CSC236H, Winter 2019
Assignment 1

Jiacheng Ge 1003795814
Yulin WANG 1003942326

1. Proof. We let our predicate be P(n):

$$(\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx) \text{ for all } x \in \mathbb{R}.$$

Base case:
When n = 1, $(\cos(x) + i\sin(x))^n = \cos(x) + i\sin(x) = \cos(1x) + i\sin(1x)$
Thus, P(1) holds.

Induction Step:
Let k $\in \mathbb{N}$, suppose k $\geq$ 1 and assume that P(k) holds.
Want to prove that P(k+1) holds.
Note that:

$$\begin{aligned}
(cos(x) + isin(x))^{k+1} &= (cos(x) + isin(x))^k \cdot (cos(x) + isin(x)) \\
&= [cos(kx) + isin(kx)] \cdot (cos(x) + isin(x)) \qquad \text{\# by I.H.} \\
&= cos(x)cos(kx) + isin(kx)cos(x) + isin(x)cos(kx) + i^2 sin(x)sin(kx) \\
&= [cos(x)cos(kx) - sin(x)sin(kx)] + i[sin(kx)cos(x) + sin(x)cos(kx)] \quad \text{\# since } i^2 = -1 \\
&= cos(x + kx) + isin(x + kx) \qquad \text{\# by trigonometric equations.} \\
&= cos[(k + 1)x] + isin[(k + 1)x]
\end{aligned}$$

Thus, P(k+1) holds.

Therefore, by the principle of simple induction, we've proved that:
$\forall n \in \mathbb{N}, n \geq 1, (\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx)$ for all $x \in \mathbb{R}$

$\square$

2. Proof. For all $n \in \mathbb{N}, n \geqslant 1$, Let $P(n)$ denote that if you start the game with a single group of n coins, then regardless of how you play the game(i.e. no matter how you choose to split up the groups), you always win a total of exactly $n(n-1)/2$ dollars.

   Base case: $n = 1$, there is only one coin, we can't split and thus can't gain any dollar.

   $\frac{1}{2} \times 1 \times (1-1) = 0$. Thus P(0) holds.

   Induction Step: Let $n \in \mathbb{N}, n \geqslant 1$. Assume $\forall 1 \leq j < n, P(j)$ holds. [I.H]

   Want to prove that: P(n) holds

   Let k be an arbitrary natural number, $1 \leq k < n$ and assume $n \geq 2$
   We split coins into two groups of k and $(n-k)$ coins respectively. So, we gain $(n-k)k$ in this round.

   Since $1 \leq k < n$ and $1 \leq n-k < n$, hence by I.H. you will gain $= \frac{k(k-1)}{2}$ from the first sub pile later and $\frac{(n-k)(n-k-1)}{2}$ from the second sub pile later.

$$\begin{aligned}
\text{total gain} &= (n-k)k + \frac{k(k-1)}{2} + \frac{(n-k)(n-k-1)}{2} && (1)\\
&= \frac{2nk - 2k^2 + k^2 - k + n^2 - nk - nk + k^2 - n + k}{2} && (2)\\
&= \frac{n^2 - n}{2} && (3)\\
&= \frac{n(n-1)}{2} && (4)
\end{aligned}$$

   Thus, $P(n)$ holds.

   Therefore, by the Principle of complete induction, $\forall n \in \mathbb{N}, n \geqslant 1$, P(n) holds.

$\square$

3. Proof. Let P(n) denote that there exists an odd natural number m and $k \in \mathbb{N}$ such that $n = 2^k \times m$
Want to prove that $\forall n \in \mathbb{N}, n \geq 1$, P(n) holds.
Assume for contradiction, there exists some $j \in \mathbb{N}, j \geq 1$, such that P(j) is false.
Let S be the set of all positive natural numbers such that $(j \in \mathbb{N}^+ \wedge j \in S) \Leftrightarrow$ (P(j) is false).
Then by assumption, S is not empty.
By definition of S, $S \in \mathbb{N}$, then by Principle of Well-Ordering, S has a minimum element b.
Claim: b cannot be odd.
Proof: Assume b is odd, we always have m = b, which is odd natural number and $k = 0 \in \mathbb{N}$, such that $b = 2^0 \cdot m$
Then b must be an even number.
So, $\frac{b}{2} \in \mathbb{N}$ and $\frac{b}{2} \geq 1$
Since b is the minimum element in set S, so $\frac{b}{2} \notin S$.
Thus, by the definition of set S, P($\frac{b}{2}$) holds,
which means that there exists an odd natural number m and $k \in \mathbb{N}$ such that $\frac{b}{2} = 2^k \times m$
Then,$b = 2 \times (\frac{b}{2}) = 2 \times 2^k \times m = 2^{k+1} \times m$, where m is odd natural number and $k \in \mathbb{N}$.
So, P(b) holds.
However, $b \in S$ and by definition of S, P(b) is false. Then contradiction!
So, the original statement is true.
Therefore, by the Principle of Well-Ordering, we have proved that$\forall n \in \mathbb{N}, n \geq 1$, there exists an odd natural number m and $k \in \mathbb{N}$ such that $n = 2^k \times m$.

□

4. Proof. $\forall$ colored tree T $\in \mathbb{C}$, let $P(T)$ denote that T contains a leaves-subtree which is a member of $\mathbb{B}$ and is monochromatic.

   Base case: let T be a single node, by definition it's q member of $\mathbb{B}$ and monochromatic. Then, $P(T)$ holds.

   Induction step: let $T_1, T_2, T_3 \in \mathbb{C}$ and have same height. Assume $P(T_1) \wedge P(T_2) \wedge P(T_3)$ (I.H)

   Let $T \in \mathbb{C}$ and has three subtrees T_1, T_2, T_3

   Want to prove that: $P(T)$ holds

   Since T_1, T_2, T_3 are monochromatic by I.H., hence we have 2 different colours and assign them to 3 trees. By Pigeonable Theorem, at least 2 of 3 trees have the same color.
   Since T_1, T_2, T_3 have same height by I.H., then their binary subtrees have same height. So, at least 2 of T_1, T_2, T_3 contain binary subtrees with same height and same colour. These two monachromatic binary subtrees plus the common root of T_1, T_2, T_3 form a new monochromatic binary subtree of T.

   Thus, P(T) holds.

   Therefore,by structural induction, $\forall$ colored tree $T \in \mathbb{C}$, P(T) holds.

□