Branching random walks, time-inhomogeneous environment, selection

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Marches aléatoires branchantes, environnement inhomogène, sélection

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dirigée par **Zhan Shi**

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À mes grand-parents,
À mon père et à ma mère,
Ici ou ailleurs.

Auprès de mon arbre,
Je vivais heureux,
J'aurais jamais dû m'éloigner d'mon arbre...
Auprès de mon arbre,
Je vivais heureux,
J'aurais jamais dû le quitter des yeux...

Marches aléatoires branchantes, environnement inhomogène, sélection

Résumé

On s'intéresse dans cette thèse au modèle de la marche aléatoire branchante, un système de particules qui évoluent au court du temps en se déplaçant et se reproduisant de façon indépendante. Le but est d'étudier le rythme auquel ces particules se déplacent, dans deux variantes particulières de marches aléatoires branchantes.

Dans la première variante, la façon dont les individus se déplacent et se reproduisent dépend du temps. Ce modèle a été introduit par Fang et Zeitouni en 2010. Nous nous intéresserons à trois types de dépendance en temps : une brusque modification du mécanisme de reproduction des individus après un temps long ; une lente évolution de ce mécanisme à une échelle macroscopique ; et des fluctuations aléatoires à chaque génération.

Dans la seconde variante, le mécanisme de reproduction est constant, mais les individus subissent un processus de sélection darwinien. La position d'un individu est interprétée comme son degré d'adaptation au milieu, et le déplacement d'un enfant par rapport à son parent représente l'héritage des gènes. Dans un tel processus, la taille maximale de la population est fixée à une certaine constante N, et à chaque étape, seuls les N plus à droite sont conservés. Ce modèle a été introduit par Brunet, Derrida, Mueller et Munier, et étudié par Bérard et Gouéré en 2010. Nous nous sommes intéressés dans un premier temps à une variante de ce modèle, qui autorise quelques grands sauts. Dans un second temps, nous avons considéré que la taille totale N de la population dépend du temps.

Mots-clefs

Marche aléatoire branchante, marche aléatoire, processus de branchement, environnement aléatoire, environnement inhomogène, somme de variables aléatoires indépendantes, sélection.

Branching random walks, time-inhomogeneous environment, selection

Abstract

In this thesis, we take interest in the branching random walk, a particles system, in which particles move and reproduce independently. The aim is to study the rhythm at which these particles invade their environment, a quantity which often reveals information on the past of the extremal individuals. We take care of two particular variants of branching random walk, that we describe below.

In the first variant, the way individuals behave evolves with time. This model has been introduced by Fang and Zeitouni in 2010. This time-dependence can be a slow evolution of the reproduction mechanism of individuals, at macroscopic scale, in which case the maximal displacement is obtained through the resolution of a convex optimization problem. A second kind of time-dependence is to sample at random, at each generation, the way individuals behave. This model has been introduced and studied in an article in collaboration with Piotr Miłos.

In the second variant, individuals endure a Darwinian selection mechanism. The position of an individual is understood as its fitness, and the displacement of a child with respect to its parent is associated to the process of heredity. In such a process, the total size of the population is fixed to some integer N, and at each step, only the N fittest individuals survive. This model was introduced by Brunet, Derrida, Mueller and Munier. In a first time, we took interest in a mechanism of reproduction which authorises some large jumps. In the second model we considered, the total size N of the population may depend on time.

Keywords

Branching random walk,random walk, branching process, random environment, time-inhomogeneous environment, sum of independent random variables, selection.

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Annexe A. Aser, Cécile, Cyril, Cyril, Cyrille, Clément, Damien, Élie, Émilien, Igor, Ilaria, Landy, Loïc, Jean-François, Joseba, Julien, Juliette, Mathias, Max, Michel, Nicolas, Olivier, Oriane, Pascal, Patricia, Pierre, Raphaël, Raoul, Romain, Thomas, Yichao, Xan, Xinxin etc.[1]

Annexe B. Bénédicte Auffray, Florence Deschamps, Zaïna Elmir, Valérie Juvé, Isabelle Mariage, Philippe Massé, Altaïr Pelissier, Maria Pochot, Jacques Portès, Josette Saman, Albane Trémeau et Laurence Vincent.

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1. et mes sincères excuses à tous ceux se trouvant dans un etc.

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Introduction

Le *processus de Galton-Watson* est l'un des plus anciens processus aléatoires introduits pour simuler l'évolution au cours du temps d'une population, et certainement l'un des plus étudiés. Dans ce modèle, les individus sont asexués et se reproduisent sans compétition. À chaque nouvelle génération, tous les individus de la génération précédente meurent en donnant naissance à un certain nombre d'enfants de façon indépendante et suivant la même loi de reproduction. D'après Kendall [Ken75], ce processus fut introduit pour la première fois par Bienaymé [Bie76] en 1845 ; et indépendamment redécouvert par Galton et Watson [GW74] en 1873 pour étudier la probabilité d'extinction des noms de famille chez les lords anglais. C'est également l'un des premiers exemples de processus de branchement connus.

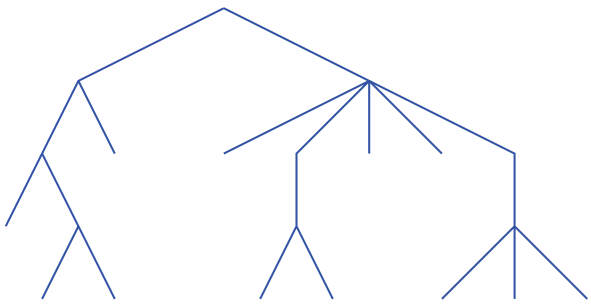


Figure 1 – L'arbre généalogique d'un processus de Galton-Watson

Francis Galton était également un cousin de Charles Darwin, qui introduisit dans son livre *On the origin of species by means of natural selection* la notion de sélection naturelle : dans la compétition pour la survie, seuls les individus les plus aptes survivent et se reproduisent. Ces individus transmettent par la même occasion leur patrimoine génétique à leurs descendants. Afin de réaliser une modélisation mathématique simple de ce phénomène, une idée peut être d'enrichir le processus de Galton-Watson avec des informations supplémentaires, qui se transmettent de parent en enfant.

Ainsi, on peut associer à chaque individu vivant dans le processus de Galton-Watson un score de *valeur sélective* ou *fitness*. Cette quantité représente le degré d'adaptation d'un individu à som milieu. Ceci influe directement sur les chances que celui-ci a de survivre et de se reproduire. Lorsqu'il se reproduit, cet individu transmet à ses enfants ce score de fitness, à un bruit aléatoire près. Ce modèle de sélection, simplifié à l'extrême, ressemble à une *marche aléatoire branchante*.

Dans une marche aléatoire branchante, on étudie l'évolution au cours du temps d'une population d'individus qui se reproduisent et se déplacent comme suit. À chaque nouvelle génération, tous les individus meurent en donnant naissance des enfants. Les enfants sont positionnés autour de leur parent au hasard selon des processus de points indépendants. En remplaçant les termes « position » par « fitness », on se ramène au modèle défini plus haut.

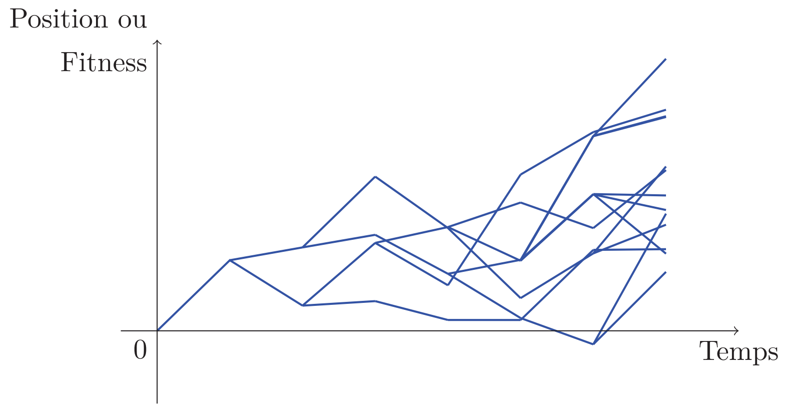


FIGURE 2 – Le graphe d'une marche aléatoire branchante

Dans cette thèse, on s'intéresse au comportement au cours du temps de certains processus de branchement, bâtis comme des variantes de la marche aléatoire branchante. Nous étudions l'impact d'une modification au cours du temps de la loi selon laquelle les individus se reproduisent d'une part ; les effets de la « sélection naturelle » sur le processus d'autre part. Les résultats portent principalement sur le comportement asymptotique du plus grand déplacement dans ces variantes de marche aléatoire branchante.

Dans un premier temps, nous listons quelques résultats bien connus sur le comportement en temps long d'une marche aléatoire branchante classique. Ensuite, nous présentons plus précisément les modèles étudiés et les notations employées, puis les résultats obtenus au cours de la thèse qui concernent la marche aléatoire branchante en environnement inhomogène ou une marche aléatoire branchante avec sélection. Nous terminons cette introduction en reliant les marches aléatoires branchantes à d'autres objets mathématiques qui ont été, ou sont encore le sujet d'études approfondies.

1 Des résultats préexistants sur les marches aléatoires branchantes

L'arbre généalogique du processus de Galton-Watson et de la marche aléatoire branchante sont identiques. Il est bien connu, depuis les travaux de Bienaymé, Galton et Watson que le processus s'éteint presque sûrement [2] si et seulement si le nombre moyen d'enfants d'un individu est inférieur ou égal à 1. Plus précisément, si on note Z_n le nombre d'enfants vivants à la génération n dans ce processus, la probabilité d'extinction est la plus petite solution q dans $[0,1]$ de l'équation $\mathbf{E}(q^{Z_1}) = q$. On peut ainsi classer les processus de Galton-Watson en processus sous-critiques, critiques et sur-critiques, pour $\mathbf{E}(Z_1)$ respectivement plus petit, égal ou plus grand que 1. Dans le cadre de cette thèse, nous supposons les processus de populations des marches aléatoires branchantes surcritique, et même régulièrement que $Z_1 \geq 1$ presque sûrement. Dans ce dernier cas, le processus de

2. C'est-à-dire qu'à partir d'un certain temps, il n'y a plus d'individus vivants.

population survit presque sûrement. Le *théorème de Kesten-Stigum* permet d'estimer la taille de la population dans le processus de Galton-Watson : si $\mathbf{E}(Z_1 \log Z_1) < +\infty$, alors elle croît à vitesse exponentielle proportionnellement à $[\mathbf{E}(Z_1)]^n$. Plus précisément il existe une variable aléatoire positive Z telle que

$$\lim_{n\to+\infty} \frac{Z_n}{[\mathbf{E}(Z_1)]^n} = Z \text{ p.s. et } \mathbf{P}(Z=0) = q.$$

Des résultats similaires ont été prouvés dans le cadre de la marche aléatoire branchante dans les années 1970. On note M_n le plus grand déplacement à l'instant n. Les travaux pionniers de Hammersley [Ham74], Kingman [Kin75] et Biggins [Big76] ont prouvé que M_n croît à vitesse balistique. En d'autre termes, il existe une constante explicite v telle que

$$\lim_{n\to+\infty} \frac{M_n}{n} = v \quad \text{p.s.}$$

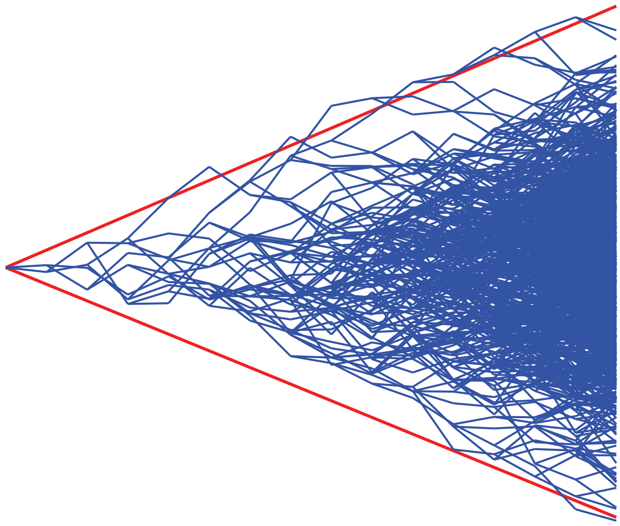


Figure 3 – Des frontières linéaires pour la marche aléatoire branchante

Par conséquent, la population de la marche aléatoire branchante envahit son environnement à vitesse balistique. De plus, il existe une fonction convexe $\kappa^* : \mathbb{R} \to \mathbb{R}$ vérifiant la propriété suivante :

- pour tout $a < v$, la taille de la population vivant à l'instant n au-dessus de na est d'ordre $e^{-n\kappa^*(a)}$;
- pour tout $a > v$, la probabilité qu'un individu existe à l'instant n au-dessus de na est d'ordre $e^{-n\kappa^*(a)}$.

En 2009, Addario-Berry et Reed [ABR09] et Hu et Shi [HS09] ont amélioré la connaissance du comportement asymptotique de M_n. Ils ont montré l'existence d'une constante $\theta > 0$ telle que

$$\limsup_{n\to+\infty} \frac{M_n - nv}{\log n} = \frac{-1}{2\theta} \quad \text{p.s.}$$
$$\liminf_{n\to+\infty} \frac{M_n - nv}{\log n} = \frac{-3}{2\theta} \quad \text{p.s.}$$
$$\left(M_n - nv + \frac{3}{2\theta}\log n, n \geq 1\right) \quad \text{est tendue.}$$

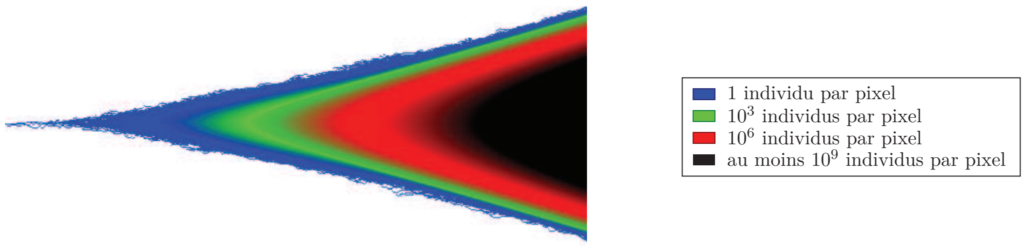


FIGURE 4 – Densité de population dans une marche aléatoire branchante

En d'autres termes, non seulement M_n est proche de $nv - \frac{3}{2\theta} \log n$ avec grande probabilité, mais cette quantité exhibe également des fluctuations presque sûres de taille logarithmique.

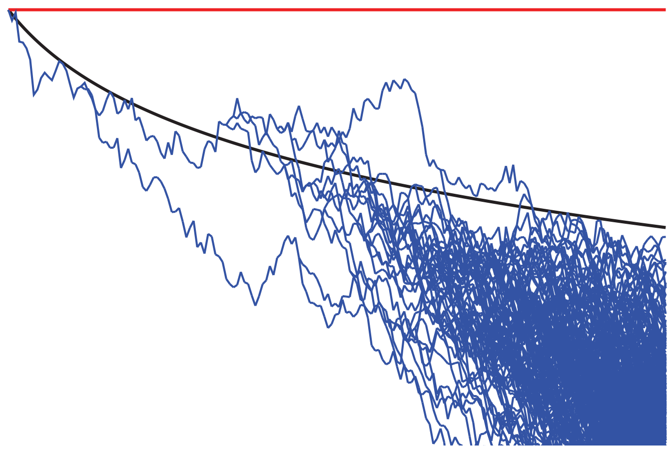


FIGURE 5 – La correction logarithmique est sujette à des fluctuations presque sûres

Le dernier terme du développement asymptotique de M_n a été obtenu par Aïdékon [Aïd13] qui a montré, sous des hypothèses dont Chen [Che14] a prouvé l'optimalité par la suite, que $M_n - nv + \frac{3}{2\theta} \log n$ converge en loi vers une variable aléatoire W. Cette loi limite s'exprime comme une variable aléatoire de Gumbel, décalée d'un coefficient aléatoire.

2 Modèles étudiés

L'objectif de cette thèse est l'étude du comportement asymptotique du plus grand déplacement dans les deux variantes suivantes de la marche aléatoire branchante :

- la marche aléatoire branchante en environnement inhomogène en temps, dans laquelle la façon dont les individus se reproduisent dépend du temps ;
- la marche aléatoire branchante avec sélection, dans laquelle à chaque étape n la taille totale de la population est fixée à une valeur N_n, et seuls les N_n individus les plus

haut placés dans le processus, interprétés comme les individus ayant la meilleure fitness, survivent.

2.1 Marche aléatoire branchante en environnement inhomogène

Ce modèle peut être vu comme une généralisation naturelle de la marche aléatoire branchante, lorsque ce processus est interprété comme l'évolution d'une population. Les conditions environnementales peuvent varier au cours du temps, ce qui peut avoir un impact sur la façon dons les individus se reproduisent ou se déplacent. Ces fluctuations peuvent avoir lieu à des échelles de temps longues, comparées à la durée de vie des individus (c.f. Chapitres 1, 2, 3). À l'inverse, l'environnement peut également être modifié à chaque nouvelle génération. C'est le cas du modèle étudié dans le Chapitre 4. Nous nous intéressons à l'effet de ces modifications d'environnement sur l'asymptotique du plus grand déplacement.

Des modèles de marche aléatoire branchante en environnement inhomogène ont été introduits par Derrida et Spohn [DS88]. En adaptant les résultats très généraux de [Big76, BK04], il est aisé de se convaincre que sous des hypothèses d'intégrabilité relativement générales, le plus grand déplacement M_n dans ce nouveau modèle reste linéaire au premier ordre.

Fang et Zeitouni ont introduit et étudié dans [FZ12a] un modèle de marche aléatoire branchante avec une interface. Dans ce modèle, les individus se reproduisent selon une première loi pendant la première moitié du temps, puis selon une seconde loi pendant la seconde moitié du temps. Dans leur modèle, tout individu vivant à la génération k fait deux enfants, qui se déplacent par rapport à leur parent de façon indépendante selon des gaussiennes centrées, de variance σ_1^2 si $k \leq n/2$ ou σ_2^2 si $k > N/2$. Ils montrent que dans ce processus, le comportement asymptotique de M_n est toujours donné par un premier ordre linéaire plus une correction logarithmique. Notamment, on observe que ce deuxième ordre logarithmique est très sensible au signe de $\sigma_2^2 - \sigma_1^2$ et subit une transition de phase lorsque ce signe change.

Nous montrons dans le Chapitre 1 un résultat similaire, pour des lois de reproduction plus générales que le cas binaire gaussien. Dans le Chapitre 2, ce modèle de marche aléatoire branchante est généralisé au cas de plusieurs interfaces. Nous montrons dans ce cas encore que l'asymptotique du plus grand déplacement est donné par un premier terme linéaire plus une correction logarithmique. Ce premier terme est obtenu en résolvant un problème d'optimisation sous contraintes, et le second ordre dépend de l'interaction de cette solution avec les contraintes.

Par la suite, un certain nombre d'articles, parmi lesquels [FZ12b, NRR14, MZ14], ont étudié des marches aléatoires branchantes évoluant dans un environnement variant à une échelle grande devant le temps de vie des individus (correspondant à des fluctuations de l'environnement à grande échelle). Dans ces modèles, on se fixe une famille de lois de reproduction $(\mathcal{L}_t, t \in [0, 1])$ ainsi que la longueur n de la marche branchante que l'on considère. Les individus présents à la génération k se reproduisent selon la loi $\mathcal{L}_{k/n}$. Dans tous les articles cités plus haut, les enfants d'un individu donné se déplacent toujours selon des variables aléatoires gaussiennes, de variance σ_t^2.

Dans le Chapitre 3, nous nous intéressons à ce modèle, tout en autorisant une grande classe de loi de reproduction. Sous l'hypothèse que la loi de reproduction évolue de façon suffisamment régulière, nous montrons qu'au premier ordre, le comportement de M_n est toujours linéaire et que la vitesse peut être calculée comme la solution d'un problème d'optimisation sous contrainte. En calculant la transformée de Laplace de l'aire sous la courbe d'un mouvement brownien conditionné à rester négatif, nous montrons que le second terme de l'asymptotique de M_n est une correction d'ordre au plus $n^{1/3}$.

Dans le Chapitre 4, nous nous intéressons à un type de marche aléatoire branchante en environnement inhomogène différent. Dans ce modèle, la loi de reproduction à chaque nouvelle génération est tirée au hasard indépendamment de la marche. Tous les individus se reproduisent ensuite en utilisant cette loi. Nous montrons que dans ce cas, le plus grand déplacement dans la marche aléatoire branchante est donné par un premier terme linéaire, des fluctuations en $n^{1/2}$ qui ne dépendent que de l'environnement, et une correction logarithmique négative propre à la marche aléatoire branchante.

2.2 Marches aléatoires branchantes avec sélection

Dans [BLSW91], il a été introduit un mécanisme de sélection dans une marche aléatoire branchante. Tous les individus dont la position devient négative sont immédiatement tués. On montre que ce processus se comporte, dans les grandes lignes, comme un processus de Galton-Watson. Ainsi, cette barrière crée une population qui meurt ou croît à vitesse exponentielle, en fonction de la loi de reproduction. Des mécanismes de sélection similaires, basés sur la position des individus ont été intensément étudiés, en particulier avec une barrière constituée d'un terme linéaire et d'une correction d'ordre $n^{1/3}$. On peut ainsi citer [FZ10, AJ11, GHS11, FHS12, Jaf12, BBS13] parmi bien d'autres.

La marche aléatoire branchante avec sélection des N individus les plus à droite, ou en plus court la N-marche aléatoire branchante, a été introduite par Brunet et Derrida dans [BD97] en temps que processus de population se déplaçant sur $\mathbb{Z}$. Ce processus a été généralisé dans [BDMM07] au cas d'un processus de population se déplaçant sur la droite réelle. La taille de la population est limitée par une constante N, et la position d'un individu est interprétée comme son degré d'adaptation à l'environnement. À chaque étape, tous les individus se reproduisent de façon indépendante, comme dans une marche aléatoire branchante. Dans un second temps, seuls les N individus les plus hauts [3] survivent et se reproduisent à l'étape suivante. Ce mécanisme de sélection est différent du précédent, dans ce nouveau modèle les individus vivants à la génération n se reproduisent de façon corrélée.

Dans ces articles, les auteurs ont conjecturé que le nuage d'individus se déplacent au cours du temps à une vitesse v_N, et de plus que lorsque $N \to +\infty$,

$$v_N = v_\infty - \frac{C}{(\log N + 3 \log \log N + O(1))^2},$$

où C est une constante explicite qui dépend de la loi de reproduction des individus. Bérard et Gouéré prouvent une première partie de cette conjecture dans [BG10], à savoir

$$v_N = v_\infty - \frac{C_1}{(\log N)^2}(1 + o(1)).$$

Maillard [Mai13] obtient par la suite des résultats plus fins, dans le cas du modèle voisin du mouvement brownien branchant avec sélection des N plus à droite, sous de bonnes hypothèses sur la configuration des individus à l'instant initial. D'autres modèles voisins ont été étudiés. Ainsi, Bérard et Maillard [BM14] ont étudié la marche aléatoire branchante avec sélection, lorsque les déplacements des enfants sont à queue lourde ; Couronné et Gerin [CG14] se sont intéressés à certaines marches aléatoires branchantes avec sélection sur $\mathbb{Z}$.

3. C'est-à-dire les N individus les plus adaptés.

Dans le Chapitre 5, nous nous intéressons au comportement du nuage d'individus, lorsque la taille de la population à l'instant n n'est pas une constante fixée, mais une quantité qui évolue au cours du temps. Plus précisément, lorsque la taille de la population croît au rythme critique $e^{an^{1/3}}$, nous calculons le comportement asymptotique des positions extrémales dans le nuage de particules. Dans le Chapitre 6, nous montrons un résultat similaire à celui de Bérard et Gouéré pour des marches aléatoires branchantes dont les déplacements autorisent quelques rares grands sauts. Pour $\alpha \in (0, 2)$, nous exhibons des marches aléatoires branchantes avec sélection telles que $v_N - v_\infty \sim C(\log N)^{-\alpha}$.

3 Notations employées

Afin de décrire avec plus de détails les résultats démontrés, nous introduisons un certain nombre de notations qui seront valables dans le reste de cette thèse. La plupart de ces notations sont rappelées dans les chapitres qui leur sont dédiés. De plus, dans chaque chapitre un index rappelle les notations spécifiques au modèle étudié.

Nous introduisons d'abord les notations d'Ulam-Harris, qui permettent de décrire les arbres généalogiques. Dans un second temps, nous présentons un certain nombre de notations liées aux processus de points, des variables aléatoires à valeurs dans l'ensemble des suites finies ou infinies de réels. Un processus de point représente alors le déplacement de l'ensemble des enfants d'un individu par rapport à leur parent. En utilisant ces deux notions, nous décrivons la loi de la marche aléatoire branchante.

3.1 Arbre plan enraciné étiqueté

On introduit tout d'abord

$$\mathcal{U}^* = \bigcap_{n \in \mathbb{N}} \mathbb{N}^n \text{ et } \mathcal{U} = \mathcal{U}^* \cup \{\emptyset\}$$

l'ensemble des suites finies d'entiers, où $\emptyset$ représente la suite vide. Un arbre sera défini comme un sous-ensemble de $\mathcal{U}$. Un élément $u \in \mathcal{U}$ symbolise un individu d'un arbre. Si $u = (u(1), \dots u(n))$ alors u est le $u(n)^{\text{ième}}$ enfant du $u(n-1)^{\text{ième}}$ enfant du ... du $u(1)^{\text{ième}}$ enfant de l'ancêtre commun $\emptyset$, que l'on appelle la racine de l'arbre.

Soit $u = (u(1), \dots u(n)) \in \mathcal{U}$, on note $|u| = n$ la génération à laquelle u appartient, avec la convention $|\emptyset| = 0$. Pour $k \leq n$, on note $u_k = (u(1), \dots u(k))$ l'ancêtre à la génération k de u, en posant $u_0 = \emptyset$. Si $u \neq \emptyset$, on note $\pi u = u_{|n|-1}$ le parent de u.

Un *arbre plan enraciné* est un sous-ensemble $\mathbf{T}$ de $\mathcal{U}$ vérifiant les propriétés suivantes.

Enracinement : la suite vide $\emptyset \in \mathbf{T}$.

Hérédité : si $u \in \mathbf{T}$, alors $\pi u \in \mathbf{T}$.

Consistance : si $u = (u(1), \dots u(n)) \in \mathbf{T}$, alors $(u(1), \dots u(n-1), v) \in \mathbf{T}$ pour tout $v \leq u(n)$.

Étant donné un arbre $\mathbf{T}$, l'ensemble $\{u \in \mathbf{T} : |u| = n\}$ est appelé la $n^{\text{ième}}$ génération de $\mathbf{T}$. Le plus grand entier tel que cet ensemble est non-vide est appelé la hauteur de l'arbre. Pour tout individu $u \in \mathbf{T}$, on notera souvent $\Omega(u) = \{v \in \mathbf{T} : \pi v = u\}$ l'ensemble des enfants de u.

Si la hauteur de l'arbre est infinie, on note $\partial \mathbf{T}$ l'ensemble des suites $(u^{(n)}) \in \mathbf{T}^{\mathbb{N}}$ vérifiant

$$\forall n \geq 0, |u^{(n)}| = n \quad \text{et} \quad \forall 0 \leq p \leq q, u^{(q)}_p = u^{(p)}.$$

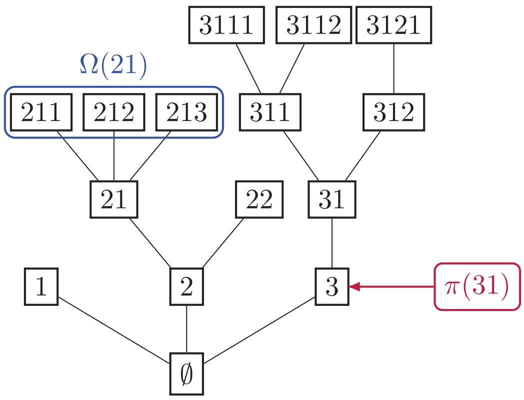


FIGURE 6 – Un arbre plan $\mathbf{T}$ de hauteur 4.

Les éléments de $\partial\mathbf{T}$ sont les branches infinies de l'arbre.

Un *arbre plan enraciné étiqueté* est la donnée d'un couple $(\mathbf{T}, V)$, où $\mathbf{T}$ est un arbre plan enraciné, et $V : \mathbf{T} \to \mathbb{R}$ est la fonction d'étiquetage. Pour $u \in \mathbf{T}$, on appelle $V(u)$ la position de u, et on pose

$$M_n = \max_{u \in \mathbf{T}:|u|=n} V(u)$$

la position de l'individu le plus haut à l'instant n dans $\mathbf{T}$.

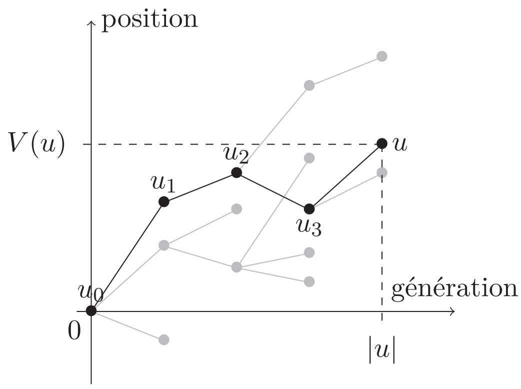


FIGURE 7 – Le graphe d'un arbre plan enraciné étiqueté $(\mathbf{T}, V)$.

Arbre de Galton-Watson. Grâce aux notations d'Ulam-Harris, on peut aisément décrire un arbre de Galton-Watson. Étant donné $(\xi_u, u \in \mathcal{U})$ un ensemble de variables aléatoires indépendantes et identiquement distribuées à valeurs dans les entiers, on écrit

$$\mathbf{T} = \left\{ u \in \mathcal{U} : \forall 1 \leq k \leq |u|, u(k) \leq \xi_{u_{k-1}} \right\}.$$

Cet ensemble est un arbre (plan, enraciné) aléatoire. On observe que si on note, pour tout $n \geq 0$, $Z_n = \#\{u \in \mathbf{T} : |u| = n\}$ le nombre d'individus vivant à la génération n dans cet arbre, le processus $(Z_n, n \geq 0)$ est un processus de Galton-Watson.

3.2 Processus de points

Un processus de points est une variable aléatoire à valeurs dans l'ensemble des mesures de comptage de $\mathbb{R}$ finies sur les compacts. Ce processus représente un ensemble de points de

$\mathbb{R}$, fini ou infini, compté avec leur multiplicité. Les processus de points que nous considérons admettent presque sûrement un plus grand élément. Par conséquent, on peut toujours écrire un processus de points $L = (\ell_1, \dots \ell_n)$, où n est le nombre aléatoire de points dans L (notons que l'on autorise $n = +\infty$ en toute généralité), et $\ell_1 \geq \ell_2 \geq \dots$ et la suite de points dans L, listée dans l'ordre décroissant (si $n = +\infty$, on pose par convention $\ell_\infty = -\infty$).

On pose $\mathcal{L}$ la loi de L. La transformée de log-Laplace de $\mathcal{L}$ est la fonction κ définie par

$$\kappa : \begin{array}{ccc} (0, +\infty) & \to & \mathbb{R} \cup \{+\infty\} \\ \theta & \mapsto & \log \mathbf{E}\left[\sum_{\ell \in L} e^{\theta \ell}\right], \end{array}$$

où $\sum_{\ell \in L}$ représente la sommation sur l'ensemble des éléments du processus de points. Une autre fonction souvent associée à $\mathcal{L}$ est sa transformée de Cramér κ^* définie par

$$\kappa^* : \begin{array}{ccc} \mathbb{R} & \mapsto & \mathbb{R} \\ a & \mapsto & \sup_{\theta > 0} \theta a - \kappa(\theta). \end{array}$$

On notera que κ et κ^* sont deux fonctions convexes semi-continues inférieurement, et de classe $\mathcal{C}^\infty$ sur l'intérieur de l'ensemble où elles sont finies. De plus, si κ est différentiable au point $\theta > 0$, on a

$$\theta \kappa'(\theta) - \kappa(\theta) = \kappa^* \left(\kappa'(\theta)\right).$$

Supposons qu'il existe $\theta > 0$ tel que $\kappa(\theta) < +\infty$, on pose

$$v = \inf_{\theta > 0} \frac{\kappa(\theta)}{\theta} = \sup\{a \in \mathbb{R} : \kappa^*(a) \leq 0\}.$$

On note $\theta^* > 0$, s'il existe, la quantité vérifiant

$$\theta^* \kappa'(\theta^*) - \kappa(\theta^*).$$

Par convexité, on a sans difficulté $v = \kappa'(\theta^*)$. Pour finir, on notera $\sigma^2 = \kappa''(\theta^*)$.

3.3 La loi de la marche aléatoire branchante

Une marche aléatoire branchante est une variable aléatoire $(\mathbf{T}, V)$ à valeurs dans l'ensemble des arbres enracinés étiquetés telle que $V(\emptyset) = 0$, et que la famille de processus de points $\{(V(v) - V(u), v \in \Omega(u)), u \in \mathbf{T}\}$ est indépendante et identiquement distribuée. On note $\mathcal{L}$ la loi de ces processus de points, que l'on appelle la loi de reproduction de la marche aléatoire branchante.

Ce processus peut être construit de la façon suivante. On se donne $\{L^u, u \in \mathcal{U}\}$ une famille de processus de points indépendants et identiquement distribués de loi $\mathcal{L}$. Pour tout $u \in \mathcal{U}$, on écrit $L^u = (\ell^u_1, \dots \ell^u_{N(u)})$, et L^u représente le processus de point du déplacement des enfants de u par rapport à leur parent. On pose alors

$$\mathbf{T} = \{\emptyset\} \cup \{u \in \mathcal{U}^* : \forall k < |u|, u(k) < N(u_{k-1})\}$$

et pour $u \in \mathbf{T}$, on écrit $V(u) = \sum_{j=1}^{|u|} \ell^{u_{j-1}}_{u(j)}$.

La filtration naturelle associée à la marche aléatoire branchante est donnée par

$$\mathcal{F}_n = \sigma\left((u, V(u)), u \in \mathbf{T} : |u| \leq n\right).$$

4 Plus grand déplacement dans des variantes de marche aléatoire branchante

Grâce aux notations de la section précédente, nous introduisons les principaux théorèmes démontrés dans cette thèse, liés à l'asymptotique du plus grand déplacement M_n dans une marche aléatoire branchante en temps inhomogène ou avec sélection. Étant donné une suite de variables aléatoires $(X_n, n \in \mathbb{N})$ et $(a_n) \in \mathbb{R}^{\mathbb{N}}$, on note

$$X_n = o_{\mathbf{P}}(a_n) \text{ si } \forall \varepsilon > 0, \lim_{n\to+\infty} \mathbf{P}\left(|X_n/a_n| \geq \varepsilon\right) = 0,$$
$$X_n = O_{\mathbf{P}}(a_n) \text{ si } \lim_{K\to+\infty} \sup_{n\in\mathbb{N}} \mathbf{P}\left(|X_n/a_n| \geq K\right) = 0.$$

4.1 Marches aléatoires branchantes en environnement inhomogène en temps

On se donne une suite $(\mathcal{L}_n, n \geq 1)$ de loi de processus de points sur $\mathbb{R}$. Une marche aléatoire branchante en environnement inhomogène est une variable aléatoire $(\mathbf{T}, V)$ à valeurs dans l'ensemble des arbres plans enracinés étiquetés telle que la famille de processus de points $\{(V(v) - V(u), v \in \mathbf{T} : \pi v = u), u \in \mathbf{T}\}$ est indépendante et pour $u \in \mathbf{T}$ la loi de $(V(v) - V(u), v \in \mathbf{T} : \pi v = u)$ est $\mathcal{L}_{|u|}$. La suite de loi $(\mathcal{L}_n, n \geq 0)$ est appelée environnement de la marche aléatoire branchante.

En d'autres termes, une marche aléatoire branchante en environnement inhomogène commence avec un unique individu situé en 0 à l'instant 0. À chaque instant n, tous les individus vivants à la génération $n-1$ meurent, en donnant naissance de façon indépendante à un certain nombre d'enfants, qui se répartissent autour de leur parent selon un processus de point de loi $\mathcal{L}_n$. Pour $n \in \mathbb{N}$, on pose κ_n la transformée de log-Laplace de $\mathcal{L}_n$ et κ_n^* la transformée de Cramér associée.

Marches aléatoires branchantes avec interfaces. On se donne des lois de processus de points $(\mathcal{L}_1, \ldots \mathcal{L}_P)$ et des réels $t_0 = 0 < t_1 < t_2 < \cdots < t_P = 1$. On fixe ensuite un entier n, la longueur de la marche aléatoire branchante avec interfaces que l'on considère. Dans ce modèle, les individus faisant partie de la génération $k \in [t_{p-1}n, t_p n)$ se reproduisent de façon indépendante selon des processus de points de loi $\mathcal{L}_p$. Les individus de la génération n meurent sans descendance.

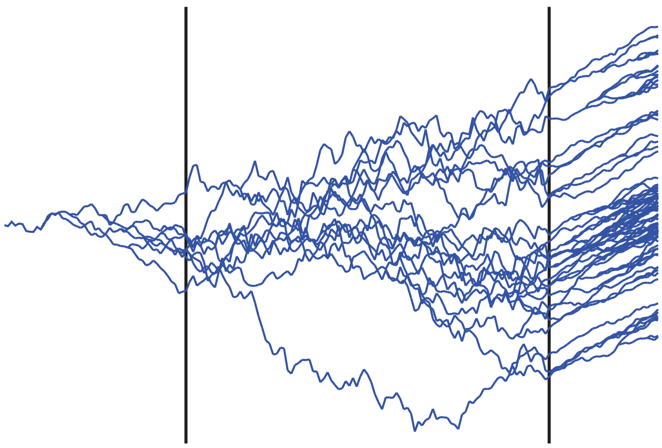


FIGURE 8 – Une marche aléatoire branchante avec deux interfaces

Dans le Chapitre 1 on s'intéresse au cas particulier $P = 2$, correspondant à la marche aléatoire branchante avec une seule interface. On montre une première généralisation du résultat de Fang et Zeitouni [FZ12a].

Théorème 1. *Soient $\mathcal{L}_1$ et $\mathcal{L}_2$ deux lois de processus de points sur $\mathbb{R}$. On suppose qu'il existe $\theta_1^*, \theta_2^* > 0$ tels que*

$$\theta_i^* \kappa_i'(\theta_i^*) - \kappa_i(\theta_i^*) = 0,$$

ainsi que certaines hypothèses d'intégrabilité supplémentaires.

Si $\theta_1^* < \theta_2^*$, alors

$$M_n = n\left(t_1 v_1 + (1 - t_1) v_2\right) - \left(\frac{3}{2\theta_1^*} + \frac{3}{2\theta_2^*}\right) \log n + O_{\mathbf{P}}(1).$$

Si $\theta_1^* = \theta_2^*$, alors

$$M_n = n\left(t_1 v_1 + (1 - t_1) v_2\right) - \frac{3}{2\theta_1^*} \log n + O_{\mathbf{P}}(1).$$

Si $\theta_1^* > \theta_2$, et s'il existe $\theta^* \in (\theta_1^*, \theta_2^*)$ tel que

$$\theta^* \left(t_1 \kappa_1'(\theta^*) + (1 - t_1)\kappa_2'(\theta^*)\right) - \left(t_1 \kappa_1(\theta^*) + (1 - t_1)\kappa_2(\theta^*)\right) = 0,$$

alors

$$M_n = n\left(t_1 \kappa_1'(\theta^*) + (1 - t_1)\kappa_2'(\theta^*)\right) - \frac{1}{2\theta^*} \log n + O_{\mathbf{P}}(1).$$

Grâce à ce théorème, on observe qu'il existe $v^{\mathrm{i}} \in \mathbb{R}$ et $\lambda^{\mathrm{i}} > 0$ vérifiant

$$M_n = n v^{\mathrm{i}} - \lambda^{\mathrm{i}} \log n + O_{\mathbf{P}}(1).$$

On note que v^{i} évolue continûment lorsque les lois de processus de points évoluent, alors que λ^{i} subit une transition de phase lorsque $\theta_2^* - \theta_1^*$ change de signe.

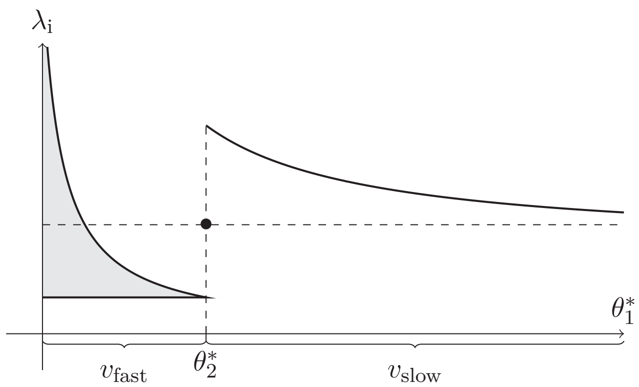


Figure 9 – Lieu des corrections logarithmiques possibles pour une marche aléatoire branchante avec une interface et vitesse associée

Dans le Chapitre 2, le Théorème 1 est étendu au cas d'un nombre arbitraire $P \geq 2$ d'interfaces. Pour ce faire, il est nécessaire de connaître la trajectoire suivie par l'individu qui réalise le plus grand déplacement au temps n. Dans ce but, on prouve tout d'abord le résultat suivant, basé sur un théorème d'existence de multiplicateurs de Lagrange.

Proposition 2. *Si* $\sup_{p\leq P, a\in\mathbb{R}} \kappa_p^*(a) < +\infty$*, il existe un unique* $a \in \mathbb{R}^P$ *tel que*

$$\sum_{p=1}^{P}(t_p - t_{p-1})a_p = \max\left\{\sum_{p=1}^{P}(t_p - t_{p-1})b_p, b \in \mathbb{R}^P : \forall q \leq P, \sum_{p=1}^{q}(t_p - t_{p-1})\kappa_p^*(b_p) \leq 0\right\},$$

vérifiant $\forall q \leq P, \sum_{p=1}^{q}(t_p - t_{p-1})\kappa_p^*(a_p) \leq 0$.

De plus, si on note $\theta_p = (\kappa_p^*)'(a_p)$ *alors*

- $\theta_1 \leq \theta_2 \leq \cdots \leq \theta_P$ *;*
- *si* $\theta_{q+1} \neq \theta_q$ *alors* $\sum_{p=1}^{q}(t_p - t_{p-1})\kappa_p^*(a_j) = 0$ *;*
- $\sum_{p=1}^{P}(t_p - t_{p-1})\kappa_p^*(a_p) = 0$.

Grâce à cette proposition, on pose $v_{\text{is}} = \sum_{p=1}^{p}(t_p - t_{p-1})a_p$. Soient $\varphi_1 < \cdots < \varphi_Q$ tels que $\{\varphi_1, \ldots, \varphi_Q\} = \{\theta_1, \ldots, \theta_P\}$ l'ensemble des valeurs distinctes prises par θ, classées dans l'ordre croissant. Pour $q \leq Q$, on pose

$$f_q = \min\{p \leq P : \theta_p = \varphi_q\} \text{ et } l_q = \max\{p \leq P : \theta_p = \varphi_q\}.$$

On écrit enfin

$$\lambda_{\text{is}} = \sum_{q=1}^{Q} \frac{1}{2\varphi_q}\left(1 + \mathbf{1}_{\left\{\kappa_{f_q}^*(a_{f_q})=0\right\}} + \mathbf{1}_{\left\{\kappa_{l_q-1}^*(a_{l_q-1})=0\right\}}\right).$$

On peut alors calculer le comportement asymptotique de M_n.

Théorème 3. *Sous de bonnes hypothèses d'intégrabilité, on a*

$$M_n = nv_{\text{is}} - \lambda_{\text{is}} \log n + O_{\mathbf{P}}(1).$$

Marches aléatoires branchantes en environnement variant. Une généralisation naturelle des processus étudiés précédemment est la suivante. On considère une famille $(\mathcal{L}_t, t \in [0,1])$ de lois de processus de points sur $\mathbb{R}$. Notons κ_t la transformée de log-Laplace de $\mathcal{L}_t$. On fixe ensuite la taille n de la marche aléatoire branchante. Les individus vivants à la génération k se reproduisent en utilisant la loi $\mathcal{L}_{k/n}$.

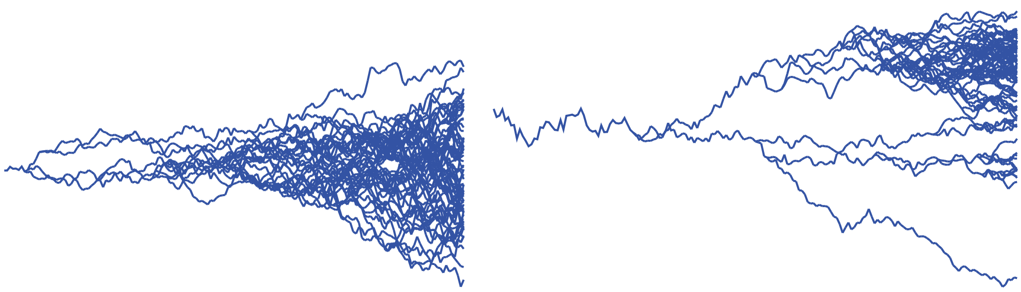


FIGURE 10 – Deux marches aléatoires branchantes en environnement variant

Lorsque $t \mapsto \mathcal{L}_t$ évolue de façon lisse, les individus se reproduisent localement comme dans une marche aléatoire branchante en temps homogène. La vitesse de cette marche aléatoire branchante est donnée par

$$v_* = \max\left\{\int_0^1 b_s ds, b \in \mathcal{C}([0,1]) : \forall t \in [0,1] \int_0^t \kappa_s^*(b_s)ds \leq 0\right\}.$$

Néanmoins, nous montrons dans le Chapitre 3 que le deuxième ordre de M_n est $n^{1/3}$. On détermine dans un premier temps le chemin suivi par l'individu réalisant le plus grand déplacement à l'instant n, un résultat similaire à la Proposition 2.

Proposition 4. *Soit a une fonction càdlàg sur $[0,1)$, on pose $\theta_t \partial_a \kappa_t^*(a_t)$. Si $(t,a) \mapsto \kappa_t^*(a)$ est de classe $\mathcal{C}^2$, a vérifie*

$$v^* = \int_0^1 a_s ds \text{ et } \forall t \in [0,1], \int_0^t \kappa_s^*(a_s) ds \leq 0,$$

si et seulement si
- *θ est strictement positive et croissante ;*
- *$\int_0^1 K^*(a)_s d\theta_s^{-1} = 0$;*
- *$\int_0^1 \kappa_s^*(a_s) ds = 0$.*

En particulier, il existe une unique solution à ce problème d'optimisation, et cette solution a est lipschitzienne.

Dans un second temps, grâce à des calculs explicites, on montre sans difficultés le résultat suivant sur la transformée de Laplace de l'aire d'un mouvement brownien restant négatif.

Lemme 5. *Soit $h : [0,1] \to [0,+\infty)$ une fonction continue. Pour tout $x < 0$, étant donné un mouvement brownien B issu de x, on a*

$$\lim_{t \to +\infty} \frac{1}{t} \log \mathbf{E}\left[\exp\left(-\int_0^t h(s/t) B_s ds\right) \mathbf{1}_{\{B_s \leq 0, s \leq t\}}\right] = \frac{\alpha_1}{2^{1/3}} \int_0^1 h_s ds,$$

où $\alpha_1 \approx -2.3381...$ est le premier zéro de la fonction Ai *d'Airy.*

Grâce à la Proposition 4 et au Lemme 5, on peut estimer M_n.

Théorème 6. *Sous de bonnes hypothèses d'intégrabilité et de régularité, on note a l'unique fonction vérifiant*

$$v^* = \int_0^1 a_s ds \text{ et } \forall t \in [0,1], \int_0^t \kappa_s^*(a_s) ds \leq 0,$$

on pose $\theta_t = \partial_a \kappa_t^*(a_t)$ et $\sigma_t^2 = \partial_\theta^2 \kappa_t(\theta_t)$. Si θ est absolument continue et admet une dérivée Riemann-intégrable $\dot{\theta}$, alors

$$M_n = n v_* + n^{1/3} \frac{\alpha_1}{2^{1/3}} \int_0^1 \frac{(\theta_s' \sigma_s)^2/3}{\theta_s} ds + o_P(n^{1/3}).$$

Marches aléatoires branchantes en environnement aléatoire. Dans ce modèle, la suite $(\mathcal{L}_n, n \in \mathbb{N})$ de lois de processus de points est une suite indépendante et identiquement distribuée. On considère la marche aléatoire branchante en environnement inhomogène dans laquelle tous les individus vivants à l'instant n se reproduisent indépendemment selon la loi $\mathcal{L}_n$.

On note $\kappa_n(\theta)$ la transformée de log-Laplace de $\mathcal{L}_n$ et $\kappa(\theta) = \mathbf{E}[\kappa_1(\theta)]$. On suppose qu'il existe $\theta^* > 0$ vérifiant

$$\theta^* \kappa'(\theta^*) - \kappa(\theta^*) = 0.$$

Des arguments classiques de marche aléatoire branchante montrent que sous de bonnes hypothèses, le plus grand déplacement à l'instant n est proche de $\sum_{j=1}^n \frac{\kappa_j(\theta^*)}{\theta^*}$. Pour déterminer le comportement asymptotique de M_n de façon plus précise, il est nécessaire d'obtenir le résultat suivant, sur la probabilité pour un mouvement brownien de rester au-dessus d'un autre mouvement brownien.

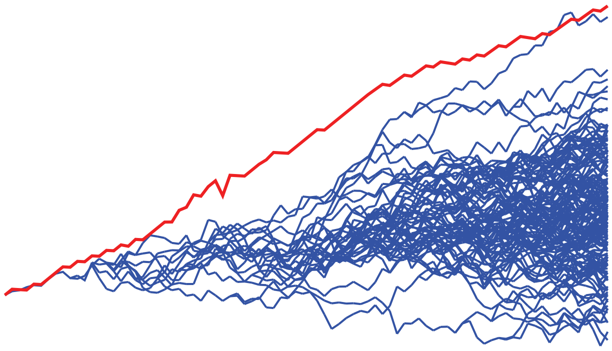


FIGURE 11 – Une marche aléatoire branchante en environnement aléatoire

Théorème 7 (Théorème du scrutin aléatoire)**.** *Soit B et W deux mouvements browniens indépendants. Il existe $\gamma : \mathbb{R} \to \mathbb{R}$ une fonction paire convexe et que $\gamma(0) = 1/2$ telle que pour tout $\beta > 0$,*

$$\lim_{t \to +\infty} \frac{1}{\log t} \log \mathbf{P}\left(B_s \geq \beta W_s - 1, s \leq t | W\right) = -\gamma(\beta) \quad \text{p.s.}$$

Grâce à ce résultat, on prouve le résultat suivant sur le plus grand déplacement dans la marche aléatoire branchante en environnement aléatoire.

Théorème 8. *On suppose que*

$$\sigma_Q^2 = (\theta^*)^2 \mathbf{E}\left[\kappa_1''(\theta^*)\right] \in (0, +\infty) \text{ et } \sigma_A^2 = \mathbf{V}\text{ar}\left[\theta^* \kappa_1'(\theta^*) - \kappa_1(\theta^*)\right] \in [0, +\infty),$$

et on note $\varphi = \frac{2}{\theta^*}\gamma\left(\frac{\sigma_A}{\sigma_Q}\right) + \frac{1}{2\theta^*}$. Sous de bonnes hypothèses d'intégrabilité, on a

$$M_n = \frac{1}{\theta^*} \sum_{j=1}^{n} \kappa_j(\theta) - \varphi \log n + o_{\mathbf{P}}(\log n).$$

On observe que le comportement asymptotique de M_n est dans ce modèle donné par un premier ordre linéaire de pente $\frac{\kappa(\theta^*)}{\theta^*}$, des fluctuations d'ordre $n^{1/2}$ qui ne dépendent que de l'environnement, et un terme logarithmique qui ne dépend pas de cet environnement.

4.2 Marches aléatoires branchantes avec sélection

Dans une marche aléatoire branchante avec sélection, on se fixe une suite d'entiers $(N_n, n \geq 1)$, qui représente la taille maximale de la population à chaque étape. A chaque instant $n \geq 1$, tous les individus vivants à la génération n meurent, en laissant des enfants placés autour de chaque parent selon des versions i.i.d. du processus de points $\mathcal{L}$. Immédiatement après, les N_{n+1} individus avec la plus grande position sont conservés, les autres sont immédiatement tués. On note M_n^N la position de l'individu le plus haut parmi les individus sélectionnés et m_n^N celle de l'individu le plus bas.

Sélection croissante. On s'intéresse dans un premier temps à une marche aléatoire branchante dans laquelle la suite des tailles de population au cours du temps est donnée par $N_n = \left\lfloor e^{an^{1/3}} \right\rfloor$, pour $a \in (0, +\infty)$. Le principal résultat du Chapitre 5 est le comportement asymptotique de M_n^N et m_n^N.

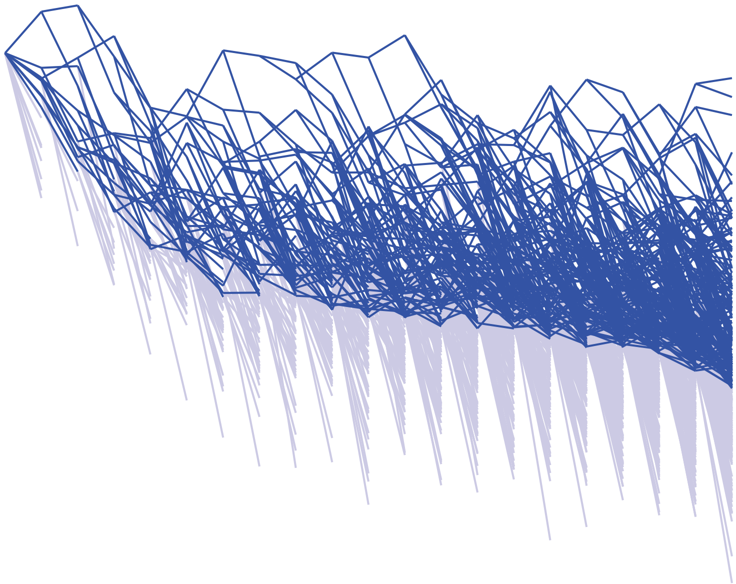


Figure 12 – Marche aléatoire branchante avec sélection des $e^{3n^{1/3}}$ individus les plus hauts à la génération n

L'un des principaux ingrédients de cette preuve est un couplage entre la marche aléatoire branchante avec sélection et une marche aléatoire branchante avec une barrière, dans laquelle les individus passant sous un niveau donné meurent.

Théorème 9. *Sous de bonnes hypothèses d'intégrabilité, on a*

$$\lim_{n\to+\infty} \frac{M_n^N - nv}{n^{1/3}} = \frac{3\pi^2\theta^*\sigma^2}{2a^2} \quad p.s.$$

$$\lim_{n\to+\infty} \frac{m_n^N - nv}{n^{1/3}} = -\frac{a}{\theta^*} - \theta^*\frac{3\pi^2\sigma^2}{2a^2} \quad p.s.$$

Sélection pour des marches à épine stable. Dans un second temps, on s'intéresse aux marches aléatoires branchantes avec sélection pour lesquelles l'*épine* (définie dans la Section 1.2.3) de la marche aléatoire branchante est dans le domaine d'attraction d'une marche aléatoire stable d'indice $\alpha \in (0,2]$. On montre dans le Chapitre 6 le résultat suivant, en reprenant et généralisant la preuve de [BG10].

Théorème 10. *Sous de bonnes hypothèses d'intégrabilité, pour tout $N \in \mathbb{N}$, on a*

$$\lim_{n\to+\infty} \frac{M_n^N}{n} = \lim_{n\to+\infty} \frac{m_n^N}{n} = v_N \ p.s.$$

De plus, il existe une fonction à variations lentes Λ et une constante $\chi > 0$ telles que

$$\lim_{N\to+\infty} (v - v_N)\frac{(\log N)^\alpha}{\Lambda(\log N)} = \chi.$$

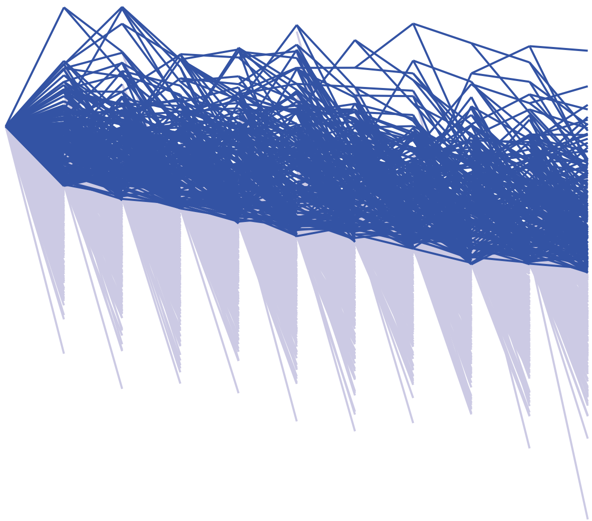


FIGURE 13 – N-Marche aléatoire branchante avec épine stable

4.3 Marche aléatoire branchante classique

Une troisième partie est consacrée à la preuve de deux résultats classiques sur les marches aléatoires branchantes en environnement homogène. Ces résultats sont prouvés sous des conditions d'intégrabilité plus fortes que celles trouvées dans la littérature. On calcule dans le Chapitre 7 l'asymptotique du plus grand déplacement dans la marche aléatoire branchante, jusqu'au terme $O_{\mathbf{P}}(1)$. On s'intéresse dans le Chapitre 8 au comportement asymptotique du plus grand déplacement consistant, défini comme

$$\Lambda_n = \min_{|u|=n} \max_{k \leq n} kv - V(u_k).$$

Il a été prouvé dans [FZ10] et [FHS12] que $\Lambda_n n^{-1/3}$ converge presque sûrement une constante explicite. Nous redémontrons ce résultat sous des conditions d'intégrabilité plus faible.

Les Chapitres 7 et 8 contiennent l'essentiel des arguments employés dans les autres chapitres de cette thèse, mais dans un cadre plus simple. Les notations et les raisonnements liés à la marche aléatoire branchante en environnement homogène gagnent en clarté par des notations plus légères. Nous espérons que leur lecture pourra éclairer quelques arguments perdus au sein de détails techniques.

5 Des modèles liés aux marches aléatoires branchantes

Nous présentons de façon très brève plusieurs objets mathématiques qui peuvent être reliés à des marches aléatoires branchantes, que l'on peut étudier en introduisant des marches aléatoires branchantes et/ou pour lesquels des techniques de preuves de marche aléatoire branchante peuvent être adaptées. La plupart du temps, ces objets sont présentés dans un cas particulier, par soucis de clarté ou de concision. Ne nous intéressant qu'à leurs liens avec les marches aléatoires branchantes, nous ne pouvons prétendre donner un aperçu complet des modèles évoqués.

Mouvement brownien branchant. Le mouvement brownien branchant peut être considéré comme la version à temps continu de la marche aléatoire branchante. C'est un processus simulant l'évolution au cours du temps d'une population sur $\mathbb{R}$. Les individus se déplacent selon des mouvements browniens indépendants. De plus, à chaque individu est associé une horloge exponentielle de paramètre 1. Lorsque cette horloge sonne, l'individu associé meurt en donnant naissance à deux enfants. Ces enfants se comportent alors comme n'importe quelle autre individu du processus, se déplacent indépendamment de façon brownienne et meurent au bout d'un temps exponentiel de paramètre 1. Le comportement du mouvement brownien branchant et de la marche aléatoire branchante sont très similaires. Un grand nombre de résultats prouvés pour la marche aléatoire branchante ont d'abord été prouvés dans le cadre du mouvement brownien branchant, cf. [Bra78, BBS13, Mai13, MZ14, Rob12].

Cascades multiplicatives de Mandelbrot. Les cascades multiplicatives de Mandelbrot ont été construites pour étudier les phénomènes d'intermittence dans la *théorie des turbulences de Kolmogorov* [Kol91b, Kol91a]. Une cascade multiplicative est, par exemple, une mesure aléatoire construite sur l'intervalle $[0, 1]$ telle que la mesure de l'intervalle dyadique $[k2^{-n}, (k + 1)2^{-n}]$ est égale en loi à la masse totale de cette mesure, multipliée par une variable aléatoire indépendante dont la loi dépend de la profondeur n de l'intervalle. Dans les articles de Mandelbrot [Man74a, Man74b, Man74c], Kahane [Kah74], Peyrière [Pey74] et Kahane et Peyrière [KP76], des marches aléatoires branchantes sont introduites pour étudier ces cascades multiplicatives.

Chaos multiplicatif gaussien. Le chaos multiplicatif gaussien est une autre mesure aléatoire, introduite par Kahane [Kah85b] en 1985. Une telle mesure peut être construite de façon informelle comme suit. Étant donné $\gamma \in \mathbb{R}$ et un champ gaussien centré $(X(x), x \in \mathbb{R}^2)$ vérifiant

$$\mathbf{E}\left[X(x)X(y)\right] \approx \log\left(\min(1, |y - x|)\right),$$

le chaos multiplicatif gaussien associé est la mesure $M_\gamma(dx) = e^{\gamma X(x) - \frac{\gamma^2}{2}\mathbf{E}(X(x)^2)}dx$. Cette mesure peut bien entendu être construite sur des espaces métriques mesurés plus généraux, et pour des noyaux non nécessairement gaussiens. Pour une présentation plus précise du chaos multiplicatif gaussien, nous nous référons à [RV14]. Le chaos multiplicatif gaussien satisfait les mêmes propriétés d'auto-similarité que les cascades de Mandelbrot. La marche aléatoire branchante peut être considérée comme un modèle-jouet de ce processus.

Équation aux dérivées partielles FKPP. Une fonction $u : [0, +\infty) \times \mathbb{R} \to [0, 1]$ satisfait l'équation de réaction-diffusion de Fisher-Kolmogorov-Petrowski-Piscounov si

$$\forall t \geq 0, \forall x \in \mathbb{R}, \frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + u(1 - u).$$

Cette équation, introduite dans [Fis37, KPP37], peut être interprétée comme l'évolution au cours du temps d'une population d'individus en interaction. Les individus se déplacent sur $\mathbb{R}$ en diffusant, et $u(t, x)$ représente la proportion d'individus présents au voisinage du point x à l'instant t, où 1 représente la constante de saturation de l'environnement. En d'autres termes, si $u(t, x) = 1$, la population cesse de croître au voisinage du point x car la mortalité due à la compétition pour les ressources contrebalance la naissance d'enfants. McKean [McK75, McK76] et Neveu [Nev88] ont noté que l'on pouvait relier l'équation FKPP avec le mouvement brownien branchant. Ainsi, si on écrit M_t le plus grand déplacement à l'instant t dans le mouvement brownien branchant, on observe que $u(t,x) = \mathbf{P}(M_t \geq x)$ satisfait l'équation FKPP ci-dessus. Un certain nombre de résultats sur les solutions de l'équation FKPP se transforment donc en résultats sur le comportement du mouvement brownien branchant et réciproquement, comme le montre par exemple l'enchaînement récent des résultats de Fang et Zeitouni [FZ12b] (mouvement brownien branchant), Nolen, Roquejoffre et Ryzhik (équation FKPP), et Maillard et Zeitouni [MZ14] (mouvement brownien branchant).

Generalised Random Energy Model. Le « Generalised Random Energy Model » ou Grem est un modèle de verre de spin introduit par Derrida [Der85]. Dans ce modèle, les particules sont corrélées selon une struture hiérarchique arborescente. Du fait de leurs nombreuses similarités, de nombreux résultats prouvés pour un modèle peuvent également être prouvés pour l'autre.

Marche aléatoire sur une marche aléatoire branchante. La marche aléatoire branchante est également un environnement de choix pour l'étude des marches aléatoires en environnement aléatoire. Dans ce modèle, un marcheur se déplace sur l'ensemble des individus présents à tout instant dans le processus. A chaque étape, le marcheur remonte vers le parent de l'individu sur lequel il se trouve avec une probabilité proportionnelle à 1, et descend vers l'un de ses enfants avec probabilité proportionnelle à l'exponentielle du déplacement relatif de l'enfant par rapport au parent (telle que définie dans [LP92]). Entre dans ce cadre, par exemple, les marches aléatoires biaisées sur les arbres de Galton-Watson [Lyo92, Lyo94].

Part I

Branching random walks in time-inhomogeneous environments

CHAPTER 1

Maximal displacement in a branching random walk through an interface

“Il paraît que la crise rend les riches plus riches et les pauvres plus pauvres. Je ne vois pas en quoi c’est une crise. Depuis que je suis petit, c’est comme ça.”

Michel Colucci, dit Coluche – Sketch "Le chômeur"

Abstract

In this chapter, we study the maximal displacement of a branching random walk in time-inhomogeneous environment. This environment consists in two macroscopic time intervals, in each of which the reproduction of individuals remains constant. This chapter provides tools helpful to the understanding of branching random walks in more general time-inhomogeneous environments. We prove here that the maximal displacement is given –as in the time-homogeneous case– by a first linear term, a negative logarithmic correction plus fluctuations of order one. This asymptotic behaviour strongly depends on the path followed by the individual who realises the maximal displacement. Furthermore, the logarithmic correction exhibit a sharp phase transition.

NOTA: This chapter is a simplified version of the article *Maximal displacement in a branching random walk through a series of interfaces* submitted to *Electronic Journal of Probabilities.* There is a single interface in the process and the text is available on arXiv:1305.6201

1.1 Introduction

A time-inhomogeneous branching random walk on $\mathbb{R}$ is a process which starts with one individual located at the origin at time 0, and evolves as follows: at each time k, every individual currently in the process dies, giving birth to a certain number of children, which are positioned around their parent according to independent versions of a point process, whose law may depend on the generation of the parent.

In 2011, Fang and Zeitouni [FZ12a] studied the asymptotic of the maximal displacement in a branching random walk defined as follows. Given $n \in \mathbb{N}$, they considered a branching random walk through an interface with length n, in which individuals split into two children, which displace around their parent according to independent Gaussian random variables. During the first $\frac{n}{2}$ unit of times, the Gaussian random variables have variance σ_1^2, while they have variance σ_2^2 between the generation $\frac{n}{2}$ and the generation n. They observed that the behaviour of such branching random walks depends on the sign of $\sigma_2^2 - \sigma_1^2$, in particular the asymptotic of the maximal displacement is a first ballistic order, and a logarithmic term, which exhibits a phase transition as σ_2^2 grows bigger than σ_1^2.

We generalize their result to a large variety of branching mechanisms. Let $t \in [0,1]$ and $\mathcal{L}_1, \mathcal{L}_2$ be two laws of point processes. For all $n \geq 1$, we study the time-inhomogeneous branching random walk $(\mathbf{T}^{(n)}, V^{(n)})$, in which individuals reproduce independently, with law $\mathcal{L}_1$ if they are alive before time tn, and with law $\mathcal{L}_2$ otherwise. In this process, M_n –the maximal displacement at time n– has again a first ballistic order, plus logarithmic corrections which again exhibit a phase-transition, and fluctuations of order 1.

We now introduce additional notation, to define more precisely time-inhomogeneous branching random walks, before stating the main result of the article, the asymptotic of M_n for a branching random walk through an interface.

1.1.1 Definition of the model and notation

The time-inhomogeneous branching random walk

We recall that $(\mathbf{T}, V) \in \mathcal{T}$ is a (plane, rooted) marked tree if $\mathbf{T}$ is a (plane, rooted) tree, and $V : \mathbf{T} \to \mathbb{R}$. For a given individual $u \in \mathbf{T}$, we write $|u|$ the generation to which u belongs. If u is not the root, then πu the parent of u and u_k the ancestor of u alive at generation k. Finally, we write $\Omega(u) = \{v \in \mathbf{T} : \pi v = u\}$ the set of children of u.

Let $(\mathcal{L}_n, n \geq 1)$ be a family of point processes laws, which we call the environment of the branching random walk. A time-inhomogeneous branching random walk with environment $(\mathcal{L}_n)$ is a random variable in $\mathcal{T}$ which we write $(\mathbf{T}, V)$. The law of this random variable is characterized by the three following properties

- $V(\emptyset) = 0$;
- $\{(V(v) - V(u), v \in \Omega(u)), u \in \mathbf{T}\}$ is a family of independent point processes;
- the point process $(V(v) - V(u), v \in \Omega(u))$ has law $\mathcal{L}_{|u|+1}$, and $\mathcal{L}_{n+1} = \delta_\emptyset$.

The branching random walk can be constructed in the following way. We write $\mathcal{U}$ the set of all finite sequences of integers –following the Ulam-Harris notations for trees– and for $u \in \mathcal{U}$, we write $|u|$ the length of u. We consider a family of independent point processes $\{L^u, u \in \mathcal{U}\}$, where L^u has law $\mathcal{L}_{|u|+1}$. For any $u \in \mathcal{U}$, we write $L^u = \left(\ell_1^u, \ldots \ell_{N(u)}^u\right)$. The plane rooted tree which represent the genealogy of the population is

$$\mathbf{T} = \{u \in \mathcal{U} : \forall k \leq |u| - 1, u(k+1) \leq N(u_k)\}.$$

In particular, we observe that the tree $\mathbf{T}$ is a –time-inhomogeneous– Galton-Watson tree, with reproduction law at generation k given by the number of points in a point process of law $\mathcal{L}_k$. We set $V(\emptyset) = 0$ and, for $u \in \mathbf{T}$ with $|u| = k$,

$$V(u) := V(\pi u) + \ell_{u(k)}^{\pi u} = \sum_{j=0}^{k-1} \ell_{u(j+1)}^{u_j}.$$

For a given $u \in \mathbf{T}$, the sequence $(V(u_0), V(u_1), \ldots V(u))$ of positions of the ancestors of u is often called the path or the trajectory of u. Finally, we write $M_n = \max_{|u|=n} V(u)$ the maximal displacement in the branching random walk.

Branching random walk through an interface

In this chapter, we take interest in a branching random walk with an interface. In this model, the environment of the time-inhomogeneous branching random walk scales at rate n, and consists in two macroscopic time-intervals, in each of which the reproduction law remains constant. Let $t \in (0,1)$ and $\mathcal{L}_1, \mathcal{L}_2$ be two laws of point processes. To avoid a discussion on the survival of the branching random walk, we assume that

$$\forall p \in \{1,2\}, \mathbf{P}(L_p = \emptyset) = 0 \quad \text{and} \quad \mathbf{E}\left[\sum_{\ell \in L_p} 1\right] > 1, \tag{1.1.1}$$

where L_p is a point process with law $\mathcal{L}_p$. For all $n \in \mathbb{N}$, we write $(\mathbf{T}^{(n)}, V^{(n)})$ a time-inhomogeneous branching random walk with length n, with environment $\mathcal{L}_1$ until time tn, $\mathcal{L}_2$ between time tn and n, and such that individuals at time n die without children. We observe in particular that $\mathbf{T}^{(n)}$ is a tree of height n.

The branching random walk $(\mathbf{T}^{(n)}, V^{(n)})$ is called branching random walk through an interface –BRWi for short– of length n. When the value of n is clear in the context, we often omit the superscripts to make the notation lighter.

Let $p \in \{1,2\}$. We write κ_p for the log-Laplace transform of $\mathcal{L}_p$. For $a \in \mathbb{R}$, we set $\kappa_p^*(a) = \sup_{\theta>0} \theta a - \kappa_p(\theta)$ its Fenchel-Legendre transform. We observe that if κ_p^* is differentiable at point a, we have

$$\kappa_p^*(a) = (\kappa_p^*)'(a)a - \kappa_p\left((\kappa_p^*)'(a)\right). \tag{1.1.2}$$

1.1.2 Assumptions and main results

Using some well-known branching random walk estimates, we build heuristics for the first order of M_n. It leads to a definition of the speed v^{i} as the solution of an optimization problem, that we study in a second time. In a third part, we finally state the main result of this chapter –the asymptotic of M_n up to stochastically bounded fluctuations.

Number of individuals at a given level in a branching random walk

We list here some classical results for (time-homogeneous) branching random walks, which can be found in [Big10]. Let $p \in \{1,2\}$, we write $(\mathbf{T}_p, V_p)$ for a branching random walk with reproduction law $\mathcal{L}_p$. If there exists $\theta > 0$ such that $\kappa_p(\theta) < +\infty$, then the speed of the branching random walk is defined by

$$\forall p \in \{1,2\}, v_p = \inf_{\theta>0} \frac{\kappa_p(\theta)}{\theta} = \sup\{a \in \mathbb{R} : \kappa_p^*(a) \leq 0\} < +\infty, \tag{1.1.3}$$

in the sense that $\lim_{n\to+\infty} \frac{\max_{|u|=n} V_p(u)}{n} = v_p$ a.s.

Additionally, the function κ_p^* is linked to the number of individuals alive at a given level at time n. As proved in [Big77a], we have

$$\begin{cases} \forall a < v_p, \lim_{n\to+\infty} \frac{1}{n} \log \sum_{|u|=n} \mathbf{1}_{\{V_p(u) \geq na\}} = -\kappa_p^*(a) \quad \text{a.s.} \\ \forall a > v_p, \lim_{n\to+\infty} \frac{1}{n} \log \mathbf{P}\left[\exists |u| = n : V_p(u) \geq na\right] = -\kappa_p^*(a). \end{cases} \tag{1.1.4}$$

Note that with high probability, there is no individual above nv_p at time n, and there is an exponentially large number of individuals at distance of order n of this maximal position. By (1.1.4), the quantity $e^{-n\kappa_p^*(a)}$ is either an approximation of the number of individuals alive at time n in a neighbourhood of na, or of the probability to observe at least one individual around na at time n, depending on the sign of $\kappa_p^*(a)$.

Heuristics for the maximal displacement

We now consider the BRWi $(\mathbf{T}, V)$. By (1.1.4), for any $a_1 < v_1$, there are approximately $e^{-nt\kappa_1^*(a_1)}$ individuals alive at time tn at position $ta_1 n$. Each of these individuals starts an independent branching random walk with reproduction law $\mathcal{L}_2$. By law of large number, there are at time n about $e^{-tn\kappa_1^*(a_1)-(1-t)n\kappa_2^*(a_2)}$ individuals around $(ta_1 + (1-t)a_2)n$, whose ancestors were close to $ta_1 n$. Therefore, there exists at least one individual to the right of nb if there exists $(a_1, a_2) \in \mathbb{R}^2$ such that

$$ta_1 + (1-t)a_2 \geq b, \; a_1 < v_1 \text{ and } t\kappa_1^*(a_1) + (1-t)\kappa_2^*(a_2) < 0.$$

We define

$$v^{\mathrm{i}} = \sup\{ta_1 + (1-t)a_2, (a_1, a_2) \in \mathbb{R}^2 : \kappa_1^*(a_1) \leq 0, t\kappa_1^*(a_1) + (1-t)\kappa_2^*(a_2) \leq 0\} \quad (1.1.5)$$

which we expect to be the speed of the BRWi. As $\kappa_1^*(v_1) \leq 0$ and $\kappa_2^*(v_2) \leq 0$ –these two functions being lower-continuous– we have immediately $tv_1 + (1-t)v_2 \leq v^{\mathrm{i}}$.

Observe that if we write

$$v^* = \sup\{ta_1 + (1-t)a_2, (a_1, a_2) \in \mathbb{R}^2 : t\kappa_1^*(a_1) + (1-t)\kappa_2^*(a_2) \leq 0\}, \quad (1.1.6)$$

we have immediately $v^{\mathrm{i}} \leq v^*$. If $v^{\mathrm{i}} = v^*$, this means that the condition $\kappa_1^*(a_1) \leq 0$ doesn't play any role in the optimization problem (1.1.5), thus the path followed by the rightmost individual until time n stays at any time $k \leq tn$ below the boundary kv_1. Otherwise, if $v^{\mathrm{i}} < v^*$, the condition $\kappa_1^*(a_1) \leq 0$ is important, and the path followed by the rightmost individual until time n has to be very close to the boundary tnv_1 at time tn. These different situations lead to different behaviours.

Around the logarithmic correction

We assume that

$$\forall p \in \{1, 2\}, \exists \theta_p > 0 : v_p = \frac{\kappa_p(\theta_p)}{\theta_p} = \mathbf{E}\left[\sum_{\ell \in L_p} \ell e^{\theta_p \ell - \kappa_p(\theta_p)}\right]. \quad (1.1.7)$$

With some additional integrability conditions (see Chapter 7), the maximal displacement at time n of $(\mathbf{T}_p, V_p)$ is of order $nv_p - \frac{3}{2\theta_p}\log n + O_{\mathbf{P}}(1)$. Moreover, by (1.1.2), if κ_p^* is differentiable at point v_p, then $(\kappa_p^*)'(v_p) = \theta_p$. As a consequence, the value of (θ_1, θ_2) is linked to the question $v^{\mathrm{i}} = v^*$ raised above. If $\kappa_1^*(a)$ and $\kappa_2^*(a)$ are both finite for any $a \in \mathbb{R}$, using careful analysis one can observe that

- if $\theta_1 < \theta_2$, then $tv_1 + (1-t)v_2 = v^{\mathrm{i}} < v^*$;
- if $\theta_1 = \theta_2$, then $tv_1 + (1-t)v_2 = v^{\mathrm{i}} = v^*$;
- if $\theta_1 > \theta_2$, then $tv_1 + (1-t)v_2 < v^{\mathrm{i}} = v^*$.

We observe that (1.1.8) is easily solved as soon as $\theta_1 \leq \theta_2$.

Proposition 1.1.1. *Under assumptions* (1.1.3) *and* (1.1.7)*, if* $\theta_1 \leq \theta_2$ *then*

$$v^{\mathrm{i}} = tv_1 + (1-t)v_2.$$

Proof. Under assumption (1.1.7), we have $\kappa_p^*(v_p) = 0$. Assuming $\theta_1 \leq \theta_2$, let $(a_1, a_2) \in \mathbb{R}^2$ be such that $\kappa_1^*(a_1) \leq 0$ and $t\kappa_1^*(a_1) + (1-t)\kappa_2^*(a_2) \leq 0$. By convexity of functions κ_p^*, we have

$$\begin{aligned}tv_1 + (1-t)v_2 - (ta_1 + (1-t)a_2) &= t(v_1 - a_1) + (1-t)(v_2 - a_2)\\ &\geq t\frac{\kappa_1^*(v_1) - \kappa_1^*(a_1)}{\theta_1} + (1-t)\frac{\kappa_2^*(v_2) - \kappa_2^*(a_2)}{\theta_2}\\ &\geq -t\frac{\kappa_1^*(a_1)}{\theta_1} - (1-t)\frac{\kappa_2^*(a_2)}{\theta_2},\end{aligned}$$

leading to

$$\begin{aligned}&tv_1 + (1-t)v_2 - (ta_1 + (1-t)a_2)\\ &\qquad\geq -t\kappa_1^*(a_1)\left(\tfrac{1}{\theta_1} - \tfrac{1}{\theta_2}\right) - (t\kappa_1^*(a_1) + (1-t)\kappa_2^*(a_2))\tfrac{1}{\theta_2} \geq 0.\end{aligned}$$

Therefore for any pair (a_1, a_2) we have $tv_1 + (1-t)v_2 \geq ta_1 + (1-t)a_2$, that yields $tv_1 + (1-t)v_2 \geq v^{\mathrm{i}}$. We conclude observing that by definition, $v^{\mathrm{i}} \geq tv_1 + (1-t)v_2$. $\square$

If $\theta_1 < \theta_2$, one would expect that in the optimization problem (1.1.5), the condition $\kappa_1^*(a_1) \leq 0$ plays a role, and the optimal path would have been higher by increasing a little a_1. This observation hints that with high probability, the rightmost individual at time n descends from one of the rightmost individuals at time tn. Therefore the maximal displacement at time n is, up to a $O_{\mathbf{P}}(1)$ the sum of the maximal displacement at time tn of $(\mathbf{T}_1, V_1)$ and the maximal displacement at time $(1-t)n$ of the branching random walk $(\mathbf{T}_2, V_2)$.

If $\theta_1 = \theta_2$, there should be no "pressure" on the path followed by the rightmost individual at time n to be within distance $O(1)$ from the boundary of the branching random walk at time tn. Nevertheless, we still expect the path to be within distance $O(n^{1/2})$ from this boundary. This has a cost $\frac{3}{2\theta_1}\log n$, thus one would expect the behaviour of the maximal displacement at time n to be similar to the one of a time-homogeneous branching random walk.

Finally, if $\theta_1 > \theta_2$, the "pressure" for the optimal path to stay close to the boundary does not exists. Moreover, the optimal path stays far from the boundary at any time $k \leq n$. As a consequence, the branching issues should not play any role in this case, and we expect the maximal displacement in this process to have a behaviour similar to the maximum of an exponentially large number of independent random walks of length n.

The asymptotic of the maximal displacement

We define here three different regimes for the BRWi, indexed by the sign of $\theta_1 - \theta_2$. The asymptotic behaviour of the maximal displacement depends on the regime, as the path followed by the rightmost individual at time n has very different features in each of them. To provide some integrability conditions, for L_p a point process with law $\mathcal{L}_p$, we define

$$X_p(\theta) = \sum_{\ell \in L_p} (1 + \ell_+)e^{\theta\ell} \quad \text{and} \quad \Sigma_p(\theta) = \sum_{\ell \in L_p} \ell^2 e^{\theta\ell}.$$

The slow regime. We assume $\theta_1 < \theta_2$. In this regime, the rightmost individual at time n descend from one of the rightmost individuals at time tn, and the following result holds.

Theorem 1.1.2. *Under the assumptions* (1.1.1), (1.1.3) *and* (1.1.7), *if* $\theta_1 < \theta_2$ *and*

$$\max_{p\in\{1,2\}} \mathbf{E}\left[\Sigma_p(\theta_p)\right] + \mathbf{E}\left[\sum_{\ell\in L_p} e^{\theta_p \ell} \log_+(X_p(\theta_p))^2\right] < +\infty$$

then

$$M_n = nv^{\mathrm{i}} - \left(\frac{3}{2\theta_1} + \frac{3}{2\theta_2}\right)\log n + O_{\mathbf{P}}(1).$$

The mean regime. If $\theta_1 = \theta_2$, the behaviour of the BRWi is very similar to the one of a classical branching random walk. In particular, the following result holds.

Theorem 1.1.3. *Under the assumption* (1.1.7), *writing* $\theta := \theta_1 = \theta_2$, *if*

$$\max_{p\in\{1,2\}} \mathbf{E}\left[\Sigma_p(\theta)\right] + \mathbf{E}\left[\sum_{\ell\in L_p} e^{\theta \ell} \log_+(X_p(\theta))^2\right] < +\infty$$

then

$$M_n = n(tv_1 + (1-t)v_2) - \frac{3}{2\theta}\log n + O_{\mathbf{P}}(1).$$

The fast regime. If $\theta_1 > \theta_2$, then the path leading to the rightmost individual at time n stays far from the boundary. We assume that

$$v_* = \inf_{\theta>0} \frac{t\kappa_1(\theta) + (1-t)\kappa_2(\theta)}{\theta} < +\infty. \tag{1.1.8}$$

Moreover, we assume there exists $\theta > 0$ such that

$$v_* = t\frac{\kappa_1(\theta)}{\theta} + (1-t)\frac{\kappa_2(\theta)}{\theta} = t\,\mathbf{E}\left[\sum_{\ell\in L_1} \ell e^{\theta\ell - \kappa_1(\theta)}\right] + (1-t)\,\mathbf{E}\left[\sum_{\ell\in L_2} \ell e^{\theta\ell-\kappa_2(\theta)}\right], \tag{1.1.9}$$

and that

$$\theta\,\mathbf{E}\left[\sum_{\ell\in L_1} \ell e^{\theta\ell-\kappa_1(\theta)}\right] - \kappa_1(\theta) < 0. \tag{1.1.10}$$

Under these assumptions, (1.1.5) can be solved once again. The following result links the quantities v^{i}, v^* and v_*.

Proposition 1.1.4. *Under assumptions* (1.1.8) *and* (1.1.9), *we have* $v^* = v_*$. *Under the additional assumption* (1.1.10), *then* $v_* = v^{\mathrm{i}}$.

Moreover, assuming (1.1.3), (1.1.7), (1.1.8) *and* (1.1.9), *if* $\theta_1 > \theta_2$, *then* (1.1.10) *holds,* $\theta \in (\theta_1, \theta_2)$ *and*

$$tv_1 + (1-t)v_2 < v^{\mathrm{i}} = v_*.$$

Proof. We write

$$a_1 = \mathbf{E}\left[\sum_{\ell\in L_1} \ell e^{\theta\ell-\kappa_1(\theta)}\right] \quad \text{and} \quad a_2 = \mathbf{E}\left[\sum_{\ell\in L_2} \ell e^{\theta\ell-\kappa_2(\theta)}\right],$$

and we observe that $\kappa_p^*(a_p) = \theta a_p - \kappa_p(\theta)$.In effect, if there exists $\varphi > \theta$ such that $\kappa_p(\varphi) < +\infty$, then a_p is the right derivative of κ_p at point θ by dominated convergence.

Moreover, by convexity of κ_p, we have $\kappa_p(\varphi) - \kappa_p(\theta) \geq (\varphi - \theta)a_p$. Similarly, if $\varphi < \theta$, $\kappa_p(\varphi) - \kappa_p(\theta) \geq (\varphi - \theta)a_p$. We conclude that

$$\theta a_p - \kappa_p(\theta) \geq \sup_{\varphi \in \mathbb{R}} \varphi a_p - \kappa_p(\varphi) = \kappa_p^*(a_p).$$

We prove in a first time that $v^* = v_*$. We first observe that

$$t\left(\theta a_1 - \kappa_1(\theta)\right) + (1-t)\left(\theta a_2 - \kappa_2(\theta)\right) = \theta v_* - t\kappa_1(\theta) - (1-t)\kappa_2(\theta) = 0.$$

Therefore, $v^* \geq v_*$. Moreover, by convexity of κ_1 and κ_2, for all $(b_1, b_2) \in \mathbb{R}^2$ such that $t\kappa_1^*(b_1) + (1-t)\kappa_2^*(b_2) \leq 0$, we have

$$\begin{aligned} ta_1 + (1-t)a_2 - (tb_1 + (1-t)b_2) &= t(a_1 - b_1) + (1-t)(a_2 - b_2) \\ &\geq t\frac{\kappa_1^*(a_1) - \kappa_1^*(b_1)}{\theta} + (1-t)\frac{\kappa_2^*(a_2) - \kappa_2^*(b_2)}{\theta} \\ &\geq t\left(\theta a_1 - \kappa_1(\theta)\right) + (1-t)\left(\theta a_2 - \kappa_2(\theta)\right) \\ &= \theta v_* - \left(t\kappa_1(\theta) + (1-t)\kappa_2(\theta)\right) = 0. \end{aligned}$$

As a consequence, optimizing in (b_1, b_2), we obtain $v_* \geq v^*$.

Moreover, under the additional assumption (1.1.10), we have $\kappa_1^*(a_1) < 0$, yielding $ta_1 + (1-t)a_2 \leq v^{\mathrm{i}}$. As $v^{\mathrm{i}} \leq v^*$, we can conclude.

We now assume (1.1.3), (1.1.7), (1.1.8) and (1.1.9), and that $\theta_2 > \theta_1$. As a Legendre transform, κ_p is a convex function, which is twice differentiable on the interior of $D_p := \{\theta \geq 0 : \kappa_p(\theta) < +\infty\}$. We write

$$f_p : \varphi \mapsto \varphi\, \mathbf{E}\left[\sum_{\ell \in L_p} \ell e^{\varphi \ell - \kappa_p(\varphi)}\right] - \kappa_p(\varphi).$$

By dominated convergence again, for any φ in the interior of D_p, $f_p(\varphi) = \varphi \kappa_p'(\varphi) - \kappa_p(\varphi)$. As a consequence, φ_p is an increasing function on its definition set.

Note that $\varphi_p(\theta_p) = \kappa_p^*(v_p) = 0$ and that $\varphi_p(\theta) = \kappa_p^*(a_p)$. As

$$t\kappa_1^*(a_1) + (1-t)\kappa_2^*(a_2) = 0,$$

if $\kappa_1^*(a_1) \geq 0$, then $\kappa_2^*(a_2) \leq 0$. As f_1 and f_2 are increasing, we have $\theta_1 \leq \theta \leq \theta_2$ which is in contradiction with $\theta_1 > \theta_2$. As a consequence, $\kappa_1^*(a_1) < 0$, thus (1.1.10) holds, and $v^{\mathrm{i}} = v_*$. Additionally, we have $\theta_2 < \theta < \theta_1$.

We prove then that $tv_1 + (1-t)v_2 < v^{\mathrm{i}}$. We observe that κ_1 is finite on $[\theta, \theta_1]$ and κ_2 is finite on $[\theta_2, \theta]$. As a consequence, for $\alpha, \beta > 0$, and $x \geq 0$ small enough, we have

$$\kappa_1^*\left(v_1 - \tfrac{\alpha}{t}x\right) = -\tfrac{\alpha}{t}\theta_1 x + O(x^2) \quad \text{and} \quad \kappa_2^*\left(v_2 + \tfrac{\beta}{1-t}x\right) = \tfrac{\beta}{1-t}\theta_2 x + O(x^2).$$

As a consequence, as soon as $\alpha\theta_1 < \beta\theta_2$, for $x \geq 0$ small enough we have

$$v_i \geq t\left(v_1 - \tfrac{\alpha}{t}x\right) + (1-t)\left(v_2 + \tfrac{\beta}{1-t}x\right) = tv_1 + (1-t)v_2 + (\beta - \alpha)x.$$

Thus, choosing $\beta > \alpha$ such that $\alpha\theta_1 < \beta\theta_2$ –which exists as $\theta_2 < \theta_1$– we obtain $v_i > tv_1 + (1-t)v_2$. $\square$

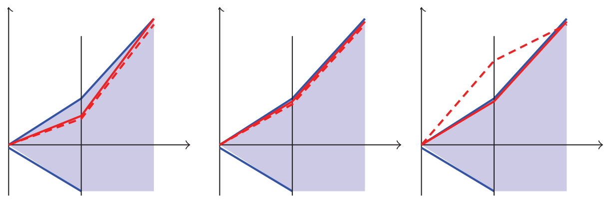
The following theorem describes the asymptotic behaviour of the maximal displacement of the branching random walk in the fast regime.

Theorem 1.1.5. *Under the assumptions* (1.1.8), (1.1.9) *and* (1.1.10), *if*

$$\max_{p\in\{1,2\}} \mathbf{E}\left[\Sigma_p(\theta)\right] + \mathbf{E}\left[\sum_{\ell\in L_p} e^{\theta\ell}\log_+\left(\sum_{\ell\in L_p} e^{\theta_p\ell}\right)\right] < +\infty$$

then

$$M_n = nv_* - \frac{1}{2\theta}\log n + O_{\mathbf{P}}(1).$$



(a) Regime $\theta_1 > \theta_2$ (b) Regime $\theta_1 = \theta_2$ (c) Regime $\theta_1 < \theta_2$

Figure 1.1: Regimes in the branching random walk with interface

Remark 1.1.6. We observe that if $\theta, \theta_1, \theta_2$ exist, then θ is always between θ_1 and θ_2, by convexity of κ_1 and κ_2. We write $M_n = nv^{\mathrm{i}} - \lambda^{\mathrm{i}}\log n + O_{\mathbf{P}}(1)$. We note that v^{i} evolves continuously with respect to $\mathcal{L}_1$ and $\mathcal{L}_2$, while

$$\lim_{\theta_2\to\theta_1^+} \lambda^{\mathrm{i}} = \frac{3}{\theta_1} > \frac{3}{2\theta_1} > \lim_{\theta_2\to\theta_1^-} \lambda^{\mathrm{i}} = \frac{1}{2\theta_1}.$$

We also observe that λ^{i} does not depend on the value of t.

In Section 1.2, we introduce a time-inhomogeneous version of the spinal decomposition, which links the computation of some moments of the branching random walk with time-inhomogeneous random walk estimates. In Section 1.3 we introduce some well-known random walk estimates, and extend these to time-inhomogeneous random walks. Finally, we bound in Section 1.4 the right tail of the maximal displacement in all three regimes, and use it in Section 1.5 to prove Theorems 1.1.2, 1.1.3 and 1.1.5.

1.2 The spinal decomposition of the time-inhomogeneous branching random walk

This section is devoted to the proof of a time-inhomogeneous version of the well-known spinal decomposition of the branching random walk. This result consists in two ways of describing the same size-biased law of the branching random walk. The spinal decomposition of a branching process has been introduced on Galton-Watson processes in [LPP95]. In [Lyo97], this result has been successfully adapted for the first time to the study of branching random walks. Until now, many results obtained in branching processes use this spinal decomposition, or its simpler version: the many-to-one lemma, first introduced by Peyrière [Pey74].

1.2.1 The size-biased law of the branching random walk

Let $(\mathcal{L}_n, n \in \mathbb{N})$ be a sequence of laws of point processes which forms the environment of the time-inhomogeneous branching random walk $(\mathbf{T}, V)$. For any $x \in \mathbb{R}$, we set $\mathbf{P}_x$ the law of $(\mathbf{T}, V + x)$ and $\mathbf{E}_x$ the corresponding expectation. We recall that, for $n \geq 1$, $\mathcal{F}_n = \sigma\left(u, V(u) : |u| \leq n\right)$ is the natural filtration of the branching random walk.

We write $\kappa_k(\theta)$ the log-Laplace transform of $\mathcal{L}_k$, and we assume there exists $\theta > 0$ such that for any $n \in \mathbb{N}$, $\kappa_n(\theta) < +\infty$. We set

$$W_n = \sum_{|u|=n} \exp\left(\theta V(u) - \sum_{k=1}^{n} \kappa_k(\theta)\right),$$

we observe that (W_n) is a $(\mathcal{F}_n)$-martingale, that $W_n \geq 0$ $\mathbf{P}_x$-a.s. and that $\mathbf{E}_x(W_n) = e^x$, therefore, we can define the law

$$\left.\overline{\mathbf{P}}_x\right|_{\mathcal{F}_n} = e^{-x} W_n \cdot \left.\mathbf{P}_x\right|_{\mathcal{F}_n}. \tag{1.2.1}$$

The spinal decomposition consists in an alternative construction of the law $\overline{\mathbf{P}}_x$, as the projection of a law on the set of planar rooted marked trees with spine, which we define below.

