CSC165H1 Problem Set 3

Yulin WANG, Qidi Zhou, Dana Zhao

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1. Special numbers

Proof by induction:

Define the predicate P(n) as F_n - 2 = $\prod_{i=0}^{n-1} F_i$,
where n is a natural number.
We want to prove that for all n $\in \mathbb{N}$ that P(n) holds.

Base Case:

n = 0. We want to prove P(0) is True.
We know that F_0 - 2 = 2^1 + 1 - 2 = 1
We also know $\prod_{i=0}^{-1} F_i = 1$ (By the definition of empty products in product notation.)
Since left hand side F_0 - 2 is equal to right hand side $\prod_{i=0}^{-1} F_i$, then P(0) holds.

Induction Step:

Let k $\in \mathbb{N}$, and assume P(k) is True that is F_k - 2 = $\prod_{i=0}^{k-1} F_i$ is True.
Now we want to show that P(k+1) holds that is F_{k+1} - 2 = $\prod_{i=0}^{k} F_i$
For right hand side, $\prod_{i=0}^{k} F_i$:

$$\begin{aligned}
\prod_{i=0}^{k} F_i &= F_0 \cdot F_1 \cdot F_2 \cdots F_{k-1} \cdot F_k \\
&= \prod_{i=0}^{k-1} F_i \cdot F_k \\
&= (F_k - 2) \cdot F_k \quad (\textit{By induction hypothesis}) \\
&= F_k^2 - 2F_k \\
&= (2^{2^k} + 1)^2 - 2(2^{2^k} + 1) \quad (From\ definition\ of\ F_n) \\
&= (2^{2^k})^2 + 2 \cdot 2^{2^k} + 1 - 2 \cdot 2^{2^k} - 2 \\
&= 2^{2^{k+1}} - 1 \quad (From\ hint)
\end{aligned}$$

For left hand side, F_{k+1} - 2:

$$\begin{aligned}
F_{k+1} - 2 &= 2^{2^{k+1}} + 1 - 2 \quad (From\ definition\ of\ F_n) \\
&= 2^{2^{k+1}} - 1
\end{aligned}$$

Therefore, left hand side is equal to right hand side, we have proven $\forall$k $\in \mathbb{N}$, P(k+1) holds.
Then we have proven $\forall$n $\in \mathbb{N}$, F_n - 2 = $\prod_{i=0}^{n-1} F_i$ by induction.

$\square$

2. Sequences

a)

$a_0 = 1 \quad a_1 = \frac{1}{2} \quad a_1 = \frac{1}{3} \quad a_1 = \frac{1}{4}$

b)

$\forall$n $\in \mathbb{N}$, $a_n = \frac{1}{n+1}$

Proof by induction:
Define the predicate P(n) as $a_n = \frac{1}{n+1}$, where n is a natural number.
We want to prove that for all n $\in \mathbb{N}$ that P(n) holds.
Base Case:
n = 0. We want to prove that P(0) is true.
Since $a_0 = \frac{1}{0+1} = 1$, so P(0) holds.
Induction Step:
Let k $\in \mathbb{N}$, and assume that P(k) is true.
That is, we assume that $a_k = \frac{1}{k+1}$.
We want to show that P(k+1) holds, that is $a_{k+1} = \frac{1}{k+2}$.
Since we know:

$$\begin{aligned} a_{k+1} &= \frac{1}{\frac{1}{a_k} + 1} \\ &= \frac{1}{(k+1)+1} \quad (\textit{By induction hypothesis}) \\ &= \frac{1}{k+2} \end{aligned}$$

Thus, P(k+1) holds.
Therefore, we have proven that $\forall$n $\in \mathbb{N}$, $a_n = \frac{1}{n+1}$ by induction.

$\square$

c)

$a_{2,0} = 2 \quad a_{2,1} = \frac{4}{3} \quad a_{2,2} = \frac{8}{7} \quad a_{2,3} = \frac{16}{15}$

$a_{3,0} = 3 \quad a_{3,1} = \frac{9}{4} \quad a_{3,2} = \frac{27}{13} \quad a_{3,3} = \frac{81}{40}$

d)

$a_{k,n} = \frac{k^{n+1}}{\frac{k^{n+1}-1}{k-1}} = \frac{k^{n+1}(k-1)}{k^{n+1}-1}$

We want to show that ∀n ∈ ℕ, ∀k ∈ ℕ, k > 1 ⇒ $a_{k,n} = \frac{k^{n+1}(k-1)}{k^{n+1}-1}$

Proof by induction:

Define the Predicate P(n) as ∀k ∈ ℕ, k > 1 ⇒ $a_{k,n} = \frac{k^{n+1}(k-1)}{k^{n+1}-1}$,
where n is a natural number.
We want to prove that for all n ∈ ℕ that P(n) holds.

