50\cdot\lceil \frac{n}{50} \rceil \land 50\cdot\lceil \frac{n}{50} \rceil \geq n) \land (\forall k \in \mathbb{N},\ 50|k \land k \geq n \Rightarrow k \geq 50\cdot\lceil \frac{n}{50} \rceil)$

Proof:
Header:
  Let $n \in \mathbb{N}$
Body:

Firstly, we want to prove that $50|50\cdot\lceil \frac{n}{50} \rceil \land 50\cdot\lceil \frac{n}{50} \rceil \geq n$
  Since $1|\lceil \frac{n}{50} \rceil$, so $50|50\cdot\lceil \frac{n}{50} \rceil$ (multiply 50 on each side)
  Since $\lceil x \rceil \geq x$, so $\lceil \frac{n}{50} \rceil \geq \frac{n}{50}$
  Thus $50\cdot\lceil \frac{n}{50} \rceil \geq n$ (multiply 50 on each side)

Therefore, we have proven that $50|50\cdot\lceil \frac{n}{50} \rceil \land 50\cdot\lceil \frac{n}{50} \rceil \geq n$

Secondly, we want to prove that $\forall k \in \mathbb{N},\ 50|k \land k \geq n \Rightarrow k \geq 50\cdot\lceil \frac{n}{50} \rceil$
  Let $k \in \mathbb{N}$
  Assume that $50|k \land k \geq n$
  Since $50|k$ represents $\exists m \in \mathbb{Z},\ k = 50m \geq n$
  So $m \geq \frac{n}{50}$ (divide 50 on each side)
  Then by Fact 1, we know $m \geq \lceil \frac{n}{50} \rceil$,
  so $50m \geq 50\cdot\lceil \frac{n}{50} \rceil$, then $k \geq 50\cdot\lceil \frac{n}{50} \rceil$

Therefore, we have proven that $\forall k \in \mathbb{N},\ 50|k \land k \geq n \Rightarrow k \geq 50\cdot\lceil \frac{n}{50} \rceil$

In conclusion, we have proven that:
$\forall n \in \mathbb{N}$, $(50 | \text{nextFifty}(n) \land \text{nextFifty}(n) \geq n) \land (\forall k \in \mathbb{N},\ 50|k \land k \geq n \Rightarrow k \geq \text{nextFifty}(n))$

$\square$

3. Divisibility

a)

Translation:
$\forall$n $\in \mathbb{N}$, n $\leq$ 2300 $\Rightarrow$ (49|n $\Leftrightarrow$ 50$\cdot$(50$\cdot\lceil\frac{n}{50}\rceil$ - n) = 50$\cdot\lceil\frac{n}{50}\rceil$)

$\Leftrightarrow$ $\forall$n $\in \mathbb{N}$, n $\leq$ 2300 $\Rightarrow$ (49|n $\Leftrightarrow$ 50$\cdot\lceil\frac{n}{50}\rceil$ - n = $\lceil\frac{n}{50}\rceil$)

Proof:
(1) Firstly, we want to prove the $\Rightarrow$ direction of the statement.
Header:
 Let n$\in \mathbb{N}$
 Assume that n $\leq$ 2300
 We want to prove that 49|n $\Rightarrow$ 50$\cdot\lceil\frac{n}{50}\rceil$ - n = $\lceil\frac{n}{50}\rceil$
Body:
 Assume that 49|n, which means:
 $\exists$k $\in \mathbb{Z}$, n = 49k, so we have:

$$\begin{aligned}
50 \cdot \lceil\frac{n}{50}\rceil - n &= 50 \cdot \lceil\frac{49k}{50}\rceil - 49k \\
&= 50 \cdot \lceil\frac{50-1}{50} \cdot k\rceil - 49k \\
&= 50 \cdot k - 49k \ (since\ n \leq 2300,\ then\ k = \frac{n}{49} \leq \frac{2300}{49} < 50,\ and\ k \in\ \mathbb{N},\ so\ by\ 2(a)) \\
&= k \\
and \quad \lceil\frac{n}{50}\rceil &= \lceil\frac{49k}{50}\rceil \\
&= \lceil\frac{50-1}{50} \cdot k\rceil \\
&= k
\end{aligned}$$