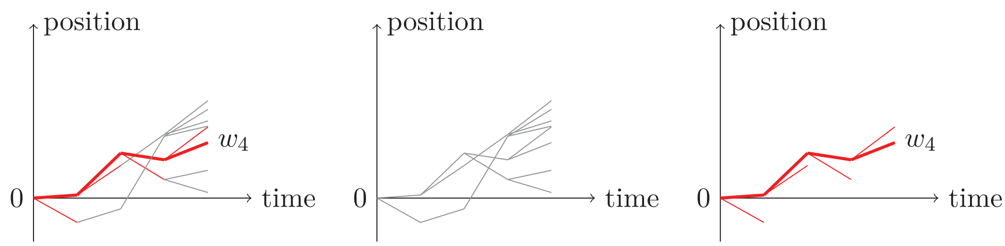
1.2.2 A law on plane rooted marked trees with spine

Let $(\mathbf{T}, V) \in \mathcal{T}$ be a marked tree, and $w \in \mathbb{N}^{\mathbb{N}}$ be a sequence of integers. We write $w_n = (w(1), \ldots w(n))$ and we assume that for all $n \in \mathbb{N}$, $w_n \in \mathbf{T}$. The triple $(\mathbf{T}, V, w)$ is then called a plane rooted marked tree with spine of length n. The spine of a tree is a distinguished path of infinite length which links the root and the boundary of the tree. The set of marked trees with spine is written $\widehat{\mathcal{T}}$. On this set, we define, for $n \in \mathbb{N}$ the filtrations

$$\mathcal{G}_n = \sigma\left((w_j, V(w_j)), j \leq n\right) \vee \sigma\left((u, V(u)), u \in \Omega(w_j), j < n\right) \quad \text{and} \quad \widehat{\mathcal{F}}_n = \mathcal{F}_n \vee \mathcal{G}_n$$

In particular $\widehat{\mathcal{F}}$ is the natural filtration of the branching random walk with spine, $\mathcal{F}$ is the information of the marked tree, obtained by forgetting about the spine, and $\mathcal{G}$ is the sigma-field of the knowledge of the spine and its children only.

Figure 1.2: The graph of a plane rooted marked tree with spine



(a) Information in $\widehat{\mathcal{F}}$ (b) Information in $\mathcal{F}$ (c) Information in $\mathcal{G}$

We now introduce a law $\widehat{\mathbf{P}}_a$ on $\widehat{\mathcal{T}}$. For any $k \in \mathbb{N}$, we write

$$\widehat{\mathcal{L}}_k = \left(\sum_{\ell \in L} e^{\theta \ell - \kappa_k(\theta)}\right) \cdot \mathcal{L}_k,$$

a law of a point process with Radon-Nikodým derivative with respect to $\mathcal{L}_k$, and we write $\widehat{L}_k = \left(\widehat{\ell}_k(j), j \leq N_k\right)$ an independent point processes of law $\widehat{\mathcal{L}}_k$. Conditionally on $(\widehat{L}_k)$, we choose, for every $k \in \mathbb{N}$, $w(k) \leq N_k$ independently at random, such that

$$\mathbf{P}\left(w(k) = h \,\middle|\, \widehat{L}_k, k \leq n\right) = \mathbf{1}_{\{h \leq N_k\}} \frac{e^{\theta \ell_k(h)}}{\sum_{j \leq N_k} e^{\theta \ell_k(j)}}.$$

We denote by w the sequence $(w(n), n \in \mathbb{N})$.

We now introduce a family of independent point processes $\{L^u, u \in \mathcal{U}, |u| \leq n\}$ such that $L^{w_k} = \widehat{L}_{k+1}$, and if $u \neq w_{|u|}$, then L^u has law $\mathcal{L}_{|u|+1}$. For any $u \in \mathcal{U}$ such that $|u| \leq n$, we write $L^u = (\ell^u_1, \ldots \ell^u_{N(u)})$. We construct the random tree

$$\mathbf{T} = \left\{u \in \mathcal{U} : \forall 1 \leq k \leq |u|, u(k) \leq N(u_{k-1})\right\},$$

and the the function

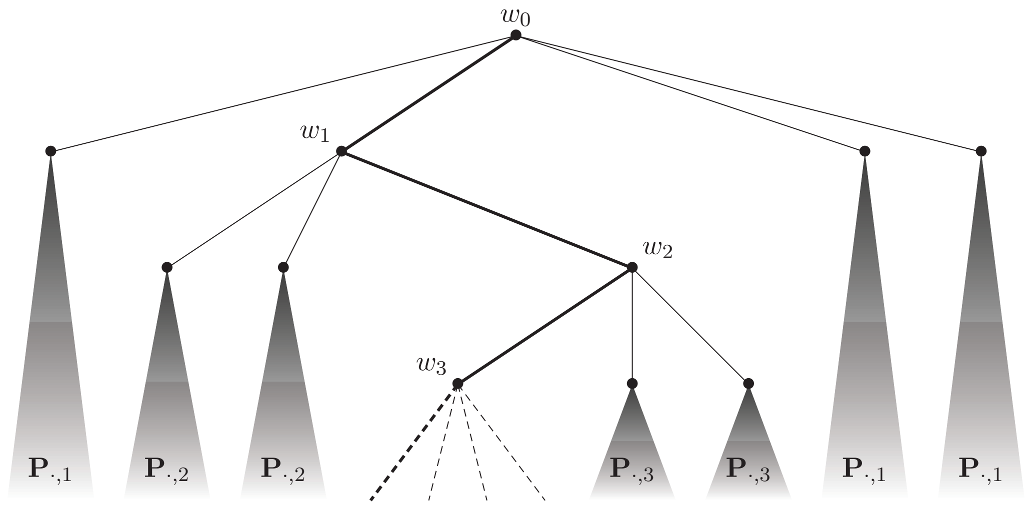
$$V : \begin{array}{rcl} \mathbf{T} & \to & \mathbb{R} \\ u & \mapsto & \sum_{k=1}^{|u|} \ell^{u_{k-1}}_{u(k)}. \end{array}$$

For all $x \in \mathbb{R}$, the law of $(\mathbf{T}, x+V, w) \in \widehat{\mathcal{T}}$ is written $\widehat{\mathbf{P}}_x$, and the corresponding expectation is $\widehat{\mathbf{E}}_x$.

This law is called branching random walk with spine, and can be constructed as a process in the following way. It starts with a unique individual positioned at x at time 0, which is the ancestral spine w_0. Then, at each time $k < n$, every individual alive at generation k die. Each of these individuals gives birth to children, which are positioned around their parent according to an independent point process. If the parent is w_k, then the law of this point process is $\widehat{\mathcal{L}}_k$, else it is $\mathcal{L}_k$. The individual w_{k+1} is then chosen at random among the children u of w_k, with probability proportional to $e^{\theta V(u)}$.

In the rest of the chapter, we write $\mathbf{P}_{x,k}$ the law of the time-inhomogeneous branching random walk starting from x with environment $(\mathcal{L}_{k+1}, \ldots \mathcal{L}_n)$. In particular, observe that conditionally on $\mathcal{G}$, the branching random walks of the descendants of children of w_k are independent, and the branching random walk of the descendants of $u \in \Omega(w_k)$ has law $\mathbf{P}_{V(u),k+1}$.

Figure 1.3: Construction of $\widehat{\mathbf{P}}$



1.2.3 The spinal decomposition

The following result, which links the laws $\widehat{\mathbf{P}}_x$ and $\overline{\mathbf{P}}_x$ is the time-inhomogeneous version of the spinal decomposition.

Proposition 1.2.1 (Spinal decomposition)**.** *For all $x \in \mathbb{R}$ and $n \geq 1$, we have*

$$\left.\overline{\mathbf{P}}_x\right|_{\mathcal{F}_n} = \left.\widehat{\mathbf{P}}_x\right|_{\mathcal{F}_n}. \tag{1.2.2}$$

Moreover, for any $n \in \mathbb{N}$ and $|u| = n$, we have

$$\widehat{\mathbf{P}}_x(w_n = u|\mathcal{F}_n) = \mathbf{1}_{\{u\in\mathbf{T}\}} \frac{\exp\left(\theta V(u) - \sum_{k=1}^n \kappa_k(\theta)\right)}{W_n}. \tag{1.2.3}$$

Proof. Let $n \in \mathbb{N}$ and $x \in \mathbb{R}$, we introduce the (non-probability) measure $\mathbf{P}^*_x$ on $\widehat{\mathcal{T}}$, in which every possible choice of spine has mass 1. More precisely, for any $\widehat{\mathcal{F}}_n$ measurable function f, we have

$$\int f(\mathbf{T}, V, w_n) d\mathbf{P}^*_x(\mathbf{T}, V, w) = \mathbf{E}_x\left[\sum_{|w|=n} f(\mathbf{T}, V, w)\right].$$

We compute in a first time by recurrence on n the Radon-Nikodým derivative of $\widehat{\mathbf{P}}_x$ with respect to $\mathbf{P}^*_x$, to prove

$$\left.\frac{d\widehat{\mathbf{P}}_x}{d\mathbf{P}^*_x}\right|_{\widehat{\mathcal{F}}_n} = \exp\left(\theta(V(w_k) - x) - \sum_{k=1}^n \kappa_k(\theta)\right). \tag{1.2.4}$$

Observe that for $n = 1$, (1.2.4) follows from the definition of $\widehat{L}_1$ and $w(1)$. In effect, writing L_1 a point process of law $\mathcal{L}_1$ and f a non-negative $\widehat{\mathcal{F}}_1$ measurable function,

$$\mathbf{E}\left[f(\widehat{L}_1, w(1))\right] = \mathbf{E}\left[\sum_{k=1}^{N_1} f(\widehat{L}_1, k) \frac{e^{\theta \ell_1(k)}}{\sum_{j=1}^{N_1} e^{\theta\ell_1(j)}}\right] = \mathbf{E}\left[\sum_{k=1}^{N_k} f(L_1, k) e^{\theta\ell_1(k) - \kappa_1(\theta)}\right].$$

We now assume (1.2.4) true for some $k \in \mathbb{N}$, and we observe that

$$\begin{aligned}
\left.\frac{d\widehat{\mathbf{P}}_x}{d\mathbf{P}^*_x}\right|_{\widehat{\mathcal{F}}_{k+1}} &= \left.\frac{d\widehat{\mathbf{P}}_x}{d\mathbf{P}^*_x}\right|_{\widehat{\mathcal{F}}_k} \times \left(\sum_{u\in\Omega(w_k)} e^{\theta(V(u) - V(w_k)) - \kappa_{k+1}(\theta)}\right) \frac{e^{-V(w_{k+1}) - V(w_k)}}{\sum_{u\in\Omega(w_k)} e^{\theta(V(u)-V(w_k)) - \kappa_{k+1}(\theta)}}\\
&= \exp\left(\theta(V(w_k) - x) - \sum_{j=1}^k \kappa_j(\theta)\right) e^{\theta(V(w_{k+1}) - V(w_k)) - \kappa_{k+1}(\theta)},
\end{aligned}$$

which proves (1.2.4).

As a consequence, for any $\mathcal{F}_n$-measurable function $f : \mathcal{T} \to \mathbb{R}_+$, we have

$$\begin{aligned}
\widehat{\mathbf{E}}_x\left[f(\mathbf{T}, V)\right] &= \int_{\widehat{\mathcal{T}}} e^{\theta(V(w_n)-x) - \sum_{j=1}^n \kappa_j(\theta)} f(\mathbf{T}, V) d\mathbf{P}^*_x(\mathbf{T}, V, w)\\
&= \mathbf{E}_x\left[f(\mathbf{T}, V) \sum_{|u|=n} e^{\theta(V(u)-x) - \sum_{j=1}^n \kappa_j(\theta)}\right]\\
&= e^{-\theta x}\, \mathbf{E}_x\left[W_n f(\mathbf{T}, V)\right)
\end{aligned}$$

therefore

$$\left.\frac{d\widehat{\mathbf{P}}_x}{d\mathbf{P}_x}\right|_{\mathcal{F}_n} = \left.\frac{d\overline{\mathbf{P}}_x}{d\mathbf{P}_x}\right|_{\mathcal{F}_n} = W_n$$

which proves (1.2.2).

Moreover, for any $\mathcal{F}_n$-measurable function $f : \mathcal{T} \to \mathbb{R}_+$ and $u \in \mathcal{U}$ with $|u| = n$, we have

$$\begin{aligned}
\widehat{\mathbf{E}}_x\left[f(\mathbf{T}, V)\mathbf{1}_{\{w_n=u\}}\right] &= \int_{\widehat{\mathcal{T}}_n} e^{\theta(V(w_n)-x)-\sum_{j=1}^n \kappa_j(\theta)} f(\mathbf{T}, V)\mathbf{1}_{\{w_n=u\}} d\mathbf{P}^*_x(\mathbf{T}, V, w)\\
&= \mathbf{E}_x\left[f(\mathbf{T}, V)\sum_{|v|=n} e^{\theta(V(v)-x)-\sum_{j=1}^n \kappa_j(\theta)}\mathbf{1}_{\{v=u\}}\right]\\
&= \overline{\mathbf{E}}_x\left[\frac{e^{\theta V(u)}}{W_n} f(\mathbf{T}, V)\mathbf{1}_{\{u\in\mathbf{T}\}}\right]\\
&= \widehat{\mathbf{E}}_x\left[\frac{e^{\theta V(u)}}{W_n} f(\mathbf{T}, V)\mathbf{1}_{\{u\in\mathbf{T}\}}\right]
\end{aligned}$$

by (1.2.2), which ends the proof of (1.2.3). □

A direct consequence of this result, which can also be proved directly by recurrence, is the well-known many-to-one lemma. This equation, known at least from the early works of Peyrière [Pey74] has been used in many forms over the last decades, and we introduce here a time-inhomogeneous version of it. Let $(X_n, n \in \mathbb{N})$ be a sequence of independent random variables such that the law of X_n is characterized, by Riesz extension theorem, by

$$\forall f \in \mathcal{C}_b, \mathbf{E}\left[f(X_n)\right] = \mathbf{E}\left[\sum_{\ell\in L_n} f(\ell)e^{\theta\ell-\kappa_n(\theta)}\right].$$

We define the time-inhomogeneous random walk S by $S_n = S_0 + \sum_{j=1}^n X_j$ that starts from $x \in \mathbb{R}$ under law $\mathbf{P}_x$. In other words, we assume $\mathbf{P}_x(S_0 = x) = 1$ for any $x \in \mathbb{R}$.

Lemma 1.2.2 (Many-to-one)**.** *For all $x \in \mathbb{R}, n \in \mathbb{N}$ and non-negative measurable function f, we have*

$$\mathbf{E}_x\left[\sum_{|u|=n} f(V(u_1), \dots V(u_n))\right] = e^{\theta x}\,\mathbf{E}_x\left[e^{-\theta S_n+\sum_{j=1}^n \kappa_j(\theta)} f(S_1, \dots S_n)\right]. \tag{1.2.5}$$

Proof. Let f a continuous bounded function and $x \in \mathbb{R}$, we have, by Proposition 1.2.1

$$\begin{aligned}
\mathbf{E}_x\left[\sum_{|u|=n} f(V(u_1), \dots V(u_n))\right] &= \overline{\mathbf{E}}_x\left[\frac{e^{\theta x}}{W_n}\sum_{|u|=n} f(V(u_1), \dots V(u_n))\right]\\
&= \widehat{\mathbf{E}}_x\left[\frac{e^{\theta x}}{W_n}\sum_{|u|=n} f(V(u_1), \dots V(u_n))\right]\\
&= \widehat{\mathbf{E}}_x\left[e^{\theta x}\sum_{|u|=n} e^{-\theta V(u)}\widehat{\mathbf{P}}(w_n = u|\mathcal{F}_n) f(V(u_1), \dots V(u_n))\right]\\
&= \widehat{\mathbf{E}}_x\left[e^{-\theta(V(w_n)-x)+\sum_{j=1}^n \kappa_j(\theta)} f(V(w_1), \dots V(w_n))\right].
\end{aligned}$$

Moreover, by definition of $\widehat{\mathbf{P}}_x$ we observe that the law of $(V(w_1), \dots, V(w_n))$ is the same as the law of $(S_1, \dots S_n)$ under $\mathbf{P}_x$, which ends the proof. $\square$

The spinal decomposition and the many-to-one lemma are used to compute moments of the number of individuals in the BRWi that stay in a given path, by using random walk estimates. The random walk estimates we use in this chapter are introduced in the next section, and extended to include time-inhomogeneous versions.

1.3 Some random walk estimates

We collect a series of well-known random walk estimates, the local limit Theorem, the ballot Theorem and the Hsu-Robbins Theorem. By classical methods, we extend these results to bound the probability for a random walk to make an excursion above a given curve. In a second time these results are extended to random walks enriched by additional random variables, correlated with the last step of the random variable. Finally, these estimates are extended to the case of "random walks through an interface", which we use with the many-to-one Lemma to prove the main theorems of the chapter.

1.3.1 Classical random walk estimates

We denote by $(T_n, n \geq 0)$ a one-dimensional centred random walk, with finite variance σ^2. We write $\mathbf{P}_x$ for the law of T such that $\mathbf{P}_x(T_0 = x) = 1$, and $\mathbf{P} = \mathbf{P}_0$. We begin with Stone's local limit theorem, which bounds the probability for a random walk to end up in an interval of finite size.

Theorem I (Stone [Sto65]). *There exists $C > 0$ such that for all $a \geq 0$ and $h \geq 0$*

$$\limsup_{n \to +\infty} n^{1/2} \sup_{|y| \geq a n^{1/2}} \mathbf{P}(T_n \in [y, y+h]) \leq C(1+h) e^{-\frac{a^2}{2\sigma^2}}.$$

Moreover, there exists $H > 0$ such that for all $a < b \in \mathbb{R}$

$$\liminf_{n \to +\infty} n^{1/2} \inf_{y \in [a n^{1/2}, b n^{1/2}]} \mathbf{P}(T_n \in [y, y+H]) > 0.$$

A similar result has been obtained in [CC13], for the random walk conditioned to stay positive.

Theorem II (Caravenna-Chaumont's local limit theorem). *Let $(r_n, n \geq 0)$ be a positive sequence such that $r_n = O(n^{1/2})$. There exists $C > 0$ such that for all $a \geq 0$, $h \geq 0$ and $y \in [0, r_n]$,*

$$\sup_{x \geq a n^{1/2}} \mathbf{P}_y(T_n \in [x, x+h] | T_j \geq 0, j \leq n) \leq C(1+h) a e^{-\frac{a^2}{2\sigma^2}} n^{-1/2}.$$

Moreover, there exists $H > 0$ such that for all $a < b \in \mathbb{R}_+$,

$$\liminf_{n \to +\infty} n^{1/2} \inf_{y \in [0, r_n]} \inf_{x \in [a n^{1/2}, b n^{1/2}]} \mathbf{P}(T_n \in [x, x+H] | T_j \geq -y, j \leq n) > 0.$$

Up to a transformation $T \mapsto T/(2H)$, which corresponds to shrinking the space by a factor $\frac{1}{2H}$, we may and will always assume that all the random walks we consider are such that the lower bound in Theorems I and II hold with $H = 1$.

The next result, often called in the literature the *ballot theorem*, compute the probability for a random walk to stay above zero. This result is stated in [Koz76], see also [ABR08] for a review article on ballot theorems.

Theorem III (Kozlov [Koz76]). *There exists $C > 0$ such that for all $n \geq 1$ and $y \geq 0$,*

$$\mathbf{P}_y(T_j \geq 0, j \leq n) \leq C(1+y)n^{-1/2}.$$

Moreover, there exists $c > 0$ such that for all $y \in [0, n^{1/2}]$

$$\mathbf{P}_y(T_j \geq 0, j \leq n) \geq c(1+y)n^{-1/2}.$$

A modification of this theorem, Theorem 3.2 of Pemantle and Peres in [PP95], expresses the probability for a random walk to stay above a boundary moving "strictly slower than $n^{1/2}$".

Theorem IV (Pemantle–Peres [PP95]). *Let $f : \mathbb{N} \to \mathbb{N}$ be an increasing positive function. The condition*

$$\sum_{n \geq 0} \frac{f_n}{n^{3/2}} < +\infty$$

is necessary and sufficient for the existence of an integer n_f such that

$$\sup_{n \in \mathbb{N}} n^{1/2} \mathbf{P}(T_j \geq -f_j, n_f \leq j \leq n) < +\infty.$$

Using Theorems III and IV, we prove two more estimates. First, a quantitative version of the probability for a random walk to stay above a curve.

Lemma 1.3.1. *Let $A > 0$ and $\alpha \in [0, 1/2)$, there exists $C > 0$ such that for any $(f_n) \in \mathbb{R}^{\mathbb{N}}$ verifying $|f_n| \leq An^\alpha$, $y \geq 0$ and $n \geq 1$, we have*

$$\mathbf{P}_y(T_j \geq -f_j, j \leq n) \leq C(1+y)n^{-1/2}.$$

Proof. Let $k \in \mathbb{N}$, we write $\tau_k = \inf\{n \geq 0 : T_n \leq -k - An^\alpha\}$. As $|f_n| \leq An^\alpha$, for any $y \in [k-1, k]$, we have $\mathbf{P}_y(T_j \geq -f_j, j \leq n) \leq \mathbf{P}(\tau_k \geq n)$. We now bound $\mathbf{P}(\tau_k \geq n)$, writing

$$\forall k \in \mathbb{N}, K_k = \sup_{n \in \mathbb{N}} n^{1/2} \mathbf{P}(\tau_k \geq n), \ K' = \sup_{y \in \mathbb{R}, n \in \mathbb{N}} (n+1)^{1/2} \mathbf{P}(T_n \in [y, y+1])$$
$$\text{and } K^* = \sup_{n \in \mathbb{N}, y \geq 0} \frac{n^{1/2}}{1+y} \mathbf{P}(T_j \leq y, j \leq n), \quad (1.3.1)$$

which are finite, by use of Theorems I, III and IV. For $k \in \mathbb{N}$, we have

$$\begin{aligned}\mathbf{P}(\tau_k \geq n) &\leq \mathbf{P}(\tau_1 \geq n) + \sum_{p=0}^{n-1} \mathbf{P}(\tau_k \geq n-p, \tau_1 = p)\\ &\leq K_1 n^{-1/2} + \sum_{p=0}^{n-1} \mathbf{P}(\tau_k \geq n-p, \tau_1 = p). \quad (1.3.2)\end{aligned}$$

Let $p < n$. Applying the Markov property at time p, we have

$$\begin{aligned}\mathbf{P}(\tau_k \geq n-p, \tau_1 = p) &\leq \mathbf{P}(\tau_1 = p) \sup_{z \in [1,k]} \mathbf{P}(T_j - z - Ap^\alpha \geq -A(p+j)^\alpha - k, j \leq (n-p))\\ &\leq \mathbf{P}(\tau_1 = p) \mathbf{P}(T_j \geq -Aj^\alpha - (k-1), j \leq (n-p))\\ &\leq \mathbf{P}(\tau_1 = p) K_{k-1} (n-p)^{-1/2}.\end{aligned}$$

We now bound $\mathbf{P}(\tau_1 = p)$. Conditioning on the p^{th} step, we have

$$\mathbf{P}(\tau_1 = p) = \mathbf{P}\left[T_p \leq -1 - Ap^\alpha, T_j \geq -1 - Aj^\alpha, j < p\right] = \mathbf{E}\left[\varphi_{p-1}(X_p)\right]$$

where we set $\varphi_p(x) = \mathbf{P}\left[T_p \leq -1 - Ap^\alpha + x, T_j \geq -1 - Aj^\alpha, j \leq p\right]$ for $x \in \mathbb{R}$ and $p \in \mathbb{N}$. By the Markov property applied at time $p' = \lfloor p/3 \rfloor$ to obtain

$$\varphi_p(x) \leq K_1 p'^{-1/2} \sup_{z \in \mathbb{R}} \mathbf{P}(T_{p-p'} + z \in [x, 0] T_j + z \geq 0, j < p - p').$$

We write $\widehat{T}_j = T_{p-p'} - T_{p-p'-j}$, which is a random walk with the same law as T. We often refer to this process as to the time-reversed random walk of T. For any $z \geq 0$,

$$\begin{aligned}
\mathbf{P}(T_{p-p'} + z \in [x,0], T_j + z \geq 0, j < p - p') \\
&\leq \mathbf{P}(\widehat{T}_{p-p'} + z \in [x,0], \widehat{T}_j \leq -x, j < p') \\
&\leq \mathbf{P}(T_j \leq -x, j \leq p') \sup_{y \in \mathbb{R}} \mathbf{P}(T_{p-2p'} \in [y, y+x]) \\
&\leq K^*(1 + x_+)(p/3)^{-1/2} K'(1 + x_+)(p/3)^{-1/2}.
\end{aligned}$$

We have $\mathbf{P}(\tau_1 = p) \leq K_1 K^* K' (p/3)^{-3/2}\, \mathbf{E}\left((1 + (X_p)_+)^2\right)$. As $\mathbf{E}(X_p^2) < +\infty$, there exists a constant $\widetilde{K} > 0$ which does not depend on k, such that (1.3.2) becomes

$$\begin{aligned}
\mathbf{P}(\tau_k \geq n) &\leq \mathbf{P}(\tau_1 \geq n) + \sum_{p=1}^{n} \mathbf{P}(\tau_1 = p, \tau_k \geq n) \\
&\leq K_1 n^{-1/2} + K_{k-1} \widetilde{K} \sum_{p=1}^{n-1} p^{-3/2} (n-p)^{-1/2} \leq K_1 n^{-1/2} + 10 \widetilde{K} K_{k-1} n^{-1/2}.
\end{aligned}$$

Thus, there exists $C > 0$ such that for all $n \geq 1$ and $k \in \mathbb{N}$, $\mathbf{P}(\tau_k \geq n) \leq C(1+k) n^{-1/2}$, which ends the proof. $\square$

We now bound from above the probability for a random walk to make an excursion.

Lemma 1.3.2. *There exists $C > 0$ such that for all $p, q \in \mathbb{N}$, $x, h \geq 0$ and $y \in \mathbb{R}$, we have*

$$\begin{aligned}
\mathbf{P}_x(T_{p+q} \in [y+h, y+h+1], T_j \geq y \mathbf{1}_{\{j > p\}}, j \leq p + q) \\
&\leq C \frac{(1+x) \wedge p^{1/2}}{p^{1/2}} \frac{1}{\max(p,q)^{1/2}} \frac{(1+h) \wedge q^{1/2}}{q^{1/2}}.
\end{aligned}$$

Proof. We denote by $p' = \lfloor p/2 \rfloor$, $q' = \lfloor q/2 \rfloor$ and by $p'' = p - p'$, $q'' = q - q'$. By the Markov property applied at time p', we have

$$\begin{aligned}
\mathbf{P}_x(T_{p+q} \in [y+h, y+h+1], T_j \geq y \mathbf{1}_{\{p < j\}}, j \leq p + q) \\
\leq \mathbf{P}_x(T_j \geq 0, j \leq p') \sup_{z \geq -x} \mathbf{P}_z(T_{p''+q} \in [y+h, y+h+1], T_j \geq y, p'' < j \leq p'' + q).
\end{aligned}$$

We denote by $\widehat{T}_k = T_{p''+q} - T_{p''+q-k}$, which is a random walk with the same law as T. We observe that, for all $z \in \mathbb{R}$

$$\begin{aligned}
\mathbf{P}_z(T_{p''+q} \in [h, h+1], T_j \geq 0, p'' < j \leq p'' + q) \\
&\leq \mathbf{P}(\widehat{T}_{p''+q} \in [z+h, z+h+1], \widehat{T}_j \leq h+1, j \leq q) \\
&\leq \mathbf{P}_{-h-1}(T_{p''+q} \in [z-1, z], T_j \leq 0, j \leq q).
\end{aligned}$$

Applying again the Markov property at time q', we deduce that

$$\begin{aligned}
&\mathbf{P}_x(T_{p+q} \in [y+h, y+h+1], T_j \geq y\mathbf{1}_{\{p<j\leq p+q\}}, j \leq p+q) \\
&\qquad \leq \underbrace{\mathbf{P}_x(T_j \geq 0, j \leq p')}_{\leq C\frac{1+x}{p^{1/2}}\wedge 1} \underbrace{\mathbf{P}_{-h-1}(T_j \leq 0, j \leq q')}_{\leq C\frac{1+h}{q^{1/2}}\wedge 1} \sup_{z\in\mathbb{R}} \underbrace{\mathbf{P}(T_{p''+q''} \in [z, z+1])}_{\leq C\frac{1}{\max(p,q)^{1/2}}}
\end{aligned}$$

using Theorems I and III. □

These two lemmas can be mixed together to bound the probability for a random walk to make an excursion above some slowly moving curve.

Lemma 1.3.3. *For any $A \geq 0$, there exists $C > 0$ such that for all $y, h \geq 0$ and $n \geq 1$, we have*

$$\begin{aligned}
&\mathbf{P}_y(T_n + A\log n \in [h, h+1], T_j \geq -A\log\tfrac{n}{n-j+1}, j \leq n) \\
&\qquad\qquad \leq C\frac{(1+y)\wedge n^{1/2}}{n^{1/2}}\frac{1}{n^{1/2}}\frac{(1+h)\wedge n^{1/2}}{n^{1/2}}.
\end{aligned}$$

Proof. Let $A \geq 0$ and $n \in \mathbb{N}$, we write $p = \lfloor n/3 \rfloor$. Applying the Markov property at time p, we have

$$\begin{aligned}
&\mathbf{P}_y(T_n - A\log n \in [h, h+1], T_j \geq -A\log\tfrac{n}{n-j+1}, j \leq n) \\
\leq &\mathbf{P}_y(T_j \geq 0, j \leq p) \sup_{z\in\mathbb{R}} \mathbf{P}_z(T_{n-p} + A\log n \in [h, h+1], T_j \geq -A\log\tfrac{n}{n-p-j+1}, j \leq n-p) \\
\leq &\mathbf{P}_y(T_j \geq 0, j \leq p)\mathbf{P}_{-h-1}(T_j \leq A\log(j+1), j \leq p) \sup_{z\in\mathbb{R}} \mathbf{P}(T_{n-2p} \in [z, z+1]),
\end{aligned}$$

by time-reversal and the Markov property applied at time p. We then apply respectively Theorem IV, Lemma 1.3.1 and Theorem I, as well as the fact that probabilities are bounded by 1 to conclude the proof. □

The lower bound in Theorem II can be used to obtain a lower bound on the probability for a random walk to make an excursion.

Lemma 1.3.4. *For all $t \in (0, 1)$, there exists $c > 0$ such that for all $n \geq 1$ large enough, $x \in [0, n^{1/2}]$ and $y \in [-n^{1/2}, n^{1/2}]$ we have*

$$\mathbf{P}_x(T_n \leq y+1, T_j \geq y\mathbf{1}_{\{j>tn\}}, j \leq n) \geq c\frac{(1+x)}{n^{3/2}}.$$

Proof. Let $t \in (0, 1)$, $n \geq 1$, $x \in [0, n^{1/2}]$ and $y \in [-n^{1/2}, n^{1/2}]$, by the Markov property applied at time $p = \lfloor tn \rfloor$, we have

$$\begin{aligned}
&\mathbf{P}_x(T_n \leq y+1, T_j \geq y\mathbf{1}_{\{j>tn\}}, j \leq n) \\
\geq &\mathbf{P}_x(T_n \leq y+1, T_p \in [3n^{1/2}, 4n^{1/2}], T_j \geq y\mathbf{1}_{\{j>p\}}, j \leq n) \\
\geq &\mathbf{P}_x(T_p \in [2n^{1/2}, 3n^{1/2}], T_j \geq 0, j \leq p) \inf_{z\in[2n^{1/2}, 4n^{1/2}]} \mathbf{P}_z(T_{n-p} \leq y+1, T_j \geq y, j \leq n-p).
\end{aligned}$$

Observe for one part that

$$\begin{aligned}
&\mathbf{P}_x(T_p \in [2n^{1/2}, 3n^{1/2}], T_j \geq 0, j \leq p)\\
&\qquad = \mathbf{P}_x(T_j \geq 0, j \leq p)\mathbf{P}_x(T_p \in [2n^{1/2}, 3n^{1/2}] | T_j \geq 0, j \leq p)\\
&\qquad \geq c\frac{1+x}{n^{1/2}} \sum_{z=\lfloor 2n^{1/2}\rfloor}^{\lfloor 3n^{1/2}\rfloor} \mathbf{P}_x(T_p \in [z, z+1] | T_j \geq 0, j \leq p)\\
&\qquad \geq c(1+x)n^{-1/2},
\end{aligned}$$

applying Theorem III then Theorem II. On the other hand, for all $z \in [2n^{1/2}, 4n^{1/2}]$, by time-reversal

$$\begin{aligned}
&\mathbf{P}_z(T_{n-p} \leq y+1, T_j \geq y, j \leq n-p)\\
&\qquad \geq \mathbf{P}(\widehat{T}_{n-p} + z \in [y, y+1], \widehat{T}_j \leq 0, j \leq n-p)\\
&\qquad \geq \inf_{\widetilde{z} \in [n^{1/2}, 4n^{1/2}]} \mathbf{P}_z(T_{n-p} \in [\widetilde{z}, \widetilde{z}+1], T_j \leq 0, j \leq n-p) \geq cn^{-1}
\end{aligned}$$

using once again Theorems II and III. □

So far, we only considered boundaries that grows slower than $n^{1/2}$. Similar estimates can be obtained for the random walk when the barrier moves at linear speed, namely the Hsu-Robbins theorem. It is used to study branching random walks in the fast regime.

Theorem V (Hsu–Robbins [HR47])**.** *For all $\varepsilon > 0$, we have*

$$\sum_{n \geq 0} \mathbf{P}(T_n \leq -n\varepsilon) < +\infty.$$

By dominated convergence theorem, Theorem V implies

$$\lim_{z \to +\infty} \sum_{n \in \mathbb{N}} \mathbf{P}(T_n \leq -n\varepsilon - z) = 0. \tag{1.3.3}$$

1.3.2 Extension to enriched random walks

We extend some of the results obtained in the previous section to a random walk enriched by extra random variables, which are correlated to the last step of the random walk. Let $((X_n, \xi_n), n \geq 0)$ be an i.i.d. sequence of random variables taking values in $\mathbb{R}^2$, such that $\mathbf{E}(X_1) = 0$, $\mathbf{E}(X_1^2) < +\infty$ and $\mathbf{E}(\xi_1^2) < +\infty$. We set $T_n = T_0 + X_1 + \cdots + X_n$, where $\mathbf{P}_x(T_0 = x) = 1$. The process $(T_n, \xi_n, n \geq 0)$ is an useful toy-model for the study of the spine of the branching random walk, defined in Proposition 1.2.1; in effect the n^{th} step of the spine w_n is only correlated with the displacement of the siblings of w_n. We begin with a lemma similar to Theorem III.

Lemma 1.3.5. *We suppose that $\mathbf{E}(X_1) = 0$, $\mathbf{E}(X_1^2) < +\infty$ and $\mathbf{E}((\xi_1)_+^2) < +\infty$. There exists $C > 0$ that does not depend on the law of ξ_1 such that for all $n \in \mathbb{N}$ and $x \geq 0$, we have*

$$\mathbf{P}_x\left[T_j \geq 0, j \leq n, \exists k \leq n : T_k \leq \xi_k\right] \leq C\frac{1+x}{n^{1/2}}\left[\mathbf{P}(\xi_1 \geq 0) + \mathbf{E}((\xi_1)_+^2)\right].$$

Proof. Let $n \in \mathbb{N}$ and $x \geq 0$. We observe that

$$\mathbf{P}_x\left[T_j \geq 0, j \leq n, \exists k \leq n : T_k \leq \xi_k\right] \leq \sum_{k=1}^{n} \underbrace{\mathbf{P}_x\left[T_k \leq \xi_k, T_j \geq 0, j \leq n\right]}_{\pi_k}.$$

Applying the the Markov property at time k, we obtain

$$\pi_k \leq \mathbf{E}_x\left[\mathbf{1}_{\{T_k \leq \xi_k\}}\mathbf{1}_{\{T_j \geq 0, j \leq k\}}\mathbf{P}_{T_k}\left(T_j \geq 0, j \leq n-k\right)\right].$$

By use of Theorem III, for all $z \in \mathbb{R}$, we have

$$\mathbf{P}_z\left[T_j \geq 0, j \leq n-k\right] \leq C(1+z)(n-k+1)^{-1/2}\mathbf{1}_{\{z \geq 0\}}.$$

Thus, writing (X, ξ) for a copy of (X_1, ξ_1) independent of $(T_n, \xi_n, n \geq 0)$, we have

$$\begin{aligned}\pi_k &\leq C(n-k+1)^{-1/2}\,\mathbf{E}_x\left[\mathbf{1}_{\{\xi_k \geq 0\}}(1+\xi_k)\mathbf{1}_{\{T_k \leq \xi_k\}}\mathbf{1}_{\{T_j \geq 0, j \leq k\}}\right]\\ &\leq C(n-k+1)^{-1/2}\,\mathbf{E}_x\left[\mathbf{1}_{\{\xi \geq 0\}}(1+\xi_+)\mathbf{1}_{\{T_{k-1} \leq \xi_+ + X_-\}}\mathbf{1}_{\{T_j \geq 0, j \leq (k-1)\}}\right]\end{aligned}$$

We bound this quantity by conditioning on the value $\zeta = \xi_+ + X_- \geq 0$, we obtain

$$\mathbf{P}_x(T_k \leq \zeta, T_j \geq 0, j \leq k) \leq \begin{cases} C\frac{(1+x)(1+\zeta^2)}{(k+1)^{3/2}} & \text{if } \zeta^2 \leq k, \text{ by Lemma 1.3.2}\\ C\frac{1+x}{(k+1)^{1/2}} & \text{otherwise, by Theorem III}\end{cases}$$

Summing these estimates, we obtain

$$\begin{aligned}\sum_{k=0}^{\min(\zeta^2, n-1)} \mathbf{P}_x\left[T_k \leq \zeta, T_j \geq 0, j \leq k\right] \leq& C(1+x)\sum_{k=0}^{\min(\zeta^2, n-1)} \frac{1}{(n-k+1)^{1/2}(k+1)^{1/2}}\\ \leq& C(1+x)(1+\zeta)n^{-1/2},\end{aligned}$$

and

$$\begin{aligned}\sum_{k=\zeta^2}^{n-1} \mathbf{P}_x\left[T_k \leq \zeta, T_j \geq 0, j \leq k\right] \leq& C(1+x)(1+\zeta^2)\sum_{k=\zeta^2}^{n-1} \frac{1}{(k+1)^{3/2}(n-k+1)^{1/2}}\\ \leq& C(1+x)(1+\zeta)n^{-1/2},\end{aligned}$$

thus

$$\sum_{k=0}^{n-1} \frac{1}{(n-k+1)^{1/2}}\mathbf{P}_x\left[T_k \leq \zeta, T_j \geq 0, j \leq k\right] \leq C(1+x)(1+\zeta)n^{-1/2}.$$

We conclude that

$$\sum_{k=1}^{n} \pi_k \leq C\frac{1+x}{n^{1/2}}\,\mathbf{E}\left[\mathbf{1}_{\{\xi \geq 0\}}(1+X_- + \xi_+)(1+\xi_+)\right] \leq C\frac{1+x}{n^{1/2}}\,\mathbf{E}(X_-^2)\left[\mathbf{P}(\xi \geq 0) + \mathbf{E}(\xi_+^2)\right]$$

by the Cauchy-Schwarz inequality, which ends the proof. □

This lemma can be extended to prove an analogue of Lemma 1.3.2.

Lemma 1.3.6. *We assume that* $\mathbf{E}(X_1) = 0$, $\mathbf{E}(X_1^2) < +\infty$ *and* $\mathbf{E}((\xi_1)_+^2) < +\infty$. *For all* $t \in (0,1)$, *there exists* $C > 0$ *that does not depend on the law of* ξ_1, *such that for all* $n \in \mathbb{N}$, $x, h \geq 0$ *and* $y \in \mathbb{R}$, *we have*

$$\mathbf{P}_x\left[T_n - y - h \in [0,1], T_j \geq y\mathbf{1}_{\{j>tn\}}, j \leq n, \exists k \leq n : T_k \leq \xi_k + y\mathbf{1}_{\{k>tn\}}\right] \leq C\frac{(1+x)(1+h)}{n^{3/2}}\left[\mathbf{P}(\xi_1 \geq 0) + \mathbf{E}((\xi_1)_+^2)\right].$$

Proof. Let $n \in \mathbb{N}$, $x, h \geq 0$ and $y \in \mathbb{R}$. We denote by $\tau = \inf\{k \geq 0 : T_k \leq \xi_k + y\mathbf{1}_{\{k>tn\}}\}$ and by $p = \lfloor tn \rfloor$. We observe that

$$\begin{aligned}\mathbf{P}_x&\left[T_n - y - h \in [0,1], T_j \geq y\mathbf{1}_{\{j>tn\}}, j \leq n, \tau \leq n\right]\\ &\leq \mathbf{P}_x\left(T_n - y - h \in [0,1], T_j \geq y\mathbf{1}_{\{j>p\}}, \tau \leq p\right)\\ &\quad + \mathbf{P}_x\left(T_n - y - h \in [0,1], T_j \geq y\mathbf{1}_{\{j>p\}}, p < \tau \leq n\right). \end{aligned} \tag{1.3.4}$$

We first take interest in the event $\{\tau \leq p\}$. Applying the Markov property at time p, we obtain

$$\mathbf{P}_x\left[T_n - y - h \in [0,1], T_j \geq y\mathbf{1}_{\{j>p\}}, \tau \leq p\right] = \mathbf{E}_x\left[\mathbf{1}_{\{T_j \geq 0, j \leq p\}}\mathbf{1}_{\{\tau \leq p\}}\varphi(T_p)\right], \tag{1.3.5}$$

where $\varphi(z) = \mathbf{P}_z\left[T_{n-p} - y - h \in [0,1], T_j \geq y, j \leq n-p\right]$ for $z \in \mathbb{R}$. Using Lemma 1.3.3, we have $\sup_{z\in\mathbb{R}} \varphi(z) \leq C(1+h)n^{-1}$, which yields

$$\begin{aligned}\mathbf{P}_x&\left(T_n - y - h \in [0,1], T_j \geq y\mathbf{1}_{\{j>p\}}, \tau \leq p\right)\\ &\leq C(1+h)n^{-1}\mathbf{P}_x\left[T_j \geq 0, j \leq p, \exists k \leq p : T_k \leq \xi_k\right]\\ &\leq C(1+x)(1+h)n^{-3/2}\left[\mathbf{P}(\xi_1 > 0) + \mathbf{E}((\xi_1)_+^2)\right]\end{aligned}$$

by use of Lemma 1.3.5.

We now take care of $\{\tau > p\}$. We denote by $\widehat{T}_j = T_n - T_{n-j}$ and $\widehat{\xi}_j = \xi_{n-j}$. We observe that $(\widehat{T}_j, \widehat{\xi}_j)_{j\leq n}$ has the same law as $(T_j, \xi_j)_{j\leq n}$ under $\mathbf{P}_0$, and

$$\begin{aligned}\mathbf{P}_x&\left[T_n - y - h \in [0,1], T_j \geq y\mathbf{1}_{\{j>p\}}, j \leq n, p \leq \tau \leq n\right]\\ &\leq \mathbf{P}_x\left[\begin{array}{l} T_n - y - h \in [0,1], T_n - T_{n-j} \leq y + h + 1 - y\mathbf{1}_{\{n-j<p\}},\\ \exists k \leq n-p : T_{n-k} \leq \xi_{n-k} + y \end{array}\right]\\ &\leq \mathbf{P}_x\left[\begin{array}{l} T_n - T_0 - y - h + x \in [0,1], T_n - T_{n-j} \leq h + 1 + y\mathbf{1}_{\{j\geq n-p\}}\\ \exists k \leq n-p : T_n - T_{n-k} \geq y + h - (\xi_{n-k} + y) \end{array}\right],\end{aligned}$$

thus, in terms of $\widehat{T}$

$$\begin{aligned}\mathbf{P}_x&\left[T_n - y - h \in [0,1], T_j \geq y\mathbf{1}_{\{j>p\}}, j \leq n, p \leq \tau \leq n\right]\\ &\leq \mathbf{P}_x\left[\begin{array}{l} \widehat{T}_n - y - h + x \in [0,1], \widehat{T}_j \leq h + 1 - y\mathbf{1}_{\{j\geq n-p\}}\\ \exists k \leq n-p : \widehat{T}_k \geq y + h - (\widehat{\xi}_k + y) \end{array}\right].\end{aligned}$$

As a consequence, using the identity in law between (T, ξ) and $(\widehat{T}, \widehat{\xi})$, we have

$$\begin{aligned}&\mathbf{P}_x\left[T_n - y - h \in [0,1], T_j \geq y\mathbf{1}_{\{j>p\}}, j \leq n, p \leq \tau \leq n\right]\\ \leq&\mathbf{P}_0\left[T_n - h - y + x \in [0,1], T_j \leq h + 1 - y\mathbf{1}_{\{j\geq n-p\}}, \exists k \leq n-p : T_k \geq h - \xi_k\right]\\ \leq&\mathbf{P}_{h+1}\left[T_n - y + x \in [-1,0], T_j \leq -y\mathbf{1}_{\{j\leq n-p\}}, \exists k \leq n-p : T_k \geq -\xi_k - 1\right].\end{aligned}$$

This quantity is bounded in (1.3.5) for the random walk $(-T, \xi)$, where the roles of x and h have been exchanged, thus similar computations give

$$\mathbf{P}_x \left[T_n - y - h \in [0,1], T_j \geq y\mathbf{1}_{\{j>p\}}, j \leq n, p < \tau \leq n\right] \leq C\frac{(1+x)(1+h)}{(n+1)^{3/2}} \left[\mathbf{P}(\xi \geq 0) + \mathbf{E}(\xi_+^2)\right]. \quad (1.3.6)$$

which ends the proof. □

We use Lemmas 1.3.5 and 1.3.6 to control at the same time the position of the spine and the number of its children.

1.3.3 Random walk through an interface

We extend the above results to time-inhomogeneous random walk. Let $(X_k^{(1)}, k \geq 1)$ and $(X_k^{(2)}, k \geq 1)$ be two independent sequences of i.i.d. random variables, centred with finite variance. For $1 \leq p < n \in \mathbb{N}$ and $k \leq n$, we write

$$S_k^{p,n} = \sum_{j=1}^{k\wedge p} X_j^{(1)} + \sum_{j=1}^{(k-p)_+} X_j^{(2)}$$

a random walk of length n through an interface. We first bound the probability that such a random walk makes an excursion.

Lemma 1.3.7. *There exists $C > 0$ such that for all $1 \leq p < n$ and $x, h \geq 0$ and $y \in \mathbb{R}$, we have*

$$\mathbf{P}\left(S_n^{p,n} \in [y+h, y+h+1], S_j^{p,n} \geq -x\mathbf{1}_{\{k\leq p\}} + y\mathbf{1}_{\{k>p\}}\right) \leq C\frac{(1+x)\wedge p^{1/2}}{p^{1/2}} \frac{1}{\max(p, n-p)^{1/2}} \frac{(1+h)\wedge(n-p)^{1/2}}{(n-p)^{1/2}}.$$

Proof. We write $S_j^{(1)} = \sum_{k=1}^{j} X_k^{(1)}$ and $S_j^{(2)} = \sum_{k=1}^{j} X_k^{(2)}$. Informally, $S^{p,n}$ is the concatenation of $S^{(1)}$ and $S^{(2)}$. Applying the Markov property at time p, we have immediately

$$\begin{aligned}
&\mathbf{P}\left(S_n^{p,n} \in [y+h, y+h+1], S_j^{p,n} \geq -x\mathbf{1}_{\{k\leq p\}} + y\mathbf{1}_{\{k>p\}}\right)\\
&\quad\leq \mathbf{P}\left(S_j^{(1)} \geq -x, j \leq p\right) \sup_{z\in\mathbb{R}} \mathbf{P}\left(S_{n-p}^{(2)} + z \in [y+h, y+h+1], S_j^{(2)} \geq y, j \leq n-p\right)\\
&\quad\leq \left(C\frac{(1+x)}{p^{1/2}} \wedge 1\right)\left(C\frac{(1+h)\wedge(n-p)^{1/2}}{(n-p)}\right),
\end{aligned}$$

by Theorem I and Lemma 1.3.2.

Moreover, if we write $\widehat{S}_j^{p,n} = \widehat{S}_n^{p,n} - \widehat{S}_{n-j}^{p,n}$, then $\widehat{S}^{p,n}$ is also a time-homogeneous random walk, with $n-p$ steps with same law as $X_1^{(2)}$ and p steps with same law as $S_1^{(1)}$. Moreover

$$\begin{aligned}
&\mathbf{P}\left(S_n^{p,n} \in [y+h, y+h+1], S_j^{p,n} \geq -x\mathbf{1}_{\{k\leq p\}} + y\mathbf{1}_{\{k>p\}}\right)\\
&\quad\leq \mathbf{P}\left(\widehat{S}_n^{p,n} \in [y+h, y+h+1], \widehat{S}_j^{p,n} \leq (h+1)\mathbf{1}_{\{k\leq n-p\}} + (y+x+h+1)\mathbf{1}_{\{k>n-p\}}\right)\\
&\quad\leq C\frac{(1+h)\wedge(n-p)^{1/2}}{(n-p)^{1/2}} \frac{(1+x)\wedge p^{1/2}}{p},
\end{aligned}$$

using the same arguments as above. Choosing the smallest upper bound concludes the proof. □

1.4 Bounding the tail of the maximal displacement

Let $(\mathbf{T}, V)$ be a BRWi of length n, and M_n be its maximal displacement at time n. We compute in this section a tight estimate for $\mathbf{P}(M_n \geq m_n + y)$ as y and $n \to +\infty$, for a suitable choice of m_n, in each of the three regimes.

1.4.1 Slow regime : strong boundary effects

We assume here that all the hypotheses of Theorem 1.1.2 are fulfilled. We write $p = \lfloor tn \rfloor$ and

$$m_n = pv_1 + (n-p)v_2 - \left(\frac{3}{2\theta_1} + \frac{3}{2\theta_2}\right)\log n \quad \text{and} \quad m_n^1 = pv_1 - \frac{3}{2\theta_1}\log n.$$

In these conditions, the rightmost individual at time n descends from one of the rightmost individuals at time tn. As a consequence, the bounds on the tail of M_n is a straightforward consequence of Aïdékon's Theorem [Aïd13] on the asymptotic behaviour of the maximal displacement in the branching random walk. We recall here a corollary of this result, of which we provide an independent proof in Chapter 7.

Theorem 1.4.1. *Let $(\mathbf{T}^h, V^h)$ be a time-homogeneous branching random walk. If there exists $\theta^* > 0$ such that $\kappa(\theta^*) := \mathbf{E}\left[\sum_{|u|=1} e^{\theta^* V^h(u)}\right] < +\infty$ and*

$$\theta^* \, \mathbf{E}\left[\sum_{|u|=1} V^h(u) e^{\theta^* V^h(u) - \kappa(\theta^*)}\right] - \kappa(\theta^*) = 0,$$

and if

$$\mathbf{E}\left[\sum_{|u|=1} V^h(u)^2 e^{\theta^* V^h(u)}\right] + \mathbf{E}\left[\left(\sum_{|u|=1} e^{\theta^* V^h(u)}\right) \log_+ \left(\sum_{|u|=1} V^h(u) e^{\theta^* V^h(u)}\right)^2\right] < +\infty$$

then there exist $c, C > 0$ such that for all $n \in \mathbb{N}$ and $y \in [0, n^{1/2}]$,

$$c(1+y)e^{-\theta^* y} \leq \mathbf{P}\left(\max_{|u|=n} V^h(u) \geq n\tfrac{\kappa(\theta^*)}{\theta^*} - \tfrac{3}{2\theta^*}\log n + y\right) \leq C(1+y)e^{-\theta^* y}.$$

The main result of the section is the following.

Lemma 1.4.2. *Under the assumptions of Theorem 1.1.2, there exist $c, C > 0$ such that for any $n \geq 1$ and $y \in [0, n^{1/2}]$, we have*

$$c(1+y)e^{-\theta_1 y} \leq \mathbf{P}(M_n \geq m_n + y) \leq C(1+y)e^{-\theta_1 y}.$$

Proof. We observe that the $p = \lfloor tn \rfloor$ first steps of the branching random walk $(\mathbf{T}, V)$ are similar to the p first steps of a branching random walk with reproduction law $\mathcal{L}_1$. Moreover, $\mathcal{L}_1$ satisfies all hypotheses of Theorem 1.4.1, therefore there exist $c, C > 0$ such that for all $n \geq 1$ and $y \in \mathbb{R}$,

$$c(1+y)e^{-\theta_1 y} \leq \mathbf{P}(\max_{|u|=p} V(u) \geq m_n^1 + y) \leq C(1+y)e^{-\theta_1 y}. \tag{1.4.1}$$

To obtain the lower bound of this lemma, we observe that the rightmost individual at time p starts from its position an independent branching random walk with reproduction law $\mathcal{L}_2$. Using again Theorem 1.4.1, there exists $r > 0$ such that with probability at least r, its rightmost descendant at time n makes a displacement of at least $(1-t)v_2 n - \frac{3}{2\theta_2}\log n$. As the maximal position at time n is greater than the position of the rightmost descendant of the rightmost individual alive at time n, we have

$$\mathbf{P}(M_n \geq m_n + y) \geq r\mathbf{P}(\max_{|u|=p} V(u) \geq m_n^1 + y) \geq c(1+y)e^{-\theta_1 y},$$

by the lower bound in (1.4.1).

We now take care of the upper bound. We write, for $k \leq p$, $f_k^{(n)} = kv_1 - \frac{3}{2\theta_1}\log\frac{p}{p-k+1}$. Note that with high probability, no individual crosses the boundary $f^{(n)}$ at any time $k \leq p$. By the Markov inequality and Lemma 1.2.2, we have

$$\begin{aligned}
&\mathbf{P}\left[\exists |u| \leq p : V(u) \geq f_{|u|}^{(n)} + y\right]\\
&\qquad \leq \sum_{k=1}^{p} \mathbf{E}\left[\sum_{|u|=k} \mathbf{1}_{\left\{V(u)\geq f_k^{(n)}+y\right\}} \mathbf{1}_{\left\{V(u_j)\leq f_j^{(n)}+y, j<k\right\}}\right]\\
&\qquad \leq \sum_{k=1}^{p} \mathbf{E}\left[e^{-\theta_1 S_k + k\kappa_1(\theta_1)} \mathbf{1}_{\left\{S_k \geq f_k^{(n)}+y, S_j\leq f_j^{(n)}+y, j<k\right\}}\right]\\
&\qquad \leq \sum_{k=1}^{p} \frac{p^{3/2}}{(p-k+1)^{3/2}} \mathbf{P}\left(S_k \geq f_k^{(n)} + y, S_j \leq f_j^{(n)} + y, j<k\right).
\end{aligned}$$

For all $k \leq p$, conditioning with respect to $S_k - S_{k-1}$, we have

$$\mathbf{P}\left(S_k \geq f_k^{(n)} + y, S_j \leq f_j^{(n)} + y, j < k\right) \leq \mathbf{E}\left[\varphi_k(S_k - S_{k-1})\right],$$

where

$$\begin{aligned}
\varphi_k(x) &= \mathbf{1}_{\{x\geq 0\}} \mathbf{P}\left(S_{k-1} \geq f_k^{(n)} + y - x, S_j \leq f_j^{(n)} + y, j \leq k-1\right)\\
&\leq \sum_{a=0}^{\lfloor x \rfloor} \mathbf{P}\left(S_{k-1} - f_{k-1}^{(n)} - y \in [-a-1, -a], S_j \leq f_j^{(n)} + y, j \leq k-1\right)\\
&\leq C\frac{(1+x)_+^2(1+y)}{(k+1)^{3/2}},
\end{aligned}$$

by Lemma 1.3.3. As $\mathbf{E}((S_k - S_{k-1})^2) = \mathbf{E}(\Sigma_1(\theta_1)) < +\infty$, we have

$$\mathbf{P}\left[\exists |u| \leq p : V(u) \geq f_{|u|}^{(n)} + y\right] \leq C(1+y)e^{-\theta_1 y}. \tag{1.4.2}$$

We now decompose the population alive at time n according to the position of its ancestor at time p. For $y, h \geq 0$, we set

$$X_n(y,h) = \sum_{|u|=p} \mathbf{1}_{\{V(u)-m_n^1-y\in[-h-1,-h]\}} \mathbf{1}_{\left\{V(u_j)\leq f_j^{(n)}+y, j\leq p\right\}}$$

the number of individuals who, staying below $f^{(n)}$ are at time p close to $m_n^1 + y - h$. By the many-to-one lemma, we have

$$\begin{aligned}
\mathbf{E}(X_n(y,h)) &= \mathbf{E}\left[e^{-\theta_1 S_p - p\kappa_1(\theta_1)} \mathbf{1}_{\{S_p - m_n^1 - y \in[-h-1,-h]\}} \mathbf{1}_{\left\{S_j \leq f_j^{(n)}+y, j\leq p\right\}}\right]\\
&\leq Cp^{3/2} e^{\theta_1(h-y)} \mathbf{P}\left(S_p - m_n^1 - y \in [-h-1, -h], S_j \leq f_j^{(n)} + y, j \leq p\right)\\
&\leq C(1+y)(1+h)e^{-\theta_1 y} e^{\theta_1 h}
\end{aligned}$$

by Lemma 1.3.3. Each one of the individuals counted in $X_n(y,h)$ starts an independent branching random walk with reproduction law $\mathcal{L}_2$. By Theorem 1.4.1, the probability that one of the descendants of an individual positioned at time p to the left of $m_n + y - h$, is at time n above $m_n + y$ is bounded from above by $C(1+h)e^{-\theta_2 h}$. As a consequence,

$$\begin{aligned}
\mathbf{P}(M_n \geq m_n + y) &\leq \mathbf{P}(\exists |u| \leq p : V(u) \geq f^{(n)}_{|u|} + y) + \sum_{h=0}^{+\infty} \mathbf{E}(X_n(y,h)) C(1+h)e^{-\theta_2 h}\\
&\leq C(1+y)e^{-\theta_1 y}\left(1 + \sum_{h=0}^{+\infty} (1+h)^2 e^{(\theta_1-\theta_2)h}\right)\\
&\leq C(1+y)e^{-\theta_1 y}
\end{aligned}$$

using (1.4.2) and the fact that $\theta_2 > \theta_1$. □

1.4.2 Mean regime : a classical branching random walk estimate

We now consider a BRWi $(\mathbf{T}, V)$ of length n, such that the hypotheses of Theorem 1.1.3 are fulfilled. We obtain in this section an asymptotic similar to the one of a time-homogeneous branching random walk. We write $\theta := \theta_1 = \theta_2$, $p = \lfloor nt \rfloor$ and

$$m_n = nv_* - \frac{3}{2\theta}\log n \quad \text{and} \quad f^{(n)}_k = (k \wedge p)v_1 + (k-p)_+ v_2 - \frac{3}{2\theta}\log\frac{n}{n-k+1}.$$

We prove in a first time a boundary estimate.

Lemma 1.4.3. *Under the assumptions of Theorem 1.1.3, there exists $C > 0$ such that for all $n \in \mathbb{N}$ and $y \geq 0$,*

$$\mathbf{P}(\exists |u| \leq n : V(u) \geq f^{(n)}_{|u|} + y) \leq C(1+y)e^{-\theta y}.$$

Proof. Let $n \in \mathbb{N}$ and $y \geq 0$, we write

$$Z_n(y) = \sum_{|u| \leq n} \mathbf{1}_{\left\{V(u) \geq f^{(n)}_{|u|} + y\right\}} \mathbf{1}_{\left\{V(u_j) < f^{(n)}_j + y, j < |u|\right\}}.$$

Applying the many-to-one lemma, we have

$$\begin{aligned}
\mathbf{E}\left[Z_n(y)\right] &= \sum_{k=1}^{n} \mathbf{E}\left[\sum_{|u|=k} \mathbf{1}_{\left\{V(u) \geq f^{(n)}_k + y\right\}} \mathbf{1}_{\left\{V(u_j) \leq f^{(n)}_j + y, j < k\right\}}\right]\\
&= \mathbf{E}\left[e^{-\theta S_k + (k\wedge p)\kappa_1(\theta) + (k-p)_+\kappa_2(\theta)} \mathbf{1}_{\left\{S_k \geq f^{(n)}_k + y\right\}} \mathbf{1}_{\left\{S_j < f^{(n)}_j + y, j < k\right\}}\right],
\end{aligned}$$

where $(S_k, k \leq n)$ is a time-inhomogeneous random walk, with step distribution verifying

$$\mathbf{E}\left[f(S_k - S_{k-1})\right] = \mathbf{E}\left[\sum_{\ell \in L_1} f(\ell) e^{\theta\ell - \kappa_1(\theta)}\right] \quad \text{for } k \leq p,$$

$$\text{and } \mathbf{E}\left[f(S_k - S_{k-1})\right] = \mathbf{E}\left[\sum_{\ell \in L_2} f(\ell) e^{\theta\ell - \kappa_2(\theta)}\right] \quad \text{for } k > p.$$

As $\kappa_1(\theta) = \theta v_1$ and $\kappa_2(\theta) = \theta v_2$, for any $k \leq n$

$$
\begin{aligned}
&\mathbf{E}\left[e^{-\theta S_k + k\kappa_1(\theta)} \mathbf{1}_{\left\{S_k \geq f_k^{(n)} + y\right\}} \mathbf{1}_{\left\{S_j < f_j^{(n)} + y, j<k\right\}}\right] \\
\leq & \frac{n^{3/2}}{(n-k+1)^{3/2}} e^{-\theta y} \mathbf{P}\left(S_k \geq f_k^{(n)} + y, S_j \leq f_j^{(n)} + y, j < k\right) \\
\leq & \frac{n^{3/2}}{(n-k+1)^{3/2}} e^{-\theta y}\, \mathbf{E}\left[\varphi_{k-1}(S_k - S_{k-1})\right]
\end{aligned}
$$

where we write $\varphi_k(x) = \mathbf{P}\left(S_k \geq f_{k+1}^{(n)} + y - x, S_j \leq f_j^{(n)} + y, j \leq k\right)$ by conditioning with respect to the last step of the random walk. Observe that if $k \leq p$, by Lemma 1.3.3 we have $\varphi_k(x) \leq C(1+y)(1+x)^2 k^{-3/2}$.

If $k > p$, writing $p' = \lfloor p/2 \rfloor$, we have

$$
\begin{aligned}
&\varphi_k(x) \\
\leq & \mathbf{P}\left(S_j \leq jv + y, j \leq p'\right) \sup_{z \in \mathbb{R}} \mathbf{P}_{p',z}\left(S_{k-p'} \geq f_{k+1}^{(n)} + y - x, S_j \leq f_{j+p'}^{(n)} + y, j \leq k - p'\right) \\
\leq & C\frac{1+y}{n^{1/2}} \sup_{z \in \mathbb{R}} \mathbf{P}_{p',z}\left(S_{k-p'} \geq f_{k+1}^{(n)} + y - x, S_j \leq f_{j+p'}^{(n)} + y, j \leq k - p'\right)
\end{aligned}
$$

by Lemma 1.3.1. Writing $\widehat{S}_j = S_k - S_{k-j}$, we have

$$
\begin{aligned}
&\mathbf{P}_{p',z}\left(S_{k-p'} \geq f_{k+1}^{(n)} + y - x, S_j \leq f_{p'+j}^{(n)} + y, j \leq k - p'\right) \\
\leq & \mathbf{P}\left(\widehat{S}_{k-p'} - f_{k+1}^{(n)} - y + z \in [-x, 0], \widehat{S}_j \geq f_{k+1}^{(n)} - f_{k-j}^{(n)} - x, j \leq k - p'\right) \\
\leq & \mathbf{P}\left(\widehat{S}_j \geq jv_2 - \tfrac{3}{2\theta}\log(n-k+j) - x, j \leq k - p\right) \sup_{z \in \mathbb{R}} \mathbf{P}\left(S_{p-p'} \in [z, z+x+C]\right) \\
\leq & C\frac{1+x}{(k-p)^{1/2}} C\frac{1+x}{n^{1/2}}
\end{aligned}
$$

by Lemma 1.3.1 and Theorem I.

As $\mathbf{E}(\Sigma_1) + \mathbf{E}(\Sigma_2) < +\infty$, we have $\mathbf{E}(S_1^2) + \mathbf{E}((S_{p+1} - S_p)^2) < +\infty$, thus

$$
\begin{aligned}
&\mathbf{E}\left[e^{-\theta S_k + k\kappa_1(\theta)} \mathbf{1}_{\left\{S_k \geq f_k^{(n)} + y\right\}} \mathbf{1}_{\left\{S_j < f_j^{(n)} + y, j<k\right\}}\right] \\
&\qquad \leq \begin{cases} C\frac{n^{3/2}}{(k+1)^{3/2}(n-k+1)^{3/2}}(1+y)e^{-\theta y}, k \leq p \\ C\frac{n^{1/2}}{(k-p)^{1/2}(n-k+1)^{3/2}}(1+y)e^{-\theta y}, k > p \end{cases}
\end{aligned}
$$

summing these estimates for $k \leq n$ gives

$$
\begin{aligned}
\mathbf{E}\left[Z_n(y)\right] &\leq C(1+y)e^{-\theta y}\left[\sum_{k=1}^{p} \frac{n^{3/2}}{(n-k+1)^{3/2} k^{3/2}} + \sum_{k=p+1}^{n} \frac{n^{3/2}}{n(k-p)^{1/2}(n-k+1)^{3/2}}\right] \\
&\leq C(1+y)e^{-\theta y}\left[\sum_{k=1}^{p} \frac{1}{k^{3/2}} \sum_{k=1}^{n-p} \frac{n^{1/2}}{k^{1/2}(n-p-k+1)^{3/2}}\right] \\
&\leq C(1+y)e^{-\theta y}.
\end{aligned}
$$

□

This lemma is enough to obtain an upper bound on $\mathbf{P}(M_n \geq m_n + y)$. To bound this quantity from below, we use a second moment argument. For all $n \in \mathbb{N}$ and $k \leq n$, we write

$$g_k^{(n)} = (k \wedge p)v_1 + (k-p)_+ v_2 - \mathbf{1}_{\{k>p\}} \frac{3}{2\theta} \log n + 1$$

a boundary close to $f^{(n)}$ but simpler to use. We prove in this section that the set

$$\mathcal{A}_n(y) = \left\{ u \in \mathbf{T}, |u| = n : V(u) \geq m_n + y, V(u_j) \leq g_j^{(n)} + y, j \leq n \right\}$$

is non-empty with high probability. To do so, we restrict ourselves to a subset of individuals which do not have too many children. We write

$$\mathcal{B}_n(z) = \left\{ u \in \mathbf{T}, |u| = n : \xi(u_j) \leq e^{\frac{\theta}{2}(V(u_j) - g_j^{(n)}) + z}, j < n \right\},$$

where $\xi(u) = \sum_{u' \in \Omega(u)} (1 + (V(u') - V(u))_+) e^{-\theta(V(u') - V(u))}$.

We set $G_n(y,z) = \mathcal{A}_n(y) \cap \mathcal{B}_n(z)$, and we compute now the first two moments of $Y_n(y,z) = \sum_{|u|=n} \mathbf{1}_{\{u \in G_n(y,z)\}}$.

Lemma 1.4.4. *Under the assumptions of Theorem 1.1.3, there exists $C > 0$ such that for all $n \in \mathbb{N}$, $y \geq 0$ and $z \geq 0$, we have*

$$\mathbf{E}\left[Y_n(y,z)^2 \right] \leq C e^z (1+y) e^{-\theta y}$$

Proof. Let $p = \lfloor tn \rfloor$, applying Proposition 1.2.1, we have

$$\begin{aligned}
\mathbf{E}(Y_n(y,z)^2) = \overline{\mathbf{E}}\left[\frac{1}{W_n} Y_n(y,z)^2 \right] &= \widehat{\mathbf{E}}\left[\frac{1}{W_n} \sum_{|u|=n} \mathbf{1}_{\{u \in G_n(y,z)\}} Y_n(y,z) \right] \\
&= \widehat{\mathbf{E}}\left[e^{-\theta V(w_n) + p\kappa_1(\theta) + (n-p)\kappa_2(\theta)} \mathbf{1}_{\{w_n \in G_n(y,z)\}} Y_n(y,z) \right].
\end{aligned}$$

Using the fact that $w_n \in \mathcal{A}_n(y) \subset G_n(y,z)$, we have

$$\mathbf{E}(Y_n(y,z)^2) \leq C n^{3/2} e^{-\theta y} \widehat{\mathbf{E}}\left[Y_n(y,z) \mathbf{1}_{\{w_n \in G_n(y,z)\}} \right].$$

We decompose $Y_n(y,z)$ along the spine, to obtain

$$Y_n(y,z) \leq \mathbf{1}_{\{w_n \in G_n(y,z)\}} + \sum_{k=0}^{n-1} \sum_{u \in \Omega(w_k)} Y_n(u,y),$$

where, for $u \in \mathbf{T}$ and $y \geq 0$, we write $Y_n(u,y) = \sum_{|u'|=n, u'>u} \mathbf{1}_{\{u' \in \mathcal{A}_n(y)\}}$.

We recall that under the law $\widehat{\mathbf{P}}$, for all $k \leq n$, the branching random walk of the children of an individual $u \in \Omega(w_k)$ has law $\mathbf{P}_{V(u),k+1}$. As a consequence, for $y \geq 0$, $k < n$ and $u \in \Omega(w_k)$,

$$\widehat{\mathbf{E}}\left[Y_n(u,y) | \mathcal{G}_n \right] = \mathbf{E}_{V(u),k+1}\left[\sum_{|u'|=n-k-1} \mathbf{1}_{\{V(u') \geq m_n + y\}} \mathbf{1}_{\left\{ V(u'_j) \leq g_{k+j+1}^{(n)} + y, j \leq n-k \right\}} \right].$$

As a consequence, by Lemma 1.2.2, we have

$$\begin{aligned}
\widehat{\mathbf{E}}\left[Y_n(u,y) | \mathcal{G}_n \right] \leq\ & C n^{3/2} e^{-\theta y} e^{\theta V(u) - (k+1 \wedge p)\kappa_1(\theta) - (k+1-p)_+ \kappa_2(\theta)} \\
& \times \mathbf{P}_{V(u),k+1}\left(S_{n-k-1} \geq m_n + y, S_j \leq g_{j+k+1}^{(n)} + y, j \leq n-k-1 \right).
\end{aligned}$$

For $k \leq p-1$ and $x \in \mathbb{R}$, applying the Markov property at time p, we have

$$\begin{aligned}
&\mathbf{P}_{x,k+1}\left(S_{n-k-1} \geq m_n + y, S_j \leq g^{(n)}_{j+k+1} + y, j \leq n-k-1\right)\\
\leq &\mathbf{P}_{x,k+1}\left(S_j \leq (k+j+1)v_1 + y + 1, j \leq p-(k+1)\right)\\
&\qquad \times \sup_{h\in\mathbb{R}} \mathbf{P}_{h,p}\left(S_{n-p} \geq (n-p)v_2 + y, S_j \leq jv_2 + y + 1, j \leq n-p\right)\\
\leq &C\frac{1+(x-g^{(n)}_{k+1}-y)_+}{(p-k)^{1/2}}\frac{1}{n}
\end{aligned}$$

by Lemma 1.3.7. Similarly, for all $k \geq p$ and $x \in \mathbb{R}$, we have

$$\mathbf{P}_{x,k+1}\left(S_{n-k-1} \geq m_n + y, S_j \leq g^{(n)}_{j+k+1} + y, j \leq n-k-1\right) \leq C\frac{1+(x-g^{(n)}_{k+1}-y)_+}{(n-k+1)^{3/2}},$$

using Lemma 1.3.2.

For all $k \leq n$, we write

$$\begin{aligned}
h_k :=\ &\widehat{\mathbf{E}}\left[\sum_{u\in\Omega(w_k)}\left(1+(V(u)-g^{(n)}_{k+1}-y)_+\right)e^{\theta(V(u)-g^{(n)}_{k+1})}\mathbf{1}_{\{w_n\in G_n(y,z)\}}\right]\\
\leq\ &\widehat{\mathbf{E}}\left[\xi(w_k)\left(1+\left(V(w_k)-g^{(n)}_{k+1}-y)_+\right)\right)e^{\theta(V(w_k)-g^{(n)}_{k+1})}\mathbf{1}_{\{w_n\in G_n(y,z)\}}\right]\\
\leq\ &e^z\widehat{\mathbf{E}}\left[\left(1+\left(V(w_k)-g^{(n)}_{k+1}-y\right)\right)e^{\frac{\theta}{2}(V(w_k)-g^{(n)}_{k+1})}\mathbf{1}_{\{w_n\in\mathcal{A}_n(y)\}}\right]
\end{aligned}$$

Decomposing this expectation with respect to the value taken by $V(w_k)$, we have

$$h_k \leq Ce^z e^{\theta y/2}\sum_{h=0}^{+\infty}(1+h)e^{-\theta h/2}\mathbf{P}\left[\begin{array}{l} S_n \geq m_n + y, S_k - g^{(n)}_{k+1} - y \in [-h-1,-h]\\ S_j \leq g^{(n)}_j + y, j \leq n\end{array}\right].$$

We apply the Markov property at time k to obtain

$$\begin{aligned}
&\mathbf{P}\left[S_n \geq m_n + y, S_k - g^{(n)}_{k+1} - y \in [-h-1,-h], S_j \leq g^{(n)}_j + y, j \leq n\right]\\
&\qquad\leq \mathbf{P}\left[S_k - g^{(n)}_{k+1} - y \in [-h-1,-h], S_j \leq g^{(n)}_j + y, j \leq k\right]\\
&\qquad\qquad\times \inf_{x\in[-h-1,-h]}\mathbf{P}_x\left[S_{n-k} \geq m_n - g^{(n)}_k, S_j \leq g^{(n)}_{k+j} - g^{(n)}_{k+1}, j \leq n-k\right].
\end{aligned}$$

If $k \leq p$ applying Lemma 1.3.2, we have

$$\mathbf{P}\left[S_k - g^{(n)}_{k+1} - y \in [-h-1,-h], S_j \leq g^{(n)}_j + y, j \leq k\right] \leq C\frac{(1+y)(1+\mathbf{1}_{\{k=p\}}\log n + h)}{(k+1)^{3/2}}$$

and applying Lemma 1.3.7,

$$\begin{aligned}
&\inf_{x\in[-h-1,-h]}\mathbf{P}_x\left[S_{n-k} \geq m_n - g^{(n)}_k, S_j \leq g^{(n)}_{k+j} - g^{(n)}_{k+1}, j \leq n-k\right]\\
&\qquad\qquad\qquad\leq C\frac{(1+\mathbf{1}_{\{k=p\}}\log n + h)}{(p-k+1)^{1/2}n}.
\end{aligned}$$

By similar arguments, if $k > p$ we have

$$\mathbf{P}\left[S_k - g^{(n)}_{k+1} - y \in [-h-1,-h], S_j \leq g^{(n)}_j + y, j \leq k\right] \leq C\frac{(1+y)(1+h)}{n(k-p+1)^{1/2}}$$

as well as

$$\inf_{x\in[-h-1,-h]} \mathbf{P}_x \left[S_{n-k} \geq m_n - g_k^{(n)}, S_j \leq g_{k+j}^{(n)} - g_{k+1}^{(n)}, j \leq n-k \right] \leq C \frac{(1+h)}{(n-k+1)^{3/2}}.$$

Therefore, if $k \leq p$, we have

$$\begin{aligned} h_k &\leq C(1+y)e^{\theta y + z} \sum_{h=0}^{+\infty} \frac{(1+h)(1+h+\mathbf{1}_{\{k=p\}}\log n)^2 e^{-\theta h/2}}{(k+1)^{3/2}(p+1-k)^{1/2}(n+1)} \\ &\leq Cz \frac{(1+y)e^{\theta y}}{n(k-p)^{1/2}(n-k+1)^{3/2}}. \end{aligned}$$

In the same way, if $k > p$,

$$\begin{aligned} h_k &\leq Ce^z e^{\theta y} \sum_{h=0}^{+\infty} \frac{(1+y)(1+h)^3 e^{-\theta h/2}}{n(k-p+1)^{1/2}(n-k+1)^{3/2}} \\ &\leq Cz \frac{(1+y)e^{\theta y}}{n(k-p)^{1/2}(n-k+1)^{3/2}}. \end{aligned}$$

As a consequence, for $k < p$, we have,

$$\begin{aligned} \widehat{\mathbf{E}} \left[\mathbf{1}_{\{w_n \in G_n(y,z)\}} \sum_{u \in \Omega(w_k)} Y_n(u,y) \right] &\leq C h_k e^{-\theta y} \frac{n^{3/2} e^{\theta g_{k+1}^{(n)} - (k+1)\kappa_1(\theta)}}{n(p+1-k)^{1/2}} \\ &\leq Ce^z(1+y) \frac{n^{3/2}}{(k+1)^{3/2} n^2 (p-k+1)}. \end{aligned}$$

Summing these estimates for $k < p$, we have

$$\begin{aligned} &\sum_{k=0}^{p-1} \widehat{\mathbf{E}} \left[\mathbf{1}_{\{w_n \in G_n(y,z)\}} \sum_{u \in \Omega(w_k)} Y_n(u,y) \right] \\ &\qquad \leq Cz(1+y) \left[n^{-3/2} \sum_{k=0}^{\lfloor p/2 \rfloor} (k+1)^{-3/2} + n^{-5/2} \sum_{k=\lfloor p/2 \rfloor + 1}^{p-1} (p-k+1)^{-1} \right] \\ &\qquad \leq Cz(1+y) n^{-3/2}. \end{aligned} \tag{1.4.3}$$

Similarly, if $k > p$, we have

$$\begin{aligned} \widehat{\mathbf{E}} \left[\mathbf{1}_{\{w_n \in G_n(y,z)\}} \sum_{u \in \Omega(w_k)} Y_n(u,y) \right] &\leq C h_k e^{-\theta y} \frac{n^{3/2} e^{\theta g_{k+1}^{(n)} - p\kappa_1(\theta) - (k+1-p)\kappa_2(\theta)}}{(n-k+1)^{3/2}} \\ &\leq Cz(1+y) \frac{1}{n(k-p)^{1/2}(n-k+1)^3}. \end{aligned}$$

Summing these estimates, we have once again

$$\sum_{k=p+1}^{n} \widehat{\mathbf{E}} \left[\mathbf{1}_{\{w_n \in G_n(y,z)\}} \sum_{u \in \Omega(w_k)} Y_n(u,y) \right] \leq Cz(1+y) n^{-3/2}. \tag{1.4.4}$$

And for $k = p$, we have

$$\widehat{\mathbf{E}} \left[\mathbf{1}_{\{w_n \in G_n(y,z)\}} \sum_{u \in \Omega(w_p)} Y_n(u,y) \right] \leq Cz(1+y) \frac{(\log n)^2}{n^3}. \tag{1.4.5}$$

We note that

$$\begin{aligned}
\widehat{\mathbf{P}}(w_n \in \mathcal{A}_n(y)) &\leq \mathbf{P}(S_n \geq m_n + y, S_j \leq g_j^{(n)} + y, j \leq n) \\
&\leq \mathbf{P}(S_j \leq v_1 j + y, j \leq p) \sup_{x \in \mathbb{R}} \mathbf{P}_{p,x}(S_n \geq v_2(n-p) + y, S_j \leq v_2 j + y + 1, j \leq n-p)
\end{aligned}$$

By Lemmas 1.3.1 and 1.3.2, this yields

$$\widehat{\mathbf{P}}(w_n \in G_n(y,z)) \leq C \frac{1+y}{n^{3/2}}. \tag{1.4.6}$$

We conclude that

$$\begin{aligned}
\mathbf{E}(Y_n(y,z)^2) &\leq C n^{3/2} e^{-\theta y} \widehat{\mathbf{E}}\left[Y_n(y,z) \mathbf{1}_{\{w_n \in G_n(y,z)\}} \right] \\
&\leq C n^{3/2} e^{-\theta y} \widehat{\mathbf{E}}\left[\left(\mathbf{1}_{\{w_n \in G_n(y,z)\}} + \sum_{k=0}^{n-1} \sum_{u \in \Omega(w_k)} Y_n(u,y) \right) \mathbf{1}_{\{w_n \in G_n(y,z)\}} \right] \\
&\leq C n^{3/2} e^{-\theta y} \left[\widehat{\mathbf{P}}(w_n \in G_n(y,z)) + \sum_{k=0}^{n-1} \widehat{\mathbf{E}}\left[\mathbf{1}_{\{w_n \in G_n(y,z)\}} \sum_{u \in \Omega(w_k)} Y_n(u,y) \right] \right] \\
&\leq C(1+y) e^{z - \theta y},
\end{aligned}$$

by (1.4.3), (1.4.4), (1.4.5) and (1.4.6). □

Lemma 1.4.5. *Under the assumptions of Theorem 1.1.3, there exist $c > 0$ and $z > 0$ such that for all $n \in \mathbb{N}$ and $y \in [0, n^{1/2}]$, we have*

$$\mathbf{E}\left[Y_n(y,z) \right] \geq c(1+y) e^{-\theta y}.$$

Proof. Let $n \geq 1$ and $y \in [0, n^{1/2}]$, applying the Proposition (1.2.1), we have

$$\begin{aligned}
\mathbf{E}(Y_n(y,z)) = \overline{\mathbf{E}}\left[\frac{1}{W_n} \sum_{|u|=n} \mathbf{1}_{\{u \in G_n(y,z)\}} \right] &= \widehat{\mathbf{E}}\left[e^{V(w_n) - p\kappa_1(\theta) - (n-p)\kappa_2(\theta)} \mathbf{1}_{\{w_n \in G_n(y,z)\}} \right] \\
&\geq n^{3/2} e^{-\theta^*(y+1)} \widehat{\mathbf{P}}(w_n \in G_n(y,z)).
\end{aligned}$$

To bound $\widehat{\mathbf{P}}(w_n \in G_n(y,z))$, we observe that

$$\widehat{\mathbf{P}}(w_n \in \mathcal{A}_n(y) \cap \mathcal{B}_n(z)) = \widehat{\mathbf{P}}(w_n \in \mathcal{A}_n(y)) - \widehat{\mathbf{P}}(w_n \in \mathcal{A}_n(y) \cap \mathcal{B}_n(z)^c).$$

We bound $\widehat{\mathbf{P}}(w_n \in \mathcal{A}_n(y))$. Applying the Markov property at time p, we have

$$\begin{aligned}
\widehat{\mathbf{P}}(w_n \in \mathcal{A}_n(y)) &= \mathbf{P}(S_n \geq m_n + y, S_j \leq g_j^{(n)} + y, j \leq n) \\
&\geq \mathbf{P}(S_p - pv_1 - y \in [3n^{1/2}, 5n^{1/2}], S_j \geq jv_1 + y, j \leq p) \\
&\quad \times \inf_{x \in [3n^{1/2}, 5n^{1/2}]} \mathbf{P}_{p,x}(S_{n-p} \geq m_n - pv_1, S_j \leq g_{p+j}^{(n)} - pv_1, j \leq n-p).
\end{aligned}$$

By Theorems II and III, we have easily

$$\begin{aligned}
&\mathbf{P}(S_p - pv_1 - y \in [3n^{1/2}, 5n^{1/2}], S_j \geq jv_1 + y, j \leq p) \\
&\qquad \geq \mathbf{P}(S_p - pv_1 \in [4n^{1/2}, 5n^{1/2}], S_j \geq jv_1 + y) \geq c \frac{1+y}{n^{1/2}}.
\end{aligned}$$

Moreover, setting $\widehat{S}_j = S_n - S_{n-j}$ and $q = \lfloor (n-p)/2 \rfloor$, we have

$$\begin{aligned}
&\inf_{x \in [3n^{1/2}, 5n^{1/2}]} \mathbf{P}_{p,x}(S_{n-p} \geq m_n - pv_1, S_j \leq g^{(n)}_{p+j} - pv_1, j \leq n-p) \\
&\quad \geq \inf_{x \in [3n^{1/2}, 5n^{1/2}]} \mathbf{P}(\widehat{S}_{n-p} - (n-p)v_2 + \tfrac{3}{2\theta} \log n \in [-x-1, -x], \widehat{S}_j \geq jv_2, j \leq n-p) \\
&\quad \geq \mathbf{P}(\widehat{S}_q \in [n^{1/2}, 2n^{1/2}], \widehat{S}_j \geq jv_2, j \leq q) \inf_{x \in [0, 7n^{1/2}]} \mathbf{P}(\widehat{S}_{n-p-q} \in [-x-1, -x]) \\
&\quad \geq \frac{1}{n^{1/2}} \times \frac{1}{n^{1/2}}
\end{aligned}$$

using Theorems II and III for one part, and Theorem I for the other part. We conclude that

$$\widehat{\mathbf{P}}(w_n \in \mathcal{A}_n(y)) \geq c \frac{1+y}{n^{3/2}}. \tag{1.4.7}$$

We now compute an upper bound of $\widehat{\mathbf{P}}(w_n \in \mathcal{A}_n(y) \cap \mathcal{B}_n(z)^c)$. We observe that

$$\begin{aligned}
\widehat{\mathbf{P}}(w_n \in \mathcal{A}_n(y) \cap \mathcal{B}_n(z)^c) &= \widehat{\mathbf{P}} \begin{pmatrix} V(w_n) \geq m_n + y, V(w_j) \leq g^{(n)}_j + y, j \leq n, \\ \exists k \leq n : \xi(w_k) \geq e^z e^{\frac{\theta}{2}(V(w_k) - g^{(n)}_k)} \end{pmatrix} \\
&\leq \pi_1 + \pi_2
\end{aligned}$$

where we write

$$\pi_1 = \widehat{\mathbf{P}} \begin{pmatrix} V(w_n) \geq m_n + y, V(w_j) \leq g^{(n)}_j + y, j \leq n, \\ \exists k \leq p : \xi(w_k) \geq e^z e^{\frac{\theta}{2}(V(w_k) - g^{(n)}_k)} \end{pmatrix} \quad \text{and}$$

$$\pi_2 = \widehat{\mathbf{P}} \begin{pmatrix} V(w_n) \geq m_n + y, V(w_j) \leq g^{(n)}_j + y, j \leq n, \\ \exists p < k \leq n : \xi(w_k) \geq e^z e^{\frac{\theta}{2}(V(w_k) - g^{(n)}_k)} \end{pmatrix}.$$

To bound π_1, applying the Markov property at time p, we have

$$\begin{aligned}
\pi_1 &\leq \widehat{\mathbf{P}} \left(V(w_j) \leq jv_1 + y, j \leq p, \exists k \leq p : \tfrac{2}{\theta} (\log \xi(w_k) - z) \geq V(w_k) - kv_1 \right) \\
&\qquad \times \sup_{x \in \mathbb{R}} \mathbf{P}_{p,x}(S_{n-p} \geq m_n + y, S_j \leq g^{(n)}_j + y, j \leq n-p) \\
&\leq C \frac{1}{n} \widehat{\mathbf{P}} \left(V(w_j) \leq jv_1 + y, j \leq p, \exists k \leq p : \tfrac{2}{\theta} (\log \xi(w_k) - z) \geq V(w_k) - kv_1 - y \right)
\end{aligned}$$

by Lemma 1.3.2. Using Lemma 1.3.5, we obtain

$$\pi_1 \leq C \frac{1+y}{n^{3/2}} \left[\widehat{\mathbf{P}} \left(\log \xi(w_0) - z \geq 0 \right) + \widehat{\mathbf{E}} \left((\log \xi(w_0) - z)_+^2 \right) \right].$$

As $\mathbf{E}(X_p (\log X_p)^2) < +\infty$, we have $\widehat{\mathbf{E}}\left((\log \xi(w_0))_+^2 \right)$, thus by dominated convergence theorem

$$\lim_{z \to +\infty} \sup_{n \in \mathbb{N}, y \geq 0} \frac{n^{3/2}}{1+y} \pi_1 = 0.$$

Similarly, for π_2, we apply the Markov property at time p again, we have

$$\pi_2 \leq \widehat{\mathbf{E}} \left(\mathbf{1}_{\{V(w_j) \leq jv_1 + y, j \leq p\}} \varphi(V(w_p)) \right)$$

where, for $x \geq 0$,

$$\varphi(x) \leq \widehat{\mathbf{P}}_{p,x} \begin{pmatrix} V(w_{n-p}) \geq m_n + y, V(w_j) \leq g^{(n)}_{p+j} + y, j \leq n-p, \\ \exists k \leq n-p : \frac{2}{\theta}\left(\log \xi(w_k) - z\right) \geq V(w_k) - g^{(n)}_{p+k} - y \end{pmatrix}$$
$$\leq C \frac{1 + (pv_1 + y - x)_+}{n^{3/2}} \left[\widehat{\mathbf{P}}(\log \xi(w_{p+1}) - z \geq 0) + \widehat{\mathbf{E}}\left((\log \xi(w_{p+1}) - z)_+^2 \right) \right]$$

by Lemma 1.3.6. As a consequence,

$$\pi_2 \leq \sum_{a=0}^{+\infty} \widehat{\mathbf{E}}\left(\mathbf{1}_{\left\{V(w_p) - pv_1 - y \in [-(a+1)n^{1/2}, -an^{1/2}]\right\}} \mathbf{1}_{\{V(w_j) \leq jv_1 + y, j \leq p\}} \varphi(V(w_p)) \right)$$
$$\leq C \left[\widehat{\mathbf{P}}(\log \xi(w_{p+1}) - z \geq 0) + \widehat{\mathbf{E}}\left((\log \xi(w_{p+1}) - z)_+^2 \right) \right]$$
$$\times \sum_{a=0}^{+\infty} \frac{1 + (a+2)n^{1/2}}{n^{3/2}} \mathbf{P}\left(S_p - pv_1 - y \in [-(a+1)n^{1/2}, -an^{1/2}], S_j \leq jv_1 + y, j \leq p \right)$$
$$\leq C \left[\widehat{\mathbf{P}}(\log \xi(w_{p+1}) - z \geq 0) + \widehat{\mathbf{E}}\left((\log \xi(w_{p+1}) - z)_+^2 \right) \right] \sum_{a=0}^{+\infty} \frac{(1+a)(1+y)}{n^{3/2}} e^{-ca^2}$$

applying Theorem II and Lemma 1.3.1. As $\mathbf{E}(X_2 (\log X_2)^2) < +\infty$, we have

$$\lim_{z \to +\infty} \sup_{n \in \mathbb{N}, y \geq 0} \frac{n^{3/2}}{1+y} \pi_2 = 0.$$

We conclude that for any $\varepsilon > 0$, there exists $z \geq 1$ such that for all $n \geq 1$ and $y \in [0, n^{1/2}]$, we have

$$\widehat{\mathbf{P}}(w_n \in \mathcal{A}_n(y) \cap \mathcal{B}_n(z)^c) \leq \varepsilon(1+y)n^{-3/2}.$$

We conclude there exists $z \geq 1$ large enough and $c > 0$ such that for any $y \geq 0$,

$$\mathbf{E}(Y_n(y,z)) \geq c(1+y)e^{-\theta y}.$$

□

Using these three lemmas, we obtain the following bound on the tail of the maximal displacement.

Lemma 1.4.6. *There exist $c, C > 0$ such that for all $n \geq 1$ and $y \in [0, n^{1/2}]$ we have*

$$c(1+y)e^{-\theta y} \leq \mathbf{P}(M_n \geq m_n + y) \leq C(1+y)e^{-\theta y}.$$

Proof. The upper bound is a direct consequence of Lemma 1.4.3, as

$$\mathbf{P}(M_n \geq m_n + y) \leq \mathbf{P}(\exists |u| \leq n : V(u) \geq f^{(n)}_{|u|} + y) \leq C(1+y)e^{-\theta y}.$$

For the lower bound, we use the Cauchy-Schwarz estimate, we fix $z > 0$ such that there exists $c > 0$ verifying

$$\mathbf{E}(Y_n(y,z)) \geq c(1+y)e^{-\theta y}$$

which exists by Lemma 1.4.5 Using Lemma 1.4.4, we have

$$\mathbf{E}(Y_n(y,z)^2) \leq C(1+y)e^{-\theta y}.$$

As a consequence, by the Cauchy-Schwarz inequality, we have

$$\mathbf{P}(M_n \geq m_n + y) \geq \mathbf{P}(Y_n(y,z) > 0) \geq \frac{\mathbf{E}(Y_n(y,z))^2}{\mathbf{E}(Y_n(y,z)^2)} \geq c(1+y)e^{-\theta y}$$

which ends the proof. □

1.4.3 Fast regime : large deviations of a random walk

We assume in this section that all the hypotheses of Theorem 1.1.5 are true. We write

$$m_n = nv_* - \frac{1}{2\theta} \log n.$$

In these conditions, the rightmost individual at time n descend from one individual at distance $O(n)$ from the rightmost position at time tn. Thus the boundary computations are of no use to obtain this asymptotic, the upper bound is easy to obtain, as the branching structure does not play a role in this asymptotic. Let

$$a_1 = \mathbf{E}\left[\sum_{\ell \in L_1} \ell e^{\theta\ell - \kappa_1(\theta)}\right] \quad \text{and} \quad a_2 = \mathbf{E}\left[\sum_{\ell \in L_2} \ell e^{\theta\ell - \kappa_2(\theta)}\right]$$

the slopes followed in each stage by the individual reaching the highest position at time n.