Base Case:

n = 0. We want to prove that P(0) is true.
Let k ∈ ℕ, and assume that k > 1.
Since $a_{k,0} = \frac{k(k-1)}{k-1}$ = k, so P(0) holds.

Induction Step:

Let m ∈ ℕ, and assume that P(m) is true.
That is, we assume that ∀k ∈ ℕ, k > 1 ⇒ $a_{k,m} = \frac{k^{m+1}(k-1)}{k^{m+1}-1}$
We want to show that P(m+1) holds.
That is ∀k ∈ ℕ, k > 1 ⇒ $a_{k,m+1} = \frac{k^{m+2}(k-1)}{k^{m+2}-1}$
Let k ∈ ℕ, and asuume that k > 1.
Since we know:

$$\begin{aligned}
a_{k,m+1} &= \frac{k}{\frac{1}{a_{k,m}} + 1} \\
&= \frac{k}{\frac{k^{m+1}-1}{k^{m+1}(k-1)} + 1} \quad (\textit{By induction hypothesis}) \\
&= \frac{k}{\frac{(k^{m+1}-1)+k^{m+1}(k-1)}{k^{m+1}(k-1)}} \\
&= \frac{k \cdot k^{m+1}(k-1)}{(k^{m+1}-1) + k^{m+1}(k-1)} \\
&= \frac{k^{m+2}(k-1)}{k^{m+1}(1+k-1) - 1} \\
&= \frac{k^{m+2}(k-1)}{k^{m+1} \cdot k - 1} \\
&= \frac{k^{m+2}(k-1)}{k^{m+2} - 1}
\end{aligned}$$

Thus, $a_{k,m+1} = \frac{k^{m+2}(k-1)}{k^{m+2}-1}$, then P(m+1) is true.

Therefore, we have proven that ∀n ∈ ℕ, ∀k ∈ ℕ, k > 1 ⇒ $a_{k,n} = \frac{k^{n+1}(k-1)}{k^{n+1}-1}$ by induction.

$\square$

3. Properties of Asymptotic Notation

a)

Proof:

Let $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$
Assume that $f \in \mathcal{O}(n)$
By definition of big-oh, that is

$$\exists c, n_o \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_o \Rightarrow f(n) \leq cn$$

We want to show that $Sum_f(n) \in \mathcal{O}(n^2)$
We have:

$$\begin{aligned}
Sum_f(n) &= \sum_{i=0}^{n} f(i) \\
&= \sum_{i=0}^{\lceil n_o \rceil - 1} f(i) + \sum_{i=\lceil n_o \rceil}^{n} f(i) \\
&\leq \textit{some constant } d + \sum_{i=\lceil n_o \rceil}^{n} ci \quad (\textit{Since } i \geq \lceil n_o \rceil \Rightarrow f(i) \leq ci) \\
&= d + cn(n - \lceil n_o \rceil + 1) \in \mathcal{O}(n^2)
\end{aligned}$$

Therefore, we have proven that for all $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$,
if $f \in \mathcal{O}(n)$, then $Sum_f(n) \in \mathcal{O}(n^2)$

$\square$

b)

Proof by induction:

We want to show that ∀n ∈ $\mathbb{N}$, $\sum_{i=1}^{2^n} \frac{1}{i} \geq \frac{n}{2}$

Define the predicate $P(n)$ as $\sum_{i=1}^{2^n} \frac{1}{i} \geq \frac{n}{2}$,

where n is a natural number.

We want to prove for all $n \in \mathbb{N}$ that $P(n)$ holds.

Base case:

$n = 0$. We want to prove that $P(0)$ is true.

Since $\sum_{i=1}^{2^0} \frac{1}{i} = \sum_{i=1}^{1} \frac{1}{i} = 1 \geq \frac{0}{2} = 0,\ \textit{then}\ P(n)$ holds.