 Thus, 50$\cdot\lceil\frac{n}{50}\rceil$ - n = $\lceil\frac{n}{50}\rceil$

Therefore, we have proven that: $\forall$n $\in \mathbb{N}$, n $\leq$ 2300 $\Rightarrow$ (49|n $\Rightarrow$ 50$\cdot$(50$\cdot\lceil\frac{n}{50}\rceil$ - n) = 50$\cdot\lceil\frac{n}{50}\rceil$)

(2) Secondly, we want to prove the $\Leftarrow$ direction of the statement.
Header:
 Let n$\in \mathbb{N}$
 Assume that n $\leq$ 2300
 We want to prove that 50$\cdot\lceil\frac{n}{50}\rceil$ - n = $\lceil\frac{n}{50}\rceil$ $\Rightarrow$ 49|n
Body:
 Assume that 50$\cdot\lceil\frac{n}{50}\rceil$ - n = $\lceil\frac{n}{50}\rceil$
 Then, 49$\cdot\lceil\frac{n}{50}\rceil$ = n
 So, $\lceil\frac{n}{50}\rceil$ = $\frac{n}{49}$ (divide 49 on each side)

Since $\lceil \frac{n}{50} \rceil \in \mathbb{Z}$, so $\frac{n}{49} \in \mathbb{Z}$, and then 49|n

Therefore, we have proven that: $\forall$n $\in \mathbb{N}$, n $\leq$ 2300 $\Rightarrow$ (50$\cdot\lceil \frac{n}{50} \rceil$ - n = $\lceil \frac{n}{50} \rceil$ $\Rightarrow$ 49|n)

In conclusion, we have proven that $\forall$n $\in \mathbb{N}$, n $\leq$ 2300 $\Rightarrow$ (49|n $\Leftrightarrow$ 50$\cdot$(50$\cdot\lceil \frac{n}{50} \rceil$ - n) = 50$\cdot\lceil \frac{n}{50} \rceil$)

$\square$

b)

In order to disprove this statement, we can prove its negation is True.
Since from 3(a) we know that:
$\forall$n $\in \mathbb{N}$, 50$\cdot$(nextFifty(n) - n) = nextFifty(n)) $\Rightarrow$ 49|n
So, we just need to disprove the $\Rightarrow$ direction of this statement.
Negation of the $\Rightarrow$ direction of the original statement:
$\exists$n $\in \mathbb{N}$, 49|n $\wedge$ (50$\cdot\lceil \frac{n}{50} \rceil$ - n $\neq$ $\lceil \frac{n}{50} \rceil$)
Proof:
Header:
Let n = 49 $\times$ 50 = 2450, then 49|n
Body:
50$\cdot\lceil \frac{n}{50} \rceil$ - n = 50$\cdot\lceil \frac{2450}{50} \rceil$ - 2450 = 0
$\lceil \frac{n}{50} \rceil$ = $\lceil \frac{2450}{50} \rceil$ = 49
So, 50$\cdot\lceil \frac{n}{50} \rceil$ - n $\neq$ $\lceil \frac{n}{50} \rceil$

Thus, $\exists$n $\in \mathbb{N}$, 49|n $\wedge$ (50$\cdot\lceil \frac{n}{50} \rceil$ - n $\neq$ $\lceil \frac{n}{50} \rceil$)

Therefore, negation of the original statement is True, then the original statement is False.

