CSC236H, Winter 2019
Assignment 3

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1(a)
$(0+1)^*(000+001+011+100+010+110+111)+\epsilon+(1+0)+(1+0)(1+0)$
Explanation: empty string, string with 1 digit, string with 2 digits and string with more than 2 digits ending with 000 or 001 or 011 or 100 or 010 or 110 or 111

1(b)
$(1+00^*11)^*(\epsilon+00^*+00^*1)$
Explanation: I found the regular expression for the language contains sub-string 010, then find the compliment of it.

1(c)
$(aa)^*(bb)^*(cc)^*+(aa)^*(bb)^*(bc)(cc)^*+(aa)^*(ab)(bb)^*(cc)^*+(aa)^*a(bb)^*c(cc)^*$
Explanation: If the sum of three integers is an even number, then it can be 3 even numbers or 2 odd numbers and 1 even numbers. $(aa)^*(bb)^*(cc)^*$ corresponds to the first situation while $(aa)^*(bb)^*(bc)(cc)^*+(aa)^*(ab)(bb)^*(cc)^*+(aa)^*a(bb)^*c(cc)^*$ are the three cases in the second situation.

2.

(a)False.

Consider counter example: Let $r = 1^*, s = \epsilon$, then $rs = 1^*\epsilon = 1^*, sr = \epsilon 1^* = 1^*$. So, $rs = sr, \mathcal{L}(rs) = \mathcal{L}(sr)$. But $r \neq s, \mathcal{L}(r) \neq \mathcal{L}(s)$. I found one counter example, so it's enough to conclude that $\mathcal{L}(rs) = \mathcal{L}(sr) \implies \mathcal{L}(r) = \mathcal{L}(s)$ is false.

(b)False.

Consider counter example: Let $r = 1^*, s = 1^*, t = \epsilon$, then $rs = 1^*1^* = 1^*, rt = 1^*\epsilon = 1^*$. So, $rs = rt, \mathcal{L}(rs) = \mathcal{L}(rt)$. But $s \neq t, \mathcal{L}(s) \neq \mathcal{L}(t)$. I found one counter example, so it's enough to conclude that $(\mathcal{L}(rs) = \mathcal{L}(rt)$ and $r \neq \emptyset) \implies \mathcal{L}(s) = \mathcal{L}(t)$ is false.

(c)True.

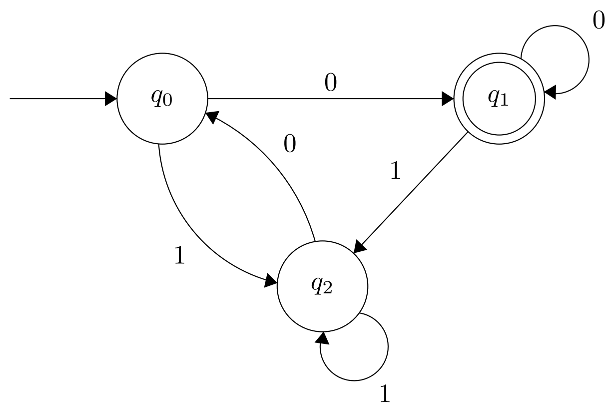
Since,

$$\begin{aligned}
(rs+r)^*r &= [r(s+\epsilon)]^*r \\
&= \{\epsilon, r, rs, rr, rsrs, ...\}r \\
&= \{r, rr, rsr, rrr, rsrsr, ...\} \\
&= r\{\epsilon, r, sr, rr, srsr, ...\} \\
&= r[(s+\epsilon)r]^* \\
&= r(sr+r)^*
\end{aligned}$$

hence,

$$\mathcal{L}((rs+r)^*r) = \mathcal{L}(r(sr+r)^*)$$

3(a)