Lemma 1.4.7. *Under the assumptions of Theorem 1.1.5, there exists $C > 0$ such that for all $n \in \mathbb{N}$ and $y \geq 0$,*

$$\mathbf{P}(M_n \geq m_n + y) \leq Ce^{-\theta y}.$$

Proof. We write $p = \lfloor tn \rfloor$. For any $y \geq 0$, we write

$$X_n(y) = \sum_{|u|=n} \mathbf{1}_{\{V(u) \in [m_n+y, m_n+y+1]\}}.$$

By Lemma 1.2.2, we have

$$\begin{aligned}\mathbf{E}(X_n(y)) &= \mathbf{E}\left[e^{-\theta S_n + p\kappa_1(\theta) + (n-p)\kappa_2(\theta)} \mathbf{1}_{\{S_n \in [m_n+y, m_n+y+1]\}}\right]\\ &\leq Ce^{-\theta y} n^{1/2} \mathbf{P}\left(S_n \in [m_n + y, m_n + y + 1]\right).\end{aligned}$$

Applying the Markov property at time p, we have

$$\mathbf{P}\left(S_n \in [m_n + y, m_n + y + 1]\right) \leq \sup_{z \in \mathbb{R}} \mathbf{P}(S^2_{n-p} \in [z, z+1]) \leq Cn^{-1/2},$$

applying Stone's local limit theorem, where S^2 is a random walk with step distribution verifying

$$\mathbf{E}\left[\sum_{\ell \in L_2} f(\ell) e^{\theta\ell - \kappa_2(\theta)}\right] = \mathbf{E}\left[f(S^2_k - S^2_{k-1})\right].$$

As a consequence,

$$\mathbf{P}(M_n \geq m_n + y) \leq \mathbf{E}(X_n(y)) \leq Ce^{-\theta y}.$$

□

To obtain a lower bound, we apply once again second moment techniques. Let $\delta > 0$ such that

$$\theta(a_1 + \delta) - \kappa_1(\theta) > \delta \quad \text{and} \quad \theta(a_2 + 2\delta) - \kappa_2(\theta) < -2\delta,$$

which exists by (1.1.10). We write $r^{(n)}_k = \delta k \mathbf{1}_{\{k \leq p\}} + \delta(n-k)\mathbf{1}_{\{k>p\}}$, and introduce an upper boundary for the set of individuals we consider

$$g^{(n)}_k = a_1(k \wedge p) + a_2(k-p)_+ + r^{(n)}_k - \mathbf{1}_{\{k>p\}} \tfrac{1}{2\theta} \log n + 1.$$

For $y \geq 0$, we set

$$\mathcal{A}_n(y) = \{u \in \mathbf{T}, |u| = n : V(u) - m_n \in [y, y+1], j \leq n\}$$

and

$$\mathcal{B}_n(y,z) = \left\{u \in \mathbf{T}, |u| = n : \xi(u_j) \leq e^{z + r_j^{(n)}}, V(u_j) \leq g_j^{(n)} + y + z, j \leq n\right\}.$$

We write $G_n(y,z) = \mathcal{A}_n(y) \cap \mathcal{B}_n(y,z)$, and $Y_n(y,z) = \sum_{|u|=n} \mathbf{1}_{\{u \in G_n(y,z)\}}$. We compute the first two moments of $Y_n(y,z)$ in the following lemma.

Lemma 1.4.8. *There exists $C > 0$ such that for all $n \in \mathbb{N}$, $y \geq 0$ and $z \geq 0$, we have*

$$\mathbf{E}\left[Y_n(y,z)^2\right] \leq Ce^{2z - \theta y}$$

Moreover, there exist $c > 0$ and $z \geq 0$ such that for all $n \in \mathbb{N}$ and $y \in [0, n^{1/2}]$, we have

$$\mathbf{E}\left[Y_n(y,z)\right] \geq ce^{-\theta y}.$$

Proof. Let $n \in \mathbb{N}$ and $y \geq 0$, we write $p = \lfloor tn \rfloor$, and compute the second moment of $Y_n(y,z)$. By Proposition 1.2.1, we have

$$\begin{aligned}
\mathbf{E}(Y_n(y,z)^2) &= \overline{\mathbf{E}}\left[\frac{1}{W_n} Y_n(y,z)^2\right] = \widehat{\mathbf{E}}\left[\frac{1}{W_n} \sum_{|u|=n} \mathbf{1}_{\{u \in G_n(y,z)\}} Y_n(y,z)\right] \\
&= \widehat{\mathbf{E}}\left[e^{-\theta V(w_n) + p\kappa_1(\theta) + (n-p)\kappa_2(\theta)} \mathbf{1}_{\{w_n \in G_n(y,z)\}} Y_n(y,z)\right].
\end{aligned}$$

Using the fact that $w_n \in \mathcal{A}_n(y) \subset G_n(y,z)$, we have

$$\begin{aligned}
\mathbf{E}(Y_n(y,z)^2) &\leq Cn^{1/2} e^{-\theta y} \widehat{\mathbf{E}}\left[Y_n(y,z) \mathbf{1}_{\{w_n \in G_n(y,z)\}}\right] \\
&\leq Cn^{1/2} e^{-\theta y}\left[\widehat{\mathbf{P}}(w_n \in G_n(y,z)) + \sum_{k=0}^{n-1} \widehat{\mathbf{E}}\left(\mathbf{1}_{\{w_n \in G_n(y,z)\}} \sum_{u \in \Omega(w_k)} Y_n(u,y)\right)\right]
\end{aligned}$$

where, for $u \in \mathbf{T}$ and $y \geq 0$, we write $Y_n(u,y) = \sum_{|u'|=n, u'>u} \mathbf{1}_{\{u' \in \mathcal{A}_n(y)\}}$. We recall that under the law $\widehat{\mathbf{P}}$, the branching random walk of the children of an individual $u \in \Omega(w_k)$ has law $\mathbf{P}_{V(u),k+1}$. As a consequence, for $y \geq 0$, $k < n$ and $u \in \Omega(w_k)$,

$$\widehat{\mathbf{E}}\left[Y_n(u,y) | \mathcal{G}_n\right] = \mathbf{E}_{V(u),k+1}\left[\sum_{|u'|=n-k-1} \mathbf{1}_{\{V(u') \geq m_n + y\}} \mathbf{1}_{\left\{V(u'_j) \leq g_{k+j+1}^{(n)} + y, j \leq n-k\right\}}\right].$$

As a consequence, by Lemma 1.2.2, we have

$$\begin{aligned}
\widehat{\mathbf{E}}\left[Y_n(u,y) | \mathcal{G}_n\right] \leq{}& Cn^{1/2} e^{-\theta y} e^{\theta V(u) - (k+1 \wedge p)\kappa_1(\theta) - (k+1-p)_+ \kappa_2(\theta)} \\
&\times \mathbf{P}_{V(u),k+1}\left(S_{n-k-1} \geq m_n + y, S_j \leq g_{j+k+1}^{(n)} + y, j \leq n-k-1\right).
\end{aligned}$$

Using the Markov property at time p if $k+1 \leq p$, for any $k < n$ we have

$$\begin{aligned}
&\mathbf{P}_{V(u),k+1}\left(S_{n-k-1} \geq m_n + y, S_j \leq g_{j+k+1}^{(n)} + y, j \leq n-k-1\right) \\
&\qquad \leq \sup_{z \in \mathbb{R}} \mathbf{P}_{0,(k+1)\vee p}\left(S_{n-k-1} \in [z, z+1]\right) \leq C(n-k+1)^{-1/2}
\end{aligned}$$

by Theorem I.

For any $k \leq n$, we compute the quantity

$$h_k := \widehat{\mathbf{E}}\left[\mathbf{1}_{\{w_n \in G_n(y,z)\}} \sum_{u \in \Omega(w_k)} e^{\theta(V(u)-g_k^{(n)})}\right].$$

By definition of $\xi(w_k)$ we have

$$\begin{aligned}
h_k &\leq C\widehat{\mathbf{E}}\left[\mathbf{1}_{\{w_n \in G_n(y,z)\}} e^{\theta(V(w_k)-g_k^{(n)})}\xi(w_k)\right]\\
&\leq Ce^{2z+r_k^{(n)}} e^{\theta y}\widehat{\mathbf{P}}\left[w_n \in \mathcal{A}_n(y)\right]\\
&\leq Ce^{2z+r_k^{(n)}} e^{\theta y}\mathbf{P}(S_n - m_n \in [y, y+1]) \leq Ce^{z+\delta r_k^{(n)}} n^{-1/2},
\end{aligned}$$

by Theorem I. We finally observe that

$$\begin{aligned}
&\mathbf{E}\left[Y_n(y,z)^2\right]\\
\leq& Cn^{1/2}e^{-\theta y}\left[\widehat{\mathbf{P}}(w_n \in \mathcal{A}_n(y)) + \sum_{k=0}^{n-1} C\widehat{\mathbf{E}}\left(\mathbf{1}_{\{w_n \in G_n(y,z)\}} \sum_{u \in \Omega(w_k)} Y_n(u,y)\right)\right]\\
\leq& Cn^{1/2}e^{-\theta y}\left[Cn^{-1/2} + Cn^{1/2}e^{-\theta y}\sum_{k=0}^{n-1} h_k \frac{e^{\theta(g_k^{(n)} - ((k+1)\wedge p)\kappa_1(\theta) - (k+1-p)_+\kappa_2(\theta))}}{(n-k+1)^{1/2}}\right]\\
\leq& Ce^{-\theta y}\left[1 + \sum_{k=0}^{p} e^{2z+(\delta-\kappa_1^*(a_1))k} + \sum_{k=p+1}^{n} \frac{e^{2z-p\kappa_1^*(a_1)+\delta(n-k)+(k-p)\kappa_2^*(a_2)}}{(n-k+1)^{1/2}}\right]\\
\leq& Ce^{2z-\theta y},
\end{aligned}$$

as $\sup_{n\in\mathbb{N}} \left|p\kappa_1^*(a_1) + (n-p)\kappa_2^*(a_2)\right| < +\infty$.

We now bound the first moment of $Y_n(y,z)$. For any $y \in [0, n^{1/2}]$, by the spinal decomposition again, we have

$$\begin{aligned}
\mathbf{E}\left[Y_n(y,z)\right] &= \widehat{\mathbf{E}}\left[e^{-\theta V(w_n)+p\kappa_1(\theta)+(n-p)\kappa_2(\theta)}\mathbf{1}_{\{w_n \in G_n(y,z)\}}\right]\\
&\geq n^{1/2}e^{-\theta(y+1)}\left[\widehat{\mathbf{P}}(w_n \in \mathcal{A}_n(y)) - \widehat{\mathbf{P}}(w_n \in \mathcal{A}_n(y) \cap \mathcal{B}_n(y,z)^c)\right].
\end{aligned}$$

We observe that $\widehat{\mathbf{P}}(w_n \in \mathcal{A}_n(y)) \geq cn^{-1/2}$ by Theorem I, moreover

$$\begin{aligned}
\widehat{\mathbf{P}}(w_n \in \mathcal{A}_n(y) \cap \mathcal{B}_n(y,z)^c) \leq& \sum_{k=0}^{n} \widehat{\mathbf{P}}(V(w_n) - m_n \in [y,y+1], V(w_k) \geq g_k^{(n)} + y)\\
&+ \sum_{k=0}^{n} \widehat{\mathbf{P}}(V(w_n) - m_n \in [y,y+1], (\log \xi(w_k) - z) \geq r_k^{(n)}).
\end{aligned}$$

For any $k \leq p$, applying the Markov property at time p and Theorem I again, we have

$$\widehat{\mathbf{P}}(V(w_n) - m_n \in [y,y+1], V(w_k) \geq g_k^{(n)} + y) \leq \frac{C}{n^{1/2}}\mathbf{P}(S_k - ka_1 \geq \delta k + z)$$

$$\widehat{\mathbf{P}}(V(w_n) - m_n \in [y,y+1], (\log \xi(w_k) - z) \geq \delta k) \leq \frac{C}{n^{1/2}}\widehat{\mathbf{P}}\left(\tfrac{\log \xi(w_0) - z}{\delta} \geq k\right),$$

therefore

$$\sum_{k=0}^{p} \widehat{\mathbf{P}}(V(w_n) - m_n \in [y, y+1], V(w_k) \geq g_k^{(n)} + y)$$
$$+ \sum_{k=0}^{p} \widehat{\mathbf{P}}(V(w_n) - m_n \in [y, y+1], (\log \xi(w_k) - z) \geq r_k^{(n)})$$
$$\leq \frac{C}{n^{1/2}} \left(\widehat{\mathbf{E}} \left[\frac{(\log \xi(w_0) - z)_+}{\delta} \right] + \sum_{k=0}^{p} \mathbf{P}(S_k \geq ka_1 + z) \right].$$

Similarly, for $k > p$, we write $\widehat{S}_j = V(w_n) - V(w_{n-j})$ and $\widehat{\xi}_j = \xi(w_{n-j})$. Applying time-reversal we have

$$\begin{aligned}
\widehat{\mathbf{P}}(V(w_n) - m_n \in [y, y+1], V(w_k) \geq g_k^{(n)} + y)&\\
&\leq \widehat{\mathbf{P}}(\widehat{S}_n - m_n \in [y, y+1], \widehat{S}_{n-k} \geq (n-k)(a_2 - \delta) - z - 1)\\
&\leq \frac{C}{n^{1/2}} \mathbf{P}(\widehat{S}_{n-k} - (n-k)ka_2 \geq -\delta(n-k) - z - 1)
\end{aligned}$$

as well as

$$\begin{aligned}
\widehat{\mathbf{P}}(V(w_n) - m_n \in [y, y+1], (\log \xi(w_k) - z) \geq \delta k)&\\
&\leq \widehat{\mathbf{P}} \left(\widehat{S}_n - m_n \in [y, y+1], \tfrac{\log \widehat{\xi}(w_{n-k}) - z}{\delta} \geq n - k \right)\\
&\leq \frac{C}{n^{1/2}} \widehat{\mathbf{P}} \left(\tfrac{\log \xi(w_{p+1}) - z}{\delta} \geq \delta(n-k) \right).
\end{aligned}$$

Consequently

$$\sum_{k=p+1}^{n} \widehat{\mathbf{P}}(V(w_n) - m_n \in [y, y+1], V(w_k) \geq g_k^{(n)} + y)$$
$$+ \sum_{k=p+1}^{n} \widehat{\mathbf{P}}(V(w_n) - m_n \in [y, y+1], (\log \xi(w_k) - z) \geq r_k^{(n)})$$
$$\leq \frac{C}{n^{1/2}} \left(\widehat{\mathbf{E}} \left[\frac{(\log \xi(w_{p+1}) - z)_+}{\delta} \right] + \sum_{k=0}^{n-p} \mathbf{P}(\widehat{S}_k \geq ka_1 + z) \right].$$

As $\mathbf{E}(X_1 \log X_1) + \mathbf{E}(X_2 \log X_2) < +\infty$, using Theorem V, and more precisely (1.3.3), we have

$$\lim_{z \to +\infty} \sup_{n \in \mathbb{N}} n^{1/2} \widehat{\mathbf{P}}(w_n \in \mathcal{A}_n(y) \cap \mathcal{B}_n(y, z)^c) = 0.$$

Consequently, for any z large enough we have $\mathbf{P}(w_n \in G_n(y, z)) \geq cn^{-1/2}$, which leads to

$$\mathbf{E}[Y_n(y, z)] \geq ce^{-y}$$

for all $n \geq 1$ and $y \in [0, n^{1/2}]$, provided that z is large enough. □

Using these two lemmas, we obtain the following asymptotic tail for the maximal displacement at time n.

Lemma 1.4.9. *There exist $c, C > 0$ such that for all $n \in \mathbb{N}$ and $y \in [0, n^{1/2}]$ we have*

$$ce^{-\theta y} \leq \mathbf{P}(M_n \geq m_n + y) \leq Ce^{-\theta y}.$$

Proof. Upper bound is obtained in Lemma 1.4.7. We apply Lemma 1.4.8 to obtain a lower bound, fixing $z > 0$ such that $\mathbf{E}(Y_n(y,z)) \geq ce^{-\theta y}$. We know that $\mathbf{E}(Y_n(y,z)^2) \leq Ce^{-\theta y}$, and apply the Cauchy-Schwarz inequality again

$$\mathbf{P}(M_n \geq m_n + y) \geq \mathbf{P}(Y_n(y,z) > 0) \geq \frac{\mathbf{E}\left[Y_n(y,z)\right]^2}{\mathbf{E}\left[Y_n(y,z)^2\right]} \geq \frac{c^2 e^{-2\theta y}}{Ce^{-\theta y}} \geq ce^{-\theta y}.$$

□

1.5 From tail estimates to the tension

The aim of this section is to obtain the tension of $M_n - m_n$, using the tail estimates obtained in the previous section. The main tool, similar for the three regimes, is the application of a cutting argument. We use the fact that the size of the population in the branching random walk alive at time k grows at exponential rate, as in a Galton-Watson process. Each one of the individuals start an independent branching random walk, and has positive probability to make a descendent to the right of m_n at time $n \gg k$, which is enough to conclude to the tension of $M_n - m_n$.

We start this section by recalling the definition of a Galton-Watson process. Let μ be a law on $\mathbb{Z}_+$, and $(X_{k,n}, (k,n) \in \mathbb{N}^2)$ an i.i.d. array of random variables with law μ. The process $(Z_n, n \in \mathbb{N})$ defined by recurrence by

$$Z_0 = 1 \quad \text{and} \quad Z_{n+1} = \sum_{k=1}^{Z_n} X_{k,n+1}$$

is a Galton-Watson process with reproduction law μ. The quantity Z_n represents the size of the population at time n, and $X_{n,k}$ the number of children of the k^{th} individual alive at time n. Galton-Watson processes have been extensively studied since their introduction by Galton and Watson in 1874 [1]. The results we use in this section can all been found in [AN04].

We write

$$f : \begin{array}{rcl} [0,1] & \to & [0,1] \\ s & \mapsto & \mathbf{E}\left[s^{X_{1,1}}\right] = \sum_{k=0}^{+\infty} \mu(k) s^k. \end{array}$$

We observe that for all $n \in \mathbb{N}$, $\mathbf{E}\left(s^{Z_n}\right) = f^n(s)$, where f^n is the n^{th} iterate of f. Moreover, if $m := \mathbf{E}(X_{1,1}) < +\infty$, then f is a $\mathcal{C}^1$ strictly increasing convex function on $[0,1]$ that verifies

$$f(0) = \mu(0), \quad f(1) = 1 \quad \text{and} \quad f'(1) = m.$$

We write q the smallest solution of the equation $f(q) = q$, it is a well-known fact that q is the probability of extinction of the Galton-Watson process, i.e. $\mathbf{P}(\exists n \in \mathbb{N} : Z_n = 0) = q$. Observe in particular that $q < 1$ if and only if $m > 1$. If $m > 1$, we also introduce $\alpha := -\frac{\log f'(q)}{\log m} \in (0, +\infty]$.

Lemma 1.5.1. *Let $(Z_n, n \geq 0)$ be a Galton-Watson process with reproduction law μ. We write $b = \min\{k \in \mathbb{Z}_+ : \mu(k) > 0\}$, $m = \mathbf{E}(Z_1) \in (1, +\infty)$ and q the smallest solution of*

1. Independently from the seminal work of Bienaymé, who also introduced and studied such a process.

the equation $\mathbf{E}(q^{Z_1}) = q$. *There exists* $C > 0$ *such that for all* $z \in (0,1)$ *and* $n \in \mathbb{N}$ *we have*

$$\mathbf{P}(Z_n \leq zm^n) \leq \begin{cases} q + Cz^{\frac{\alpha}{\alpha+1}} & \text{if } b = 0 \\ Cz^{\alpha} & \text{if } b = 1 \\ \exp\left[-Cz^{-\frac{\log b}{\log m - \log b}}\right] & \text{if } b \geq 2. \end{cases}$$

A more precise computation of the left tail of the Galton-Watson process can be found in [FW07].

Proof. We write $s_0 = \frac{q+1}{2}$, and for all $k \in \mathbb{Z}$, $s_k = f^k(s_0)$ –where negative iterations are understood as iterations of f^{-1}. Using the properties of f, there exists $C_- > 0$ such that $1 - s_k \sim_{k \to -\infty} C_- m^k$. Moreover, if $\mu(0) + \mu(1) > 0$, there exists $C_+ > 0$ such that $s_k - q \sim_{k \to +\infty} C_+ f'(q)^k$. Otherwise,

$$s_k = f^{(b)}(0)^{\frac{b^k}{b-1} + o(b^k)} \quad \text{as } k \to +\infty$$

where $f^{(b)}(0) = b!\mu(b)$ is the b^{th} derivative of f at point 0.

Observe that for any $z < m^{-n}$, we have $\mathbf{P}(Z_n \leq zm^n) = \mathbf{P}(Z_n = 0) \leq 1$, therefore, we assume in the rest of the proof that $z \geq m^{-n}$. By the Markov inequality, we have, for all $z \in (m^{-n}, 1)$ and $s \in (0,1)$,

$$\mathbf{P}(Z_n \leq zm^n) = \mathbf{P}(s^{Z_n} \geq s^{zm^n}) \leq \frac{\mathbf{E}(s^{Z_n})}{s^{zm^n}} = \frac{f^n(s)}{s^{zm^n}}.$$

In particular, for $s = s_{k-n}$, we have $\mathbf{P}(Z_n \leq zm^n) \leq \frac{s_k}{(s_{k-n})^{zm^n}}$, the rest of the proof consists in choosing the optimal k in this equation, depending on the value of b.

If $b = 0$, we choose $k = \frac{-\log z}{\log m - \log f'(q)}$ which grows to $+\infty$ as $z \to 0$, while $k - n \to -\infty$. As a consequence, there exists $c > 0$ such that for all $n \geq 1$ and $z \geq m^{-n}$,

$$(s_{k-n})^{-zm^n} \leq \exp\left(Czm^k\right).$$

As $\lim_{z \to 0} zm^k = 0$, we conclude there exists $C > 0$ such that for all $n \geq 1$ and $z \geq m^{-n}$,

$$\mathbf{P}(Z_n \leq zm^n) \leq q + Cf'(q)^{\frac{-\log z}{\log m - \log f'(q)}} + Czm^k = q + Cz^{-\frac{\log f'(q)}{\log m - \log f'(q)}} = q + Cz^{\frac{\alpha}{\alpha+1}}.$$

Similarly, if $b = 1$, then $q = 0$ and $f'(0) = \mu(1)$. We set $k = \frac{-\log z}{\log m}$, there exists $C > 0$ such that for all $n \geq 1$ and $z \geq m^{-n}$, we have

$$\mathbf{P}(Z_n \leq zm^n) \leq C\mu(1)^{-\frac{\log z}{\log m}} \leq Cz^{-\frac{\log \mu(1)}{\log m}}.$$

Finally, if $b \geq 2$, we choose $k = -\frac{\log z}{\log m - \log b}$, there exists $c > 0$ (small enough) such that

$$\mathbf{P}(Z_n \leq zm^n) \leq \exp\left[-cz^{-\frac{\log b}{\log m - \log b}}\right],$$

which ends the proof. $\square$

Proof of Theorems 1.1.2, 1.1.3 and 1.1.5. Let $(\mathbf{T}, V)$ be a branching random walk. We set

$$m_n = \begin{cases} n(tv_1 + (1-t)v_2) - \left(\frac{3}{2\theta_1} + \frac{3}{2\theta_2}\right)\log n & \text{for Theorem 1.1.2} \\ n(tv_1 + (1-t)v_2) - \frac{3}{2\theta}\log n & \text{for Theorem 1.1.3} \\ n(ta_1 + (1-t)a_2) - \frac{1}{2\theta}\log n & \text{for Theorem 1.1.5.} \end{cases}$$

We choose $\varphi > 0$ such that applying Lemmas 1.4.2, 1.4.6 or 1.4.9 depending on the case, there exist $c, C > 0$ verifying for all $y \in [0, n^{1/2}]$

$$ce^{-\varphi y} \leq \mathbf{P}(M_n \geq m_n + y) \leq C(1+y)e^{-\varphi y}. \tag{1.5.1}$$

As a consequence, we have easily

$$\lim_{y\to+\infty} \sup_{n\in\mathbb{N}} \mathbf{P}(M_n \geq m_n + y) = 0.$$

We now turn to a lower bound. Let L_1 be a point process with law $\mathcal{L}_1$. For all $h \geq 0$, we write $N_h = \sum_{\ell\in L_1} \mathbf{1}_{\{\ell\geq -h\}}$ and μ_h the law of N_h the number of children of a given individual which makes a displacement greater than $-h$. We write

$$f_h = \mathbf{E}(s^{N_h}) \quad \text{and} \quad f = \mathbf{E}\left[s^N\right],$$

where $N = \#L_1$ is the total number of elements in L_1. By monotone convergence, we have $f_h(s) \underset{h\to+\infty}{\longrightarrow} f(s)$ for all $s \in [0,1]$. In particular, q_h the smallest solution of $f_h(q_h) = q_h$ converge, as $h \to +\infty$ to 0 the smallest solution of the equation $f(q) = q$ (by (1.1.1)).

Moreover, $\mathbf{E}(N) > 1$, we choose h large enough such that $\mathbf{E}(N_h) > \varrho^2 > 1$. Applying Lemma 1.5.1, we have

$$\mathbf{P}\left(\sum_{|u|=k} \mathbf{1}_{\{V(u)\geq -kh\}} \leq \varrho^k\right) \leq q_h + C\beta^k$$

for some $\beta < 1$. As a consequence, for all $\varepsilon > 0$, there exists h large enough and k large enough such that for all $n \in \mathbb{N}$ such that $p \geq k$, we have

$$\mathbf{P}\left(\sum_{|u|=k} \mathbf{1}_{\{V(u)\geq -kh\}} \leq \varrho^k\right) \leq 2\varepsilon.$$

We now consider $(\mathbf{T}^{(n)}, V^{(n)})$ to be a BRWi of length n, such that $tn > k$. Let $q \in \mathbb{N}$ such that $\lfloor tq \rfloor = p - k$. For every individual alive at time k in $\mathbf{T}^{(n)}$, we write

$$\mathbf{T}^u = \{v \in \mathbf{T}^{(n)} : v > u, |v| \leq q + k\},$$

and we observe that $(\mathbf{T}^u, V^{(n)} - V^{(n)}(u))$ has the same law as $(\mathbf{T}^{(q)}, V^{(q)})$. Thus, using the lower bound of (1.5.1), we have

$$\mathbf{P}\left(\exists |u| = k + q : V^{(n)}(u) \geq m_q - kh\right) \geq 1 - 2\varepsilon - (1-c)^{\varrho^k}.$$

Consequently, for all $\varepsilon > 0$, there exists $h > 0$ and $k \geq 1$ chosen large enough, such that for all $n > k/t$ we have

$$\mathbf{P}\left(\exists |u| = k + q : V^{(n)}(u) \geq m_q - kh\right) \geq 1 - 3\varepsilon.$$

We observe there exists $C > 0$ such that $n - q \leq Ck$. Therefore, we consider ε, h, k being fixed. There exists $y_1 > 0$ such that for all $n > k/t$

$$\mathbf{P}\left(\exists |u| = k + q : V^{(n)}(u) \geq m_n - y_1\right) \geq 1 - 3\varepsilon,$$

moreover, if there exists an individual u alive at time $k+q$ such that $V^{(n)}(u) \geq m_n - y_1$, as a point process with law $\mathcal{L}_2$ is always non-empty, we can consider v the rightmost child of the rightmost child of ... of the rightmost child of u, which lives at generation n. There exists obviously $y_2 > 0$ large enough such that

$$\sup_{n \in \mathbb{N}} \mathbf{P}(V^{(n)}(v) - V^{(n)}(u) \leq -y_2) \leq \varepsilon,$$

as a consequence,

$$\mathbf{P}(M_n \geq m_n - y_1 - y_2) \geq 1 - 4\varepsilon.$$

We conclude that

$$\lim_{y \to +\infty} \sup_{n \in \mathbb{N}} \mathbf{P}(M_n \geq m_n - y) = 0,$$

which ends the proof of the tension of M_n. $\square$

CHAPTER 2

Maximal displacement in a branching random walk through interfaces

Τα παντα ρει.

Héraclite d'Éphèse

Abstract

We generalize in this chapter the results obtained in Chapter 1, allowing any finite number of interfaces. We compute the asymptotic of the maximal displacement of a branching random walk in a time-inhomogeneous environment, which consists in a sequence of macroscopic time intervals, in each of which the law of reproduction remains constant. In this model again, the asymptotic consists again in a first ballistic order, given by the solution of an optimization problem under constraints, a negative logarithmic correction, plus stochastically bounded fluctuations.

NOTA: This chapter is a slightly modified version of the article *Maximal displacement in a branching random walk through interfaces* submitted to *Electronic Journal of Probabilities* in May 2013.

2.1 Introduction

In the previous chapter, we considered time-inhomogeneous branching random walks defined in the following manner. We fix $t \in (0, 1)$ and $\mathcal{L}_1, \mathcal{L}_2$ two point processes laws. Given $n \in \mathbb{N}$, the BRWi (branching random walk with an interface) of length n starts with a unique individual located at 0 at time 0. At time 1, this individual dies, giving birth to children which are positioned on $\mathbb{R}$ according to a point process with law $\mathcal{L}_1$. These individuals form the first generation of the process. Then, at each time $k < n$, every individual alive at generation k dies, giving birth to children, positioned according to independent point processes shifted by the position of their ancestor. The law of the point processes is $\mathcal{L}_1$ if $k \leq tn$, and $\mathcal{L}_2$ if $tn < k \leq n$. At time n, individuals die with no children. We proved in the previous chapter that under some mild integrability assumptions, the maximal displacement at time n of the BRWi M_n is given by a first ballistic order, plus a negative logarithmic correction and stochastically bounded fluctuations.

These results can be generalized to a time-inhomogeneous environment that changes more than once. We set $P > 0$ an integer, $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_P = 1$ forming a partition of $[0,1]$ and $(\mathcal{L}_p, 1 \leq p \leq P)$ a family of laws of point processes. Let $n \in \mathbb{N}$, the time-inhomogeneous branching random walk we study is defined as follows: every individual in the process at generation $k \in [n\alpha_{p-1}, n\alpha_p)$ reproduces according to the law $\mathcal{L}_p$, independently of all other reproduction events in the process. We prove, under mild integrability conditions, that in such a branching random walk trough interfaces (BRWis for short) the asymptotic of the maximal displacement is again given by a first ballistic order plus logarithmic corrections and stochastically bounded fluctuations. The speed of the BRWis is obtained as the maximum of an optimization problem under constraints. The logarithmic correction strongly depends on the way the optimal solution interacts with the constraints, as this interaction has an influence on the path followed by the rightmost individual at time n.

We recall that c, C are two positive constants, respectively small enough and large enough, which may change from line to line, and depend only on the law of the random variables we consider. For a given sequence of random variables $(X_n, n \geq 1)$, we write $X_n = O_{\mathbf{P}}(1)$ if the sequence is tensed, i.e.

$$\lim_{K \to +\infty} \sup_{n \geq 1} \mathbf{P}(|X_n| \geq K) = 0.$$

Moreover, we always assume the convention $\max \emptyset = -\infty$ and $\min \emptyset = +\infty$, and for $u \in \mathbb{R}$, we write $u_+ = \max(u, 0)$, and $\log_+(u) = (\log u)_+$. Finally, $\mathcal{C}_b$ is the set of continuous bounded functions on $\mathbb{R}$.

In the rest of the introduction, we give a formal definition of the BRWis in Section 2.1.1, and describe an heuristic for the asymptotic for the value of M_n in Section 2.1.2. Finally, we give in Section 2.1.3 the asymptotic of the maximal displacement at time n, and compare this result with the one we obtained for the BRWi in the previous chapter.

2.1.1 Definition of the branching random walk through interfaces and notation

We recall that $(\mathbf{T}, V) \in \mathcal{T}$ is a (plane, rooted) marked tree if $\mathbf{T}$ is a (plane, rooted) tree, and $V : \mathbf{T} \to \mathbb{R}$. For a given individual $u \in \mathbf{T}$, we write $|u|$ the generation to which u belongs. If u is not the root, then πu is the parent of u. For $k \leq |u|$, we write u_k the ancestor of u in generation k. Finally, we write $\Omega(u) = \{v \in \mathbf{T} : \pi v = u\}$ the set of children of u.

In this chapter, we take interest in BRWis. In this model, the time-inhomogeneous environment consists in a sequence of macroscopic stages. We set $P \in \mathbb{N}$ the number of such stages, $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_P = 1$ the times at which the interfaces are placed, and $(\mathcal{L}_p, p \leq P)$ a P-uple of laws of point processes. We assume that the point processes are never empty and have supercritical offspring i.e.

$$\forall p \leq P, \mathbf{P}(L_p = \emptyset) = 0 \quad \text{and} \quad \mathbf{E}\left(\sum_{\ell \in L_p} 1\right) > 1, \tag{2.1.1}$$

where L_p is a point process with law $\mathcal{L}_p$.

For all $p \leq P$ and $\theta \geq 0$, we write $\kappa_p(\theta) = \log \mathbf{E}\left[\sum_{\ell \in L_p} e^{\theta \ell}\right]$ the log-Laplace transform of $\mathcal{L}_p$, and for all $a \in \mathbb{R}$, $\kappa_p^*(a) = \sup_{\theta \geq 0} \left[\theta a - \kappa_p(\theta)\right]$ its Fenchel-Legendre transform. We recall that the Fenchel-Legendre transform is an involution on the set of convex functions. Moreover, if $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is a convex function, and f^* its Fenchel-Legendre transform, if f^* is differentiable at point x, we have

$$f^*(x) = (f^*)'(x)x - f\left((f^*)'(x)\right). \tag{2.1.2}$$

For $n \in \mathbb{N}$ and $p \leq P$, we write $\alpha_p^{(n)} = \lfloor n\alpha_p \rfloor$. The BRWis of length n is a branching random walk in time-inhomogeneous environment, in which individuals alive at generation k reproduce with law $\mathcal{L}_p$ for all $\alpha_{p-1}^{(n)} \leq k < \alpha_p^{(n)}$. In other words, this is a random marked tree $(\mathbf{T}^{(n)}, V^{(n)})$ of length n verifying

- $V^{(n)}(\emptyset) = 0$;
- $\left\{\left(V^{(n)}(v) - V^{(n)}(u), v \in \Omega(u)\right), u \in \mathbf{T}^{(n)}\right\}$ is a family of independent point processes;
- $\left(V^{(n)}(v) - V^{(n)}(u), v \in \Omega(u)\right)$ has law $\mathcal{L}_p$ if $\alpha_{p-1}^{(n)} \leq |u| < \alpha_p^{(n)}$.

When the value of n is clear in the context, we often omit the superscripts to make the notations lighter.

Remark 2.1.1. By splitting the first time-interval into three pieces, we may always assume that the number P of distinct point processes we consider is greater than or equal to 3 in the rest of the article. This remark is discussed more precisely in the introduction statement of Section 2.4.

2.1.2 Heuristics for the asymptotic of the maximal displacement

We fix an integer P, a sequence $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_P = 1$ and $(\mathcal{L}_p, p \leq P)$ the laws of points processes of the BRWis we consider.

Some well-known time-homogeneous branching random walk estimates

We recall once again the following classical branching random walk results, which can be found in [Big10]. Let $p \leq P$, we consider a time-homogeneous branching random walk $(\mathbf{T}_p, V_p)$ in which individuals reproduce according to $\mathcal{L}_p$. We write $M_{p,n}$ its maximal displacement at time n. If there exists $\theta > 0$ such that $\kappa_p(\theta) < +\infty$, we write

$$v_p = \inf_{\theta > 0} \frac{\kappa_p(\theta)}{\theta} = \sup\{a \in \mathbb{R} : \kappa^*(a) \leq 0\} \tag{2.1.3}$$

which is the speed of the branching random walk, i.e. $\lim_{n\to+\infty} \frac{M_{p,n}}{n} = v_p$ a.s. Under the assumption

$$\forall p \leq P, \exists \overline{\theta}_p \in \mathbb{R}_+ : \overline{\theta}_p \kappa_p'(\overline{\theta}_p) - \kappa_p(\overline{\theta}_p) = 0, \tag{2.1.4}$$

we have $v_p = \kappa_p'(\overline{\theta}_p)$. Moreover, the function κ_p^* is linked to the density of individuals alive at time n around some point. As proved in [Big77a], we have

$$\begin{cases} \forall a < v_p, \lim_{n\to+\infty} \frac{1}{n} \log \sum_{|u|=n} \mathbf{1}_{\{V_p(u) \geq na\}} = -\kappa_p^*(a) \quad \text{a.s.} \\ \forall a > v_p, \lim_{n\to+\infty} \frac{1}{n} \log \mathbf{P}\left[\exists |u| = n : V_p(u) \geq na\right] = -\kappa_p^*(a). \end{cases} \tag{2.1.5}$$

For all $\varepsilon > 0$, with high probability, there is no individual above $n(v_p + \varepsilon)$ at time n, and there is an exponentially large number of individuals above $n(v_p - \varepsilon)$. Equation (2.1.5) gives that $e^{-n\kappa^*(a)}$ is either an approximation of the number of individuals alive at time n in a neighbourhood of na, or of the probability to observe at least one individual around na at time n.

The speed of the BRWis

We now consider the BRWis $(\mathbf{T}, V)$. Given $\mathbf{a} = (a_p, p \leq P) \in \mathbb{R}^P$ –in the rest of this chapter, we write in bold letters real P-uples– we take interest in the number of individuals alive at time n, whose path is close to $a_1\alpha_1^{(n)}$ at time $\alpha_1^{(n)}$, $a_1\alpha_1^{(n)} + a_2(\alpha_2^{(n)} - \alpha_1^{(n)})$ at time $\alpha_2^{(n)}$, and for all $p \leq P$, close to $\sum_{k=1}^{p} a_k(\alpha_k^{(n)} - \alpha_{k-1}^{(n)})$ at time $\alpha_p^{(n)}$. If such an individual exists, we say that it "follows the path driven by $\mathbf{a}$".

Using (2.1.5), we know there are $e^{-\alpha_1^{(n)}\kappa_1^*(a_1)}$ individuals alive at time $\alpha_1^{(n)}$ around $\alpha_1^{(n)}a_1$ if $\kappa_1^*(a_1) < 0$, and none otherwise. Each one of these individuals –if any– starts an independent branching random walk from $\alpha_1^{(n)}a_1$, therefore by the law of large numbers, we expect $e^{-\alpha_1^{(n)}\kappa_1^*(a_1)-(\alpha_2^{(n)}-\alpha_1^{(n)})\kappa_2^*(a_2)}$ individuals alive at time $\alpha_2^{(n)}$ at the wanted position. More generally, writing

$$K^* : \begin{array}{ccc} \mathbb{R}^P & \to & \mathbb{R}^P \\ \mathbf{a} & \mapsto & \left(\sum_{q=1}^{p}(\alpha_q - \alpha_{q-1})\kappa_1^*(a_q), p \leq P\right) \end{array}$$

the rate function associated to the BRWis; the expected number of individuals that followed the path driven by $\mathbf{a}$ is $e^{-nK^*(\mathbf{a})}$.

Observe that if for all $p \leq P$, $K^*(\mathbf{a})_p < 0$, we obtain that with high probability, the number of individuals that followed the path driven by $\mathbf{a}$ is strictly positive, thus the maximal displacement at time n is greater than $\sum_{p=1}^{P}(\alpha_p - \alpha_{p-1})a_p$. On the other hand, if there exists $p_0 \leq P$ such that $K^*(\mathbf{a})_{p_0} > 0$, then with high probability, there is no individual that followed the path driven by $\mathbf{a}$ until time $\alpha_{p_0}^{(n)}$, therefore no individual at time n that followed the path driven by $\mathbf{a}$.

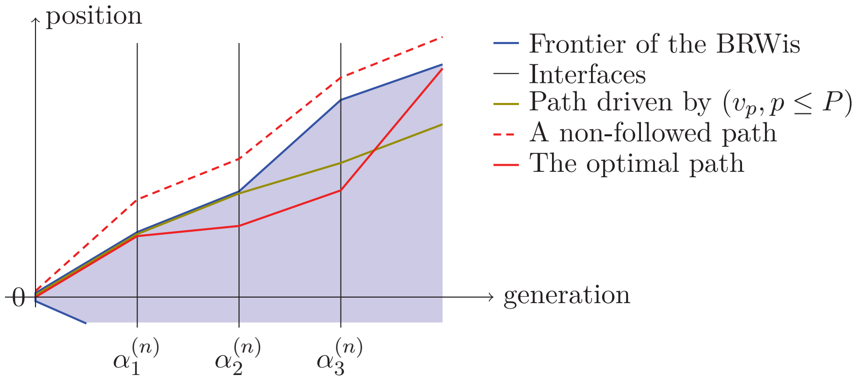


Figure 2.1: Finding the optimal path

We set $\mathcal{R} = \left\{\mathbf{a} \in \mathbb{R}^P : \forall p \leq P, K^*(\mathbf{a})_p \leq 0\right\}$. Following the heuristic, we expect to find individuals alive in the process at time n around position nu if and only if $u = \sum(\alpha_p - \alpha_{p-1})a_p$ for some $\mathbf{a} \in \mathcal{R}$. As a consequence, we write

$$v_{\text{is}} = \sup_{\mathbf{a} \in \mathcal{R}} \sum_{p=1}^{P}(\alpha_p - \alpha_{p-1})a_p \tag{2.1.6}$$

which is the conjectured speed for the maximal displacement in the BRWis.

The optimization problem

According to this heuristic, if the BRWis verifies

$$\exists \mathbf{a} \in \mathcal{R} : v_{\text{is}} = \sum_{p=1}^{P} (\alpha_p - \alpha_{p-1}) a_p, \tag{2.1.7}$$

then the rightmost individual at time n followed the path driven by the optimal solution $\mathbf{a}$. Under the additional assumption

$$\forall p \leq P, \forall a \in \mathbb{R}, \kappa_p^* \text{ is differentiable at point } a \text{ or } \kappa_p^*(a) = +\infty, \tag{2.1.8}$$

this optimal solution satisfies some interesting properties. To guarantee existence and/or uniqueness of the solutions of (2.1.7), we need to introduce another hypothesis,

$$\forall p \leq P, \kappa_p(0) \in (0, +\infty) \quad \text{and} \quad \kappa_p'(0) \text{ exists.} \tag{2.1.9}$$

Proposition 2.1.2. *If point processes $\mathcal{L}_1, \ldots \mathcal{L}_P$ verify* (2.1.1)*, under assumption* (2.1.8)*, $\mathbf{a} \in \mathcal{R}$ is a solution of* (2.1.7) *if and only if, writing $\theta_p = \left(\kappa_p^*\right)' (a_p)$, we have*

(P1) *$\boldsymbol{\theta}$ is non-decreasing and positive;*

(P2) *if $K^*(\mathbf{a})_p < 0$, then $\theta_{p+1} = \theta_p$;*

(P3) $K^*(\mathbf{a})_P = 0$.

Under the conditions (2.1.4) *and* (2.1.8)*, there exists at most one solution to* (2.1.7)*.*
Under the conditions (2.1.8) *and* (2.1.9)*, there exists at least one solution to* (2.1.7)*.*

The proof of this result, which is a direct application of the theory of Lagrange multipliers, is postponed to Section 2.B. Observe that, despite this would be a natural candidate, the path driven by $\mathbf{v} := (v_1, \ldots, v_P)$ is not always the optimal solution –as in the BRWi case, $tv_1 + (1-t)v_2$ is not always the correct speed. For example, if there exists $p \leq P - 1$ such that $\overline{\theta}_p > \overline{\theta}_{p+1}$, Proposition 2.1.2 proves that $\mathbf{v}$ is not the solution. Loosely speaking, in this case, the path of the rightmost individual at time n does not stay close to the boundary of the branching random walk at all time.

Remark 2.1.3. On the other hand, under the assumptions (2.1.4) and (2.1.8), if $\overline{\theta}$ is positive and non-decreasing, then $\mathbf{v}$ is indeed the optimal solution. In this case, $\mathbf{v}$ satisfies the first assumption of Proposition 2.1.2, and the two others are easy since $K^*(\mathbf{v})_p = 0$ for all $p \leq P$. This situation corresponds, in Gaussian settings, to branching random walks with decreasing variance. In this situation, the ancestors of the rightmost individual at time n were at any time $k < n$ within range $O(n^{1/2})$ from the boundary of the BRWis.

On the logarithmic correction

In this section, we use the previous heuristic to build a conjecture on the value of the logarithmic correction of the BRWis. For a time-homogeneous branching random walk with reproduction law $\mathcal{L}_p$, under assumption (2.1.4) and some additional integrability conditions –as proved in Chapter 7– we have

$$M_n^{(p)} = nv_p - \frac{3}{2\overline{\theta}_p} \log n + O_{\mathbf{P}}(1),$$

and the $-\frac{3}{2}\log n$ second order comes from the following estimate for a random walk with finite variance,

$$\log \mathbf{P}\left[S_n \leq \mathbf{E}(S_n) + 1, S_j \geq \mathbf{E}(S_j), j \leq n\right] \approx -\frac{3}{2}\log n.$$

This is due to the fact that, if u is the rightmost individual alive at time n, then for all $k \leq n$, $V(u_k) \leq kv_p$, thus the path $(V(u_0), V(u_1), \ldots V(u))$ forms an excursion, as it has been underlined in [AS10].

A similar condition holds for BRWis, the path leading to the rightmost individual at time n stays at any time below the boundary of the branching random walk. If $K^*(\mathbf{a})_p = 0$, the optimal path is at distance $o(n)$ from the boundary of the branching random walk at time $\alpha_p^{(n)}$. Moreover, for all $p < P$ such that $\theta_{p+1} > \theta_p$, we prove in Section 2.4.3 that the ancestor at time $\alpha_p^{(n)}$ of the rightmost individual at time n was within distance $O(1)$ from the boundary [1]. As a result, the logarithmic correction in the BRWis is a sum of terms related to the probability for a random walk to stay below the boundary of the branching random walk, and hit at time n this boundary.

From now on, $\mathbf{a}$ stands for the optimal solution of (2.1.6), and $\theta_p = (\kappa_p^*)'(a_p)$. We write $T = \#\{\theta_p, p \leq P\}$ the number of values taken by $\boldsymbol{\theta}$ and $\varphi_1 < \varphi_2 < \cdots < \varphi_T$ these distinct values, listed in the increasing order. For any $t \leq T$, let $f_t = \min\{p \leq P : \theta_p = \varphi_t\}$ and $l_t = \max\{p \leq P : \theta_p = \varphi_t\}$. Observe that for all $p \in [f_t, l_t]$, we have $\theta_p = \varphi_t$. Finally, we write

$$\lambda = \sum_{t=1}^{T} \frac{1}{2\varphi_t}\left[\mathbf{1}_{\left\{K^*(\mathbf{a})_{f_t}=0\right\}} + 1 + \mathbf{1}_{\left\{K^*(\mathbf{a})_{l_t-1}=0\right\}}\right] \tag{2.1.10}$$

with the convention $K^*(\mathbf{a})_0 = 0$. If $K^*(\mathbf{a})_{f_t} = 0$, then between times $\alpha_{f_t-1}^{(n)}$ and $\alpha_{f_t}^{(n)}$, the optimal path stays close to the boundary of the BRWis, which has a cost of order $-\frac{1}{2}\log n$ by the ballot theorem –see Section 2.3. The fact that at time $\alpha_{l_t}^{(n)}$, the optimal path is within a widows of size $O(1)$ has also cost of order $-\frac{1}{2}\log n$ by local limit theorem. Finally, if $K^*(\mathbf{a})_{l_t-1}$ then the optimal path stays close to the boundary again, between times $\alpha_{l_t-1}^{(n)}$ and $\alpha_{l_t}^{(n)}$.

We prove in the rest of this chapter that under some good integrability conditions $M_n \approx vn - \lambda\log n$. We observe that $\lambda \geq \frac{1}{2\varphi_1} > 0$. Moreover, if $P = T = 1$, then $\lambda = \frac{3}{2\varphi_1}$, which is consistent with the results of Hu–Shi [HS09], Addario-Berry–Reed [ABR09] and Chapter 7.

2.1.3 The asymptotic of the maximal displacement in the BRWis

We recall that $\mathbf{a}$ is the maximal solution of (2.1.7). We write

$$B = \left\{p \leq P : K^*(\mathbf{a})_{p-1} = K^*(\mathbf{a})_p = 0\right\}, \tag{2.1.11}$$

for all $k \in \cup_{p\in B}[\alpha_{p-1}^{(n)}, \alpha_p^{(n)}]$, the path leading to the the rightmost individual at time n is within distance $o(n)$ from the boundary of the branching random walk. For all $p \leq P$, we introduce the random variable

$$X_p = \sum_{\ell \in L_p} e^{\theta_p \ell}, \tag{2.1.12}$$

1. Similarly to what happens in a BRWi in the slow regime, see Chapter 1.

and the following integrability conditions for the point processes:

$$\sup_{p \leq P} \mathbf{E}\left[\sum_{\ell \in \mathcal{L}_p} \ell^2 e^{\theta_p \ell}\right] < +\infty, \tag{2.1.13}$$

$$\sup_{p \in B} \mathbf{E}\left[X_p \left(\log_+ X_p\right)^2\right] < +\infty \tag{2.1.14}$$

$$\sup_{p \in B^c} \mathbf{E}\left[X_p (\log_+ X_p)\right] < +\infty. \tag{2.1.15}$$

The following theorem is the main result of this chapter.

Theorem 2.1.4. *If all the point processes $\mathcal{L}_1, \cdots \mathcal{L}_p$ satisfy* (2.1.1)*, under assumptions* (2.1.7)*,* (2.1.8)*,* (2.1.13)*,* (2.1.14) *and* (2.1.15)*, we have*

$$M_n = n v_{\text{is}} - \lambda \log n + O_{\mathbf{P}}(1).$$

To prove this theorem, we bound in a first time the tail of M_n, obtaining the following result.

Theorem 2.1.5. *If point processes $\mathcal{L}_1, \ldots \mathcal{L}_P$ verify* (2.1.1)*, under assumptions* (2.1.7)*,* (2.1.8) *and* (2.1.13)*, there exists $C > 0$ such that for all $n \in \mathbb{N}$ and $y \geq 0$,*

$$\mathbf{P}(M_n \geq n v_{\text{is}} - \lambda \log n + y) \leq C(1 + y \mathbf{1}_B(1)) e^{-\theta_1 y}.$$

Moreover, under the additional assumptions (2.1.14) *and* (2.1.15)*, there exists $c > 0$ such that for all $n \in \mathbb{N}$ and $y \in [0, n^{1/2}]$,*

$$\mathbf{P}(M_n \geq n v_{\text{is}} - \lambda \log n + y) \geq c(1 + y \mathbf{1}_B(1)) e^{-\theta_1 y}.$$

Before proving these two theorems, we apply them to BRWi, and obtain once again the three regimes described in the previous chapter.

Application to the BRWi

Theorem 2.1.4 is consistent with Theorems 1.1.2, 1.1.3 and 1.1.2 of Chapter 1. We consider here a BRWis with a single interface $(\mathbf{T}, V)$, or in other words, such that $P = 2$. We set two point processes $\mathcal{L}_1$ and $\mathcal{L}_2$ verifying (2.1.1), and $\alpha_1 \in (0,1)$. We assume (2.1.4), i.e. there exist $\overline{\theta}_1, \overline{\theta}_2$ such that for all $i \in \{1, 2\}$,

$$\overline{\theta}_i \kappa_i'(\overline{\theta}_i) - \kappa_i(\overline{\theta}_i) = 0.$$

We also suppose there exists $\theta > 0$ such that κ_1 and κ_2 are differentiable at point θ and

$$\theta(\alpha_1 \kappa_1'(\theta) + (1 - \alpha_1)\kappa_2'(\theta)) - (\alpha_1 \kappa_1(\theta) + (1 - \alpha_1)\kappa_2(\theta)) = 0. \tag{2.1.16}$$

For all $i \in \{1, 2\}$, κ_i is a convex function on $\{\varphi > 0 : \kappa_i(\varphi) < +\infty\}$, which is twice differentiable on the interior of this set. As a consequence, $\varphi \mapsto \varphi \kappa_i'(\varphi) - \kappa_i(\varphi)$ is a decreasing function. Thus, θ is always between $\overline{\theta}_1$ and $\overline{\theta}_2$. We write $v_{\text{fast}} = \alpha_1 \kappa_1'(\theta) + (1 - \alpha_1)\kappa_2'(\theta)$ and $v_{\text{slow}} = \alpha_1 \kappa_1'(\overline{\theta}_1) + (1 - \alpha_1)\kappa_2'(\overline{\theta}_2)$. Observe that v_{slow} is the sum of the speeds of a branching random walk with reproduction $\mathcal{L}_1$ of length $n\alpha_1$ with one with reproduction $\mathcal{L}_2$ and length $n\alpha_2$.

Applying Theorem 2.1.4 and using Proposition 2.1.2, under (2.1.13), (2.1.14) and (2.1.15), one of the following alternative is true.

- If $\overline{\theta}_1 > \overline{\theta}_2$, then $\theta \in (\overline{\theta}_2, \overline{\theta}_1)$, $v_{\text{slow}} < v_{\text{fast}}$ and
$$M_n = nv_{\text{fast}} - \frac{1}{2\theta} \log n + O_{\mathbf{P}}(1),$$
in which case the optimal path is at time $\alpha_1 n$ at distance $O(n)$ from the boundary of the branching random walk.
- If $\overline{\theta_1} = \overline{\theta}_2$, then $\theta = \overline{\theta}_1 = \overline{\theta}_2$, $v_{\text{slow}} = v_{\text{fast}}$ and
$$M_n = nv_{\text{fast}} - \frac{3}{2\theta} \log n + O_{\mathbf{P}}(1),$$
and the process behaves similarly to time-homogeneous branching random walk, thus the path followed by to the rightmost individual at time n is at time $\alpha_1 n$ within distance $O(\sqrt{n})$ from the boundary of the branching random walk.
- If $\overline{\theta}_1 < \overline{\theta}_2$, then $v_{\text{slow}} < v_{\text{fast}}$ and
$$M_n = nv_{\text{slow}} - \left[\frac{3}{2\overline{\theta}_1} + \frac{3}{2\overline{\theta}_2}\right] \log n + O_{\mathbf{P}}(1),$$
in other words, the logarithmic corrections add up, and the rightmost individual at time n descend from one of the rightmost individuals alive at time $\alpha_1 n$.

Under assumption (2.1.8), using Lagrange theorem [2] we have

$$\begin{aligned}
v_{\text{fast}} &= \sup \left\{\alpha_1 a_1 + (1-\alpha_1)a_2 : \alpha_1 \kappa_1^*(a_1) + (1-\alpha_1)\kappa_2^*(a_2) \leq 0\right\},\\
v_{\text{slow}} &= \sup \left\{\alpha_1 a_1 + (1-\alpha_1)a_2 : \alpha_1 \kappa_1^*(a_1) \leq 0, \alpha_1 \kappa_1^*(a_1) + (1-\alpha_1)\kappa_2^*(a_2) \leq 0\right\}.
\end{aligned}$$

Therefore, a branching random walk goes at speed v_{slow} if the condition $\kappa_1^*(a_1) \leq 0$ modifies the solution of (2.1.7). If this is the case, it means that the "theoretical optimal path" would cross the boundary of the branching random walk, thus no individual could follow it. Under these circumstances, an individual which is at time $\alpha_1 n$ close to the rightmost position has an important advantage to breed the rightmost descendant at time n. Otherwise, at time $\alpha_1 n$, there is a large number of individuals around $\alpha_1 a_1 n$, each of which having small probability to be the rightmost individual. Therefore the logarithmic correction is similar to the one obtained computing the maximal displacement of a large number of independent random walks. Although the speed varies continuously as $\overline{\theta}_1$ grows bigger than $\overline{\theta}_2$, the logarithmic correction exhibits a phase transition, as observed in Chapter 1.

We now prove Theorems 2.1.4 and 2.1.5 for a general BRWis. The organisation of the chapter is very similar to the one in Chapter 1. In Section 2.2, we recall the spinal decomposition for a time-inhomogeneous branching random walks. In Section 2.3, we extend the random walks estimates of Chapter 1 to a random walk with any number of interfaces. In Section 2.4, we prove Theorem 2.1.5 by recurrence, then use it to prove Theorem 2.1.4. Finally, we prove in Section 2.A the random walk estimates described in Section 2.3 and Proposition 2.1.2 in Section 2.B.

2.2 The spinal decomposition

We recall in this section the time-inhomogeneous version of the spinal decomposition of the branching random walk [3]. We give two ways of describing a size-biased version of

2. See Appendix 2.B.
3. 6:00 – "I Got You Babe".

the law of the branching random walk. After its introduction to the study of Galton-Watson processes in [LPP95], this method has been adapted to branching random walks in [Lyo97], and to general branching Markov processes in [BK04].

2.2.1 The size-biased law of the branching random walk

Let $n \geq 1$ and $(\mathcal{L}_k, k \leq n)$ be a sequence of point processes laws which forms the environment of a time-inhomogeneous branching random walk $(\mathbf{T}, V)$. For all $x \in \mathbb{R}$ we set $\mathbf{P}_x$ the law on $\mathcal{T}$ of the marked tree $(\mathbf{T}, V + x)$, and $\mathbf{E}_x$ the corresponding expectation.

We write $\kappa_k(\theta)$ the log-Laplace transform of $\mathcal{L}_k$ and we assume there exists $\theta > 0$ such that for all $k \leq n$ we have $\kappa_k(\theta) < +\infty$. Let

$$W_n = \sum_{|u|=n} \exp\left(\theta V(u) - \sum_{j=1}^{n} \kappa_j(\theta)\right).$$

We observe that $W_n \geq 0, \mathbf{P}_x -$ a.s. and $\mathbf{E}_x(W_n) = e^x$. Therefore, we can define the law $\overline{\mathbf{P}}$ on the set of marked trees of height n by

$$\overline{\mathbf{P}}_x = e^{-x} W_n \cdot \mathbf{P}_x. \tag{2.2.1}$$

The spinal decomposition consists of an alternative construction of the law $\overline{\mathbf{P}}_a$, as the projection of a law on the set of planar rooted marked trees with spine, which we define below.

2.2.2 A law on plane rooted marked trees with spine

Let $(\mathbf{T}, V)$ be a marked tree of height n, and $w \in \{u \in \mathbf{T} : |u| = n\}$ an individual alive at the n^{th} generation. The triplet $(\mathbf{T}, V, w)$ is then called a plane rooted marked tree with spine of length n. The spine of a tree is a distinguished path of length n linking the root and the last generation. The set of marked trees with spine of height n is written $\widehat{\mathcal{T}}_n$. On this set, we define the three following filtrations,

$$\forall k \leq n, \widehat{\mathcal{F}}_k = \sigma\left(u, V(u), u \in \mathbf{T}, |u| \leq k\right) \vee \sigma(w_j, j \leq k) \quad \text{and} \quad \widehat{\mathcal{F}} = \widehat{\mathcal{F}}_n$$
$$\forall k \leq n, \mathcal{F}_k = \sigma\left(u, V(u) : u \in \mathbf{T}, |u| \leq k\right) \quad \text{and} \quad \mathcal{F} = \mathcal{F}_n$$
$$\forall k \leq n, \mathcal{G}_k = \sigma\left(w_j, V(w_j) : j \leq k\right) \vee \sigma\left(u, V(u), u \in \Omega(w_j), j < k\right) \quad \text{and} \quad \mathcal{G} = \mathcal{G}_n.$$

The filtration $\mathcal{F}$ is the information of the marked tree, obtained by forgetting the spine, $\mathcal{G}$ is the sigma-field of the knowledge of the spine and its children only, and $\widehat{\mathcal{F}} = \mathcal{F} \vee \mathcal{G}$ is the natural filtration of the branching random walk with spine.

We introduce a law $\widehat{\mathbf{P}}_a$ on $\widehat{\mathcal{T}}_n$. For any $k \leq n$, we write

$$\widehat{\mathcal{L}}_k = \left(\sum_{\ell \in L} e^{\theta \ell - \kappa_k(\theta)}\right) \cdot \mathcal{L}_k,$$

a law of a point process with Radon-Nikodým derivative with respect to $\mathcal{L}_k$, and we write $\widehat{L}_k = \left(\widehat{\ell}_k(j), j \leq N_k\right)$ an independent point processes of law $\widehat{\mathcal{L}}_k$. Conditionally on $(\widehat{L}_k, k \leq n)$, we choose, for every $k \leq n$, $w(k) \leq N_k$ independently at random, such that

$$\mathbf{P}\left(w(k) = h \middle| \widehat{L}_k, k \leq n\right) = \mathbf{1}_{\{h \leq N_k\}} \frac{e^{\theta \ell_k(h)}}{\sum_{j \leq N_k} e^{\theta \ell_k(j)}}.$$

We denote by $w_n \in \mathcal{U}$ the sequence $(w(1), \dots w(n))$.

We consider a family $\{L^u, u \in \mathcal{U}, |u| \leq n\}$ of independent point processes such that $L^{w_k} = \widehat{L}_{k+1}$, and if $u \neq w_{|u|}$, then L^u has law $\mathcal{L}_{|u|+1}$. For any $u \in \mathcal{U}$ such that $|u| \leq n$, we write $L^u = (\ell^u_1, \dots \ell^u_{N(u)})$. We construct the tree

$$\mathbf{T} = \{u \in \mathcal{U} : |u| \leq n, \forall 1 \leq k \leq |u|, u(k) \leq N(u_{k-1})\},$$

and the the function

$$V : \begin{array}{ccc} \mathbf{T} & \to & \mathbb{R} \\ u & \mapsto & \sum_{k=1}^{|u|} \ell^{u_{k-1}}_{u(k)}. \end{array}$$

For all $x \in \mathbb{R}$, the law of $(\mathbf{T}, x + V, w_n) \in \widehat{\mathcal{T}}_n$ is written $\widehat{\mathbf{P}}_x$, and the corresponding expectation is $\widehat{\mathbf{E}}_x$.

The marked tree with spine $(\mathbf{T}, x + V, w_n)$ is called a branching random walk with spine, and can be constructed as a process in the following manner. It starts with a unique individual positioned at x at time 0, which is the ancestral spine w_0. Then, at each time $k < n$, every individual alive at generation k dies. Each of these individuals gives birth to children, which are positioned around their parent according to an independent point process. If the parent is w_k, then the law of this point process is $\widehat{\mathcal{L}}_k$, otherwise it is $\mathcal{L}_k$. The individual w_{k+1} is then chosen at random among the children u of w_k, with probability proportional to $e^{\theta V(u)}$. At time n, individuals die without children.

In the rest of the article, we write $\mathbf{P}_{x,k}$ the law of the time-inhomogeneous branching random walk of length $n - k$ starting from x with environment $(\mathcal{L}_{k+1}, \dots \mathcal{L}_n)$. Observe that conditionally on $\mathcal{G}$, the branching random walks of the descendants of the children of w_k are independent, and the branching random walk of the children of $u \in \Omega(w_k)$ has law $\mathbf{P}_{V(u),k+1}$.

2.2.3 The spinal decomposition

The following result, which links the laws $\widehat{\mathbf{P}}_x$ and $\overline{\mathbf{P}}_x$, is the time-inhomogeneous version of the spinal decomposition, proved in Chapter 1.

Proposition 2.2.1 (Spinal decomposition)**.** *For all $x \in \mathbb{R}$, we have*

$$\overline{\mathbf{P}}_x = \widehat{\mathbf{P}}_x\Big|_{\mathcal{F}}. \tag{2.2.2}$$

Moreover, for any $|u| = n$, we have

$$\widehat{\mathbf{P}}_x(w_n = u|\mathcal{F}) = \frac{\exp\left(\theta V(u) - \sum_{k=1}^n \kappa_k(\theta)\right)}{W_n} \mathbf{1}_{\{u \in \mathbf{T}\}}. \tag{2.2.3}$$

A straightforward consequence of this result is the well-known many-to-one lemma. This equation, known at least from the early works of Peyrière [Pey74] has been used in many forms over the last decades, and we introduce here a time-inhomogeneous version of it, proved again in Chapter 1.

Lemma 2.2.2 (Many-to-one)**.** *We define an independent sequence of random variables $(X_k, k \leq n)$ by*

$$\forall k \leq n, \forall x \in \mathbb{R}, \mathbf{P}\left[X_k \leq x\right] = \mathbf{E}\left[\sum_{\ell \in L_k} \mathbf{1}_{\{\ell \leq x\}} e^{\theta \ell - \kappa_k(\theta)}\right],$$

and we write $S_k = S_0 + \sum_{j=1}^k X_j$ for $k \leq n$, where $\mathbf{P}_x(S_0 = x) = 1$. For all $x \in \mathbb{R}, k \leq n$ and f a measurable non-negative function, we have

$$\mathbf{E}_x\left[\sum_{|u|=k} f(V(u_1), \dots V(u_k))\right] = e^{\theta x}\,\mathbf{E}_x\left[e^{-\theta S_k + \sum_{j=1}^k \kappa_j(\theta)} f(S_1, \dots S_k)\right]. \tag{2.2.4}$$

The many-to-one lemma and the spinal decompositions enable to compute moments of an additive functional of the branching random walk, by using random walk estimates. These estimates are introduced in the next section, and extended to include time-inhomogeneous versions, and the control of random variables correlated to the last step.

2.3 Some random walk estimates

We recall the random walk estimates introduced in Chapter 1, and extend them to bound similar events on random walks through a sequence of interfaces. We denote by $(T_n, n \geq 0)$ a one-dimensional centred random walk, with finite variance σ^2. We begin with Stone's local limit theorem [Sto65]. There exists $C > 0$ such that for all $a \geq 0$ and $h \geq 0$, we have

$$\limsup_{n\to+\infty} n^{1/2} \sup_{|y|\geq an^{1/2}} \mathbf{P}(T_n \in [y, y+h]) \leq C(1+h)e^{-\frac{a^2}{2\sigma^2}}. \tag{2.3.1}$$

Moreover, there exists $H > 0$ such that for all $a < b \in \mathbb{R}$

$$\liminf_{n\to+\infty} n^{1/2} \inf_{y\in[an^{1/2}, bn^{1/2}]} \mathbf{P}(T_n \in [y, y+H]) > 0. \tag{2.3.2}$$

We continue with Caravenna–Chaumont's local limit theorem [CC13]. Let $(r_n, n \geq 0)$ be a positive sequence such that $r_n = O(n^{1/2})$. There exists $C > 0$ such that for all $a \geq 0$ and $h \geq 0$,

$$\limsup_{n\to+\infty} n^{1/2} \sup_{y\in[0,r_n]} \sup_{x\geq an^{1/2}} \mathbf{P}(T_n \in [x, x+h] | T_j \geq -y, j \leq n) \leq C(1+h)ae^{-\frac{a^2}{2\sigma^2}}. \tag{2.3.3}$$

Moreover, there exists $H > 0$ such that for all $a < b \in \mathbb{R}_+$,

$$\liminf_{n\to+\infty} n^{1/2} \inf_{y\in[0,r_n]} \inf_{x\in[an^{1/2}, bn^{1/2}]} \mathbf{P}(T_n \in [x, x+H] | T_j \geq -y, j \leq n) > 0. \tag{2.3.4}$$

Once again, up to a transformation $T \mapsto T/(2H)$, we assume in the rest of this chapter that all the random walks we consider are such that (2.3.2) and (2.3.4) hold with $H = 1$. The Hsu–Robbins theorem, introduced in Chapter 1 is useful to bound the probability for a random walk to stay below a linear boundary, for all $\varepsilon 0$

$$\sum_{n\geq 0} \mathbf{P}(T_n \leq -n\varepsilon) < +\infty. \tag{2.3.5}$$

We next recall extension of Kozlov's and Pemantle–Peres' ballot theorem [Koz76], proved in Chapter 1. For all $A \geq 0$ and $\alpha \in [0, 1/2)$, there exists $C > 0$ such that for all $n \geq 1$ and $y \geq 0$,

$$\mathbf{P}(T_j \geq -y - Aj^\alpha, j \leq n) \leq C(1+y)n^{-1/2}, \tag{2.3.6}$$

moreover, there exists $c > 0$ such that for all $n \geq 1$ and $y \in [0, n^{1/2}]$

$$\mathbf{P}(T_j \geq -y, j \leq n) \geq c(1+y)n^{-1/2}. \tag{2.3.7}$$

Mixing (2.3.6) and (2.3.1), we proved in Chapter 1 that there exists $C > 0$ such that for all $x, h \geq 0$ and $y \in \mathbb{R}$,

$$\begin{aligned}\mathbf{P}(T_{p+q} \in [y+h, y+h+1], T_j \geq -x + y\mathbf{1}_{\{j>p\}}, j \leq n)&\\ \leq C\frac{(1+x)\wedge p^{1/2}}{p^{1/2}}\frac{1}{\max(p,q)^{1/2}}\frac{(1+h)\wedge q^{1/2}}{q^{1/2}}&,\end{aligned} \tag{2.3.8}$$

and using (2.3.7) and (2.3.2), that there exists $c > 0$ such that for all $n \geq 1$ large enough, $x \in [0, n^{1/2}]$ and $y \in [-n^{1/2}, n^{1/2}]$,

$$\mathbf{P}_x(T_n \leq y+1, T_j \geq y\mathbf{1}_{\{j>n/2\}}, j \leq n) \geq c\frac{(1+x)}{n^{3/2}}. \tag{2.3.9}$$

This result also holds for excursions above bended curves. For all $A \geq 0$ there exists $C > 0$ such that for all $n \in \mathbb{N}$ and $y, h \geq 0$

$$\begin{aligned}\mathbf{P}(T_n + A\log n \in [h-y, h-y+1], T_j \geq -A\log \tfrac{n}{n-j+1} - y, j \leq n)&\\ \leq C\frac{((1+y)\wedge n^{1/2})((1+h)\wedge n^{1/2})}{n^{3/2}}&.\end{aligned} \tag{2.3.10}$$

We sum up all the previous random walks estimates into two lemmas, that bound from above and from below the probability, for a random walk through interfaces, to make an excursion above a given curve. Let $p, q, r \in \mathbb{N}$, $(X_k)_{k\in\mathbb{N}}$ and $(\widetilde{X}_k)_{k\in\mathbb{N}}$ be two independent families of i.i.d. random variables, with mean 0 and finite variance, and $(Y_n)_{n\geq 0}$ be a family of independent random variables. We write $n = p + q + r$ and define the time-inhomogeneous random walk $(S_k, k \leq n)$ as follows:

$$S_k = \sum_{j=1}^{\min\{k,p\}} X_j + \sum_{j=1}^{\min\{k-p,q\}} Y_j + \sum_{j=1}^{\min\{k-p-q,r\}} \widetilde{X}_j.$$

Let $A \in \mathbb{R}$, and $x, y \in \mathbb{R}_+$, $h \in \mathbb{R}$, we denote by

$$\Gamma^{A,1}(x,y,h) = \{s \in \mathbb{R}^n : \forall k \leq p, s_k \geq -x\} \quad \text{and}$$
$$\Gamma^{A,3}(x,y,h) = \{s \in \mathbb{R}^n : \forall k \in [n-r,n], s_k \geq y + A\log \tfrac{n}{n-k+1}\}$$

the sets of trajectories staying above $-x$ during the p initial steps and above a logarithmic boundary during the r last steps. The next lemma is proved in Section 2.A.1.

Lemma 2.3.1. *For all $A \in \mathbb{R}$ and $F \subset \{1,3\}$, there exists $C > 0$ such that for all $p, q, r \in \mathbb{N}$, $x, y \in \mathbb{R}_+$ and $h \in \mathbb{R}$, we have*

$$\begin{aligned}\mathbf{P}\left[S_n + A\log n \in [y+h, y+h+1], (S_k, k \leq n) \in \bigcap_{f\in F}\Gamma^{A,f}(x,y,h)\right]&\\ \leq C\frac{1+y\mathbf{1}_F(1)}{p^{\mathbf{1}_F(1)/2}}\frac{1}{\max(p,r)^{1/2}}\frac{1+h_+\mathbf{1}_F(3)}{r^{\mathbf{1}_F(3)/2}}&.\end{aligned}$$

We now bound from below a similar event, for a random walk through interfaces S, defined as follows: given $X^{(1)}, \dots X^{(P)}$ real-valued centred random walks with finite variance, the process S is a sum of independent random variables, such that the law of $S_{k+1} - S_k$ is the same as $X^{(p)}$ for all $p \in [\alpha^{(n)}_{p-1}, \alpha^{(n)}_p)$. For $F \subset \{1,3\}$ and $x, y, \delta \in \mathbb{R}_+$, we write

$$\Upsilon^F(x,y,\delta) = \left\{ s \in \mathbb{R}^n : \begin{array}{l} \forall k \leq \alpha^{(n)}_1, s_k \geq -x\mathbf{1}_{\{1 \in F\}} - \delta k \mathbf{1}_{\{1 \notin F\}} \\ \forall k \in (\alpha^{(n)}_1, \alpha^n_{P-1}], s_k \geq 0 \\ \forall k \in (\alpha^n_{P-1}, n], s_k \geq y\mathbf{1}_{\{3 \in F\}} - \delta(n-k)\mathbf{1}_{\{3 \notin F\}} \end{array} \right\}.$$

Lemma 2.3.2. *There exists $c > 0$ such that for all $n \geq 1$ large enough, $F \subset \{1,3\}$, $x \in [0, n^{1/2}]$, $y \in [-n^{1/2}, n^{1/2}]$ and $\delta > 0$*

$$\mathbf{P}(S_n \leq y+1, S \in \Upsilon^F(x,y,\delta)) \geq c \frac{1 + x\mathbf{1}_F(1)}{n^{\mathbf{1}_F(1)/2}} \frac{1}{n^{1/2}} \frac{1}{n^{\mathbf{1}_F(3)/2}}.$$

This lemma is proved in Section 2.A.2.

Finally, we recall the upper bounds for enriched random walks, obtained in Chapter 1. Given $(X_n, \xi_n, n \in \mathbb{N})$ i.i.d. random variables such that $\mathbf{E}(X_1) = 0$, $\mathbf{E}(X_1^2) < +\infty$ and $\mathbf{E}((\xi_1)_+^2) < +\infty$, we write $T_n = \sum_{k=1}^n X_k$. For all $t \in (0,1)$, there exists $C > 0$ that does not depend on the law of ξ_1 such that for all $n \geq 1$, $x, h \geq 0$ and $y \in \mathbb{R}$, we have

$$\mathbf{P}\left[T_j \geq -x, j \leq n, \exists k \leq n : T_k \leq \xi_k - x\right] \leq C \frac{1+x}{n^{1/2}} \left[\mathbf{P}(\xi_1 \geq 0) + \mathbf{E}((\xi_1)_+^2)\right], \tag{2.3.11}$$

as well as

$$\mathbf{P}\left[T_n - x - y - h \in [0,1], T_j \geq -x + y\mathbf{1}_{\{j > tn\}}, j \leq n, \exists k \leq n : T_k \leq \xi_k + y\mathbf{1}_{\{k > tn\}} - x\right] \\ \leq C \frac{(1+x)(1+h)}{n^{3/2}} \left[\mathbf{P}(\xi_1 \geq 0) + \mathbf{E}((\xi_1)_+^2)\right]. \tag{2.3.12}$$

We introduce an additional result, which is to Hsu–Robbins theorem what (2.3.11) and (2.3.12) are respectively to (2.3.6) and (2.3.8).

Lemma 2.3.3. *We suppose that $\mathbf{E}(X_1) = 0$, $\mathbf{E}(X_1^2) < +\infty$ and $\mathbf{E}((\xi_1)_+) < +\infty$. Let $\varepsilon > 0$, there exists $C > 0$ that does not depend on the law of ξ_1 such that for all $x, z \geq 0$ and $n \in \mathbb{N}$*

$$\mathbf{P}\left[T_j \geq -x - \varepsilon j, j \leq n, \exists k \leq n : T_k \leq -x - \varepsilon k + \xi_k\right] \\ \leq C \left[\frac{\mathbf{E}\left[(\xi + z)_+\right]}{\varepsilon}\right] + \mathbf{E}\left[\sum_{n \geq 0} \mathbf{1}_{\{T_n \leq -x-z-n\varepsilon/2\}}\right].$$

Proof. We observe that for all $z \geq 0$,

$$\mathbf{P}\left[T_j \geq -x - \varepsilon j, j \leq n, \exists k \leq n : T_k \leq -x - \varepsilon k + \xi_k\right] \leq \sum_{k=1}^n \mathbf{P}(T_k \leq -x - \varepsilon k + \xi_k),$$

moreover

$$\mathbf{P}(T_k \leq -x - \varepsilon k + \xi_k) \leq \mathbf{P}(T_k \leq -x - z - \varepsilon k/2) + \mathbf{P}(\xi_k \geq \varepsilon k/2 - z),$$

thus

$$\begin{aligned}
&\mathbf{P}\left[T_j \geq -\varepsilon j - x, j \leq n, \exists k \leq n : T_k \leq -\varepsilon k - x + \xi_k\right] \\
&\qquad \leq \sum_{k=1}^{n} \mathbf{P}\left[T_k \leq -\varepsilon j/2 + z - x\right] + \sum_{k=1}^{n} \mathbf{P}(\xi_k \geq \varepsilon k/2 - z) \\
&\qquad\qquad \leq \mathbf{E}\left[\sum_{k=1}^{+\infty} \mathbf{1}_{\{T_j \leq -z-\varepsilon j/2\}}\right] + 2\frac{\mathbf{E}((\xi+z)_+)}{\varepsilon},
\end{aligned}$$

ending the proof. □

2.4 Bounding the tail of the maximal displacement

We consider a BRWis $(\mathbf{T}, V)$ of length n. The sequence of point processes can freely be replaced by $(\mathcal{L}_1, \mathcal{L}_1, \mathcal{L}_1, \mathcal{L}_2, \ldots \mathcal{L}_P)$ and the sequence of positions of the interfaces be replaced by $0 = \alpha_0 < \alpha_1/3 < 2\alpha_1/3 < \alpha_1 < \alpha_2 < \ldots < \alpha_P = 1$. Thus, we assume without loss of generality in this section that $P \geq 3$. For $p \leq P$ and $\theta > 0$, we write $\kappa_p(\theta) = \log \mathbf{E}\left[\sum_{\ell \in L_p} e^{\theta \ell}\right]$ the log-Laplace transform of $\mathcal{L}_p$. We write M_n the maximal displacement at time n of the BRWis.

We prove in a first time Theorem 2.1.5, using the decomposition of the BRWis obtained thanks to Proposition 2.1.2. According to this result, if $\mathbf{a}$ is the solution of (2.1.7), and $\theta_p = (\kappa_p^*)'(a_p)$, the sequence $\boldsymbol{\theta}$ is non-decreasing, and takes a finite number T of values. We proceed by induction on T. The next section proves Theorem 2.1.5 for a BRWis such that $T = 1$. In Section 2.4.2 we prove the induction hypothesis, and Section 2.4.3 derives Theorem 2.1.4 from Theorem 2.1.5.

2.4.1 The case of a mono-parameter branching random walk

We consider in a first time a BRWis $(\mathbf{T}, V)$ satisfying additional assumptions that guarantee the sequence $\boldsymbol{\theta}$ to be constant. For all $\varphi \in \mathbb{R}_+$, we write

$$E_p(\varphi) = \sum_{q=1}^{p} (\alpha_q - \alpha_{q-1})(\varphi \kappa_q'(\varphi) - \kappa_q(\varphi)).$$

We assume there exists $\theta > 0$ such that

$$\forall p \leq P, E_p(\theta) \leq 0 \quad \text{and} \quad E_P(\theta) = 0. \tag{2.4.1}$$

We write $a_p = \kappa_p'(\theta)$ and $B = \{p \leq P : E_p = E_{p-1} = 0\}$. By (2.1.2), $\mathbf{a} \in \mathcal{R}$, and by Proposition 2.1.2, $\mathbf{a}$ is the solution of (2.1.7). With these notations, we have

$$v_{\mathrm{is}} = \sum_{p=1}^{P} (\alpha_p - \alpha_{p-1}) a_p \quad \text{and} \quad \lambda = \frac{1}{2\theta}\left(1 + \mathbf{1}_B(1) + \mathbf{1}_B(P)\right). \tag{2.4.2}$$

Theorem 2.4.1. *Under assumptions* (2.1.13) *and* (2.4.1)*, there exists $C > 0$ such that for all $n \geq 1$ and $y \geq 0$,*

$$\mathbf{P}(M_n \geq n v_{\mathrm{is}} - \lambda \log n + y) \leq C(1 + y\mathbf{1}_B(1)) e^{-\theta y}.$$

Moreover, under the additional assumptions (2.1.14) *and* (2.1.15)*, there exists $c > 0$ such that for all $n \geq 1$ and $y \in [0, \sqrt{n}]$,*

$$\mathbf{P}(M_n \geq n v_{\mathrm{is}} - \lambda \log n + y) \geq c(1 + y\mathbf{1}_B(1)) e^{-\theta y}.$$

We write $m_n = nv_{\text{is}} - \lambda \log n$. To obtain the upper bound, we prove the existence of a boundary that no individual crosses with high probability. Then, we bound from above and from below the first two moments of the number of individuals who stayed below the boundary, and end at time n close to m_n.

We denote by

$$K_k^{(n)} = \sum_{p=1}^P \sum_{j=1}^k \kappa_p(\theta)\mathbf{1}_{\left\{j\in[\alpha_{p-1}^{(n)},\alpha_p^{(n)})\right\}}$$

and by $\overline{a}_k^{(n)} = \sum_{p=1}^P \sum_{j=1}^k a_p \mathbf{1}_{\left\{j\in[\alpha_{p-1}^{(n)},\alpha_p^{(n)})\right\}}$ the path followed by the rightmost individual. Using Equation (2.1.2), we observe that

$$\theta\overline{a}_k^{(n)} - K_k^{(n)} = \sum_{p=1}^P \kappa_p^*(a_p) \sum_{j=1}^k \mathbf{1}_{\left\{j\in[\alpha_{p-1}^{(n)},\alpha_p^{(n)})\right\}}, \tag{2.4.3}$$

in particular, if $\varphi_t = \int_0^t \sum_{p=1}^P \kappa_p^*(a_p)\mathbf{1}_{\{s\in[\alpha_{p-1},\alpha_p)\}}ds$, by Riemann integration

$$\sup_{n\geq 0}\sup_{k\leq n}\left|\theta\overline{a}_k^{(n)} - K_k^{(n)} - n\varphi_{\frac{k}{n}}\right| < +\infty. \tag{2.4.4}$$

A boundary for the branching random walk

We prove in a first time that if $1 \in B$, then with high probability, there is no individual to the right of $a_1 k$ at any time $k \leq \alpha_1^{(n)}$.

Lemma 2.4.2. *Under assumption* (2.4.1)*, if* $1 \in B$*, then for all* $y \geq 0$ *and* $n \in \mathbb{N}$*,*

$$\mathbf{P}(\exists u \in \mathbf{T}, |u| \leq \alpha_1^{(n)} : V(u) \geq a_1|u| + y) \leq e^{-\theta y}.$$

Proof. Let $y \geq 0$ and $n \geq 1$, for $k \leq \alpha_1^{(n)}$, we write

$$Z_k^{(n)} = \sum_{|u|=k} \mathbf{1}_{\{V(u)\geq a_1 k+y\}}\mathbf{1}_{\{V(u_j)\leq a_1 j+y, j\leq k\}},$$

the number of individuals who are for the first time at time k above the curve $a_1 k + y$. By use of (2.2.4), we have

$$\mathbf{E}(Z_k^{(n)}) = \mathbf{E}\left[e^{-\theta S_k + k\kappa_1(\theta)}\mathbf{1}_{\{S_k\geq ka_1+y\}}\mathbf{1}_{\{S_j\leq ja_1+y, j<k\}}\right],$$

where S is a random walk with mean

$$\mathbf{E}\left[\sum_{\ell\in L_1} \ell e^{\theta\ell-\kappa_1(\theta)}\right] = \kappa_1'(\theta) = a_1,$$

and finite variance, thanks to (2.1.13). Moreover, as $1 \in B$, we have $E_1 = \theta a_1 - \kappa_1(\theta) = 0$, therefore

$$\mathbf{E}(Z_k^{(n)}) \leq e^{-\theta y}\mathbf{P}(S_k \geq ka_1 + y, S_j \leq ja_1 + y, j < k).$$

As a consequence, by Markov inequality, we have

$$\begin{aligned}
\mathbf{P}(\exists u \in \mathbf{T}, |u| \leq \alpha_1^{(n)} : V(u) \geq a_1|u| + y) &\leq \sum_{k=1}^{\alpha_1^{(n)}} \mathbf{E}(Z_k^{(n)})\\
&\leq e^{-\theta y}\sum_{k=1}^n \mathbf{P}(S_k \geq ka_1 + y, S_j \leq ja_1 + y, j < k)\\
&\leq e^{-\theta y}\mathbf{P}(\exists k \leq n : S_k \geq ka_1 + y),
\end{aligned}$$

which ends the proof. $\square$

We now compute, if $P \in B$, the probability there exists at some time $k \geq \alpha_{P-1}^{(n)}$ an individual above some well-chosen curve. To do so, we denote by

$$r_k^{(n)} = a_P(k-n) + \frac{3}{2\theta}\log(n-k+1).$$

We add a piece of notation to describe the boundary of the branching random walk. We write

$$F^{(n)} = \bigcup_{p \in B \cap \{1,P\}} \left[\alpha_{p-1}^{(n)}, \alpha_p^{(n)}\right],$$

$F_k^{(n)} = F^{(n)} \cap [0,k]$ and, for $j \in F^{(n)}$,

$$f_j^{(n)} = a_1 j \mathbf{1}_{\left\{j \leq \alpha_1^{(n)}\right\}} + \left(m_n + r_k^{(n)}\right)\mathbf{1}_{\left\{j \geq \alpha_{P-1}^{(n)}\right\}}.$$

The following estimate holds.

Lemma 2.4.3. *Under assumptions* (2.1.13) *and* (2.4.1), *if $P \in B$, there exists $C > 0$ such that for all $y \geq 0$ and $n \in \mathbb{N}$,*

$$\mathbf{P}\left[\exists |u| > \alpha_{P-1}^{(n)} : V(u) \geq m_n + r_k^{(n)} + y\right] \leq C(1 + y\mathbf{1}_B(1))e^{-\theta y}.$$

Proof. We assume in a first time that $1 \notin B$. We have $\lambda = \frac{1}{\theta}$ and

$$\begin{aligned}
&\mathbf{P}\left[\exists |u| > \alpha_{P-1}^{(n)} : V(u) \geq m_n + r_k^{(n)} + y\right]\\
\leq &\mathbf{E}\left[\sum_{|u| \geq \alpha_{P-1}^{(n)}} \mathbf{1}_{\left\{V(u) \geq m_n + r_{|u|}^{(n)} + y\right\}} \mathbf{1}_{\left\{V(u_j) \leq m_n + r_j^{(n)} + y, \alpha_{P-1}^{(n)} \leq j < k\right\}}\right]\\
\leq &\sum_{k=\alpha_{P-1}^{(n)}}^{n} \mathbf{E}\left[e^{-\theta S_k + K_k^{(n)}} \mathbf{1}_{\left\{S_k \geq m_n + r_k^{(n)} + y, S_j \leq m_n + r_j^{(n)} + y, \alpha_{P-1}^{(n)} \leq j < k\right\}}\right]\\
\leq &\sum_{k=\alpha_{P-1}^{(n)}}^{n} \frac{C n^{\theta\lambda} e^{-\theta y}}{(n-k+1)^{3/2}} \mathbf{P}\left(S_k \geq m_n + r_k^{(n)} + y, S_j \leq m_n + r_j^{(n)} + y, \alpha_{P-1}^{(n)} \leq j < k\right),
\end{aligned} \tag{2.4.5}$$

by (2.2.4) and (2.4.4). By conditioning with respect to $S_k - S_{k-1}$, we have

$$\mathbf{P}\left(S_k \geq m_n + r_k^{(n)} + y, S_j \leq m_n + r_j^{(n)} + y, \alpha_{P-1}^{(n)} \leq j < k\right) = \mathbf{E}\left[\varphi_k(S_k - S_{k-1} - a_1)\right],$$

writing for $x \in \mathbb{R}$,

$$\begin{aligned}
&\varphi_k(x)\\
=&\mathbf{P}(S_{k-1} \geq m_n + r_k^{(n)} + y - x, S_j \leq m_n + r_j^{(n)} + y, \alpha_{P-1}^{(n)} \leq j \leq k-1)\\
=&\sum_{h=0}^{+\infty} \mathbf{P}(S_{k-1} - m_n - r_k^{(n)} - y - h \in [h, h+1), S_j \leq m_n + r_j^{(n)} + y, \alpha_{P-1}^{(n)} \leq j \leq k-1)\\
\leq &\sum_{h=0}^{\lfloor x \rfloor} C \frac{1+h}{n^{1/2}(k - \alpha_{P-1}^{(n)})^{1/2}}\\
\leq &C \frac{(1+x_+)^2}{n^{1/2}(k - \alpha_{P-1}^{(n)})^{1/2}},
\end{aligned}$$

by use of Lemma 2.3.1. Thus, by (2.1.13),

$$\mathbf{P}\left(S_k \geq m_n + r_k^{(n)} + y, S_j \leq m_n + r_j^{(n)} + y, \alpha_{P-1}^{(n)} \leq j < k\right) \leq \frac{C}{n^{1/2}(k-\alpha_{P-1}^{(n)})^{1/2}}.$$

As a consequence (2.4.5) becomes

$$\mathbf{P}\left[\exists |u| > \alpha_{P-1}^{(n)} : V(u) \geq m_n + r_k^{(n)} + y\right] \leq \sum_{k=\alpha_{P-1}^{(n)}}^{n} Ce^{-\theta y} \frac{n^{1/2}}{(k-\alpha_{P-1}^{(n)}+1)^{1/2}(n-k+1)^{3/2}} \leq Ce^{-\theta y}.$$

In a second time, if $1 \in B$, then $\lambda = \frac{3}{2\theta}$. We have

$$\begin{aligned}\mathbf{P}\left[\exists |u| > \alpha_{P-1}^{(n)} : V(u) \geq m_n + r_k^{(n)} + y\right] &\leq \mathbf{P}\left[\exists |u| \leq \alpha_1^{(n)} : V(u) \geq a_1|u| + y\right] \\ &\quad + \mathbf{P}\left[\exists |u| \geq \alpha_{P-1}^{(n)} : V(u) \geq f_k^{(n)} + y, V(u_j) \leq f_j^{(n)} + y, j \in F_{k-1}^{(n)}\right].\end{aligned}$$

Using Lemma 2.4.2, we only need to bound the second part of this inequality. Then, by (2.2.4), we have, for $k \geq \alpha_{P-1}^{(n)}$

$$\begin{aligned}&\mathbf{E}\left[\sum_{|u|=k} \mathbf{1}_{\left\{V(u)\geq f_k^{(n)}+y\right\}} \mathbf{1}_{\left\{V(u_j)\leq f_j^{(n)}+y, j\in F_{k-1}^{(n)}\right\}}\right] \\ &\quad \leq \mathbf{E}\left[e^{-\theta S_k + K_k^{(n)}} \mathbf{1}_{\left\{S_k\geq f_k^{(n)}+y\right\}} \mathbf{1}_{\left\{S_j\leq f_j^{(n)}+y, j\in F_{k-1}^{(n)}\right\}}\right] \\ &\quad \leq C\frac{n^{\theta\lambda}}{(n-k+1)^{3/2}} e^{-\theta y} \mathbf{P}(S_k \geq f_k^{(n)} + y, S_j \leq f_j^{(n)} + y, j \in F_{k-1}^{(n)}) \\ &\quad \leq C(1+y)e^{-\theta y} \frac{n^{3/2}}{(k-\alpha_{P-1}^{(n)}+1)^{3/2}(n-k+1)^{3/2}},\end{aligned}$$

using again Lemma 2.3.1, and conditioning with respect to the last step of the random walk. Thus, by Markov inequality, we have

$$\begin{aligned}&\mathbf{P}\left[\exists |u| \geq \alpha_{P-1}^{(n)} : V(u) \geq f_k^{(n)} + y, V(u_j) \leq f_j^{(n)} + y, j \in F_{k-1}^{(n)}\right] \\ &\quad \leq C(1+y)e^{-\theta y} \sum_{k=\alpha_{P-1}^{(n)}}^{n} \frac{n^{3/2}}{(k-\alpha_{P-1}^{(n)}+1)^{3/2}(n-k+1)^{3/2}} \leq C(1+y)e^{-\theta y},\end{aligned}$$

which ends the proof. □

These two lemmas imply that with high probability, there is no individual above $f^{(n)}+y$ at any time in $F^{(n)}$. To complete the proof of the upper bound for the tail distribution of M_n, we compute the number of individuals who, travelling below that boundary, are at time n in a neighbourhood of m_n. We write

$$X^{(n)}(y,h) = \sum_{|u|=n} \mathbf{1}_{\{V(u)-m_n-y\in[-h,-h+1]\}} \mathbf{1}_{\left\{V(u_j)\leq f_j^{(n)}+y, j\in F^{(n)}\right\}}.$$

Lemma 2.4.4. *Under assumptions* (2.1.13) *and* (2.4.1)*, there exists $C > 0$ such that for all $n \geq 1$, $y \in \mathbb{R}_+$ and $h \in \mathbb{R}$, we have*

$$\mathbf{E}(X^{(n)}(y,h)) \leq C(1 + y\mathbf{1}_B(1))(1 + h_+\mathbf{1}_B(P))e^{-\theta(y-h)}.$$

Proof. We observe that if $P \in B$ and $h < -1$, then $X^{(n)}(y,h) = 0$. Otherwise, using Equation (2.2.4), we have

$$\begin{aligned}\mathbf{E}(X^{(n)}(y,h)) &= \mathbf{E}\left[e^{-\theta S_n + K_n^{(n)}}\mathbf{1}_{\{S_n - m_n - y \in [-h,-h+1]\}}\mathbf{1}_{\left\{S_j \leq f_j^{(n)}+y, j \in F^{(n)}\right\}}\right]\\ &\leq Cn^{\theta\lambda}e^{-\theta(y-h)}\mathbf{P}\left(S_n - m_n - y \in [-h,-h+1], S_j \leq f_j^{(n)} + y, j \in F^{(n)}\right)\end{aligned}$$

by Equation 2.4.4. Applying Lemma 2.3.1, we obtain

$$\begin{aligned}&\mathbf{P}\left(S_n - f_n^{(n)} - y \in [-h,-h+1], S_j \leq f_j^{(n)} + y, j \in F^{(n)}\right)\\ &\qquad\qquad \leq C\frac{(1 + y\mathbf{1}_B(1))(1 + h_+\mathbf{1}_B(P))}{(n+1)^{(1+\mathbf{1}_B(1)+\mathbf{1}_B(P))/2}}.\end{aligned}$$

□

These lemmas can be used to obtain a tight upper bound for $\mathbf{P}(M_n \geq nv_{\text{is}} - \lambda\log n + y)$.