$\square$

4. Functions

a)

$\exists k \in \mathbb{R}, \forall x \in \mathbb{N}, f(x) \leq k$

b)

Translation:
$\forall f_1, f_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}, (\exists k_1 \in \mathbb{R}, \forall x \in \mathbb{N}, f_1(x) \leq k_1) \wedge (\exists k_2 \in \mathbb{R}, \forall x \in \mathbb{N}, f_2(x) \leq k_2)$
$\Rightarrow (\exists k_3 \in \mathbb{R}, \forall x \in \mathbb{N}, (f_1 + f_2)(x) \leq k_3)$

Proof:
Header:
Let $f_1, f_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}$
Assume that $(\exists k_1 \in \mathbb{R}, \forall x \in \mathbb{N}, f_1(x) \leq k_1) \wedge (\exists k_2 \in \mathbb{R}, \forall x \in \mathbb{N}, f_2(x) \leq k_2)$
We want to prove that $\exists k_3 \in \mathbb{R}, \forall x \in \mathbb{N}, (f_1 + f_2)(x) \leq k_3$

Body:
Let $x \in \mathbb{N}$
Let $k_3 = k_1 + k_2$
Since $0 \leq f_1(x) \leq k_1$ and $0 \leq f_2(x) \leq k_2$
So, $0 \leq f_1(x) + f_2(x) \leq k_1 + k_2 = k_3$
Thus, $(f_1 + f_2)(x) \leq k_3$

Therefore, we have proven that:
$\forall f_1, f_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}, (\exists k_1 \in \mathbb{R}, \forall x \in \mathbb{N}, f_1(x) \leq k_1) \wedge (\exists k_2 \in \mathbb{R}, \forall x \in \mathbb{N}, f_2(x) \leq k_2)$
$\Rightarrow (\exists k_3 \in \mathbb{R}, \forall x \in \mathbb{N}, (f_1 + f_2)(x) \leq k_3)$

$\square$

c)

Translation:
$\forall f_1, f_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}$, ($\exists k_3 \in \mathbb{R}$, $\forall$x $\in \mathbb{N}$, $(f_1 + f_2)$(x) $\leq k_3$)
$\Rightarrow$ ($\exists k_1 \in \mathbb{R}$, $\forall$x $\in \mathbb{N}$, $f_1(x) \leq k_1$) $\wedge$ ($\exists k_2 \in \mathbb{R}$, $\forall$x $\in \mathbb{N}$, $f_2(x) \leq k_2$)

Proof:

Header:

Let $f_1, f_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}$
Assume that $\exists k_3 \in \mathbb{R}$, $\forall$x $\in \mathbb{N}$, $(f_1 + f_2)$(x) $\leq k_3$
We want to prove that ($\exists k_1 \in \mathbb{R}$, $\forall$x $\in \mathbb{N}$, $f_1(x) \leq k_1$) $\wedge$ ($\exists k_2 \in \mathbb{R}$, $\forall$x $\in \mathbb{N}$, $f_2(x) \leq k_2$)

Boby:

Let x $\in \mathbb{N}$
Let $k_1 = \frac{k_3}{2}$ - 1
Let $k_2 = \frac{k_3}{2} + 1$
Since, $(f_1 + f_2)$(x) $\leq k_3$
So, $f_1(x) + f_2(x) \leq k_3$
Then, $f_1(x) \leq k_3$ - $f_2(x) \leq k_3$ - $k_2 = k_3$ - $(\frac{k_3}{2} + 1) = \frac{k_3}{2}$ - $1 = k_1$
Thus, $f_1(x) \leq k_1$
And, $f_2(x) \leq k_3$ - $f_1(x) \leq k_3$ - $k_1 = k_3$ - $(\frac{k_3}{2}$ - $1) = \frac{k_3}{2} + 1 = k_2$
Thus, $f_2(x) \leq k_2$

Therefore, we have proven that:
$\forall f_1, f_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}$, ($\exists k_3 \in \mathbb{R}$, $\forall$x $\in \mathbb{N}$, $(f_1 + f_2)$(x) $\leq k_3$)
$\Rightarrow$ ($\exists k_1 \in \mathbb{R}$, $\forall$x $\in \mathbb{N}$, $f_1(x) \leq k_1$) $\wedge$ ($\exists k_2 \in \mathbb{R}$, $\forall$x $\in \mathbb{N}$, $f_2(x) \leq k_2$)

$\square$