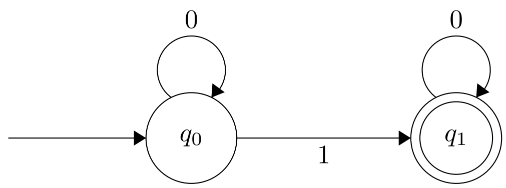


Here are the state invariants

$$P(x) = \delta^*(q_0, x) = \begin{cases} q_0 & \text{if } x \text{ is an empty string, } val(x) \text{ is } null\text{, or if } x \text{ ends with 10, } (val(x) \bmod 4) = 2 \\ q_1 & \text{if } (val(x) \bmod 4) = 0\text{, and } x \text{ ends with 0} \\ q_2 & \text{if } (val(x) \bmod 4) = 1 \text{ or } (val(x) \bmod 4) = 3\text{, and } x \text{ ends with 1} \end{cases}$$

where $x \in \{0, 1\}^*$
The initial state is q_0. The only accepting state is q_1.

3(b)

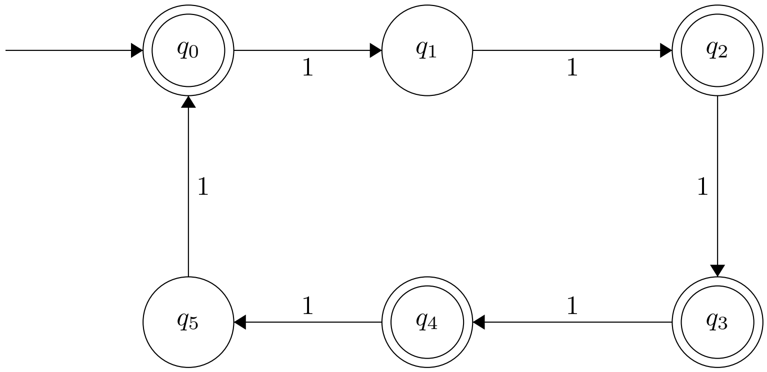


Here are the state invariants

$$P(x) = \delta^*(q_0, x) = \begin{cases} q_0 & \text{if } x \text{ is empty or only contains 0, and } val(x) \text{ is not a power of 2} \\ q_1 & \text{if } x \text{ contains one 1, and } val(x) \text{ is a power of 2} \end{cases}$$

where $x \in \{0, 1\}^*$
The initial state is q_0. The only accepting state is q_1.

3(c)



Let $k \in \mathbb{N}$, k is the number of "1" in x.
Here are the state invariants

$$P(x) = \delta^*(q_0, x) = \begin{cases} q_0 & \text{if } x \text{ is empty or contains only (6m) 1s, where } k = (6m) \text{ is a multiple of 2 or 3} \\ q_1 & \text{if } x \text{ contains only (6m+1) 1s, } k = (6m+1) \\ q_2 & \text{if } x \text{ contains only (6m+2) 1s, where } k = (6m+2) \text{ is a multiple of 2} \\ q_3 & \text{if } x \text{ contains only (6m+3) 1s, where } k = (6m+3) \text{ is a multiple of 3} \\ q_4 & \text{if } x \text{ contains only (6m+4) 1s, where } k = (6m+4) \text{ is a multiple of 2} \\ q_5 & \text{if } x \text{ contains only (6m+5) 1s, } k = (6m+5) \end{cases}$$

where $x \in \{0,1\}^*$ and $m \in \mathbb{N}$
The initial state is q_0. The only accepting states are q_0, q_2, q_3 and q_4.

5(a)

(1) strings in $\{a, b, c\}^*$ are mapped to 0100 by g: aca bcb acb bca

(2) a regular expression R such that $\mathcal{L}(R) = g^{-1}(\mathcal{L}(0^*100^*))$: $(a+b)^*c(a+b)^*$

5(b)

Step1: Show the construction of a DFA for $h^{-1}(L)$

Suppose $L \subseteq \Sigma^*$ is regular, then there exists a DFA, such that $L = \mathcal{L}(A)$.
Let this DFA be $A =< Q, \Sigma, \delta_A, q_0, F >$
Let $C =< Q, \Gamma, \delta_C, q_0, F >$ be a DFA, where for all $q \in Q$ and $a \in \Gamma$, $\delta_C(q, a) = \delta_A^*(q, h(a))$
For all $w \in \Gamma^*$, let $P(w)$ denote the assertion that $\delta_C^*(q_0, w) = \delta_A^*(q_0, h(w))$
Prove that $\forall w \in \Gamma^*, P(w)$ holds by structural induction.