Corollary 2.4.5. *Under assumptions* (2.1.13) *and* (2.4.1)*, there exists $C > 0$ such that for all $y \geq 0$ and $n \in \mathbb{N}$, we have*

$$\mathbf{P}(M_n \geq nv_{\text{is}} - \lambda\log n + y) \leq C(1 + y\mathbf{1}_B(1))e^{-\theta y}.$$

Proof. Let $y \geq 0$ and $n \in \mathbb{N}$, we have

$$\mathbf{P}(M_n \geq nv_{\text{is}} - \lambda\log n + y) \leq \mathbf{P}\left(\exists |u| \in F^{(n)} : V(u) \geq f_{|u|}^{(n)} + y\right) + \sum_{h=0}^{+\infty}\mathbf{E}\left(X^{(n)}(y,-h)\right).$$

Using Lemmas 2.4.2 and 2.4.3, we have

$$\mathbf{P}\left(\exists |u| \in F^{(n)} : V(u) \geq f_{|u|}^{(n)} + y\right) \leq C(1 + y\mathbf{1}_B(1))e^{-\theta y}$$

and applying Lemma 2.4.4, we obtain

$$\sum_{h=0}^{+\infty}\mathbf{E}(X^{(n)}(y,-h)) \leq C(1 + y\mathbf{1}_B(1))e^{-\theta y}\sum_{h=0}^{+\infty}e^{-\theta h} \leq C(1 + y\mathbf{1}_B(1))e^{-\theta y}.$$

□

Lower bound through a second moment computation

To bound from below $\mathbf{P}(M_n \geq m_n + y)$, we bound from below the probability there exists an individual alive at time n, which stayed an any time $k \leq n$ below some curve $g^{(n)}$ defined below and is at time n above m_n. We write $B^{(n)} = \cup_{p\in B}(\alpha_{p-1}^{(n)}, \alpha_p^{(n)}]$ the set of times such that the optimal path is close to the boundary of the BRWis. We choose $\delta > 0$

small enough such that $3\theta\delta < \min_{p \in B^c} -E_p(\theta)$. For all $n \geq 1$, $p \leq P$ and $k \in (\alpha_{p-1}^{(n)}, \alpha_p^{(n)}]$ we define

$$g_k^{(n)} = 1 + \begin{cases} \overline{a}_k^{(n)} - \mathbf{1}_{\{p=P\}} \lambda \log n & \text{if} \quad E_p(\theta) = E_{p-1}(\theta) = 0 \\ \overline{a}_k^{(n)} + (k - \alpha_{p-1}^{(n)})\delta & \text{if} \quad E_{p-1}(\theta) = 0, E_p(\theta) < 0 \\ \overline{a}_k^{(n)} + (\alpha_p^{(n)} - k)\delta & \text{if} \quad E_p(\theta) = 0, E_{p-1}(\theta) < 0 \\ \overline{a}_k^{(n)} + \delta n & \text{otherwise.} \end{cases} \tag{2.4.6}$$

With this definition, using (2.4.4), we have,

$$\theta g_k^{(n)} - K_k^{(n)} \leq C + \begin{cases} -\mathbf{1}_{\{p=P\}} \theta \lambda \log n & E_p(\theta) = E_{p-1}(\theta) = 0 \\ -\delta(k - \alpha_{p-1}^{(n)}) & \text{if} \quad E_{p-1}(\theta) = 0, E_p(\theta) < 0 \\ -\delta(\alpha_{p+1}^{(n)} - k) - \mathbf{1}_{\{p=P\}} \theta \lambda \log n & \text{if} \quad E_{p-1}(\theta) < 0, E_p(\theta) = 0 \\ -\delta n & \text{if} \quad E_{p-1}(\theta) > 0, E_p(\theta) > 0. \end{cases} \tag{2.4.7}$$

We prove in the rest of the section that the set

$$\mathcal{A}_n(y) = \left\{ u \in \mathbf{T} : V(u) \geq m_n + y, V(u_j) \leq g_j^{(n)} + y, j \leq n \right\}$$

is non-empty. To do so, we restrict this set to individuals with a constraint on their reproduction. For $u \in \mathbf{T}$, we denote by

$$\xi(u) = \sum_{u' \in \Omega(u)} \left(1 + (V(u') - V(u))_+ \mathbf{1}_{\left\{|u| \in B^{(n)}+1\right\}} \right) e^{\theta(V(u') - V(u))}$$

a quantity closely related to the spread of the offspring of u. We write, for $z > 0$ and $p \leq P$

$$\mathcal{B}_n(z) = \left\{ u \in \mathbf{T} : |u| = n, \xi(u_j) \leq z e^{-\frac{\theta}{2}\left[V(u_j) - g_j^{(n)}\right]}, j < n \right\},$$

and we consider the set $G_n(y,z) = \mathcal{A}_n(y) \cap \mathcal{B}_n(z)$. We compute the first two moments of

$$Y_n(y,z) = \sum_{|u|=n} \mathbf{1}_{\{u \in G_n(y,z)\}},$$

to bound from below $\mathbf{P}(Y_n(y,z) \geq 1)$, using the Cauchy-Schwarz inequality. We begin with an upper bound of the second moment of Y_n.

Lemma 2.4.6. *Under assumptions* (2.1.13) *and* (2.4.1), *there exists $C > 0$ such that for all $y \geq 0$, $z > 0$ and $n \in \mathbb{N}$, we have*

$$\mathbf{E}(Y_n(y,z)^2) \leq Cz(1 + y\mathbf{1}_B(1))e^{-\theta y}.$$

Proof. Applying Lemma 2.2.1, we have

$$\begin{aligned} \mathbf{E}(Y_n(y,z)^2) &= \overline{\mathbf{E}}\left[\frac{1}{W_n} Y_n(y,z)^2\right] = \widehat{\mathbf{E}}\left[\frac{1}{W_n} \sum_{|u|=n} \mathbf{1}_{\{u \in G_n(y,z)\}} Y_n(y,z)\right] \\ &= \widehat{\mathbf{E}}\left[e^{-\theta V(w_n) + K_n^{(n)}} \mathbf{1}_{\{w_n \in G_n(y,z)\}} Y_n(y,z)\right]. \end{aligned}$$

Using the fact that $w_n \in \mathcal{A}_n(y) \subset G_n(y,z)$, we have

$$\mathbf{E}(Y_n(y,z)^2) \leq C n^{\theta\lambda} e^{-\theta y} \widehat{\mathbf{E}}\left[Y_n(y,z) \mathbf{1}_{\{w_n \in G_n(y,z)\}} \right].$$

We decompose $Y_n(y,z)$ along the spine, to obtain

$$Y_n(y,z) \leq \mathbf{1}_{\{w_n \in G_n(y,z)\}} + \sum_{k=0}^{n-1} \sum_{u \in \Omega(w_k)} Y_n(u,y),$$

where, for $u \in \mathbf{T}$ and $y \geq 0$, we write $Y_n(u,y) = \sum_{|u'|=n, u'>u} \mathbf{1}_{\{u' \in \mathcal{A}_n(y)\}}$. Let $k < n$. We recall that conditionally on $\mathcal{G}_n$, the branching random walks of the descendants of distinct children $u, v \in \Omega(w_k)$ are independent. Moreover, the branching random walk extracted from an individual $u \in \Omega(w_k)$ has law $\mathbf{P}_{V(u),k+1}$. As a consequence, for $y \geq 0$, $k < n$ and $u \in \Omega(w_k)$,

$$\widehat{\mathbf{E}}\left[Y_n(u,y) | \mathcal{G}_n \right] = \mathbf{E}_{V(u),k+1}\left[\sum_{|u'|=n-k-1} \mathbf{1}_{\{V(u') \geq m_n + y\}} \mathbf{1}_{\left\{ V(u'_j) \leq g^{(n)}_{k+j+1} + y, j \leq n-k \right\}} \right].$$

We use (2.2.4) and (2.4.4) to obtain

$$\begin{aligned} &\widehat{\mathbf{E}}\left[Y_n(u,y) | \mathcal{G}_n \right] \\ &\leq C n^{\lambda\theta} e^{\theta V(u) - K^{(n)}_{k+1} - \theta y} \mathbf{P}_{V(u),k+1}\left(S_{n-k-1} \geq m_n + y, S_j \leq g^{(n)}_{j+k+1} + y, j \leq n-k-1 \right). \end{aligned}$$

We now apply Lemma 2.3.1. For all $p \leq P$ and $k \in [\alpha^{(n)}_{p-1}, \alpha^{(n)}_p)$, we have

$$\begin{aligned} &\mathbf{P}_{V(u),k+1}(S_{n-k-1} \geq m_n + y, S_j \leq g^{(n)}_{j+k+1} + y, j \leq n-k-1) \\ &\qquad \leq \begin{cases} C \frac{1 + (g^{(n)}_{k+1} + y - V(u))_+ \mathbf{1}_B(p)}{(\alpha^{(n)}_p - k + 1)^{\mathbf{1}_B(p)/2} n^{(1 + \mathbf{1}_B(p))/2}} & \text{if} \quad p < P-1 \\ C \frac{1 + (g^{(n)}_{k+1} + y - V(u))_+ \mathbf{1}_B(P)}{(n-k+1)^{1/2 + \mathbf{1}_B(P)}} & \text{if} \quad p = P. \end{cases} \end{aligned} \tag{2.4.8}$$

Let $p \leq P$ and $k \in [\alpha^{(n)}_{p-1}, \alpha^{(n)}_p)$, we compute the quantity

$$h_k := \widehat{\mathbf{E}}\left[\mathbf{1}_{\{w_n \in G_n(y,z)\}} \sum_{u \in \Omega(w_k)} (1 + (g^{(n)}_{k+1} + y - V(u))_+ \mathbf{1}_B(p)) e^{\theta V(u) - g^{(n)}_{k+1}} \right].$$

Using (2.4.3), the definition of $\xi(w_k)$ and the fact $x \mapsto x_+$ is Lipschitz, we have

$$\begin{aligned} h_k &\leq C \widehat{\mathbf{E}}\left[e^{\theta(V(w_k) - g^{(n)}_k)} (1 + (g^{(n)}_k + y - V(w_k)_+) \xi(w_k) \mathbf{1}_{\{w_n \in G_n(y,z)\}} \right] \\ &\leq C z \widehat{\mathbf{E}}\left[e^{\frac{\theta}{2}(V(w_k) - g^{(n)}_k)} (1 + (g^{(n)}_k + y - V(w_k)_+) \xi(w_k) \mathbf{1}_{\{w_n \in \mathcal{A}_n(y,z)\}} \right] \end{aligned}$$

as $w_n \in \mathcal{B}_n(z)$. Decomposing this expectation with respect tot the value taken by $V(w_k)$, we obtain

$$h_k \leq C z e^{\theta y} \sum_{i=0}^{+\infty} (1+i) e^{-\theta i/2} \mathbf{P}\left[\begin{array}{l} S_n \geq m_n + y, S_k - g^{(n)}_k - y \in [-i-1, -i], \\ S_j \leq g^{(n)}_j + y, j \in B^{(n)} \end{array} \right].$$

We apply the Markov property at time k and Lemma 2.3.1 to obtain, if $p \in B$

$$h_k \leq \begin{cases} Cz\frac{(1+y)e^{\theta y}}{k^{3/2}(\alpha_1^{(n)}-k+1)^{1/2}}\frac{n^{1/2}}{n^{\theta\lambda}} & \text{if} \quad p = 1 \\ Cz\frac{(1+y\mathbf{1}_B(1))e^{\theta y}}{(k-\alpha_{p-1}^{(n)})^{1/2}(\alpha_p^{(n)}-k+1)^{1/2}n^{1/2}}\frac{1}{n^{\theta\lambda}} & \text{if} \quad 1 < p < P \\ Cz\frac{(1+y\mathbf{1}_B(1))e^{\theta y}}{(k-\alpha_{P-1}^{(n)}+1)^{1/2}(n-k+1)^{3/2}}\frac{1}{n^{\theta\lambda}} & \text{if} \quad p = P. \end{cases} \tag{2.4.9}$$

In the same way, if $p \notin B$, we have

$$h_k \leq \begin{cases} Cze^{\theta y}\frac{1}{k^{1/2}}\frac{1}{n^{\theta\lambda}} & \text{if} \quad k < \alpha_1^{(n)} \\ Cze^{\theta y}\frac{1+y\mathbf{1}_B(1)}{n^{1/2}}\frac{1}{n^{\theta\lambda}} & \text{if} \quad \alpha_1^{(n)} \leq k < \alpha_{P-1}^{(n)} \\ Cze^{\theta y}\frac{1+y\mathbf{1}_B(1)}{(n-k+1)^{1/2}}\frac{1}{n^{\theta\lambda}} & \text{otherwise}, \end{cases} \tag{2.4.10}$$

applying again Lemma 2.3.1.

For $p \leq P$ we denote by

$$\begin{aligned} H_p &:= \sum_{k=\alpha_{p-1}^{(n)}}^{\alpha_p^{(n)}-1} \widehat{\mathbf{E}}\left[\mathbf{1}_{\{w_n \in G_n(y,z)\}} \sum_{u \in \Omega(w_k)} Y_n(u,y)\right] \\ &\leq C \sum_{k=\alpha_{p-1}^{(n)}}^{\alpha_p^{(n)}-1} h_k e^{\theta g_{k+1}^{(n)} - K_{k+1}^{(n)}}. \end{aligned}$$

Using (2.4.3), and summing the estimates (2.4.8), (2.4.9) and (2.4.10), we obtain that $H_p \leq Cz(1+y\mathbf{1}_B(1))e^{-\theta y}$ for all $p \leq P$. To conclude this proof, we observe that

$$\mathbf{E}(Y_n(y,z)^2) \leq \sum_{p=1}^{P} H_p + Cn^{\theta\lambda}e^{-\theta y}\mathbf{P}(w_n \in G_n(y,z)) \leq Cz(1+y\mathbf{1}_B(1))e^{-\theta y},$$

as Lemma 2.3.1 implies $\mathbf{P}(w_n \in G_n(y,z)) \leq C(1+y\mathbf{1}_B(1))n^{-\theta\lambda}$. □

We now prove the following result, a lower bound on the first moment of $Y_n(y,B)$.

Lemma 2.4.7. *Under assumptions* (2.1.13), (2.1.14), (2.1.15) *and* (2.4.1), *there exists* $c > 0$ *and* $z > 0$ *such that for all* $n \geq 0$ *and* $y \in [0, \sqrt{n}]$, *we have*

$$\mathbf{E}(Y_n(y,z)) \geq c(1+\mathbf{1}_B(1)y)e^{-\theta y}.$$

Proof. Using Lemma 2.2.1, we have

$$\mathbf{E}(Y_n(y,z)) = \widehat{\mathbf{E}}\left[e^{-\theta V(w_n)+K_n^{(n)}}\mathbf{1}_{\{w_n \in G_n(y,z)\}}\right] \geq cn^{\theta\lambda}e^{-\theta y}\widehat{\mathbf{P}}(w_n \in G_n(y,z)).$$

To bound $\widehat{\mathbf{P}}(w_n \in G_n(y,z))$, we observe that

$$\widehat{\mathbf{P}}(w_n \in G_n(y,z)) = \widehat{\mathbf{P}}(w_n \in \mathcal{A}_n(y)) - \widehat{\mathbf{P}}(w_n \in \mathcal{A}_n(y) \cap \mathcal{B}_n(z)^c).$$

Moreover, by Lemma 2.3.2, for all $n \geq 1$ and $y \in [0, \sqrt{n}]$

$$\widehat{\mathbf{P}}(w_n \in \mathcal{A}_n(y)) = \mathbf{P}(S_n \geq m_n + y, S_j \leq g_j^{(n)} + y, j \leq n) \geq c(1+y\mathbf{1}_B(1))n^{-\theta\lambda}.$$

Therefore, we only need to bound from above $\widehat{\mathbf{P}}(w_n \in \mathcal{A}_n(y) \cap \mathcal{B}_n(z)^c)$ for $z > 0$ large enough.

We write

$$\tau^{(n)}(z) = \inf\left\{k \leq n : \xi(w_k) \geq z \exp\left(-\frac{\theta}{2}\left[V(w_k) - g_k^{(n)}\right]\right)\right\},$$

and, for $p \leq P$, $\pi_p = \widehat{\mathbf{P}}\left(w_n \in \mathcal{A}_n(y), \tau^{(n)}(z) \in (\alpha_{p-1}^{(n)}, \alpha_p^{(n)}]\right)$. We introduce, for $p \leq P$ the random variables

$$(\xi_p, \Delta_p) \stackrel{(d)}{=} (V(w_{k+1} - V(w_k), \xi(w_k)) \quad \text{for } k \in [\alpha_{p-1}^{(n)}, \alpha_p^{(n)}).$$

Let (ξ_n^p, Δ_n^p) be i.i.d random variables with the same law as (ξ_p, Δ_p). We introduce the random walk $T_n^p = \Delta_1^p + \cdots + \Delta_n^p$. Keeping in mind (2.3.12) and Lemma 2.3.3, we define for $p \in B$ and $z \geq 1$ the functions

$$\chi_p(z) = \widehat{\mathbf{E}}\left[\left(1 + (\log_+(\xi_p) - \Delta_p - \log z)_+\right)^2 \mathbf{1}_{\{\xi_p \geq z\}}\right]$$

and, for $p \in B^c$

$$\widetilde{\chi}_p(z) = \widehat{\mathbf{E}}\left[\frac{(\log_+(\xi_p) - \Delta_p - \log z/2)_+}{\delta}\right] + \begin{cases} \mathbf{E}\left[\sum_{k=0}^{+\infty} \mathbf{1}_{\left\{T_k^p \geq (\delta k + \log z)/2\right\}}\right] & \text{if } \quad E_p(\theta) < 0 \\ \mathbf{E}\left[\sum_{k=0}^{+\infty} \mathbf{1}_{\left\{T_k^p \leq -(\delta k + \log z)/2\right\}}\right] & \text{if } \quad E_p(\theta) = 0, E_{p-1}(\theta) < 0. \end{cases}$$

First, if $p = 1$, we apply the Markov property at time $\alpha_1^{(n)}$ and Lemma 2.3.1 to obtain

$$\pi_1 \leq C \frac{1}{n^{(1+\mathbf{1}_B(P))/2}} \mathbf{P}\left(T_j^1 \leq g_j^{(n)} + y, j \leq \alpha_1^{(n)}, \exists k \leq \alpha_1^{(n)} : \xi_k^1 \geq z e^{-\theta/2(T_k^1 - g_k^{(n)})}\right).$$

As a consequence, if $1 \in B$, we apply (2.3.11) to obtain $\pi_1 \leq C \frac{1+y}{n^{\theta\lambda}} \chi_1(z)$; and if $1 \notin B$, then $E_1 < 0$ so, applying Lemma 2.3.3 we have $\pi_1 \leq C \frac{1}{n^{\theta\lambda}} \widetilde{\chi}_1(z)$.

We now suppose that $1 < p < P$. Applying the Markov property at times $\alpha_p^{(n)}$ and $\alpha_{p-1}^{(n)}$, we have

$$\pi_p \leq \frac{C}{n^{(1+\mathbf{1}_B(P))/2}} \widehat{\mathbf{E}}\left[\mathbf{1}_{\left\{V(w_j) \leq g_j^{(n)} + y, j \leq \alpha_{p-1}^{(n)}\right\}} \varphi_p\left(V(w_{\alpha_{p-1}^{(n)}})\right)\right], \tag{2.4.11}$$

where we write, for $s \in \mathbb{R}$

$$\varphi_p(s) = \mathbf{P}_s\left[T_j^p \leq g_{\alpha_{p-1}^{(n)}+j}^{(n)} + y, j \leq \alpha_p^{(n)} - \alpha_{p-1}^{(n)}, \tau^{(n)}(z) \in (\alpha_{p-1}^{(n)}, \alpha_p^{(n)}]\right].$$

If $p \in B$, applying (2.3.11), we have $\varphi_p(s) \leq \frac{1+y+s}{n^{1/2}} \chi_p(z)$, and, by (2.3.3),

$$\sup_{n \in \mathbb{N}} \frac{1}{n^{1/2}} \mathbf{E}\left[\left|S_{\alpha_{p-1}^{(n)}} - \overline{a}_{\alpha_{p-1}^{(n)}}^{(n)}\right| \middle| S_j \leq g_j^{(n)} + y, j \leq \alpha_{p-1}^{(n)}\right] < +\infty.$$

Then, by Lemma 2.3.1, as $y \leq \sqrt{n}$, we have $\pi_p \leq \frac{C(1+y\mathbf{1}_B(1))}{n^{\theta\lambda}} \chi_p(z)$.

In the same way, if $p \notin B$, we use Lemma 2.3.3, and time-reversal when $E_p(\theta) = 0$ and $E_{p-1}(\theta) < 0$– to have $\varphi_p(s) \leq \widetilde{\chi}_p(z)$, which, thanks to (2.4.11) leads to $\pi_p \leq \frac{1+y\mathbf{1}_B(1)}{n^{\theta\lambda}} \widetilde{\chi}_p(z)$.

We finally take care of the case $p = P$. If $P \in B$, we apply the Markov property and (2.3.12) to obtain

$$\begin{aligned}
\pi_P &\leq C\,\mathbf{E}\left[\frac{1+\left(S_{\alpha^{(n)}_{P-1}} - a^{(n)}_{\alpha^{(n)}_{P-1}} + y\right)_+}{n^{3/2}} \mathbf{1}_{\left\{S_j \leq g^{(n)}_j + y, j \leq \alpha^{(n)}_{P-1}\right\}}\right]\\
&\leq C\frac{1+y\mathbf{1}_B(1)}{n^{\theta\lambda}}\chi_P(z).
\end{aligned}$$

If $P \notin B$, we use the time-reversal, then Lemma 2.3.3 to obtain

$$\begin{aligned}
\pi_P &\leq C\widetilde{\chi}_P(z) \sup_{h\in\mathbb{R}} \mathbf{P}\left[S_{\alpha^{(n)}_{p-1}} \in [h,h+1], S_j \leq g^{(n)}_j + y, j \leq \alpha^{(n)}_{P-1}\right]\\
&\leq C\frac{1+y\mathbf{1}_B(1)}{n^{\theta\lambda}}\widetilde{\chi}_P(z).
\end{aligned}$$

We conclude there exists $C > 0$ such that

$$\widehat{\mathbf{P}}(w_n \in \mathcal{A}^{(n)}(y) \cap \mathcal{B}^{(n)}(z)^c) \leq C\frac{1+y\mathbf{1}_B(1)}{n^{\theta\lambda}}\left[\sum_{p\in B}\chi_p(z) + \sum_{p\in B^c}\widetilde{\chi}_p(z)\right].$$

If $p \in B$, by (2.1.14) and (2.1.13), $\mathbf{E}((\log \xi_p - \Delta_p)^2) < +\infty$. In the same way, if $p \notin B$, using (2.1.15) and (2.1.13) again, we have $\mathbf{E}((\log \xi_p - \Delta_p)_+) < +\infty$. Applying the dominated convergence theorem, we have

$$\lim_{z\to+\infty}\left[\sum_{p\in B}\chi_p(z) + \sum_{p\in B^c}\chi_p(z)\right] = 0.$$

Consequently, there exists $z \geq 0$ such that $\widehat{\mathbf{P}}(w_n \in \mathcal{A}^{(n)}(y) \cap \mathcal{B}^{(n)}(z)^c) \leq c\frac{(1+y\mathbf{1}_B(1))}{2n^{\theta\lambda}}$, therefore,

$$\begin{aligned}
\widehat{\mathbf{P}}(w_n \in \mathcal{A}^{(n)}(y) \cap \mathcal{B}^{(n)}(z)) &\geq \widehat{\mathbf{P}}(w_n \in \mathcal{A}^{(n)}(y)) - \widehat{\mathbf{P}}(w_n \in \mathcal{A}^{(n)}(y) \cap \mathcal{B}^{(n)}(z)^c)\\
&\geq c/2\frac{1+y\mathbf{1}_B(1)}{n^{\theta\lambda}},
\end{aligned}$$

which ends the proof. □

Using these two lemmas, we obtain a lower bound on M_n.

Lower bound in Theorem 2.4.1. By Lemma 2.4.7, there exist $c > 0$ and $z > 0$ such that for all $n \geq 1$ and $y \in [0, \sqrt{n}]$, we have

$$\mathbf{E}(Y_n(y,z)) \geq c(1+y\mathbf{1}_B(1))e^{-\theta y}.$$

Thus, using Lemma 2.4.6 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\mathbf{P}(Y_n(y,z) \geq 1) &\geq \frac{\mathbf{E}(Y_n(y,z))^2}{\mathbf{E}(Y_n(y,z)^2)} \geq \frac{\left(c(1+y\mathbf{1}_B(1))e^{-\theta y}\right)^2}{Cz(1+y\mathbf{1}_B(1))e^{-\theta y}}\\
&\geq c(1+y\mathbf{1}_B(1))e^{-\theta y}.
\end{aligned}$$

□

2.4.2 Extension to the multi-parameter branching random walk

In this section, we extend Theorem 2.4.1 to BRWis such that $\boldsymbol{\theta}$ is non-constant, reasoning by induction on the number T of different values taken by the sequence.

Proof of Theorem 2.1.5. If $T = 1$, then the branching random walk satisfies all the hypotheses of Theorem 2.4.1, with optimal path $\mathbf{a}$, and parameter $\theta = \varphi_1$, by Proposition 2.1.2. The initiation of the recurrence is then given by Theorem 2.4.1. Therefore, we only need to prove the induction hypothesis.

Let $T \in \mathbb{N}$, we assume that for all BRWis such that $\#\{\theta_p, p \leq P\} < T$, Theorem 2.1.5 holds. For $n \in \mathbb{N}$, we now consider a BRWis $(\mathbf{T}^{(n)}, V^{(n)})$ of length n. We write $\mathbf{a}$ the optimal solution of Proposition 2.1.2, and $\theta_p = \kappa_p'(a_p)$. We assume that $T = \#\{\theta_p, p \leq P\}$, and write $\varphi_1 < \varphi_2 < \cdots < \varphi_T$ these values, listed in the increasing order. For any $t \leq T$, let $f_t = \min\{p \leq P : \theta_p = \varphi_t\}$ and $l_t = \max\{p \leq P : \theta_p = \varphi_t\}$. Finally, we write v_{is} and λ the speed and correction as defined in (2.1.6) and (2.1.10), and $m_n = nv_{\text{is}} - \lambda \log n$ the expected position of the maximal displacement M_n. We now divide this BRWis into two parts, before and after the first time α_{l_1} such that $\theta_{l_1+1} > \theta_{l_1}$.

We write $l = l_1$ and

$$v_1 = \sum_{p=1}^{l} (\alpha_p - \alpha_{p-1}) a_p \quad \text{and} \quad \lambda_1 = \frac{1}{2\varphi_1} \left(1 + \mathbf{1}_B(1) + \mathbf{1}_B(l)\right).$$

We denote by $\mathbf{T}_1^{(n)} = \{u \in \mathbf{T} : |u| \leq \alpha_l n\}$ the tree cut at generation $n_1 = \lfloor \alpha_l n \rfloor$. By Proposition 2.1.2, we observe that $(\mathbf{T}_1^{(n)}, V^{(n)}_{|\mathbf{T}_1^{(n)}})$ is a BRWis which satisfies the hypotheses of Theorem 2.4.1, with parameter $\theta := \varphi_1$. Therefore, if we write $m_n^1 = v_1 n - \lambda_1 \log n$ and $M_n^1 = \max_{|u|=n_1} V(u)$, there exist $c, C > 0$ such that for all $n \in \mathbb{N}$ large enough and $y \in [0, n^{1/2}]$, we have

$$c(1 + y\mathbf{1}_{\left\{\kappa_1^*(a_1)=0\right\}})e^{-\varphi_1 y} \leq \mathbf{P}(M_n^1 \geq m_n^1 + y) \leq C(1 + y\mathbf{1}_{\left\{\kappa_1^*(a_1)=0\right\}})e^{-\varphi_1 y}.$$

We now consider a branching random walk $(\mathbf{T}^{(n)}_{\text{tail}}, V^{(n)}_{\text{tail}})$ of law $\mathbf{P}_{n_1,0}$, which has the law of the branching random walk of the descendants of any individual alive at time n_1. We write $n_{\text{tail}} = n - n_1$ the length of this BRWis, and

$$v_{\text{tail}} = v - v_1, \quad \lambda_{\text{tail}} = \lambda - \lambda_1 \quad \text{and} \quad m_n^{\text{tail}} = v_{\text{tail}} n - \lambda_{\text{tail}} \log n.$$

By Proposition 2.1.2 again, this marked tree is a BRWis and its optimal path is the path driven by $(a_{l+1}, \ldots, a_P)$. Moreover, $\#\{\theta_p, l < p \leq P\} = T - 1 < T$. Therefore, by the induction hypothesis, writing $M_n^{\text{tail}} = \max_{|u|=n_{\text{tail}}} V^{\text{tail}}(u)$, there exist $c, C > 0$ such that for all $n \in \mathbb{N}$ large enough and $y \in [0, n^{1/2}]$, we have

$$ce^{-\varphi_2 y} \leq \mathbf{P}(M_n^{\text{tail}} \geq m_n^{\text{tail}} + y) \leq C(1 + y)e^{-\varphi_2 y}.$$

To obtain the lower bound of Theorem 2.1.5, we observe that if $M_n^1 \geq m_n^1 + y$, and if one of the descendants of the rightmost individual at time n_1 makes a displacement greater than m_n^{tail}, then $M_n \geq m_n + y$. Therefore

$$\mathbf{P}(M_n \geq nv_{\text{is}} - \lambda \log n + y) \geq c(1 + y\mathbf{1}_B(1))e^{-\varphi_1 l}$$

for $n \in \mathbb{N}$ large enough and $y \in [0, n^{1/2}]$. To obtain an upper bound for $\mathbf{P}(M_n \geq m_n + y)$, we decompose the n^{th} generation of the branching random walk with respect to the position of their ancestors alive at time n_1. We write

$$X^{(n)}(y,h) = \sum_{|u|=n_1} \mathbf{1}_{\left\{V(u_j) \leq f_j^{(n_1)} + y, j \in C_1^{(n_1)}\right\}} \mathbf{1}_{\{V(u) - m_n^1 - y \in [-h-1, -h]\}},$$

and, by union bound and the Markov property, we have

$$\begin{aligned}\mathbf{P}(M_n \geq m_n + y) \leq \mathbf{P}(\exists |u| \in C_1^{(n_1)} : V(x) \geq f_k^{(n_1)} + r_k^{(n)} + y) \\ + \sum_{h=0}^{+\infty} \mathbf{E}(X^{(n)}(y,h)) \mathbf{P}(M_n^{\text{tail}} \geq m_n^{\text{tail}} + h).\end{aligned}$$

As a consequence, applying Lemma 2.4.4 and the upper bound of Theorem 2.4.1,

$$\begin{aligned}\mathbf{P}(M_n \geq m_n + y) &\leq C(1 + y\mathbf{1}_{\left\{\kappa_1^*(a_1)=0\right\}}) e^{-\varphi_1 y} \left[1 + \sum_{h=0}^{+\infty} (1+h) e^{(\varphi_1 - \varphi_2)h}\right] \\ &\leq C(1 + y\mathbf{1}_{\left\{\kappa_1^*(a_1)=0\right\}}) e^{-\varphi_1 y},\end{aligned}$$

which gives the correct upper bound. □

2.4.3 Proof of Theorem 2.1.4

Using Theorem 2.1.5, we are able to obtain Theorem 2.1.4. To do so, we strengthen the estimate $\mathbf{P}(M_n \geq nv_{\text{is}} - \lambda \log n) > c > 0$ in

$$\lim_{y \to -\infty} \liminf_{n \to +\infty} \mathbf{P}(M_n \geq nv_{\text{is}} - \lambda \log n) = 1,$$

using a standard cutting argument. Loosely speaking, with high probability, there will be a large number of individuals alive at generation k, each of which having positive probability to make a descendant that displaced more than m_n, which is enough to conclude.

Proof of Theorem 2.1.4. Let $(\mathbf{T}, V)$ be a BRWis of length n, satisfying all hypotheses of Theorem 2.1.4. To prove that the sequence $(M_n - m_n)$ is tight, we prove that

$$\lim_{K \to +\infty} \sup_{n \in \mathbb{N}} \mathbf{P}(|M_n - m_n| \geq K) = 0.$$

By Theorem 2.1.5, there exists $C > 0$ such that

$$\sup_{n \in \mathbb{N}} \mathbf{P}(M_n \geq m_n + K) \leq C(1+K)e^{-\varphi_1 K},$$

therefore the upper bound is easy to obtain.

We now turn to the lower bound. Applying Theorem 2.1.5, there exists $c_1 > 0$ such that

$$\inf_{n \in \mathbb{N}} \mathbf{P}(M_n \geq m_n) \geq c_1.$$

Let L_1 be a point process of law $\mathcal{L}_1$. By (2.1.1), there exists $h > 0$ and $N \in \mathbb{N}$ such that

$$m = \mathbf{E}\left[\max\left(N, \sum_{\ell \in L_1} \mathbf{1}_{\{\ell \geq -h\}}\right)\right] > 1.$$

We write μ the law of $\max\left(N, \sum_{\ell \in L_1} \mathbf{1}_{\{\ell \geq -h\}}\right)$, and $(Z_n, n \geq 0)$ a Galton-Watson process with reproduction law μ. Observe we can couple (Z_n) and a branching random walk $(\mathbf{T}_1, V_1)$ with reproduction law $\mathcal{L}_1$ such that for any $n \in \mathbb{N}$, $\sum_{|u|=n} \mathbf{1}_{\{V_1(u) \geq -nh\}} \geq Z_n$. By standard Galton-Watson processes theory, there exists $c_2 > 0$ and $\delta > 0$ such that $\inf_{k \in \mathbb{N}} \mathbf{P}(Z_k \geq \delta m^k) > c_2$.

Let $\varepsilon > 0$ and $R > 0$ be such that $(1 - c_1)^R \leq \varepsilon$. We now choose $k \in \mathbb{N}$ such that $\delta m^k \geq R$. For any $n \in \mathbb{N}$, we write $u_n = (1, \ldots 1) \in \mathcal{U}$. By (2.1.1), for all $n \in \mathbb{N}$ we have $u_n \in \mathbf{T}$. We write τ the first time n such that u_n has a sibling at distance smaller than h, and this child has at least R descendants alive at time $n + k$ which displaced less that $-kh$. According to the previous computations, τ is stochastically dominated by a Geometric random variable. Therefore, it exists $\tau_0 \in \mathbb{N}$ such that $\mathbf{P}(\tau > \tau_0) < \varepsilon$.

Therefore, with probability at least $1-2\varepsilon$, there are at least R individuals alive at some time before $\tau_0 + k$, all of which are above $\inf_{j \leq \tau_0} V(u_j) - kh$. Each of these individuals u starts an independent BRWis with law $\mathbf{P}_{k,V(u)}$, thus, using Theorem 2.1.5, there exists $y > 0$ such that, for all $n \geq 1$ large enough

$$\mathbf{P}(M_{n + \frac{k+\tau_0}{\alpha_1}} \geq m_n - y) \geq 1 - 4\varepsilon$$

which ends the proof of the lower bound. □

2.A Time-inhomogeneous random walk estimates

In this section, we prove the random walk estimates we defined in Section 2.3.

2.A.1 Proof of Lemma 2.3.1

We recall here the notations of Lemma 2.3.1. Let $p, q, r \in \mathbb{N}$, set $n = p + q + r$. The time-inhomogeneous random walk S consists of p steps of independent centred random walk with finite variance, q steps of independent random variables, then r steps of another centred random variable with finite variance.

Let $A \in \mathbb{R}$, and $x, y \in \mathbb{R}_+$, $h \in \mathbb{R}$, we denote by

$$\Gamma^{A,1}(x, y, h) = \{s \in \mathbb{R}^n : \forall k \leq p, s_k \geq -x\}$$

the set of trajectories staying above $-x$ during the initial steps, and by

$$\Gamma^{A,3}(x, y, h) = \left\{s \in \mathbb{R}^n : \forall k \in [n - r, n], s_k \geq y + A \log \tfrac{n}{n-k+1}\right\}.$$

Proof. Let $A > 0$, $p, q, r \in \mathbb{N}$, $y \geq 0$ and $h \in \mathbb{R}$. Without loss of generality, we can assume that both p and r are even (by changing q in $q + 1$ or $q + 2$).

If $F = \emptyset$, Lemma 2.3.1 is an easy consequence of (2.3.1).

If $F = \{1\}$, applying the Markov property at time $p/2$, we obtain

$$\begin{aligned} &\mathbf{P}\left[S_n + A \log n \in [y + h, y + h + 1], (S_k, k \leq n) \in \Gamma^{A,1}(x, y, h)\right] \\ &\qquad \leq \mathbf{P}\left(S_j \geq -x, j \leq p/2\right) \sup_{z \in \mathbb{R}} \mathbf{P}\left(S_n - S_{p/2} \in [z, z + 1]\right) \leq C \frac{1 + x}{p^{1/2}} \frac{1}{\max(p, r)^{1/2}}, \end{aligned}$$

using (2.3.7) and (2.3.1). If $F = \{3\}$, we apply time-reversal, let $\widehat{S}_j = S_n - S_{n-j}$, we have

$$\begin{aligned}
&\mathbf{P}\left[S_n + A\log n - y - h \in [0,1], S_j \geq y + A\log \tfrac{n}{n-j+1}, n-r \leq j \leq n\right]\\
&\qquad\leq \mathbf{P}\left[\widehat{S}_n + A\log n - y - h \in [0,1], \widehat{S}_j \leq h + 1 - A\log(j+1), j \leq r\right]\\
&\qquad\qquad\leq C\frac{1+h_+}{r^{1/2}}\frac{1}{\max(p,r)^{1/2}}
\end{aligned}$$

by the same arguments as above.

Finally, if $F = \{1,3\}$, applying Markov property at time $p/2$, and time-reversal

$$\begin{aligned}
&\mathbf{P}\left[(S_k, k \leq n) \in \Gamma^{A,1}(x,y,h) \cap \Gamma^{A,3}(x,y,h)\right]\\
&\quad\leq \mathbf{P}\left[S_j \geq -x, j \leq p/2\right] \sup_{z\in\mathbb{R}} \mathbf{P}\left[\widehat{S}_{n-p/2} \in [z,z+1], \widehat{S}_j \leq h+1-A\log(j+1), j \leq r\right].
\end{aligned}$$

Using the same arguments as above, we obtain

$$\mathbf{P}\left[(S_k, k \leq n) \in \Gamma^{A,1}(x,y,h) \cap \Gamma^{A,3}(x,y,h)\right] \leq C\frac{1+x}{p^{1/2}}\frac{1}{\max(p,r)^{1/2}}\frac{1+h_+}{r^{1/2}}.$$

□

2.A.2 Proof of Lemma 2.3.2

We consider a collection of independent random variables $(X_n^p, n \geq 0, p \leq P)$, with, for all $p \leq P$, $(X_n^p, n \geq 0)$ an i.i.d. sequence of real-valued centred random variables with finite variance. Let $n \geq 1$, we write, for $k \leq n$, $S_k = \sum_{p=1}^P \sum_{j=1}^k X_j \mathbf{1}_{\left\{j \in (\alpha_{p-1}^{(n)}, \alpha_p^{(n)}]\right\}}$. For $F \subset \{1,3\}$ and $x, y, \delta \in \mathbb{R}_+$, we write

$$\Upsilon^F(x,y,\delta) = \left\{ s \in \mathbb{R}^n : \begin{array}{l} \forall k \leq \alpha_1^{(n)}, s_k \geq -x\mathbf{1}_{\{1\in F\}} - \delta k \mathbf{1}_{\{1\notin F\}},\\ \forall k \in (\alpha_1^{(n)}, \alpha_{P-1}^{(n)}], s_k \geq 0\\ \forall k \in (\alpha_{p-1}^{(n)}, n], s_k \geq y\mathbf{1}_{\{3\in F\}} - \delta(n-k)\mathbf{1}_{\{3\notin F\}} \end{array} \right\}.$$

We now prove there exists $c > 0$ such that for any $F \subset \{1,3\}$, for all $x \in [0, n^{1/2}]$, $y \in [-n^{1/2}, n^{1/2}]$ and $\delta > 0$

$$\mathbf{P}\left[S_n \leq y+1, S \in \Upsilon^F(x,y,\delta)\right] \geq c\frac{1+x\mathbf{1}_F(1)}{n^{\mathbf{1}_F(1)/2}}\frac{1}{n^{1/2}}\frac{1}{n^{\mathbf{1}_F(3)/2}}.$$

Proof of Lemma 2.3.2. Let $n \geq 1$, $x, |y| \in [0, n^{1/2}]$ and $\delta > 0$. We denote by

$$\Omega^F(\delta,y) = \left\{ s \in \mathbb{R}^{n-\alpha_1^{(n)}} : \begin{array}{l} \forall k \leq \alpha_{P-1}^{(n)} - \alpha_1^{(n)}, s_k \geq 0\\ \forall k \in (\alpha_{p-1}^{(n)}, n], s_k \geq y\mathbf{1}_{\{3\in F\}} - \delta(n-k)\mathbf{1}_{\{3\notin F\}} \end{array} \right\}.$$

Applying the Markov property at time $\alpha_1^{(n)}$, we have

$$\begin{aligned}
&\mathbf{P}\left[S_n \leq y+1, S \in \Upsilon^F(x,y,\delta)\right]\\
&\qquad= \mathbf{E}\left[\mathbf{1}_{\left\{S_j \geq -x\mathbf{1}_{\{1\in F\}} - \delta k\mathbf{1}_{\{1\notin F\}}\right\}} \mathbf{P}_{\alpha_1^{(n)}, S\alpha_1^{(n)}}\left(S_{n-\alpha_1^{(n)}} \leq y+1, S \in \Omega^F(\delta,y)\right)\right].
\end{aligned}$$

On the one hand, if $1 \in F$, we have

$$\begin{aligned}\mathbf{P}\left[S_n \leq y+1, S \in \Upsilon^F(x,y,\delta)\right] \\ &\geq \mathbf{P}\left(S_j \geq -x, S_{\alpha_1^{(n)}} \in [3n^{1/2}, 4n^{1/2}]\right) \\ &\qquad \times \inf_{u \in [3n^{1/2}, 4n^{1/2}]} \mathbf{P}_{\alpha_1^{(n)},u}\left(S_{n-\alpha_1^{(n)}} \leq y+1, S \in \Omega^F(\delta,y)\right).\end{aligned}$$

Using (2.3.4) and (2.3.7), we have

$$\mathbf{P}\left(S_j \geq -x, S_{\alpha_1^{(n)}} \in [3n^{1/2}, 4n^{1/2}]\right) \geq \frac{c(1+x)}{n^{1/2}}.$$

On the other hand, if $1 \notin F$, for all $h > 3$

$$\begin{aligned}\mathbf{P}\left[S_n \leq y+1, S \in \Upsilon^F(x,y,\delta)\right] \\ &\geq \mathbf{P}\left(S_j \geq -\delta k, \left|S_{\alpha_1^{(n)}}\right| \in [3n^{1/2}, hn^{1/2}]\right) \\ &\qquad \times \inf_{u \in [3n^{1/2}, hn^{1/2}]} \mathbf{P}_{\alpha_1^{(n)},u}\left(S_{n-\alpha_1^{(n)}} \leq y+1, S \in \Omega^F(\delta,y)\right).\end{aligned}$$

By (2.3.5), we have $\mathbf{P}(\forall n \in \mathbb{N}, S_n \geq -\delta n) > 0$. Thus, writing $\lambda^{(n)} = \left\lfloor \alpha_1^{(n)}/2 \right\rfloor$, by central limit theorem, there exists $c > 0$ and $h > 0$ such that for all $n \geq 1$ large enough

$$\mathbf{P}\left(S_j \geq -\delta j, j \leq \lambda^{(n)}, S_{\lambda^{(n)}} \in [-h\sqrt{n}, h\sqrt{n}]\right) \geq c.$$

Moreover, by the Donsker theorem

$$\liminf_{n\to+\infty} \inf_{|z| \leq h\sqrt{n}} \mathbf{P}_z\left(S_j \geq -2h\sqrt{n}, S_{\lambda^{(n)}} \in [3\sqrt{n}, 4\sqrt{n}]\right) > 0.$$

As a consequence, we have

$$P\left(S_j \geq -x\mathbf{1}_{\{1\in F\}} - \delta k \mathbf{1}_{\{1 \notin F\}}, S_{\alpha_1^{(n)}} \in [3n^{1/2}, 4n^{1/2}]\right) \geq \frac{c(1 + x\mathbf{1}_F(1))}{n^{\mathbf{1}_F(1)/2}}.$$

We now apply time-reversal, for $k \leq n$, let $\widehat{S}_k = S_n - S_{n-k}$, we have

$$\begin{aligned}\inf_{z \in [3n^{1/2}, 4n^{1/2}]} \mathbf{P}_{\alpha_1^{(n)},z}\left(S_{n-\alpha_1^{(n)}} \leq y+1, S \in \Omega^F(\delta,y)\right) \\ \geq \inf_{u \in [2n^{1/2}, 5n^{1/2}]} \mathbf{P}\left[\begin{array}{c}\widehat{S}_{n-\alpha_1^{(n)}} \in [u,u+1], \widehat{S}_j \geq -\delta n \mathbf{1}_{\{3 \notin F\}}, j \leq n - \alpha_{P-1}^{(n)} \\ \widehat{S}_j \geq n^{1/2}, j \leq n - \alpha_1^{(n)}\end{array}\right].\end{aligned}$$

We write $\overline{S}_k = \widehat{S}_{n-\alpha_{P-1}^{(n)}+k} - \widehat{S}_{n-\alpha_{P-1}^{(n)}}$, we apply again the Markov property at time $n - \alpha_{P-1}^{(n)}$

$$\begin{aligned}\inf_{z \in [3n^{1/2}, 4n^{1/2}]} \mathbf{P}_{\alpha_1^{(n)},z}\left(S_{n-\alpha_1^{(n)}} \leq y+1, S \in \Omega^F(\delta,y)\right) \\ \geq \frac{c}{n^{\mathbf{1}_F(3)/2}} \inf_{z \in [0, 10n^{1/2}]} \mathbf{P}\left[\min_{j \leq \alpha_{P-1}^{(n)} - \alpha_1^{(n)}} \overline{S}_j \geq -n^{1/2}, \overline{S}_{\alpha_{P-1}^{(n)} - \alpha_1^{(n)}} \in [z, z+1]\right]\end{aligned}$$

using the same tools as above. Finally

$$\mathbf{P}\left[\min_{j\leq\alpha^{(n)}_{P-1}-\alpha^{(n)}_1}\overline{S}_j \geq -n^{1/2}, \overline{S}_{\alpha^{(n)}_{P-1}-\alpha^{(n)}_1} \in [z,z+1]\right] \geq \frac{c}{n^{1/2}},$$

using (2.3.4) and the Donsker theorem, implying

$$\inf_{n\in\mathbb{N}}\mathbf{P}\left[\min_{j\leq\alpha^{(n)}_{P-1}-\alpha^{(n)}_1}\overline{S}_j \geq -n^{1/2}\right] > 0.$$

□

2.B Lagrange multipliers for the optimization problem

In this section, for any $\mathbf{h},\mathbf{k}\in\mathbb{R}^P$, we write $\mathbf{h}.\mathbf{k}=\sum_{p=1}^P h_pk_p$ the usual scalar product in $\mathbb{R}^P$. Moreover, if $f:\mathbb{R}^P\to\mathbb{R}$ is differentiable at point $\mathbf{h}$, we write $\nabla f(\mathbf{h}) = (\partial_1 f(\mathbf{h}),\ldots \partial_P f(\mathbf{h}))$ the gradient of f.

We study in this section the optimization problem consisting of finding $\mathbf{a}\in\mathcal{R}$ such that

$$\sum_{p=1}^P(\alpha_p-\alpha_{p-1})\,a_p = \sup\left\{\sum_{p=1}^P(\alpha_p-\alpha_{p-1})\,b_p : \mathbf{b}\in\mathcal{R}\right\}. \tag{2.B.1}$$

Equation (2.B.1) is a problem of optimization under constraint the $\mathbf{a}\in\mathcal{R}$. To obtain a solution, we use an existence of Lagrange multipliers theorem. The version we use here is stated in [Kur76], for Banach spaces.

Theorem VI (Existence of Lagrange multipliers)**.** *Let $P,Q\in\mathbb{N}$. We denote by U an open subset of $\mathbb{R}^P$, J a differentiable function $U\to\mathbb{R}$ and $g=(g_1,\ldots g_Q)$ a differentiable function $U\to\mathbb{R}^Q$. Let R be a close convex cone in $\mathbb{R}^Q$ i.e. a close subset of $\mathbb{R}^Q$ such that*

$$\forall x,y\in R, \forall\lambda,\mu\in\mathbb{R}_+, \lambda x+\mu y\in R.$$

If $\mathbf{a}\in\mathbb{R}^P$ verifies $g(a)\in R$ and

$$J(\mathbf{a}) = \sup\left\{J(\mathbf{b}), \mathbf{b}\in\mathbb{R}^p : g(\mathbf{b})\in R\right\},$$

and if the differential of g at point $\mathbf{a}$ is a surjection, then there exist non-negative Lagrange multipliers $\lambda_1,\ldots\lambda_Q$ verifying the following properties.

(L1) *For all $\mathbf{h}\in\mathbb{R}^P$, $\nabla J(\mathbf{a}).\mathbf{h}=\sum_{q=1}^Q\lambda_q(\nabla g_q(a).\mathbf{h})$.*

(L2) *For all $h\in R$, $\sum_{q=1}^Q\lambda_q h_q\leq 0$;*

(L3) *$\sum_{q=1}^Q\lambda_q g_q(\mathbf{a})=0$.*

Using this theorem, we prove Proposition 2.1.2. We start by proving that if $\mathbf{a}$ satisfies some specific properties, then $\mathbf{a}$ is the solution to (2.B.1).

Lemma 2.B.1. *Under assumptions* (2.1.1) *and* (2.1.8)*, $\mathbf{a}\in\mathcal{R}$ is a solution of* (2.B.1) *if and only if, writing $\theta_p=\left(\kappa^*_p\right)'(a_p)$, we have*

(P1) *$\boldsymbol{\theta}$ is non-decreasing and positive;*

(P2) *if $K^*(\mathbf{a})_p<0$, then $\theta_{p+1}=\theta_p$;*

(P3) $K^*(\mathbf{a})_P = 0$.

Proof. For $\mathbf{b} \in \mathbb{R}^P$, we denote by

$$J(\mathbf{b}) = \sum_{p=1}^{P} (\alpha_p - \alpha_{p-1}) b_p, \quad R = \{\mathbf{k} \in \mathbb{R}^P : k_p \leq 0, p \leq P\},$$

and we write, $\theta_p(\mathbf{b}) = (\kappa_p^*)'(b_p)$.

We assume in a first time that $\mathbf{a} \in \mathcal{R}$ is a solution of (2.B.1), in which case

$$J(\mathbf{a}) = \sup \left\{ J(\mathbf{b}), \mathbf{b} \in \mathbb{R}^P : K^*(\mathbf{b}) \in R \right\}. \tag{2.B.2}$$

The function J is linear thus differentiable, and assumption (2.1.8) implies that K^* is differentiable at point $\mathbf{a}$. For $\mathbf{h} \in \mathbb{R}^P$, we have $\nabla J(\mathbf{a}).\mathbf{h} = \sum_{p=1}^P (\alpha_p - \alpha_{p-1}) h_p$, and $\nabla K^*(\mathbf{a})_p.\mathbf{h} = (\alpha_p - \alpha_{p-1}) \theta_p(\mathbf{a}) h_p$.

To prove that K^* has a surjective differential, it is enough to prove that for all $p \leq P$, $\theta_p(\mathbf{a}) \neq 0$. Let $p \leq P$ be the smallest value such that $\theta_p(\mathbf{a}) = 0$. Observe that in this case, $\kappa_p^*(a_p) < 0$ by (2.1.1), thus we can increase a little a_p and stay in $\mathcal{R}$ as soon as we decrease a little a_{p-1} –or a_P if $p = 1$, in which case same proof would work with few modifications. For $\varepsilon > 0$ and $q \leq P$, we write $\mathbf{a}_q^\varepsilon = \mathbf{a}_q - \varepsilon \mathbf{1}_{\{q=p-1\}} + \varepsilon^{2/3} \mathbf{1}_{\{q=p\}}$. We observe that, for all $\varepsilon > 0$ small enough,

$$\begin{aligned} K^*(\mathbf{a}^\varepsilon)_{p-1} &= K^*(\mathbf{a}^\varepsilon)_{p-2} + (\alpha_{p-1} - \alpha_{p-2}) \kappa_{p-1}^*(a_{p-1} - \varepsilon) \\ &\leq K^*(\mathbf{a})_{p-2} + (\alpha_{p-1} - \alpha_{p-2}) \kappa_{p-1}^*(a_{p-1}) - (\alpha_{p-1} - \alpha_{p-2}) \theta_{p-1}(\mathbf{a}) \varepsilon + O(\varepsilon^2) \\ &\leq K^*(\mathbf{a})_{p-1} - (\alpha_{p-1} - \alpha_{p-2}) \theta_{p-1}(\mathbf{a}) \varepsilon + O(\varepsilon^2) \end{aligned}$$

and

$$\begin{aligned} K^*(\mathbf{a}^\varepsilon)_p &\leq K^*(\mathbf{a})_{p-1} + (\alpha_p - \alpha_{p-1}) \kappa_p^*(a_p + \varepsilon^{2/3}) \\ &\leq K^*(\mathbf{a})_{p-1} - (\alpha_{p-1} - \alpha_{p-2}) \theta_{p-1}(\mathbf{a}) \varepsilon + (\alpha_p - \alpha_{p-1}) \kappa_p^*(a_p) + O(\varepsilon^{4/3}) \\ &\leq K^*(\mathbf{a})_p - (\alpha_{p-1} - \alpha_{p-2}) \theta_{p-1}(\mathbf{a}) \varepsilon + O(\varepsilon^{4/3}), \end{aligned}$$

thus, for $\varepsilon > 0$ small enough, $\mathbf{a}^\varepsilon \in \mathcal{R}$ and $\sum_{p=1}^P (\alpha_p - \alpha_{p-1}) a_p^\varepsilon > \sum_{p=1}^P (\alpha_p - \alpha_{p-1}) a_p$, which is inconsistent with the fact that $\mathbf{a}$ is the optimal solution of (2.B.1).

Therefore, by Theorem VI, there exist non-negative $\lambda_1, \ldots \lambda_P$ such that

(L1) $\forall \mathbf{h} \in \mathbb{R}^P$, $\nabla J(a).\mathbf{h} = \sum_{p=1}^P \lambda_p \nabla K^*(\mathbf{a})_p.\mathbf{h}$;

(L2) $\forall \mathbf{h} \in R$, $\sum_{p=1}^P \lambda_p h_p \leq 0$;

(L3) $\sum_{p=1}^P \lambda_p K^*(\mathbf{a})_p = 0$.

We observe that Condition (L1) can be rewritten $\forall p \leq P, \lambda_p \theta_p(\mathbf{a}) = 1$, therefore $\theta_p(\mathbf{a}) = \frac{1}{\lambda_p}$. We define $\mathbf{h}^p \in \mathbb{R}^P$ such that $h_j^p = -\mathbf{1}_{\{j=p\}} + \mathbf{1}_{\{j=p+1\}}$. Condition (L2) applied to $\mathbf{h}^p \in R$ implies that λ is non-increasing, thus $\boldsymbol{\theta}$ is non-decreasing; which gives (P1). Finally, we rewrite Condition (L3) as follows, by discrete integration by part

$$0 = \sum_{p=1}^P \lambda_p K^*(\mathbf{a})_p = \underbrace{\lambda_P K^*(\mathbf{a})_P}_{\leq 0} - \sum_{p=1}^{P-1} \underbrace{(\lambda_{p+1} - \lambda_p) K^*(\mathbf{a})_p}_{\geq 0},$$

therefore Condition (P3) ($K^*(\mathbf{a})_P = 0$) is verified; and if $\lambda_{p+1} \neq \lambda_p$, then $K^*(\mathbf{a})_p = 0$, which implies (P2).

We now suppose that $\mathbf{a} \in \mathcal{R}$ verifies Conditions (P1), (P2) and (P3) and we prove that for all $\mathbf{b} \in \mathcal{R}$,

$$\sum_{p=1}^{P}(\alpha_p - \alpha_{p-1})a_p \geq \sum_{p=1}^{P}(\alpha_p - \alpha_{p-1})b_p. \tag{2.B.3}$$

To do so, we use the fact that functions κ_p^* are convex and differentiable at point $\mathbf{a}$, therefore, for all $x \in \mathbb{R}$, $\kappa_p^*(x) \geq \kappa_p^*(a_p) + \theta_p(x - a_p)$. As a consequence, we have

$$\begin{aligned}\sum_{p=1}^{P}(\alpha_p - \alpha_{p-1})(a_p - b_p) &\geq \sum_{p=1}^{P} \frac{\kappa_p^*(a_p) - \kappa_p^*(b_p)}{\theta_p}(\alpha_p - \alpha_{p-1})\\ &\geq (K^*(\mathbf{a})_P - K^*(\mathbf{b})_P)\frac{1}{\theta_P} - \sum_{p=1}^{P-1}\left(\frac{1}{\theta_{p+1}} - \frac{1}{\theta_p}\right)(K^*(\mathbf{a})_p - K^*(\mathbf{b})_p)\end{aligned}$$

by discrete integration by part. By the specific properties of a, we have

$$K^*(\mathbf{a})_P\frac{1}{\theta_P} - \sum_{p=1}^{P-1}\left(\frac{1}{\theta_{p+1}} - \frac{1}{\theta_p}\right)K^*(\mathbf{a})_p = 0,$$

thus

$$\sum_{p=1}^{P}(\alpha_p - \alpha_{p-1})(a_p - b_p) \geq -\frac{K^*(\mathbf{b})_P}{\theta_P} + \sum_{p=1}^{P-1}\left(\frac{1}{\theta_{p+1}} - \frac{1}{\theta_p}\right)K^*(\mathbf{b})_p \geq 0,$$

as $\boldsymbol{\theta}$ is non-decreasing an $K^*(\mathbf{b})$ non-positive. Optimizing (2.B.3) over $\mathbf{b} \in \mathcal{R}$ gives us

$$\sum_{p=1}^{P}(\alpha_p - \alpha_{p-1})a_p \geq v_{\text{is}}$$

which ends the proof. □

We now prove the uniqueness of the solution of (2.B.1).

Lemma 2.B.2. *If for all $p \leq P$, κ_p is finite on an open subset of* $[0, +\infty)$, *then there is at most one solution to* (2.B.1).

Proof. The uniqueness of the solution in an easy consequence of the strict convexity of $(\kappa_p^*, p \leq P)$. Let $\mathbf{a}$ and $\mathbf{b}$ be two different solutions to (2.B.1), there exists a largest $p \leq P$ such that $a_p \neq b_p$. Then, writing $\mathbf{c} = \frac{\mathbf{a}+\mathbf{b}}{2}$, we have

$$\forall q \geq p, K^*(\mathbf{c})_q < \frac{K^*(\mathbf{a})_q + K^*(\mathbf{b})_q}{2} \leq 0.$$

Thus, by continuity of K^*, $\mathbf{c}$ is in the interior of $\mathcal{R}$, then we can increase a little c_p, and the path driven by $(\mathbf{c} + \varepsilon\mathbf{1}_{\{.=p\}})$ goes farther than both $\mathbf{a}$ and $\mathbf{b}$, which is a contradiction. □

Finally, we prove the existence of such a solution when the mean number of children of an individual in the BRWis is finite.

Lemma 2.B.3. *Under the assumptions* (2.1.8) *and* (2.1.9), *there exists at least a solution to* (2.B.1).

Proof. If $\kappa_p(0) < +\infty$, then $\inf_{\mathbb{R}} \kappa_p^* = -\kappa_p(0)$ and the minimum is reached at $\kappa_p'(0)$. As κ_p^* are bounded from below, for all $p \leq P$ there exists $x_p \geq 0$ such that

$$(\alpha_p - \alpha_{p-1})\kappa_p^*(x_p) + \sum_{q \neq p}(\alpha_q - \alpha_{q-1}) \inf_{\mathbb{R}} \kappa_q^* > 0.$$

Therefore, writing $X = \mathcal{R} \cap \prod_{p \leq P}[\kappa_p'(0), x_p]$, we have

$$\sup_{\mathbf{b} \in \mathcal{R}} \sum (\alpha_p - \alpha_{p-1}) b_p = \sup_{\mathbf{b} \in X} \sum (\alpha_p - \alpha_{p-1}) b_p.$$

But, X being compact, this supremum is in fact a maximum. There exists $\mathbf{a} \in X$ such that

$$\sum (\alpha_p - \alpha_{p-1}) a_p = \sup_{\mathbf{b} \in \mathcal{R}} \sum (\alpha_p - \alpha_{p-1}) b_p$$

which ends the proof. □

2.C Notation

- *Point processes*
  - $\mathcal{L}_p$: law of a point process;
  - L_p: point process with law $\mathcal{L}_p$;
  - κ_p: log-Laplace transform of $\mathcal{L}_p$;
  - κ_p^*: Fenchel-Legendre transform of $\mathcal{L}_p$;
  - X_p: defined in 2.1.12.
- *Generic marked tree*
  - $\mathbf{T}$: genealogical tree of the process;
  - $u \in \mathbf{T}$: individual in the process;
  - $V(u)$: position of the individual u;
  - $|u|$: generation at which u belongs;
  - u_k: ancestor at generation k of u;
  - $\emptyset$: initial ancestor of the process;
  - if $u \neq \emptyset$, πu: parent of u;
  - $\Omega(u)$: set of the children of u;
  - $M_n = \max_{|u|=n} V(u)$ maximal displacement at the n^{th} generation in $(\mathbf{T}, V)$.
- *Branching random walk with reproduction law $\mathcal{L}_p$*
  - $v_p = \inf_{\theta > 0} \frac{\kappa_p(\theta)}{\theta}$: speed of the branching random walk, such that $\frac{M_n}{n} \to v_p$ a.s.;
  - $\overline{\theta}_p$ critical parameter such that $\overline{\theta}_p v_p - \kappa_p(\overline{\theta}_p) = 0$.
- *Branching random walk through interfaces*
  - P: number of distinct phases in the process;
  - $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_P = 1$: position of the interfaces;
  - $\alpha_p^{(n)} = \lfloor n\alpha_p \rfloor$: position of the p^{th} interface for the BRWis of length n;
  - $\overline{a}_k^{(n)} = \sum_{p=1}^P \sum_{j=1}^k \mathbf{1}_{\left\{j \in (\alpha_{p-1}^{(n)}, \alpha_p^{(n)}]\right\}}$ path driven by $\mathbf{a} := (a_1, \ldots a_p) \in \mathbb{R}^P$;
  - u "follows path $\overline{a}^{(n)}$" if $\forall k \leq |u|$, $|V(u_k) - \overline{a}_k^{(n)}| \leq n^{1/2}$;
  - $K^*(\mathbf{a})_p = \sum_{q=1}^p (\alpha_q - \alpha_{q-1}) a_q$: rate function associated to the BRWis;
  - $\mathcal{R} = \left\{\mathbf{a} \in \mathbb{R}^P : \forall p \leq P, K^*(\mathbf{a})_p \leq 0\right\}$.
- *The optimal path*
  - $v_{\text{is}} = \max_{\mathbf{b} \in \mathcal{R}} \sum_{p=1}^P (\alpha_p - \alpha_{p-1}) b_p$: speed of the BRWis;

- $\mathbf{a} \in \mathcal{R}$ such that $\sum_{p=1}^P (\alpha_p - \alpha_{p-1}) a_p = v_{\text{is}}$: optimal speed profile;
- $\theta_p = (\kappa_p^*)'(a_p)$;
- $T = \#\{\theta_p, p \leq P\}$: number of different values taken by $\boldsymbol{\theta}$;
- $\varphi_1 < \varphi_2 < \cdots < \varphi_T$: different values taken by $\boldsymbol{\theta}$;
- $f_t = \min\{k \leq P : \theta_k = \varphi_t\}$ and $l_t = \max\{k \leq P : \theta_k = \varphi_t\}$;
- $\lambda = \sum_{t=1}^T \frac{1}{2\varphi_t} \left[\mathbf{1}_{\left\{K^*(\mathbf{a})_{f_t}=0\right\}} + 1 + \mathbf{1}_{\left\{K^*(\mathbf{a})_{l_t-1}=0\right\}} \right]$: log-correction of the BRWis;
- $B = \{p \leq P : K^*(\mathbf{a})_{p-1} = K^*(\mathbf{a})_p = 0\}$.

* *Spinal decomposition*
  - $W_n = \sum_{|u|=n} e^{\theta V(u) - \sum_{k=1}^n \kappa_k(\theta)}$: the additive martingale with parameter θ;
  - $\mathbf{P}_{k,x}$: law of the time-inhomogeneous branching random walk with environment $(\mathcal{L}_k, \mathcal{L}_{k+1}, \ldots)$;
  - $\overline{\mathbf{P}}_{k,x} = W_n \cdot \mathbf{P}_{k,x}$: size-biased law of $\mathbf{P}_{k,x}$;
  - $\widehat{\mathbf{P}}_{k,x}$: law of the branching random walk with spine;
  - w: spine of the branching random walk;
  - $\mathcal{F}_n = \sigma(u, V(u), |u| \leq n)$: filtration of the branching random walk;
  - $\mathcal{G}_n = \sigma(w_k, V(w_k), k \leq n) \vee \sigma(u, V(u), u \in \Omega(w_k), k < n)$: filtration of the spine;
  - $\widehat{\mathcal{F}}_n = \mathcal{F}_n \vee \mathcal{G}_n$: filtration of the branching random walk with spine;
  - Spinal decomposition: Proposition 2.2.1;
  - Many-to-one lemma: Lemma 2.2.2.
* *Random walks*
  - (T_n): random walk with finite variance;
  - (S_n): random walk through interfaces;
  - Time-reversal: replacement of S by $(\widehat{S}_n = S_n - S_{n-k}, k \leq n)$.
* *Branching random walk estimates*
  - $m_n = n v_{\text{is}} - \lambda \log n$;
  - $E_p(\varphi) = \sum_{q=1}^p (\alpha_q - \alpha_{q-1})(\varphi \kappa_p'(\varphi) - \kappa_q(\varphi))$;
  - $K_k^{(n)} = \sum_{p=1}^P \kappa_p(\theta) \sum_{j=1}^k \mathbf{1}_{\left\{j \in (\alpha_{p-1}^{(n)}, \alpha_p^{(n)}]\right\}}$;
  - $r_k^{(n)} = a_P(k-n) + \frac{3}{2\theta} \log(n-k+1)$;
  - $B^{(n)} = \bigcup_{p \in B} (\alpha_{p-1}^{(n)}, \alpha_p^{(n)}]$ and $F^{(n)} = \bigcup_{p \in B \cap \{1,P\}} [\alpha_{p-1}^{(n)}, \alpha_p^{(n)}]$;
  - $f_j^{(n)} = a_1 j \mathbf{1}_{\left\{j \leq \alpha_1^{(n)}\right\}} + \left(m_n + r_k^{(n)}\right) \mathbf{1}_{\left\{j \geq \alpha_{P-1}^{(n)}\right\}}$: upper boundary;
  - $X^{(n)}(y,h) = \sum_{|u|=n} \mathbf{1}_{\{V(u)-m_n-y \in [-h,-h+1]\}} \mathbf{1}_{\left\{V(u_j) \leq f_j^{(n)} + y, j \in F^{(n)}\right\}}$;
  - $g_k^{(n)}$: another boundary, defined in (2.4.6);
  - $\xi(u) = \sum_{u' \in \Omega(u)} (1 + (V(u') - V(u))_+ \mathbf{1}_{\left\{|u| \in B^{(n)}\right\}}) e^{\theta(V(u') - V(u))}$;
  - $\mathcal{A}_n(y) = \left\{u \in \mathbf{T} : V(u) \geq m_n + y, V(u_j) \leq g_j^{(n)} + y, j \leq n\right\}$;
  - $\mathcal{B}_n(z) = \left\{u \in \mathbf{T} : |u| = n, \xi(u_j) \leq z e^{-\frac{\theta}{2}\left[V(u_j) - g_j^{(n)}\right]}\right\}$;
  - $G_n(y,z) = \mathcal{A}_n(y) \cap \mathcal{B}_n(z)$ and $Y_n(y,z) = \#G_n(y,z)$.

CHAPTER 3

The maximal displacement of a branching random walk in slowly varying environment

"Winter is coming."

Georges R. R. Martin – Game of thrones

Abstract

We consider in this chapter a BRWtie in which the time-inhomogeneous environment evolves smoothly at a time scale of order n. The asymptotic behaviour of the maximal displacement in this process consists in a first ballistic order plus a negative correction of order $n^{1/3}$. The speed of the process is obtained, as in the previous chapter, as the solution of an optimization problem under constraints. The second term comes from time-inhomogeneous random walk estimates. This result partially answers a conjecture of Fang and Zeitouni [FZ12b]. We also obtain in this chapter the asymptotic behaviour of the consistent maximal displacement with respect to the optimal path.

Nota: This chapter is a slightly modified version of the article *Maximal displacement in a branching random walk through a series of interfaces* accepted for publication in *Stochastic Processes and Applications*, doi 10.1016/j.spa.2015.05.011. Available on arXiv:1307.4496.

3.1 Introduction

The time-inhomogeneous branching random walk on $\mathbb{R}$ studied in this chapter is defined as follow. Let $(\mathcal{L}_t, t \in [0,1])$ be a family of point processes, that we call the environment of the branching random walk. We consider a branching random walk $(\mathbf{T}^{(n)}, V^{(n)})$ in which individuals alive at generation $k < n$ reproduce independently according to point processes of law $\mathcal{L}_{k/n}$. Individuals alive at generation n have no children. We call such a process a branching random walk in large scale time inhomogeneous environment (abbreviated as BRWls).

The study of the maximal displacement in a time-inhomogeneous branching Brownian motion, the continuous time counterpart of the branching random walk, with smoothly varying environment has been started in [FZ12b]. In this process individuals split into 2 children at rate 1, and move according to independent Gaussian diffusion with variance

$\sigma^2_{t/n}$ at time $t \in [0, n]$. Fang and Zeitouni conjectured that under mild hypotheses, there exists a constant v^* and a function g verifying

$$-\infty < \liminf_{n\to+\infty} \frac{g(n)}{n^{1/3}} \leq \limsup_{n\to+\infty} \frac{g(n)}{n^{1/3}} \leq 0$$

such that the sequence $(M_n - nv^* - g(n), n \geq 1)$ is tensed. They proved this result for smoothly decreasing variance. Using PDE techniques, Nolen, Roquejoffre and Ryzhik [NRR14] established, again in the case of decreasing variances, that $g(n) = l^* n^{1/3} + O(\log n)$ for some explicit constant l^*. Maillard and Zeitouni [MZ14] proved, independently from our result, that $g(n) = l^* n^{1/3} - c_1 \log n$, for some explicit c_1. The techniques they used for their proofs are similar to the ones presented here, based on first and second moment computations of number of individuals staying in a given path and the study of a partial differential equation (see Appendix 3.A).

In this chapter, we prove that for a large class of time-inhomogeneous branching random walks, $M_n - nv^* \sim_{n\to+\infty} l^* n^{1/3}$ in probability for some explicit constants v^* and l^*. Conversely to previous articles in the domain, the displacements we authorize are non necessarily Gaussian, and the law of the number of children may be correlated with the displacement and depend on the time. More importantly, we do not restrict ourselves to (an hypothesis similar to) decreasing variance. However assuming a decreasing variance remains interesting as in this case quantities such as v^* and l^* admit a closed expression.

We do not prove in this chapter there exists a function such that $(M_n - nv^* - g(n))$ is tight, thus we do not exactly answer to the conjecture of Fang and Zeitouni. However Fang [Fan12] proved the sequence (M_n) shifted by its median is tight for a large class of *generalized branching random walks*. This class does not exactly covers the class of time-inhomogeneous branching random walks we consider, but on the non-trivial intersection, the conjecture is then proved applying Theorem 3.1.3.

To address the fact the displacements are non Gaussian, we use the Sakhanenko estimate [Sak84], which couples sums of independent random variables with a Brownian motion. The non-monotonicity of the variance leads to additional concerns. We discuss in Section 3.1.2 a formula for $\lim_{n\to+\infty} \frac{M_n}{n}$, expressed as the solution of an optimization problem under constraints (3.1.6). This equation is solved in Section 3.4, using some analytical tools such as the existence of Lagrange multipliers in Banach spaces described in [Kur76]. Solving this problem, an increasing function appears that replaces the inverse of the variance in computations of [NRR14] and [MZ14]. We finally use Brownian estimates and the many-to-one lemma to compute moments of an additive functional of the BRWls.

Notation In this chapter, c, C stand for two positive constants[1], respectively small enough and large enough, which may change from line to line, and depend only on the law of the random variables we consider. We assume the convention $\max \emptyset = -\infty$ and $\min \emptyset = +\infty$. For $x \in \mathbb{R}$, we write $x_+ = \max(x, 0)$, $x_- = \max(-x, 0)$ and $\log_+(x) = (\log x)_+$. For any function $f : [0, 1] \to \mathbb{R}$, we say that f is *Riemann-integrable* if

$$\liminf_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} \min_{s\in[\frac{k-1}{n},\frac{k+2}{n}]} f_s = \limsup_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} \max_{s\in[\frac{k-1}{n},\frac{k+2}{n}]} f_s,$$

and this common value is written $\int_0^1 f_s ds$. In particular, a Riemann-integrable function is bounded. A subset $F \subset [0, 1]$ is said to be *Riemann-integrable* if $\mathbf{1}_F$ is Riemann-integrable.

1. Whereas everywhere else, C is for cookie.

For example, an open subset of $(0,1)$ of Lebesgue measure $1/2$ that contains all rational numbers is not Riemann-integrable. Finally, if A is a measurable event, we write $\mathbf{E}(\cdot; A)$ for $\mathbf{E}(\cdot \mathbf{1}_A)$. An index of notation is available in Appendix 3.B

The rest of the introduction is organised as follows. We start with some branching random walk notation in Section 3.1.1. We describe in Section 3.1.2 the optimization problem that gives the speed of the time-inhomogeneous branching random walk. In Section 3.1.3, we state the main result of this article: the asymptotic of the maximal displacement in a time-inhomogeneous branching random walk. We also introduce another quantity of interest for the branching random walk: the consistent maximal displacement with respect to the optimal path in Section 3.1.4. Finally, in Section 3.1.5, we introduce some of the random walk estimates that are used to compute moments of the branching random walk.

3.1.1 Branching random walk notation

We consider $(\mathcal{L}_t, t \in [0,1])$ a family of laws of point processes. Let $t \in [0,1]$, we write L_t for a point process with law $\mathcal{L}_t$. For $\theta > 0$, we denote by $\kappa_t(\theta) = \log \mathbf{E}\left[\sum_{\ell \in L_t} e^{\theta \ell}\right]$ the log-Laplace transform of θ and for $a \in \mathbb{R}$ by $\kappa_t^*(a) = \sup_{\theta>0}[\theta a - \kappa_t(\theta)]$ its Fenchel-Legendre transform. We recall the following elementary fact: if κ_t^* is differentiable at point a, then setting $\theta = \partial_a \kappa_t^*(a)$, we have

$$\theta a - \kappa_t(\theta) = \kappa_t^*(a). \tag{3.1.1}$$

The *branching random walk of length n with large scale time-inhomogeneous environment* $(\mathcal{L}_t, t \in [0,1])$ is the marked tree $(\mathbf{T}^{(n)}, V^{(n)})$ such that $\{L^u, u \in \mathbf{T}^{(n)}\}$ forms a family of independent point processes, where L^u has law $\mathcal{L}_{\frac{|u|+1}{n}}$ if $|u| < n$, and is empty otherwise. In particular, $\mathbf{T}^{(n)}$ is the (time-inhomogeneous) Galton-Watson tree of the genealogy of this process. When the value of n is clear in the context, we often omit the superscript, to lighten notation.

We consider processes that never get extinct, and have supercritical offspring above a given straight line with slope p. We introduce this (strong) supercritical assumption

$$\forall t \in [0,1], \mathbf{P}(L_t = \emptyset) = 0 \quad \text{and} \quad \exists p \in \mathbb{R} : \inf_{t \in [0,1]} \mathbf{P}(\#\{\ell \in L_t : \ell \geq p\} \geq 2) > 0. \tag{3.1.2}$$

A weaker supercritical assumption is enough for most of the results proved in this chapter, but this stronger version is technically convenient to obtain concentration inequalities for the maximal displacement. It is also helpful to guarantee the existence of a solution to the optimization problem that defines v^*.

We also make some assumptions on the regularity of the function $t \mapsto \mathcal{L}_t$. We write

$$D = \{(t,\theta) \in [0,1] \times [0,+\infty) : \kappa_t(\theta) < +\infty\} \text{ and } D^* = \{(t,a) : \kappa_t^*(a) < +\infty\}, \tag{3.1.3}$$

and we assume that D and D^* are non-empty, that D (resp. D^*) is open in $[0,1] \times [0,+\infty)$ (resp. $[0,1] \times \mathbb{R}$) and that

$$\kappa \in \mathcal{C}^{1,2}(D) \text{ and } \kappa^* \in \mathcal{C}^{1,2}(D^*). \tag{3.1.4}$$

These regularity assumptions are used to ensure the solution of the optimization problem defining v^* is regular. If (3.1.4) is verified, then the maximum of L_t has at least exponential tails, and $\mathbf{P}(\max L_t = \operatorname{esssup} \max L_t) = 0$. We do not claim these assumptions to be optimal, but they are sufficient to define v^*.

For example, a finite number of i.i.d. random variables with exponential left tails satisfy the above condition. Conversely, heavy tailed random variables, or if the maximum of the point process verifies

$$\mathbf{P}(\max\{\ell \in L\} \geq x) \sim_{x\to+\infty} x^{-1-\varepsilon} e^{-x}$$

for some $\varepsilon \in (0, 2)$ do not satisfy (3.1.3).

3.1.2 The optimization problem

We write $\mathcal{C}$ for the set of continuous functions, and $\mathcal{D}$ for the set of càdlàg[2] functions on $[0, 1]$ which are continuous at point 1. To a function $b \in \mathcal{D}$, we associate the path of length n defined for $k \leq n$ by $\overline{b}^{(n)}_k = \sum_{j=1}^k b_{j/n}$. We say that b is the *speed profile* of the path $\overline{b}^{(n)}$, and we introduce

$$K^* : \begin{array}{ccc} \mathcal{D} & \longrightarrow & \mathcal{C} \\ b & \longmapsto & \left(\int_0^t \kappa^*_s(b_s)ds, t \in [0,1]\right). \end{array}$$

By standard computations on branching random walks (see, e.g. [Big10]), for any $t \in [0, 1]$, the mean number of individuals that follow the path $\overline{b}^{(n)}$ until time tn i.e., that stay at all time within distance $\sqrt{n}$ from the path, verifies

$$\frac{1}{n} \log \mathbf{E}\left[\sum_{|u|=\lfloor nt \rfloor} \mathbf{1}_{\left\{\left|V(u_k)-\overline{b}^{(n)}_{k/n}\right|<\sqrt{n}, k\leq nt\right\}}\right] \approx_{n\to+\infty} -K^*(b)_t.$$

Therefore, $e^{-nK^*(b)_t}$ is a good approximation of the number of individuals that stay close to the path $\overline{b}^{(n)}$ until time tn.

If there exists $t_0 \in (0, 1)$ such that $K^*(b)_{t_0} > 0$, by Markov property, with high probability there is no individual who stayed close to this path until time nt_0. Consequently no individual in BRWls followed path $\overline{b}^{(n)}$ until time n. Conversely, if for all $t \in [0, 1]$, $K^*(b)_t \leq 0$, one would expect to find with positive probability at least one individual at time n to the right of $\overline{b}^{(n)}$. Following this heuristic, we introduce

$$v^* = \sup\left\{\int_0^1 b_s ds, b \in \mathcal{D} : \forall t \in [0, 1], K^*(b)_t \leq 0\right\}. \tag{3.1.5}$$

We expect nv^* to be the highest terminal point in the set of paths that are followed with positive probability by individuals in the branching random walk. Therefore we expect that $\lim_{n\to+\infty} \frac{M_n}{n} = v^*$ in probability.

We are interested in the path that realises the maximum in (3.1.5). We define the optimization problem under constraints

$$\exists a \in \mathcal{D} : v^* = \int_0^1 a_s ds \text{ and } \forall t \in [0, 1], K^*(a)_t \leq 0. \tag{3.1.6}$$

We say that a is a solution to (3.1.6) if $\int_0^1 a_s ds = 0$ and $K^*(a)$ is non-positive. Describing such a path gives the second order correction. In effect, as highlighted for regular branching random walks in [AS10], the second order of the asymptotic of M_n is linked to the probability for a random walk to follow this optimal path.

2. Right-continuous with left limits at each point.

Proposition 3.1.1. *Under the assumptions* (3.1.2) *and* (3.1.4)*, there exists a unique solution a to* (3.1.6)*, and a and θ are Lipschitz.*

Moreover, a is a solution to (3.1.6) *if and only if, setting $\theta_t = \partial_a \kappa^*_t(a_t)$, we have*

(P1) *θ is positive and non-decreasing;*

(P2) *$K^*(a)_1 = 0$;*

(P3) *$\int_0^1 K^*(a)_s d\theta_s^{-1} = 0$.*

This result is proved in Section 3.4. The path a solution to (3.1.6) is called *the optimal speed profil*, and $\overline{a}$ is called *the optimal path.* This optimization problem is similar to the one solved for the GREM by Bovier and Kurkova [BK07].

3.1.3 Asymptotic of the maximal displacement

Under the assumptions (3.1.4) and (3.1.6), let a be the optimal speed profile characterised by Proposition 3.1.1. For $t \in [0, 1]$ we denote by

$$\theta_t = \partial_a \kappa^*_t(a_t) \quad \text{and} \quad \sigma_t^2 = \partial^2_\theta \kappa_t(\theta_t). \tag{3.1.7}$$

To obtain the asymptotic of the maximal displacement, we introduce the following regularity assumptions:

$$\theta \text{ is absolutely continuous, with a Riemann-integrable derivative } \dot{\theta}, \tag{3.1.8}$$

$$\{t \in [0, 1] : K^*_t(a) = 0\} \text{ is Riemann-integrable.} \tag{3.1.9}$$

Finally, we make the following second order integrability assumption:

$$\sup_{t \in [0,1]} \mathbf{E}\left[\left(\sum_{\ell \in L_t} e^{\theta_t \ell_t} \right)^2 \right] < +\infty. \tag{3.1.10}$$

Remark 3.1.2. This last integrability condition is not optimal. Using the spinal decomposition as well as estimates on random walks enriched by random variables depending only on the last step, as in the previous chapters would lead to an integrability condition of the form $\mathbf{E}(X(\log X)^2) < +\infty$ instead of (3.1.10). However, this assumption considerably simplifies the proofs.

The main result of this article is the following.

Theorem 3.1.3 (Maximal displacement in the BRWls)**.** *We assume* (3.1.2)*,* (3.1.4)*,* (3.1.8)*,* (3.1.9) *and* (3.1.10) *are verified. We write α_1 for the largest zero of the Airy function of first kind*[3] *and we set*

$$l^* = \frac{\alpha_1}{2^{1/3}} \int_0^1 \frac{(\dot{\theta}_s \sigma_s)^{2/3}}{\theta_s} ds \leq 0. \tag{3.1.11}$$

For any $l > 0$ we have,

$$\lim_{n \to +\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\left(M_n \geq nv^* + (l^* + l) n^{1/3} \right) = -\theta_0 l,$$

and for any $\varepsilon > 0$,

$$\limsup_{n \to +\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\left(\left| M_n - nv^* - l^* n^{1/3} \right| \geq \varepsilon n^{1/3} \right) < 0.$$

3. Recall that $\alpha_1 \approx -2.3381...$

This theorem is proved in Section 3.5. The presence of the largest zero of the Airy function of first kind is closely related to the asymptotic of the Laplace transform of the area under a Brownian motion B starting from 1 staying positive,

$$\mathbf{E}_1\left(e^{-\int_0^t B_s ds}; B_s \geq 0, s \leq t\right) \approx_{t\to+\infty} e^{\frac{\alpha_1}{2^{1/3}}t+o(t)}.$$

The fact that the second order of M_n is $n^{1/3}$ can be explained as follows: when θ is strictly increasing at time t, the optimal path has to stay very close to the boundary of the branching random walk at time nt. If θ is strictly increasing on $[0,1]$, the optimal path has to stay close to the boundary of the branching random walk. This $n^{1/3}$ second order is then similar to the asymptotic of the consistent minimal displacement for the time-homogeneous branching random walk, obtained in [FZ10, FHS12].

3.1.4 Consistent maximal displacement

The arguments we develop for the proof of Theorem 3.1.3 can be extended to obtain the asymptotic behaviour of the consistent maximal displacement with respect to the optimal path in the BRWls, which we define now. For $n \in \mathbb{N}$ and $u \in \mathbf{T}^{(n)}$, we denote by

$$\Lambda(u) = \max_{k\leq|u|}\left[\overline{a}^{(n)}_k - V(u_k)\right],$$

the maximal distance between the optimal path and the position of an ancestor of individual u. The consistent maximal displacement with respect to the optimal path is defined by

$$\Lambda_n = \min_{u\in\mathbf{T}^{(n)},|u|=n} \Lambda(u). \tag{3.1.12}$$

This quantity correspond to the smallest distance from the optimal path $\overline{a}$ at which one can put a killing barrier, bellow which individuals get killed, such that the system survives until time n. The consistent maximal displacement has been studied for time-homogeneous branching random walks in [FZ10, FHS12]. In the case of BRWls, the following result holds.

Theorem 3.1.4 (Consistent maximal displacement). *Under the assumptions* (3.1.2), (3.1.4)*,* (3.1.8), (3.1.9) *and* (3.1.10)*, there exists $\lambda^* \leq -l^*$, defined in* (3.5.13) *such that for any $\lambda \in (0,\lambda^*)$,*

$$\lim_{n\to+\infty}\frac{1}{n^{1/3}}\log\mathbf{P}\left(\Lambda_n \leq (\lambda^*-\lambda)n^{1/3}\right) = -\theta_0\lambda,$$

and for any $\varepsilon > 0$,

$$\limsup_{n\to+\infty}\frac{1}{n^{1/3}}\log\mathbf{P}(|\Lambda_n - \lambda^* n^{1/3}| \geq \varepsilon n^{1/3}) < 0.$$

Remark 3.1.5. If $u \in \mathbf{T}^{(n)}$ verifies $V(u) = M_n$, then $\Lambda(u) \leq nv^* - M_n$. As a consequence, the inequality $\Lambda_n \leq nv^* - M_n$ holds almost surely, thus $\lambda^* \leq -l^*$, as soon as these quantities are well-defined.

In Theorem 3.1.4, we give the asymptotic of the consistent maximal displacement with respect to the optimal path. However, this is not the only path one may choose to consider. For example, one can choose the "natural speed path", in which the speed profile is a function $v \in \mathcal{C}$ defined by $v_t = \inf_{\theta>0} \frac{\kappa_t(\theta)}{\theta}$. Note that v_t is the speed of a time-homogeneous branching random walk with reproduction law $\mathcal{L}_t$. This path is interesting, as it is the unique path such that for all $t \in [0,1]$, $K^*(v)_t = 0$. Consequently, for any $\lambda > 0$, the number of individuals that stay above $\overline{v}^{(n)} - \lambda n^{1/3}$ at all time is at most of order $e^{O(n^{1/3})}$ with high probability.

In Section 3.3, we provide a new time-inhomogeneous version of the many-to-one lemma, linking additive moments of the branching random walk with time-inhomogeneous random walk estimates. To prove Theorems 3.1.3 and 3.1.4, we use random walk estimates that are proved in Section 3.2.

3.1.5 Airy functions and random walk estimates

We introduce a few basic property on Airy functions, that can be found in [AS64]. The *Airy function of first kind* Ai can be defined, for $x \in \mathbb{R}$, by the improper integral

$$\mathrm{Ai}(x) = \frac{1}{\pi} \lim_{t\to+\infty} \int_0^t \cos\left(\tfrac{s^3}{3} + xs\right) ds, \tag{3.1.13}$$

and the *Airy function of second kind* Bi by

$$\mathrm{Bi}(x) = \frac{1}{\pi} \lim_{t\to+\infty} \int_0^t \exp\left(-\tfrac{s^3}{3} + xs\right) + \sin\left(\tfrac{s^3}{3} + xs\right) ds. \tag{3.1.14}$$

These two functions form a basis of the space of solutions of

$$\forall x \in \mathbb{R}, y''(x) - xy(x) = 0,$$

and verify $\lim_{x\to+\infty} \mathrm{Ai}(x) = 0$ and $\lim_{x\to+\infty} \mathrm{Bi}(x) = +\infty$. The equation $\mathrm{Ai}(x) = 0$ has an infinitely countable number of solutions, which are negative with no accumulation points, that we list in the decreasing order: $0 > \alpha_1 > \alpha_2 > \cdots$.

The Laplace transform of the area below a random walk, or a Brownian motion, conditioned to stay positive admits an asymptotic behaviour linked to the largest zero of Ai, as proved by Darling [Dar83], Louchard [Lou84] and Takács [Tak92]. This result still holds in time-inhomogeneous settings. Let $(X_{n,k}, n \geq 1, k \leq n)$ be a triangular array of independent centred random variables. We assume that

$$\exists \sigma \in \mathcal{C}([0,1],(0,+\infty)) : \forall n \in \mathbb{N}, \forall k \leq n, \mathbf{E}(X_{n,k}^2) = \sigma_{k/n}^2, \tag{3.1.15}$$

$$\exists \mu > 0 : \mathbf{E}\left[e^{\mu|X_{n,k}|}\right] < +\infty. \tag{3.1.16}$$

We write $S_k^{(n)} = \sum_{j=1}^k X_{n,j}$ for the time-inhomogeneous random walk.

Theorem 3.1.6 (Time-inhomogeneous Takács estimate)**.** *Under* (3.1.15) *and* (3.1.16), *for any continuous function g such that $g(0) > 0$ and any absolutely continuous increasing function h with a Riemann-integrable derivative $\dot{h}$, we have*

$$\lim_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{E}\left[\exp\left(-\sum_{j=1}^n (h_{j/n} - h_{(j-1)/n}) S_j^{(n)}\right); S_j \leq g_{j/n} n^{1/3}, j \leq n\right] = \int_0^1 \left(\dot{h}_s g_s + \frac{a_1}{2^{1/3}} (\dot{h}_s \sigma_s)^{2/3}\right) ds.$$

This result is, in some sense, similar to the Mogul'skiĭ estimate [Mog74], which gives the asymptotic of the probability for a random walk to stay in an interval of length $n^{1/3}$. A time-inhomogeneous version of this result, with an additional exponential weight, holds again. To state this result, we introduce a function Ψ, defined in the following lemma.

Lemma 3.1.7. *Let B be a Brownian motion. There exists a unique convex function $\Psi : \mathbb{R} \to \mathbb{R}$ such that for all $h \in \mathbb{R}$*

$$\lim_{t\to+\infty} \frac{1}{t} \log \sup_{x\in[0,1]} \mathbf{E}_x \left[e^{-h\int_0^t B_s ds} ; B_s \in [0,1], s \in [0,t] \right] = \Psi(h). \tag{3.1.17}$$

Remark 3.1.8. We show in Section 3.A.2 that Ψ admits the following alternative definition:

$$\forall h > 0, \Psi(h) = \frac{h^{2/3}}{2^{1/3}} \sup \left\{ \lambda \leq 0 : \mathrm{Ai}\,(\lambda)\,\mathrm{Bi}\left(\lambda + (2h)^{1/3}\right) - \mathrm{Bi}\,(\lambda)\,\mathrm{Ai}\left(\lambda + (2h)^{1/3}\right) = 0 \right\},$$

and prove that Ψ verifies $\Psi(0) = -\frac{\pi^2}{2}$, $\Psi(h) \sim_{h\to+\infty} \alpha_1 \frac{h^{2/3}}{2^{1/3}}$ and $\Psi(h) - \Psi(-h) = -h$ for all $h \in \mathbb{R}$.

Proof of Lemma 3.1.7. For $h \in \mathbb{R}$ and $t \geq 0$, we write

$$\Psi_t(h) = \frac{1}{t} \log \sup_{x\in[0,1]} \mathbf{E}_x \left[e^{h\int_0^t B_s ds} ; B_s \in [0,1], s \in [0,t] \right].$$

As $B_s \in [0,1]$, we have trivially $|\Psi_t(h)| \leq |h| < +\infty$. Let $0 \leq t_1 \leq t_2$ and $x \in [0,1]$, by the Markov property

$$\begin{aligned}
&\mathbf{E}_x \left[e^{h\int_0^{t_1+t_2} B_s ds} ; B_s \in [0,1], s \in [0, t_1+t_2] \right] \\
=&\mathbf{E}_x \left[e^{h\int_0^{t_1} B_s ds} \mathbf{E}_{B_{t_1}} \left[e^{h\int_0^{t_2} B_s ds} ; B_s \in [0,1], s \in [0,t_2] \right] ; B_s \in [0,1], s \in [0,t_1] \right] \\
\leq& e^{t_2 \Psi_{t_2}(h)} \mathbf{E}_x \left[e^{h\int_0^{t_1} B_s ds} ; B_s \in [0,1], s \in [0,t_1] \right] \\
\leq& e^{t_1 \Psi_{t_1}(h)} e^{t_2 \Psi_{t_2}(h)}.
\end{aligned}$$

As a consequence, for all $h \in \mathbb{R}$, $(t\Psi_t(h), t \geq 0)$ is a sub-additive function, therefore

$$\lim_{t\to+\infty} \Psi_t(h) = \inf_{t\geq 0} \Psi_t(h) =: \Psi(h).$$

In particular, for all $h \in \mathbb{R}$, we have $|\Psi(h)| \leq |h| < +\infty$.

