1 Written Part

1(a) K^2HW

If the filter is not separable, then the process of performing a convolution requires $K \cdot K$ operations per pixel, and since the image has $H \cdot W$ pixels in total, so the number of operations would be K^2HW.

1(b) $2KHW$

If the filter is separable, this operation could be speed up by first performing a $1D$ horizontal convolution followed by a $1D$ vertical convolution, then the process of performing a convolution requires $K + K$ operations per pixel, and since the image has $H \cdot W$ pixels in total, so the number of operations would be $2KHW$.

2(a) A vertical derivative, $\frac{\partial G(x,y)}{\partial y}$, of a Gaussian filter G is a separable filter for both isotropic and anisotropic cases.

2(a).01 isotropic Gaussian filter

$G(x,y) = \frac{1}{2\pi\sigma^2}e^{-\frac{x^2+y^2}{2\sigma^2}}$

$\frac{\partial G(x,y)}{\partial y} = -\frac{y}{2\pi\sigma^4}e^{-\frac{x^2+y^2}{2\sigma^2}} = [-\frac{1}{\sigma^2\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma^2}}] \cdot [\frac{y}{\sigma^2\sqrt{2\pi}}e^{-\frac{y^2}{2\sigma^2}}]$

Since the above function could be converted into $f(x) \cdot f(y)$, so it's a separable filter.

2(a).02 anisotropic Gaussian filter

$G(x,y) = \frac{1}{2\pi\sigma_x\sigma_y}e^{-(\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2})}$

$\frac{\partial G(x,y)}{\partial y} = -\frac{y}{2\pi\sigma_x\sigma_y^3}e^{-(\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2})} = [-\frac{1}{\sigma_x\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma_x^2}}] \cdot [\frac{y}{\sigma_y^3\sqrt{2\pi}}e^{-\frac{y^2}{2\sigma_y^2}}]$

Since the above function could be converted into $f(x) \cdot f(y)$, so it's a separable filter.

2(b) A Laplacian of Gaussians (LoG) is NOT a separable filter.

The LoG is the sum of the second derivatives of the Gussian, $LoG = \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2}$, which could not be directly separated into two 1D kernels, thus it is not a separable filter.

2(c) We could apply the singular value decomposition (SVD) on this 3×3 filter and get $F = U\Sigma V^\intercal$, which as a sum of matrix multiplications, each of a single row/column from the $U \in \mathbb{R}^{3\times3}$ and $V \in \mathbb{R}^{3\times3}$, multiplied by the diagonal entry from $\Sigma \in \mathbb{R}^{3\times3}$. The matrix Σ contains entries, which called "singular values", on the diagonal only and they are sorted from the highest to the lowest value. If only one singular value is non-zero, which means that it is a rank 1 matrix and then this filter is separable.

3 Cross correlation is NOT commutative.

Proof:

By the definition of cross correlation, we have:

$Corr[f, g]_t = Corr[g, f]_{-t}$

which means $f(t) * g(t) = [g(\bar{t}) * f(\bar{t})](-t)$

but it needs to satisfy $f(t) * g(t) = g(t) * f(t)$ to be commutative,

so that the cross correlation is not commutative.

4 Convolve the image with a Gaussian filter with $\sigma = \sqrt{5}$

$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2} = \sqrt{1^2 + 2^2} = \sqrt{5}$

Proof:

Want to show that $G(x, y|\sigma) = G(x, y|\sqrt{\sigma_1^2 + \sigma_2^2}) = G(x, y|\sigma_1) \cdot G(x, y|\sigma_2)$

Note that if X_1, X_2 are two independent random variables and $X_1 \sim N(0, \sigma_1^2 I)$ and $X_2 \sim N(0, \sigma_2^2 I)$ with two probability density functions G_1 and G_2 respectively, then $X = X_1 + X_2$ would have a density function $G = G_1 \cdot G_2$. And given that $X \sim N(0, (\sigma_1^2 + \sigma_2^2)I)$, we have the probability density function (PDF) of variable X:

$$
\begin{aligned}
f_X(X) &= \frac{1}{2\pi\sqrt{|\Sigma|}} exp\{-\frac{1}{2}X^\intercal \Sigma^{-1} X\} \\
&= \frac{1}{2\pi(\sigma_1^2 + \sigma_2^2)} exp\{-\frac{X^\intercal X}{2(\sigma_1^2 + \sigma_2^2)}\} \\
&= \frac{1}{2\pi(\sigma_1^2 + \sigma_2^2)} exp\{-\frac{x^2 + y^2}{2(\sigma_1^2 + \sigma_2^2)}\} \\
&= G(x, y|\sqrt{\sigma_1^2 + \sigma_2^2}) \\
&= G(x, y|\sigma)
\end{aligned}
$$

where $|\Sigma| = |(\sigma_1^2 + \sigma_2^2)I| = (\sigma_1^2 + \sigma_2^2)^2, X = (x, y)$

Therefore, we've proved that $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$.

2 Coding Part

3 Coding Part Continued