Base case: Since $\delta_C^*(q_0, \epsilon) = q_0$ and $\delta_A^*(q_0, h(\epsilon)) = \delta_A^*(q_0, \epsilon) = q_0$ #by definition of h in Hint, $h(\epsilon) = \epsilon$
So, $\delta_C^*(q_0, \epsilon) = \delta_A^*(q_0, h(\epsilon))$, then $P(\epsilon)$ holds.

Induction Step: Let $w = ax$, where $a \in \Gamma$ and $x \in \Gamma^*$
Assume that P(x) holds, which is $\delta_C^*(q_0, x) = \delta_A^*(q_0, h(x))$
Then, we have:

$$\begin{aligned}
\delta_C^*(q_0, w) &= \delta_C(\delta_C^*(q_0, x), a) && \text{by definition of } \delta_C^* \\
&= \delta_C(\delta_A^*(q_0, h(x)), a) && \text{by IH} \\
&= \delta_A^*(\delta_A^*(q_0, h(x)), h(a)) && \text{by definition of } \delta_C,\ \delta_C(q, a) = \delta_A^*(q, h(a)) \\
&= \delta_A^*(q_0, h(x) \cdot h(a)) && \text{by property of } \delta_A^* \\
&= \delta_A^*(q_0, h(w)) && \text{by definition of h in Hint, } h(x \cdot a) = h(x) \cdot h(a)
\end{aligned}$$

Then, $\delta_C^*(q_0, w) = \delta_A^*(q_0, h(w))$, so P(w) holds.
Thus, by the principle of structural induction, $\forall w \in \Gamma^*, P(w)$ holds.

Step2: Prove the correctness of the DFA

$$\begin{aligned}
\mathcal{L}(C) &= \{w \in \Gamma^* | \delta_C^*(q_0, w) \in F\} \\
&= \{w \in \Gamma^* | \delta_A^*(q_0, h(w)) \in F\} && \text{since } \delta_C^*(q_0, w) = \delta_A^*(q_0, h(w)) \\
&= \{w \in \Gamma^* | h(w) \in \mathcal{L}(A)\} && \text{by definition of A} \\
&= \{w \in \Gamma^* | h(w) \in L\} && \text{since } \mathcal{L}(A) = L \\
&= h^{-1}(L) && \text{by definition of } h^{-1}(L)
\end{aligned}$$

Therefore, $h^{-1}(L)$ is regular.

□

5(c)

For contradiction, assume that $L_2 = \{0^i 1 0^i | i \geq 1\}$ is regular.
Then $g^{-1}(L_2) = \{(a+b)^i c (a+b)^{i-1} | i \geq 1\}$ is also regular. #by solution from part(b)
where g is the homomorphism defined in part(a).
Since regular languages are closed under intersection.
So, $g^{-1}(L_2) \cap \mathcal{L}(a^* c b^*) = \{a^i c b^{i-1} | i \geq 1\}$ is regular.
Define a homomorphism $f : \{a, b, c\}^* \rightarrow \{a, b\}^*$ such that $f(a) = a, f(b) = b, f(c) = b$
Then, since regular languages are closed under homomorphisms,
so $L_1 = f(g^{-1}(L_2) \cap \mathcal{L}(a^* c b^*)) = \{a^i b b^{i-1} | i \geq 1\} = \{a^i b^i | i \geq 1\}$ is regular.
Since this is contradict to our assumption($L1$ is not regular), so we have proved that $L(2)$ is not regular by contradiction.

$\square$