We now prove that Ψ is a convex function on $\mathbb{R}$, thus continuous. By the Hölder inequality, for all $\lambda \in [0,1]$, $(h_1, h_2) \in \mathbb{R}^2$, $x \in [0,1]$ and $t \geq 0$, we have

$$\begin{aligned}
&\mathbf{E}_x \left[e^{(\lambda h_1 + (1-\lambda) h_2) \int_0^t B_s ds} ; B_s \in [0,1], s \in [0,t] \right] \\
\leq& \mathbf{E}_x \left[\left(e^{\lambda h_1 \int_0^t B_s ds} \mathbf{1}_{\{B_s \in [0,1], s \in [0,t]\}} \right)^{\frac{1}{\lambda}} \right]^{\lambda} \mathbf{E}_x \left[\left(e^{(1-\lambda) h_2 \int_0^t B_s ds} \mathbf{1}_{\{B_s \in [0,1], s \in [0,t]\}} \right)^{\frac{1}{1-\lambda}} \right]^{1-\lambda} \\
\leq& \mathbf{E}_x \left[e^{h_1 \int_0^t B_s ds} ; B_s \in [0,1], s \in [0,t] \right]^{\lambda} \mathbf{E}_x \left[e^{h_2 \int_0^t B_s ds} ; B_s \in [0,1], s \in [0,t] \right]^{1-\lambda} \\
\leq& e^{t\lambda \Psi_t(h_1)} e^{t(1-\lambda)\Psi_t(h_2)},
\end{aligned}$$

therefore, for all $t \geq 0$, Ψ_t is a convex function. As a consequence

$$\limsup_{t\to+\infty} \Psi_t(\lambda h_1 + (1-\lambda) h_2) \leq \lambda \limsup_{t\to+\infty} \Psi_t(h_1) + (1-\lambda) \limsup_{t\to+\infty} \Psi_t(h_2),$$

which proves that Ψ is convex, thus continuous. $\square$

Theorem 3.1.9 (Time-inhomogeneous Mogul'skiĭ estimate)**.** *Under assumptions* (3.1.15) *and* (3.1.16)*, for any pair of continuous functions $f < g$ such that $f(0) < 0 < g(0)$ and any absolutely continuous function h with a Riemann-integrable derivative $\dot{h}$, we have*

$$\lim_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{E}\left[\exp\left(\sum_{j=1}^{n}(h_{j/n} - h_{(j-1)/n})S_j^{(n)}\right) ; \frac{S_j}{n^{1/3}} \in [f_{j/n}, g_{j/n}], j \leq n\right]$$
$$= \int_0^1 \left(\dot{h}_s g_s + \frac{\sigma_s^2}{(g_s - f_s)^2}\Psi\left(\tfrac{(g_s-f_s)^3}{\sigma_s^2}\dot{h}_s\right)\right) ds.$$

The rest of the chapter is organised as follows. Theorems 3.1.6 and 3.1.9 are unified and proved in Section 3.2. These results are used in Section 3.3 to compute some branching random walk estimates, useful to bound the probability there exists an individual staying in a given path until time n. We study (3.1.6) in Section 3.4, proving in particular Proposition 3.1.1. Using the particular structure of the optimal path, we prove Theorems 3.1.3 and 3.1.4 in Section 3.5. In Appendix 3.A we prove a Brownian motion equivalent of Theorems 3.1.6 and 3.1.9, that were used in Section 3.2.

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3.2 Random walk estimates

We consider an array $(X_{n,k}, n \geq 1, k \leq n)$ of independent centred random variables, such that there exist $\sigma \in \mathcal{C}([0,1], (0,+\infty))$ and $\mu \in (0,+\infty)$ verifying (3.1.15) and (3.1.16). We write $S_k^{(n)} = S_0^{(n)} + \sum_{j=1}^{k} X_{n,j}$ for the time-inhomogeneous random walk of length n, such that $\mathbf{P}_x(S_0^{(n)} = x) = 1$. Let $\mathbf{E}_x$ be the expectation corresponding to the probability $\mathbf{P}_x$. Let h be a continuous function on $[0,1]$ such that

$$h \text{ is absolutely continuous, with Riemann-integrable derivative } \dot{h}. \tag{3.2.1}$$

The main result of this section is the computation of the asymptotic behaviour of the Laplace transform of the integral of $S^{(n)}$ with respect to h, as $n \to +\infty$, on the event that $S^{(n)}$ stays in a given path, that is defined in (3.2.4).

Let f and g be two continuous functions on $[0,1]$ such that $f < g$ and $f(0) < 0 < g(0)$, and F and G be two Riemann-integrable subsets of $[0,1]$ (i.e., such that $\mathbf{1}_F$ and $\mathbf{1}_G$ are Riemann-integrable). We assume that

$$\{t \in [0,1] : \dot{h}_t < 0\} \subset F \quad \text{and} \quad \{t \in [0,1] : \dot{h}_t > 0\} \subset G. \tag{3.2.2}$$

Interval F (respectively G) represent the set of times at which the barrier f (resp. g) is put below (resp. above) the path of the time-inhomogeneous random walk. Consequently, (3.2.2) implies that when there is no barrier below, the Laplace exponent is non-negative, so that the random walk does not "escape" to $-\infty$ (resp. $+\infty$) with high probability.

For $n \geq 1$, we introduce the $\frac{1}{n}^{\text{th}}$ approximation of F and G, defined by

$$F_n = \left\{1 \leq k \leq n : \left[\tfrac{k}{n}, \tfrac{k+1}{n}\right] \cap F \neq \emptyset\right\}, \ G_n = \left\{0 \leq k \leq n : \left[\tfrac{k}{n}, \tfrac{k+1}{n}\right] \cap G \neq \emptyset\right\}. \tag{3.2.3}$$

The path followed by the random walk of length n is defined, for $0 \leq j \leq n$, by

$$I_n(j) = \begin{cases} [f_{j/n} n^{1/3}, g_{j/n} n^{1/3}] & \text{if } j \in F_n \cap G_n, \\ [f_{j/n}, +\infty) & \text{if } j \in F_n \cap G_n^c, \\ (-\infty, g_{j/n} n^{1/3}] & \text{if } j \in F_n^c \cap G_n, \\ \mathbb{R} & \text{otherwise.} \end{cases} \quad (3.2.4)$$

The random walk $S^{(n)}$ follows the path $I^{(n)}$ if $\geq f_{k/n} n^{1/3}$ at any time $k \in F_n$, and $S_k^{(n)} \leq g_{k/n} n^{1/3}$ at any time $k \in G_n$. Choosing F and G in an appropriate way, we obtain Theorem 3.1.6 or Theorem 3.1.9.

We introduce the quantity

$$H_{f,g}^{F,G} = \int_0^1 \dot{h}_s g_s ds + \int_{F \cap G} \frac{\sigma_s^2}{(g_s - f_s)^2} \Psi\left(\tfrac{(g_s - f_s)^3}{\sigma_s^2} \dot{h}_s \right) ds$$
$$+ \int_{F^c \cap G} \frac{\alpha_1}{2^{1/3}} (\dot{h}_s \sigma_s)^{2/3} ds + \int_{G \cap F^c} \left(\dot{h}_s (f_s - g_s) + \frac{\alpha_1}{2^{1/3}} (-\dot{h}_s \sigma_s)^{2/3} \right) ds, \quad (3.2.5)$$

where Ψ is the function defined by (3.1.17). The first integral in this definition enables to "center" the path interval in a way that g is replaced by 0. The integral term over $F \cap G$ comes from the set of times in which the random walk is blocked in an interval of finite length as in Theorem 3.1.9, and the last two integral terms correspond to paths with only one boundary, above or below the random walk respectively.

The rest of the section is devoted to the proof of the following result.

Theorem 3.2.1. *Under the assumptions* (3.1.15) *and* (3.1.16)*, for any continuous function h satisfying* (3.2.1) *and pair of continuous functions $f < g$ such that $f(0) < 0 < g(0)$, for any Riemann-integrable $F, G \subset [0,1]$ such that* (3.2.2) *holds, we have*

$$\limsup_{n \to +\infty} \frac{1}{n^{1/3}} \sup_{x \in \mathbb{R}} \log \mathbf{E}_x \left[e^{\sum_{j=1}^n (h_{(j+1)/n} - h_{j/n}) S_j^{(n)}} ; S_j^{(n)} \in I_j^{(n)}, j \leq n \right] = H_{f,g}^{F,G}(1). \quad (3.2.6)$$

Moreover, setting $\widetilde{I}_j^{(n)} = I_j^{(n)} \cap [-n^{2/3}, n^{2/3}]$, for all $f_1 < a < b < g_1$ we have

$$\liminf_{n \to +\infty} \frac{1}{n^{1/3}} \log \mathbf{E}_0 \left[e^{\sum_{j=1}^n (h_{(j+1)/n} - h_{j/n}) S_j^{(n)}} \mathbf{1}_{\left\{ S_n^{(n)} \in [a n^{1/3}, b n^{1/3}] \right\}} ; S_j^{(n)} \in \widetilde{I}_j^{(n)}, j \leq n \right]$$
$$= H_{f,g}^{F,G}. \quad (3.2.7)$$

Remark 3.2.2. Observe that when (3.2.2) does not hold, the correct rate of growth of the expectations in (3.2.6) and (3.2.7) is $e^{O(n)}$, instead of the order $e^{O(n^{1/3})}$.

To prove this theorem, we decompose the time interval $[0, n]$ into A intervals, each of length $\frac{n}{A}$. On these smaller intervals, the functions f, g and $\dot{h}$ can be approached by constants. These intervals are divide into $\frac{n^{1/3}}{tA}$ subintervals of length $tn^{2/3}$. On these subintervals, the time-inhomogeneous random walk can be approached by a Brownian motion. The corresponding quantities are explicitly computed using the Feynman-Kac formula. Letting n, t then A grow to $+\infty$, we conclude the proof of Theorem 3.2.1. We give in Section 3.2.1 the asymptotic of the area under a Brownian motion constrained to stay non-negative or in an interval, and use the Sakhanenko exponential inequality in Section 3.2.2 to quantify the approximation of a random walk by a Brownian motion, before proving Theorem 3.2.1 in Section 3.2.3.

3.2.1 Brownian estimates through the Feynman-Kac formula

The asymptotic behaviour of the Laplace transform of the area under a Brownian motion, constrained to stay non-negative or in an interval, is proved in Appendix 3.A. In this section, $(B_t, t \geq 0)$ is a standard Brownian motion, which starts at position $x \in \mathbb{R}$ at time 0 under the law $\mathbf{P}_x$. We give the main results that are used in the next section to compute similar quantities for time-inhomogeneous random walks. First, for a Brownian motion that stay non-negative:

Lemma 3.2.3. *For all $h > 0$, $0 < a < b$ and $0 < a' < b'$, we have*

$$\lim_{t\to+\infty} \frac{1}{t} \log \sup_{x\in\mathbb{R}} \mathbf{E}_x \left[e^{-h\int_0^t B_s ds}; B_s \geq 0, s \leq t \right]$$
$$= \lim_{t\to+\infty} \frac{1}{t} \log \inf_{x\in[a,b]} \mathbf{E}_x \left[e^{-h\int_0^t B_s ds} \mathbf{1}_{\{B_t \in [a',b']\}}; B_s \geq 0, s \leq t \right] = \frac{\alpha_1}{2^{1/3}} h^{2/3}. \quad (3.2.8)$$

A similar estimate holds for a Brownian motion constrained to stay in the interval $[0, 1]$:

Lemma 3.2.4. *Let B be a Brownian motion. For all $h \in \mathbb{R}$, $0 < a < b < 1$ and $0 < a' < b' < 1$, we have*

$$\lim_{t\to+\infty} \frac{1}{t} \sup_{x\in[0,1]} \log \mathbf{E}_x \left[e^{-h\int_0^t B_s ds}; B_s \in [0, 1], s \leq t \right]$$
$$= \lim_{t\to+\infty} \frac{1}{t} \inf_{x\in[a,b]} \log \mathbf{E}_x \left[e^{-h\int_0^t B_s ds} \mathbf{1}_{\{B_t \in [a',b']\}}; B_s \in [0, 1], s \leq t \right] = \Psi(h). \quad (3.2.9)$$

Moreover, for all $h > 0$, we have

$$\Psi(h) = \frac{h^{2/3}}{2^{1/3}} \sup \left\{ \lambda \leq 0 : \mathrm{Ai}(\lambda) \,\mathrm{Bi}\left(\lambda + (2h)^{1/3} \right) - \mathrm{Bi}(\lambda) \,\mathrm{Ai}\left(\lambda + (2h)^{1/3} \right) = 0 \right\}. \quad (3.2.10)$$

We also have $\Psi(0) = -\frac{\pi^2}{2}$, $\lim_{h\to+\infty} \frac{\Psi(h)}{h^{2/3}} = \frac{\alpha_1}{2^{1/3}}$ and, for $h \in \mathbb{R}$, $\Psi(h) - \Psi(-h) = h$.

3.2.2 From a Brownian motion to a random walk

We use the Sakhanenko exponential inequality to extend the Brownian estimates to time-inhomogeneous random walks. We obtain here the correct $n^{1/3}$ order, but non-optimal upper and lower bounds. These results are used in the next section to prove Theorem 3.2.1.

Theorem VII (Sakhanenko exponential inequality [Sak84])**.** *Let $X = (X_1, \ldots X_n)$ be a sequence of independent centred random variables. We suppose there exists $\lambda > 0$ such that for all $j \leq n$*

$$\lambda \mathbf{E} \left(|X_j|^3 e^{\lambda |X_j|} \right) \leq \mathbf{E} \left(X_j^2 \right). \quad (3.2.11)$$

We can construct a sequence $\widetilde{X} = (\widetilde{X}_1, \ldots \widetilde{X}_n)$ with the same law as X; and Y a sequence of centred Gaussian random variables with the same covariance as $\widetilde{X}$ such that for some universal constant C_0 and all $n \geq 1$

$$\mathbf{E} \left[\exp(C_0 \lambda \Delta_n) \right] \leq 1 + \lambda \sqrt{\sum_{j=1}^n \mathbf{V}\mathrm{ar}(X_j)},$$

where $\Delta_n = \max_{j\leq n} \left| \sum_{k=1}^j \widetilde{X}_k - Y_k \right|$.

Using this theorem, we couple a time-inhomogeneous random walk with a Brownian motion such a way that they stay at distance $O(\log n)$ with high probability. Technically, to prove Theorem 3.2.1, we simply need a uniform control on $\mathbf{P}(\Delta_n \geq \varepsilon n^{1/3})$. The polynomial Sakhanenko inequality would be enough, that only impose a uniform bound on the third moment of the array of random variables instead of (3.1.16). However in the context of branching random walks, exponential integrability conditions are needed to guarantee the regularity of the optimal path (see Section 3.1.2).

Let $(X_{n,k}, n \in \mathbb{N}, k \leq n)$ be a triangular array of independent centred random variables, such that there exists a continuous positive function σ^2 verifying

$$\forall n \in \mathbb{N}, k \leq n, \mathbf{E}\left[X_{n,k}^2\right] = \sigma_{k/n}^2. \tag{3.2.12}$$

We set $\underline{\sigma} = \min_{t\in[0,1]} \sigma_t > 0$ and $\overline{\sigma} = \max_{t\in[0,1]} \sigma_t < +\infty$. We also assume that

$$\exists \lambda > 0 : \sup_{n\in\mathbb{N}, k\leq n} \mathbf{E}\left(e^{\lambda|X_{n,k}|}\right) < +\infty. \tag{3.2.13}$$

Note there exists $C > 0$ such that for any $\mu < \lambda/2$ and $x \geq 0$, $x^3 e^{\mu x} \leq Ce^{\lambda x}$. Thus (3.2.13) implies

$$\exists \mu > 0 : \sup_{n\geq 1, k\leq n} \mu\, \mathbf{E}\left(|X_{n,j}|^3 e^{\mu|X_{n,j}|}\right) \leq \underline{\sigma}^2. \tag{3.2.14}$$

In the first instance, we bound from above the asymptotic of the Laplace transform of the area under the time-inhomogeneous random walk $S_k^{(n)} = \sum_{j=1}^k X_{n,j}$.

Lemma 3.2.5. *We assume* (3.2.12) *and* (3.2.13) *are verified. For all $h > 0$, we have*

$$\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \sup_{x\in\mathbb{R}} \mathbf{E}_x\left[e^{-\frac{h}{n}\sum_{j=0}^{n-1} S_j^{(n)}}; S_j^{(n)} \geq 0, j \leq n\right] \leq \frac{\alpha_1}{2^{1/3}}(h\underline{\sigma})^{2/3}. \tag{3.2.15}$$

For all $h \in \mathbb{R}$ and $r > 0$, we have

$$\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \sup_{x\in\mathbb{R}} \mathbf{E}_x\left[e^{-\frac{h}{n}\sum_{j=0}^{n-1} S_j^{(n)}}; S_j^{(n)} \in [0, rn^{1/3}]\right] \leq \frac{\underline{\sigma}^2}{r^2}\Psi\left(\frac{r^3}{\underline{\sigma}^2}h\right). \tag{3.2.16}$$

Proof. In this proof, we assume $h \geq 0$ (and $h > 0$ if $r = +\infty$). The result for $h < 0$ in (3.2.16) can be deduced by symmetry and the formula $\Psi(h) - \Psi(-h) = -h$.

For all $r \in [0, +\infty)$, we write $f(r) = \frac{\underline{\sigma}^2}{r^2}\Psi\left(\frac{r^3}{\underline{\sigma}^2}h\right)$ and $f(+\infty) = \frac{\alpha_1}{2^{1/3}}(h\underline{\sigma})^{2/3}$. For all $x \in \mathbb{R}$, we use the convention $+\infty + x = x + \infty = +\infty$. By Lemmas 3.2.3 and 3.2.4, for all $r \in [0, +\infty]$, we have

$$\limsup_{t\to+\infty} \frac{1}{t} \log \sup_{x\in\mathbb{R}} \mathbf{E}_x\left[e^{-h\int_0^t B_{\underline{\sigma}^2 s} ds}; B_{\underline{\sigma}^2 s} \in [0, r], s \leq t\right] \leq f(r), \tag{3.2.17}$$

using the scaling property of the Brownian motion.

Let $A \in \mathbb{N}$ and $n \in \mathbb{N}$, we write $T = \left\lceil An^{2/3}\right\rceil$ and $K = \lfloor n/T \rfloor$. For all $k \leq K$, we write $m_k = kT$; applying the Markov property at time $m_K, m_{K-1}, \ldots m_1$, we have

$$\sup_{x\geq 0} \mathbf{E}_x\left[e^{-\frac{h}{n}\sum_{j=0}^{n-1} S_j^{(n)}}; S_j^{(n)} \in [0, rn^{1/3}], j \leq n\right]$$
$$\leq \prod_{k=0}^{K-1} \sup_{x\in\mathbb{R}} \mathbf{E}_x\left[e^{-\frac{h}{n}\sum_{j=0}^{n-1} S_j^{(n,k)}}; S_j^{(n,k)} \in [0, rn^{1/3}], j \leq T\right], \tag{3.2.18}$$

where we write $S^{(n,k)}_j = S^{(n)}_0 + S^{(n)}_{m_k+j} - S^{(n)}_{m_k}$ for the time-inhomogeneous random walk starting at time m_k and at position x under $\mathbf{P}_x$. We now bound, uniformly in $k < K$, the quantity

$$E^{(n)}_k(r) = \sup_{x\in\mathbb{R}} \mathbf{E}_x \left[e^{-\frac{h}{n}\sum_{j=0}^{T-1} S^{(n,k)}_j} ; S^{(n,k)}_j \in [0, rn^{1/3}], j \leq T \right].$$

Let $k < K$, we write $t^k_j = \sum_{i=kT+1}^{kT+j} \sigma^2_{j/n}$. We apply Theorem VII, by (3.2.12) and (3.2.13), there exist Brownian motions $B^{(k)}$ such that, denoting by $\widetilde{S}^{(n,k)}$ a random walk with same law as $S^{(n,k)}$ and $\Delta^k_n = \max_{j\leq T} \left| B^{(k)}_{t^k_j} - \widetilde{S}^{(n,k)}_j \right|$, there exists $\mu > 0$ such that for all $\varepsilon > 0$, $n \geq 1$ and $k \leq K$,

$$\mathbf{P}\left(\Delta^k_n \geq \varepsilon n^{1/3}\right) \leq e^{-C_0\mu\varepsilon n^{1/3}} \mathbf{E}\left(e^{C_0\mu\Delta^k_n}\right) \leq e^{-C_0\mu\varepsilon n^{1/3}} \left(1 + \mu\overline{\sigma}A^{1/2}n^{1/3}\right),$$

where we used (3.2.13) (thus (3.2.14)) and the exponential Markov inequality. Note in particular that for all $\varepsilon > 0$, $\mathbf{P}(\Delta^k_n \geq \varepsilon n^{1/3})$ converges to 0 as $n \to +\infty$, uniformly in $k \leq K$. As a consequence, for all $\varepsilon > 0$

$$\begin{aligned} E^{(n)}_k(r) &= \sup_{x\in\mathbb{R}} \mathbf{E}_x \left[e^{-\frac{h}{n}\sum_{j=0}^{T-1} \widetilde{S}^{(n,k)}_j} ; \widetilde{S}^{(n,k)}_j \in [0, rn^{1/3}], j \leq T \right] \\ &\leq \sup_{x\in\mathbb{R}} \mathbf{E}_x \left[e^{-\frac{h}{n}\sum_{j=0}^{T-1} \widetilde{S}^{(n,k)}_j} \mathbf{1}_{\left\{\Delta^k_n \leq \varepsilon n^{1/3}\right\}} ; \widetilde{S}^{(n,k)}_j \in [0, rn^{1/3}], j \leq T \right] + \mathbf{P}\left(\Delta^k_n \geq \varepsilon n^{1/3}\right). \end{aligned}$$

Moreover,

$$\begin{aligned} &\sup_{x\in\mathbb{R}} \mathbf{E}_x \left[e^{-\frac{h}{n}\sum_{j=0}^{T-1} \widetilde{S}^{(n,k)}_j} \mathbf{1}_{\left\{\Delta^k_n \leq \varepsilon n^{1/3}\right\}} ; \widetilde{S}^{(n,k)}_j \in [0, rn^{1/3}], j \leq T \right] \\ &\qquad \leq \sup_{x\in\mathbb{R}} \mathbf{E}_x \left[e^{h\frac{T}{n}\Delta^k_n - \frac{h}{n}\sum_{j=0}^{T-1} B_{t^k_j}} \mathbf{1}_{\left\{\Delta^k_n \leq \varepsilon n^{1/3}\right\}} ; B_{t^k_j} \in [-\Delta^k_n, rn^{1/3} + \Delta^k_n], j \leq T \right] \\ &\qquad \leq \sup_{x\in\mathbb{R}} \mathbf{E}_x \left[e^{h\frac{T}{n}\varepsilon n^{1/3} - \frac{h}{n}\sum_{j=0}^{T-1} B_{t^k_j}} ; B_{t^k_j} \in [-\varepsilon n^{1/3}, (r+\varepsilon)n^{1/3}], j \leq T \right] \\ &\qquad \leq e^{3hA\varepsilon} \widetilde{E}^{(n)}_k(r + 2\varepsilon), \end{aligned}$$

setting $\widetilde{E}^{(n)}_k(r) = \sup_{x\in\mathbb{R}} \mathbf{E}_x \left[e^{-\frac{h}{n}\sum_{j=0}^{T-1} B_{t^k_j}} ; B_{t^k_j} \in [0, rn^{1/3}], j \leq T \right]$ for all $r \in [0, +\infty]$. We set $\tau^k_j = n^{-2/3} t^k_j$; by the scaling property of the Brownian motion, we have

$$\widetilde{E}^{(n)}_k(r) = \sup_{x\in\mathbb{R}} \mathbf{E}_x \left[e^{-\frac{h}{n^{2/3}}\sum_{j=0}^{T-1} B_{\tau^k_j}} ; B_{\tau^k_j} \in [0, r], j \leq T \right].$$

We now replace the sum in $\widetilde{E}$ by an integral: we set

$$\omega_{n,A} = \sup_{|t-s|\leq 2An^{-1/3}} \left|\sigma^2_t - \sigma^2_s\right| \quad \text{and} \quad \Omega_{n,A} = \sup_{\substack{s,t \leq 2\overline{\sigma}^2 A + \omega_{n,A} \\ |t-s| \leq 2A\omega_{n,A} + \overline{\sigma}^2 n^{-1/3}}} \left|B_t - B_s\right|.$$

For all $k < K$ and $j \leq T$, we have

$$\left|\tau^k_j - j\sigma^2_{kT/n} n^{-2/3}\right| \leq n^{-2/3} \sum_{i=m_k+1}^{m_k+j} \left|\sigma^2_{i/n} - \sigma^2_{kT/n}\right| \leq 2A\omega_{n,A},$$

and $\sup_{s\in[j\underline{\sigma}^2/n,(j+1)\underline{\sigma}^2/n]}\left|B_s - B_{t_j^k}\right| \leq \Omega_{n,A}$. As a consequence, for all $\varepsilon > 0$, we obtain

$$\begin{aligned}\widetilde{E}_k^{(n)}(r) &\leq \sup_{x\in\mathbb{R}} \mathbf{E}_x\left[e^{-\frac{h}{n^{2/3}}\sum_{j=0}^{T-1} B_{\tau_j^k}}\mathbf{1}_{\left\{\Omega_{n,A}\leq\varepsilon\right\}}; B_{\tau_j^k}\in[0,r], j\leq T\right] + \mathbf{P}(\Omega_{n,A}\geq\varepsilon)\\ &\leq e^{3hA\varepsilon}\sup_{x\in\mathbb{R}}\mathbf{E}_x\left[e^{-h\int_0^A B_{\underline{\sigma}^2 s}ds}; B_{\underline{\sigma}^2 s}\in[0,(r+2\varepsilon)], s\leq A\right] + \mathbf{P}(\Omega_{n,A}\geq\varepsilon).\end{aligned}$$

We set $\overline{E}^A(r) = \sup_{x\in\mathbb{R}}\mathbf{E}_x\left[e^{-h\int_0^A B_{\underline{\sigma}^2 s}ds}; B_{\underline{\sigma}^2 s}\in[0,r], s\leq A\right]$. As B is continuous, we have $\lim_{n\to+\infty}\mathbf{P}(\Omega_{n,A}\geq\varepsilon) = 0$ uniformly in $k < K$. Therefore (3.2.18) leads to

$$\begin{aligned}&\limsup_{n\to+\infty}\frac{1}{n^{1/3}}\log\sup_{x\geq 0}\mathbf{E}_x\left[e^{-\frac{h}{n}\sum_{j=0}^{n-1}S_j^{(n)}}; S_j^{(n)}\in[0,rn^{1/3}], j\leq n\right]\\ &\qquad\leq \limsup_{n\to+\infty}\frac{K}{n^{1/3}}\max_{k\leq K}\log E_k^{(n)}(r)\\ &\qquad\leq \frac{1}{A}\limsup_{n\to+\infty}\left[3hA\varepsilon + \max_{k\leq K}\log\left(\widetilde{E}_k^{(n)}(r+2\varepsilon) + \mathbf{P}\left(\Delta_n^k\geq\varepsilon n^{1/3}\right)\right)\right]\\ &\qquad\leq 6h\varepsilon + \limsup_{n\to+\infty}\log\left[\overline{E}^A(r+4\varepsilon) + \mathbf{P}(\Omega_{n,A}\geq\varepsilon) + \max_{k\leq K}\mathbf{P}\left(\Delta_n^k\geq\varepsilon n^{1/3}\right)\right]\\ &\leq 6h\varepsilon + \frac{1}{A}\log\overline{E}^A(r+4\varepsilon).\end{aligned}$$

We now use (3.2.17), letting $A\to+\infty$, and thereby letting $\varepsilon\to 0$, this yields

$$\limsup_{n\to+\infty}\frac{1}{n^{1/3}}\log\sup_{x\geq 0}\mathbf{E}_x\left[e^{-\frac{h}{n}\sum_{j=0}^{n-1}S_j^{(n)}}; S_j^{(n)}\in[0,rn^{1/3}], j\leq n\right]\leq f(r),$$

which ends the proof. □

Next, we derive lower bounds with similar computations. To lighten notations, we set $I_{a,b}^{(n)} = [an^{1/3}, bn^{1/3}]$.

Lemma 3.2.6. *We assume* (3.2.12) *and* (3.2.13)*. For all $h > 0$, $0 < a < b$ and $0 < a' < b'$, we have*

$$\liminf_{n\to+\infty}\frac{1}{n^{1/3}}\log\inf_{x\in I_{a,b}^{(n)}}\mathbf{E}_x\left[e^{-\frac{h}{n}\sum_{j=0}^{n-1}S_j^{(n)}}\mathbf{1}_{\left\{S_n\in I_{a',b'}^{(n)}\right\}}; S_j^{(n)}\geq 0, j\leq n\right]\geq\frac{\alpha_1}{2^{1/3}}(h\overline{\sigma})^{2/3},\tag{3.2.19}$$

and for all $h\in\mathbb{R}$, $r > 0$, $0 < a < b < r$ and $0 < a' < b' < r$, we have

$$\liminf_{n\to+\infty}\frac{1}{n^{1/3}}\log\inf_{x\in I_{a,b}^{(n)}}\mathbf{E}_x\left[e^{-\frac{h}{n}\sum_{j=0}^{n-1}S_j^{(n)}}\mathbf{1}_{\left\{S_n\in I_{a',b'}^{(n)}\right\}}; S_j^{(n)}\in I_{0,r}^{(n)}, j\leq n\right]\geq\frac{\overline{\sigma}^2}{r^2}\Psi\left(\frac{r^3}{\overline{\sigma}^2}h\right).\tag{3.2.20}$$

Proof. We once again assume $h\geq 0$; as if $h < 0$ we can deduce (3.2.20) by symmetry and the formula $\Psi(h) - \Psi(-h) = h$. We write, for all $r\in[0,+\infty)$, $f(r) = \frac{\overline{\sigma}^2}{r^2}\Psi\left(\frac{r^3}{\overline{\sigma}^2}h\right)$ and $f(+\infty) = \frac{\alpha_1}{2^{1/3}}(h\overline{\sigma})^{2/3}$. By Lemmas 3.2.3 and 3.2.4, for all $r\in[0,+\infty]$, $0 < a < b < r$ and $0 < a' < b' < r$, we have

$$\liminf_{t\to+\infty}\frac{1}{t}\log\inf_{x\in[a,b]}\mathbf{E}_x\left[e^{-h\int_0^t B_{\overline{\sigma}^2 s}ds}\mathbf{1}_{\{B_t\in[a',b']\}}; B_s\in[0,r], s\leq t\right]\geq f(r).\tag{3.2.21}$$

We choose $u \in (a', b')$ and $\delta > 0$ such that $(u - 3\delta, u + 3\delta) \subset (a', b')$, and we introduce $J_\delta^{(n)} = I^{(n)}(u-\delta, u+\delta)$. We decompose again $[0, n]$ into subintervals of length of order $n^{2/3}$. Let $A \in \mathbb{N}$ and $n \in \mathbb{N}$, we write $T = \left\lfloor An^{2/3} \right\rfloor$ and $K = \lfloor n/T \rfloor$. For all $k \leq K$, we set again $m_k = kT$, for all $r \in [0, +\infty]$, applying the Markov property at times $m_K, m_{K-1}, \ldots m_1$ leads to

$$\begin{aligned}
&\inf_{x \in I_{a,b}^{(n)}} \mathbf{E}_x \left[e^{-\frac{h}{n} \sum_{j=0}^{n-1} S_j^{(n)}} \mathbf{1}_{\left\{ S_n \in I_{a',b'}^{(n)} \right\}} ; S_j^{(n)} \in I_{0,r}^{(n)}, j \leq n \right] \\
&\quad \geq \inf_{x \in I_{a,b}^{(n)}} \mathbf{E}_x \left[e^{-\frac{h}{n} \sum_{j=0}^{T-1} S_j^{(n)}} \mathbf{1}_{\left\{ S_T^{(n)} \in J_\delta^{(n)} \right\}} ; S_j^{(n)} \in I_{0,r}^{(n)}, j \leq T \right] \\
&\qquad \times \prod_{k=1}^{K-1} \inf_{x \in J_\delta^{(n)}} \mathbf{E}_x \left[e^{-\frac{h}{n} \sum_{j=0}^{T-1} S_j^{(n,k)}} \mathbf{1}_{\left\{ S_T^{(n,k)} \in J_\delta^{(n)} \right\}} ; S_j^{(n,k)} \in I_{0,r}^{(n)}, j \leq T \right] \\
&\qquad \times \inf_{x \in J_\delta^{(n)}} \mathbf{E}_x \left[e^{-\frac{h}{n} \sum_{j=0}^{n-KT} S_j^{(n,K)}} ; S_j^{(n,k)} \in I_{a',b'}^{(n)}, j \leq n - KT \right], \qquad (3.2.22)
\end{aligned}$$

where $S_j^{(n,k)} = S_0^{(n)} + S_{m_k+j}^{(n)} - S_{m_k}^{(n)}$. Let $0 < a < b < r$ and $0 < a' < b' < r$, we set $\varepsilon > 0$ such that $a > 8\varepsilon$, $r - b > 8\varepsilon$ and $b' - a' > 8\varepsilon$. We bound uniformly in k the quantity

$$E_k^{(n)}(r) = \inf_{x \in I_{a,b}^{(n)}} \mathbf{E}_x \left[e^{-\frac{h}{n} \sum_{j=0}^{T-1} S_j^{(n,k)}} \mathbf{1}_{\left\{ S_T^{(n,k)} \in I_{a',b'}^{(n)} \right\}} ; S_j^{(n,k)} \in I_{0,r}^{(n)}, j \leq T \right].$$

To do so, we set once again, for $k < K$, $t_j^k = \sum_{i=kT+1}^{kT+j} \sigma_{j/n}^2$. By Theorem VII, we introduce a Brownian motion B such that, denoting by $\widetilde{S}^{(n,k)}$ a random walk with the same law as $S^{(n,k)}$ and setting $\Delta_n^k = \max_{j \leq T} \left| B_{t_j^k} - \widetilde{S}_j^{(n,k)} \right|$, for all $\varepsilon > 0$, by (3.2.14) and the exponential Markov inequality we get

$$\sup_{k \leq K} \mathbf{P}\left(\Delta_n^k \geq \varepsilon n^{1/3} \right) \leq e^{-C_0 \mu \varepsilon n^{1/3}} \left(1 + \mu \overline{\sigma} A^{1/2} n^{1/3} \right),$$

which converges to 0, uniformly in k, as $n \to +\infty$. As a consequence, for all $\varepsilon > 0$ and $k < K$,

$$\begin{aligned}
&E_k^{(n)}(r) \\
&= \inf_{x \in I_{a,b}^{(n)}} \mathbf{E}_x \left[e^{-\frac{h}{n} \sum_{j=0}^{T-1} \widetilde{S}_j^{(n,k)}} \mathbf{1}_{\left\{ \widetilde{S}_T^{(n,k)} \in I_{a',b'}^{(n)} \right\}} ; \widetilde{S}_j^{(n,k)} \in I_{0,r}^{(n)}, j \leq T \right] \\
&\geq \inf_{x \in I_{a,b}^{(n)}} \mathbf{E}_x \left[e^{-\frac{h}{n} \sum_{j=0}^{T-1} \widetilde{S}_j^{(n,k)}} \mathbf{1}_{\left\{ \widetilde{S}_T^{(n,k)} \in I_{a',b'}^{(n)} \right\}} \mathbf{1}_{\left\{ \Delta_n^k \leq \varepsilon n^{1/3} \right\}} ; \widetilde{S}_j^{(n,k)} \in I_{0,r}^{(n)}, j \leq T \right] \\
&\geq \inf_{x \in I_{a-\varepsilon,b+\varepsilon}^{(n)}} e^{-3hA\varepsilon} \mathbf{E}_x \left[e^{-\frac{h}{n} \sum_{j=0}^{T-1} B_{t_j^k}} \mathbf{1}_{\left\{ B_{t_T^k} \in I_{a'+\varepsilon,b'-\varepsilon}^{(n)} \right\}} \mathbf{1}_{\left\{ \Delta_n^k \leq \varepsilon n^{1/3} \right\}} ; B_{t_j^k} \in I_{\varepsilon,r-\varepsilon}^{(n)}, j \leq T \right] \\
&\geq e^{-3hA\varepsilon} \left(\widetilde{E}_k^{(n)}(r - 2\varepsilon) - \mathbf{P}(\Delta_n^k \geq \varepsilon n^{1/3}) \right),
\end{aligned}$$

where we set

$$\widetilde{E}_k^{(n)}(r) = \inf_{x \in [a-2\varepsilon, b+2\varepsilon]} \mathbf{E}_x \left[e^{-\frac{h}{n^{2/3}} \sum_{j=0}^{T-1} B_{\tau_j^k}} \mathbf{1}_{\left\{ B_{\tau_T^k} \in [a'+2\varepsilon, b'-2\varepsilon] \right\}} ; B_s \in [0, r], s \leq \tau_T^k \right],$$

and $\tau_j^k = t_j^k n^{-2/3}$. We also set

$$\omega_{n,A} = \sup_{|t-s|\leq 2An^{-1/3}} \left|\sigma_t^2 - \sigma_s^2\right| \quad \text{and} \quad \Omega_{n,A} = \sup_{\substack{s,t \leq 2\overline{\sigma}^2 A + \omega_{n,A} \\ |t-s| \leq 2A\omega_{n,A} + \overline{\sigma}^2 n^{-1/3}}} \left|B_t - B_s\right|,$$

so that for all $k < K$ and $j \leq T$, we have

$$\left|\tau_j^k - j\sigma_{kT/n}^2 n^{-2/3}\right| \leq n^{-2/3} \sum_{i=m_k+1}^{m_k+j} \left|\sigma_{i/n}^2 - \sigma_{kT/n}^2\right| \leq 2A\omega_{n,A},$$

and $\sup_{s \in [\underline{\sigma}^2 \frac{j}{n}, \underline{\sigma}^2 \frac{j+1}{n}]} \left|B_s - B_{t_j^k}\right| \leq \Omega_{n,A}$. As a consequence,

$$\begin{aligned}\widetilde{E}_k^{(n)}(r) e^{3hA\varepsilon} \geq \\ \inf_{x \in [a-4\varepsilon, b+4\varepsilon]} \mathbf{E}_x &\left[e^{-h\int_0^A B_{\underline{\sigma}^2 s} ds} \mathbf{1}_{\left\{B_{\underline{\sigma}^2 A} \in [a'+4\varepsilon, b'-4\varepsilon]\right\}} ; B_{\underline{\sigma}^2 s} \in [0, r-2\varepsilon], s \leq A \right] \\ &- \mathbf{P}(\Omega_{n,A} \geq \varepsilon).\end{aligned}$$

This last estimate gives a lower bound for $E_k^{(n)}(r)$ which is uniform in $k \leq K$. As a consequence, (3.2.22) yields

$$\liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \inf_{x \in I_{a,b}^{(n)}} \mathbf{E}_x \left[e^{-\frac{h}{n}\sum_{j=0}^{n-1} S_j^{(n)}} \mathbf{1}_{\left\{S_n \in I_{a',b'}^{(n)}\right\}} ; S_j^{(n)} \in I_{0,r}^{(n)}, j \leq n \right] \geq$$

$$-6h\varepsilon + \frac{1}{A} \log \inf_{x \in [a-4\varepsilon, b+4\varepsilon]} \mathbf{E}_x \left[e^{-h\int_0^A B_{\underline{\sigma}^2 s} ds} \mathbf{1}_{\left\{B_{\underline{\sigma}^2 A} \in [a'+4\varepsilon, b'-4\varepsilon]\right\}} ; B_{\underline{\sigma}^2 s} \in [0, r-4\varepsilon], s \leq A \right].$$

Letting $A \to +\infty$, then $\varepsilon \to 0$ leads to

$$\liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \inf_{x \in I_{a,b}^{(n)}} \mathbf{E}_x \left[e^{-\frac{h}{n}\sum_{j=0}^{n-1} S_j^{(n)}} \mathbf{1}_{\left\{S_n \in I_{a',b'}^{(n)}\right\}} ; S_j^{(n)} \in I_{0,r}^{(n)}, j \leq n \right] \geq f(r),$$

which ends the proof. □

3.2.3 Proof of Theorem 3.2.1

We prove Theorem 3.2.1 by decomposing $[0, n]$ into A intervals of length n/A, and apply Lemmas 3.2.5 and 3.2.6.

Proof of Theorem 3.2.1. Let $n \in \mathbb{N}$ and $A \in \mathbb{N}$. For $0 \leq a \leq A$, we write $m_a = \lfloor na/A \rfloor$, and $d_a = m_{a+1} - m_a$.

Upper bound in (3.2.6). We apply the Markov property at times $m_{A-1}, m_{A-2}, \ldots m_1$, to see that

$$\begin{aligned}&\sup_{x \in I_0^{(n)}} \mathbf{E}_x \left[e^{\sum_{j=0}^{n-1} (h_{(j+1)/n} - h_{j/n}) S_j^{(n)}} ; S_j^{(n)} \in I_j^{(n)}, j \leq n \right] \\ &\qquad \leq \prod_{a=0}^{A-1} \underbrace{\sup_{x \in I_{m_a}^{(n)}} \mathbf{E}_x \left[e^{\sum_{j=0}^{d_a-1} (h_{(m_a+j+1)/n} - h_{(m_a+j)/n}) S_j^{(n,a)}} ; S_j^{(n,a)} \in I_{m_a+j}^{(n)}, j \leq d_a \right]}_{R_{a,A}^{(n)}},\end{aligned}$$

where $S_j^{(n,a)} = S_0^{(n)} + S_{m_a+j}^{(n)} - S_{m_a}^{(n)}$ is the time-inhomogeneous random walk starting at time m_a and position x. Letting $n \to +\infty$, this yields

$$\limsup_{n\to+\infty} \sup_{x\in\mathbb{R}} \frac{1}{n^{1/3}} \log \mathbf{E}_x \left[e^{\sum_{j=1}^n (h_{(j+1)/n} - h_{j/n}) S_j^{(n)}} ; S_j^{(n)} \in I_j^{(n)}, j \leq n \right] \leq \sum_{a=0}^{A-1} \limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log R_{a,A}^{(n)}. \quad (3.2.23)$$

To bound $R_{a,A}^{(n)}$, we replace functions f, g and $\dot{h}$ by constants. We set, for all $A \in \mathbb{N}$ and $a \leq A$,

$$\overline{h}_{a,A} = \sup_{t\in[\frac{a-1}{A},\frac{a+2}{A}]} \dot{h}_t, \quad \underline{h}_{a,A} = \inf_{t\in[\frac{a-1}{A},\frac{a+2}{A}]} \dot{h}_t,$$

$$g_{a,A} = \sup_{t\in[\frac{a-1}{A},\frac{a+2}{A}]} g_t, \quad f_{a,A} = \inf_{t\in[\frac{a-1}{A},\frac{a+2}{A}]} f_t \quad \text{and} \quad \sigma_{a,A} = \inf_{t\in[\frac{a-1}{A},\frac{a+2}{A}]} \sigma_s.$$

Observe that for all $n \in \mathbb{N}$ and $k \leq n$, by (3.2.2), if $h_{(k+1)/n} > h_{k/n}$, then $k \in G_n$, and if $h_{(k+1)/n} < h_{k/n}$, then $k \in F_n$. Consequently, for all $x \in I_k^{(n)}$,

$$(h_{(k+1)/n} - h_{k/n})x \leq (h_{(k+1)/n} - h_{k/n})_+ g_{k/n} n^{1/3} - (h_{k/n} - h_{(k+1)/n})_+ f_{k/n} n^{1/3}. \quad (3.2.24)$$

We now bound from above $R_{a,A}^{(n)}$ in four different ways, depending on the presence of the boundaries f and g.

First, for all $a < A$, by (3.2.24), we have

$$R_{a,A}^{(n)} \leq \exp\left(\sum_{j=m_a}^{m_{a+1}-1} (h_{(j+1)/n} - h_{j/n})_+ g_{j/n} n^{1/3} - (h_{j/n} - h_{(j+1)/n})_+ f_{k/n} n^{1/3} \right),$$

and thus,

$$\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log R_{a,A}^{(n)} \leq \int_{a/A}^{(a+1)/A} (\dot{h}_s)_+ g_s - (\dot{h}_s)_- f_s ds = \int_{a/A}^{(a+1)/A} \dot{h}_s g_s - (\dot{h}_s)_- (f_s - g_s) ds. \quad (3.2.25)$$

This crude estimate can be improved as follows. If $\underline{h}_{a,A} > 0$, then $[\frac{a}{A}, \frac{a+1}{A}] \subset G$ and the upper bound $g_{k/n} n^{1/3}$ of the path is present at all times $k \in [m_a, m_{a+1}]$. As a consequence, (3.2.24) becomes

$$\forall k \in [m_a, m_{a+1}), \sup_{x\in I_k^{(n)}} (h_{(k+1)/n} - h_{k/n})x \leq (h_{(k+1)/n} - h_{k/n}) g_{a,A} n^{1/3} + \frac{1}{n} \underline{h}_{a,A}(x - g_{a,A} n^{1/3}). \quad (3.2.26)$$

We have

$$\begin{aligned} R_{a,A}^{(n)} &= \sup_{x\in I_{m_a}^{(n)}} \mathbf{E}_x \left[e^{\sum_{j=0}^{d_a-1} (h_{(m_a+j+1)/n} - h_{(m_a+j)/n}) S_j^{(n),a}} ; S_j^{(n),a} \in I_{m_a+j}^{(n)}, j \leq d_a \right] \\ &\leq e^{\sum_{j=m_a}^{m_{a+1}-1} (h_{(j+1)/n} - h_{j/n}) g_{a,A} n^{1/3}} \\ &\qquad \times \sup_{x\in I_{m_a}^{(n)}} \mathbf{E}_x \left[e^{\frac{1}{n} \sum_{j=0}^{d_a-1} \underline{h}_{a,A}(S_j^{(n),a} - g_{a,A} n^{1/3})} ; S_j^{(n),a} \in I_{m_a+j}^{(n)}, j \leq d_a \right] \\ &\leq e^{(h_{m_{a+1}/n} - h_{m_a/n}) g_{a,A} n^{1/3}} \sup_{x\leq 0} \mathbf{E}_x \left[e^{\frac{1}{n} \sum_{j=0}^{d_a-1} \underline{h}_{a,A} S_j^{(n),a}} ; S_{m_a+j}^{(n),a} \leq 0, j \leq d_a \right]. \end{aligned}$$

Letting $n \to +\infty$, we have

$$\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log R_{a,A}^{(n)} \leq (h_{(a+1)/A} - h_{a/A}) g_{a,A} \\ + \limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \sup_{x\leq 0} \mathbf{E}_x \left[e^{\frac{\underline{h}_{a,A}}{n} \sum_{j=0}^{d_a-1} S_j^{(n),a}} ; S_j^{(n),a} \leq 0, j \leq d_a \right].$$

As $d_a \sim_{n\to+\infty} n/A$, by (3.2.15),

$$\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \sup_{x\leq 0} \mathbf{E}_x \left[e^{\frac{\underline{h}_{a,A}}{A(d_a+1)} \sum_{j=0}^{d_a-1} S_j^{(n),a}} ; S_j^{(n),a} \leq 0, j \leq d_a \right] \\ \leq \frac{1}{A^{1/3}} \frac{\alpha_1}{2^{1/3}} \left(\tfrac{1}{A} \underline{h}_{a,A} \sigma_{a,A} \right)^{2/3} = \frac{\alpha_1}{2^{1/3} A} \left(\underline{h}_{a,A} \sigma_{a,A} \right)^{2/3}.$$

We conclude that

$$\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log R_{a,A}^{(n)} \leq (h_{(a+1)/A} - h_{a/A}) g_{a,A} + \frac{\alpha_1}{2^{1/3} A} \left(\underline{h}_{a,A} \sigma_{a,A} \right)^{2/3}. \tag{3.2.27}$$

By symmetry, if $\overline{h}_{a,A} < 0$, then $[\frac{a}{A}, \frac{a+1}{A}] \subset F$, $h_{(k+1)/n} < h_{k/n}$ and the lower bound of the path is present at all time, which leads to

$$\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log R_{a,A}^{(n)} \leq (h_{(a+1)/A} - h_{a/A}) f_{a,A} + \frac{\alpha_1}{2^{1/3} A} \left(-\overline{h}_{a,A} \sigma_{a,A} \right)^{2/3}. \tag{3.2.28}$$

Fourth and the smallest upper bound; if $[\frac{a}{A}, \frac{a+1}{A}] \subset F \cap G$, then both bounds of the path are present at any time in $[m_a, m_{a+1}]$, and, by (3.2.26), setting $r_{a,A} = g_{a,A} - f_{a,A}$,

$$R_{a,A}^{(n)} \leq e^{(h_{m_{a+1}/n} - h_{m_a/n}) g_{a,A} n^{1/3}} \\ \times \sup_{x\in[r_{a,A}n^{1/3},0]} \mathbf{E}_x \left[e^{\frac{1}{n} \sum_{j=0}^{d_a-1} \underline{h}_{a,A} S_j^{(n),a}} ; S_j^{(n),a} \in [-r_{a,A} n^{1/3}, 0], j \leq d_a \right].$$

We conclude that

$$\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log R_{a,A}^{(n)} \leq (h_{(a+1)/A} - h_{a/A}) g_{a,A} \\ + \limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \sup_{x\in[-r_{a,A}n^{1/3},0]} \mathbf{E}_x \left[e^{\frac{\underline{h}_{a,A}}{n} \sum_{j=0}^{d_a-1} S_j^{(n),a}} ; S_j^{(n),a} \in [-r_{a,A} n^{1/3}, 0], j \leq d_a \right].$$

Applying then (3.2.16), this yields

$$\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \sup_{x\in[-r_{a,A}n^{1/3},0]} \mathbf{E}_x \left[e^{\frac{\underline{h}_{a,A}}{n} \sum_{j=0}^{d_a-1} S_j^{(n),a}} ; S_j^{(n),a} \in [-r_{a,A} n^{1/3}, 0], j \leq d_a \right] \\ \leq \frac{\sigma_{a,A}^2}{A r_{a,A}^2} \Psi \left(\frac{r_{a,A}^3}{\sigma_{a,A}^2} \underline{h}_{a,A} \right),$$

which yields

$$\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log R_{a,A}^{(n)} \leq (h_{(a+1)/A} - h_{a/A}) g_{a,A} + \frac{\sigma_{a,A}^2}{A (g_{a,A} - f_{a,A})^2} \Psi \left(\frac{(g_{a,A} - f_{a,A})^3}{\sigma_{a,A}^2} \underline{h}_{a,A} \right). \tag{3.2.29}$$

We now let A grow to $+\infty$ in (3.2.23). By Riemann-integrability of F, G and $\dot{h}$, we have

$$\limsup_{A\to+\infty} \sum_{\substack{0\leq a<A\\ [\frac{a}{A},\frac{a+1}{A}]\subset F\cap G}} \left[(h_{(a+1)/A} - h_{a/A})g_{a,A} + \frac{\sigma_{a,A}^2}{A(g_{a,A}-f_{a,A})^2}\Psi\left(\frac{(g_{a,A}-f_{a,A})^3}{\sigma_{a,A}^2}\underline{h}_{a,A}\right)\right]$$
$$\leq \int_{F\cap G} \dot{h}_s g_s + \frac{\sigma_s^2}{(g_s-f_s)^2}\Psi\left(\frac{(g_s-f_s)^3}{\sigma_s^2}\dot{h}_s\right) ds. \quad (3.2.30)$$

Similarly, using the fact that $\dot{h}$ is non-negative on F^c, and non-positive on G^c, (3.2.27) and (3.2.28) lead respectively to

$$\limsup_{A\to+\infty} \sum_{\substack{0\leq a<A\\ \underline{h}_{a,A}>0,[\frac{a}{A},\frac{a+1}{A}]\not\subset F\cap G}} \left[(h_{(a+1)/A} - h_{a/A})g_{a,A} + \frac{\alpha_1}{A2^{1/3}}\left(\underline{h}_{a,A}\sigma_{a,A}\right)^{2/3}\right]$$
$$\leq \int_{F^c\cap G} \dot{h}_s g_s + \frac{\alpha_1}{2^{1/3}}\left(\dot{h}_s\sigma_s\right)^{2/3} ds, \quad (3.2.31)$$

and to

$$\limsup_{A\to+\infty} \sum_{\substack{0\leq a<A\\ \overline{h}_{a,A}<0,[\frac{a}{A},\frac{a+1}{A}]\not\subset F\cap G}} \left[(h_{(a+1)/A} - h_{a/A})f_{a,A} + \frac{\alpha_1}{A2^{1/3}}\left(-\overline{h}_{a,A}\sigma_{a,A}\right)^{2/3}\right]$$
$$\leq \int_{F^c\cap G} \dot{h}_s g_s + \dot{h}_s(f_s-g_s) + \frac{\alpha_1}{2^{1/3}}\left(-\dot{h}_s\sigma_s\right)^{2/3} ds. \quad (3.2.32)$$

Finally, by (3.2.25), (3.2.30), (3.2.31) and (3.2.32), letting $A\to+\infty$, (3.2.23) yields

$$\limsup_{n\to+\infty}\sup_{x\in\mathbb{R}} \frac{1}{n^{1/3}}\log \mathbf{E}_x\left[e^{\sum_{j=1}^n (h_{(j+1)/n}-h_{j/n})S_j^{(n)}}; S_j^{(n)}\in I_j^{(n)}, j\leq n\right] \leq H_{f,g}^{F,G}.$$

Lower bound in (3.2.7). We now take care of the lower bound. We start by fixing $H>0$, and we write

$$I_j^{(n,H)} = I_j^{(n)} \cap [-Hn^{1/3}, Hn^{1/3}],$$

letting H grow to $+\infty$ at the end of the proof. We only need (3.2.20) here.

We choose $k\in\mathcal{C}([0,1])$ a continuous function such that $k_0=0$ and $k_1\in(a,b)$ and $\varepsilon>0$ such that for all $t\in[0,1]$, $k_t\in[f_t+4\varepsilon, g_t-4\varepsilon]$ and $k_1\in[a+4\varepsilon, b-4\varepsilon]$. We set

$$J_a^{(n)} = \left[(k_{a/A}-\varepsilon)n^{1/3}, (k_{a/A}+\varepsilon)n^{1/3}\right].$$

We apply the Markov property at times $m_{A-1},\dots m_1$, only considering random walk paths that are in interval $J_a^{(n)}$ at any time m_a. For all $n\geq 1$ large enough, we have

$$\mathbf{E}\left[e^{\sum_{j=0}^{n-1}(h_{(j+1)/n}-h_{j/n})S_j^{(n)}}\mathbf{1}_{\left\{\frac{S_n^{(n)}}{n^{1/3}}\in[a',b']\right\}}; S_j^{(n)}\in\widetilde{I}_j^{(n)}, j\leq n\right]$$
$$\geq \prod_{a=0}^{A-1}\inf_{x\in I_{m_a}^{(n)}}\mathbf{E}_x\left[e^{\sum_{j=0}^{d_a-1}(h_{(m_a+j+1)/n}-h_{(m_a+j)/n})S_j^{(n,a)}}\mathbf{1}_{\left\{S_{d_a}^{(n,a)}\in J_{a+1}^{(n)}\right\}}; S_j^{(n,a)}\in I_{m_a+j}^{(n,H)}, j\leq d_a\right]$$
$$=: \prod_{a=0}^{A-1}\widetilde{R}_{a,A}^{(n)},$$

with the same random walk notation as in the previous paragraph. Therefore,

$$\liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{E}\left[e^{\sum_{j=1}^{n}(h_{(j+1)/n}-h_{j/n})S_j^{(n)}}; S_j^{(n)} \in I_j^{(n)}, j \leq n\right] \geq \sum_{a=0}^{A-1} \liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \widetilde{R}_{a,A}^{(n)}. \tag{3.2.33}$$

We now bound from below $\widetilde{R}_{a,A}^{(n)}$, replacing functions f, g and $\dot{h}$ by constants. We write here

$$f_{a,A} = \sup_{t\in[\frac{a-1}{A},\frac{a+2}{A}]} f_t, \quad g_{a,A} = \inf_{t\in[\frac{a-1}{A},\frac{a+2}{A}]} g_t \quad \text{and} \quad \sigma_{a,A} = \inf_{t\in[\frac{a-1}{A},\frac{a+2}{A}]} \sigma_t,$$

keeping notations $\overline{h}_{a,A}$ and $\underline{h}_{a,A}$ as above. We assume $A > 0$ is chosen large enough such that

$$\sup_{|t-s|\leq \frac{2}{A}} |f_t - f_s| + |g_t - g_s| + |k_t - k_s| \leq \varepsilon.$$

We first observe that $[f_{a,A}n^{1/3}, g_{a,A}n^{1/3}] \subset I_j^{(n,H)}$ for all $j \in [m_a, m_{a+1}]$, therefore, writing $r_{a,A} = g_{a,A} - f_{a,A}$,

$$\widetilde{R}_{a,A}^{(n)} \geq e^{(h_{m_{a+1}/n}-h_{m_a/n})g_{a,A}n^{1/3}}$$
$$\times \inf_{x\in J_a^{(n)}} \mathbf{E}_x\left[e^{\frac{\overline{h}_{a,A}}{n}\sum_{j=0}^{d_a-1} S_j^{(n,a)}} \mathbf{1}_{\left\{S_{d_a}^{(n,a)} \in J_{a+1}^{(n)}\right\}}; S_j^{(n,a)} \in [-r_{a,A}n^{1/3}, 0], j \leq d_a\right].$$

Thus, by (3.2.20), we have

$$\liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \widetilde{R}_{a,A}^{(n)} \geq (h_{(a+1)/A} - h_{a/A})g_{a,A} + \frac{\sigma_{a,A}^2}{A(g_{a,A} - f_{a,A})^2}\Psi\left(\frac{(g_{a,A}-f_{a,A})^3}{\sigma_{a,A}^2}\overline{h}_{a,A}\right). \tag{3.2.34}$$

This lower bound can be improved, if $[\frac{a}{A}, \frac{a+1}{A}] \subset F^c$, in which case for all $j \in [m_a, m_{a+1}]$ we have $[-Hn^{1/3}, g_{a,A}n^{1/3}] \subset I_j^{(n,H)}$. Thus

$$\widetilde{R}_{a,A}^{(n)} \geq e^{(h_{m_{a+1}/n}-h_{m_a/n})g_{a,A}n^{1/3}}$$
$$\times \inf_{x\in J_a^{(n)}} \mathbf{E}_x\left[e^{\frac{\overline{h}_{a,A}}{n}\sum_{j=0}^{d_a-1} S_j^{(n,a)}} \mathbf{1}_{\left\{S_{d_a}^{(n,a)} \in J_{a+1}^{(n)}\right\}}; S_j^{(n,a)} \in [-(H - g_{a,A})n^{1/3}, 0], j \leq d_a\right],$$

which leads to

$$\liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \widetilde{R}_{a,A}^{(n)} \geq (h_{(a+1)/A}-h_{a/A})g_{a,A}+\frac{\sigma_{a,A}^2}{A(g_{a,A} + H)^2}\Psi\left(\frac{(g_{a,A}+H)^3}{\sigma_{a,A}^2}\overline{h}_{a,A}\right). \tag{3.2.35}$$

By symmetry, if $[\frac{a}{A}, \frac{a+1}{A}] \subset G^c$, we have

$$\liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \widetilde{R}_{a,A}^{(n)} \geq (h_{(a+1)/A} - h_{a/A})f_{a,A} + \frac{\sigma_{a,A}^2}{A(H - f_{a,A})^2}\Psi\left(-\frac{(H-f_{a,A})^3}{\sigma_{a,A}^2}\underline{h}_{a,A}\right). \tag{3.2.36}$$

As a consequence, letting $A \to +\infty$, by Riemann-integrability of F, G and $\dot{h}$, (3.2.34) leads to

$$\liminf_{A\to+\infty} \sum_{\substack{0\leq a\leq A \\ [\frac{a}{A},\frac{a+1}{A}]\cap F\cap G\neq\emptyset}} \liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \widetilde{R}_{a,A}^{(n)} \geq \int_{F\cap G} \dot{h}_t g_t + \frac{\sigma_t^2}{(g_t - f_t)^2}\Psi\left(\frac{(g_t-f_t)^3}{\sigma_t^2}\dot{h}_t\right) dt. \tag{3.2.37}$$

Similarly, (3.2.35) gives

$$\liminf_{A\to+\infty} \sum_{\substack{0\leq a\leq A\\ [\frac{a}{A},\frac{a+1}{A}]\subset F^c}} \liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \widetilde{R}^{(n)}_{a,A} \geq \int_{F^c} \dot{h}_t g_t + \frac{\sigma_t^2}{(g_t+H)^2}\Psi\left(\tfrac{(g_t+H)^3}{\sigma_t^2}\dot{h}_t\right) dt, \quad (3.2.38)$$

and (3.2.36) gives

$$\liminf_{A\to+\infty} \sum_{\substack{0\leq a\leq A\\ [\frac{a}{A},\frac{a+1}{A}]\subset F\cap G^c}} \liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \widetilde{R}^{(n)}_{a,A}$$
$$\geq \int_{F\cap G^c} \dot{h}_t g_t + \dot{h}_t(f_t-g_t) + \frac{\sigma_t^2}{(H-f_t)^2}\Psi\left(-\tfrac{(H-f_t)^3}{\sigma_t^2}\dot{h}_t\right) dt. \quad (3.2.39)$$

Finally, we recall that

$$\lim_{H\to+\infty} \frac{1}{H^{2/3}}\Psi(H) = \frac{\alpha_1}{2^{1/3}}.$$

As $\dot{h}$ is non-negative on G^c and null on $F^c\cap G^c$, by dominated convergence, we have

$$\lim_{H\to+\infty} \int_{F^c} \frac{\sigma_s^2}{(g_s+H)^2}\Psi\left(\tfrac{(g_s+H)^3}{\sigma_s^2}\dot{h}_s\right) ds = \int_{F^c} \frac{\alpha_1}{2^{1/3}}(\dot{h}_s\sigma_s)^{2/3} ds,$$

and as $\dot{h}$ is non-positive on F^c, we have similarly,

$$\lim_{H\to+\infty} \int_{F\cap G^c} \frac{\sigma_s^2}{(H-f_s)^2}\Psi\left(-\tfrac{(H-f_s)^3}{\sigma_s^2}\dot{h}_s\right) ds = \int_{F\cap G^c} \frac{\alpha_1}{2^{1/3}}(-\dot{h}_s\sigma_s)^{2/3} ds.$$

Consequently, letting n, then A, then H grow to $+\infty$ –observe that ε, given it is small enough, does not have any impact on the asymptotic– we have

$$\liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{E}_0\left[e^{\sum_{j=0}^{n-1}(h_{(j+1)/n}-h_{j/n})S_j^{(n)}} \mathbf{1}_{\left\{S_n^{(n)}\in[an^{1/3},bn^{1/3}]\right\}} ; S_j^{(n)} \in \widetilde{I}_j^{(n)}, j\leq n\right]$$
$$\geq H^{F,G}_{f,g}.$$

Conclusion. Using the fact that

$$\sup_{x\in\mathbb{R}} \mathbf{E}_x\left[e^{\sum_{j=0}^{n-1}(h_{(j+1)/n}-h_{j/n})S_j^{(n)}} ; S_j^{(n)} \in I_j^{(n)}, j\leq n\right]$$
$$\geq \mathbf{E}_0\left[e^{\sum_{j=1}^{n}(h_{(j+1)/n}-h_{j/n})S_j^{(n)}} \mathbf{1}_{\left\{S_n^{(n)}\in[an^{1/3},bn^{1/3}]\right\}} ; S_j^{(n)} \in \widetilde{I}_j^{(n)}, j\leq n\right],$$

the two inequalities we obtained above allow to conclude the proof. □

3.3 The many-to-one lemma and branching random walk estimates

In this section, we introduce a time-inhomogeneous version of the many-to-one lemma, that links some additive moments of the branching random walk with the random walk estimates obtained in the previous section. Using the well-established method in the branching random walk theory (see e.g. the previous chapters, [Aïd13, AJ11, AS10, FZ12a, FZ12b, GHS11, HR11, HS09, MZ14] and a lot of others) that consists in proving the existence of a boundary via a first moment method, then bounding the tail distribution of maximal displacement below this boundary by estimation of first and second moments of the number of individuals below this boundary, and the Cauchy-Schwarz inequality. The boundary is determined by a differential equation, which is solved in Section 5.

3.3.1 Branching random walk notations and the many-to-one lemma

The many-to-one lemma can be traced back at least to the early works of Peyrière [Pey74] and Kahane and Peyrière [KP76]. This result has been used under many forms in the past years, extended to branching Markov processes in [BK04]. This is a very powerful tool that has been used to obtain different branching random walk estimates, see e.g. [Aïd13, AJ11, AS10, FZ12a, FZ12b, HR11]. We introduce some additional branching random walk notation in a first time.

Let $(\mathbf{T}, V)$ be a BRWls of length n with environment $(\mathcal{L}_t, t \in [0,1])$. We recall that $\mathbf{T}$ is a tree of height n and that for any $u \in \mathbf{T}$, $|u|$ is the generation to which u belongs, u_k the ancestor of u at generation k and $V(u)$ the position of u. We introduce, for $k \leq n$, $\mathcal{F}_k = \sigma\left((u, V(u)), |u| \leq k\right)$ the σ-field generated by the branching random walk up to generation k.

Let $y \in \mathbb{R}$ and $k \leq n$. We denote by $\mathbf{P}_{k,y}$ the law of the time-inhomogeneous branching random walk $(\mathbf{T}^k, V^k)$ where $\mathbf{T}^k$ is a tree of length $n-k$, such that the family of point processes $\{L^{u'}, u' \in \mathbf{T}^k, |u'| \leq n-k-1\}$ is independent, with $L^{u'}$ of law $\mathcal{L}_{(|u'|+k+1)/n}$. With this definition, we observe that conditionally on $\mathcal{F}_k$, for every individual $u \in \mathbf{T}$ alive at generation k, the subtree $\mathbf{T}^u$ of $\mathbf{T}$ rooted at u, with marks $V_{|\mathbf{T}^u}$ is a time-inhomogeneous branching random walk with law $\mathbf{P}_{|u|,V(u)}$, independent of the rest of the branching random walk $(\mathbf{T}\backslash\mathbf{T}^u, V_{|\mathbf{T}\backslash\mathbf{T}^u})$.

We introduce φ a continuous positive function on $[0,1]$ such that

$$\forall t \in [0,1], \kappa_t(\varphi_t) < +\infty, \tag{3.3.1}$$

and set, for $t \in [0,1]$

$$b_t = \partial_\theta \kappa_t(\varphi_t) \quad \text{and} \quad \sigma_t^2 = \partial^2_\theta \kappa_t(\varphi_t). \tag{3.3.2}$$

Let $(X_{n,k}, n \geq 1, k \leq n)$ be a triangular array of independent random variables such that for all $n \geq 1$, $k \leq n$ and $x \in \mathbb{R}$, we have

$$\mathbf{P}(X_{n,k} \leq x) = \mathbf{E}\left[\sum_{\ell \in L_{k/n}} \mathbf{1}_{\{\ell \leq x\}} e^{\varphi_{k/n}\ell - \kappa_{k/n}(\varphi_{k/n})}\right],$$

where $L_{k/n}$ is a point process of law $\mathcal{L}_{k/n}$. By (3.1.4) and (3.3.2), we have

$$\mathbf{E}(X_{n,k}) = b_{k/n} \quad \text{and} \quad \mathbf{E}\left((X_{n,k} - b_{k/n})^2\right) = \sigma^2_{k/n}.$$

For $k \leq n$, we denote by $S_k = \sum_{j=1}^k X_{n,j}$ the time-inhomogeneous random walk associated to φ, by $\overline{b}^{(n)}_k = \sum_{j=1}^k b_{j/n}$, by $\widetilde{S}_k = S_k - \overline{b}^{(n)}_k$ the centred version of this random walk and by

$$E_k := \sum_{j=1}^k \varphi_{j/n} b_{j/n} - \kappa_{j/n}(\varphi_{j/n}) = \sum_{j=1}^k \kappa^*_{j/n}(b_{j/n}), \tag{3.3.3}$$

by (3.1.1). Under $\mathbf{P}_{k,y}$, $(S_j, j \leq n-k)$ and $\left(y + \sum_{i=k+1}^{k+j+1} X_{n,i}, j \leq n-k\right)$ have the same law.

Lemma 3.3.1 (Many-to-one lemma)**.** *Let $n \geq 1$ and $k \leq n$. Under assumption* (3.3.1)*, for any measurable non-negative function f, we have*

$$\mathbf{E}\left(\sum_{|u|=k} f(V(u_j), j \leq k)\right) = e^{-E_k}\,\mathbf{E}\left[e^{-\varphi_{k/n}\widetilde{S}_k + \sum_{j=0}^{k-1}(\varphi_{(j+1)/n}-\varphi_{j/n})\widetilde{S}_j} f(S_j, j \leq k)\right].$$

Remark 3.3.2. As an immediate corollary of the many-to-one lemma, we have, for $p \leq n$, $y \in \mathbb{R}$ and $k \leq n-p$,

$$\begin{aligned}\mathbf{E}_{p,y}&\left(\sum_{|u|=k} f(V(u_j), j \leq k)\right)\\ &= e^{E_p - E_{k+p}} e^{\varphi_{p/n} y}\,\mathbf{E}_{p,y}\left[e^{-\varphi_{(k+p)/n}\widetilde{S}_k + \sum_{j=p}^{p+k-1}(\varphi_{(j+1)/n}-\varphi_{j/n})\widetilde{S}_{j-p}} f(S_j, j \leq k)\right].\end{aligned}$$

Proof. Let $n \geq 1$, $k \leq n$ and f non-negative and measurable, we prove by induction on $k \leq n$ that

$$\mathbf{E}\left(\sum_{|u|=k} f(V(u_j), j \leq k)\right) = \mathbf{E}\left[e^{-\sum_{j=1}^{k}\varphi_{j/n}X_{n,j} - \kappa_{j/n}(\varphi_{j/n})} f(S_j, j \leq k)\right].$$

We first observe that if $k=1$, by definition of $X_{n,1}$, we have

$$\mathbf{E}\left(\sum_{|u|=1} f(V(u))\right) = \mathbf{E}\left[e^{-\varphi_{1/n}X_{n,1} + \kappa_{1/n}(\varphi_{1/n})} f(X_{n,1})\right].$$

Let $k \geq 2$. By conditioning on $\mathcal{F}_{k-1}$, we have

$$\begin{aligned}\mathbf{E}\left(\sum_{|u|=k} f(V(u_j), j \leq k)\right) &= \mathbf{E}\left[\sum_{|u|=k-1}\sum_{u' \in \Omega(u)} f(V(u'_j), j \leq k)\right]\\ &= \mathbf{E}\left(\sum_{|u|=k-1} g(V(u_j), j \leq k-1)\right),\end{aligned}$$

where, for $(x_j, j \leq k-1) \in \mathbb{R}^{k-1}$,

$$\begin{aligned}g(x_j, j \leq k-1) &= \mathbf{E}\left[\sum_{\ell \in L_{k/n}} f(x_1, \dots x_{k-1}, x_{k-1}+\ell)\right]\\ &= \mathbf{E}\left[e^{-\varphi_{k/n}X_{n,k} + \kappa_{k/n}(\varphi_{k/n})} f(x_1, \dots x_{k-1}, x_{k-1}+X_{n,k})\right].\end{aligned}$$

Using the induction hypothesis, we conclude that

$$\begin{aligned}\mathbf{E}\left(\sum_{|u|=k} f(V(u_j), j \leq k)\right) &= \mathbf{E}\left[e^{-\sum_{j=1}^{k}\varphi_{j/n}X_{n,j} - \kappa_{j/n}(\varphi_{j/n})} f(S_j, j \leq k)\right]\\ &= e^{-E_k}\,\mathbf{E}\left[e^{-\sum_{j=1}^{k}\varphi_{j/n}(X_{n,j}-b_{j/n})} f(S_j, j \leq k)\right].\end{aligned}$$

Finally, we modify the exponential weight by the Abel transform,

$$\sum_{j=1}^{k} \varphi_{j/n}(X_{n,j} - b_j) = \sum_{j=1}^{k} \varphi_{j/n}(\widetilde{S}_j - \widetilde{S}_{j-1}) = \sum_{j=1}^{k} \varphi_{j/n}\widetilde{S}_j - \sum_{j=1}^{k} \varphi_{j/n}\widetilde{S}_{j-1}$$
$$= \sum_{j=1}^{k} \varphi_{j/n}\widetilde{S}_j - \sum_{j=1}^{k-1} \varphi_{(j+1)/n}\widetilde{S}_j = \varphi_{k/n}\widetilde{S}_k - \sum_{j=1}^{k-1}(\varphi_{(j+1)/n} - \varphi_{j/n})\widetilde{S}_j,$$

which ends the proof. □

3.3.2 Number of individuals staying along a path

In this section, we bound some quantities related to the number of individuals that stay along a path. We start with an upper bound of the expected number of individuals that stay in the path until some time $k \leq n$, and then exit the path by the upper boundary. Subsequently, we bound the probability that there exists an individual that stays in the path until time n. We then compute the first two moments of the number of such individuals, and apply the Cauchy-Schwarz inequality to conclude. We assume in this section that

$$\varphi \text{ is absolutely continuous, with a Riemann-integrable derivative } \dot{\varphi}, \tag{3.3.4}$$

as we plan to apply Theorem 3.2.1 with function $h = \varphi$. Under this assumption, φ is Lipschitz, thus so is b. As a consequence, we have

$$\sup_{\substack{n\in\mathbb{N}\\k\leq n}} \sup_{t\in[\frac{k-1}{n},\frac{k+2}{n}]} |E_k - nK^*(b)_t| < +\infty \quad \text{and} \quad \sup_{\substack{n\in\mathbb{N}\\k\leq n}} \sup_{t\in[\frac{k-1}{n},\frac{k+2}{n}]} \left|\overline{b}_k^{(n)} - n\int_0^t b_s ds\right| < +\infty. \tag{3.3.5}$$

Let $f < g$ be two continuous functions such that $f(0) < 0 < g(0)$, and F and G two Riemann-integrable subsets of $[0,1]$ such that

$$\{t \in [0,1] : \dot{\varphi}_t < 0\} \subset F \quad \text{and} \quad \{t \in [0,1] : \dot{\varphi}_t > 0\} \subset G. \tag{3.3.6}$$

We write, for $t \in [0,1]$

$$H_t^{F,G}(f,g,\varphi) = \int_0^t \dot{\varphi}_s g_s ds + \int_0^t \mathbf{1}_{F\cap G}(s)\frac{\sigma_s^2}{(g_s - f_s)^2}\Psi\left(\tfrac{(g_s-f_s)^3}{\sigma_s^2}\dot{\varphi}_s\right) ds$$
$$+ \int_0^t \mathbf{1}_{F^c\cap G}(s)\frac{a_1}{2^{1/3}}(\dot{\varphi}_s\sigma_s)^{2/3} + \mathbf{1}_{F\cap G^c}\left(\dot{\varphi}_s(f_s - g_s) + \frac{a_1}{2^{1/3}}(-\dot{\varphi}_s\sigma_s)^{2/3}\right) ds. \tag{3.3.7}$$

We keep notation of Section 3.2: F_n and G_n are the subsets of $\{0,\dots n-1\}$ defined in (3.2.3), and the path $I_k^{(n)}$ as defined in (3.2.4). We are interested in the individuals u alive at generation n such that for all $k \leq n$, $V(u_k) - \overline{b}_k^{(n)} \in I_k^{(n)}$.

A boundary estimate

We compute the number of individuals that stayed in $\overline{b}^{(n)} + I^{(n)}$ until some time $k-1$ and then crossed the upper boundary $\overline{b}_k^{(n)} + g_{k/n}n^{1/3}$ of the path at time $k \in G_n$. We denote by

$$\mathcal{A}_n^{F,G}(f,g) = \left\{u \in \mathbf{T}, |u| \in G_n : V(u) - \overline{b}_{|u|}^{(n)} > g_{|u|/n}n^{1/3}, V(u_j) - \overline{b}_j^{(n)} \in I_j^{(n)}, j < |u|\right\},$$

the set of such individuals, and by $A_n^{F,G}(f,g) = \#\mathcal{A}_n^{F,G}(f,g)$.

Lemma 3.3.3. *Under the assumptions* (3.1.4), (3.3.1), (3.3.4) *and* (3.3.6), *if we have* $G \subset \{t \in [0,1] : K^*(b)_t = 0\}$ *then,*

$$\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{E}(A_n^{F,G}(f,g)) \leq \sup_{t\in[0,1]} \left[H_t^{F,G}(f,g,\varphi) - \varphi_t g_t\right].$$

Remark 3.3.4. Observe that in order to use this lemma, we need to assume that

$$\{t \in [0,1] : \dot{\varphi}_t > 0\} \subset G \subset \{t \in [0,1] : K^*(b)_t = 0\},$$

we cannot consider paths of speed profile b such that the associated parameter φ increases at a time when there is an exponentially large number of individuals following the path. For such paths, the mean of A_n grows exponentially fast.

Proof. By (3.3.1) and Lemma 3.3.1, we have

$$\begin{aligned}
&\mathbf{E}(A_n^{F,G}(f,g))\\
&= \sum_{k\in G_n} \mathbf{E}\left[\sum_{|u|=k} \mathbf{1}_{\left\{V(u)-\bar{b}_k^{(n)} > g_{k/n} n^{1/3}\right\}} \mathbf{1}_{\left\{V(u_j)-\bar{b}_j^{(n)} \in I_j^{(n)}, j<k\right\}}\right]\\
&= \sum_{k\in G_n} e^{-E_k}\, \mathbf{E}\left[e^{-\varphi_{k/n}\widetilde{S}_k + \sum_{j=0}^{k-1}(\varphi_{(j+1)/n}-\varphi_{j/n})\widetilde{S}_j} \mathbf{1}_{\left\{\widetilde{S}_k > g_{k/n}n^{1/3}\right\}} \mathbf{1}_{\left\{\widetilde{S}_j \in I_j^{(n)}, j<k\right\}}\right].
\end{aligned}$$

For all $k \in G_n$, there exists $t \in [k/n,(k+1)/n]$ such that $t \in G$, thus $K^*(b)_t = 0$. By (3.3.5), this implies that $\sup_{n\in\mathbb{N},k\in G_n} E_k < +\infty$, hence

$$\begin{aligned}
&\mathbf{E}(A_n^{F,G}(f,g))\\
&\quad\leq C \sum_{k\in G_n} e^{-\varphi_{k/n} g_{k/n} n^{1/3}}\, \mathbf{E}\left[e^{\sum_{j=0}^{k-1}(\varphi_{(j+1)/n}-\varphi_{j/n})\widetilde{S}_j} \mathbf{1}_{\left\{\widetilde{S}_k > g_{k/n}n^{1/3}\right\}} \mathbf{1}_{\left\{\widetilde{S}_j \in I_j^{(n)}, j<k\right\}}\right].
\end{aligned}$$

As (3.3.6) is verified, similarly to (3.2.24), for all $k \leq n$ and $x \in I_k^{(n)}$, we have

$$(\varphi_{(k+1)/n} - \varphi_{k/n})x \leq (\varphi_{(k+1)/n} - \varphi_{k/n})_+ g_{k/n} n^{1/3} - (\varphi_{k/n} - \varphi_{(k+1)/n})_+ f_{k/n} n^{1/3}. \quad (3.3.8)$$

In particular, $(\varphi_{(k+1)/n} - \varphi_{k/n})x \leq \left|\varphi_{(k+1)/n} - \varphi_{k/n}\right| (\|f\|_\infty + \|g\|_\infty)$. Let $A > 0$ be a large integer. For $a < A$, we set $m_a = \lfloor an/A \rfloor$ and

$$\underline{g}_{a,A} = \inf\left\{g_t, t \in \left[\tfrac{a-1}{A}, \tfrac{a+2}{A}\right]\right\}, \quad \underline{\varphi}_{a,A} = \inf\left\{\varphi_t, t \in \left[\tfrac{a-1}{A}, \tfrac{a+2}{A}\right]\right\}$$
$$\text{and} \quad d_{a,A} = (\|f\|_\infty + \|g\|_\infty) \int_{(a-1)/A}^{(a+2)/A} |\dot{\varphi}_s| ds.$$

For $k \in (m_a, m_{a+1}]$, applying the Markov property at time m_a, we have

$$\mathbf{E}\left[e^{\sum_{j=0}^{k-1}(\varphi_{(j+1)/n}-\varphi_{j/n})\widetilde{S}_j} \mathbf{1}_{\left\{\widetilde{S}_k > g_{k/n}n^{1/3}\right\}} \mathbf{1}_{\left\{\widetilde{S}_j \in I_j^{(n)}, j<k\right\}}\right] \leq \exp\left(d_{a,A} n^{1/3}\right) \Phi_{a,A}^{(n)},$$

where $\Phi^{(n)}_{a,A} = \mathbf{E}\left[e^{\sum_{j=1}^{m_a} (\varphi_{(j+1)/n} - \varphi_{j/n}) \widetilde{S}_j} \mathbf{1}_{\left\{ \widetilde{S}_j \in I^{(n)}_j, j \leq m_a \right\}} \right]$. We observe that $\widetilde{S}$ is a centred random walk which, by (3.3.2), verifies (3.1.15) with variance function σ^2. Moreover, as

$$\mathbf{E}\left[e^{\mu |X_{n,k}|}\right] \leq \mathbf{E}\left[e^{\mu X_{n,k}} + e^{-\mu X_{n,k}}\right]$$
$$\leq \mathbf{E}\left[\sum_{\ell \in L_{k/n}} e^{(\varphi_t + \mu)\ell - \kappa_t(\varphi_t)} + e^{(\varphi_t - \mu)\ell - \kappa_t(\varphi_t)} \right] \leq e^{\kappa_t(\varphi_t + \mu) - \kappa_t(\varphi_t)} + e^{\kappa_t(\varphi_t - \mu) - \kappa_t(\varphi_t)},$$

by (3.1.4), there exists $\mu > 0$ such that $\sup_{n \in \mathbb{N}, k \leq n} \mathbf{E}\left[e^{\mu |X_{n,k}|}\right] < +\infty$ and (3.1.16) is verified. For all $a \leq A$, we apply Theorem 3.2.1, to $h_t = \varphi_{t \wedge a/A}$, functions f and g and intervals F and G stopped at time a/A, to obtain

$$\limsup_{n \to +\infty} \frac{\log \Phi^{(n)}_{a,A}}{n^{1/3}} = H^{F,G}_{a/A}(f, g, \varphi).$$

We observe that

$$\mathbf{E}(A^{F,G}_n(f, g)) \leq C \sum_{a=0}^{A-1} \frac{n}{A} \exp\left(\left(d_{a,A} - \underline{\varphi}_{a,A} \underline{g}_{a,A} \right) n^{1/3} \right) \Phi^{(n)}_{a,A}.$$

Letting $n \to +\infty$, we have

$$\limsup_{n \to +\infty} \frac{\log E(A_n(f, g))}{n^{1/3}} \leq \max_{a<A} H^{F,G}_{a/A}(f, g, \varphi) - \underline{\varphi}_{a,A} \underline{g}_{a,A} + d_{a,A}.$$

By uniform continuity of K, g, φ, and as $\lim_{A \to +\infty} \max_{a<A} d_{a,A} = 0$, letting $A \to +\infty$, we have

$$\limsup_{n \to +\infty} \frac{\log E(A^{F,G}_n(f, g))}{n^{1/3}} \leq \sup_{t \in [0,1]} \left[H^{F,G}_t(f, g, \varphi) - \varphi_t g_t \right].$$

□

Lemma 3.3.3 is used to obtain an upper bound for the maximal displacement among individuals that stay above $\overline{b}^{(n)}_k + n^{1/3} f_{k/n}$ at any time $k \in F_n$. If $H^{F,G}_t(f, g, \varphi) < \varphi_t g_t$ for all $t \in [0, 1]$, then with high probability, no individual crosses the boundary $\overline{b}^{(n)}_k + n^{1/3} g_{k/n}$ at time $k \in (G \cup \{1\})_n$. In particular, there is at time n no individual above $\overline{b}^{(n)}_n + g_1 n^{1/3}$. If we choose g and G in a proper manner, the upper bound obtained here is tight.

Concentration estimate by a second moment method

We take interest in the number of individuals which stay at any time $k \leq n$ in $\overline{b}^{(n)}_k + I^{(n)}_k$. For all $0 < x < g_1 - f_1$, we set

$$\mathcal{B}^{F,G}_n(f, g, x) = \left\{ |u| = n : V(u_j) - \overline{b}^{(n)}_j \in \widetilde{I}^{(n)}_j, j \leq n, V(u) - \overline{b}^{(n)}_n \geq (g_1 - x) n^{1/3} \right\},$$

where $\widetilde{I}^{(n)}_j = I^{(n)}_j \cap [-n^{2/3}, n^{2/3}]$. We denote by $B^{F,G}_n(f, g, x) = \#\mathcal{B}^{F,G}_n(f, g, x)$. In order to bound from above the probability that $\mathcal{B}_n \neq \emptyset$, we compute the mean of B_n.

Lemma 3.3.5. *We assume* (3.1.4), (3.3.1), (3.3.4) *and* (3.3.6)*. If* $K^*(b)_1 = 0$ *then*

$$\lim_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{E}(B_n^{F,G}(f,g,x)) = H_1^{F,G}(f,g,\varphi) - \varphi_1(g_1 - x).$$

Proof. Observe that, as $K^*(b)_1 = 0$, by (3.3.5) $|E_n|$ is bounded by a constant uniformly in $n \in \mathbb{N}$. Using the many-to-one lemma, we have

$$\begin{aligned}\mathbf{E}(B_n^{F,G}(f,g,x)) &= e^{-E_n}\,\mathbf{E}\left[e^{-\varphi_1 \widetilde{S}_n + \sum_{j=1}^n (\varphi_{(j+1)/n} - \varphi_{j/n})\widetilde{S}_j} \mathbf{1}_{\left\{\widetilde{S}_j \in \widetilde{I}_j^{(n)}, j \leq n\right\}} \mathbf{1}_{\left\{\widetilde{S}_n \geq (g_1 - x)n^{1/3}\right\}}\right]\\ &\leq C e^{-\varphi_1(g_1-x)n^{1/3}}\,\mathbf{E}\left[e^{\sum_{j=1}^n (\varphi_{(j+1)/n} - \varphi_{j/n})\widetilde{S}_j} \mathbf{1}_{\left\{\widetilde{S}_j \in I_j^{(n)}, j \leq n\right\}}\right].\end{aligned}$$

Therefore applying Theorem 3.2.1, we have

$$\limsup_{n\to+\infty} \frac{\log \mathbf{E}(B_n^{F,G}(f,g,x))}{n^{1/3}} = H_1^{F,G}(f,g,\varphi) - \varphi_1(g_1 - x).$$

We now compute a lower bound for $\mathbf{E}(B_n)$. Applying the many-to-one lemma, for all $\varepsilon > 0$ we have

$$\begin{aligned}&\mathbf{E}(B_n^{F,G}(f,g,x))\\ &\quad\geq e^{-E_n}\,\mathbf{E}\left[e^{-\varphi_1 \widetilde{S}_n + \sum_{j=1}^n (\varphi_{(j+1)/n} - \varphi_{j/n})\widetilde{S}_j} \mathbf{1}_{\left\{\widetilde{S}_j \in \widetilde{I}_j^{(n)}, j \leq n\right\}} \mathbf{1}_{\left\{\widetilde{S}_n - (g_1 - x)n^{1/3} \in [0, \varepsilon n^{1/3}]\right\}}\right]\\ &\geq c e^{-\varphi_1(g_1 - x + \varepsilon)n^{1/3}}\,\mathbf{E}\left[e^{\sum_{j=1}^n (\varphi_{(j+1)/n} - \varphi_{j/n})\widetilde{S}_j} \mathbf{1}_{\left\{\widetilde{S}_j \in \widetilde{I}_j^{(n)}, j \leq n\right\}} \mathbf{1}_{\left\{\widetilde{S}_n - (g_1 - x)n^{1/3} \in [0, \varepsilon n^{1/3}]\right\}}\right].\end{aligned}$$

Applying Theorem 3.2.1 again, we have

$$\liminf_{n\to+\infty} \frac{\log \mathbf{E}\left(B_n^{F,G}(f,g,x)\right)}{n^{1/3}} \geq H_1^{F,G}(f,g,\varphi) - \varphi_1(g_1 - x + \varepsilon).$$

Letting $\varepsilon \to 0$ concludes the proof. □

To obtain a lower bound for $\mathbf{P}(\mathcal{B}_n \neq \emptyset)$, we compute an upper bound for the second moment of B_n. We assume

$$\sup_{t\in[0,1]} \mathbf{E}\left[\left(\sum_{\ell \in L_t} e^{\varphi_t \ell}\right)^2\right] < +\infty \tag{3.3.9}$$

which enables to bound the second moment of B_n.

Lemma 3.3.6. *Under the assumptions* (3.1.4), (3.3.1), (3.3.4), (3.3.6) *and* (3.3.9)*, if* $G = [0,1]$*,* $K^*(b)_1 = 0$ *and for all* $t \in [0,1]$*,* $K^*(b)_t \leq 0$*, then*

$$\begin{aligned}&\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{E}\left(B_n^{F,G}(f,g,x)^2\right)\\ &\qquad\leq 2\left[H_1^{F,G}(f,g,\varphi) - \varphi_1(g_1 - x)\right] - \inf_{t\in[0,1]}\left[H_t^{F,G}(f,g,\varphi) - \varphi_t g_t\right].\end{aligned}$$

Proof. In order to estimate the second moment of B_n, we decompose the set of pairs of individuals $(u, u') \in \mathbf{T}^2$ according to their most recent common ancestor $u \wedge u'$ as follows:

$$\begin{aligned}\mathbf{E}\left[B_n^{F,G}(f,g,x)^2\right] &= \mathbf{E}\left[\sum_{|u|=|u'|=n} \mathbf{1}_{\left\{u\in\mathcal{B}_n^{F,G}(f,g,x)\right\}}\mathbf{1}_{\left\{u'\in\mathcal{B}_n^{F,G}(f,g,x)\right\}}\right]\\ &= \sum_{k=0}^{n}\mathbf{E}\left[\sum_{\substack{|u|=|v|=n\\ |u\wedge u'|=k}} \mathbf{1}_{\left\{u\in\mathcal{B}_n^{F,G}(f,g,x)\right\}}\mathbf{1}_{\left\{u'\in\mathcal{B}_n^{F,G}(f,g,x)\right\}}\right].\end{aligned}$$

Therefore writing, for $u' \in \mathbf{T}, \Lambda(u') = \sum_{|u|=n, u>u'} \mathbf{1}_{\left\{u\in\mathcal{B}_n^{F,G}(f,g,x)\right\}}$ the number of descendants of u' which are in $\mathcal{B}_n$, we have

$$\begin{aligned}&\mathbf{E}\left[B_n^{F,G}(f,g,x)^2\right]\\ &\quad= \mathbf{E}\left[B_n^{F,G}(f,g,x)\right] + \sum_{k=0}^{n-1}\mathbf{E}\left[\sum_{|u|=k}\mathbf{1}_{\left\{V(u_j)-\bar{b}_j^{(n)}\in\widetilde{I}_j^{(n)}, j\le k\right\}}\sum_{u_1\neq u_2\in\Omega(u)}\Lambda(u_1)\Lambda(u_2)\right].\end{aligned}$$

We observe that for any two distinct individuals $|u_1| = |u_2| = k$, conditionally to $\mathcal{F}_k$, the quantities $\Lambda(u_1)$ and $\Lambda(u_2)$ are independent.