Induction Step:

Let $k \in \mathbb{N}$, and assume that $P(k)$ is true.

That is, we assume that $\sum_{i=1}^{2^k} \frac{1}{i} \geq \frac{k}{2}$

We want to show that $P(k+1)$ holds, that is $\sum_{i=1}^{2^{k+1}} \frac{1}{i} \geq \frac{k+1}{2}$

We have:

$$\begin{aligned}
\sum_{i=1}^{2^{k+1}} \frac{1}{i} &= \sum_{i=1}^{2^k} \frac{1}{i} + \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i} \\
&\geq \frac{k}{2} + \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i} \\
&\geq \frac{k}{2} + \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{2^{k+1}} \\
&= \frac{k}{2} + (2 \cdot 2^k - (2^k + 1) + 1) \cdot \frac{1}{2^{k+1}} \\
&= \frac{k}{2} + 2^k \cdot \frac{1}{2^{k+1}} \\
&= \frac{k}{2} + \frac{1}{2} = \frac{k+1}{2}
\end{aligned}$$

Thus, $\sum_{i=1}^{2^{k+1}} \frac{1}{i} \geq \frac{k+1}{2}$

Therefore, we have proven that $\forall n \in \mathbb{N},\ \sum_{i=1}^{2^n} \frac{1}{i} \geq \frac{n}{2}$

□

c)

Negation of the original statement:

$\exists$ f, g: $\mathbb{N} \to \mathbb{R}^{\geq 0}$, $f(n) \in \mathcal{O}(g(n)) \land Sum_f(n) \notin \mathcal{O}(n \cdot g(n))$

$\equiv \exists$ f, g: $\mathbb{N} \to \mathbb{R}^{\geq 0}$, $(\exists c_1, n_o \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_o \Rightarrow f(n) \leq c_1 g(n))$

$\land (\forall c_2, n_1 \in \mathbb{R}^+, \exists n \in \mathbb{N}, n \geq n_1 \land Sum_f(n) > c_2 n g(n))$

We want to prove its negation to disprove it.

Proof:

Let f(n) = $\frac{1}{n+1}$, and g(n) = $\frac{1}{n}$

(1) We want to show that $\exists c_1, n_o \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_o \Rightarrow f(n) \leq c_1 g(n)$

Let $c_1 = 1 \in \mathbb{R}^+$ and $n_0 = 1 \in \mathbb{R}^+$

Let n $\in \mathbb{N}$, and assume that n $\geq n_0$, then n ≥ 1

When n ≥ 1, f(n) = $\frac{1}{n+1} \leq c_1$g(n) = $\frac{1}{n}$

Thus, $f(n) \in \mathcal{O}(g(n))$

(2) We want to show that $\forall c_2, n_1 \in \mathbb{R}^+, \exists n \in \mathbb{N}, n \geq n_1 \land Sum_f(n) > c_2 n g(n)$

Let c_2, $n_1 \in \mathbb{R}^+$

Let n = max{ n_1, $2^{2c_2} - 1$ }+ 1 $\in \mathbb{N}$

Then, we have:

$$\begin{aligned}
Sum_f(n) &= \sum_{i=0}^{n} \frac{1}{i+1} \\
&= \sum_{i=1}^{n+1} \frac{1}{i} \\
&= \sum_{i=1}^{2^{\log_2(n+1)}} \frac{1}{i} \\
&\geq \frac{\log_2(n+1)}{2} \quad \textit{(By } \forall n \in \mathbb{N}, \sum_{i=1}^{2^n} \frac{1}{i} \geq \frac{n}{2} \textit{ in question 3b)} \\
&> \frac{\log_2(2^{2c_2} - 1 + 1)}{2} \quad \textit{(Since } n > 2^{2c_2} - 1) \\
&= \frac{2c_2}{2} \\
&= c_2 \\
&= c_2 \cdot n \cdot \frac{1}{n} \\
&= c_2 n g(n)
\end{aligned}$$

Thus, $Sum_f(n) \notin \mathcal{O}(n \cdot g(n))$

Therefore, by (1) and (2), we have proven that:

$\exists$ f, g: $\mathbb{N} \to \mathbb{R}^{\geq 0}$, $f(n) \in \mathcal{O}(g(n)) \land Sum_f(n) \notin \mathcal{O}(n \cdot g(n))$

Hence, we have disproven the original statement.

$\square$