By the Markov property applied at time k, for all $u' \in \mathbf{T}$ with $|u'| = k$, we have

$$\begin{aligned}&\mathbf{E}\left[\Lambda(u')\middle|\mathcal{F}_k\right]\\ &\quad= \mathbf{E}_{k,V(u')}\left[\sum_{|u|=n-k}\mathbf{1}_{\left\{V(u)-\bar{b}_n^{(n)}\ge(g_1-x)n^{1/3}\right\}}\mathbf{1}_{\left\{V(u_j)-\bar{b}_{j+k}^{(n)}\in\widetilde{I}_j^{(n)}, j\le n-k\right\}}\middle|\mathcal{F}_k\right]\\ &\quad= \exp\left(-E_n + E_k + \varphi_{k/n}(V(u') - \bar{b}_k^{(n)})\right)\\ &\qquad\times\mathbf{E}_{k,V(u')}\left[e^{-\varphi_1\widetilde{S}_{n-k}+\sum_{j=0}^{n-k-1}\Delta\varphi_{n,k+j}\widetilde{S}_j}\mathbf{1}_{\left\{\widetilde{S}_j\in\widetilde{I}_{j+k}^{(n)}, j\le n-k\right\}}\mathbf{1}_{\left\{\widetilde{S}_{n-k}\ge(g_1-x)n^{1/3}\right\}}\right],\end{aligned}$$

using the many-to-one lemma. Therefore,

$$\begin{aligned}\mathbf{E}\left[\Lambda(u')\middle|\mathcal{F}_k\right] \le C\exp\left(E_k + \varphi_{k/n}(V(u') - \bar{b}_k^{(n)}) - \varphi_1(g_1-x)n^{1/3}\right)&\\ \times\mathbf{E}_{k,V(u')}\left[e^{\sum_{j=0}^{n-k-1}\Delta\varphi_{n,j+k}\widetilde{S}_j}\mathbf{1}_{\left\{\widetilde{S}_j\in I_{j+k}^{(n)}, j\le n-k\right\}}\right]&.\end{aligned}$$

Let $A > 0$ be a large integer, and for $a \le A$, let $m_a = \lfloor an/A\rfloor$. We introduce

$$\Phi_{a,A}^{\text{start}} = \mathbf{E}\left[\exp\left(\sum_{j=0}^{m_a-1}(\varphi_{(j+1)/n} - \varphi_{j/n})\widetilde{S}_j\right)\mathbf{1}_{\left\{\widetilde{S}_j\in\widetilde{I}_j^{(n)}, j\le m_a\right\}}\right] \quad\text{and}$$

$$\Phi_{a,A}^{\text{end}} = \sup_{y\in\mathbb{R}}\mathbf{E}_{m_a,y}\left[\exp\left(\sum_{j=0}^{n-m_a-1}\Delta\varphi_{n,m_a+j}\widetilde{S}_j\right)\mathbf{1}_{\left\{\widetilde{S}_j\in I_{m_a+j}^{(n)}, j\le n-m_a\right\}}\right].$$

By Theorem 3.2.1, we have

$$\limsup_{n\to+\infty}\frac{\log\Phi_{a,A}^{\text{start}}}{n^{1/3}} = H_{a/A}^{F,G}(f,g,\varphi) \quad\text{and}\quad \limsup_{n\to+\infty}\frac{\log\Phi_{a,A}^{\text{end}}}{n^{1/3}} = H_1^{F,G}(f,g,\varphi) - H_{a/A}^{F,G}(f,g,\varphi).$$

Moreover, using the same estimates as in Lemma 3.3.3, and setting

$$\overline{g}_{a,A} = \sup\left\{g_t, t \in \left[\tfrac{a-1}{A}, \tfrac{a+1}{A}\right]\right\}, \quad \overline{\varphi}_{a,A} = \sup\left\{\varphi_t, t \in \left[\tfrac{a-1}{A}, \tfrac{a+1}{A}\right]\right\}$$
$$\text{and} \quad d_{a,A} = \int_{(a-1)/A}^{(a+1)/A} |\dot{\varphi}_s| ds (\|f\|_\infty + \|g\|_\infty),$$

for all $k \in [m_a, m_{a+1})$, applying the Markov property at time m_{a+1}, we have

$$\mathbf{E}\left[\Lambda(u')|\mathcal{F}_k\right] \leq C e^{E_k + \varphi_{k/n}(V(u') - \overline{b}^{(n)}_k)} \exp\left(\left(d_{a,A} - \varphi_1(g_1 - x)\right) n^{1/3}\right) \Phi^{\text{end}}_{a+1,A}. \tag{3.3.10}$$

We observe that for all $u \in \mathbf{T}$ with $|u| = k$ and $V(u) \in \widetilde{I}^{(n)}_k$ we have

$$\mathbf{E}\left[\left.\sum_{u_1 \neq u_2 \in \Omega(u)} e^{\varphi_{(k+1)/n}(V(u_1) + V(u_2))}\right| \mathcal{F}_k\right] \leq e^{2\varphi_{(k+1)/n} V(u)} \mathbf{E}\left[\left(\sum_{\ell \in L_{(k+1)/n}} e^{\varphi_{(k+1)/n}\ell}\right)^2\right]$$
$$\leq C e^{2\varphi_{k/n} V(u)} e^{n^{2/3}|\varphi_{(k+1)/n} - \varphi_{k/n}|} \leq C e^{2\varphi_{k/n} V(u)}, \tag{3.3.11}$$

using (3.3.9) and the fact that φ is Lipschitz. We now bound, for $k \in [m_a, m_{a+1})$

$$\mathbf{E}\left[\sum_{|u|=k} e^{2\varphi_{k/n}(V(u) - \overline{b}^{(n)}_k)} \mathbf{1}_{\left\{V(u_j) - \overline{b}^{(n)}_j \in \widetilde{I}^{(n)}_j, j \leq k\right\}}\right]$$
$$= \mathbf{E}\left[e^{\varphi_{k/n}\widetilde{S}_k + \sum_{j=0}^{k-1}(\varphi_{(j+1)/n} - \varphi_{j/n})\widetilde{S}_j} \mathbf{1}_{\left\{\widetilde{S}_j \in \widetilde{I}^{(n)}_j, j \leq k\right\}}\right], \tag{3.3.12}$$

using Lemma 3.3.1. As $\sup_{t \in [0,1]} K^*(b)_t \leq 0$ and by (3.3.5), E_k is bounded from above uniformly in $n \in \mathbb{N}$ and $k \leq n$. As $G_n = \{0, \ldots, n\}$, for all $n \in \mathbb{N}$ large enough and $k \in [m_a, m_{a+1})$, applying the Markov property at time $m_a n$, it yields

$$\mathbf{E}\left[\sum_{|u|=k} e^{2\varphi_{k/n}(V(u) - \overline{b}^{(n)}_k)} \mathbf{1}_{\left\{V(u_j) \in \widetilde{I}^{(n)}_j, j \leq k\right\}}\right] \leq \exp\left(\left(\overline{\varphi}_{a,A}\overline{g}_{a,A} + d_{a,A}\right) n^{1/3}\right) \Phi^{\text{start}}_{a,A}. \tag{3.3.13}$$

Finally, combining (3.3.10) with (3.3.11) and (3.3.13), for all $n \geq 1$ large enough and $k \in [m_a, m_{a+1})$,

$$\mathbf{E}\left[\sum_{|u|=k} \mathbf{1}_{\left\{V(u_j) - \overline{b}^{(n)}_j \in \widetilde{I}^{(n)}_j, j \leq k\right\}} \sum_{u_1 \neq u_2 \in \Omega(u)} \Lambda(u_1)\Lambda(u_2)\right]$$
$$\leq C \exp\left[n^{1/3}\left(-2\varphi_1(g_1 - x) + \overline{\varphi}_{a,A}\overline{g}_{a,A} + 3d_{a,A}\right)\right] \Phi^{\text{start}}_{a,A} \left(\Phi^{\text{end}}_{a+1,A}\right)^2,$$

thus

$$\limsup_{n \to +\infty} \frac{1}{n^{1/3}} \log \sum_{k=0}^{n-1} \mathbf{E}\left[\sum_{|u|=k} \mathbf{1}_{\left\{V(u_j) - \overline{b}^{(n)}_j \in \widetilde{I}^{(n)}_j, j \leq k\right\}} \sum_{u_1 \neq u_2 \in \Omega(u)} \Lambda(u_1)\Lambda(u_2)\right]$$
$$\leq 2\left(H^{F,G}_1(f, g, \varphi) - (g_1 - x)\right) - \min_{a < A} 2H^{F,G}_{\frac{a+1}{A}}(f, g, \varphi) - H^{F,G}_{\frac{a}{A}}(f, g, \varphi) - \overline{\varphi}_{a,A}\overline{g}_{a,A} - 3d_{a,A}.$$

Letting $A \to +\infty$, and using Lemma 3.3.5, we obtain

$$\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{E}(B_n(f,g)^2) \\ \leq 2\left(H_1^{F,G}(f,g,\varphi) - (g_1 - x)\right) - \inf_{t\in[0,1]} \left(H_t^{F,G}(f,g,\varphi) - \varphi_t g_t\right).$$

□

Using the previous two lemmas, we can bound from below the probability that there exists an individual that follows the path $\bar{b}^{(n)} + I^{(n)}$.

Lemma 3.3.7. *Under the assumptions* (3.1.4)*,* (3.3.1)*,* (3.3.4)*,* (3.3.6) *and* (3.3.9)*, if*

$$K^*(b)_1 = \sup_{t\in[0,1]} K^*(b)_t = 0,$$

then for all $x < g_1$

$$\liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}(\mathcal{B}_n^{F,G}(f,g,x) \neq \emptyset) \geq \inf_{t\in[0,1]} \left(H_t^{F,G}(f,g,\varphi) - \varphi_t g_t\right). \tag{3.3.14}$$

Proof. We first assume that $G = [0,1]$. Since $B_n \in \mathbb{Z}_+$ a.s, we have

$$\mathbf{P}(\mathcal{B}_n^{F,G}(f,g,x) \neq \emptyset) = \mathbf{P}(B_n^{F,G}(f,g,x) > 0) \geq \frac{\mathbf{E}(B_n^{F,G}(f,g,x))^2}{\mathbf{E}(B_n^{F,G}(f,g,x)^2)},$$

using the Cauchy-Schwarz inequality. As a consequence,

$$\liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}(\mathcal{B}_n^{F,G}(f,g,x) \neq \emptyset) \\ \geq 2 \liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{E}(B_n^{F,G}(f,g,x)) - \limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{E}\left(B_n^{F,G}(f,g,x)^2\right) \\ \geq \inf_{t\in[0,1]} \left(H_t^{F,G}(f,g,\varphi) - \varphi_t g_t\right).$$

We then extend this estimate for G a Riemann-integrable subset of $[0,1]$, that we can, without loss of generality, choose closed –as the Lebesgue measure of the boundary of a Riemann-integrable set is null. According to (3.3.6), $\{\dot{\varphi} > 0\} \subset G$. We set, for $H > 0$

$$g_t^H = \max\left\{g_t, -\|g\|_\infty + Hd(t,G)\right\}.$$

Observe that g^H is an increasing sequence of functions, that are equal to g on G and increase to $+\infty$ on G^c. For all $n \in \mathbb{N}$, $x \in [f_1, g_1]$ and $H > 0$, we have

$$\mathcal{B}_n^{F,[0,1]}(f, g^H, x) \subset \mathcal{B}_n^{F,G}(f,g,x).$$

As a consequence,

$$\liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}(\mathcal{B}_n^{F,G}(f,g,x) \neq \emptyset) \geq \lim_{H\to+\infty} \liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}(\mathcal{B}_n^{F,[0,1]}(f,g^H,x) \neq \emptyset) \\ \geq \lim_{H\to+\infty} \inf_{t\in[0,1]} \left(H_t^{F,[0,1]}(f,g^H,\varphi) - \varphi_t g_t^H\right).$$

By Lemma 3.2.4, we have $\Psi(h) \sim_{h\to+\infty} \frac{\alpha_1}{2^{1/3}} h^{2/3}$. Thus, using (3.3.6), this yields

$$\liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}(\mathcal{B}_n^{F,G}(f,g,x) \neq \emptyset) \geq \inf_{t\in[0,1]} \left(H_t^{F,G}(f,g,\varphi) - \varphi_t g_t\right).$$

□

Remark 3.3.8. Observe that the inequality in Lemma 3.3.7 is sharp when

$$\inf_{t\in[0,1]} \left(H_t^{F,G}(f,g,\varphi) - \varphi_t g_t \right) = H_1^{F,G}(f,g,\varphi) - \varphi_1 g_1.$$

3.4 Identification of the optimal path

We denote by $\mathcal{R} = \{b \in \mathcal{D} : \forall t \in [0,1], K^*(b)_t \leq 0\}$. In this section, we take interest in functions $a \in \mathcal{R}$ that verify

$$\int_0^1 a_s ds = \sup \left\{ \int_0^1 b_s ds, b \in \mathcal{R} \right\}, \tag{3.4.1}$$

i.e. which are solution of (3.1.6). This equation is an optimisation problem under constraints. Information on its solution can be obtained using a theorem of existence of Lagrange multipliers in Banach spaces.

Let E, F be two Banach spaces, a function $f : E \to F$ is said to be differentiable at $u \in E$ if there exists a linear continuous mapping $D_u f : E \to F$ called its *Fréchet derivative at* u, verifying

$$f(u+h) = f(u) + D_u f(h) + o(\|h\|), \quad \|h\| \to 0, \quad h \in E.$$

A set K is a *closed convex cone* of F if it is a closed subset of F such that

$$\forall x, y \in K, \forall \lambda, \mu \in [0,+\infty)^2, \lambda x + \mu y \in K.$$

Finally, we set F^* the set of linear continuous mappings from F to $\mathbb{R}$. We now introduce a result on the existence of Lagrange multipliers in Banach spaces obtained in [Kur76], Theorem 4.5.

Theorem VIII (Kurcyusz [Kur76])**.** *Let E, F be two Banach spaces, $J : E \to \mathbb{R}$, $g : E \to F$ and K be a closed convex cone of F. If $\widehat{u}$ verifies*

$$J(\widehat{u}) = \max\{J(u), u \in E : g(u) \in K\} \quad \text{and} \quad g(\widehat{u}) \in K,$$

and if J and g are both differentiable at $\widehat{u}$, and $D_{\widehat{u}} g$ is a bijection, then there exists $\lambda \in F^*$ such that

$$\forall h \in E, D_{\widehat{u}} J(h) = \lambda^* \left[D_{\widehat{u}} g(h) \right] \tag{3.4.2}$$

$$\forall h \in K, \lambda^*(h) \leq 0 \tag{3.4.3}$$

$$\lambda^*(g(\widehat{u})) = 0. \tag{3.4.4}$$

We first introduce the *natural speed path* of the branching random walk, which is the path driven by $(v_t, t \in [0,1])$.

Lemma 3.4.1. *Under the assumptions* (3.1.2) *and* (3.1.4)*, there exists a unique $v \in \mathcal{R}$ such that for all $t \in [0,1]$, $\kappa_t^*(v_t) = 0$. Moreover, for all $t \in [0,1]$, $\overline{\theta}_t := \partial_a \kappa_t^*(v_t) > 0$, and v and $\overline{\theta}$ are $\mathcal{C}_1$ function.*

Proof. For all $t \in [0,1]$, as κ_t^* is the Fenchel-Legendre transform of κ_t, we have

$$\inf_{a\in\mathbb{R}} \kappa_t^*(a) = -\kappa_t(0) < 0.$$

Moreover, $a \mapsto \kappa_t^*(a)$ is convex, continuous on the interior of its definition set and increasing. By (3.1.4), we have $\kappa^*(a) \to +\infty$ when a increases to $\sup\{b \in \mathbb{R} : \kappa_t^*(b) < +\infty\}$. As a consequence, by continuity, there exists $x \in \mathbb{R}$ such that $\kappa_t^*(x) = 0$. Furthermore, as $\inf_{a\in\mathbb{R}} \kappa_t^*(a) < 0$, κ_t^* is strictly increasing at point x. Therefore the point $v_t = x$ is uniquely determined, and $\overline{\theta}_t = \partial_a \kappa_t^*(x)$ at point x is positive. Finally, $v \in \mathcal{C}_1$ by the implicit function theorem; thus so is $\overline{\theta}$, by composition with $\partial_a \kappa^*$. $\square$

We now observe that if a is a solution of (3.1.6), then a is a regular point of $\mathcal{R}$ –i.e. we can apply Theorem VIII.

Lemma 3.4.2. *Under the assumptions* (3.1.2) *and* (3.1.4)*, if a is a solution of* (3.1.6)*, then for all $t \in [0,1]$, $\partial_a \kappa_t^*(a_t) > 0$.*

Proof. Let $a \in \mathcal{R}$ be a solution of (3.1.6). For $t \in [0,1]$, we set $\theta_t = \partial_a \kappa_t^*(a_t)$. We observe that $\theta \in \mathcal{D}$ is non-negative.

We first assume that for all $t \in [0,1]$, $\theta_t = 0$, in which case $\kappa_t^*(a_t)$ is the minimal value of κ_t^*. By (3.1.2), we have $\inf_{t\in[0,1]} \kappa_t(0) > 0$, thus $\sup_{t\in[0,1]} \inf_{a\in\mathbb{R}} \kappa_t^*(a) < 0$. As a consequence, by continuity, there exists $x \in \mathcal{D}$ such that for all $t \in [0,1]$, $\kappa_t^*(a_t + x_t) \leq 0$. We have $a + x \in \mathcal{R}$ and $\int_0^1 a_s + x_s ds > \int_0^1 a_s ds$, which contradicts a is a solution of (3.1.6).

We now assume that θ is non-identically null, but there exists $t \in [0,1]$ such that $\theta_t = 0$. We start with the case $\theta_0 > \varepsilon > 0$. As $\theta \in \mathcal{D}$, there exists $t > 0$ and $\delta > 0$ such that $\inf_{s\in[0,\delta]} \theta_s > \varepsilon$ and $\sup_{s\in[t,t+\delta]} \theta_s < \varepsilon/3$. For $x > 0$, we set $a^x = a - x\mathbf{1}_{[0,\delta]} + 2x\mathbf{1}_{[t,t+\delta]}$. We observe that uniformly for $s \in [0,1]$, as $x \to 0$

$$K^*(a^x)_s \leq K^*(a)_s - x\varepsilon s \wedge \delta + \tfrac{2}{3} x\varepsilon (s-t)_+ \wedge \delta + O(x^2).$$

There exists $x > 0$ small enough such that $a^x \in \mathcal{R}$ and $\int a^x > \int a$, which contradicts again the fact that a is a solution of (3.1.6).

Finally, we assume that $\theta_0 = 0$. In this case, as $\partial_t K^*(a)_0 < 0$, there exists $\delta > 0$ such that $K^*(a)_t < -\delta t$ for any $t \leq \delta$. Therefore, there exists $t > 0$ such that for all $0 < s \leq t$, $K^*(a)_s < 0$, and $\theta_t > 0$. For any $\theta_t > \varepsilon > 0$, there exists $\delta' > 0$ such that for any $s < \delta'$, we have $\theta_s < \varepsilon/3$ and for all $s \in [t, t+\delta']$, $\theta_s > 2\varepsilon$. Therefore, setting $a^x = a + 2x\mathbf{1}_{[0,\delta)} - x\mathbf{1}_{[t,t+\delta)}$, as $x \to 0$, uniformly in $s \in [0,1]$, we have

$$K^*(a^x) \leq K^*(a)_s + \frac{2}{3} x\varepsilon (s \wedge \delta') - x\varepsilon((s-t)_+ \wedge \delta) + O(x^2),$$

so for $x > 0$ small enough we have $a^x \in \mathcal{R}$. Moreover $\int_0^1 a^x > \int_0^1 a$ which, once again, contradicts the fact that a is a solution of (3.1.6). $\square$

Applying Theorem VIII, and using the previous lemma, we prove Proposition 3.1.1.

Proof of Proposition 3.1.1. We first consider a function $a \in \mathcal{R}$ that verifies

$$\int_0^1 a_s ds = \sup \left\{ \int_0^1 b_s ds, b \in \mathcal{R} \right\},$$

i.e. such that a is a solution of (3.1.6). We set $\theta_t = \partial_a \kappa_t^*(a_t)$, and observe that $\theta \in \mathcal{D}$.

We introduce $J : b \mapsto \int_0^1 b_s ds$ and $g : b \mapsto (\kappa_s^*(b_s), s \in [0,1])$. These functions are differentiable at point a, and for $h \in \mathcal{D}$, we have $D_a J(h) = \int_0^1 h_s ds$ and $D_a g(h)_t = \theta_t h_t$. We denote by

$$K = \left\{ h \in \mathcal{D} : \forall t \in [0,1], \int_0^t h_s ds \leq 0 \right\},$$

which is a closed convex cone of $\mathcal{D}$. Using Lemma 3.4.2, we have $\theta_t > 0$ for all $t \in [0,1]$, thus $D_a g$ is a bijection.

By Theorem VIII, there exists $\lambda^* \in \mathcal{D}^*$ –which is a measure by the Riesz representation theorem– such that

$$\forall h \in \mathcal{D}, \int_0^1 h_s ds = \int_0^1 D_a g(h)_s \lambda^*(ds) \tag{3.4.5}$$

$$\forall h \in K, \int_0^1 h_s \lambda^*(ds) \leq 0 \tag{3.4.6}$$

$$\int_0^1 g(a)_s \lambda^*(ds) = 0. \tag{3.4.7}$$

We observe easily that (3.4.5) implies that λ^* admits a Radon-Nikodým derivative with respect to the Lebesgue measure, and that $\frac{\lambda^*(ds)}{ds} = \frac{1}{\theta_s}$. As a consequence, we can rewrite (3.4.6) as

$$\forall h \in K, \int_0^1 h_s \frac{ds}{\theta_s} \leq 0.$$

We set $f_t = \int_0^t \frac{ds}{\theta_s}$, for all $s, t \in [0,1]$, and $\mu \in (0,1)$, by (3.4.6), we have

$$\mu f_t + (1-\mu) f_s - f_{\mu t + (1-\mu)s} = \int_0^1 \left(\mu \mathbf{1}_{\{u<t\}} + (1-\mu)\mathbf{1}_{\{u<s\}} - \mathbf{1}_{\{u<\mu t+(1-\mu)s\}} \right) \frac{du}{\theta_u} \leq 0.$$

As a consequence, f is concave. In particular, its right derivative function $\frac{1}{\theta}$ is non-increasing. Consequently θ is non-decreasing.

The last equation (3.4.7) gives

$$0 = \int_0^1 \kappa_s^*(a_s)\lambda^*(ds) = \int_0^1 \kappa_s^*(a_s)\theta_s^{-1} ds = K^*(a)_1 \frac{1}{\theta_1} - \int_0^1 K^*(a)_s d\theta_s^{-1},$$

by Stieltjès integration by part. But for all $t \in [0,1]$, $K^*(a)_t \leq 0$, and $\frac{1}{\theta}$ is non-increasing. This yields

$$K^*(a)_1 = 0 \quad \text{and} \quad \int_0^1 K^*(a)_s d\theta_s^{-1} = 0.$$

In particular, as $a \in \mathcal{R}$, θ increases on $\{t \in [0,1] : K^*(a)_t = 0\}$.

Conversely, we consider a function $a \in \mathcal{R}$ such that, setting $\theta_t = \partial_a \kappa_t^*(a_t)$, we have

- θ is non-decreasing ;
- $K^*(a)_1 = 0$;
- $\int_0^1 K^*(a)_s d\theta_s^{-1} = 0$.

Our aim is to prove that $\int_0^1 a_s ds = v^*$, by observing that $\int_0^1 a_s ds \geq \int_0^1 b_s ds$ for all $b \in \mathcal{R}$. By convexity of κ_t^*, we have, for all $t \in [0,1]$

$$\kappa_t^*(b_t) \geq \kappa_t^*(a_t) + \theta_t(b_t - a_t),$$

and integrating with respect to t, we obtain

$$\begin{aligned}\int_0^1 a_t - b_t dt &\geq \int_0^1 \frac{\kappa_t^*(a_t) - \kappa_t^*(b_t)}{\theta_t} dt \\ &\leq K^*(a)_1 - K^*(b)_1 - \int_0^1 \left(K^*(a)_t - K^*(b)_t\right) d\theta_t^{-1},\end{aligned}$$

by Stieltljès integration by parts. Using the specific properties of a, we get

$$\int_0^1 a_t - b_t dt \leq -K^*(b)_1 + \int_0^1 K^*(b)_t d\theta_t^{-1}.$$

As $K^*(b)$ is non-positive, and θ^{-1} is non-increasing, we conclude that the left-hand side is non-positive, which leads to $\int_0^1 a_s ds \geq \int_0^1 b_s ds$. Optimizing this inequality over $b \in \mathcal{R}$ proves that a is a solution of (3.1.6).

We now prove that if a is a solution of (3.1.6), then a is continuous. Observe that as $a = \partial_\theta \kappa_t(\theta_t)$ and θ is non-decreasing, it admits a right and left limit at each point. We assume there exists $t \in (0,1)$ such that $a_t \neq a_{t-}$, i.e. such that a jumps at time t. Then, $\theta_t \neq \theta_{t-}$ by continuity of $\partial_a \kappa^*$ on D^*. As $\int_0^1 K^*(a)_s d\theta_s^{-1} = 0$ and $d\theta^{-1}$ has an atom at point t, thus $K^*(a)_t = 0$.

Therefore, if a jumps at time t, then the continuous function $s \mapsto K^*(a)_s$ with right and left derivatives at each point, bounded from above by 0, hits a local maximum at time t. Its left derivative $\kappa^*_t(a_{t-})$ is then non-negative and its right derivative $\kappa^*_t(a_t)$ non-positive. As $\kappa^* t$ is a non-decreasing function, we obtain $a_{t-} \geq a_t$.

Moreover, by convexity of κ^*_t, $x \mapsto \partial_a \kappa^*_t(x)$ is also non-decreasing, and as a consequence $\theta_{t-} \geq \theta_t$, which is a contradiction with the hypothesis $\theta_{t-} \neq \theta_t$ and θ non-decreasing. We conclude that a (and θ) is continuous as a càdlàg function with no jump.

We now assume there exists another solution $b \in \mathcal{R}$ to (3.1.6). Using the previous computations, we have $\int_0^1 K^*(b)_s d\theta_s^{-1} = 0$, and b is continuous. As a consequence, denoting by T the support of $d\theta^{-1}$, for all $t \in T$, $K^*(b)_t = 0$. Moreover, $K^*(b)$ is a $\mathcal{C}^1$ function, with a local maximum at time t, thus $\kappa^*_t(b_t) = 0$, or in other words, $b_t = v_t$, by Lemma 3.4.2.

Consequently, if we write $\varphi_t = \partial_a \kappa^*_t(b_t)$, we know from previous results that φ is continuous and increasing. Furthermore, φ increases only on T, and $\varphi_t = \partial_a \kappa^*_t(v_t) = \overline{\theta}_t$. For all $t \in [0,1]$, we set $\sigma_t = \sup\{s \leq t : s \in T\}$ and $\tau_t = \inf\{s \geq t : s \in T\}$. If σ and τ are finite then

$$\overline{\theta}_{\sigma_t} = \varphi_{\sigma_t} = \varphi_t = \varphi_{\tau_t} = \overline{\theta}_{\tau_t}.$$

As a is also a solution of (3.1.6), we have $\overline{\theta}_{\sigma_t} = \theta_t = \overline{\theta}_{\tau_t}$, therefore $\theta = \varphi$. As a consequence, we have

$$a_t = \partial_\theta \kappa_t(\theta_t) = \partial_\theta \kappa_t(\varphi_t) = b_t,$$

which proves the uniqueness of the solution.

We prove that a and θ are Lipschitz functions. For all $t \in [0,1]$, $\int_0^t \kappa^*_s(a_s) ds \leq 0$, and $\int_0^t \kappa^*_s(a_s) d\theta_s^{-1} = 0$. In particular, this means that $\kappa^*_t(a_t)$ vanishes $d\theta_t^{-1}$-almost everywhere, thus $\theta_t = \overline{\theta}_t$, $d\theta_t^{-1}$-almost everywhere. By continuity of θ and $\overline{\theta}$, these functions are identical on T. In addition, for all $s < t$ such that $(s,t) \subset [0,1] \backslash T$, we have $\int_s^t d\theta_u^{-1} = 0$, hence $\theta_t = \theta_s$, which proves that θ is constant on $[0,1]\backslash T$. As a result, for all $s < t \in [0,1]$, we have $\theta_t = \theta_s$ if $(s,t) \subset [0,1]\backslash T$, otherwise

$$\theta_s = \inf_{u \geq s, u \in T} \overline{\theta}_u \quad \text{and} \quad \theta_t = \inf_{u \leq t, u \in T} \overline{\theta}_u,$$

In consequence $|\theta_t - \theta_s| \leq \sup_{r,r' \in [s,t]} |\overline{\theta}_r - \overline{\theta}_{r'}|$. As $\overline{\theta}$ is $\mathcal{C}^1$ on $[0,1]$, $\overline{\theta}$ and θ are Lipschitz functions. As $a_t = \partial_\theta \kappa_t(\theta_t)$, the function a is also Lipschitz.

Finally, we prove the existence of a solution to (3.1.6). To do so, we reformulate this optimization problem in terms of an optimization problem for θ. The aim is to find a positive function $\theta \in \mathcal{C}$ such that

$$\int_0^1 \partial_\theta \kappa_t(\theta_t) dt = \max\left\{ \int_0^1 \partial_\theta \kappa_t(\varphi_t) dt : \varphi \in \mathcal{C}, \forall t \in [0,1], E(\varphi)_t < +\infty \right\}, \qquad (3.4.8)$$

where $E(\varphi)_t = \int_0^t \varphi_s \partial_\theta \kappa_t(\varphi_t) - \kappa_t(\varphi_t)$. By Theorem VIII, if θ exists, then it verifies

- θ non-decreasing;
- $E(\theta)_1 = 0$;
- $\forall t \in [0,1], \int_0^t E(\theta)_s d\theta_s^{-1} = 0$.

Using these three properties, we have $\theta = \overline{\theta}$ on the support of the measure $d\theta^{-1}$. Moreover, as $E_0(\theta) = E_1(\theta) = 0$ and E_t is non-positive, we observe that $E_t(\theta)$ is locally non-increasing in the neighbourhood of 0 and locally non-decreasing in the neighbourhood of 1, in particular

$$\theta_0 \partial_\theta \kappa_0(\theta_0) - \kappa_0(\theta_0) \leq 0 \quad \text{and} \quad \theta_1 \partial_\theta \kappa_1(\theta_1) - \kappa_1(\theta_1) \geq 0.$$

As for all $t \in [0,1]$, the function $\varphi \mapsto \varphi \partial_\theta \kappa_t(\varphi) - \kappa_t(\varphi)$ is increasing, we conclude that $\theta_0 \leq \overline{\theta}_0$ and $\theta_1 \geq \overline{\theta}_1$. As a consequence, $T = \{t \in [0,1] : \theta_t = \overline{\theta}_t\}$ is non-empty, and, setting $\sigma_t = \sup\{s \leq t : s \in T\}$ and $\tau_t = \inf\{s \geq t : s \in T\}$ we have $\theta_t = \overline{\theta}_{\sigma_t}$ if $\sigma_t > -\infty$ and $\theta_t = \overline{\theta}_{\tau_t}$ if $\tau_t < +\infty$.

We write

$$\Theta = \left\{ \theta \in \mathcal{C} : \theta \text{ non-decreasing}, \theta_0 \geq 0, \forall t \in [0,1], \int_0^t E_s(\theta) d\theta_s^{-1} = 0 \text{ and } E_t(\theta) \leq 0 \right\}.$$

This set is uniformly equicontinuous and bounded, thus by Arzelà-Ascoli theorem, it is compact. It is non-empty as for all $\varepsilon > 0$ small enough, the function $t \mapsto \varepsilon$ belongs to Θ. We write θ a maximizer of $\int_0^1 \partial_\theta \kappa_s(\theta_s) ds$ on Θ.

By continuity, if $E(\theta)_1 < 0$, then we can increase a little θ in the neighbourhood of 1, thus θ is non-optimal. As a result, θ is non-decreasing, verifies $E(\theta)_1 = 0$ and $\int E(\theta)_s d\theta_s^{-1} = 0$, which proves that $a = \partial_\theta \kappa(\theta)$ is a solution of (3.1.6). □

The previous proof gives some characteristics of the unique solution a of (3.1.6). In particular, if we set $\theta_t = \partial_a \kappa_t^*(a_t)$, we know that θ is positive, non-decreasing, and that on the support of the measure $d\theta^{-1}$, θ and $\overline{\theta}$ are identical. Consequently, the optimal speed path of the branching random walk verifies the following property: while in the bulk of the branching random walk, it follows an equipotential line, and when close to the boundary it follows the natural speed path.

For some time-inhomogeneous environments, (3.1.6) can be solved explicitly. This is the case, for example, when the function $t \mapsto \overline{\theta}_t$ is monotone. A time-inhomogeneous environment such that $\overline{\theta}$ is increasing behaves as the branching Brownian motion with decreasing variance, studied in [FZ12b, NRR14, MZ14] and verifies $a = v$. If $\overline{\theta}$ is decreasing, then θ is constant, and Theorem 3.1.3 is non-optimal: for a wide class of processes, the correct correction is logarithmic.

Lemma 3.4.3. *We assume* (3.1.2) *and* (3.1.4)*.*

- *If $\overline{\theta}$ is non-decreasing, then $a = v$ (and $\overline{\theta} = \theta$).*
- *If $\overline{\theta}$ is non-increasing, then there exists $\theta \in [0, +\infty)$ such that $a_t = \partial_\theta \kappa_t(\theta)$.*
- *If $\overline{\theta}$ is non-increasing on* $[0, 1/2]$ *and non-decreasing on* $[1/2, 1]$*, then there exists* $t \in [1/2, 1]$ *such that*
$$\forall s \in [0,1], \partial_a \kappa_s^*(a_s) = \overline{\theta}_{s \vee t}.$$

Proof. We first assume that $\overline{\theta}$ is a non-decreasing function. As $K^*(v)_t = 0$ for all $t \in [0,1]$, we have

- $\overline{\theta}$ non-decreasing;
- $\int_0^t K^*(v)_t d\overline{\theta}_t^{-1} = 0$;

- $K^*(v)_1 = 0$;

which, by Proposition 3.1.1 implies that v is the solution of (3.1.6).

We now denote by a the solution of (3.1.6), and by $\theta_t = \partial_a \kappa^*_t(a_t)$. Let T be the support of the measure of $d\theta_t^{-1}$, we know by Proposition 3.1.1 that θ is non-decreasing and equal to $\overline{\theta}_t$ when it is increasing. In particular, we have

$$\frac{1}{\theta_t} = \frac{1}{\theta_0} + \int_0^t d\theta_s^{-1} = \frac{1}{\theta_0} + \int_0^t \mathbf{1}_{\{s\in T\}} d\theta_s^{-1} = \frac{1}{\theta_0} + \int_0^t \mathbf{1}_{\{s\in T\}} d\overline{\theta}_s^{-1}.$$

As a consequence, if $\overline{\theta}$ is non-increasing on $[0,t]$, then $\int_0^u \mathbf{1}_{\{s\in T\}} d\overline{\theta}_s^{-1} \geq 0$ for all $u \leq t$. As θ^{-1} is non-increasing, we conclude that $\int_0^u \mathbf{1}_{\{s\in T\}} d\overline{\theta}_u^{-1} = 0$, and $\theta_u = \theta_0$. In particular, in the non-increasing case, we conclude that θ is a constant.

In the mixed case, we have just shown that θ is constant up to time $1/2$. We set $u = \inf\{t > 1/2 : \overline{\theta}_t = \theta_0\}$. Since $\theta = \overline{\theta}$ on T, we know that $T \cap [1/2, u) = \emptyset$. Hence θ is constant up to point u. For $t > u$, as $\overline{\theta}$ increases, we have

$$\frac{1}{\theta_t} = \frac{1}{\theta_0} + \int_0^t \mathbf{1}_{\{s\in T\}} d\overline{\theta}_s^{-1} = \frac{1}{\overline{\theta}_u} + \int_u^t \mathbf{1}_{\{s\in T\}} d\overline{\theta}_s^{-1} \geq \frac{1}{\overline{\theta}_t},$$

which yields $\overline{\theta}_t \geq \theta_t$. We now observe that $K^*(a)_1 = 0$, thus $K^*(a)$ attains a local maximum at time 1, and its left derivative $\kappa^*_1(a_1)$ is non-negative. This implies that $\theta_1 \geq \overline{\theta}_1$. If there exists $s > u$ such that $\theta_s < \overline{\theta}_s$, then $T \cap [s, 1] = \emptyset$, and $\theta_1 = \theta_s < \overline{\theta}_s \leq \overline{\theta}_1$, which contradicts the previous statement. In consequence, for $t \geq u$, we have $\theta_t = \overline{\theta}_t$, which ends the proof of the mixed case. $\square$

3.5 Maximal and consistent maximal displacements

We apply the estimates obtained in the previous section to compute the asymptotic behaviour of some quantities of interest for the BRWls. In Section 3.5.1, we take interest in the maximal displacement in a BRWls with selection. In Section 3.5.2, we obtain a formula for the consistent maximal displacement with respect to a given path. If we apply these estimates in a particular case, we prove Theorems 3.1.3 and 3.1.4.

3.5.1 Maximal displacement in a branching random walk with selection

We first define the *maximal displacement in a branching random walk with selection*, which is the position of the rightmost individual among those alive at generation n that stayed above a prescribed curve. We consider a positive function φ that satisfies (3.3.1) and (3.3.4). We introduce functions b and σ according to (3.3.2). Let f be a continuous function on $[0,1]$ with $f(0) < 0$, and F be a Riemann-integrable subset of $[0,1]$. The set of individuals we consider is

$$\mathcal{W}_n^\varphi(f,F) = \left\{ |u| \leq n : \forall j \in F_n, V(u_j) \geq \overline{b}_j^{(n)} + f_{j/n} n^{1/3} \right\}.$$

This set is the tree of the BRWls with selection $(\mathcal{W}_n^\varphi(f,F), V_{|\mathcal{W}_n^\varphi(f,F)})$, in which every individual u alive at time $k \in F_n$ at a position below $\overline{b}_k^{(n)} + f_{k/n} n^{1/3}$ is immediately killed, as well as all its descendants. Its maximal displacement at time n is denoted by

$$M_n^\varphi(f,F) = \max \left\{ V(u), u \in \mathcal{W}_n^\varphi(f,F) : |u| = n \right\}.$$

To apply the results of the previous section, we assume here that b satisfies

$$\sup_{t\in[0,1]} K^*(b)_t = 0 = K^*(b)_1; \tag{3.5.1}$$

in other words, there exists individuals that follow the path with speed profile b with positive probability, and at time 1, there are $e^{o(n)}$ of those individuals. We set G the set of zeros of $K^*(b)$, and we assume that

$$G = \{t \in [0,1] : K^*(b)_t = 0\} \text{ is Riemann-integrable.} \tag{3.5.2}$$

For $\lambda \in \mathbb{R}$, we set $g^\lambda \in \mathcal{C}([0,t_\lambda))$ the function solution of

$$\forall t \in [0,t_\lambda), \varphi_t g^\lambda_t - H^{F,G}_t(f, g^\lambda, \varphi) = \varphi_0 \lambda. \tag{3.5.3}$$

To study $M^\varphi_n(f,F)$, we first prove the existence of a unique maximal $t_\lambda \in (0,1]$, and a function g^λ solution of (3.5.3). We recall the following theorem of Carathéodory, that can be found in [Fil88].

Theorem IX (Existence and uniqueness of solutions of Carathéodory's ordinary differential equation)**.** *Let $0 \leq t_1 < t_2 \leq 1$, $x_1 < x_2$, $M > 0$ and $f : [t_1,t_2] \times [x_1,x_2] \to [-M,M]$ a bounded function. Let $t_0 \in [t_1,t_2]$ and $x_0 \in [x_1,x_2]$, we consider the differential equation consisting in finding $t > 0$ and a continuous function $\gamma : [t_0, t_0+t] \to \mathbb{R}$ such that*

$$\forall s \in [t_0, t_0+t], \gamma(s) = x_0 + \int_{t_0}^s f(u,\gamma(u))du. \tag{3.5.4}$$

If for all $x \in [x_1,x_2]$, $t \mapsto f(t,x)$ is measurable and for all $t \in [t_1,t_2]$, $x \mapsto f(t,x)$ is continuous, then for all (t_0,x_0) there exists $t \geq \min(t_2 - t_0, \frac{x_2-x_0}{M}, \frac{x_0-x_1}{M})$ and γ that satisfy (3.5.4)*.*

If additionally, there exists $L > 0$ such that for all $x,y \in [x_1,x_2]$ and $t \in [t_1,t_2]$, $|f(t,x) - f(t,y)| \leq L|x-y|$, then for every pair of solutions (t,γ) and $(\widetilde{t},\widetilde{\gamma})$ of (3.5.4)*, we have*

$$\forall s \leq \min(t,\widetilde{t}), \gamma(s) = \widetilde{\gamma}(s).$$

Consequently, there exists a unique solution defined on a maximal interval $[t_0, t_0 + t_{\max}]$.

We use this theorem to prove there exists a unique solution g to (3.5.3).

Lemma 3.5.1. *Let f be a continuous function, φ that verifies* (3.3.4)*, and F, G two Riemann-integrable subsets of* $[0,1]$*. For all $\lambda > f_0$, there exists a unique $t_\lambda \in [0,1]$ and a unique continuous function defined on $[0,t_\lambda]$ such that for all $t < t_\lambda$, we have*

$$g^\lambda_t > f_t \quad \text{and} \quad \varphi_t g^\lambda_t = \varphi_0\lambda + H^{F,G}_s(f, g^\lambda, \varphi).$$

Moreover, there exists λ_c such that for all $\lambda > \lambda_c$, $t_\lambda = 1$ and $\lambda \mapsto g^\lambda$ is continuous with respect to the uniform norm and strictly increasing.

Proof. Let f be a continuous function, and F be a Riemann-integrable subset of $[0,1]$, we set

$$D = \{(t,x) \in [0,1] \times \mathbb{R} : \text{if } t \in F, \text{ then } x > \varphi_t f_t\},$$

and, for $(t,x) \in D$,

$$\Phi(t,x) = \frac{\dot{\varphi}_t}{\varphi_t}x + \mathbf{1}_{F\cap G}(t)\frac{\sigma_t^2}{(\frac{x}{\varphi_t} - f_t)^2}\Psi\left(\frac{(\frac{x}{\varphi_t}-f_t)^3}{\sigma_t^2}\dot{\varphi}_t\right)$$
$$\mathbf{1}_{F^c\cap G}(t)\frac{a_1}{2^{1/3}}(\dot{\varphi}_t\sigma_t)^{2/3} + \mathbf{1}_{F\cap G^c}\left(\dot{\varphi}_t(f_t - \tfrac{x}{\varphi_t}) + \frac{a_1}{2^{1/3}}(-\dot{\varphi}_t\sigma_t)^{2/3}\right).$$

For all $\lambda > f_0$, we introduce

$$\Gamma^\lambda = \left\{ (t,h), t \in [0,1], h \in \mathcal{C}([0,t]) : \forall s \leq t, h_s = \varphi_0 \lambda + \int_0^s \Phi(u, h_u) du \right\},$$

the set of functions such that $g = \frac{h}{\varphi}$ is a solution of (3.5.3).

We observe that for all $[t_1,t_2] \times [x_1,x_2] \subset D$, $\Phi_{|[t_1,t_2]\times[x_1,x_2]}$ is measurable with respect to t, and uniformly Lipschitz with respect to x. As a consequence, by Theorem IX, for all $(t_0,x_0) \in D$, there exists $t > 0$ such that there exists a unique function $h \in \mathcal{C}([0,t])$ satisfying

$$\forall s \leq t, h_s = x_0 + \int_{t_0}^s \Phi(u, h_u) du. \tag{3.5.5}$$

Using this result, we first prove that Γ^λ is a set of consistent functions. Indeed, let (t_1,h^1) and (t_2,h^2) be two elements of Γ^λ, and let $\tau = \inf\{s \leq \min(t_1,t_2) : h^1_s \neq h^2_s\}$. We observe that if $\tau < \min(t_1,t_2)$, then by continuity of h^1 and h^2, we have $h^1_\tau = h^2_\tau$. Furthermore, $s \mapsto h^1_{\tau+s}$ and $s \mapsto h^2_{\tau+s}$ are two different functions satisfying (3.5.5) with $t_0 = \tau$ and $x_0 = h^1_\tau = h^2_\tau$, which contradicts the uniqueness of the solution. We conclude that $\tau \geq \min(t_1,t_2)$, every pair of functions in Γ^λ are consistent up to the first terminal point.

We define $t_\lambda = \max\left\{ t \in [0,1] : \exists h \in \mathcal{C}([0,t]), (t,h) \in \Gamma^\lambda \right\}$. We have $t_\lambda > 0$ by the existence of a local solution starting at time 0 and position $\varphi_0\lambda$. For $s < t_\lambda$, we write $h^\lambda_s = h_s$, where $(s,h_s) \in \Gamma^\lambda$. By definition,

$$\forall s < t_\lambda, h^\lambda_s = \varphi_0\lambda + \int_0^s \Phi(u, h^\lambda_u) du.$$

By local uniqueness of the solution, if there exists $t \in (0,1)$ such that $h^\lambda_t = h^{\lambda'}_t$, then for all $s \leq t$, $h^\lambda_s = h^{\lambda'}_s$, and in particular $\lambda = \lambda'$. We deduce that for all $\lambda < \lambda'$, if $s < \min(t_\lambda, t'_\lambda)$ then $h^\lambda_s < h^{\lambda'}_s$.

Moreover, as there exist C_1 and $C_2 > 0$ such that for all $t \in [0,1]$ and $x > C_1$, $\Phi(t,x) < C_2$, we have $\limsup_{t \to t_\lambda} h^\lambda_t < +\infty$. Hence, if $\lambda < \lambda'$ and $t_\lambda > t'_\lambda$, if

$$x_0 \in \left[\liminf_{t \to t_\lambda} h^\lambda_t, \limsup_{t \to t_\lambda} h^\lambda_t \right],$$

then as $x_0 > h^\lambda_{t_{\lambda'}}$, we can extend $h^{\lambda'}$ on $[t_{\lambda'}, t_{\lambda'} + \delta]$, which contradicts the fact that $t_{\lambda'}$ is maximal. We conclude that $t_{\lambda'} \geq t_\lambda$.

If $\lambda' > \lambda > \lambda_c$, the functions h^λ and $h^{\lambda'}$ are defined on $[0,1]$. Moreover, the set

$$H^{\lambda,\lambda'} = \left\{ (t,x) \in [0,1] \times \mathbb{R} : x \in [h^\lambda_t, h^{\lambda'}_t] \right\},$$

is a compact subset of D, that can be paved by a finite number of rectangles in D. As a consequence, there exists $L > 0$ such that

$$\forall t \in [0,1], \forall x, x' : (t,x) \in H^{\lambda,\lambda'}, (t,x') \in H^{\lambda,\lambda'}, |\Phi(t,x) - \Phi(t,x')| \leq L|x - x'|.$$

As for all $\mu \in [\lambda, \lambda']$, $(t, h^\mu_t) \in H^{\lambda,\lambda'}$, we observe that

$$\begin{aligned} \left| h^\mu_t - h^{\mu'}_t \right| &\leq |\mu - \mu'| + \int_0^t \left| \Phi(s, h^\mu_s) - \Phi(s, h^{\mu'}_s) \right| ds \\ &\leq |\mu - \mu'| + L \int_0^t \left| h^\mu_s - h^{\mu'} a_s \right| ds. \end{aligned}$$

Applying the Gronwall inequality, for all $\mu, \mu' \in [\lambda, \lambda']$, we have

$$\left\| h^{\mu} - h^{\mu'} \right\|_{\infty} \leq |\mu - \mu'| \, e^{L}$$

which proves that $\lambda \mapsto h^\lambda$ is continuous with respect to the uniform norm.

Finally, there exist C_0 and $C_1 > 0$ such that for all $t \in [0,1]$ and $x \geq C_0$, we have $\Phi(t,x) \geq -C_1$. Therefore, for all $\lambda \geq C_0 + C_1 + \|\varphi f\|_\infty$, for all $t \in [0,1]$, $h^\lambda_t \geq \|\varphi f\|_\infty$, and $t_\lambda = 1$. We set $\lambda_c = \inf\{\lambda \in \mathbb{R} : t_\lambda = 1\}$, and we conclude the proof by observing that $g^\lambda = \frac{h^\lambda}{\varphi^\lambda}$ is the solution of (3.5.3). □

Lemma 3.5.2. *Under the assumptions* (3.1.4), (3.3.1), (3.3.4), (3.3.6), (3.3.9) *and* (3.5.1), *for all* $\lambda > \max(0, \lambda_c)$, *we have*

$$\lim_{n \to +\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\left(M^\varphi_n(f,F) \geq \overline{b}^{(n)}_n + g^\lambda_1 n^{1/3} \right) = -\varphi_0 \lambda.$$

Proof. To obtain an upper bound, we recall that $1 \in G$, as $K^*(b)_1 = 0$ by (3.5.1). Let $\lambda > \max(0, \lambda_c)$, we set $g = g^\lambda$ the unique solution of (3.5.3). We observe that

$$\mathbf{P}\left(M^\varphi_n(f,F) \geq \overline{b}^{(n)}_n + g^\lambda_1 n^{1/3} \right) \leq \mathbf{P}\left(\mathcal{A}^{F,G}_n(f,g) \neq \emptyset \right) \leq \mathbf{E}\left(A^{F,G}_n(f,g) \right).$$

Therefore, by Lemma 3.3.3, we have

$$\limsup_{n \to +\infty} \frac{1}{n^{1/3}} \log \mathbf{P}(M^\varphi_n(f,F) \geq \overline{b}^{(n)}_n + g^\lambda_1 n^{1/3}) \leq \sup_{t \in [0,1]} H^{F,G}_t(f,g,\varphi) - \varphi_t g_t = -\varphi_0 \lambda.$$

When $\lambda' > \lambda$, we have $g^{\lambda'}_1 > g_1$, and

$$\mathbf{P}\left(M^\varphi_n(f,F) \geq \overline{b}^{(n)}_n + g^\lambda_1 n^{1/3} \right) \geq \mathbf{P}\left(\mathcal{B}^{F,G}_n(f, g^{\lambda'}, g^{\lambda'}_1 - g_1) \neq \emptyset \right).$$

Consequently, using Lemma 3.3.7, we have

$$\liminf_{n \to +\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\left(M^\varphi_n(f,F) \geq \overline{b}^{(n)}_n + g^\lambda_1 n^{1/3} \right) \geq \sup_{t \in [0,1]} H^{F,G}_t(f,g,\varphi) - \varphi_t g_t = -\varphi_0 \lambda'.$$

Letting λ' decrease to λ, we have

$$\lim_{n \to +\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\left(M^\varphi_n(f,F) \geq \overline{b}^{(n)}_n + g^\lambda_1 n^{1/3} \right) = -\varphi_0 \lambda.$$

□

The previous lemma gives an estimate of the right tail of $M^\varphi_n(f,F)$ for any $f \in \mathcal{C}$ and Riemann-integrable set $F \subset [0,1]$. Note that to obtain this estimate, we do not need the assumption (3.1.2) of supercritical reproduction, however (3.5.1) implies that

$$\inf_{t \in [0,1]} \liminf_{n \to +\infty} \frac{1}{n} \log \mathbf{E}\left[\#\{u \in \mathbf{T}^{(n)}, |u| = \lfloor tn \rfloor\} \right] \geq 0,$$

which is a weaker supercriticality condition. Assuming (3.1.2), we strengthen Lemma 3.5.2 to prove a concentration estimate for $M^\varphi_n(f,F)$ around $\overline{b}^{(n)}_n + g^0_1 n^{1/3}$.

Lemma 3.5.3 (Concentration inequality)**.** *Under the assumptions* (3.1.2)*,* (3.1.4)*,* (3.3.1)*,* (3.3.4)*,* (3.3.6)*,* (3.3.9) *and* (3.5.1)*, if $\lambda_c > 0$, then for all $\varepsilon > 0$, we have*

$$\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\left(\left| M_n^\varphi(f,F) - \bar{b}_n^{(n)} - g_1^0 n^{1/3} \right| \geq \varepsilon n^{1/3} \right) < 0.$$

Proof. We set $g = g^0$ the solution of (3.5.3) for $\lambda = 0$. We observe that for all $\varepsilon > 0$ and $t \in [0,1]$, we have

$$H_t^{F,G}(f, g+\varepsilon, \varphi) - \varphi_t(g_t + \varepsilon) < 0.$$

Consequently, for all $\varepsilon > 0$, we have

$$\begin{aligned}\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\left(M_n^\varphi(f,F) \geq \bar{b}_n^{(n)} + (g_1+\varepsilon)n^{1/3} \right)&\\ \leq \limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\left(\mathcal{A}_n^{F,G\cup\{1\}}(f, g+\varepsilon) \neq \emptyset \right)&\\ \leq \sup_{t\in[0,1]} H_t^{F,G}(f, g+\varepsilon, \varphi) - \varphi_t(g_t+\varepsilon) < 0,&\end{aligned}$$

by Lemma 3.3.3.

To obtain a lower bound, we need to strengthen the tail estimate of $M_n^\varphi(f,F)$. Using (3.1.2), the size of the population in the branching random walk increases at exponential rate. We set $p \in \mathbb{R}$ and $\varrho > 0$ such that $\varrho = \inf_{t\in[0,1]} \mathbf{P}(\#\{\ell \in L_t : \ell \geq p\} \geq 2)$. We can assume, without loss of generality, that $p < b_0$. We couple the BRWls with a Galton-Watson process $(Z_n, n \geq 0)$ with $Z_0 = 1$, and reproduction law defined by $\mathbf{P}(Z_1 = 2) = 1 - \mathbf{P}(Z_1 = 1) = \varrho$; in a way that for any $\eta > 0$,

$$\#\{u \in \mathbf{T}^{(n)} : |u| = \left\lfloor \eta n^{1/3} \right\rfloor, V(u) \geq \eta p n^{1/3}\} \geq Z_k \text{ a.s. for } n \text{ large enough.}$$

By standard Galton-Watson theory (see, e.g. [FW07]), there exists $\alpha > 0$ such that

$$\limsup_{n\to+\infty} \frac{1}{n} \log \mathbf{P}\left(Z_n \leq e^{\alpha n} \right) < 0.$$

Consequently, with high probability, there are at least $e^{\alpha k}$ individuals to the right of pk at any time $k \leq n$.

Let $\varepsilon > 0$ and $\eta > 0$, we set $k = \left\lfloor \eta n^{1/3} \right\rfloor$. Applying the Markov property at time k, we have

$$\mathbf{P}\left(M_n \leq m_n - \varepsilon n^{1/3} \right) \leq \mathbf{P}\left(Z_k \leq e^{\alpha k} \right) + \left[1 - \mathbf{P}_{k,kp}\left(M_{n-k} \geq m_n - \varepsilon n^{1/3} \right) \right]^{e^{\alpha k}}.$$

As a consequence

$$\begin{aligned}&\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\left(M_n \leq m_n - \varepsilon n^{1/3} \right)\\ &\leq \max\left\{ \limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\left(Z_k \leq e^{\alpha k} \right), - \liminf_{n\to+\infty} \frac{e^{\alpha k}}{n^{1/3}} \mathbf{P}_{k,kp}\left(M_{n-k} \geq m_n - \varepsilon n^{1/3} \right) \right\}.\end{aligned}$$

To conclude the proof, we need to prove that

$$\liminf_{n\to+\infty} e^{\alpha\eta n^{1/3}} \mathbf{P}_{k,0}\left(M_{n-k} \geq \bar{b}_n^{(n)} + (g_1 - \varepsilon)n^{1/3} - kp \right) > 0. \tag{3.5.6}$$

Let $\delta > 0$, we choose $\eta = \frac{\varepsilon}{b_0 - p} + \delta$, we have

$$\begin{aligned}
&\liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}_{k,0}\left(M_{n-k} \geq \overline{b}^{(n)}_n + (g_1 - \varepsilon)n^{1/3} - kp\right)\\
=&\liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}_{k,0}\left(M_{n-k} \geq \overline{b}^{(n)}_n - \overline{b}^{(n)}_k + g_1 + \delta n^{1/3}\right)\\
\leq& -\varphi_0\lambda_\delta,
\end{aligned}$$

by applying Lemma 3.5.2, where λ_δ is the solution of the equation $g_1^{\lambda_\delta} = \delta$. Here, we implicitly used the fact that the estimate obtained in Lemma 3.5.2 is true uniformly in $k \in [0, \eta n^{1/3}]$. This is due to the fact that this is also true for Theorem 3.2.1. Finally, letting $\delta \to 0$, we have $\lambda_\delta \to 0$, hence

$$\liminf_{n\to+\infty} \frac{e^{\alpha k}}{n^{1/3}} \mathbf{P}_{k,kp}\left(M_{n-k} \geq m_n - \varepsilon n^{1/3}\right) = +\infty,$$

which concludes the proof. □

Proof of Theorem 3.1.3

We denote by a the solution of (3.1.6) and by θ the function defined by $\theta_t = \partial_a \kappa^*_t(a_t)$. We assume that (3.1.4) is verified, i.e. θ is absolutely continuous with a Riemann-integrable derivative $\dot{h}$. For all $n \in \mathbb{N}$ and $k \leq n$, we set $\overline{a}^{(n)}_k = \sum_{j=1}^k a_{j/n}$. We recall that

$$l^* = \frac{\alpha_1}{2^{1/3}} \int_0^1 \frac{(\dot{\theta}_s\sigma_s)^{2/3}}{\theta_s} ds,$$

where α_1 is the largest zero of the Airy function of first kind.

Proof of Theorem 3.1.3. With the previous notation, we have $M_n = M^\theta_n(0, \emptyset)$. By Proposition 3.1.1, a satisfies (3.5.1), θ is non-decreasing and increases only on G. As a consequence, (3.3.6) is verified, and (3.5.4) can be written, for $\lambda \in \mathbb{R}$,

$$\forall t \in [0,1], \theta_t g^\lambda_t = \theta_0\lambda + \int_0^t \dot{\theta}_s g^\lambda_s + \frac{\alpha_1}{2^{1/3}}(\dot{\theta}_s\sigma_s)^{2/3} ds. \tag{3.5.7}$$

By integration by parts, $g^\lambda_t = \lambda + \int_0^t \frac{\alpha_1}{2^{1/3}}(\dot{\theta}_s\sigma_s)^{2/3}ds$. In particular, $g^\lambda_1 = \lambda + l^*$. As a consequence, applying Lemma 3.5.2 to $\lambda = l$, we have

$$\lim_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}(M_n \geq \overline{a}^{(n)}_n + (l^* + l)n^{1/3}) = -\theta_0 l.$$

Similarly, using Lemma 3.5.3, for all $\varepsilon > 0$,

$$\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\left(\left|M_n - \overline{a}^{(n)}_n - l^* n^{1/3}\right| \geq \varepsilon n^{1/3}\right) < 0.$$

As a is a Lipschitz function, we have

$$\overline{a}^{(n)}_n = \sum_{j=1}^n a_{j/n} = n \underbrace{\int_0^1 a_s ds}_{v^*} + O(1).$$

This concludes the proof. □

Mixing Lemma 3.4.3 and Theorem 3.1.3, we obtain an explicit asymptotic for the maximal displacement, in some particular cases. If $\overline{\theta}$ is non-decreasing, then $\theta = \overline{\theta}$. As a result, setting

$$\overline{l}^* = \frac{\alpha_1}{2^{1/3}} \int_0^1 \frac{\left(\dot{\overline{\theta}}_s \sigma_s\right)^{2/3}}{\overline{\theta}_s} ds,$$

we have $M_n = n \int_0^1 v_s ds + \overline{l}^* n^{1/3} + o(n^{1/3})$ in probability.

Remark 3.5.4. Let $\sigma \in \mathcal{C}^2$ be a positive decreasing function. For $t \in [0,1]$, we define the point process $L_t = (\ell_t^1, \ell_t^2)$ with ℓ_t^1, ℓ_t^2 two i.i.d. centred Gaussian random variables with variance σ_t. We consider the BRWls with environment $(\mathcal{L}_t, t \in [0,1])$. We have $\overline{\theta}_t = \frac{\sqrt{2\log 2}}{\sigma_t}$, which is increasing. Consequently, by Theorem 3.1.3 and Lemma 3.4.3

$$M_n = n\sqrt{2\log 2} \int_0^1 \sigma_s ds + n^{1/3} \frac{\alpha_1}{2^{1/3}(2\log 2)^{1/6}} \int_0^1 (-\sigma_s')^{2/3} \sigma_s^{1/3} ds + o_{\mathbf{P}}(n^{1/3}),$$

which is consistent with the results obtained in [MZ14] and [NRR14].

Similarly, if $\overline{\theta}$ is non-increasing, then θ is constant. Applying Theorem 3.1.3, we have $M_n = n \int_0^1 a_s ds + o(n^{1/3})$, and the correct second order is logarithmic.

3.5.2 Consistent maximal displacement with respect to a given path

Let φ be a continuous positive function, we write $b_t = \partial_\theta \kappa_t(\varphi_t)$, and we assume that b satisfies (3.5.1). We take interest in the consistent maximal displacement with respect to the path with speed profile b, defined by

$$\Lambda_n^\varphi = \min_{|u|=n} \max_{k \leq n} \left(\overline{b}_k^{(n)} - V(u_k)\right). \tag{3.5.8}$$

In other words, this is the smallest number such that, killing every individual at generation k and in a position below $\overline{b}_k^{(n)} - \Lambda_n^\varphi$, an individual remains alive until time n.

We set, for $u \in \mathbf{T}$, $\Lambda^\varphi(u) = \max_{k \leq |u|} \left(\overline{b}_k^{(n)} - V(u_k)\right)$ the maximal delay of individual u. In particular, with the definition of Section 3.3, for all $\mu \geq 0$, we have

$$M_n^\varphi(-\mu, [0,1]) = \max\left\{V(u), |u| = n, \Lambda^\varphi(u) \leq \mu n^{1/3}\right\}, \tag{3.5.9}$$

in particular $M_n^\varphi(-\mu, [0,1]) > -\infty \iff \Lambda_n^\varphi \leq \mu n^{1/3}$.

For $\lambda, \mu > 0$, we denote by $g^{\lambda,\mu}$ the solution of

$$\varphi_t g_t = \varphi_0 \lambda + \int_0^t \dot{\varphi}_s g_s + \mathbf{1}_{\{K^*(b)_s = 0\}} \frac{\sigma_s^2}{(g_s+\mu)^2} \Psi\left(\tfrac{(g_s+\mu)^3}{\sigma_s^2} \dot{\varphi}_s\right) ds. \tag{3.5.10}$$

Using the particular structure of this differential equation, for all $\lambda, \mu > 0$, we have $g^{\lambda,\mu} = g^{\lambda+\mu,0} - \mu$. Indeed, let $\lambda > 0$ and $\mu > 0$, and let $g = g^{\lambda+\mu,0} - \mu$. By definition, the differential equation satisfied by $g + \mu$ is

$$\varphi_t(g_t + \mu) = \varphi_0(\lambda + \mu) + \int_0^t \dot{\varphi}_s(g_s+\mu) + \mathbf{1}_{\{K^*(b)_s = 0\}} \frac{\sigma_s^2}{(g_s+\mu)^2} \Psi\left(\tfrac{(g_s+\mu)^3}{\sigma_s^2} \dot{\varphi}_s\right) ds$$
$$\varphi_t g_t = \varphi_0 \lambda + \int_0^t \dot{\varphi}_s g_s + \mathbf{1}_{\{K^*(b)_s = 0\}} \frac{\sigma_s^2}{(g_s+\mu)^2} \Psi\left(\tfrac{(g_s+\mu)^3}{\sigma_s^2} \dot{\varphi}_s\right) ds,$$

and by uniqueness of the solution of the equation, we have $g = g^{\lambda,\mu}$.

For $\lambda > 0$, we set $g^\lambda = g^{\lambda,0}$. We observe that if $\{\dot{\varphi}_t > 0\} \subset \{K^*(b)_t = 0\}$, then, for all $\lambda \geq 0$, g^λ is a decreasing function. As $\lambda \mapsto g^\lambda$ is strictly increasing and continuous, there exists a unique non-negative λ^* that verifies

$$g_1^{\lambda^*} = 0. \tag{3.5.11}$$

Alternatively, λ^* can be defined as $\widetilde{g}_1/\varphi_0$, where $\widetilde{g}$ is the unique solution of the differential equation

$$\forall t \in [0,1), \varphi_t \widetilde{g}_t = -\int_t^1 \dot{\varphi}_s g_s + \mathbf{1}_{\{K^*(b)_s=0\}} \frac{\sigma_s^2}{\widetilde{g}_s^2} \Psi\left(\tfrac{\widetilde{g}_s^3}{\sigma_s^2}\dot{\varphi}_s\right) ds.$$

Lemma 3.5.5 (Asymptotic of the consistent maximal displacement)**.** *Under the assumptions* (3.1.4)*,* (3.3.1)*,* (3.3.4)*,* (3.3.6)*,* (3.3.9) *and* (3.5.1)*, for all $\lambda < \lambda^*$, we have*

$$\lim_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\left(\Lambda_n^\varphi \leq (\lambda^* - \lambda)n^{1/3}\right) = -\varphi_0\lambda.$$

Moreover, for all $\varepsilon > 0$,

$$\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\left(\left|\Lambda_n^\varphi - \lambda^* n^{1/3}\right| \geq \varepsilon n^{1/3}\right) < 0.$$

Proof. Let $\lambda \in (0, \lambda^*)$, we set $g_t = g_t^{\lambda^*}$. Note first that $\Lambda_n^\varphi \leq \lambda n^{1/3}$ if and only if there exists an individual u alive at generation n such that $\Lambda^\varphi(u) \leq \lambda n^{1/3}$. To bound this quantity from above, we observe that such an individual either crosses $\bar{b}^{(n)}_{.} + n^{1/3}(g_{./n} - \lambda + \varepsilon)$ at some time before n, or stays below this boundary until time n. Consequently, for all $\varepsilon > 0$, we have

$$\begin{aligned}
&\mathbf{P}\left(\mathcal{B}_n^{[0,1],G}(-\lambda, g - \lambda + \varepsilon, -\lambda) \neq \emptyset\right)\\
&\qquad \leq \mathbf{P}\left(\Lambda_n^\varphi \leq \lambda n^{1/3}\right)\\
&\qquad\quad \leq \mathbf{P}\left(\mathcal{A}_n^{[0,1],G}(-\lambda, g - \lambda + \varepsilon) \neq \emptyset\right) + \mathbf{P}\left(\mathcal{B}_n^{[0,1],G}(-\lambda, g - \lambda + \varepsilon, -\lambda\right).
\end{aligned}$$

Using Lemma 3.3.3, Lemma 3.3.5 and Lemma 3.3.7, and letting $\varepsilon \to 0$, we conclude

$$\lim_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\left(\Lambda_n^\varphi \leq \lambda n^{1/3}\right) = -\varphi_0(\lambda^* - \lambda).$$

Finally, to bound $\mathbf{P}(\Lambda_n^\varphi \geq (\lambda^* + \varepsilon)n^{1/3})$ we apply (3.5.9), and we get

$$\mathbf{P}(\Lambda_n^\varphi \geq (\lambda^* + \varepsilon)n^{1/3}) = \mathbf{P}(M_n(-\lambda^* - \varepsilon, [0,1]) = -\infty).$$

By Lemma 3.5.3, we conclude that

$$\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}(\Lambda_n^\varphi \geq (\lambda^* + \varepsilon)n^{1/3}) < 0.$$

□

Proof of Theorem 3.1.4

We prove Theorem 3.1.4, applying Lemma 3.5.5 to $\Lambda_n = \Lambda_n^\theta$.

Proof of Theorem 3.1.4. We denote by $G = \{t \in [0,1] : K^*(a)_t = 0\}$. For $\lambda > 0$, we let g^λ be the solution of the equation

$$\theta_t g_t = \theta_0 \lambda + \int_0^t \dot{\theta}_s g_s + \mathbf{1}_G(s) \frac{\sigma_s^2}{g_s^2} \Psi\left(\tfrac{g_s^3}{\sigma_s^2} \dot{\theta}_s \right) ds, \tag{3.5.12}$$

and λ^* be the unique non-negative real number that verifies

$$g_1^{\lambda^*} = 0. \tag{3.5.13}$$

By Proposition 3.1.1, a satisfies (3.5.1) and $\{\theta_t > 0\} \subset \{K^*(b)_t = 0\}$. By Lemma 3.5.5, for all $\lambda \in (0, \lambda^*)$, we have

$$\lim_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\left[\Lambda_n^\theta \leq (\lambda^* - \lambda) n^{1/3} \right] = -\theta_0 \lambda,$$

and for all $\varepsilon > 0$, we have

$$\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\left[\left| \Lambda_n^\varphi - \lambda^* n^{1/3} \right| > \varepsilon n^{1/3} \right] < 0.$$

$\square$

In a similar way, we can compute the consistent maximal displacement with respect to the path with speed profile v, which is $\Lambda_n^{\overline{\theta}}$. We denote by $\overline{g}^\lambda$ the solution of the equation

$$\overline{\theta}_t g_t = \overline{\theta}_0 \lambda + \int_0^t \dot{\overline{\theta}}_s g_s + \frac{\sigma_s^2}{g_s^2} \Psi\left(\tfrac{g_s^3}{\sigma_s^2} \dot{\overline{\theta}}_s \right) ds,$$

and by $\overline{\lambda}^*$ the solution of $g_1^{\overline{\lambda}^*} = 0$. By Lemma 3.5.5, for all $0 \leq l \leq \overline{\lambda}^*$,

$$\lim_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\left(\Lambda_n^\varphi \leq (\lambda^* - \lambda) n^{1/3} \right) = -\varphi_0 \lambda,$$

and for all $\varepsilon > 0$, we have

$$\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\left[\left| \Lambda_n^{\overline{\theta}} - \overline{\lambda}^* n^{1/3} \right| > \varepsilon n^{1/3} \right] < 0.$$

Consistent maximal displacement of the time-homogeneous branching random walk

We consider $(\mathbf{T}, V)$ a time-homogeneous branching random walk, with reproduction law $\mathcal{L}$. We denote by κ the Laplace transform of $\mathcal{L}$. The optimal speed profile is a constant $v = \inf_{\theta>0} \frac{\kappa(\theta)}{\theta}$, and we set $\theta^* = \kappa'(\theta^*)$ and $\sigma^2 = \kappa''(\theta^*)$. The equation (3.5.12) can be written in the simpler form

$$\theta^* g_t^\lambda = \theta^* \lambda + \int_0^t \frac{\sigma^2}{(g_s^\lambda)^2} \Psi(0) ds.$$

As $\Psi(0) = -\frac{\pi^2}{2}$, the solution of this differential equation is $g_t^\lambda = \left(\lambda^3 - t \frac{3\pi^2\sigma^2}{2\theta^*} \right)^{1/3}$.

For $\Lambda_n = \min_{|u|=n} \max_{k\leq n} (kv - V(u_k))$, applying Theorem 3.1.4, and the Borel-Cantelli lemma, we obtain

$$\lim_{n\to+\infty} \frac{\Lambda_n}{n^{1/3}} = \left(\frac{3\pi^2\sigma^2}{2\theta^*}\right)^{1/3} \quad \text{a.s.}$$

This result is similar to the one obtained in [FZ10] and [FHS12].

More generally, if $(\mathbf{T}, V)$ is a BRWls such that $\overline{\theta}$ is non-increasing, then θ is a constant, and for all $\varepsilon > 0$,

$$\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\left[\left|\Lambda_n - \left(\frac{3\pi^2\sigma^2}{2\theta}\int_0^1 \mathbf{1}_{\{K^*(a)_s=0\}} ds\right)^{1/3} n^{1/3}\right| \geq \varepsilon n^{1/3}\right].$$

3.A Airy facts and Brownian motion estimates

In Section 3.A.1, using some Airy functions –introduced in Section 3.1.5– facts, the Feynman-Kac formula and PDE analysis, we compute the asymptotic of the Laplace transform of the area under a Brownian motion constrained to stay positive, proving Lemma 3.2.3. Adding some Sturm-Liouville theory, we obtain by similar arguments Lemma 3.2.4 in Section 3.A.2. In all this section, B stands for a standard Brownian motion, starting from x under law $\mathbf{P}_x$.

3.A.1 Asymptotic of the Laplace transform of the area under a Brownian motion constrained to stay non-negative

In this section, we write $\mathrm{L}^2 = \mathrm{L}^2([0, +\infty))$ for the set of square-integrable measurable functions on $[0, +\infty)$. This space L^2 can be seen as a Hilbert space, when equipped with the scalar product

$$\langle f, g\rangle = \int_0^{+\infty} f(x)g(x)dx.$$

We denote by $\mathcal{C}_0^2 = \mathcal{C}_0^2([0, +\infty))$ the set of twice differentiable functions w with a continuous second derivative, such that $w(0) = \lim_{x\to+\infty} w(x) = 0$. Finally, for any continuous function w, $\|w\|_\infty = \sup_{x\geq 0} |w(x)|$. The main result of the section is:

For all $h > 0$, $0 < a < b$ and $0 < a' < b'$, we have

$$\begin{aligned}\lim_{t\to+\infty} \frac{1}{t} \log \sup_{x\in\mathbb{R}} \mathbf{E}_x\left[e^{-h\int_0^t B_s ds}; B_s \geq 0, s\leq t\right]\\ = \lim_{t\to+\infty} \frac{1}{t} \log \inf_{x\in[a,b]} \mathbf{E}_x\left[e^{-h\int_0^t B_s ds}\mathbf{1}_{\{B_t\in[a',b']\}}; B_s \geq 0, s\leq t\right] = \frac{\alpha_1}{2^{1/3}} h^{2/3}. \quad (3.A.1)\end{aligned}$$

We recall that $(\alpha_n, n\in\mathbb{N})$ is the set of zeros of Ai, listed in the decreasing order. We start with some results on the Airy function Ai, defined in (3.1.13).

Lemma 3.A.1. *For $n \in \mathbb{N}$ and $x \geq 0$, we set*

$$\psi_n(x) = \mathrm{Ai}(x+\alpha_n)\left(\int_{\alpha_n}^{+\infty} \mathrm{Ai}(y)dy\right)^{-1/2}. \quad (3.A.2)$$

The following properties hold:

- $(\psi_n, n\in\mathbb{N})$ *forms an orthogonal basis of* L^2*;*

- $\lim_{n\to+\infty} \alpha_n n^{-2/3} = -\frac{3\pi}{2}$;
- *for all* $\lambda \in \mathbb{R}$ *and* $\psi \in \mathcal{C}^2$, *if*

$$\begin{cases} \forall x > 0, \psi''(x) - x\psi(x) = \lambda\psi(x) \\ \psi(0) = \lim_{x\to+\infty} \psi(x) = 0, \end{cases} \tag{3.A.3}$$

then either $\psi = 0$, *or there exist* $n \in \mathbb{N}$ *and* $c \in \mathbb{R}$ *such that* $\lambda = \alpha_n$ *and* $\psi = c\psi_n$.

Proof. The fact that $\lim_{n\to+\infty} \alpha_n n^{-2/3} = -\frac{3\pi}{2}$ and that $(\psi_n, n \in \mathbb{N})$ is an orthogonal basis of L^2 can be found in [VS10]. We now consider (λ, ψ) a solution of (3.A.3). In particular ψ verifies

$$\forall x > 0, \psi''(x) - (x + \lambda)\psi(x) = 0.$$

By definition of Ai and Bi, there exist c_1, c_2 such that $\psi(x) = c_1\mathrm{Ai}(x + \lambda) + c_2\mathrm{Bi}(x + \lambda)$. As $\lim_{x\to+\infty} \psi(x) = 0$, we have $c_2 = 0$, and as $\psi(0) = 0$, either $c_1 = 0$, or $\mathrm{Ai}(\lambda) = 0$. We conclude that either $\psi = 0$, or λ is a zero of Ai, in which case $\psi(x) = c_1\psi_n(x)$ for some $n \in \mathbb{N}$. □

As α_1 is the largest zero of Ai, note that the eigenfunction ψ_1 corresponding to the largest eigenvector α_1 is non-negative on $[0, +\infty)$, and is positive on $(0, +\infty)$.

For $h > 0$ and $n \in \mathbb{N}$, we define $\psi_n^h = (2h)^{1/6}\psi_n((2h)^{1/3}x)$. By Lemma 3.A.1, the sequence $(\psi_n^h, n \in \mathbb{N})$ forms an orthonormal basis of L^2. With this lemma, we can prove the following preliminary result.

Lemma 3.A.2. *Let* $h > 0$ *and* $u_0 \in \mathcal{C}_0^2 \cap \mathrm{L}^2$, *such that* $u_0', u_0'' \in \mathrm{L}^2$ *and* $\|u_0''\|_\infty < +\infty$. *We define, for* $t \geq 0$ *and* $x \geq 0$

$$u(t, x) = \mathbf{E}_x\left[u_0(B_t)e^{-h\int_0^t B_s ds}; B_s \geq 0, s \in [0, t]\right].$$

We have

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} \left| e^{-\frac{h^{2/3}}{2^{1/3}}\alpha_1 t} u(t, x) - \langle u_0, \psi_1^h\rangle\psi_1^h(x) \right| = 0. \tag{3.A.4}$$

Proof. Let $h > 0$, by the Feynman-Kac formula (see e.g. [KS91], Theorem 5.7.6), u is the unique solution of the equation

$$\begin{cases} \forall t > 0, \forall x > 0, \partial_t u(t, x) = \frac{1}{2}\partial_x^2 u(t, x) - hxu(t, x) \\ \forall x \geq 0, u(0, x) = u_0(x) \\ \forall t \geq 0, u(t, 0) = \lim_{x\to+\infty} u(t, x) = 0. \end{cases} \tag{3.A.5}$$

We define the operator

$$\mathcal{G}^h : \begin{array}{ccc} \mathcal{C}_0^2 & \to & \mathcal{C} \\ w & \mapsto & \left(x \mapsto \frac{1}{2}w''(x) - hxw(x), x \in [0, +\infty)\right), \end{array}$$

By definition of Ai and of the ψ_n^h, we have

$$\mathcal{G}^h\psi_n^h = \frac{h^{2/3}}{2^{1/3}}\alpha_n\psi_n^h.$$

thus (ψ_n^h) forms an orthogonal basis of eigenfunctions of $\mathcal{G}^h$.

There exists $C > 0$ such that for any $x \geq 0$, $\mathrm{Ai}(x) + \mathrm{Ai}'(x) \leq C(1 + x^{1/4})e^{-2x^{2/3}/3}$ (see e.g. [VS10]). For any $w \in \mathcal{C}_0^2 \cap \mathrm{L}^2$ such that w' and w'' are bounded, by integration by parts

$$\begin{aligned}
\langle \mathcal{G}^h w, \psi_n^h \rangle &= \frac{1}{2} \int_0^{+\infty} w''(x)\psi_n^h(x)dx - h \int_0^{+\infty} xw(x)\psi_n^h(x)dx \\
&= \frac{1}{2} \int_0^{+\infty} w(x)(\psi_n^h)''(x)dx - h \int_0^{+\infty} xw(x)\psi_n^h(x)dx \\
&= \int_0^{+\infty} w(x)(\mathcal{G}^h \psi_n^h)(x)dx \\
&= \frac{h^{2/3}}{2^{1/3}} \alpha_n \langle w, \psi_n^h \rangle.
\end{aligned}$$

Therefore, decomposing w with respect to the basis (ψ_n^h), we have

$$\langle \mathcal{G}^h w, w \rangle = \langle \mathcal{G}^h w, \sum_{n=1}^{+\infty} \langle \psi_n^h, w \rangle \psi_n^h \rangle = \sum_{n=1}^{+\infty} \langle w, \psi_n^h \rangle \langle \mathcal{G}^h w, \psi_n^h \rangle = \frac{h^{2/3}}{2^{1/3}} \sum_{n=1}^{+\infty} \alpha_n \langle w, \psi_n^h \rangle^2.$$

As (α_n) is a decreasing sequence, we have

$$\langle \mathcal{G}^h w, w \rangle \leq \frac{h^{2/3}}{2^{1/3}} \sum_{n=1}^{+\infty} \alpha_1 \langle w, \psi_n^h \rangle^2 \leq \frac{h^{2/3}}{2^{1/3}} \alpha_1 \langle w, w \rangle. \tag{3.A.6}$$

If $\langle w, \psi_n^h \rangle = 0$, the inequality can be improved in

$$\langle \mathcal{G}^h w, w \rangle \leq \frac{h^{2/3}}{2^{1/3}} \sum_{n=2}^{+\infty} \alpha_2 \langle w, \psi_n^h \rangle^2 \leq \frac{h^{2/3}}{2^{1/3}} \alpha_2 \langle w, w \rangle. \tag{3.A.7}$$

Using these results, we now prove (3.A.4). For $x \geq 0$ and $t \geq 0$, we define

$$v(t, x) = e^{-\frac{h^{2/3}}{2^{1/3}} \alpha_1 t} u(t, x) - \langle u_0, \psi_1^h \rangle \psi_1^h.$$

We observe first that for all $t \geq 0$, $\langle v(t, \cdot), \psi_1^h \rangle = 0$. Indeed, we have $\langle v(0, \cdot), \psi_1^h \rangle = 0$ by definition, and deriving with respect to t, we have

$$\begin{aligned}
\partial_t \langle v(t, \cdot), \psi_1^h \rangle &= -\frac{h^{2/3}}{2^{1/3}} \alpha_1 e^{-\frac{h^{2/3}}{2^{1/3}} \alpha_1 t} \langle u(t, x), \psi_1^h \rangle + e^{-\frac{h^{2/3}}{2^{1/3}} \alpha_1 t} \langle \partial_t u(t, x), \psi_1^h \rangle \\
&= -\frac{h^{2/3}}{2^{1/3}} \alpha_1 e^{-\frac{h^{2/3}}{2^{1/3}} \alpha_1 t} \langle u(t, x), \psi_1^h \rangle + e^{-\frac{h^{2/3}}{2^{1/3}} \alpha_1 t} \langle \mathcal{G}^h u(t, x), \psi_1^h \rangle \\
&= -\frac{h^{2/3}}{2^{1/3}} \alpha_1 e^{-\frac{h^{2/3}}{2^{1/3}} \alpha_1 t} \langle u(t, x), \psi_1^h \rangle + \frac{h^{2/3}}{2^{1/3}} \alpha_1 e^{-\frac{h^{2/3}}{2^{1/3}} \alpha_1 t} \langle u(t, x), \psi_1^h \rangle \\
&= 0.
\end{aligned}$$

We now prove that the non-negative, finite functions

$$J_1(t) = \int_0^{+\infty} |v(t, x)|^2 dx \quad \text{and} \quad J_2(t) = \int_0^{+\infty} |\partial_x v(t, x)|^2 dx,$$

are decreasing, and converge to 0 as $t \to +\infty$. We observe first that

$$\begin{aligned}
\partial_t J_1(t) &= \int_0^{+\infty} 2v(t,x)\partial_t v(t,x)dx \\
&= \int_0^{+\infty} 2v(t,x)\left[-\frac{h^{2/3}}{2^{1/3}}\alpha_1 e^{-\frac{h^{2/3}}{2^{1/3}}\alpha_1 t}u(t,x) + e^{-\frac{h^{2/3}}{2^{1/3}}\alpha_1 t}\partial_t u(t,x)\right]dx \\
&= \int_0^{+\infty} 2v(t,x)\left[-\frac{h^{2/3}}{2^{1/3}}\alpha_1 v(t,x) + \mathcal{G}^h v(t,x)\right]dx \\
&= 2\left[-\frac{h^{2/3}}{2^{1/3}}\alpha_1\langle v(t,\cdot),v(t,\cdot)\rangle + \langle v(t,\cdot),\mathcal{G}^h v(t,\cdot)\rangle\right],
\end{aligned}$$

and as $\langle v(t,\cdot),\psi_1^h\rangle = 0$, by (3.A.7)

$$\partial_t J_1(t) \le (2h)^{2/3}(\alpha_2-\alpha_1)J_1(t) \le -cJ_1(t).$$

By Grönwall inequality, $J_1(t)$ decreases to 0 as $t \to +\infty$ exponentially fast. Similarly, we have $J_2(0) < +\infty$, and

$$\begin{aligned}
\partial_t J_2(t) &= \int_0^{+\infty} 2\partial_x v(t,x)\partial_t\partial_x v(t,x)dx = 2\int_0^{+\infty}\partial_x v(t,x)\partial_x\partial_t v(t,x)dx \\
=&2\int_0^{+\infty}\partial_x v(t,x)\left[-\frac{h^{2/3}}{2^{1/3}}\alpha_1\partial_x v(t,x) + \partial_x\mathcal{G}^h v(t,x)\right]dx \\
=&-(2h)^{2/3}\alpha_1\langle\partial_x v(t,\cdot),\partial_x v(t,\cdot)\rangle + 2\langle\partial_x v(t,\cdot),\mathcal{G}^h\partial_x v(t,\cdot)\rangle - 2h\underbrace{\int_0^{+\infty} v(t,x)\partial_x v(t,x)dx}_{\lim_{x\to+\infty} v(t,x)^2 - v(t,0)^2},
\end{aligned}$$

by integration by parts and (3.A.6). Thus, $\partial_t J_2(t) \le 0$. As J_2 is non-negative and decreasing, this function converges, as $t \to +\infty$, to $J_2(+\infty) \ge 0$. Moreover, we can write the derivative of J_1 as follows

$$\begin{aligned}
\partial_t J_1(t) &= 2\left[-\frac{h^{2/3}}{2^{1/3}}\alpha_1\langle v(t,\cdot),v(t,\cdot)\rangle + \langle v(t,\cdot),\mathcal{G}^h v(t,\cdot)\rangle\right] \\
&= -(2h)^{2/3}\alpha_1 J_1(t) + \int_0^{+\infty} v(t,x)\partial_x^2 v(t,x)dx - \int_0^{+\infty} hxv(t,x)^2dx \\
&\le -(2h)^{2/3}\alpha_1 J_1(t) - \int_0^{+\infty}\left(\partial_x v(t,x)\right)^2dx \\
&\le -J_2(t) - (2h)^{2/3}\alpha_1 J_1(t).
\end{aligned}$$

As $\lim_{t\to+\infty} J_1(t) = 0$, if $\lim_{t\to+\infty} J_2(t) = J_2(+\infty) > 0$, we would conclude that the derivative of J_1 stays negative and bounded away from 0 for large t. This would indicate that J_1 decreases to $-\infty$, contradicting the fact that $\lim_{t\to+\infty} J_1(t) = 0$. We conclude that $\lim_{t\to+\infty} J_2(t) = 0$.

As J_1 and J_2 decrease to 0, we have

$$\lim_{t\to+\infty}\int_0^{+\infty} |v(t,x)|^2 + |\partial_x v(t,x)|^2 dx = 0,$$

which means that $v(t,\cdot)$ converges to 0 in H^1 norm, as $t \to +\infty$. By Sobolev injection in dimension 1, there exists $C > 0$ such that

$$\|v(t,\cdot)\|_{+\infty} \le C\int_0^{+\infty} |v(t,x)|^2 + |\partial_x v(t,x)|^2 dx,$$

which proves (3.A.4). □

This lemma can be easily extended to authorize any bounded starting function u_0.

Corollary 3.A.3. *Let $h > 0$ and u_0 be a measurable bounded function. Setting, for $x \geq 0$ and $t \geq 0$*

$$u(t,x) = \mathbf{E}_x \left[u_0(B_t) e^{-h \int_0^t B_s ds}; B_s \geq 0, s \in [0,t] \right],$$

we have

$$\lim_{t \to +\infty} \sup_{x \in \mathbb{R}} \left| e^{-\frac{h^{2/3}}{2^{1/3}} \alpha_1 t} u(t,x) - \langle u_0, \psi_1^h \rangle \psi_1^h(x) \right| = 0. \tag{3.A.8}$$

Proof. Let u_0 be a measurable bounded function. We introduce, for $x \geq 0$ and $\varepsilon > 0$

$$u_\varepsilon(x) = u(\varepsilon, x) = \mathbf{E}_x \left[u_0(B_\varepsilon) e^{-h \int_0^\varepsilon B_s ds}; B_s \geq 0, s \in [0,1] \right].$$

Observe that by the Markov property, for all $t \geq \varepsilon$, we have

$$\begin{aligned} &u(t,x) \\ =& \mathbf{E}_x \left[u_0(B_t) e^{-h \int_0^t B_s ds}; B_s \geq 0, s \in [0,t] \right] \\ =& \mathbf{E}_x \left[\mathbf{E}_{B_{t-\varepsilon}} \left[u_0(B_\varepsilon) e^{-h \int_0^\varepsilon B_s ds}; B_s \geq 0, s \in [0,\varepsilon] \right] e^{-h \int_0^{t-\varepsilon} B_s ds}; B_s \geq 0, s \in [0, t-\varepsilon] \right] \\ =& \mathbf{E}_x \left[u_\varepsilon(B_{t-\varepsilon}) e^{-h \int_0^{t-\varepsilon} B_s ds}; s \in [0, t-\varepsilon] \right]. \end{aligned}$$

Therefore, $u(t,x) = u_\varepsilon(t-\varepsilon, x)$, wher $u_\varepsilon(t,x) = \mathbf{E}_x \left[u_\varepsilon(B_t) e^{-h \int_0^t B_s ds}; s \in [0,t] \right]$.

As $\int_0^\varepsilon B_s ds$ is, under the law $\mathbf{P}_x$, a Gaussian random variable with mean εx and variance $\varepsilon^3/3$, we have

$$|u_\varepsilon(x)| \leq \|u_0\|_\infty \mathbf{E}_x \left[e^{-h \int_0^\varepsilon B_s ds} \right] \leq \|u_0\|_\infty e^{-\varepsilon h x} e^{\frac{h^2 \varepsilon^3}{6}}.$$

Moreover, as $h > 0$, by the Ballot lemma,

$$|u_\varepsilon(x)| \leq \|u_0\|_\infty \mathbf{P}_x \left[B_s \geq 0, s \in [0,\varepsilon] \right] \leq C \varepsilon^{-1/2} x.$$

For any $\varepsilon > 0$, there exists $C > 0$ such that for any $x \geq 0$, $u_\varepsilon(x) \leq Cx \wedge e^{-hx\varepsilon}$. Therefore, we can find sequences $(v^{(n)})$ and $(w^{(n)})$ of functions in $\mathcal{C}_0^2 \cap \mathrm{L}^2$, such that $(v^{(n)})', (v^{(n)})'', (w^{(n)})'$ and $(w^{(n)})''$ are in L^2, that their second derivatives are bounded, and that for all $x \in \mathbb{R}$,

$$w^{(n)} \leq u_\varepsilon \leq w^{(n)} + \frac{1}{n} \quad \text{and} \quad v^{(n)} - \frac{1}{n} \leq u_\varepsilon \leq v^{(n)}.$$

For $n \in \mathbb{N}$, $x \geq 0$ and $t \geq 0$, we set

$$v^{(n)}(t,x) = \mathbf{E}_x \left[v^{(n)}(B_t) e^{-\int_0^t B_s ds}; B_s \geq 0, s \in [0,t] \right] \quad \text{and}$$
$$w^{(n)}(t,x) = \mathbf{E}_x \left[w^{(n)}(B_t) e^{-\int_0^t B_s ds}; B_s \geq 0, s \in [0,t] \right].$$

Note that for all $x \geq 0$ and $t \geq 0$ we have $w^{(n)}(t,x) \leq u_\varepsilon(t,x) \leq v^{(n)}(t,x)$. Moreover, by Lemma 3.A.2, we have

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} \left| e^{-\frac{h^{2/3}}{2^{1/3}}\alpha_1 t} v^{(n)}(t,x) - \langle v^{(n)}, \psi_1^h \rangle \psi_1^h(x) \right| = 0,$$

$$\text{and} \quad \lim_{t\to+\infty} \sup_{x\in\mathbb{R}} \left| e^{-\frac{h^{2/3}}{2^{1/3}}\alpha_1 t} w^{(n)}(t,x) - \langle w^{(n)}, \psi_1^h \rangle \psi_1^h(x) \right| = 0.$$

By the dominated convergence theorem, we have

$$\lim_{n\to+\infty} \langle w^{(n)}, \psi_1^h \rangle = \lim_{n\to+\infty} \langle v^{(n)}, \psi_1^h \rangle = \langle u_\varepsilon, \psi_1^h \rangle.$$

As a result, letting t, then $n \to +\infty$, this yields

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} \left| e^{-\frac{h^{2/3}}{2^{1/3}}\alpha_1 t} u_\varepsilon(t,x) - \langle u_\varepsilon, \psi_1^h \rangle \psi_1^h(x) \right| = 0.$$

For almost every $x \geq 0$, letting $\varepsilon \to 0$, we have $u_\varepsilon(x) \to u_0(x)$, and thus by dominated convergence theorem again, $\lim_{\varepsilon\to 0} \langle u_\varepsilon, \psi_1^h \rangle = \langle u_0, \psi_1^h \rangle$. As $u(t,x) = u_\varepsilon(t-\varepsilon, x)$, we conclude that

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} \left| e^{-\frac{h^{2/3}}{2^{1/3}}\alpha_1 t} u(t,x) - \langle u_0, \psi_1^h \rangle \psi_1^h(x) \right| = 0.$$

□

This last corollary is enough to prove the exponential decay of the Laplace transform of the area under a Brownian motion constrained to stay positive.

Proof of Lemma 3.2.3. Let $h > 0$, we set

$$u(t,x) = \mathbf{E}_x \left[e^{-h\int_0^t B_s ds}; B_s \geq 0, s \in [0,t] \right],$$

and $\mu_h = \int_0^{+\infty} \psi_1^h(x)dx < +\infty$. By Corollary 3.A.3, we have

$$\lim_{t\to+\infty} \sup_{x\in[0,+\infty)} \left| e^{-\frac{h^{2/3}}{2^{1/3}}\alpha_1 t} u(t,x) - \mu_h \psi_1^h(x) \right| = 0.$$

As ψ_1^h is bounded, we have

$$\limsup_{t\to+\infty} \sup_{x\geq 0} \frac{1}{t} \log u(t,x) = \frac{h^{2/3}}{2^{1/3}} \alpha_1. \tag{3.A.9}$$

Similarly, for $0 < a < b$ and $0 < a' < b'$, we set

$$\widetilde{u}(t,x) = \mathbf{E}_x \left[\mathbf{1}_{\{B_t \in [a',b']\}} e^{-h\int_0^t B_s ds}; B_s \geq 0, s \in [0,t] \right]$$

and $\widetilde{\mu}_h = \int_{a'}^{b'} \psi_1^h(x)dx > 0$. By Corollary 3.A.3 again, we have

$$\lim_{t\to+\infty} \sup_{x\geq 0} \left| e^{-\frac{h^{2/3}}{2^{1/3}}\alpha_1 t} \widetilde{u}(t,x) - \widetilde{\mu}_h \psi_1^h(x) \right| = 0.$$

In particular, as $\inf_{x\in[a,b]}\psi_1^h > 0$, we have

$$\liminf_{t\to+\infty}\inf_{x\in[a,b]}\frac{1}{t}\log\widetilde{u}(t,x)=\frac{h^{2/3}}{2^{1/3}}\alpha_1. \tag{3.A.10}$$

As $\widetilde{u}\le u$, mixing (3.A.9) and (3.A.10), we conclude that

$$\lim_{t\to+\infty}\sup_{x\ge 0}\frac{1}{t}\log u(t,x)=\lim_{t\to+\infty}\inf_{x\in[a,b]}\frac{1}{t}\log\widetilde{u}(t,x)=\frac{h^{2/3}}{2^{1/3}}\alpha_1.$$

□

3.A.2 The area under a Brownian motion constrained to stay in an interval

The main result of this section is that:
For all $h\in\mathbb{R}$, $0<a<b<1$ and $0<a'<b'<1$, we have

$$\begin{aligned}&\lim_{t\to+\infty}\frac{1}{t}\sup_{x\in[0,1]}\log\mathbf{E}_x\left[e^{-h\int_0^t B_s ds};B_s\in[0,1],s\le t\right]\\&=\lim_{t\to+\infty}\frac{1}{t}\inf_{x\in[a,b]}\log\mathbf{E}_x\left[e^{-h\int_0^t B_s ds}\mathbf{1}_{\{B_t\in[a',b']\}};B_s\in[0,1],s\le t\right]=\Psi(h),\end{aligned} \tag{3.A.11}$$

where Ψ is the function defined in (3.1.17).

The organisation, the results and techniques of the section are very similar to Section 3.A.1, with a few exceptions. First, to exhibit an orthonormal basis of eigenfunctions, we need some additional Sturm-Liouville theory, that can be found in [Zet05]. Secondly, we work on $[0,1]$, which is a compact set. This lightens the analysis of the PDE obtained while proving Lemma 3.A.5.

In this section, we write $\mathrm{L}^2=\mathrm{L}^2([0,1])$ for the set of square-integrable measurable functions on $[0,1]$, equipped with the scalar product

$$\langle f,g\rangle=\int_0^1 f(x)g(x)dx.$$

Moreover, we write $\mathcal{C}_0^2=\mathcal{C}_0^2([0,1])$ for the set of continuous, twice differentiable functions w on $[0,1]$ such that $w(0)=w(1)=0$. Finally, for any continuous function w, $\|w\|_\infty=\sup_{x\in[0,1]}|w(x)|$ and $\|w\|_2=\langle w,w\rangle^{1/2}$. We introduce in a first time a new specific orthogonal basis of $[0,1]$.

Lemma 3.A.4. *Let $h>0$. The set of zeros of the function*

$$\lambda\mapsto\mathrm{Ai}(\lambda)\mathrm{Bi}(\lambda+(2h)^{1/3})-\mathrm{Ai}(\lambda+(2h)^{1/3})\mathrm{Bi}(\lambda)$$

is countable and bounded from above by 0, that are listed in the decreasing order as follows: $\lambda_1^h>\lambda_2^h>\cdots$. In particular, we have

$$\lambda_1^h=\sup\left\{\lambda\le 0:\mathrm{Ai}(\lambda)\mathrm{Bi}(\lambda+(2h)^{1/3})=\mathrm{Ai}(\lambda+(2h)^{1/3})\mathrm{Bi}(\lambda)\right\}. \tag{3.A.12}$$

Additionally, for $n\in\mathbb{N}$ and $x\in[0,1]$, we define

$$\varphi_n^h(x)=\frac{\mathrm{Ai}\left(\lambda_n^h\right)\mathrm{Bi}\left(\lambda_n^h+(2h)^{1/3}x\right)-\mathrm{Ai}\left(\lambda_n^h+(2h)^{1/3}x\right)\mathrm{Bi}\left(\lambda_n^h\right)}{\left\|\mathrm{Ai}\left(\lambda_n^h\right)\mathrm{Bi}\left(\lambda_n^h+\cdot\right)-\mathrm{Ai}\left(\lambda_n^h+\cdot\right)\mathrm{Bi}\left(\lambda_n^h\right)\right\|_2}. \tag{3.A.13}$$

The following properties are verified:

- $(\varphi_n^h, n \in \mathbb{N})$ *forms an orthogonal basis of* L^2*;*
- $\lim_{n\to+\infty} \lambda_n^h n^{-2} = -\frac{\pi^2}{2}$*;*
- *for all* $\mu \in \mathbb{R}$ *and* $\varphi \in \mathcal{C}_0^2$*, if*

$$\begin{cases} \forall x \in (0,1), \frac{1}{2}\varphi''(x) - hx\varphi(x) = \mu\varphi(x) \\ \varphi(0) = \varphi(1) = 0, \end{cases} \tag{3.A.14}$$

then either $\varphi = 0$*, or there exist* $n \in \mathbb{N}$ *and* $c \in \mathbb{R}$ *such that* $\mu = \frac{h^{2/3}}{2^{1/3}}\lambda_n^h$ *and* $\varphi = c\varphi_n^h$*.*

Proof. We consider equation (3.A.14). This is a Sturm-Liouville problem with separated boundary conditions, that satisfies the hypotheses of Theorem 4.6.2 of [Zet05]. Therefore, there is an infinite but countable number of real numbers $(\mu_n^h, n \in \mathbb{N})$ such that the differential equation

$$\begin{cases} \forall x \in (0,1), \frac{1}{2}\varphi''(x) - hx\varphi(x) = \mu_n^h\varphi(x) \\ \varphi(0) = \varphi(1) = 0, \end{cases}$$

admit non-zero solutions. For all $n \in \mathbb{N}$, we write φ_n^h for one of such solutions normalized so that $\left\|\varphi_n^h\right\|_2 = 1$. For every solution (λ, φ) of (3.A.14), there exist $n \in \mathbb{N}$ and $c \in \mathbb{R}$ such that $\lambda = \mu_n^h$ and $\varphi = c\varphi_n^h$. Moreover, since $(\varphi_n^h, n \in \mathbb{N})$ forms an orthonormal basis of L^2. By Theorem 4.3.1. of [Zet05], we have $\lim_{n\to+\infty} \lambda_n^h n^{-2} = -\frac{\pi^2}{2}$.

We now identify (μ_n^h) and (φ_n^h). By the definition of Airy functions, given $\mu \in \mathbb{R}$, the solutions of

$$\begin{cases} \frac{1}{2}\varphi''(x) - hx\varphi(x) = \mu\varphi(x) \\ \varphi(0) = 0, \end{cases}$$

are, up to a multiplicative constant

$$x \mapsto \mathrm{Bi}\left(\tfrac{2^{1/3}}{h^{2/3}}\mu\right) \mathrm{Ai}\left(\tfrac{2^{1/3}}{h^{2/3}}\mu + (2h)^{1/3}x\right) - \mathrm{Ai}\left(\tfrac{2^{1/3}}{h^{2/3}}\mu\right) \mathrm{Bi}\left(\tfrac{2^{1/3}}{h^{2/3}}\mu + (2h)^{1/3}x\right).$$

This function is null at point $x = 1$ if and only if

$$\mathrm{Bi}\left(\tfrac{2^{1/3}}{h^{2/3}}\mu\right) \mathrm{Ai}\left(\tfrac{2^{1/3}}{h^{2/3}}\mu + (2h)^{1/3}\right) - \mathrm{Ai}\left(\tfrac{2^{1/3}}{h^{2/3}}\mu\right) \mathrm{Bi}\left(\tfrac{2^{1/3}}{h^{2/3}}\mu + (2h)^{1/3}\right) = 0.$$

Therefore, the zeros of

$$\lambda \mapsto \mathrm{Ai}(\lambda)\mathrm{Bi}(\lambda + (2h)^{1/3}) - \mathrm{Ai}(\lambda + (2h)^{1/3})\mathrm{Bi}(\lambda),$$

can be listed in the decreasing order as follows: $\lambda_1^h > \lambda_2^h > \ldots$, and we have $\lambda_n^h = \frac{2^{1/3}}{h^{2/3}}\mu_n^h$. Moreover, we conclude that the eigenfunction φ_n^h described above is proportional to

$$x \mapsto \mathrm{Ai}\left(\lambda_n^h\right) \mathrm{Bi}\left(\lambda_n^h + (2h)^{1/3}x\right) - \mathrm{Ai}\left(\lambda_n^h + (2h)^{1/3}x\right) \mathrm{Bi}\left(\lambda_n^h\right),$$

and has L^2 norm 1, which validates the formula (3.A.13).

We have left to prove that for all $n \geq 1$, $\lambda_n^h < 0$. To do so, we observe that if (μ, φ) is a solution of (3.A.14), we have

$$\begin{aligned} \mu \int_0^1 \varphi(x)^2 dx &= \int_0^1 \varphi(x)\frac{1}{2}\partial_x^2\varphi(x) - \int_0^1 x\varphi(x)^2 dx \\ &= -\frac{1}{2}\int_0^1 (\partial_x\varphi(x))^2 dx - h\int_0^1 x\varphi(x)^2 dx < 0, \end{aligned}$$

which proves that for all $n \in \mathbb{N}$, $\mu_n^h < 0$, so $\lambda_1^h < 0$ which justifies (3.A.12). $\square$

We observe that once again, the eigenfunction φ_1^h corresponding to the largest eigenvector $\frac{h^{2/3}}{2^{1/3}}\lambda_n^h$ is a non-negative function on $[0,1]$, and positive on $(0,1)$.

Using this lemma, we can obtain a precise asymptotic of the Laplace transform of the area under a Brownian motion.

Lemma 3.A.5. *Let $h > 0$ and $u_0 \in \mathcal{C}^2([0,1])$ such that $u_0(0) = u_0(1) = 0$. We define, for $t \geq 0$ and $x \geq 0$*

$$u(t,x) = \mathbf{E}_x\left[u_0(B_t)e^{-h\int_0^t B_s ds}; B_s \in [0,1], s \in [0,t]\right].$$

We have

$$\lim_{t\to+\infty}\sup_{x\in\mathbb{R}}\left|e^{-\frac{h^{2/3}}{2^{1/3}}\lambda_1^h t}u(t,x) - \langle u_0, \varphi_1^h\rangle\varphi_1^h(x)\right| = 0. \tag{3.A.15}$$

Proof. This proof is very similar to the proof of Lemma 3.A.2. For $h > 0$, by the Feynman-Kac formula, u is the unique solution of the equation

$$\begin{cases}\forall t > 0, \forall x \in (0,1), \partial_t u(t,x) = \frac{1}{2}\partial_x^2 u(t,x) - hxu(t,x)\\ \forall x \in [0,1], u(0,x) = u_0(x)\\ \forall t \geq 0, u(t,0) = \lim_{x\to+\infty} u(t,x) = 0.\end{cases} \tag{3.A.16}$$

We define the operator

$$\mathcal{G}^h : \begin{array}{ccc}\mathcal{C}_0^2 & \to & \mathcal{C}\\ w & \mapsto & \left(x \mapsto \frac{1}{2}w''(x) - hxw(x), x \in [0,1]\right),\end{array}$$

By Lemma 3.A.4, we know that (φ_n^h) forms an orthogonal basis of L^2 consisting of eigenvectors of $\mathcal{G}^h$. In particular, for all $n \in \mathbb{N}$,

$$\mathcal{G}^h\varphi_n^h = \frac{h^{2/3}}{2^{1/3}}\lambda_n^h\varphi_n^h.$$

For all $w \in \mathcal{C}_0^2$, by integration by parts, we have

$$\begin{aligned}\langle\mathcal{G}^h w, \varphi_n^h\rangle &= \frac{1}{2}\int_0^1 w''(x)\varphi_n^h(x)dx - h\int_0^{+\infty} xw(x)\varphi_n^h(x)dx\\ &= \frac{1}{2}\int_0^1 w(x)(\varphi_n^h)''(x)dx - h\int_0^{+\infty} xw(x)\varphi_n^h(x)dx\\ &= \int_0^1 w(x)(\mathcal{G}^h\varphi_n^h)(x)dx\\ &= \frac{h^{2/3}}{2^{1/3}}\lambda_n^h\langle w, \varphi_n^h\rangle.\end{aligned}$$

Therefore, decomposing w with respect to the basis (φ_n^h), we obtain

$$\langle\mathcal{G}^h w, w\rangle = \langle\mathcal{G}^h w, \sum_{n=1}^{+\infty}\langle\varphi_n^h, w\rangle\varphi_n^h\rangle = \sum_{n=1}^{+\infty}\langle w, \varphi_n^h\rangle\langle\mathcal{G}^h w, \varphi_n^h\rangle = \sum_{n=1}^{+\infty}\frac{h^{2/3}}{2^{1/3}}\lambda_n^h\langle w, \varphi_n^h\rangle^2.$$

As (λ_n^h) is a decreasing sequence, we get

$$\langle\mathcal{G}^h w, w\rangle \leq \sum_{n=1}^{+\infty}\frac{h^{2/3}}{2^{1/3}}\lambda_1^h\langle w, \varphi_n^h\rangle^2 \leq \frac{h^{2/3}}{2^{1/3}}\lambda_1^h\langle w, w\rangle. \tag{3.A.17}$$

In addition, if $\langle w, \varphi_n^h \rangle = 0$, the inequality can be strengthened in

$$\langle \mathcal{G}^h w, w \rangle \leq \sum_{n=2}^{+\infty} \frac{h^{2/3}}{2^{1/3}} \lambda_2^h \langle w, \varphi_n^h \rangle^2 \leq \frac{h^{2/3}}{2^{1/3}} \lambda_2^h \langle w, w \rangle. \tag{3.A.18}$$

Using these results, we prove (3.A.15). For $x \in [0,1]$ and $t \geq 0$, we set

$$v(t,x) = e^{-\frac{h^{2/3}}{2^{1/3}}\lambda_1^h t} u(t,x) - \langle u_0, \varphi_1^h \rangle \varphi_1^h.$$

We observe first that $\langle v(0,\cdot), \varphi_1^h \rangle = 0$ by definition, and that for all $t \geq 0$,

$$\begin{aligned}
\partial_t \langle v(t,\cdot), \varphi_1^h \rangle &= -\frac{h^{2/3}}{2^{1/3}} \lambda_1^h e^{-\frac{h^{2/3}}{2^{1/3}}\lambda_1^h t} \langle u(t,x), \varphi_1^h \rangle + e^{-\frac{h^{2/3}}{2^{1/3}}\lambda_1^h t} \langle \partial_t u(t,x), \varphi_1^h \rangle \\
&= -\frac{h^{2/3}}{2^{1/3}} \lambda_1^h e^{-\frac{h^{2/3}}{2^{1/3}}\lambda_1^h t} \langle u(t,x), \varphi_1^h \rangle + e^{-\frac{h^{2/3}}{2^{1/3}}\lambda_1^h t} \langle \mathcal{G}^h u(t,x), \varphi_1^h \rangle \\
&= -\frac{h^{2/3}}{2^{1/3}} \lambda_1^h e^{-\frac{h^{2/3}}{2^{1/3}}\lambda_1^h t} \langle u(t,x), \varphi_1^h \rangle + \frac{h^{2/3}}{2^{1/3}} \lambda_1^h e^{-\frac{h^{2/3}}{2^{1/3}}\lambda_1^h t} \langle u(t,x), \varphi_1^h \rangle \\
&= 0.
\end{aligned}$$

which proves that for all $t \geq 0$, $\langle v(t,\cdot), \varphi_1^h \rangle = 0$.

We now prove that the functions

$$J_1(t) = \int_0^1 |v(t,x)|^2 dx \quad \text{and} \quad J_2(t) = \int_0^1 |\partial_x v(t,x)|^2 dx,$$

are non-negative, decreasing, and converge to 0 as $t \to +\infty$. Note that

$$\begin{aligned}
\partial_t J_1(t) &= \int_0^1 2v(t,x) \partial_t v(t,x) dx \\
&= \int_0^1 2v(t,x) \left[-\frac{h^{2/3}}{2^{1/3}} \lambda_1^h e^{-\frac{h^{2/3}}{2^{1/3}}\lambda_1^h t} u(t,x) + e^{-\frac{h^{2/3}}{2^{1/3}}\lambda_1^h t} \partial_t u(t,x) \right] dx \\
&= \int_0^1 2v(t,x) \left[-\frac{h^{2/3}}{2^{1/3}} \lambda_1^h v(t,x) + \mathcal{G}^h v(t,x) \right] dx \\
&= 2 \left[-\frac{h^{2/3}}{2^{1/3}} \lambda_1^h \langle v(t,\cdot), v(t,\cdot) \rangle + \langle v(t,\cdot), \mathcal{G}^h v(t,\cdot) \rangle \right],
\end{aligned}$$

and as $\langle v(t,\cdot), \varphi_1^h \rangle = 0$, we have

$$\partial_t J_1(t) \leq (2h)^{2/3} (\lambda_2^h - \lambda_1^h) J_1(t).$$

Therefore, $J_1(t)$ decreases to 0 as $t \to +\infty$. Similarly,

$$\begin{aligned}
\partial_t J_2(t) &= \int_0^1 2\partial_x v(t,x) \partial_t \partial_x v(t,x) dx \\
&= 2 \int_0^1 \partial_x v(t,x) \partial_x \partial_t v(t,x) dx \\
&= 2 \int_0^1 \partial_x v(t,x) \left[-\frac{h^{2/3}}{2^{1/3}} \lambda_1^h \partial_x v(t,x) + \partial_x \mathcal{G}^h v(t,x) \right] dx \\
&= -\frac{h^{2/3}}{2^{1/3}} \lambda_1^h \langle \partial_x v(t,\cdot), \partial_x v(t,\cdot) \rangle + 2 \langle \partial_x v(t,\cdot), \mathcal{G}^h \partial_x v(t,\cdot) \rangle - 2h \underbrace{\int_0^1 v(t,x) \partial_x v(t,x) dx}_{v(t,1)^2 - v(t,0)^2} \\
&\leq 0.
\end{aligned}$$

As a consequence, J_2 is decreasing and non-negative, is therefore convergent, as $t \to +\infty$ to $J_2(+\infty)$. In addition, we can bound the derivative of J_1 as follows

$$\begin{aligned}
\partial_t J_1(t) &= 2\left[-\frac{h^{2/3}}{2^{1/3}}\lambda_1^h \langle v(t,\cdot), v(t,\cdot)\rangle + \langle v(t,\cdot), \mathcal{G}^h v(t,\cdot)\rangle\right] \\
&= -(2h)^{2/3}\lambda_1^h J_1(t) + \int_0^1 v(t,x)\partial_x^2 v(t,x)dx - \int_0^1 hxv(t,x)^2 dx \\
&\le -(2h)^{2/3}\lambda_1^h J_1(t) - \int_0^1 (\partial_x v(t,x))^2\, dx \\
&\le -J_2(t) - (2h)^{2/3}\lambda_1^h J_1(t).
\end{aligned}$$

As $J_1(t) \to 0$, if $J_2(t) \to J_2(+\infty) > 0$, then the derivative of J_1 stays negative and bounded away from 0, which would indicate that J_1 decreases to $-\infty$, contradicting the fact that $J_1 \ge 0$. We conclude that $\lim_{t\to+\infty} J_2(t) = 0$.

Finally, by Cauchy-Schwarz inequality, for all $x \in [0,1]$, we have

$$|v(t,x)| \le \int_0^x |\partial_x v(t,x)|dx \le x^{1/2}\left(\int_0^x |\partial_x v(t,x)|^2 dx\right)^{1/2} \le J_2(t),$$

so $\lim_{t\to+\infty} \|v(t,\cdot)\|_\infty = 0$, which proves (3.A.4). □

This lemma can be easily extended to authorize more general starting function u_0.

Corollary 3.A.6. *Let $h > 0$ and u_0 be a measurable bounded function. Setting, for $x \ge 0$ and $t \ge 0$*

$$u(t,x) = \mathbf{E}_x\left[u_0(B_t)e^{-h\int_0^t B_s ds}; B_s \in [0,1], s \in [0,t]\right],$$

we have

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} \left| e^{-\frac{h^{2/3}}{2^{1/3}}\lambda_1^h t} u(t,x) - \langle u_0, \varphi_1^h\rangle \varphi_1^h(x)\right| = 0. \tag{3.A.19}$$

Proof. Let u_0 be a measurable bounded function. We define, for $x \ge 0$ and $\varepsilon > 0$

$$u_\varepsilon(x) = u(\varepsilon, x) = \mathbf{E}_x\left[u_0(B_\varepsilon)e^{-h\int_0^\varepsilon B_s ds}; B_s \in [0,1], s \in [0,1]\right].$$

Observe that by the Markov property, for all $t \ge \varepsilon$, we have

$$\begin{aligned}
u(t,x) &= \mathbf{E}_x\left[u_0(B_t)e^{-h\int_0^t B_s ds}; B_s \in [0,1], s \in [0,t]\right] \\
&= \mathbf{E}_x\left[u_\varepsilon(B_{t-\varepsilon})e^{-h\int_0^{t-\varepsilon} B_s ds}; B_s \in [0,1], s \in [0,t-\varepsilon]\right].
\end{aligned}$$

Therefore, $u(t,x) = u_\varepsilon(t-\varepsilon, x)$, setting $u_\varepsilon(t,x) = \mathbf{E}_x\left[u_\varepsilon(B_t)e^{-h\int_0^t B_s ds}; s \in [0,t]\right]$.

As $h > 0$, for all $\varepsilon > 0$ and $x \in [0,1]$, we have

$$\begin{aligned}
|u_\varepsilon(x)| &\le \|u_0\|_\infty \mathbf{P}_x\left[B_s \in [0,1], s \in [0,\varepsilon]\right] \\
&\le C\max\left(\mathbf{P}_x\left(B_s \ge 0, s \in [0,\varepsilon]\right), \mathbf{P}_x\left(B_s \le 1, s \in [0,\varepsilon]\right)\right) \\
&\le C\varepsilon^{-1/2}\max(x, 1-x).
\end{aligned}$$

As a consequence, we can find sequences $(v^{(n)})$ and $(w^{(n)})$ of functions in $\mathcal{C}_0^2$ such that for all $x \in \mathbb{R}$

$$w^{(n)} \leq u_\varepsilon \leq w^{(n)} + \frac{1}{n} \quad \text{and} \quad v^{(n)} - \frac{1}{n} \leq u_\varepsilon \leq v^{(n)}.$$

For $n \in \mathbb{N}$, $x \geq 0$ and $t \geq 0$, we denote by

$$v^{(n)}(t,x) = \mathbf{E}_x \left[v^{(n)}(x) e^{-\int_0^t B_s ds}; B_s \in [0,1], s \in [0,t] \right]$$

$$\text{and by} \quad w^{(n)}(t,x) = \mathbf{E}_x \left[w^{(n)}(x) e^{-\int_0^t B_s ds}; B_s \in [0,1], s \in [0,t] \right].$$

For all $x \geq 0$ and $t \geq 0$ we have $w^{(n)}(t,x) \leq u_\varepsilon(t,x) \leq v^{(n)}(t,x)$. Moreover Lemma 3.A.5 gives

$$\lim_{t \to +\infty} \sup_{x \in \mathbb{R}} \left| e^{-\frac{h^{2/3}}{2^{1/3}} \lambda_1^h t} v^{(n)}(t,x) - \langle v^{(n)}, \varphi_1^h \rangle \varphi_1^h(x) \right| = 0.$$

$$\lim_{t \to +\infty} \sup_{x \in \mathbb{R}} \left| e^{-\frac{h^{2/3}}{2^{1/3}} \lambda_1^h t} w^{(n)}(t,x) - \langle w^{(n)}, \varphi_1^h \rangle \varphi_1^h(x) \right| = 0.$$

By the dominated convergence theorem, we have

$$\lim_{n \to +\infty} \langle w^{(n)}, \varphi_1^h \rangle = \lim_{n \to +\infty} \langle v^{(n)}, \varphi_1^h \rangle = \langle u_\varepsilon, \varphi_1^h \rangle,$$

which yields

$$\lim_{t \to +\infty} \sup_{x \in \mathbb{R}} \left| e^{-\frac{h^{2/3}}{2^{1/3}} \lambda_1^h t} u_\varepsilon(t,x) - \langle u_\varepsilon, \varphi_1^h \rangle \varphi_1^h(x) \right| = 0.$$

For all $x \geq 0$, we have $u_\varepsilon(x) \to u_0(x)$ as $\varepsilon \to 0$. By dominated convergence theorem again

$$\lim_{\varepsilon \to 0} \langle u_\varepsilon, \varphi_1^h \rangle = \langle u_0, \varphi_1^h \rangle,$$

as $u(t,x) = u_\varepsilon(t - \varepsilon, x)$, we conclude that

$$\lim_{t \to +\infty} \sup_{x \in \mathbb{R}} \left| e^{-\frac{h^{2/3}}{2^{1/3}} \lambda_1^h t} u(t,x) - \langle u_0, \varphi_1^h \rangle \varphi_1^h(x) \right| = 0.$$

□

Proof of Lemma 3.2.4. We begin with the case $h > 0$. We set

$$u(t,x) = \mathbf{E}_x \left[e^{-h \int_0^t B_s ds}; B_s \in [0,1], s \in [0,t] \right]$$

and $\mu_h = \int_0^1 \varphi_1^h(x) dx < +\infty$. By Corollary 3.A.3, we have

$$\lim_{t \to +\infty} \sup_{x \geq 0} \left| e^{-\frac{h^{2/3}}{2^{1/3}} \lambda_1^h t} u(t,x) - \mu_h \varphi_1^h(x) \right| = 0.$$

As φ_1^h is bounded,

$$\limsup_{t \to +\infty} \sup_{x \geq 0} \frac{1}{t} \log u(t,x) = 2^{-1/3} h^{2/3} \lambda_1^h. \tag{3.A.20}$$

Similarly, for $0 < a < b < 1$ and $0 < a' < b' < 1$, we set

$$\widetilde{u}(t,x) = \mathbf{E}_x\left[\mathbf{1}_{\{B_t \in [a',b']\}} e^{-h\int_0^t B_s ds}; B_s \in [0,1], s \in [0,t]\right],$$

and $\widetilde{\mu}_h = \int_{a'}^{b'} \varphi_1^h(x)dx > 0$. Corollary 3.A.3 implies that

$$\lim_{t\to+\infty} \sup_{x\geq 0} \left| e^{-\frac{h^{2/3}}{2^{1/3}}\lambda_1^h t}\widetilde{u}(t,x) - \widetilde{\mu}_h \varphi_1^h(x)\right| = 0.$$

In particular, as $\inf_{x\in[a,b]} \varphi_1^h > 0$, we have

$$\liminf_{t\to+\infty} \inf_{x\in[a,b]} \frac{1}{t} \log \widetilde{u}(t,x) = \frac{h^{2/3}}{2^{1/3}}\lambda_1^h. \tag{3.A.21}$$

Using the fact that $\widetilde{u} \leq u$, (3.A.20) and (3.A.21) lead to

$$\lim_{t\to+\infty} \sup_{x\geq 0} \frac{1}{t} \log u(t,x) = \lim_{t\to+\infty} \inf_{x\in[a,b]} \frac{1}{t} \log \widetilde{u}(t,x) = \frac{h^{2/3}}{2^{1/3}}\lambda_1^h.$$

Moreover, by definition of Ψ, for all $h > 0$ we have $\Psi(h) = \frac{h^{2/3}}{2^{1/3}}\lambda_1^h$, and (3.2.10) is a consequence of the definition of λ_1^h. Ussing this alternative definition and the implicit function theorem, we observe immediately that Ψ is differentiable on $(0,+\infty)$. We compute the asymptotic of Ψ as h increases to $+\infty$ and we get

$$\frac{\Psi(h)}{h^{2/3}} = 2^{-1/3}\lambda_1^h = 2^{-1/3} \sup\left\{x \in \mathbb{R} : \mathrm{Bi}\,(\lambda)\,\mathrm{Ai}\left(\lambda + (2h)^{1/3}\right) = \mathrm{Ai}\,(\lambda)\,\mathrm{Bi}\left(\lambda + (2h)^{1/3}\right)\right\}.$$

We recall that $\lambda_1^h < 0$, we have $\mathrm{Ai}(\lambda_1^h) = \frac{\mathrm{Bi}(\lambda)\mathrm{Ai}\left(\lambda+(2h)^{1/3}\right)}{\mathrm{Bi}\left(\lambda+(2h)^{1/3}\right)}$ and $\log \mathrm{Bi}(x) \sim_{x\to+\infty} x^{2/3}$. Consequently,

$$\lim_{h\to+\infty} \sup_{\lambda\geq\alpha_2} \frac{\mathrm{Bi}\,(\lambda)\,\mathrm{Ai}\left(\lambda+(2h)^{1/3}\right)}{\mathrm{Bi}\,(\lambda+(2h)^{1/3})} = 0.$$

For all $\varepsilon > 0$, there exists $\delta > 0$ small enough such that

$$\left\{x \in \left[\tfrac{\alpha_1+\alpha_2}{2}, 0\right] : |\mathrm{Ai}(x)| \leq \delta\right\} \subset [\alpha_1 - \varepsilon, \alpha_1 + \varepsilon].$$

Moreover, there exists $H > 0$ such that $\sup_{h\geq H,\lambda\geq\alpha_2}\left|\frac{\mathrm{Bi}(\lambda)\mathrm{Ai}\left(\lambda+(2h)^{1/3}\right)}{\mathrm{Bi}\left(\lambda+(2h)^{1/3}\right)}\right| \leq \delta$. As a result, for all $h \geq H$, $\left|\lambda_1^h - \alpha_1\right| \leq \varepsilon$, which is enough to conclude that $\lim_{h\to+\infty} \frac{\Psi(h)}{h^{2/3}} = 2^{-1/3}\alpha_1$.

We now observe that if $h < 0$, then

$$\begin{aligned}
&\mathbf{E}_x\left[u(B_t)e^{-h\int_0^t B_s ds}; B_s \in [0,1], s \in [0,t]\right]\\
&= \mathbf{E}_x\left[u(B_t)e^{-h\int_0^t (B_s-1+1)ds}; B_s - 1 \in [-1,0], s \in [0,t]\right]\\
&= e^{-ht}\,\mathbf{E}_{x-1}\left[u(B_t+1)e^{-h\int_0^t B_s ds}; B_s \in [-1,0], s \in [0,t]\right]\\
&= e^{-ht}\,\mathbf{E}_{1-x}\left[u(1-B_t)e^{h\int_0^t B_s ds}; B_s \in [0,1], s \in [0,t]\right].
\end{aligned}$$

For all $0 < a < b < 1$ and $0 < a' < b' < 1$, we have

$$\lim_{t\to+\infty} \sup_{x\in[0,1]} \frac{1}{t} \log \mathbf{E}_x \left[e^{-h\int_0^t B_s ds}; B_s \in [0,1], s \in [0,t] \right]$$
$$= \lim_{t\to+\infty} \inf_{x\in[a,b]} \frac{1}{t} \log \mathbf{E}_x \left[\mathbf{1}_{\{B_t\in[a',b']\}} e^{-h\int_0^t B_s ds}; B_s \in [0,1], s \in [0,t] \right] = -h + \Psi(-h).$$

We also conclude that for $h < 0$, $\Psi(h) = \Psi(-h) - h$, and in particular, Ψ is differentiable on $(-\infty, 0)$. Moreover, at point 0, its right derivative and its left derivative do not match, $\Psi'_l(0) = \Psi'_r(0) + 1$.

Finally, we take care of the case $h = 0$. By [IM74],

$$\begin{aligned}
\mathbf{E}_x \left[\mathbf{1}_{\{B_t\in[a,b]\}} e^{-0\int_0^t B_s ds}; B_s \in [0,1], s \in [0,t] \right] \\
&= \mathbf{P}_x \left[B_t \in [a,b], B_s \in [0,1], s \in [0,t] \right] \\
&= \int_a^b 2 \sum_{n=1}^{+\infty} e^{-n^2 \frac{\pi^2}{2} t} \sin(n\pi x) \sin(n\pi z) dz \\
&= 2 \sum_{n=1}^{+\infty} \sin(n\pi x) \frac{\cos(n\pi a) - \cos(n\pi b)}{n\pi} e^{-n^2 \frac{\pi^2}{2} t}.
\end{aligned}$$

As a consequence,

$$\lim_{t\to+\infty} \sup_{x\in[0,1]} \frac{1}{t} \log \mathbf{P}_x \left[B_s \in [0,1], s \in [0,t] \right]$$
$$= \lim_{t\to+\infty} \inf_{x\in[a,b]} \frac{1}{t} \log \mathbf{P}_x \left[B_t \in [a',b'], B_s \in [0,1], s \in [0,t] \right] = \Psi(0) = -\frac{\pi^2}{2}.$$

□

3.B Notation

- *Point processes*
  - $\mathcal{L}_t$: law of a point process;
  - L_t: point process with law $\mathcal{L}_t$;
  - κ_t: log-Laplace transform of $\mathcal{L}_t$;
  - κ_t^*: Fenchel-Legendre transform of $\mathcal{L}_t$.
- *Paths*
  - $\mathcal{C}$: set of continuous functions on $[0,1]$;
  - $\mathcal{D}$: set of càdlàg functions on $[0,1]$, continuous at point 1;
  - $\overline{b}^{(n)}_k = \sum_{j=1}^k b_{j/n}$: path of speed profile $b \in \mathcal{D}$;
  - $K^*(b)_t = \int_0^t \kappa_s^*(b_s) ds$: energy associated to the path of speed profile b;
  - $\varphi_t = \partial_a \kappa_t^*(b_t)$: parameter associated to the path of speed profile b;
  - $E(\varphi)_t = \int_0^t \varphi_s \partial_\theta \kappa_s(\varphi_s) - \kappa_s(\varphi_s) ds$: quantity equal to $K^*(b)_t$, energy associated to the path of parameter function φ;
  - $\mathcal{R} = \{b \in \mathcal{D} : \forall t \in [0,1], K^*(b)_t \leq 0\}$: set of speed profiles b such that $\overline{b}^{(n)}$ is followed until time n by at least one individual with positive probability.
- *Branching random walk*
  - $\mathbf{T}$: genealogical tree of the process;

- $u \in \mathbf{T}$: individual in the process;
- $V(u)$: position of the individual u;
- $|u|$: generation at which u belongs;
- u_k: ancestor of u at generation k;
- $\emptyset$: initial ancestor of the process;
- if $u \neq \emptyset$, πu: parent of u;
- $\Omega(u)$: set of children of u;
- $L^u = (V(v) - V(u), v \in \Omega(u))$: point process of the displacement of the children;
- $M_n = \max_{|u|=n} V(u)$ maximal displacement at the n^{th} generation in $(\mathbf{T}, V)$;
- $\Lambda_n = \min_{|u|=n} \max_{k \leq n} \overline{b}^{(n)}_k - V(u_k)$: consistent maximal displacement with respect to the path $\overline{b}^{(n)}$;
- $\mathcal{W}^{\varphi}_n = \{u \in \mathbf{T} : \forall k \in F_n, V(u_k) \geq \overline{b}^{(n)}_k + f_{k/n} n^{1/3}\}$: tree of a BRWls with selection above the curve $\overline{b}^{(n)}_{.} + n^{1/3} f_{./n}$ at times in F_n.

- *The optimal path*
  - $v^* = \sup_{b \in \mathcal{R}} \int_0^1 b_s ds$: speed of the BRWls;
  - $a \in \mathcal{R}$ such that $\int_0^1 a_s ds = v^*$: optimal speed profile;
  - $\theta_t = \partial_a \kappa^*_s(a_s)$: parameter of the optimal path;
  - $\sigma^2_t = \partial^2_\theta \kappa_t(\theta_t)$: variance of individuals following the optimal path;
  - $\dot{\theta}$: Radon-Nikodým derivative of $d\theta$ with respect to the Lebesgue measure;
  - $l^* = \alpha_1 2^{-1/3} \int_0^1 \frac{(\dot{\theta}_s \sigma_s)^{2/3}}{\theta_s} ds$: $n^{1/3}$ correction of the maximal displacement;
  - $v_t = \inf_{\theta > 0} \frac{\kappa_t(\theta)}{\theta}$: natural speed profile.
- *Airy functions*
  - $\mathrm{Ai}(x) = \frac{1}{\pi} \int_0^{+\infty} \cos\left(\frac{s^3}{3} + xs\right) ds$: Airy function of the first kind;
  - $\mathrm{Bi}(x) = \frac{1}{\pi} \int_0^{+\infty} \exp\left(-\frac{s^3}{3} + xs\right) + \sin\left(\frac{s^3}{3} + xs\right) ds$: Airy function of the second kind;
  - (α_n): zeros of Ai, listed in the decreasing order;
  - $\Psi(h) = \lim_{t \to +\infty} \frac{1}{t} \log \sup_{x \in [0,1]} \mathbf{E}_x \left[e^{-h \int_0^t B_s ds}; B_s \in [0,1], s \in [0,t] \right]$.
- *Random walk estimates*
  - $(X_{n,k}, n \in \mathbb{N}, k \leq n)$: array of independent random variables;
  - $S^{(n)}_k = x + \sum_{j=1}^k X_{n,j}$: time-inhomogeneous random walk starting from x;
  - Given $f, g \in \mathcal{C}$, and $0 \leq j \leq n$,

$$I_n(j) = \begin{cases} \left[f_{j/n} n^{1/3}, g_{j/n} n^{1/3}\right] & \text{if } j \in F_n \cap G_n, \\ \left[f_{j/n}, +\infty\right[& \text{if } j \in F_n \cap G_n^c, \\ \left]-\infty, g_{j/n} n^{1/3}\right] & \text{if } j \in F_n^c \cap G_n, \\ \mathbb{R} & \text{otherwise;} \end{cases}$$

  - $\widetilde{I}^{(n)}_j = I^{(n)}_j \cap [-n^{2/3}, n^{2/3}]$.
- *Many-to-one lemma*
  - $\mathbf{P}_{k,x}$: law of the BRWls of length $n-k$ with environment $(\mathcal{L}_{(k+j)/n}, j \leq n-k)$;
  - $\mathcal{F}_k = \sigma(u, V(u), |u| \leq k)$: filtration of the branching random walk;
  - Many-to-one lemma: Lemma 3.3.1.
- *Random walk estimates*
  - $\mathcal{A}^{F,G}_n(f,g)$: individuals staying in the path $\overline{b}^{(n)} + I^{(n)}$ until some time then exiting by the upper boundary;

- $\mathcal{B}_n^{F,G}(f,g,x)$: individuals that stayed in the path $\overline{b}^{(n)} + I^{(n)}$ at any time $k \leq n$, that are above $\overline{b}_n^{(n)} + (g_1 - x)n^{1/3}$ at time n.

CHAPTER 4

The maximal displacement of a branching random walk in random environment

"Un homme tirait au sort toutes ses décisions.
Il ne lui arriva pas plus de mal qu'aux autres qui réfléchissent."

Paul Valéry – Choses tues.

Abstract

The behaviour of the tip of supercritical branching random walk (BRW) has been a subject of intense studies for a long time. But only recently, starting with the work of [FZ12a], the case of time-inhomogeneous branching has gained focus. The main contribution of this chapter is to analyse a model in which the branching law at the n^{th} generation is a random variable $\mathcal{L}_n$ sampled independently from a distribution of point measures (representing the displacement of the children). We present an asymptotic of the maximal displacement at time n up to a logarithmic term. The first, ballistic, order is established for a much more general class of models. As a tool, we derive a result of independent interest concerning the probability for a Brownian motion to stay above another Brownian motion path.

NOTA: This chapter is based on a joint work in collaboration with Piotr Miłos [1], currently unpublished.

4.1 Introduction

We recall that a branching random walk in random environment on $\mathbb{R}$ is a model defined as follows. Let $(\mathcal{L}_n, n \in \mathbb{N})$ be a sequence of point processes laws [2] that we call the environment of the branching random walk. It starts with one individual located at the origin at time 0. This individual dies at time 1 giving birth to children, positioned according to a realisation a point process of law $\mathcal{L}_1$. Similarly, at each time n every individual alive at generation $n-1$ dies giving birth to children. The position of the children with respect to their parent are given by an independent realisation of a point process with law $\mathcal{L}_n$. We denote by $\mathbf{T}$ the (random) genealogical tree of the process.

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2. I.e., probability distributions on $\bigcup_{k\in\overline{\mathbb{N}}} \mathbb{R}^k$.

For a given individual $u \in \mathbf{T}$ we write $V(u) \in \mathbb{R}$ for the position of u and $|u|$ for the generation at which u is alive. The couple $(\mathbf{T}, V)$ is called the branching random walk in time-inhomogeneous environment.

We assume the time-inhomogeneous Galton-Watson tree to be *supercritical* (i.e. the number of individuals alive at generation n grows exponentially fast), that does not become extinct (we assume that the number of offspring is always at least 1). We take interest in the maximal displacement at time n of $(\mathbf{T}, V)$, defined by

$$M_n = \max_{u \in \mathbf{T}:|u|=n} V(u). \tag{4.1.1}$$

In this chapter, the environment of the branching random walk is sampled randomly. More precisely, we set $(\mathcal{L}_n, n \in \mathbb{N})$ a sequence of i.i.d. random laws of point processes. A *branching random walk in random environment* (BRWre) is a branching random walk with the time-inhomogeneous environment $(\mathcal{L}_n, n \in \mathbb{N})$. Conditionally on this sequence, we write $\mathbb{P}_{\mathcal{L}}$ for the law of this BRWre $(\mathbf{T}, V)$ and $\mathrm{E}_{\mathcal{L}}$ for the corresponding expectation. The joint probability of the environment and the branching random walk is written $\mathbf{P}$, with the corresponding expectation $\mathbf{E}$. For the clarity of exposition, we present the most important results deferring to the subsequent sections number of generalisations which require additional notation.

Some notation. Given a sequence $(x_n) \in \mathbb{R}^{\mathbb{N}}$, we recall that $O_{\mathbf{P}}(x_n)$ is a sequence of random variables $(X_n, n \in \mathbb{N})$ verifying

$$\forall \varepsilon > 0, \exists K > 0 : \sup_{n \in \mathbb{N}} \mathbf{P}\left(X_n \geq K x_n\right) \leq \varepsilon.$$

Similarly, $o_{\mathbf{P}}(x_n)$ denotes a generic sequence of random variables $(X_n, n \in \mathbb{N})$ such that $\frac{X_n}{x_n} \to 0$ in $\mathbf{P}$-probability. Moreover C and c stand for two positive constants respectively large enough and small enough, that may change from line to line.

To ensure the non-extinction of the BRWre $(\mathbf{T}, V)$, we assume that

$$\mathbb{P}_{\mathcal{L}}\left(\{u \in \mathbf{T} : |u| = 1\} = \emptyset\right) = 0 \quad \text{a.s.} \tag{4.1.2}$$

and using [BM08], a sufficient condition for the random tree $\mathbf{T}$ to be supercritical is

$$\exists C > 0 : \mathrm{E}_{\mathcal{L}}\left(\#\left\{u \in \mathbf{T} : |u| = 1\right\}^2\right) \leq C \text{ a.s. and } \mathbf{E}\left(\log \mathrm{E}_{\mathcal{L}}(\#\{|u| = 1\})\right) > 0. \tag{4.1.3}$$

Under (4.1.2) and (4.1.3), by [BM08, Theorem 1.1] we have

$$\liminf_{n \to +\infty} \left(\#\{u \in \mathbf{T} : |u| = n\}\right)^{1/n} \geq \exp\left[\mathbf{E}\left[\log \mathrm{E}_{\mathcal{L}}(\#\{|u| = 1\})\right]\right] \quad \mathbb{P}_{\mathcal{L}} - \text{a.s.} \tag{4.1.4}$$

For $n \in \mathbb{N}$, we introduce the log-Laplace transform $\kappa_n : \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$ of $\mathcal{L}_n$, defined for $\theta \in \mathbb{R}_+$ by

$$\kappa_n(\theta) = \log \mathrm{E}_{\mathcal{L}}\left(\sum_{\ell \in L_n} e^{\theta \ell}\right), \tag{4.1.5}$$

where L_n is a point process on $\mathbb{R}$ distributed according to the law $\mathcal{L}_n$. As the point process L_n is a.s. non-empty we have $\kappa_n(\theta) > -\infty$ a.s. For a fixed $\theta > 0$, $(\kappa_n(\theta), n \in \mathbb{N})$ is an i.i.d. sequence of random variables under law $\mathbf{P}$. We assume that $\mathbf{E}\left(\kappa_1(\theta)_-\right) < +\infty$ for all $\theta > 0$, and we define $\kappa : (0, +\infty) \to \mathbb{R} \cup \{+\infty\}$ by $\kappa(\theta) = \mathbf{E}\left(\kappa_n(\theta)\right)$.

As κ_1 is the log-Laplace transform of a measure on $\mathbb{R}$, κ is a convex function, that is $\mathcal{C}^\infty$ on the interior of the interval $\{\theta > 0 : \kappa(\theta) < +\infty\}$. Assuming this interval is non-empty, we define

$$v = \inf_{\theta>0} \frac{\kappa(\theta)}{\theta}, \tag{4.1.6}$$

and we assume there exists $\theta^* > 0$ such that κ is differentiable at point θ^* (implying that κ is finite in a neighbourhood of θ^*) and

$$\theta^* \kappa'(\theta^*) - \kappa(\theta^*) = 0. \tag{4.1.7}$$

Under this assumption, by dominated convergence, we have $v = \kappa'(\theta^*) = \mathbf{E}\left[\kappa_1'(\theta^*)\right]$. We introduce two variance terms, that we assume to be finite

$$\sigma_Q^2 := (\theta^*)^2 \, \mathbf{E}\left[\kappa_1''(\theta^*)\right] \in (0, +\infty) \quad \text{and} \quad \sigma_A^2 := \mathbf{V}\mathrm{ar}\left[\theta^* \kappa_1'(\theta^*) - \kappa_1(\theta^*)\right] \in [0, +\infty). \tag{4.1.8}$$

To compute the asymptotic behaviour of M_n we introduce two additional integrability assumptions:

$$\mathbf{E}\left[\mathrm{E}_{\mathcal{L}}\left[\left(\sum_{\ell \in L_1} \left(1 + e^{\theta^* \ell}\right)\right)^2\right]\right] < +\infty, \tag{4.1.9}$$

as well as

$$\exists \mu > 0 : \mu \mathrm{E}_{\mathcal{L}}\left[\sum_{\ell \in L_1} |\ell|^3 \left(e^{(\theta^*+\mu)\ell} + e^{(\theta^*-\mu)\ell}\right)\right] \leq \mathrm{E}_{\mathcal{L}}\left[\sum_{\ell \in L_1} \ell^2 e^{\theta^* \ell}\right] \quad \text{a.s.} \tag{4.1.10}$$

When the reproduction law does not depend on the time, under mild integrability assumptions, Hu and Shi [HS09] and Addario-Berry and Reed [ABR09] independently proved the existence of a second order logarithmic correction to first order behaviour of M_n. Our aim is to extend the scope of this result to the random environment setting. We introduce $K_n = \sum_{k=1}^n \kappa_k(\theta^*)$. This quantity depends only of the environment. Being a random walk, we have $K_n = nv + O_{\mathbf{P}}(n^{1/2})$. The main result of this article is

Theorem 4.1.1. *Under the assumptions* (4.1.2)*,* (4.1.3)*,* (4.1.7)*,* (4.1.8)*,* (4.1.9) *and* (4.1.10)*, there exists $\varphi \geq \frac{3}{2\theta^*}$, defined in* (4.1.11) *below, such that*

$$\lim_{n\to+\infty} \mathbb{P}_{\mathcal{L}}\left[M_n - \tfrac{1}{\theta^*} K_n \geq -\beta \log n\right] = \begin{cases} 1 & \text{if } \beta > \varphi \\ 0 & \text{if } \beta < \varphi \end{cases} \text{ in probability.}$$

A direct consequence of this theorem is the asymptotic behaviour of M_n under law $\mathbf{P}$.

Corollary 4.1.2. *Under the assumptions of Theorem 4.1.1, we have*

$$\lim_{n\to+\infty} \frac{M_n - \frac{1}{\theta^*} K_n}{\log n} = -\varphi \text{ in } \mathbf{P} - \text{probability.}$$

Most likely, the convergence cannot be strengthen to $M_n = \frac{1}{\theta^*} K_n - \varphi \log n + o_{\mathbb{P}_{\mathcal{L}}}(\log n)$ $\mathbf{P}$-a.s. We expect the median of M_n under law $\mathbb{P}$ to exhibit some non-trivial $\log n$-scale fluctuations. This fact is discussed in more details in Section 4.3. We also expect that as soon as $\sigma_A^2 > 0$, the random environment of the branching random walk slows down the process. In other words, the constant φ is expected to be strictly greater than $\frac{3}{2\theta^*}$ the logarithmic correction of the time-homogeneous branching random walk.

It is well-known, see for example [AS10], that the constant $\frac{3}{2}$ of the branching random walk is directly related to the $\frac{3}{2}$ exponent of the ballot theorem problem[3] i.e., for any centred random walk (S_n) with finite variance, there exists $A > 0$ and $C > 0$ such that

$$\mathbf{P}(S_n \leq A, S_j \geq -1, j \leq n) \sim Cn^{-3/2}.$$

In analysis of our model, studying a random environment version of ballot theorem comes out naturally. We believe that this result is of independent interest. We state here a toy-model version of the main results (Theorem 4.2.3 for the Brownian motion and Theorem 4.3.1 for the random walk in random environment) which involve additional notation.

Theorem 4.1.3. *Let B and W be two independent Brownian motions. There exists an even convex function $\gamma : \mathbb{R} \mapsto \mathbb{R}_+$ such that for any $\beta \in \mathbb{R}$ we have*

$$\lim_{t \to +\infty} \frac{\log \mathbf{P}\left(B_s \geq \beta W_s - 1, s \leq t | W\right)}{\log t} = -\gamma(\beta) \ a.s.$$

Moreover, $\inf_{\beta \in \mathbb{R}} \gamma(\beta) = \gamma(0) = \frac{1}{2}$.

The case $\beta = 0$ reduces to the classical ballot theorem. The constant φ from Theorem 4.1.1 can be expressed using the function γ. We write

$$\varphi = \frac{2}{\theta^*}\gamma\left(\tfrac{\sigma_A}{\sigma_Q}\right) + \frac{1}{2\theta^*}. \tag{4.1.11}$$

In the case of a classical branching random walk, we have $\sigma_A^2 = 0$ thus $\varphi = \frac{3}{2\theta^*}$, which is consistent with the results of Hu and Shi, and Addario-Berry and Reed.

The rest of the paper is organised as follows. The next section is devoted to the proof of Theorem 4.1.3, based on Kingman's subadditive ergodic theorem. Section 4.3 extends the random ballot theorem to random walks in random environment, using the Sakhanenko exponential inequality, that links a sum of independent random variables with a Brownian motion. We use this result to prove Theorem 4.1.1 in Section 4.4, applying a classical tool of branching processes theory: the many-to-one lemma. It links the computation of additive moments of the branching random walk with random walk estimates.

4.2 Asymptotic behaviour of the probability for a Brownian motion to stay above another Brownian motion

Let B and W be two Brownian motions. We study in this section the asymptotic behaviour as $t \to +\infty$ of the probability for B to stay at any time above βW, conditionally on the Brownian path W. We first prove the convergence in Theorem 4.1.3, that the probability in question almost surely behaves as $t^{-\gamma(\beta)+o(1)}$. Secondly, taking this result as input, we prove that it holds for a class of perturbed Brownian motions. This is used in the next section to extend this random ballot theorem to random walks in random environment. This section is concluded establishing some properties of γ.

To study these Brownian estimates, we use the Fortuin–Kasteleyn–Ginibre (FKG) inequality for the Brownian motion, that we define now. We set $f \preccurlyeq g$ if for all $0 \leq s_1 \leq s_2 \leq t$, $f(s_2) - f(s_1) \leq g(s_2) - g(s_1)$. We say that a subset Γ of the set of continuous functions of $[0, t]$ is an increasing event if for any pair of continuous functions (f, g),

$$f \preccurlyeq g \text{ and } f \in \Gamma \Rightarrow g \in \Gamma. \tag{4.2.1}$$

3. For review on ballot theorems, one can look at [ABR08].

The (strong) FKG inequality for the Brownian motion, proved in [Bar05], states that increasing events are positively correlated, i.e. if Γ and Γ' are two increasing events, then

$$\mathbf{P}\left((B_s, s \in [0,t]) \in \Gamma \cap \Gamma'\right) \geq \mathbf{P}\left((B_s, s \in [0,t]) \in \Gamma\right) \mathbf{P}\left((B_s, s \in [0,t]) \in \Gamma'\right). \tag{4.2.2}$$

Note that for continuous functions f that verify $f(0) = 0$, the order $\preccurlyeq$ is weaker than the order $\leq$ defined by $f \leq g$ if for all $s \leq t$, $f(s) \leq g(s)$. Therefore, if Γ verifies

$$f \leq g \text{ and } f \in \Gamma \Rightarrow g \in \Gamma, \tag{4.2.3}$$

then $\Gamma \cap \{f \in \mathcal{C} : f(0) = 0\}$ is an increasing event, and satisfies FKG inequality.

Lemma 4.2.1. *Let B and W be two independent Brownian motions, there exists a function $\gamma : \mathbb{R} \to \mathbb{R}_+$ such that for any $\beta \in \mathbb{R}$ and any $a \geq 0$, we have*

$$\lim_{t \to +\infty} \frac{\log \mathbf{P}\left(B_t \geq a t^{1/2} + \beta W_t, B_s + 1 \geq \beta W_s, 0 \leq s \leq t \middle| W\right)}{\log t} = -\gamma(\beta) \text{ a.s. and in } L^1.$$

Proof. We first note that if $\beta = 0$, then

$$\mathbf{P}\left(B_s + 1 \geq \beta W_s, 0 \leq s \leq t \middle| W\right) = \mathbf{P}\left(B_s \geq -1, s \leq t\right).$$

As $\lim_{t\to+\infty} \frac{1}{\log t} \log \mathbf{P}(B_s \geq -1, s \leq t) = -\frac{1}{2}$, $\gamma(0)$ exists and is equal to $\frac{1}{2}$. Moreover, by standard Brownian estimate, for any $a > 0$ we have

$$\lim_{t \to +\infty} \frac{1}{\log t} \log \mathbf{P}\left(B_t \geq a t^{1/2}, B_s \geq -1, 0 \leq s \leq t\right) = -\frac{1}{2} \text{ a.s. and in } L^1.$$

We now assume that $\beta \neq 0$. For $0 \leq s < t$, we introduce

$$p_{s,t} = \mathbf{P}\left(\begin{array}{l} B_{e^s+u} - B_{e^s} \geq \beta(W_{e^s+u} - W_{e^s}) + e^{s/2}, u \leq e^t - e^s \\ B_{e^t} - B_{e^s} \geq \beta(W_{e^t} - W_{e^s}) + e^{t/2} \end{array} \middle| W \right).$$

Note that

$$p_{0,\log t} \overset{(d)}{=} \mathbf{P}\left(B_{t-1} \geq \beta(W_t - W_1) + t^{1/2}, B_u + 1 \geq \beta(W_{u+1} - W_1), u \leq t-1 \middle| W\right),$$

a quantity closely related to the one we aim to bound.

For all $0 < s < t$, applying the Markov property at time $e^s - 1$ we have

$$p_{0,t} \geq p_{0,s} p_{s,t}. \tag{4.2.4}$$

Moreover, by the scaling property of the Brownian motion,

$$p_{s,t} \overset{(d)}{=} p_{0,t-s}, \tag{4.2.5}$$

and by independence of the increments of the Brownian motion, for all $0 \leq s < t$ and $0 \leq s' < t'$:

$$\text{if } [s,t] \cap [s',t'] = \emptyset, \text{ then } p_{s,t} \text{ and } p_{s',t'} \text{ are independent.} \tag{4.2.6}$$

We now prove that $\mathbf{E}(-\log p_{0,1}) < +\infty$. Writing $T = e - 1$, we observe that by the inversion property of the Brownian motion,

$$\begin{aligned} \mathbf{P}\left(\exists t \leq T : W_t \geq a t^{1/3}\right) &= \mathbf{P}\left(\exists t \leq T : W_{\frac{1}{t}} \geq a t^{-2/3}\right) = \mathbf{P}\left(\exists t \geq T^{-1} : W_t \geq a t^{2/3}\right) \\ &\leq \sum_{i \geq 1} \mathbf{P}\left(\exists t \in \left[\tfrac{i}{T}, \tfrac{i+1}{T}\right] : W_t \geq a i^{2/3}\right) \leq \sum_{i \geq 1} \exp\left[-C \frac{a^2 i^{4/3}}{i}\right] \\ &\leq c \exp\left(-C a^{\alpha}\right), \end{aligned}$$

for some $c, C, \alpha > 0$. In particular, writing $X = \sup_{t\in[0,T]} \frac{W_t}{t^{1/3}}$, we have $\mathbf{P}(X \geq a) \leq e^{-Ca^\alpha}$. Thus,

$$\begin{aligned}
&\mathbf{E}(-\log p_{0,1}) \\
&= \sum_{a=0}^{+\infty} \mathbf{E}\left[-\log p_{0,1} \mathbf{1}_{\{\beta X \in (a,a+1]\}}\right] \\
&\leq \sum_{a=0}^{+\infty} \mathbf{E}\left[-\log \mathbf{P}(B_T \geq (a+1)T^{1/3} + e^{T/2}, B_s \geq (a+1)s^{1/3} - 1, s \leq T) \mathbf{1}_{\{\beta X \in (a,a+1]\}}\right] \\
&\leq \sum_{a=0}^{+\infty} -\log \mathbf{P}\left(B_T \geq (a+1)T^{1/3} + e^{T/2}, B_s \geq (a+1)s^{1/3} - 1, s \leq T\right) \mathbf{P}(\beta X \geq a).
\end{aligned}$$

Moreover, there exists $c', C' > 0$ such that

$$\mathbf{P}\left(B_T \geq aT^{1/3} + e^{T/2}, B_s \geq as^{1/3} - 1, s \leq T\right) \geq c' e^{-C'a^5},$$

which leads to

$$\mathbf{E}(-\log p_{0,1}) \leq \sum_{a=0}^{+\infty} C' a^5 e^{-C\left(\frac{a}{\beta}\right)^\alpha} < +\infty. \tag{4.2.7}$$

As a result, by Kingman's subadditive ergodic theorem (see e.g. [Kal02, Theorem 10.22]), (4.2.4), (4.2.5), (4.2.6) and (4.2.7) imply there exists $\gamma(\beta) \in \mathbb{R}_+$ such that

$$\lim_{t\to+\infty} \frac{-\log p_{0,t}}{t} = \gamma(\beta) \text{ a.s. and in } L^1. \tag{4.2.8}$$

We use (4.2.8) to obtain the asymptotic behaviour, as $t \to +\infty$, of

$$\widetilde{p}_{0,t} = \mathbf{P}\left(B_{u+1} - B_1 + 1 \geq \beta(W_{u+1} - W_1), u \leq e^t - 1 \middle| W\right).$$

To do so, we apply the FKG inequality for the Brownian motion, we have

$$\widetilde{p}_{0,t} \times \mathbf{P}\left(B_{e^t} - B_1 \geq \beta(W_{e^t} - W_1) + e^{t/2} \middle| W\right) \leq p_{0,t} \leq \widetilde{p}_{0,t}. \tag{4.2.9}$$

Moreover, bounding the right tail of the Gaussian random variable we have

$$\lim_{t\to+\infty} \frac{\log \mathbf{P}\left(B_{e^t} - B_1 + 1 \geq \beta(W_{e^t} - W_1) + e^{t/2} \middle| W\right)}{t} = 0 \text{ a.s. and in } L^1. \tag{4.2.10}$$

As a consequence, (4.2.8) and (4.2.9) yield

$$\lim_{t\to+\infty} \frac{\log \widetilde{p}_{0,t}}{t} = -\gamma(\beta) \text{ a.s. and in } L^1. \tag{4.2.11}$$

Let $a > 0$, we use once again the FKG inequality to compute the asymptotic behaviour of

$$\widetilde{p}^a_{0,t} = \mathbf{P}\left(\begin{array}{c} B_{u+1} - B_1 + 1 \geq \beta(W_{u+1} - W_1), u \leq e^t - 1 \\ B_{e^t} - B_1 \geq ae^{t/2} + \beta(W_{e^t} - W_1) \end{array} \middle| W\right).$$

We have

$$\widetilde{p}^a_{0,t} \leq \widetilde{p}_{0,t} \leq \frac{\widetilde{p}^a_{0,t}}{\mathbf{P}\left(B_{e^t} - B_1 \geq \beta(W_{e^t} - W_1) + ae^{t/2} \middle| W\right)}.$$

By (4.2.10), we have $\log \mathbf{P}\left(B_{e^t} - B_1 \geq \beta(W_{e^t} - W_1) + ae^{t/2} \middle| W \right) / \log t \to 0$ a.s. therefore (4.2.11) yield

$$\lim_{t\to+\infty} \frac{\log \widetilde{p}^a_{0,t}}{t} = -\gamma(\beta) \text{ a.s. and in } L^1.$$

□

In a second time, we add an upper bound on the terminal value of $B_t - \beta W_t$ in the event.

Lemma 4.2.2. *Let B and W be two independent Brownian motions, for any $\beta > 0$ and $0 \leq a < b \leq +\infty$ we have*

$$\lim_{t\to+\infty} \frac{1}{\log t} \log \mathbf{P}\left(B_t - \beta W_t \in [at^{1/2}, bt^{1/2}], B_s + 1 \geq \beta W_s, 0 \leq s \leq t \middle| W \right) = -\gamma(\beta) \textit{ in probability.}$$

Proof. Let $\beta \in \mathbb{R}$. We first observe that for any $\lambda > 0$,

$$\mathbf{P}\left(B_t - \beta W_t \geq \lambda (t \log t)^{1/2} \middle| W \right) \leq \exp\left(-\frac{(\beta W_t + \lambda(t\log t)^{1/2})^2}{t} \right).$$

Consequently, we have

$$\limsup_{t\to+\infty} \frac{1}{\log t} \log \mathbf{P}\left(B_t - \beta W_t \geq \lambda (t\log t)^{1/2} \middle| W \right) \leq -\lambda^2 \quad \text{a.s.}$$

Therefore, for any $\lambda > \gamma(\beta)^2$, we have

$$\lim_{n\to+\infty} \frac{1}{\log t} \log \mathbf{P}\left(B_t - \beta W_t \in [at^{1/2}, \lambda(t\log t)^{1/2}], B_s + 1 \geq \beta W_s, 0 \leq s \leq t \middle| W \right) = -\gamma(\beta) \text{ a.s.}$$

Using the scaling probability of the Brownian motion and the fact that for any $\varepsilon > 0$,

$$\mathbf{P}\left(B_1 + 1 > \beta W_1, B_s + \varepsilon > \beta W_s \middle| W \right) > 0,$$

we conclude that for any $\lambda > 0$

$$\lim_{n\to+\infty} \frac{1}{\log t} \log \mathbf{P}\left(B_t - \beta W_t \in [at^{1/2}, \lambda(t\log t)^{1/2}], B_s + 1 \geq \beta W_s, 0 \leq s \leq t \middle| W \right) = -\gamma(\beta) \text{ a.s.}$$

We now use the fact that with high probability, $|W_{(1+\varepsilon)t} - W_t| \leq A(\varepsilon t)^{1/2}$, and that

$$\inf_{x\in[at^{1/2},\lambda(t\log t)^{1/2}]} \inf_{y\in[-A(\varepsilon t)^{1/2},A(\varepsilon t)^{1/2}]} \mathbf{P}(B_{\varepsilon t} + x \in y + [at^{1/2}, bt^{1/2}]) \geq \exp\left(\frac{\lambda^2 t \log t}{\varepsilon t} \right) (\log t)^{-1},$$

to obtain, choosing A large enough, $\varepsilon > 0$ small enough then $\lambda > 0$ small enough

$$\lim_{n\to+\infty} \frac{1}{\log t} \log \mathbf{P}\left(B_t - \beta W_t \in [at^{1/2}, bt^{1/2}], B_s + 1 \geq \beta W_s, 0 \leq s \leq t \middle| W \right) = -\gamma(\beta) \text{ in probability.}$$

□

The convergence studied in Lemma 4.2.1 turns out to be stable under various perturbations, e.g. adding $o(t^1/2)$ function to a Brownian motion. We formalize it in the next theorem which is one of main results of the section and is an essential tool for studying random walks version of the problem in the next section.

Theorem 4.2.3 (Random ballot theorem for the Brownian motion)**.** *Let B and W be two independent Brownian motions. We set $f : \mathbb{R}_+ \to \mathbb{R}$, $g : \mathbb{R}_+ \to [1, +\infty)$ and $h : \mathbb{R}_+ \to \mathbb{R}_+$ to be continuous functions such that there exists $\alpha < 1/2$ fulfilling $f(0) = 0$, $|f(t)| = O(t^\alpha)$ and $\log g(t), \log h(t) = o(\log t)$ as $t \to +\infty$. For any $\beta \in \mathbb{R}$ and $0 \leq a < b \leq +\infty$ we have*

$$\lim_{t\to+\infty} \frac{1}{\log t} \log \mathbf{P}\left[g(t) + B_s \geq \beta W_s + f(s), h(t) \leq s \leq t\right] = -\gamma(\beta) \text{ in probability.} \tag{4.2.12}$$

Proof. We note there exist $A > 0$, $\alpha < 1/2$ such that for any $t \geq 0$, $|f(t)| \leq 1/2 + At^\alpha$, therefore

$$\begin{aligned}
&\mathbf{P}\left(B_s + 1 \geq W_s + 1/2 + As^\alpha, 0 \leq s \leq t |\, W\right)\\
&\qquad\leq \mathbf{P}\left(g(t) + B_s \geq \beta W_s + f(s), h(t) \leq s \leq t |\, W\right)\\
&\qquad\qquad\leq \mathbf{P}\left(1 + B_s \geq \beta W_s - 1/2 - As^\alpha, \widetilde{h}(t) \leq s \leq t \,\middle|\, W\right),
\end{aligned}$$

where $\widetilde{h}(t) = \max(h(t), g(t)^{\frac{1}{\alpha}})$ verifies again $\log \widetilde{h}(t) = o(\log t)$ as $t \to +\infty$. Consequently, to prove (4.2.12), it is enough to prove that, for all $A > 0$, $\alpha < \frac{1}{2}$, $0 < a < b < +\infty$ and $h(t) = e^{o(\log t)}$, for any $\varepsilon > 0$ we have

$$\lim_{t\to+\infty} \mathbf{P}\left(\log \mathbf{P}(3/2 + B_s \geq \beta W_s - As^\alpha, h(t) \leq s \leq t|W) \geq -(\gamma(\beta) - \varepsilon) \log t\right) = 0 \tag{4.2.13}$$

as well as

$$\lim_{t\to+\infty} \mathbf{P}\left(\log \mathbf{P}\left(\begin{matrix} 1/2 + B_s \geq \beta W_s + As^\alpha, s \leq t \\ B_t - \beta W_t \in \left[at^{1/2}, bt^{1/2}\right] \end{matrix} \,\middle|\, W\right) \leq -(\gamma(\beta) + \varepsilon) \log t\right) = 0. \tag{4.2.14}$$

For $t \geq 0$, we write $r(t) = \frac{A}{\beta} t^\alpha$, and

$$Z_t = \exp\left[\int_0^t r'(s) dW_s - \frac{1}{2} \int_0^t r'(s)^2 ds\right].$$

As $\int_{\mathbb{R}_+} r'(s)^2 ds < +\infty$, $(Z_t, t \geq 0)$ is a uniformly integrable non-negative martingale. We denote by Z_∞ its a.s. limit. By the Girsanov theorem, under probability $\mathbf{Q} := Z_\infty \cdot \mathbf{P}$, the process $(W_t - r(t), t \geq 0)$ is a standard Brownian motion. Moreover, under $\mathbf{Q}$, $(W_t - r(t))$ is independent of B. By definition, $\mathbf{Q}$ is absolutely continuous with respect to $\mathbf{P}$. Moreover, as under $\mathbf{Q}$, (Z_t^{-1}) is an uniformly integrable martingale, we have $\mathbf{P} = Z_\infty^{-1} \cdot \mathbf{Q}$, which proves that $\mathbf{Q}$ and $\mathbf{P}$ are equivalent measures.

We have

$$\begin{aligned}
&\mathbf{Q}(B_t - \beta(W_t - r(t)) \in \left[at^{1/2}, bt^{1/2}\right], B_s + 1/2 \geq \beta(W_s - r(s)), 0 \leq s \leq t|W)\\
&\qquad= \mathbf{P}(B_t - \beta W_t \in \left[at^{1/2}, bt^{1/2}\right], B_s + 1/2 \geq \beta W_s, 0 \leq s \leq t|W).
\end{aligned}$$

By Lemma 4.2.2, we have

$$\begin{aligned}
&\lim_{t\to+\infty} \frac{\log \mathbf{Q}(B_t - \beta(W_t - r(t)) \in \left[at^{1/2}, bt^{1/2}\right], 1/2 + B_s \geq \beta(W_s - r(s)), s \leq t|W)}{\log t}\\
&\qquad= -\gamma(\beta) \text{ in } \mathbf{Q}\text{-probability.}
\end{aligned}$$

As $\mathbf{P}$ is absolutely continuous with respect to $\mathbf{Q}$, we obtain (4.2.14).

To prove (4.2.13), we use once again the strong FKG inequality,

$$\begin{aligned}\mathbf{Q}\left(1+B_s \geq \beta(W_s - r(s)), 0 \leq s \leq t | W\right) \\ \geq \mathbf{Q}\left(1+B_s \geq \beta(W_s - r(s)), 0 \leq s \leq h(t) | W\right) \\ \times \mathbf{Q}\left(1+B_s \geq \beta(W_s - r(s)), h(t) \leq s \leq t | W\right).\end{aligned}$$

We recall that under $\mathbf{Q}$,the process $(W_t - r(t), t \geq 0)$ is a standard Brownian motion. Therefore,

$$\begin{aligned}\limsup_{t\to+\infty} \frac{1}{\log t} \log \mathbf{Q}(1+B_s \geq \beta W_s - A(s+1)^\alpha, h(t) \leq s \leq t | W) \\ \leq \limsup_{t\to+\infty} \frac{1}{\log t} \log \mathbf{Q}(1+B_s \geq \beta(W_s - r(s)), 0 \leq s \leq t | W) \\ - \liminf_{t\to+\infty} \frac{1}{\log t} \log \mathbf{Q}(1+B_s \geq \beta(W_s - r(s)), 0 \leq s \leq h(t)) \\ \leq -\gamma(\beta), \quad \mathbf{Q}\text{-a.s.},\end{aligned}$$

applying Lemma 4.2.1. As $\mathbf{Q}$ is absolutely continuous with respect to $\mathbf{P}$, this inequality also holds $\mathbf{P}$-a.s, concluding the proof. □

We study some of the properties of the function $\beta \mapsto \gamma(\beta)$ introduced in Lemma 4.2.1.

Lemma 4.2.4. *The function $\beta \mapsto \gamma(\beta)$ is convex, even and $\gamma(0) = \inf_{\beta\in\mathbb{R}} \gamma(\beta) = 1/2$.*

Proof. The bound from below by $\frac{1}{2}$ is a direct [4] consequence of the Jensen inequality. We have

$$\begin{aligned}\mathbf{E}\left(-\log \mathbf{P}(B_s + 1 \geq \beta W_s, 0 \leq s \leq t | W)\right) \\ \leq -\log \mathbf{E}\left(\mathbf{P}(B_s + 1 \geq \beta W_s, 0 \leq s \leq t | W)\right) \sim_{t\to+\infty} \frac{\log t}{2}.\end{aligned}$$

As a consequence, letting $t \to +\infty$, we have

$$\gamma(\beta) = \lim_{t\to+\infty} \frac{1}{\log t} \mathbf{E}\left(-\log \mathbf{P}(B_s + 1 \geq \beta W_s, 0 \leq s \leq t | W)\right) \geq \frac{1}{2}.$$

This lower bound is tight, indeed

$$\gamma(0) = \lim_{t\to+\infty} \frac{-\log \mathbf{P}(B_s \geq -1, s \leq t)}{\log t} = \frac{1}{2}.$$

Let $\beta \in \mathbb{R}$, using the symmetry of the Brownian motion W, we have

$$\left(\mathbf{P}(B_s \geq -\beta W_s + 1, s \leq t | W), t \geq 0\right) \overset{(d)}{=} \left(\mathbf{P}(B_s \geq \beta W_s + 1, s \leq t | W), t \geq 0\right),$$

which proves that $\gamma(\beta) = \gamma(-\beta)$.

Finally, we use the log-concavity of the Gaussian measure, for any pair of continuous functions (f, g) and $n \in \mathbb{N}$, we have

$$\begin{aligned}\mathbf{P}\left(B_{tj/n} \geq f(tj/n), j \leq n\right) \mathbf{P}\left(B_{tj/n} \geq g(tj/n), j \leq n\right) \\ \leq \mathbf{P}\left(B_{tj/n} \geq \frac{f(tj/n) + g(tj/n)}{2}, j \leq n\right)^2.\end{aligned}$$

4. And unnecessary, as the minimum of an even convex function is its value at 0.

Letting $n \to +\infty$, we have

$$\mathbf{P}(B_s \geq f(s), 0 \leq s \leq t)\mathbf{P}(B_s \geq g(s), 0 \leq s \leq t) \leq \mathbf{P}\left(B_s \geq \frac{f(s)+g(s)}{2}, 0 \leq s \leq t\right)^2.$$

As a consequence, for all $\beta_1, \beta_2 \in \mathbb{R}$, we have

$$\mathbf{P}(B_s \geq 1 + \frac{\beta_1+\beta_2}{2} W_s, 0 \leq s \leq t | W)^2 \\ \geq \mathbf{P}(B_s \geq 1 + \beta_1 W_s, s \leq t | W)\mathbf{P}(B_s \geq 1 + \beta_2 W_s, s \leq t | W),$$

letting $t \to +\infty$, this leads

$$\gamma\left(\frac{\beta_1+\beta_2}{2}\right) \leq \frac{\gamma(\beta_1)+\gamma(\beta_2)}{2},$$

which proves that γ is convex. □

We observe that Theorem 4.1.3 follows easily combining Lemma 4.2.1 and Lemma 4.2.4.

Anticipating the next section and the problem of Brownian excursions above a Brownian motion, we compute the probability for a Brownian motion B to stay above the curve $s \mapsto W_t - W_{t-s}$, that is a Brownian motion seen backward, and the end part of an excursion. We observe this quantity exhibits almost sure fluctuations on *log*-scale.

Lemma 4.2.5. *For all $\beta \in \mathbb{R}$, we have*

$$\lim_{t\to+\infty} \frac{1}{\log t} \log \mathbf{P}\left(B_s + 1 \geq \beta(W_t - W_{t-s}), s \leq t | W\right) = -\gamma(\beta) \quad \textit{in probability} \tag{4.2.15}$$

$$\limsup_{t\to+\infty} \frac{1}{\log t} \log \mathbf{P}\left(B_s + 1 \geq \beta(W_t - W_{t-s}), s \leq t | W\right) \geq -1/2 \quad \textit{a.s.} \tag{4.2.16}$$

$$\liminf_{t\to+\infty} \frac{1}{\log t} \log \mathbf{P}\left(B_s + 1 \geq \beta(W_t - W_{t-s}), s \leq t | W\right) \leq -\max\left(1/2, \beta^2/2\right) \quad \textit{a.s.} \tag{4.2.17}$$

The constants we obtain here in (4.2.16) and (4.2.17) are far from optimal. For example, (4.2.17) gives no information for $\beta \in (-1,1)$.

Proof. We first observe that

$$\mathbf{P}\left(B_s + 1 \geq W_t - W_{t-s}, s \leq t | W\right) \overset{(d)}{=} \mathbf{P}\left(B_s + 1 \geq W_s, s \leq t | W\right).$$

Thus (4.2.15) follows by Lemma 4.2.1. This limit does not coincides with limsup and the liminf as the behaviour of the Brownian motion seen backwards near the origin fluctuates much more.

We set $I_t = \inf_{s \leq t} W_s$. It is well-known that $\lim_{t\to+\infty} I_t = -\infty$ and $W_t = I_t$ infinitely often. Let $t > 0$ be such that $I_t = W_t$ then $\sup_{s \leq t} W_t - W_{t-s} \leq 0$. As a consequence,

$$\mathbf{P}\left(B_s + 1 \geq W_t - W_{t-s}, s \leq t | W\right) \geq \mathbf{P}(B_s + 1 \geq 0, s \leq t).$$

Now (4.2.16) follows by the classical ballot theorem.

We now prove (4.2.17). Let $\beta > 0$, applying the Girsanov theorem, for all $a > 0$ and $t > 0$ we have

$$Ce^{-\frac{a^2}{2}t} \geq \mathbf{P}(W_t - W_{t-s} \geq as - 1/(2\beta), s \leq t) \geq ct^{-1/2}e^{-\frac{a^2}{2}t}. \tag{4.2.18}$$

As a consequence, for any $\lambda > 0$, we have

$$\begin{aligned}
&\mathbf{P}\left(\exists u \in [\lambda \log t, t] : \forall s \leq \lambda \log t, W_u - W_{u-s} \geq as - 1/(2\beta)\right)\\
&\qquad \geq \mathbf{P}\left(\exists j \leq \left\lfloor \tfrac{t}{\lambda \log t} \right\rfloor : \forall s \leq \lambda \log t, W_{j\lambda \log t} - W_{j\lambda \log t - s} \geq as - 1/(2\beta)\right)\\
&\qquad \geq 1 - \left(1 - \mathbf{P}\left(W_{\lambda \log t} - W_{\lambda \log t - s} \geq as - 1/(2\beta), s \leq \lambda \log t\right)\right)^{\left\lfloor \frac{t}{\lambda \log t} \right\rfloor}.
\end{aligned}$$

By (4.2.18), we obtain that for all $\lambda < \frac{a^2}{2}$, there exists infinitely many times t such that for $s \leq \lambda \log t$, $W_t - W_{t-s} \geq as - 1/(2\beta)$. For any such t we have

$$\mathbf{P}\left(B_s + 1 \geq W_t - W_{t-s}, s \leq t | W\right) \leq \mathbf{P}\left(B_s \geq \beta as - 1/2, s \leq \lambda \log t\right) \leq C e^{-\beta^2 \frac{a^2}{2} \lambda \log t}.$$

We conclude that for all $\varepsilon > 0$, there exists infinitely many times t such that

$$\frac{1}{\log t} \log \mathbf{P}\left(B_s + 1 \geq W_t - W_{t-s}, s \leq t | W\right) \leq -(1-\varepsilon)\frac{\beta^2}{2},$$

proving (4.2.17). □

4.3 Ballot theorem for a random walk in random environment

The exponent of polynomial decay $\gamma(\beta)$ found in Section 4.2 admit some level of uniformity. It is valid not only for Brownian motions but also for a class of random walks in random environment. The connexion between the ballot theorem for random walks in random environment and the Brownian motion problem is explained in Remark 4.3.3.

We now define the random walk in random (time) environment we consider. We write $(\mu_n, n \in \mathbb{N})$ for a sequence of i.i.d. random probability measures on $\mathbb{R}$. Conditionally on this sequence, we introduce a sequence (X_n) of independent random variables, with X_n of law μ_n. We denote by $S_n = \sum_{j=1}^n X_j$ the random walk in random environment. We write $\mathbb{P}_\mu$ for the law of $(S_n, n \geq 0)$ conditionally on the sequence $(\mu_n, n \in \mathbb{N})$, and $\mathbf{P}$ for the joint law of S and the environment μ. The corresponding expectations are respectively written E_μ and $\mathbf{E}$. We introduce the integrability assumption

$$\mathbf{E}\left[\mathrm{E}_\mu(S_1)\right] = 0,\ \sigma_Q^2 := \mathbf{E}\left[\mathrm{E}_\mu(S_1^2) - \mathrm{E}_\mu(S_1)^2\right] \in (0, +\infty)$$
$$\text{and } \sigma_A^2 := \mathbf{E}\left[\mathrm{E}_\mu(S_1)^2\right] \in [0, +\infty). \quad (4.3.1)$$

We observe that σ_A^2 is the variance of $\mathrm{E}_\mu(X_1)$ the quenched expectation of the random walk, while σ_Q^2 is the expected variance of the law μ_1. These definitions share similarities with the quantities defined in (4.1.8). The main result of the section is the ballot theorem for a random walk in a random time-environment.

Theorem 4.3.1 (Random ballot theorem)**.** *We assume* (4.3.1) *and there exists* $\lambda > 0$ *such that*

$$\lambda \mathrm{E}_\mu\left[|X|^3 e^{\lambda |X|}\right] \leq \mathrm{E}_\mu\left[X^2\right] \quad a.s. \quad (4.3.2)$$

We set $(b_n) \in \mathbb{R}_+^{\mathbb{N}}$ *and* $\alpha < 1/2$*. If* $\lim_{n\to+\infty} \frac{\log b_n}{\log n} = 0$ *then*

$$\limsup_{n\to+\infty} \frac{1}{\log n} \log \sup_{z \in [0, b_n]} \mathbb{P}_\mu(S_j \geq -z - j^\alpha, j \leq n) \leq -\gamma\left(\tfrac{\sigma_A}{\sigma_Q}\right) \quad a.s. \quad (4.3.3)$$

Additionally, if $\liminf_{n\to+\infty} \frac{b_n}{\log n} > 0$*, then for all* $0 \leq a < b$*,*

$$\liminf_{n\to+\infty} \frac{1}{\log n} \log \inf_{z\geq b_n} \mathbb{P}_\mu\left(S_n \in \left[an^{1/2}, bn^{1/2}\right], S_j \geq -z + j^\alpha, j \leq n\right) \geq -\gamma\left(\tfrac{\sigma_A}{\sigma_Q}\right) \ a.s. \tag{4.3.4}$$

Remark 4.3.2. The condition $\liminf b_n/\log n > 0$ we introduce to prove (4.3.4) is non-optimal, but simplifies the proof. In greater generality, we may introduce the event

$$\mathcal{A}_y = \{\exists n \in \mathbb{N} : \mathbb{P}_\mu(S_j \geq -y, j \leq n) = 0\}.$$

One may observe that $\mathbf{P}(\mathcal{A}_y) \in [0,1)$ and that on $\mathcal{A}_y^c$, the probability $\mathbb{P}_\mu(S_j \geq -y, j \leq n)$ decreases as $n^{-\gamma\left(\frac{\sigma_A}{\sigma_Q}\right)+o(1)}$. Moreover, by simple random walk theory, almost surely, there exists $y > 0$ large enough such that $\mathcal{A}_y^c$ holds.

Remark 4.3.3. The link between Theorems 4.3.1 and 4.1.3 can be expressed as follows. We note that S can be decomposed as $S_n = S_n - \mathrm{E}_\mu(S_n) + \mathrm{E}_\mu(S_n)$. As $\mathrm{E}_\mu(S_n)$ is a random walk, it can be approached by a Brownian motion with variance σ_A^2. Then $S_n - \mathrm{E}_\mu(S_n)$ is a sum of independent centred random variables, that behaves as a Brownian motion with variance σ_Q^2.

The two main tools of the proof of Theorem 4.3.1 are the following theorems, that couple Brownian motions with sum of independent random variables. We first introduce the KMT coupling, discovered by Komlós, Major and Tusnády, that links a random walk with a standard Brownian motion.

Theorem 4.3.4 (Komlós, Major, Tusnády [KMT76])**.** *Let X be a random variable such that*

$$\mathbf{E}(X) = 0, \quad \sigma^2 := \mathbf{E}(X^2) \in [0, +\infty) \quad \text{and} \quad \exists \alpha > 0 : \mathbf{E}\left[e^{\alpha|X|}\right] < +\infty. \tag{4.3.5}$$

There exist positive numbers λ, C, D, i.i.d. random variables (X_n) with the same law as X and i.i.d. standard Gaussian random variables (Z_n) such that, writing

$$\Delta_n = \max_{k\leq n} \left|\sum_{j=1}^k X_j - \sigma Z_j\right|,$$

we have

$$\forall x \in \mathbb{R}_+, \forall n \in \mathbb{N}, \mathbf{P}(\Delta_n \geq x + D\log n) \leq Ce^{-\lambda x}.$$

Note that under this integrability assumption, by Jensen inequality, the random variable $\mathrm{E}_\mu(X_1)$ satisfies (4.3.5), with $\sigma^2 = \sigma_A^2$. This result has been extended by Sakhanenko to sums of independent random variables with non identical law.

Theorem 4.3.5 (Sakhanenko [Sak84])**.** *Let $(X_1, \ldots, X_n)$ be independent random variables, we assume there exists $\lambda > 0$ such that*

$$\forall k \leq n, \lambda\, \mathbf{E}\left[|X_k|^3 e^{\lambda|X_k|}\right] \leq \mathbf{E}\left[|X_k|^2\right] < +\infty.$$

There exists random variables $(\widetilde{X}_1, \ldots, \widetilde{X}_n)$ with the same law as $(X_1, \ldots, X_n)$ and independent Gaussian random variables $(Z_1, \ldots, Z_n)$ with same means and variances as $(X_1, \ldots X_n)$ such that, writing

$$\Delta_n = \max_{k\leq n} \left|\sum_{j=1}^k \widetilde{X}_j - Z_j\right|,$$

there exists a universal constant $C_0 > 0$ *such that*

$$\mathbf{E}\left[e^{C_0\lambda\Delta_n}\right] \leq 1 + \lambda\sqrt{\sum_{j=1}^n \mathrm{Var}(X_j)}.$$

Proof of Theorem 4.3.1. We consider the random walk in random environment (S_n). For any $n \in \mathbb{N}$, we introduce

$$M_n = \mathrm{E}_\mu(S_n) \quad \text{and} \quad \Sigma_n^2 = \mathrm{E}_\mu(S_n^2) - M_n^2. \tag{4.3.6}$$

The process (M_n) is the random walk of the mean of (S_n), and is centred. The process (Σ_n^2) is the random walk of the variance of (S_n). By law of large numbers, (Σ_n^2) converges to $+\infty$ at ballistic speed σ_A^2. We now construct two independent Brownian motions B and W such that we may replace $\mathbb{P}_\mu(S_j \geq 0, j \leq n)$ by $\mathbf{P}(\sigma_Q B_s + 1 \geq \sigma_A W_s, s \leq n)$.

By Theorem 4.3.4, we couple the random walk $(M_n, n \geq 0)$ with a Brownian motion W such that, writing $\Delta_n = \sup_{k \leq n} |M_k - W_k|$, there exists λ, C, D such that

$$\forall x \geq 0, \forall n \in \mathbb{N}, \mathbf{P}(\Delta_n \geq D \log n + x) \leq Ce^{-\lambda x}. \tag{4.3.7}$$

Slightly abusing notation we use the same notation $\mathbf{P}$ for the probability space on which the random walk (M_n) is coupled with a Brownian motion such that Theorem holds. Conditionally on the sequence (μ_n) and the coupled Brownian motion W, we introduce independent random variables $(X_n, n \geq 1)$, where X_n has law μ_n and set $S_n = \sum_{j=1}^n X_j$.

Conditionally on this construction, we consider the process $(S_n - M_n, n \in \mathbb{N})$, which is a sum of independent centred random variables. By Theorem 4.3.5, we introduce[5] a Brownian motion B (independent of W) such that, writing $\widetilde{\Delta}_n = \sup_{k \leq n} |S_k - M_k - B_{\Sigma_k}|$, we have

$$\mathrm{E}_\mu\left[e^{C_0\lambda\widetilde{\Delta}_n}\right] \leq 1 + \lambda\sqrt{\Sigma_n} \quad \text{a.s.} \tag{4.3.8}$$

We introduce the event

$$\mathcal{B}_n = \left\{\Delta_n \geq (\log n)^2\right\} \cup \left\{\exists j \leq n : \left|\Sigma_j - j\sigma_A^2\right| \geq (\log n)^2 + j^{2/3}\right\} \\ \cup \left\{\exists j \leq n : \sup_{|t| \leq 1} \left|W_{\sigma_A^2 j} - W_{\sigma_A^2 j + t}\right| \geq (\log n)^2\right\}.$$

By (4.3.7), and the Borel-Cantelli lemma, $\lim_{n\to+\infty} \mathbf{1}_{\mathcal{B}_n^c} = 1$ a.s. Moreover, by (4.3.8), we have

$$\liminf_{n\to+\infty} -\frac{\log \mathbb{P}_\mu(\widetilde{\Delta}_n \geq (\log n)^2)}{\log n} = +\infty. \tag{4.3.9}$$

Consequently

$$\limsup_{n\to+\infty} \frac{1}{\log n} \log \sup_{z\in[0,b_n]} \mathbb{P}_\mu(S_j \geq -z - j^\alpha, j \leq n) \\ \leq \limsup_{n\to+\infty} \frac{\mathbf{1}_{\mathcal{B}_n^c}}{\log n} \log \mathbb{P}_\mu\left(S_j \geq -b_n - j^\alpha, j \leq n, \widetilde{\Delta}_n \leq (\log n)^2\right).$$

5. Up to enlarging once again the probability space.

We adhere to the convention that j, k iterate over natural numbers and s, t over real numbers. By the definition of $\mathcal{B}_n$, we have

$$\begin{aligned}\mathbf{1}_{\mathcal{B}_n^c}\mathbb{P}_\mu&\left(S_j \geq -b_n - j^\alpha, j \leq n, \widetilde{\Delta}_n \leq (\log n)^2\right)\\ &\leq \mathbf{1}_{\mathcal{B}_n^c}\mathbb{P}_\mu\left(B_{\Sigma_j} + 2(\log n)^2 \geq -b_n - j^\alpha - W_{\sigma_A^2 j}, j \leq n\right)\\ &\leq \mathbf{1}_{\mathcal{B}_n^c}\mathbb{P}_\mu\left(B_{\Sigma_{\lfloor s\rfloor}} \geq -b_n - j^\alpha - 3(\log n)^2 - W_{\sigma_A^2 s}, s \leq n\right).\end{aligned}$$

Moreover, on $\mathcal{B}_n^c$, $\left|\Sigma_j - j\sigma_Q^2\right| \leq (\log n)^2 + j^{2/3}$. Using the union bound and standard Gaussian calculations, we observe that

$$\liminf_{n\to+\infty} \frac{-1}{\log n} \log \mathbb{P}_\mu\left(\exists j \leq n, \exists |t| \leq (\log n)^2 + j^{2/3} : B_{j\sigma_Q^2} - B_{j\sigma_Q^2+t} \geq j^{2/5} + (\log n)^2\right) = +\infty, \quad (4.3.10)$$

which is enough to conclude that

$$\begin{aligned}\limsup_{n\to+\infty} \frac{1}{\log n} \log \sup_{z\in[0,b_n]} \mathbb{P}_\mu(S_j \geq -z - j^\alpha, j \leq n)&\\ \leq \limsup_{n\to+\infty} \frac{1}{\log n} \log \mathbb{P}_\mu\left(B_{\sigma_Q^2 s} \geq W_{\sigma_A^2 s} - b_n - 4(\log n)^2 - (s^{2/5} + s^\alpha), s \leq n\right)&\\ = -\gamma\left(\tfrac{\sigma_A}{\sigma_Q}\right) \text{ a.s.}&\end{aligned}$$

applying Lemma 4.2.1.

The lower bound holds using similar arguments. We assume that for any $n \in \mathbb{N}$, $b_n \geq 5(\log n)^2$. In this case, for $0 \leq a < b$ and $n \geq 1$, we have

$$\begin{aligned}\inf_{z\geq b_n} \mathbb{P}_\mu&\left(S_n \in \left[an^{1/2}, bn^{1/2}\right], S_j \geq -z + j^\alpha, j \leq n\right)\\ &\geq \mathbb{P}_\mu\left(S_n \in \left[an^{1/2}, bn^{1/2}\right], S_j \geq -5(\log n)^2 + j^\alpha, j \leq n\right)\\ &\geq \mathbb{P}_\mu\left(S_n \in \left[an^{1/2}, bn^{1/2}\right], (S_j - M_j) \geq -M_j - 5(\log n)^2 + j^\alpha, j \leq n\right)\\ &\geq \mathbf{1}_{\mathcal{B}_n^c}\mathbb{P}_\mu\left(\begin{array}{l} B_{\Sigma_n} + W_{\sigma_A^2 n} \in \left[an^{1/2} + 2(\log n)^2, bn^{1/2} - 2(\log n)^2\right]\\ B_{\Sigma_j} \geq -W_{\sigma_A^2 j} - 3(\log n)^2 + j^\alpha, j \leq n, \widetilde{\Delta}_n \leq (\log n)^2\end{array}\right).\end{aligned}$$

Moreover, keeping (4.3.9) in mind, we notice that

$$\begin{aligned}\mathbb{P}_\mu&\left(\begin{array}{l} B_{\Sigma_n} + W_{\sigma_A^2 n} \in \left[an^{1/2} + 2(\log n)^2, bn^{1/2} - 2(\log n)^2\right]\\ B_{\Sigma_j} \geq -W_{\sigma_A^2 j} - 3(\log n)^2 + j^\alpha, j \leq n, \widetilde{\Delta}_n \leq (\log n)^2\end{array}\right)\\ &\geq \mathbb{P}_\mu\left(\begin{array}{l} B_{\Sigma_n} + W_{\sigma_A^2 n} \in \left[an^{1/2} + 2(\log n)^2, bn^{1/2} - 2(\log n)^2\right]\\ B_{\Sigma_j} \geq -W_{\sigma_A^2 j} - 3(\log n)^2 + j^\alpha, j \leq n\end{array}\right)\\ &\qquad - \mathbb{P}_\mu\left(\widetilde{\Delta}_n \leq (\log n)^2\right).\end{aligned}$$

By (4.3.10) and Lemma 4.2.2, we have

$$\lim_{n\to+\infty} \frac{1}{\log n} \log \inf_{z\geq 0} \mathbb{P}_\mu\left(S_n \in [an^{1/2}, bn^{1/2}], S_j \geq -z + j^\alpha, j \leq n\right)$$
$$\geq \lim_{n\to+\infty} \frac{\mathbf{1}_{\mathcal{B}_n^c}}{\log n} \log \mathbb{P}_\mu \begin{pmatrix} B_{\sigma_Q^2 n} + W_{\sigma_A^2 n} \in \left[an^{1/2} + 4(\log n)^2, bn^{1/2} - 4(\log n)^2\right], \\ B_{\sigma_Q^2 s} \geq -W_{\sigma_A^2 s} - (\log n)^2 + s^{2/5} + s^\alpha, s \leq n \end{pmatrix}$$
$$= -\gamma\left(\tfrac{\sigma_A}{\sigma_Q}\right) \text{ in probability.} \quad (4.3.11)$$

Finally, we choose $\varepsilon > 0$ such that $\liminf_{n\to+\infty} \frac{b_n}{\log n} > \varepsilon$. For all $n \geq 1$ large enough, and $0 \leq a < b$ we have

$$\inf_{z\geq b_n} \mathbb{P}_\mu\left(S_n \in [an^{1/2}, bn^{1/2}], S_j \geq -z + j^\alpha, j \leq n\right)$$
$$\geq \mathbb{P}_\mu\left(S_n \in [an^{1/2}, bn^{1/2}], S_j \geq -\varepsilon \log n + j^\alpha, j \leq n\right).$$

We set $p = \left\lfloor e^{(\varepsilon \log n)^{1/2}} \right\rfloor$, applying the Markov property at time p, for all $\delta > 0$ small enough and $n \geq 1$ large enough, we have

$$\mathbb{P}_\mu\left(S_n \in \left[an^{1/2}, bn^{1/2}\right], S_j \geq -\varepsilon \log n + j^\alpha, j \leq n\right)$$
$$\geq \mathbb{P}_\mu\left(S_p \geq\in [p^{1/2}, 2p^{1/2}], S_j \geq -5(\log p)^2, j \leq p\right)$$
$$\times \mathbb{P}_\mu\left(S_n \in \left[(a+\delta)n^{1/2}, (b-\delta)n^{1/2}\right], S_j \geq -(5\log n)^2 + (j+p)^\alpha, j \leq n-p\right).$$

Applying (4.3.11) twice, we observe the log of the first probability is a.s. negligible with respect to $\log n$ and

$$\lim_{n\to+\infty} \inf_{z\geq b_n} \frac{\log \mathbb{P}_\mu\left(S_n \geq \varrho n^{1/2}, S_j \geq -\varepsilon \log n + j^\alpha, j \leq n\right)}{\log n} \geq -\gamma\left(\tfrac{\sigma_A}{\sigma_Q}\right) \text{ in probability.}$$

□

The upper bound of Theorem 4.3.1 can be strengthened to hold wih some level of uniformity with respect to the environment. More precisely, for a given environment $(\mu_n, n \in \mathbb{N})$ and $k \in \mathbb{N}$, we denote by $\mathbb{P}_\mu^k$ the law of the random walk with random environment $(\mu_{k+j}, j \in \mathbb{N})$, shifted in time by k. Alternatively, one can see the law of $(S_j, j \geq 0)$ under law $\mathbf{P}^k$ as the law of $(S_{k+j} - S_k, j \geq 0)$ under law $\mathbf{P}$. We prove in the next lemma that the previous lemma holds uniformly in $k \leq e^{o(\log n)}$.

Lemma 4.3.6. *We assume that* (4.3.1) *and* (4.3.2) *hold. Let* $(b_n), (t_n) \in \mathbb{R}_+^{\mathbb{N}}$ *be such that* $\lim_{n\to+\infty} \frac{\log b_n}{\log n} = \lim_{n\to+\infty} \frac{\log t_n}{\log n} = 0$ *and* $\alpha \in [0, 1/2)$. *We have*

$$\lim_{n\to+\infty} \sup_{k\leq t_n} \frac{\mathbb{P}^k\left(S_j \geq -b_n - j^\alpha, j \leq n\right)}{\log n} = -\gamma(\beta) \textit{ in probability.}$$

Proof. Note the lower bound of this lemma is a direct consequence of Theorem 4.3.1. We assume, without loss of generality, that $\lim_{n\to+\infty} t_n = +\infty$. Let $k \leq t_n$, applying the Markov property at time k, we have

$$\mathbb{P}_\mu^k\left(S_j \geq -b_n - j^\alpha, j \leq b\right) \mathbb{P}_\mu\left(S_j \geq -\log t_n, j \leq k\right)$$
$$\leq \mathbb{P}_\mu\left(S_j \geq -b_n - \log t_n - j^\alpha, j \leq n+k\right).$$

As a consequence, uniformly in $k \leq t_n$, we have

$$\mathbb{P}_\mu^k \left(S_j \geq -b_n - j^\alpha, j \leq b \right) \leq \frac{\mathbb{P}_\mu \left(S_j \geq -b_n - \log t_n - j^\alpha, j \leq n \right)}{\mathbb{P}_\mu \left(S_j \geq -\log t_n, j \leq t_n \right)}.$$

By Theorem 4.3.1, we have

$$\lim_{n\to+\infty} \frac{\log \mathbb{P}_\mu \left(S_j \geq -b_n - j^\alpha, j \leq n \right)}{\log n} = -\gamma(\beta) \text{ in probability,}$$

as well as

$$\lim_{n\to+\infty} \frac{\log \mathbb{P} \left(S_j \geq -\log t_n, j \leq t_n \right)}{\log b_n} = -\gamma(\beta) \text{in probability,}$$

which concludes the proof. □

This almost sure exponent for the random ballot theorem can be used to obtain another exponent of interest, which correspond to the probability of observation of an excursion of length n. For a classical centred random walk with finite variance, we have

$$\mathbf{P}(S_n \leq 1, S_j \geq 0, j \leq n) \approx_{n\to+\infty} n^{-3/2}.$$

It can be explained as follows. An excursion of length n can be divided into three parts. The one between 0 and $n/3$ is a random walk required to stay positive, which happens with probability $n^{-1/2}$. Similarly, the end part between $2n/3$ and n seen backward is a random walk required to stay negative, which again happens with probability $n^{-1/2}$ factor. Finally, the part between $n/3$ and $2n/3$ joins these segments, which by the local CLT costs another $n^{-1/2}$.

Using similar arguments on the random walk in random environment suggests that writing

$$\lambda := 2\gamma\left(\tfrac{\sigma_A}{\sigma_Q}\right) + \frac{1}{2}, \tag{4.3.12}$$

we have $\mathbb{P}_\mu \left(S_n \leq 1, S_j \geq 0, j \leq n \right) \approx n^{-\lambda}$. However, working with the random walk seen from backward leads to additional difficulties. As already observed in Lemma 4.2.5, the convergence for a backward random walk holds only in probability and cannot be improved to an a.s. result.

Lemma 4.3.7. *We assume that* (4.3.1) *and* (4.3.2) *hold. Let* $(t_n) \in \mathbb{R}_+^{\mathbb{N}}$ *be such that* $\lim_{n\to+\infty} \frac{\log t_n}{\log n} = 0$. *For any* $\varepsilon > 0$, *we have*

$$\lim_{n\to+\infty} \mathbf{P}\left(\sup_{-t_n \leq y < x \leq 0} \sup_{k,k' \leq t_n} \frac{\log \mathbb{P}_\mu^k \left(S_{n-k-k'} \leq x, S_j \geq y, j \leq n-k-k' \right)}{\log n} \geq \varepsilon - \lambda \right) = 0.$$

Proof. Without loss of generality, we assume that $\lim_{n\to+\infty} \frac{t_n}{\log n} = +\infty$. Let $n \in \mathbb{N}$, we set $p = \lfloor n/3 \rfloor$. We choose $-t_n \leq x \leq y \leq 0$ and $k \leq t_n$, $k' \leq t_n$. Applying the Markov property at time $p - k - k'$, we have

$$\begin{aligned}
&\mathbb{P}_\mu^k \left(S_{n-k-k'} \geq x, S_j \leq y, j \leq n \right)\\
&\qquad \leq \mathbb{P}_\mu^k \left(S_j \leq t_n, j \leq p-k \right) \sup_{z\in\mathbb{R}} \mathbb{P}_\mu^p \left(S_{n-p-k'} + z \geq x, S_j + z \leq y, j \leq n-p-k' \right).
\end{aligned}$$

We introduce, for $j \leq n-p-k'$ the time-reversed random walk $\widehat{S}_j^{(k')} = S_{n-p-k'} - S_{n-p-k'-j}$. Note that for all fixed n, $((\widehat{S}_j^{(k')}, j \leq n-p-k'), k' \leq t_n)$ has the same law as $((S_{k+j} - S_k, j \leq n-p-k), k \leq t_n)$. We apply the Markov property at time $p-k'$ to obtain

$$\begin{aligned}&\mathbb{P}_\mu^p\left(S_{n-p-k'} + z \geq x, S_j + z \leq y, j \leq n-p\right)\\&\qquad\qquad \leq \mathbb{P}_\mu^p\left(\widehat{S}_j^{(k')} \geq -t_n, j \leq p-k'\right) \sup_{h \in \mathbb{R}} \mathbb{P}_\mu^p\left(S_{n-2p} \in [h, h+y-x]\right).\end{aligned}$$

We conclude that

$$\begin{aligned}&\sup_{-t_n \leq y < x \leq 0} \sup_{k,k' \leq t_n} \mathbb{P}_\mu^k\left(S_{n-k-k'} \leq x, S_j \geq y, j \leq n-k-k'\right)\\&\qquad \leq \sup_{k \leq t_n} \mathbb{P}_\mu^k\left(S_j \geq -t_n, j \leq p-t_n\right) \sup_{k' \leq t_n} \mathbb{P}_\mu^p\left(\widehat{S}_j^{(k')} \leq t_n, j \leq p-t_n\right)\\&\qquad\qquad\qquad \times \sup_{h \in \mathbb{R}} \mathbb{P}_\mu^p\left(S_{n-2p} \in [h, h+t_n]\right).\end{aligned}$$

By Theorem 4.3.1, we have

$$\lim_{n \to +\infty} \frac{\log \sup_{k \leq t_n} \mathbb{P}_\mu^k(S_j \leq t_n, j \leq p)}{\log n} = -\gamma\left(\tfrac{\sigma_A}{\sigma_Q}\right) \text{ in probability.}$$

Moreover, as $\widehat{S}$ has the same law as S thus

$$\lim_{n \to +\infty} \frac{\log \sup_{k' \leq t_n} \mathbb{P}_\mu^p\left(\widehat{S}_j^{(k')} \leq t_n, j \leq p-t_n\right)}{\log n} = -\gamma\left(\tfrac{\sigma_A}{\sigma_Q}\right) \text{ in probability.}$$

Finally, applying again Theorem 4.3.5, by (4.3.2) we have

$$\limsup_{n \to +\infty} \frac{\log \sup_{h \in \mathbb{R}} \mathbb{P}_\mu^p\left(S_{n-2p} \in [h, h+t_n]\right)}{\log n} = -\frac{1}{2} \quad \text{a.s.}$$

Combining the last three estimates concludes the proof. □

We now derive a lower bound, using a similar reasoning.

Lemma 4.3.8. *We assume that (4.3.1) and (4.3.2) hold. Let $(a_n), (b_n) \in \mathbb{R}_+^{\mathbb{N}}$ be such that $a_n \leq b_n$, $\liminf_{n \to +\infty} \frac{a_n}{\log n} > 0$ and $\lim_{n \to +\infty} \frac{\log b_n}{\log n} = 0$ and $\alpha \in [0, 1/2)$. For $k \leq n$, we write $r_{n,k} = \min(k, n-k)^\alpha$. For all $\varepsilon > 0$, we have*

$$\lim_{n \to +\infty} \mathbf{P}\left(\inf_{-b_n \leq y < x - a_n \leq -2a_n} \frac{\log \mathbf{P}\left(S_n \leq x, S_j \geq y + r_{n,j}, j \leq n\right)}{\log n} \leq -\varepsilon - \lambda\right) = 0.$$

Proof. Let $n \in \mathbb{N}$, we set $p = \lfloor n/3 \rfloor$ and $\widehat{S}_j = S_{n-p} - S_{n-p-j}$. Applying the Markov property at times p and $n-p$, for all $-b_n \leq y \leq x - a_n \leq -2a_n$ and $\varrho > 0$, we have

$$\begin{aligned}&\mathbb{P}_\mu\left(S_n \leq x, S_j \geq y + r_{n,j}, j \leq n\right)\\&\qquad \geq \mathbb{P}_\mu\left(S_p \in \left[\varrho n^{1/2}, 2\varrho n^{1/2}\right], S_j \geq -a_n + j^\alpha, j \leq p\right)\\&\qquad\quad \times \mathbb{P}_\mu^p\left(\widehat{S}_p \in \left[-2\varrho n^{1/2}, -\varrho n^{1/2}\right], \widehat{S}_j \leq a_n - j^\alpha, j \leq p\right)\\&\qquad\quad \times \inf_{z \in [0, 2\varrho n^{1/2}]} \mathbb{P}_\mu^p\left(S_{n-2p} \in [z, z+a_n], S_j \geq -\varrho n^{1/2}, j \leq n-2p\right).\end{aligned}$$

Using Theorem 4.3.1, we have

$$\lim_{n\to+\infty} \frac{1}{\log n} \log \mathbb{P}_\mu \left(S_p \in \left[\varrho n^{1/2}, 2\varrho n^{1/2}\right], S_j \geq -a_n + j^\alpha, j \leq p \right) = -\gamma\left(\tfrac{\sigma_A}{\sigma_Q}\right) \quad \text{a.s.}$$

as well as

$$\lim_{n\to+\infty} \frac{1}{\log n} \log \mathbb{P}^p_\mu \left(\widehat{S}_p \in \left[-2\varrho n^{1/2}, -\varrho n^{1/2}\right], \widehat{S}_j \leq a_n - j^\alpha, j \leq p \right) = -\gamma\left(\tfrac{\sigma_A}{\sigma_Q}\right) \quad \text{a.s.}$$

Finally, using the Sakhanenko coupling of the random walk in random environment, we have

$$\lim_{n\to+\infty} \frac{1}{\log n} \log \mathbb{P}^p_\mu \left(S_{n-2p} \in [z, z+a_n], S_j \geq -\varrho n^{1/2}, j \leq n-2p \right) = -\frac{1}{2},$$

which concludes the proof. □

Combining Lemma 4.3.8 and Lemma 4.3.7, for a random walk in random environment that satisfies (4.3.1) and (4.3.2), we have

$$\lim_{n\to+\infty} \frac{1}{\log n} \log \mathbb{P}_\mu \left(S_n \leq 0, S_j \geq -\delta \log n, j \leq n \right) = -\lambda \quad \text{in probability.} \tag{4.3.13}$$

4.4 Maximal displacement for the branching random walk in random environment

We use the random walk estimates presented in Section 4.3 to obtain bounds on the maximal displacement for the branching random walk in random environment. As often, the proof to compute the asymptotic behaviour of M_n the maximal displacement at time n, is based on the computation of the asymptotic behaviour of $\mathbb{P}_\mathcal{L}(M_n \geq y)$ as $n, y \to +\infty$. To obtain an upper bound, we exhibit a border the random walk does not cross with high probability; to obtain a lower bound, we compute first two moments of the number of individuals who stayed below this border at any time before n, and are at time n above a given level.

4.4.1 The many-to-one lemma

We introduce the celebrated many-to-one lemma, which expresses expectation of additive functionals of branching random walks by functionals of random walks. It has been essential in various studies of extremal behaviour of branching random walks. It can be traced down to the early works of Peyrière [Pey74] and Kahane and Peyrière [KP76]. Many variations and modifications of this concept have been introduced, see e.g. [BK04]. In this article, we use a time-inhomogeneous version of this lemma, that can be found in Chapter 1. For all $n \geq 1$, we write L_n for a point process with law $\mathcal{L}_n$, and we define the probability measure μ_n by

$$\mu_n((-\infty, x]) = \mathrm{E}_\mathcal{L}\left[\sum_{\ell \in L_n} \mathbf{1}_{\{\ell \leq x\}} e^{\theta^* \ell - \kappa_n(\theta^*)} \right].$$

Let $(X_n, n \in \mathbb{N})$ be a sequence of independent random variables, where X_n has law μ_n. We set $S_n = \sum_{j=1}^n X_j$. From now on, the law $\mathbb{P}_\mathcal{L}$ stand for the joint law of the BRWre

$(\mathbf{T}, V)$ and the random walk in random environment S, conditionally on the environment $\mathcal{L}$.

The many-to-one lemma is expressed as follows: for any $n \in \mathbb{N}$ and any measurable bounded function f, we have

$$\mathrm{E}_{\mathcal{L}}\left[\sum_{|u|=n} f(V(u_1), \dots, V(u_n))\right] = \mathrm{E}_{\mathcal{L}}\left[e^{-\theta^* S_n + \sum_{k=1}^{n} \kappa_j(\theta^*)} f(S_1, \dots S_n)\right] \quad \text{a.s.} \tag{4.4.1}$$

It is useful to consider a shifted version of (4.4.1). For $k \in \mathbb{N}$ we consider the environment $(\mathcal{L}_{j+k}, j \in \mathbb{N})$. The definitions of branching random walks and random walks above are still valid, we denote the corresponding probability function by $\mathbb{P}^k_{\mathcal{L}}$. In this scenario (4.4.1) writes as

$$\mathrm{E}^k_{\mathcal{L}}\left[\sum_{|u|=n} f(V(u_1), \dots, V(u_n))\right] = \mathrm{E}^k_{\mathcal{L}}\left[e^{-\theta^* S_n + \sum_{j=k+1}^{k+n} \kappa_j(\theta^*)} f(S_1, \dots S_n)\right] \quad \text{a.s.} \tag{4.4.2}$$

For $n \in \mathbb{N}$, we introduce

$$K_n = \sum_{j=1}^{n} \kappa_j(\theta^*) \quad \text{and} \quad T_n = \theta^* S_n - K_n. \tag{4.4.3}$$

By (4.1.7), we have $\mathbf{E}(T_n) = 0$. Moreover, by (4.1.8), we have

$$\sigma_Q^2 = \mathbf{E}\left[\mathrm{E}_{\mathcal{L}}(T_1^2) - \mathrm{E}_{\mathcal{L}}(T_1)^2\right] \quad \text{and} \quad \sigma_A^2 = \mathbf{E}\left[\mathrm{E}_{\mathcal{L}}(T_1)^2\right],$$

and by (4.1.10), the random walk in random environment (T_n) satisfies (4.3.2). As a consequence, Theorem 4.3.1 and similar results apply to (T_n).

4.4.2 Proof of the upper bound of Theorem 4.1.1

In this section, we bound the probability that $M_n \geq \frac{1}{\theta^*} K_n - \varphi \log n$. To do so, we observe that with high probability, no individual of the branching random walk crosses the border $n \mapsto \frac{1}{\theta^*} K_n + \log n$ for large $n \geq 0$.

Lemma 4.4.1. *We assume that for all $n \in \mathbb{N}$, $\kappa_n(\theta^*) < +\infty$ a.s. For all $y \geq 0$, we have*

$$\mathbf{P}_{\mathcal{L}}\left(\exists u \in \mathbf{T} : V(u) \geq \tfrac{1}{\theta^*} K_{|u|} + y\right) \leq e^{-\theta^* y}.$$

Proof. Let $y \geq 0$, we denote by

$$Z(y) = \sum_{u \in \mathbf{T}} \mathbf{1}_{\left\{V(u) \geq \frac{K_{|u|}}{\theta^*} + y\right\}} \mathbf{1}_{\left\{V(u_j) < \frac{K_j}{\theta^*} + y, j < |u|\right\}},$$

the number of individuals that cross the line $n \mapsto \frac{K_n}{\theta^*} + y$ for the first time. Using the Markov inequality, we have

$$\mathbb{P}_{\mathcal{L}}\left(\exists u \in \mathbf{T} : V(u) \geq \frac{K_{|u|}}{\theta^*} + y\right) = \mathbb{P}_{\mathcal{L}}\left(Z(y) > 0\right) \leq \mathrm{E}_{\mathcal{L}}\left[Z(y)\right].$$

By (4.4.1) and (4.4.3), we have

$$\begin{aligned}
\mathrm{E}_{\mathcal{L}}(Z(y)) &= \sum_{n=1}^{+\infty} \mathrm{E}_{\mathcal{L}}\left[\sum_{|u|=n} \mathbf{1}_{\{\theta^* V(u)-K_n \geq \theta^* y\}} \mathbf{1}_{\{\theta^* V(u_j)-K_j<\theta^* y, j<n\}}\right] \\
&= \sum_{n=1}^{+\infty} \mathrm{E}_{\mathcal{L}}\left[e^{-\theta^* S_n+K_n} \mathbf{1}_{\{\theta^* S_n-K_n \geq \theta^* y\}} \mathbf{1}_{\{\theta^* S_j-K_j<\theta^* y, j<n\}}\right] \\
&\leq \sum_{n=1}^{+\infty} e^{-\theta^* y} \mathbb{P}_{\mathcal{L}}\left(T_n \geq \theta^* y, T_j<\theta^* y, j<n\right) \\
&\leq e^{-\theta^* y} \mathbb{P}_{\mathcal{L}}\left(\exists n \in \mathbb{N}: T_n \geq \theta^* y\right) \leq e^{-\theta^* y} .
\end{aligned}$$

□

We then partition the set of individuals that are higher that $\frac{K_n}{\theta^*} - \beta \log n$ into two subsets: the set of individuals that crossed $n \mapsto \frac{K_n}{\theta^*} + y$, and the set of individuals that made an excursion of length n below this curve. This leads to the following lemma, that proves the upper bound of Theorem 4.1.1.

Lemma 4.4.2. *We assume that* (4.1.7)*,* (4.1.8) *and* (4.1.10) *hold. For any $\beta < \varphi$, we have*

$$\mathbb{P}_{\mathcal{L}}\left(M_n \geq \tfrac{K_n}{\theta^*} - \beta \log n\right) \to 0 \quad \textit{in probability.}$$

Proof. Let $n \in \mathbb{N}$. We start noticing that by Lemma 4.4.1, we have

$$\mathbb{P}_{\mathcal{L}}\left(\exists u \in \mathbf{T} : V(u) \geq \frac{K_{|u|} + \log n}{\theta^*}\right) \leq n^{-1}.$$

For $\beta > 0$, we set $Y_n(\beta) = \sum_{|u|=n} \mathbf{1}_{\{\theta^* V(u) - K_n \geq -\beta\theta^* \log n\}} \mathbf{1}_{\{\theta^* V(u_j) - K_j \leq \log n, j \leq n\}}$. We observe that

$$\begin{aligned}
&\mathbb{P}_{\mathcal{L}}\left(M_n \geq \frac{K_n}{\theta^*} - \beta \log n\right) \\
&\qquad \leq \mathbb{P}_{\mathcal{L}}\left(\exists u \in \mathbf{T} : V(u) \geq \frac{K_{|u|} + \log n}{\theta^*}\right) + \mathbb{P}_{\mathcal{L}}(Y_n(\beta) > 0) \\
&\qquad \leq n^{-1} + \mathrm{E}_{\mathcal{L}}\left(Y_n(\beta)\right). \qquad (4.4.4)
\end{aligned}$$

We apply (4.4.1), we have

$$\begin{aligned}
\mathrm{E}_{\mathcal{L}}\left(Y_n(\beta)\right) &= \mathrm{E}_{\mathcal{L}}\left[e^{-\theta^* S_n + K_n} \mathbf{1}_{\{\theta^* S_n - K_n \geq -\beta\theta^* \log n\}} \mathbf{1}_{\{\theta^* S_j - K_j \leq \log n, j \leq n\}}\right] \\
&\leq n^{\beta\theta^*} \mathbb{P}_{\mathcal{L}}\left(T_n \geq -\beta\theta^* \log n, T_j \leq \log n, j \leq n\right).
\end{aligned}$$

Applying Lemma 4.3.7, we obtain

$$\lim_{n \to +\infty} \frac{\log \mathbb{P}_{\mathcal{L}}\left(T_n \geq -\beta\theta^* \log n, T_j \leq \log n, j \leq n\right)}{\log n} = -\frac{1}{2} - 2\gamma\left(\tfrac{\sigma_A}{\sigma_Q}\right) = \theta^* \varphi \text{ in probability.}$$

Thus, for any $\varepsilon > 0$, we have

$$\lim_{n \to +\infty} \mathbf{P}\left[\mathbb{P}_{\mathcal{L}}\left(T_n \geq -\beta\theta^* \log n, T_j \leq \log n, j \leq n\right) \geq n^{-\theta^* \varphi + \varepsilon}\right] = 0,$$

therefore

$$\lim_{n \to +\infty} \mathbf{P}\left[\mathrm{E}_{\mathcal{L}}\left[Y_n(\beta)\right] \geq n^{\theta^*(\beta - \varphi) + \varepsilon}\right] = 0.$$

As $\beta < \varphi$, choosing $\varepsilon > 0$ small enough we obtain that $\mathrm{E}_{\mathcal{L}}\left[Y_n(\beta)\right]$ converge to 0 in probability. By (4.4.4), we conclude that $\mathbf{P}(M_n \geq K_n - \beta \log n)$ converges to 0 in probability, which ends the proof. □

4.4.3 Proof of the lower bound of Theorem 4.1.1

We now prove that with high probability, $M_n \geq \frac{K_n}{\theta^*} - \varphi \log n$. To do so, we bound from below the probability there exists an individual above $\frac{K_n}{\theta^*} - \varphi \log n$ at time n, that stayed at any time $k \leq n$ away from the boundary $k \mapsto \frac{K_k}{\theta^*}$.

Lemma 4.4.3. *We assume that* (4.1.9), (4.1.7), (4.1.8) *and* (4.1.10) *hold. For any* $\varepsilon > 0$, *we have*

$$\lim_{n \to +\infty} \mathbf{P} \left[\mathbb{P}_{\mathcal{L}} \left(M_n \geq \frac{K_n}{\theta^*} - \varphi \log n \right) \leq n^{-\varepsilon} \right] = 0.$$

Proof. Let $n \in \mathbb{N}$ and $\delta > 0$. For $k \leq n$ we set

$$r_{n,k} = \begin{cases} k^{1/3} - 2\delta \log n & \text{if} \quad k < n/2 \\ (n-k)^{1/3} - (2\delta - \varphi) \log n & \text{otherwise.} \end{cases}$$

We introduce

$$X_n(\delta) = \sum_{|u|=n} \mathbf{1}_{\left\{ V(u) - \frac{K_n}{\theta^*} + \varphi \log n \in [0, \delta \log n] \right\}} \mathbf{1}_{\left\{ V(u_j) \leq \frac{K_j}{\theta^*} - r_{n,j} \right\}}.$$

Our aim is to to bound from below $\mathbb{P}_{\mathcal{L}}(X_n(\delta) \geq 1)$. To this end we utilize the second moment method.

We first bound from below $\mathrm{E}_{\mathcal{L}}(X_n(\delta))$. Applying the many-to-one lemma, we have

$$\begin{aligned} \mathrm{E}_{\mathcal{L}} \left(X_n(\delta) \right) &= \mathrm{E}_{\mathcal{L}} \left[e^{-T_n} \mathbf{1}_{\{T_n + \theta^* \varphi \log n \in [0, \theta^* \delta \log n]\}} \mathbf{1}_{\{T_j \leq -\theta^* r_{n,j}, j \leq n\}} \right] \\ &\geq n^{\theta^* \varphi - \theta^* \delta} \mathbb{P}_{\mathcal{L}} \left(T_n + \theta^* \varphi \log n \in [0, \delta \theta^* \log n], T_j \leq -\theta^* r_{n,j}, j \leq n \right). \end{aligned}$$

Applying Lemma 4.3.8, for any $\varepsilon > 0$, we have

$$\lim_{n \to +\infty} \mathbf{P} \left(\mathrm{E}_{\mathcal{L}} \left(X_n(\delta) \right) \leq n^{-\theta^* \delta - \varepsilon} \right) = 0. \tag{4.4.5}$$

We then bound from above $\mathrm{E}\left[X_n(\delta)^2 \right]$. We note that $X_n(\delta)^2$ is the number of pairs of individuals that are at time n in a neighbourhood of $\frac{K_n}{\theta^*} - \varphi \log n$, and stayed at any time $k \leq n$ at distance at least $r_{n,k}$ from $\frac{K_k}{\theta^*}$. For $u^1, u^2 \in \mathbf{T}$, we set $u^1 \wedge u^2$ the most recent common ancestor of u^1 and u^2. We then partition $X_n(\delta)^2$ according to the generation of the most recent common ancestor. We write

$$X_n(\delta)^2 = \sum_{k=0}^{n} \sum_{|u|=k} \sum_{\substack{|u^1|=|u^2|=n \\ u^1 \wedge u^2 = u}} \mathbf{1}_{\left\{ V(u^i) - \frac{K_n}{\theta^*} + \varphi \log n \in [0, \delta \log n], V(u^i_j) \leq \frac{K_j}{\theta^*} - r_{n,j}, i \in \{1,2\} \right\}}.$$

Notice that

$$\sum_{|u|=n} \sum_{\substack{|u^1|=|u^2|=n \\ u^1 \wedge u^2 = u}} \mathbf{1}_{\left\{ V(u^i) - \frac{K_n}{\theta^*} + \varphi \log n \in [0, \delta \log n], V(u^i_j) \leq \frac{K_j}{\theta^*} - r_{n,j}, i \in \{1,2\} \right\}} = X_n(\delta).$$

By similar calculations as the ones leading to (4.4.5), we use the many-to-one lemma and Lemma 4.3.7 to prove that for any $\varepsilon > 0$,

$$\lim_{n \to +\infty} \mathbf{P} \left[\mathrm{E}_{\mathcal{L}}(X_n(\delta)) \geq n^{\varepsilon} \right] = 0.$$

Let $k < n$, we set

$$\Lambda_k = \sum_{|u|=k} \sum_{\substack{|u^1|=|u^2|=n\\ u^1\wedge u^2=u}} \mathbf{1}_{\left\{V^i(u)-\frac{K_n}{\theta^*}+\varphi\log n\in[0,\delta\log n], V(u^i_j)\leq \frac{K_j}{\theta^*}-r_{n,j}, i\in\{1,2\}\right\}}.$$

We denote by $\mathcal{F}_k = \sigma(u, V(u), |u| \leq k)$. Applying the Markov property at time $k+1$, we have

$$\mathrm{E}_{\mathcal{L}}\left[\Lambda_k \middle| \mathcal{F}_{k+1}\right] \leq \sum_{|u|=k} \mathbf{1}_{\left\{V(u_j)\leq\frac{K_j}{\theta^*}-r_{n,j}, j\leq k\right\}} \sum_{\substack{|u^1|=|u^2|=k+1\\ u^1\wedge u^2=u}} f_{k+1}(V(u^1))f_{k+1}(V(u^2)),$$

where we set, for $k < n$ and $x \in \mathbb{R}$,

$$f_{k+1}(x) =$$
$$\mathrm{E}^{k+1}_{\mathcal{L}}\left[\sum_{|u|=n-k-1} \mathbf{1}_{\left\{V(u)+x-\frac{K_n}{\theta^*}+\varphi\log n\in[0,\delta\log n], V(u_j)+x\leq\frac{K_{k+j+1}}{\theta^*}-r_{n,k+j+1}, j\leq n-k-1\right\}}\right].$$

Note that if $x \geq \frac{K_{k+1}}{\theta^*} - r_{n,k+1}$, then $f_{k+1}(x) = 0$.

We set $b_n = (\log n)^6$, and consider $k > b_n$ in a first time. Applying (4.4.2), we obtain the following upper bound for f_{k+1}, for $k+1 \geq b_n$:

$$\begin{aligned}
&f_{k+1}(x)\\
\leq&\mathbf{1}_{\left\{x\leq\frac{K_{k+1}}{\theta^*}-r_{n,k+1}\right\}} \mathrm{E}^{k+1}_{\mathcal{L}}\left[e^{-\theta^* S_{n-k-1}-\sum_{j=k+2}^n \kappa_j(\theta^*)} \mathbf{1}_{\left\{S_{n-k-1}+x-\frac{K_n}{\theta^*}+\varphi\theta^*\log n\in[0,\delta\log n]\right\}}\right]\\
\leq&\mathbf{1}_{\left\{x\leq\frac{K_{k+1}}{\theta^*}-r_{n,k+1}\right\}} n^{\theta^*\varphi} e^{\theta^* x - K_{k+1}}
\end{aligned}$$

As a consequence, we have

$$\mathrm{E}_{\mathcal{L}}\left[\Lambda_k\right] \leq n^{2\theta^*\varphi}\mathrm{E}_{\mathcal{L}}\left[\sum_{|u|=k} e^{(\theta^* V(u_k)-K_k)} \mathbf{1}_{\left\{V(u_j)\leq\frac{K_j}{\theta^*}-r_{n,j}, j\leq k\right\}} \sum_{\ell,\ell'\in L_{k+1}} e^{\theta^*(\ell+\ell')-2\kappa_{k+1}(\theta)}\right]$$

where L_{k+1} is a point process with law $\mathcal{L}_{k+1}$, independent from $(\mathbf{T}, V)$. We set

$$\Phi_n = \max_{k\leq n} \mathrm{E}_{\mathcal{L}}\left[\left(\sum_{\ell\in L_{k+1}}\left(1+e^{\theta^*\ell-\kappa_{k+1}(\theta)}\right)\right)^2\right].$$

By (4.1.9), this is the maximum of n i.i.d. random variables with finite mean, therefore, for all $\eta > 0$, there exists $x > 0$ such that

$$\sup_{n\in\mathbb{N}} \mathbf{P}\left(\Phi_n \geq nx\right) \leq \eta. \tag{4.4.6}$$

Applying (4.4.1), we have

$$\mathrm{E}_{\mathcal{L}}\left[\Lambda_k\right] \leq \Phi_n n^{2\theta^*\varphi}\mathrm{E}_{\mathcal{L}}\left[e^{T_k}\mathbf{1}_{\{T_j\leq-\theta^* r_{n,j}, j\leq k\}}\right].$$

Note that if $k \leq n - b_n$, then $r_{n,k} \geq (\log n)^2 - 2\delta\log n$. Therefore for any $n \geq 1$ large enough

$$\sup_{k\in[b_n, n-b_n]} \mathrm{E}_{\mathcal{L}}\left[\Lambda_k\right] \leq \Phi_n e^{-\theta^*(\log n)^2/2}. \tag{4.4.7}$$

If $k \geq n - b_n$, we still have $r_{n,k} \geq (\varphi - 2\delta)\log n$. We set

$$\Phi_n^{\mathrm{end}} = \max_{k \in [n-b_n,n]} \mathrm{E}_{\mathcal{L}}\left[\left(\sum_{\ell \in L_{k+1}} \left(1 + e^{\theta^*\ell - \kappa_{k+1}(\theta)}\right)\right)^2\right].$$

Dividing the expectation between the event $\{T_k \geq -b_n\}$ and $\{T_k < -b_n\}$, for any $n \geq 1$ large enough we have

$$\mathrm{E}_{\mathcal{L}}\left[\Lambda_k\right] \leq \Phi_n^{\mathrm{end}} n^{\theta^*(\varphi+2\delta)} \mathbb{P}_{\mathcal{L}}\left[T_k \geq -b_n, T_j \leq -\theta^* r_{n,j}, j \leq k\right] + \Phi_n^{\mathrm{end}} e^{-\theta^*(\log n)^2/2}.$$

We introduce

$$P_n^{\mathrm{end}} = \sup_{k \geq n-b_n} \mathbb{P}_{\mathcal{L}}\left[T_k \geq -b_n, T_j \leq -\theta^* r_{n,j}, j \leq k\right].$$

By Lemma 4.3.7, for any $\varepsilon > 0$, we have

$$\lim_{n \to +\infty} \mathbf{P}\left[P_n^{\mathrm{end}} \geq n^{-\theta^*\varphi + \varepsilon}\right] = 0,$$

yielding

$$\mathrm{E}_{\mathcal{L}}\left[\Lambda_k\right] \leq \Phi_n^{\mathrm{end}}\left[n^{\theta^*(\varphi+2\delta)} P_n^{\mathrm{end}} + e^{-\theta^*(\log n)^2/2}\right]. \tag{4.4.8}$$

In a second time, we bound $\mathrm{E}_{\mathcal{L}}[\Lambda_k]$ for $k \leq b_n$. By (4.4.2), as $r_{n,j} \geq \delta \log n$, we have

$$\begin{aligned} f_{k+1}(x) \leq & e^{-K_{k+1}} \mathrm{E}_{\mathcal{L}}^k\left[e^{-\theta^* S_{n-k-1} + K_n} \mathbf{1}_{\left\{\substack{\theta^*(S_{n-k-1}+x) - K_n - \theta^*\varphi \log n \in [0, \delta \log n] \\ \theta^*(S_j + x) \leq K_j + \delta \log n, j \leq n-k-1}\right\}}\right] \\ \leq & n^{\theta^*\varphi} e^{\theta^* x - K_{k+1}} \times \mathbb{P}_{\mathcal{L}}^k\left[\begin{array}{c} \theta^*(S_{n-k-1} + x) - K_n - \theta^*\varphi \log n \in [0, \delta \log n] \\ \theta^*(S_j + x) \leq K_j + \delta \log n, j \leq n-k-1 \end{array}\right]. \end{aligned}$$

We set

$$\begin{aligned} \Phi_n^{\mathrm{start}} &= \max_{k \leq b_n} \mathrm{E}_{\mathcal{L}}\left[\left(\sum_{\ell \in L_{k+1}} \left(1 + e^{\theta^*\ell - \kappa_{k+1}(\theta)}\right)\right)^2\right] \\ P_n^{\mathrm{start}} &= \sup_{k \leq b_n} \sup_{\theta^* x - K_{k+1} \geq -b_n} \mathbb{P}_{\mathcal{L}}^k\left[\begin{array}{c} \theta^*(S_{n-k-1} + x) - K_n - \theta^*\varphi \log n \in [0, \delta \log n], \\ \theta^*(S_j + x) \leq K_j + \delta \log n, j \leq n-k-1 \end{array}\right], \end{aligned}$$

and we recall that by Lemma 4.3.7, for all $\varepsilon > 0$, we have

$$\lim_{n \to +\infty} \mathbf{P}\left[P_n^{\mathrm{start}} \geq n^{-\theta^*\varphi + \varepsilon}\right] = 0.$$

For all $k + 1 \leq b_n$, we have

$$f_{k+1}(x) \leq n^{\theta^*\varphi} e^{\theta^* x - K_{k+1}} \left(P_n^{\mathrm{start}} \mathbf{1}_{\{\theta^* x - K_{k+1} \geq -b_n\}} + \mathbf{1}_{\{\theta^* x - K_{k+1} \leq -b_n\}}\right).$$

Consequently, applying (4.4.1), we have

$$\begin{aligned} \mathrm{E}_{\mathcal{L}}\left[\Lambda_k\right] \leq 2 P_n^{\mathrm{start}} n^{2\theta^*\varphi} \mathrm{E}_{\mathcal{L}}\left[e^{T_k} \mathbf{1}_{\{T_k \leq 2\theta^*\delta \log n\}}\right] \mathrm{E}_{\mathcal{L}}\left[\left(\sum_{\ell \in L_{k+1}} e^{\theta^*\ell - \kappa_{k+1}(\theta^*)}\right)^2\right] \\ + 2 n^{2\theta^*\varphi} e^{-b_n} \mathrm{E}_{\mathcal{L}}\left[\# L_{k+1}^2\right]. \end{aligned}$$

We conclude that, for all $n \geq 1$ large enough,

$$\mathrm{E}_{\mathcal{L}}\left[\Lambda_k\right] \leq \Phi_n^{\text{start}} \left[2 P_n^{\text{start}} n^{2\theta^*(\varphi+\delta)} + e^{-\theta^*(\log n)^2}\right]. \tag{4.4.9}$$

We conclude that for all $n \geq 1$ large enough, by (4.4.7), (4.4.8) and (4.4.9), we have

$$\begin{aligned}\mathrm{E}_{\mathcal{L}}\left[X_n(\delta)^2\right] \leq \mathrm{E}_{\mathcal{L}}\left[X_n(\delta)\right] + 2b_n\Phi_n^{\text{start}}\left(n^{2\theta^*(\varphi+\delta)}P_n^{\text{start}} + e^{-\theta^*(\log n)^2}\right)\\ + n\Phi_n e^{-\theta^*(\log n)^2/2} + b_n\Phi_n^{\text{end}}\left(n^{\theta^*(\varphi+2\delta)}P_n^{\text{end}} + e^{-\theta^*(\log n)^2/2}\right).\end{aligned}$$

By (4.4.6), for any $\eta > 0$, there exists $x \geq 0$ such that for any $n \in \mathbb{N}$,

$$\mathbf{P}\left[\Phi_n^{\text{stat}} \geq x b_n\right] + \mathbf{P}\left[\Phi_n^{\text{end}} \geq x b_n\right] + \mathbf{P}\left[\Phi_n \geq xn\right] \leq \eta.$$

We conclude that for any $\eta > 0$ and $\varepsilon > 0$, we have

$$\limsup_{n\to+\infty} \mathbf{P}\left(\mathrm{E}_{\mathcal{L}}\left[X_n(\delta)^2\right] \geq n^{2\theta^*\delta+\varepsilon}\right) \leq \eta.$$

Letting $\eta \to 0$, we have

$$\lim_{n\to+\infty} \mathbf{P}\left(\mathrm{E}_{\mathcal{L}}\left[X_n(\delta)^2\right] \geq n^{2\theta^*\delta+\varepsilon}\right) = 0. \tag{4.4.10}$$

By the Cauchy-Schwarz inequality, for any $\delta > 0$ we have

$$\mathbb{P}_{\mathcal{L}}\left(M_n \geq \frac{K_n}{\theta^*} - \varphi\log n\right) \geq \mathbb{P}_{\mathcal{L}}\left(X_n(\delta) \geq 1\right) \geq \frac{\mathrm{E}_{\mathcal{L}}\left[X_n(\delta)^2\right]}{\mathrm{E}_{\mathcal{L}}\left[X_n(\delta)\right]^2}.$$

Applying (4.4.5) and (4.4.10), we have

$$\lim_{n\to+\infty} \mathbf{P}\left(\mathrm{E}_{\mathcal{L}}\left[X_n(\delta)^2\right] \geq n^{2\theta^*\delta+\varepsilon} \text{ or } \mathrm{E}_{\mathcal{L}}\left[X_n(\delta)\right] \leq n^{-\theta^*\delta-\varepsilon}\right) = 0.$$

We obtain

$$\lim_{n\to+\infty} \mathbf{P}\left(\mathbb{P}_{\mathcal{L}}\left(M_n \geq \frac{K_n}{\theta^*} - \varphi\log n\right) \leq n^{-4\theta^*\delta-3\varepsilon}\right) = 0,$$

choosing $\varepsilon, \delta > 0$ small enough, we conclude the proof. $\square$

Proof of Theorem 4.1.1. Lemmas 4.4.2 and 4.4.3 can be used to prove Theorem 4.1.1. By Lemma 4.4.2 for $\beta < \varphi$, $\mathbb{P}_{\mathcal{L}}\left(M_n \geq \frac{K_n}{\theta^*} - \beta\log n\right)$ converges to 0 in probability.

We are now left to prove that if $\beta > \varphi$, then

$$\lim_{n\to+\infty} \mathbb{P}_{\mathcal{L}}\left(M_n \geq \frac{K_n}{\theta^*} - \beta\log n\right) = 1 \quad \text{in probability.}$$

To do so, we use, for the first time in this chapter, the assumptions (4.1.2) and (4.1.3). We obtain there exists $\varrho > 1$ such that $\mathbb{P}_{\mathcal{L}}$-a.s.

$$\liminf_{n\to+\infty} \#\{|u| = n\}^{1/n} > \varrho.$$

Let $n \in \mathbb{N}$ and $k \leq n$, we use the Markov property at time k: with high probability, there are at least ϱ^k individuals, each of which being below $\frac{K_k}{\theta^*} + y$ with high probability by

Lemma 4.4.1, that start an independent branching random walk in random environment with law $\mathbb{P}^k_{\mathcal{L}}$. Therefore,

$$\mathbb{P}_{\mathcal{L}}\left(M_n \leq \frac{K_n}{\theta^*} - \beta \log n\right) \leq \mathrm{E}_{\mathcal{L}}\left(\prod_{|u|=k} \varphi_{n,k}(V(u))\right),$$

where $\varphi_{n,k}(x) = \mathbb{P}^k_{\mathcal{L}}\left(M_{n-k} \leq \frac{K_n}{\theta^*} - \beta \log n - x\right)$. Note this function is increasing, thus, for all $y \geq 0$

$$\begin{aligned}
&\mathbb{P}_{\mathcal{L}}\left(M_n \leq \frac{K_n}{\theta^*} - \beta \log n\right)\\
\leq&\mathbb{P}_{\mathcal{L}}\left(\exists |u| = k : \theta^* V(u) - K_k \leq y\right) + \mathrm{E}_{\mathcal{L}}\left(\prod_{|u|=k} \varphi_{n,k}(\frac{K_k}{\theta^*} + y)\right)\\
\leq&e^{-y} + \mathrm{E}_{\mathcal{L}}\left[\left(\mathbb{P}^k_{\mathcal{L}}\left(M_{n-k} \leq \frac{K_n - K_k}{\theta^*} - \beta \log n + y\right)\right)^{\#\{|u|=k\}}\right].
\end{aligned}$$

Let $\varepsilon > 0$, we set $k = \varepsilon \log n$ and obtain

$$\begin{aligned}
&\mathbb{P}_{\mathcal{L}}\left(M_n \leq \frac{K_n}{\theta^*} - \beta \log n\right)\\
\leq&e^{-y} + \mathbb{P}_{\mathcal{L}}\left[\#\{|u| = k\} \leq \varrho^k\right] + \mathrm{E}_{\mathcal{L}}\left[\left(\mathbb{P}^k_{\mathcal{L}}\left(M_{n-k} \leq \frac{K_n - K_k}{\theta^*} - \beta \log n + y\right)\right)^{n^{\varepsilon \log \varrho}}\right].
\end{aligned}$$

For any $n \geq 1$ large enough, we have

$$\begin{aligned}
&\left(\mathbb{P}^k_{\mathcal{L}}\left(M_{n-k} \leq \frac{K_n - K_k}{\theta^*} - \beta \log n + y\right)\right)^{n^{\varepsilon \log \varrho}}\\
&\qquad\qquad \leq \left(\mathbb{P}^k_{\mathcal{L}}\left(M_{n-k} \leq \frac{K_n - K_k}{\theta^*} - \varphi \log(n-k)\right)\right)^{n^{\varepsilon \log \varrho}},
\end{aligned}$$

and as $\mathbb{P}^k_{\mathcal{L}}\left(M_{n-k} \geq \frac{K_n - K_k}{\theta^*} - \varphi \log(n-k)\right) \overset{(d)}{=} \mathbb{P}_{\mathcal{L}}\left(M_{n-k} \geq \frac{K_{n-k}}{\theta^*} - \varphi \log(n-k)\right)$, we have

$$\lim_{n \to +\infty} \left(1 - \mathbb{P}^k_{\mathcal{L}}\left(M_{n-k} \geq \frac{K_n - K_k}{\theta^*} - \beta \log n + y\right)\right)^{n^{\varepsilon \log \varrho}} = 0 \text{ in probability.}$$

By dominated convergence, we conclude that

$$\limsup_{n \to +\infty} \mathbf{P}\left[\mathbb{P}_{\mathcal{L}}\left(M_n \leq \frac{K_n}{\theta^*} - \beta \log n\right) \geq e^{-y}\right] = 0.$$

This estimate holding for every $y > 0$, we conclude that

$$\lim_{n \to +\infty} \mathbb{P}_{\mathcal{L}}\left(M_n \leq \frac{K_n}{\theta^*} - \beta \log n\right) = 0 \quad \text{in probability,}$$

which ends the proof of Theorem 4.1.1. $\square$

Part II

Branching random walk with selection

CHAPTER 5

The branching random walk with increasing selection

"One general law, leading to the advancement of all organic beings, namely, multiply, vary, let the strongest live and the weakest die."

Charles Darwin – The Origin of Species

Abstract

We consider in this chapter a branching-selection particle system on the real line. In this model the total size of the population at time n is limited by $\exp\left(an^{1/3}\right)$. At each step n, every individual dies while reproducing independently, making children around their current position according to i.i.d. point processes. Only the $\exp\left(a(n+1)^{1/3}\right)$ rightmost children survive to form the $(n+1)^{\text{th}}$ generation. This process can be seen as a generalisation of the branching random walk with selection of the N rightmost individuals, introduced by Brunet and Derrida in [BD97]. We obtain the asymptotic behaviour of position of the extremal particles alive at time n by coupling this process with a branching random walk with a killing boundary.

Nota: This chapter is a slight modification of the article *Branching random walk with selection at critical rate*, available on arXiv:1502.07390.

5.1 Introduction

Let $\mathcal{L}$ be the law of a point process on $\mathbb{R}$. A branching random walk on $\mathbb{R}$ with reproduction law $\mathcal{L}$ is a particle process defined as follows: it starts at time 0 with a unique individual $\emptyset$ positioned at 0. At time 1, this individual dies giving birth to children which are positioned according to a point process of law $\mathcal{L}$. Then at each time $k \in \mathbb{N}$, each individual in the process dies, giving birth to children which are positioned according to i.i.d. point processes of law $\mathcal{L}$, shifted by the position of their parent. We denote by $\mathbf{T}$ the genealogical tree of the process, encoded with the Ulam-Harris notation. Note that $\mathbf{T}$ is a Galton-Watson tree. For a given individual $u \in \mathbf{T}$, we write $V(u) \in \mathbb{R}$ for the position of u, and $|u| \in \mathbb{Z}_+$ for the generation of u. If u is not the initial individual, we denote by πu the parent of u. The marked Galton-Watson tree $(\mathbf{T}, V)$ is the branching random walk on $\mathbb{R}$ with reproduction law $\mathcal{L}$.

Let L be a point process with law $\mathcal{L}$. In this chapter again, we assume the Galton-Watson tree $\mathbf{T}$ never get extinct and is supercritical, i.e.

$$\mathbf{P}\left(\#L = 0\right) = 0 \quad \text{and} \quad \mathbf{E}\left[\#L\right] > 1. \tag{5.1.1}$$

We also assume the branching random walk $(-V, \mathbf{T})$ to be in the so-called boundary case, with the terminology of [BK04]:

$$\mathbf{E}\left[\sum_{\ell \in L} e^{\ell}\right] = 1, \ \ \mathbf{E}\left[\sum_{\ell \in L} \ell e^{\ell}\right] = 0 \ \text{ and } \ \sigma^2 := \mathbf{E}\left[\sum_{\ell \in L} \ell^2 e^{\ell}\right] < +\infty. \tag{5.1.2}$$

Under mild assumptions, discussed in [Jaf12, Appendix A], there exists an affine transformation mapping a branching random walk into a branching random walk in the boundary case. We impose that

$$\mathbf{E}\left[\sum_{\ell \in L} e^{\ell} \log\left(\sum_{\ell' \in L} e^{\ell' - \ell}\right)^2\right] < +\infty. \tag{5.1.3}$$

Under slightly stronger integrability conditions, Aïdékon [Aïd13] proved that

$$\max_{|u|=n} V(u) + \frac{3}{2}\log n \underset{n \to +\infty}{\Longrightarrow} W,$$

where W is a random shift of a negative Gumble distribution.

In [BD97], Brunet and Derrida described a discrete-time particle system [1] on $\mathbb{Z}$ in which the total size of the population remains constant equal to N. At each time k, individuals alive reproduce in the same way as in a branching random walk, but only the N rightmost individuals are kept alive to form the $(k+1)^{\text{th}}$ generation. This process is called the N-branching random walk. They conjectured that the cloud of particles in the process moves at some deterministic speed v_N, satisfying

$$v_N = -\frac{\pi^2\sigma^2}{2(\log N)^2}\left(1 + \frac{(6 + o(1))\log\log N}{\log N}\right) \quad \text{as } N \to +\infty.$$

Bérard and Gouéré [BG10] proved that in a N-branching random walk satisfying some stronger integrability conditions, the cloud of particles moves at linear speed v_N on $\mathbb{R}$, i.e. writing m_n^N, M_n^N respectively the minimal and maximal position at time n, we have

$$\lim_{n \to +\infty} \frac{M_n^N}{n} = \lim_{n \to +\infty} \frac{m_n^N}{n} = v_N \text{ a.s. and } \lim_{N \to +\infty} (\log N)^2 v_N = -\frac{\pi^2\sigma^2}{2},$$

partially proving the Brunet-Derrida conjecture.

We introduce a similar model of branching-selection process. We set $\varphi : \mathbb{N} \to \mathbb{N}$, and we consider a process with selection of the $\varphi(n)$ rightmost individuals at generation n. More precisely we define $\mathbf{T}^\varphi$ as a non-empty subtree of $\mathbf{T}$, such that $\emptyset \in \mathbf{T}^\varphi$ and the generation $k \in \mathbb{N}$ is composed of the $\varphi(k)$ children of $\{u \in \mathbf{T}^\varphi : |u| = k-1\}$ with largest positions, with ties broken uniformly at random [2]. The marked tree $(\mathbf{T}^\varphi, V)$ is the branching random walk with selection of the $\varphi(n)$ rightmost individuals at time n. We write

$$m_n^\varphi = \min_{u \in \mathbf{T}^\varphi, |u|=n} V(u) \quad \text{and} \quad M_n^\varphi = \max_{u \in \mathbf{T}^\varphi, |u|=n} V(u). \tag{5.1.4}$$

The main result of the chapter is the following.

1. Extended in [BDMM07] to a particle system on $\mathbb{R}$.
2. Or in any other predictable fashion.

Theorem 5.1.1. *Let $a > 0$, we set $\varphi(n) = \left\lfloor \exp\left(an^{1/3}\right)\right\rfloor$. Under assumptions* (5.1.1), (5.1.2) *and* (5.1.3) *we have*

$$M_n^\varphi \sim_{n\to+\infty} -\frac{3\pi^2\sigma^2}{2a^2} n^{1/3} \quad \text{a.s.} \tag{5.1.5}$$

$$m_n^\varphi \sim -\left(\frac{3\pi^2\sigma^2}{2a^2} n^{1/3} + a\right) n^{1/3} \quad \text{a.s.} \tag{5.1.6}$$

We prove Theorem 5.1.1 using a coupling between the branching random walk with selection and a branching random walk with a killing boundary, introduced in [BG10]. We also provide in this chapter the asymptotic behaviour of the extremal positions in a branching random walk with a killing boundary; and the asymptotic behaviour of the extremal positions in a branching random walk with selection of the $\left\lfloor e^{h_{k/n} n^{1/3}} \right\rfloor$ at time $k \leq n$, where h is a positive continuous function.

We consider in this chapter populations with $e^{an^{1/3}}$ individuals on the interval of time $[0, n]$. This rate of growth is in some sense critical. More precisely in [BDMM07], the branching random walk with selection of the N rightmost individuals is conjectured to typically behave at the time scale $(\log N)^3$. This observation has been confirmed by the results of [BG10, BBS13, Mai13]. Using methods similar to the ones developed here, or in [BG10], one can prove that the maximal displacement in a branching random walk with selection of the e^{an^α} rightmost individuals behaves as $-\frac{\pi^2\sigma^2}{2(1-2\alpha)a^2} n^{1-2\alpha}$ for $\alpha < 1/2$. If $\alpha > 1/2$, it is expected that the behaviour of the maximal displacement in the branching random walk with selection is similar to the one of the classical branching random walk, of order $\log n$.

We recall that c, C stand for positive constants, respectively small enough and large enough, which may change from line to line and depend only on the law of the processes we consider. Moreover, the set $\{|u| = n\}$ represents the set of individuals alive at the n^{th} generation in a generic branching random walk $(\mathbf{T}, V)$ with reproduction law $\mathcal{L}$.

The rest of the chapter is organised as follows. In Section 5.2, we introduce the spinal decomposition of the branching random walk, the Mogul'skiĭ small deviation estimate and lower bounds on the total size of the population in a Galton-Watson process. Using these results, we study in Section 5.3 the behaviour of a branching random walk with a killing boundary. Section 5.4 is devoted to the study of branching random walks with selection, that we use to prove Theorem 5.1.1.

5.2 Some useful lemmas

5.2.1 The spinal decomposition of the branching random walk

For any $a \in \mathbb{R}$, we write $\mathbf{P}_a$ for the probability distribution of $(\mathbf{T}, V + a)$ the branching random walk with initial individual positioned at a, and $\mathbf{E}_a$ for the corresponding expectation. To shorten notation, we set $\mathbf{P} = \mathbf{P}_0$ and $\mathbf{E} = \mathbf{E}_0$. We write $\mathcal{F}_n = \sigma(u, V(u), |u| \leq n)$ for the natural filtration on the set of marked trees. Let $W_n = \sum_{|u|=n} e^{V(u)}$. By (5.1.2), we observe that (W_n) is a non-negative martingale with respect to the filtration $(\mathcal{F}_n)$. We define a new probability measure $\overline{\mathbf{P}}_a$ on $\mathcal{F}_\infty$ such that for all $n \in \mathbb{N}$,

$$\left.\frac{d\overline{\mathbf{P}}_a}{d\mathbf{P}_a}\right|_{\mathcal{F}_n} = e^{-a} W_n. \tag{5.2.1}$$

We write $\overline{\mathbf{E}}_a$ for the corresponding expectation and $\overline{\mathbf{P}} = \overline{\mathbf{P}}_0$, $\overline{\mathbf{E}} = \overline{\mathbf{E}}_0$. The so-called spinal decomposition, introduced in branching processes by Lyons, Pemantle and Peres in [LPP95], and extended to branching random walks by Lyons in [Lyo97] gives an alternative construction of the measure $\overline{\mathbf{P}}_a$, by introducing a special individual with modified reproduction law.

Let L be a point process with law $\mathcal{L}$, we introduce the law $\widehat{\mathcal{L}}$ defined by

$$\frac{d\widehat{\mathcal{L}}}{d\mathcal{L}}(L) = \sum_{\ell \in L} e^{\ell}. \tag{5.2.2}$$

We describe a probability measure $\widehat{\mathbf{P}}_a$ on the set of marked trees with spine $(\mathbf{T}, V, w)$, where $(\mathbf{T}, V)$ is a marked tree, and $w = (w_n, n \in \mathbb{N})$ is a sequence of individuals such that for any $n \in \mathbb{N}$, $w_n \in \mathbf{T}$, $|w_n| = n$ and $\pi w_n = w_{n-1}$. The ray w is called the spine of the branching random walk.

Under law $\widehat{\mathbf{P}}_a$, the process starts at time 0 with a unique individual $w_0 = \emptyset$ located at position a. It generates its children according to a point process of law $\widehat{\mathcal{L}}$. Individual w_1 is chosen at random among the children u of w_0 with probability proportional to $e^{V(u)}$. At each time $n \in \mathbb{N}$, every individual u in the n^{th} generation die, giving independently birth to children according to the measure $\mathcal{L}$ if $u \neq w_n$ and $\widehat{\mathcal{L}}$ if $u = w_n$. Finally, w_{n+1} is chosen at random among the children v of w_n with probability proportional to $e^{V(v)}$.

Proposition 5.2.1 (Spinal decomposition)**.** *Under assumption* (5.1.2)*, for all $n \in \mathbb{N}$, we have*

$$\left.\widehat{\mathbf{P}}_a\right|_{\mathcal{F}_n} = \left.\overline{\mathbf{P}}_a\right|_{\mathcal{F}_n}.$$

Moreover, for all $u \in \mathbf{T}$ such that $|u| = n$,

$$\widehat{\mathbf{P}}_a\left(w_n = u \middle| \mathcal{F}_n\right) = \frac{e^{V(u)}}{W_n},$$

and $(V(w_n), n \geq 0)$ is a centred random walk starting from a with variance σ^2

This proposition in particular implies the following result, often called in the literature the many-to-one lemma, which has been introduced for the first time by Kahane and Peyrière in [KP76, Pey74], and links additive moments of the branching random walks with random walk estimates.

Lemma 5.2.2 (Many-to-one lemma)**.** *There exists a centred random walk $(S_n, n \geq 0)$, starting from a under $\mathbf{P}_a$, with variance σ^2 such that for any $n \geq 1$ and any measurable non-negative function g, we have*

$$\mathbf{E}_a\left[\sum_{|u|=n} g(V(u_1), \cdots V(u_n))\right] = \mathbf{E}_a\left[e^{a-S_n} g(S_1, \cdots S_n)\right]. \tag{5.2.3}$$

Proof. We use Proposition 5.2.1 to compute

$$\begin{aligned}
\mathbf{E}_a\left[\sum_{|u|=n} g(V(u_1), \cdots V(u_n))\right] &= \overline{\mathbf{E}}_a\left[\frac{e^a}{W_n}\sum_{|z|=n} g(V(u_1), \cdots V(u_n))\right]\\
&= \widehat{\mathbf{E}}_a\left[e^a \sum_{|u|=n} \mathbf{1}_{\{u=w_n\}} e^{-V(u)} g(V(u_1), \cdots V(u_n))\right]\\
&= \widehat{\mathbf{E}}_a\left[e^{a-V(w_n)} g(V(w_1), \cdots, V(w_n))\right].
\end{aligned}$$

Therefore we define the random walk S under $\mathbf{P}_a$ as a process with the same law as $(V(\omega_n), n \geq 0)$ under $\widehat{\mathbf{P}}_a$, which ends the proof. Note that for any continuous bounded function,

$$\mathbf{E}_a(f(S_1 - a)) = \mathbf{E}\left[\sum_{\ell \in L} e^{\ell} f(\ell)\right].$$

□

Using the many-to-one lemma, to compute the number of individuals in a branching random walk who stay in a well-chosen path, we only need to understand the probability for a random walk to stay in this path. This is what is done in the next section.

5.2.2 Small deviation estimate and variations

The following theorem gives asymptotic bounds for the probability for a random walk to have small deviations, i.e., to stay until time n within distance significantly smaller than $\sqrt{n}$ from the origin. Let $(S_n, n \geq 0)$ be a centred random walk on $\mathbb{R}$ with finite variance σ^2. We assume that for any $x \in \mathbb{R}$, $\mathbf{P}_x(S_0 = x) = 1$ and we set $\mathbf{P} = \mathbf{P}_0$.

Theorem 5.2.3 (Mogul'skiĭ estimate [Mog74])**.** *Let $f < g$ be continuous functions on $[0,1]$ such that $f_0 < 0 < g_0$ and (a_n) a sequence of positive numbers such that*

$$\lim_{n \to +\infty} a_n = +\infty \quad \text{and} \quad \lim_{n \to +\infty} \frac{a_n^2}{n} = 0.$$

For any $f_1 \leq x < y \leq g_1$, we have

$$\lim_{n \to +\infty} \frac{a_n^2}{n} \log \mathbf{P}\left[\frac{S_n}{a_n} \in [x,y], \frac{S_j}{a_n} \in \left[f_{j/n}, g_{j/n}\right], j \leq n\right] = -\frac{\pi^2 \sigma^2}{2} \int_0^1 \frac{ds}{(g_s - f_s)^2}. \tag{5.2.4}$$

In the rest of this chapter, we use some modifications of the Mogul'skiĭ theorem, choosing $a_n = n^{1/3}$. We start with a straightforward corollary: the Mogul'skiĭ theorem holds uniformly with respect to the starting point.

Corollary 5.2.4. *Let $f < g$ be continuous functions on $[0,1]$ such that $f_0 < g_0$ and (a_n) a sequence of positive numbers such that*

$$\lim_{n \to +\infty} a_n = +\infty \quad \text{and} \quad \lim_{n \to +\infty} \frac{a_n^2}{n} = 0.$$

For any $f_1 \leq x < y \leq g_1$, we have

$$\lim_{n \to +\infty} \frac{a_n^2}{n} \log \sup_{z \in \mathbb{R}} \mathbf{P}_{z a_n}\left[\frac{S_n}{a_n} \in [x,y], \frac{S_j}{a_n} \in \left[f_{j/n}, g_{j/n}\right], j \leq n\right] = -\frac{\pi^2 \sigma^2}{2} \int_0^1 \frac{ds}{(g_s - f_s)^2}. \tag{5.2.5}$$

Proof. We observe that

$$\sup_{z \in \mathbb{R}} \mathbf{P}_{z a_n}\left[\frac{S_n}{a_n} \in [x,y], \frac{S_j}{a_n} \in \left[f_{j/n}, g_{j/n}\right], j \leq n\right] \\ \geq \mathbf{P}_{a_n \frac{f_0+g_0}{2}}\left[\frac{S_n}{a_n} \in [x,y], \frac{S_j}{a_n} \in \left[f_{j/n}, g_{j/n}\right], j \leq n\right].$$

Therefore, applying Theorem 5.2.3, we have

$$\liminf_{n\to+\infty} \frac{a_n^2}{n} \log \sup_{z\in\mathbb{R}} \mathbf{P}_{za_n}\left[\frac{S_n}{a_n} \in [x,y], \frac{S_j}{a_n} \in \left[f_{j/n}, g_{j/n}\right], j \leq n\right] \geq -\frac{\pi^2\sigma^2}{2}\int_0^1 \frac{ds}{(g_s-f_s)^2}.$$

We choose $\delta > 0$, and set $M = \left\lceil \frac{g_0-f_0}{\delta} \right\rceil$. We observe that

$$\mathbf{P}_{za_n}\left[\frac{S_n}{a_n} \in [x,y], \frac{S_j}{a_n} \in \left[f_{j/n}, g_{j/n}\right], j \leq n\right] = 0,$$

thus

$$\begin{aligned}
&\sup_{z\in\mathbb{R}} \mathbf{P}_{za_n}\left[\frac{S_n}{a_n} \in [x,y], \frac{S_j}{a_n} \in \left[f_{j/n}, g_{j/n}\right], j \leq n\right] \\
&= \max_{0\leq k\leq M-1} \sup_{z\in[f_0+k\delta, f_0+(k+1)\delta]} \mathbf{P}_{za_n}\left[\frac{S_n}{a_n} \in [x,y], \frac{S_j}{a_n} \in \left[f_{j/n}, g_{j/n}\right], j \leq n\right] \\
&\leq \max_{0\leq k\leq M-1} \mathbf{P}_{a_n(f_0+k\delta)}\left[\frac{S_n}{a_n} \in [x,y+\delta], \frac{S_j}{a_n} \in \left[f_{j/n}, g_{j/n}+\delta\right], j \leq n\right].
\end{aligned}$$

As a consequence, we have

$$\limsup_{n\to+\infty} \frac{a_n^2}{n} \log \sup_{z\in\mathbb{R}} \mathbf{P}_{za_n}\left[\frac{S_n}{a_n} \in [x,y], \frac{S_j}{a_n} \in \left[f_{j/n}, g_{j/n}\right], j \leq n\right] \leq -\frac{\pi^2\sigma^2}{2}\int_0^1 \frac{ds}{(g_s-f_s+\delta)^2}.$$

Letting $\delta \to 0$ ends the proof. $\square$

We present a more involved result on enriched random walks, a useful toy-model to study the spine of the branching random walk. The following lemma is proved by mocking the original proof of Mogul'skiĭ.

Lemma 5.2.5 (Mogul'skiĭ estimate for spine)**.** *Let $((X_j,\xi_j), j \in \mathbb{N})$ be an i.i.d. sequence of random variables taking values in $\mathbb{R}\times\mathbb{R}_+$, such that*

$$\mathbf{E}(X_1) = 0 \quad \textit{and} \quad \sigma^2 := \mathbf{E}(X_1^2) < +\infty.$$

We write $S_n = \sum_{j=1}^n X_j$ and $E_n = \{\xi_j \leq n, j \leq n\}$. Let $(a_n) \in \mathbb{R}_+^{\mathbb{N}}$ be such that

$$\lim_{n\to+\infty} a_n = +\infty, \quad \lim_{n\to+\infty} \frac{a_n^2}{n} = 0 \quad \textit{and} \quad \lim_{n\to+\infty} a_n^2 \mathbf{P}(\xi_1 \geq n) = 0.$$

Let $f < g$ be two continuous functions. For all $f_0 < x < y < g_0$ and $f_1 < x' < y' < g_1$, we have

$$\lim_{n\to+\infty} \frac{a_n^2}{n} \inf_{z\in[x,y]} \log \mathbf{P}_{za_n}\left(\frac{S_n}{a_n} \in [x',y'] \frac{S_j}{a_n} \in \left[f_{j/n}, g_{j/n}\right], j \leq n, E_n\right) = -\frac{\pi^2\sigma^2}{2}\int_0^1 \frac{ds}{(g_s-f_s)^2}.$$

Proof. For any $z \in [x,y]$, we have

$$\mathbf{P}_{za_n}\left(\frac{S_n}{a_n} \in [x',y'], \frac{S_j}{a_n} \in \left[f_{j/n}, g_{j/n}\right], j \leq n, E_n\right) \leq \sup_{h\in\mathbb{R}} \mathbf{P}_{ha_n}\left(\frac{S_j}{a_n} \in \left[f_{j/n}, g_{j/n}\right], j \leq n\right).$$

So the upper bound in this lemma is a direct consequence of Corollary 5.2.4. We now consider the lower bound.

We suppose in a first time that f and g are two constants. Let $n \geq 1$, $f < x < y < g$ and $f < x' < y' < g$, we bound from below the quantity

$$P_{x,y}^{x',y'}(f,g) = \inf_{z \in [x,y]} \mathbf{P}_{za_n}\left(\frac{S_n}{a_n} \in [x',y'], \frac{S_j}{a_n} \in [f,g], j \leq n, E_n\right).$$

Setting $A \in \mathbb{N}$ and $r_n = \lfloor Aa_n^2 \rfloor$, we divide $[0,n]$ into $K = \left\lfloor \frac{n}{r_n} \right\rfloor$ intervals of length r_n. For $k \leq K$, we write $m_k = kr_n$, and $m_{K+1} = n$. By restriction to the set of trajectories verifying $S_{m_k} \in [x'a_n, y'a_n]$, and applying the Markov property at time $m_K, \ldots m_1$, and restricting to trajectories which are at any time m_k in $[x'a_n, y'a_n]$, we have

$$P_{x,y}^{x',y'}(f,g) \geq \pi_{x,y}^{x',y'}(f,g)\left(\pi_{x',y'}^{x',y'}\right)^K, \tag{5.2.6}$$

writing

$$\pi_{x,y}^{x',y'}(f,g) = \inf_{z \in [x,y]} \mathbf{P}_{za_n}\left(\frac{S_{r_n}}{a_n} \in [x',y'], \frac{S_j}{a_n} \in [f,g], j \leq r_n, E_{r_n}\right).$$

Let $\delta > 0$ chosen small enough such that $M = \left\lceil \frac{y-x}{\delta} \right\rceil \geq 3$ we observe easily that

$$\begin{aligned}
\pi_{x,y}^{x',y'}(f,g) &\geq \min_{0 \leq m \leq M} \pi_{x+m\delta, x+(m+1)\delta}^{x',y'}(f,g) \\
&\geq \min_{0 \leq m \leq M} \pi_{x,x}^{x'-(m-1)\delta, y-(m+1)\delta}(f-(m-1)\delta, g-(m+1)\delta).
\end{aligned} \tag{5.2.7}$$

Moreover, we have

$$\begin{aligned}
\pi_{x,x}^{x',y'}(f,g) &= \mathbf{P}_{xa_n}\left(\frac{S_{r_n}}{a_n} \in [x',y'], \frac{S_j}{a_n} \in [f,g], E_{r_n}\right) \\
&\geq \mathbf{P}_{xa_n}\left(\frac{S_{r_n}}{a_n} \in [x',y'], \frac{S_j}{a_n} \in [f,g]\right) - r_n \mathbf{P}(\xi_1 \geq n).
\end{aligned}$$

Using the Donsker theorem [Don51], $\left(\frac{S_{\lfloor r_n t \rfloor}}{a_n}, t \in [0,1]\right)$ converges, under law $\mathbf{P}_{xa_n}$, as n grows to infinity to a Brownian motion with variance $\sigma\sqrt{A}$ starting from x. In particular

$$\liminf_{n \to +\infty} \pi_{x,x}^{x',y'}(f,g) \geq \mathbf{P}_x(B_{A\sigma^2} \in (x',y'), B_u \in (f,g), u \leq A\sigma^2).$$

Using (5.2.7), we have

$$\liminf_{n \to +\infty} \pi_{x,y}^{x',y'}(f,g) \geq \min_{0 \leq m \leq M} \mathbf{P}_{x+m\delta}(B_{A\sigma^2} \in (x'+\delta, y'-\delta), B_u \in (f+\delta, g-\delta), u \leq A\sigma^2).$$

As a consequence, recalling that $K \sim \frac{n}{Aa_n^2}$, (5.2.6) leads to

$$\begin{aligned}
&\liminf_{n \to +\infty} \frac{a_n^2}{n} \log P_{x,y}^{x',y'}(f,g) \geq \\
&\qquad \frac{1}{A} \min_{0 \leq m \leq M} \log \mathbf{P}_{x+m\delta}(B_{A\sigma^2} \in (x'+\delta, y'-\delta), B_u \in (f+\delta, g-\delta), u \leq A\sigma^2).
\end{aligned} \tag{5.2.8}$$

According to Karatzas and Shreve [KS91], probability $\mathbf{P}_x(B_t \in (x',y'), B_s \in (f,g), s \leq t)$ is exactly computable, and

$$\lim_{t \to +\infty} \frac{1}{t} \log_x \mathbf{P}(B_t \in (x',y'), B_s \in (f,g), s \leq t) = -\frac{\pi^2}{2(g-f)^2}.$$

Letting $A \to +\infty$ then $\delta \to 0$, (5.2.8) becomes

$$\liminf_{n\to+\infty} \frac{a_n^2}{n} \log P_{x,y}^{x',y'}(f,g) \geq -\frac{\pi^2\sigma^2}{2(g-f)^2}. \tag{5.2.9}$$

We now take care of the general case. Let $f < g$ be two continuous functions such that $f_0 < 0 < g_0$. We write $h_t = \frac{f_t+g_t}{2}$. Let $\varepsilon > 0$ be such that

$$12\varepsilon \leq \inf_{t\in[0,1]} g_t - f_t$$

and $A \in \mathbb{N}$ such that

$$\sup_{|t-s|\leq \frac{2}{A}} |f_t - f_s| + |g_t - g_s| + |h_t - h_s| \leq \varepsilon.$$

For any $a \leq A$, we write $m_a = \lfloor an/A \rfloor$,

$$I_{a,A} = [f_{a/A} + \varepsilon, g_{a/A} - \varepsilon] \quad \text{and} \quad J_{a,A} = [h_{a/A} - \varepsilon, h_{a/A} + \varepsilon],$$

except $J_{0,A} = [x, y]$ and $J_{A,A} = [x', y']$.

We apply the Markov property at times $m_{A-1}, \ldots, m_1$, we have

$$\begin{aligned}\inf_{z\in J_{0,A}} \mathbf{P}_{za_n}\left(\frac{S_j}{a_n} \in \left[f_{j/n}, g_{j/n}\right], j \leq n, E_n\right)&\\ \geq \prod_{a=0}^{A-1} \inf_{z\in J_{a,A}} \mathbf{P}_{za_n}&\left(\frac{S_{m_{a+1}}}{a_n} \in J_{a+1,A}, E_{m_{a+1}-m_a} \frac{S_j}{a_n} \in I_{a,A}, j \leq m_{a+1} - m_a\right).\end{aligned}$$

Applying equation (5.2.9), we conclude

$$\begin{aligned}\liminf_{n\to+\infty} \frac{a_n^2}{n} \log \inf_{z\in J_{0,A}} \mathbf{P}_{za_n}\left(\frac{S_j}{a_n} \in \left[f_{j/n}, g_{j/n}\right] \quad \text{and} \quad \xi_j \leq n, j \leq n\right)&\\ \geq -\frac{1}{A}\sum_{a=0}^{A-1} \frac{\pi^2\sigma^2}{2(g_{a,A} - f_{a,A} - 2\varepsilon)^2}&.\end{aligned}$$

Letting $\varepsilon \to 0$ then $A \to +\infty$, we conclude the proof. $\square$

Lemma 5.2.5 is extended in the following fashion, to take into account functions g such that $g(0) = 0$.

Corollary 5.2.6. *Let $((X_j, \xi_j), j \in \mathbb{N})$ be an i.i.d. sequence of random variables taking values in $\mathbb{R} \times \mathbb{R}_+$ such that*

$$\mathbf{E}(X_1) = 0 \quad \textit{and} \quad \sigma^2 := \mathbf{E}(X_1^2) < +\infty.$$

We write $S_n = \sum_{j=1}^n X_j$ and $E_n = \{\xi_j \leq n, j \leq n\}$. Let $(a_n) \in \mathbb{R}_+^{\mathbb{N}}$ verifying

$$\lim_{n\to+\infty} a_n = +\infty, \quad \limsup_{n\to+\infty} \frac{a_n^3}{n} < +\infty \quad \textit{and} \quad \lim_{n\to+\infty} a_n^2 \mathbf{P}(\xi_1 \geq n) = 0.$$

Let $f < g$ be two continuous functions such that $f_0 < 0$ and $\liminf_{t\to 0} \frac{g_t}{t} > -\infty$. For any $f_1 \leq x' < y' \leq g_1$, we have

$$\lim_{n\to+\infty} \frac{a_n^2}{n} \log \mathbf{P}\left(\frac{S_n}{a_n} \in [x', y'], \frac{S_j}{a_n} \in [f_{j/n}, g_{j/n}], j \leq n, E_n\right) = -\frac{\pi^2\sigma^2}{2}\int_0^1 \frac{ds}{(g_s - f_s)^2}.$$

Proof. Let $d > 0$ be such that for all $t \in [0,1]$, $g(t) \geq -dt$. We set $x < y < 0$ and $A > 0$ verifying $\mathbf{P}(X_1 \in [x,y], \xi_1 \leq A) > 0$. For any $\delta > 0$, we set $N = \lfloor \delta a_n \rfloor$. Applying the Markov property at time N, for any $n \in \mathbb{N}$ large enough, we have

$$\mathbf{P}\left(\frac{S_n}{a_n} \in [x',y'], \frac{S_j}{a_n} \in \left[f_{j/n}, g_{j/n}\right], j \leq n, E_n\right) \geq \mathbf{P}\left(S_j \in [jx, jy], j \leq N, E_N\right)$$
$$\times \inf_{z \in [2\delta x, \delta y/2]} \mathbf{P}_{za_n}\left(\frac{S_{n-N}}{a_n} \in [x',y'], \frac{S_{j-N}}{a_n} \in \left[f_{\frac{j+N}{n}}, g_{\frac{j+N}{n}}\right], j \leq n-N, E_{n-N}\right)$$

with $\mathbf{P}\left(S_j \in [jx, jy], j \leq N, E_N\right) \geq \mathbf{P}\left(X_1 \in [x,y], \xi_1 \leq A\right)^N$. As $\limsup_{n\to+\infty} \frac{a_n^3}{n} < +\infty$, we have

$$\liminf_{n\to+\infty} \frac{a_n^2}{n} \log \mathbf{P}\left(\frac{S_n}{a_n} \in [x',y'], \frac{S_j}{a_n} \in [f_{j/n}, g_{j/n}], j \leq n, E_n\right)$$
$$\geq \liminf_{n\to+\infty} \frac{a_n^2}{n} \inf_{z \in [2\delta x, \delta y/2]} \mathbf{P}_{za_n}\left(\begin{array}{l} \frac{S_{j-N}}{a_n} \in \left[f_{\frac{j+N}{n}}, g_{\frac{j+N}{n}}\right], j \leq n-N \\ \frac{S_{n-N}}{a_n} \in [x',y'], E_{n-N} \end{array}\right).$$

Consequently, applying Lemma 5.2.5 and letting $\delta \to 0$, we have

$$\liminf_{n\to+\infty} \frac{a_n^2}{n} \log \mathbf{P}\left(\frac{S_n}{a_n} \in [x',y'], \frac{S_j}{a_n} \in [f_{j/n}, g_{j/n}], j \leq n, E_n\right) \geq -\frac{\pi^2\sigma^2}{2}\int_0^1 \frac{ds}{(g_s - f_s)^2}.$$

The upper bound is a direct consequence of Corollary 5.2.4. □

5.2.3 Lower bounds for the total size of the population above a barrier

To prove Theorem 5.1.1, we need an almost sure lower bound on the size of the population in a branching random walk that stay above a given boundary. We obtain this lower bound using Lemma 1.5.1, that bounds from below the size of the population in a supercritical Galton-Watson process. This lemma is recalled below.

Lemma 5.2.7. *Let $(Z_n, n \geq 0)$ be a Galton-Watson process with reproduction law μ. We write $b = \min\{k \in \mathbb{Z}_+ : \mu(k) > 0\}$, $m = \mathbf{E}(Z_1) \in (1, +\infty)$ and q the smallest solution of the equation $\mathbf{E}(q^{Z_1}) = q$. There exists $C > 0$ such that for all $z \in (0,1)$ and $n \in \mathbb{N}$ we have*

$$\mathbf{P}(Z_n \leq zm^n) \leq \begin{cases} q + Cz^{\frac{\alpha}{\alpha+1}} & \text{if } b = 0 \\ Cz^\alpha & \text{if } b = 1 \\ \exp\left[-Cz^{-\frac{\log b}{\log m - \log b}}\right] & \text{if } b \geq 2. \end{cases}$$

This result is used to obtain a lower bound on the size of the population in a branching random walk above a given position.

Lemma 5.2.8. *Under assumptions* (5.1.1) *and* (5.1.3)*, there exist $a > 0$ and $\varrho > 1$ such that a.s. for $n \geq 1$ large enough*

$$\#\left\{|u| = n : \forall j \leq n, V(u_j) \geq -na\right\} \geq \varrho^n.$$

Proof. As $\lim_{a\to+\infty} \mathbf{E}\left[\sum_{|u|=1} \mathbf{1}_{\{V(u) \geq -a\}}\right] = \mathbf{E}\left[\sum_{|u|=1} 1\right]$, by (5.1.1), there exists $a > 0$ such that $\varrho_1 := \mathbf{E}\left[\sum_{|u|=1} \mathbf{1}_{\{V(u) \leq a\}}\right] > 1$. We write $N = \sum_{|u|=1} \mathbf{1}_{\{V(u) \geq -a\}}$. By (5.1.3), we have $\mathbf{E}(N) < +\infty$. One can easily couple a Galton-Watson process Z with reproduction law N with the branching random walk $(\mathbf{T}, V)$ in a way that

$$\sum_{|u|=n} \mathbf{1}_{\{\forall j \leq n, V(u_j) \geq -ja\}} \geq Z_n.$$

We write $p := \mathbf{P}(\forall n \in \mathbb{N}, Z_n > 0) > 0$ for the survival probability of this Galton-Watson process.

For $n \in \mathbb{N}$, we write $\widetilde{Z}_n$ for the number of individuals with an infinite number of descendants. Conditionally on the survival of Z, the process $(\widetilde{Z}_n, n \geq 0)$ is a supercritical Galton-Watson process that survives almost surely (see e.g. [AN04]). Applying Lemma 5.2.7, there exists $\varrho > 1$ such that

$$\mathbf{P}(\widetilde{Z}_n \leq \varrho^n) \leq \varrho^{-n}.$$

Applying the Borel-Cantelli lemma, a.s. for any $n \geq 1$ large enough $\widetilde{Z}_n \geq \varrho^n$.

We introduce a sequence of individuals $(u_n) \in \mathbf{T}^{\mathbb{N}}$ such that $|u_n| = n$, $u_0 = \emptyset$ and u_{n+1} is the leftmost child of u_n, with ties broken uniformly at random. We write $q = \mathbf{P}(N \geq 2)$ for the probability that u_n has at least two children, both of them above $-a$. We introduce the random time T defined as the smallest $k \in \mathbb{N}$ such that the second leftmost child v of u_k is above $-a$, and the Galton-Watson process coupled with the branching random walk rooted at v survives. We observe that T is stochastically bounded by a geometric random variable with parameter pq, and that conditionally on T, the Galton-Watson tree that survives has the same law as $\widetilde{Z}$.

Thanks to these observations, we note that $T < +\infty$ and $\inf_{j \leq T} V(u_j) > -\infty$. For any $n \geq 1$ large enough such that $T < n$ and $\inf_{j \leq T} V(u_j) \geq -na$ we have

$$\# \{u \in \mathbf{T} : |u| = 2n, \forall j \leq n, V(u_j) \geq -3na\} \geq \varrho^n,$$

as desired. □

5.3 Branching random walk with a killing boundary at critical rate

In this section, we study the behaviour of a branching random walk on $\mathbb{R}$ in which individuals below a given barrier are killed. Given a continuous function $f \in \mathcal{C}([0,1])$ such that $\limsup_{t \to 0} \frac{f_t}{t} < +\infty$ and $n \in \mathbb{N}$, for any $k \leq n$ every individual alive at generation k below level $f_{k/n} n^{1/3}$ are removed, as well as all their descendants. Let $(\mathbf{T}, V)$ be a branching random walk, we denote by

$$\mathbf{T}_f^{(n)} = \left\{ u \in \mathbf{T} : |u| \leq n, \forall j \leq |u|, V(u_j) \geq n^{1/3} f(k/n) \right\},$$

and note that $\mathbf{T}_f^{(n)}$ is a random tree. The process $(\mathbf{T}_f^{(n)}, V)$, called branching random walk with a killing boundary, has been introduced in [AJ11, Jaf12], where the question of survival of the process is studied.

We compute the survival probability of $\mathbf{T}_f^{(n)}$, and provide a bound on the size of the population in $\mathbf{T}_f^{(n)}$ at any time $k \leq n$. We start finding a function g such that with high probability, no individual alive at generation $k \in \mathbf{T}_f^{(n)}$ is above $n^{1/3} g_{k/n}$. We then compute the first and second moments of the number of individuals in $\mathbf{T}$ that stay at any time $k \leq n$ between $n^{1/3} f_{k/n}$ and $n^{1/3} g_{k/n}$.

With a careful choice of functions f and g, one can compute the asymptotic behaviour of the consistent maximal displacement at time n, which is [FZ10, Theorem 1] and [FHS12, Theorem 1.4]; or the asymptotic behaviour as $\varepsilon \to 0$ of the probability there exists an individual in the branching random walk staying at any time $n \in \mathbb{N}$ above $-\varepsilon n$, which is [GHS11, Theorem 1.2]. We present these results respectively in Theorem 5.3.7 and Theorem 5.3.8, with weaker integrability conditions than in the seminal articles.

5.3.1 Number of individuals in a given path

For any two continuous functions $f < g$, we denote by

$$H_t(f,g) = \frac{\pi^2\sigma^2}{2}\int_0^t \frac{ds}{(g_s - f_s)^2}.$$

For $n \geq 1$ and $k \leq n$, we write $I_k^{(n)} = [f_{k/n}n^{1/3}, g_{k/n}n^{1/3}]$. We compute in a first time the number of individuals in $\mathbf{T}_f^{(n)}$ that crosses for the first time at some time $k \leq n$ the boundary $g_{k/n}n^{1/3}$. We set

$$Y_{f,g}^{(n)} = \sum_{u \in \mathbf{T}_f^{(n)}} \mathbf{1}_{\left\{V(u) > g_{|u|/n}n^{1/3}\right\}} \mathbf{1}_{\left\{V(u_j) \leq g_{j/n}n^{1/3}, j < |u|\right\}}.$$

Lemma 5.3.1. *Let $f \leq g$ such that $f_0 \leq 0 \leq g_0$. Under assumptions* (5.1.1) *and* (5.1.2),

$$\limsup_{n\to+\infty} n^{-1/3} \log \mathbf{E}\left[Y_{f,g}^{(n)}\right] \leq -\inf_{t\in[0,1]} g_t + H_t(f,g). \tag{5.3.1}$$

Proof. Using Lemma 5.2.2, we have

$$\begin{aligned}
\mathbf{E}\left[Y_{f,g}^{(n)}\right] &= \sum_{k=1}^n \mathbf{E}\left[\sum_{|u|=k} \mathbf{1}_{\left\{V(u)\geq g_{k/n}n^{1/3}\right\}} \mathbf{1}_{\left\{V(u_j)\in I_j^{(n)}, j<k\right\}}\right] \\
&= \sum_{k=1}^n \mathbf{E}\left[e^{-S_k} \mathbf{1}_{\left\{S_k \geq g_{k/n}n^{1/3}\right\}} \mathbf{1}_{\left\{S_j \in I_j^{(n)}, j<k\right\}}\right] \\
&\leq \sum_{k=1}^n e^{-n^{1/3}g_{k/n}} \mathbf{P}\left(S_j \in I^{(n)}j_j, j < k\right).
\end{aligned}$$

Let $\delta > 0$, we set $I_k^{(n),\delta} = \left[(f_{k/n} - \delta)n^{1/3}, (g_{k/n} + \delta)n^{1/3}\right]$. Let $A \in \mathbb{N}$, for $a \leq A$ we write $m_a = \lfloor na/A \rfloor$ and $\underline{g}_{a,A} = \inf_{s\in[a/A,(a+1)/A]} g_s$. Applying the Markov property at time m_a, for any $k > m_a$, we have

$$e^{-n^{1/3}g_{k/n}} \mathbf{P}\left(S_j \in I^{(n)}j_j, j < k\right) \leq e^{-n^{1/3}\underline{g}_{a,A}} \mathbf{P}\left(S_j \in I^{(n)}, \delta_j, j \leq m_a\right).$$

Applying Theorem 5.2.3, we have

$$\limsup_{n\to+\infty} n^{-1/3} \log \mathbf{E}\left[Y_{f,g}^{(n)}\right] \leq \max_{a<A} -\underline{g}_{a,A} - H_{a/A}(f-\delta, g+\delta).$$

Letting $\delta \to 0$ and $A \to +\infty$, we conclude that

$$\limsup_{n\to+\infty} n^{-1/3} \log \mathbf{E}\left[Y_{f,g}^{(n)}\right] \leq \sup_{t\in[0,1]} -g_t - H_t(f,g).$$

□

Using this lemma, we note that if $\inf_{t\in[0,1]} g_t + H_t(f,g) \geq \delta$, then with high probability no individual in $\mathbf{T}_f^{(n)}$ crosses the curve $g_{./n}n^{1/3}$ with probability at least $1 - e^{-\delta n^{1/3}}$. In a second time, we take interest in the number of individuals that stays between $f_{./n}n^{1/3}$ and $g_{./n}n^{1/3}$. For any $f_1 \leq x < y \leq g_1$, we set

$$Z_{f,g}^{(n)}(x,y) = \sum_{|u|=n} \mathbf{1}_{\left\{V(u)\in[xn^{1/3},yn^{1/3}]\right\}} \mathbf{1}_{\left\{V(u_j)\in I_j^{(n)}, j\leq n\right\}}.$$

Lemma 5.3.2. *Let $f < g$ be such that $\liminf_{t\to 0} \frac{g_t}{t} > -\infty$ and $\limsup_{t\to 0} \frac{f_t}{t} < +\infty$. Under assumptions* (5.1.1) *and* (5.1.2)*, we have*

$$\lim_{n\to+\infty} n^{-1/3} \log \mathbf{E}\left(Z_{f,g}^{(n)}(x,y)\right) = -(x + H_1(f,g)).$$

Proof. Applying (5.2.3), we have

$$\mathbf{E}\left(Z_{f,g}^{(n)}(x,y)\right) = \mathbf{E}\left[e^{-S_n} \mathbf{1}_{\left\{S_n\in[xn^{1/3},yn^{1/3}]\right\}} \mathbf{1}_{\left\{S_j\in I_j^{(n)}, j\leq n\right\}}\right],$$

which yields

$$\mathbf{E}\left(Z_{f,g}^{(n)}(x,y)\right) \leq e^{-xn^{1/3}} \mathbf{P}\left(S_n \in [xn^{1/3}, yn^{1/3}], S_j \in I_j^{(n)}, j \leq n\right). \tag{5.3.2}$$

Moreover, note that for any $\varepsilon > 0$, $Z_{f,g}^{(n)}(x,y) \geq Z_{f,g}^{(n)}(x, x+\varepsilon)$, and we have

$$\mathbf{E}(Z_{f,g}^{(n)}(x,y) \geq e^{-(x+\varepsilon)n^{1/3}} \mathbf{P}\left(S_n \in [xn^{1/3}, (x+\varepsilon)n^{1/3}], S_j \in I_j^{(n)}, j \leq n\right). \tag{5.3.3}$$

As $f < g$, $\liminf_{t\to 0} \frac{g_t}{t} > -\infty$ and $\limsup_{t\to 0} \frac{f_t}{t} < +\infty$, either $f_0 < 0$ or $g_0 > 0$. Consequently, applying Corollary 5.2.6, for any $f_1 \leq x' < y' \leq g_1$ we have

$$\lim_{n\to+\infty} n^{-1/3} \log \mathbf{P}\left(S_n \in [x'n^{1/3}, y'n^{1/3}], S_j \in I_j^{(n)}, j \leq n\right) = -H_1(f,g).$$

Therefore, (5.3.2) yields

$$\limsup_{n\to+\infty} n^{-1/3} \log \mathbf{E}(Z_{f,g}^{(n)}(x,y)) \leq -x - H_1(f,g)$$

and (5.3.3) yields

$$\liminf_{n\to+\infty} n^{-1/3} \log \mathbf{E}(Z_{f,g}^{(n)}(x,y)) \geq -x - \varepsilon - H_1(f,g).$$

Letting $\varepsilon \to 0$ concludes the proof. □

Lemma 5.3.2 is used to bound from above the number of individuals in $\mathbf{T}_f^{(n)}$ who are at time n in a given interval. To compute a lower bound we use a second moment concentration estimate. To successfully bound from above the second moment, we are led to restrict the set of individuals we consider to individuals who have "not too many siblings" in the following sense. For $u \in \mathbf{T}$, we set

$$\xi(u) = \log\left(1 + \sum_{v\in\Upsilon(u)} e^{V(v)-V(u)}\right)$$

where $\Upsilon(u)$ is the set of siblings of u, except u itself. In other words,

$$\Upsilon(u) = \{v \in \mathbf{T} : \pi v = \pi u, v \neq u\}.$$

For $\delta > 0$ and $f_1 \leq x < y \leq g_1$, we write

$$\widetilde{Z}^{(n)}_{f,g}(x, y, \delta) = \sum_{|u|=n} \mathbf{1}_{\left\{V(u)\in[xn^{1/3},yn^{1/3}]\right\}} \mathbf{1}_{\left\{V(u_j)\in I_j^{(n)}, \xi(u_j)\leq \delta n^{1/3}, j\leq n\right\}},$$

and observe that for any $\delta > 0$, $\widetilde{Z}^{(n)}_{f,g}(x, y, \delta) \leq Z^{(n)}_{f,g}(x, y)$.

Lemma 5.3.3. *Let $f < g$ be such that $\liminf_{t\to 0} \frac{g_t}{t} > -\infty$ and $\limsup_{t\to 0} \frac{f_t}{t} < +\infty$. Under assumptions* (5.1.1), (5.1.2) *and* (5.1.3), *for any $f_1 \leq x < y \leq g_1$ and $\delta > 0$ we have*

$$\liminf_{n\to+\infty} n^{-1/3} \log \mathbf{E}(\widetilde{Z}^{(n)}_{f,g}(x, y, \delta)) \geq -(x + H_1(f, g)), \tag{5.3.4}$$

$$\limsup_{n\to+\infty} n^{-1/3} \log \mathbf{E}\left[\left(\widetilde{Z}^{(n)}_{f,g}(x, y, \delta)\right)^2\right] \leq -2(x+H_1(f, g))+\delta+\sup_{t\in[0,1]} g_t+H_t(f, g). \tag{5.3.5}$$

Proof. For any $\varepsilon > 0$, applying Proposition 5.2.1 we have

$$\begin{aligned}
&\mathbf{E}\left[\widetilde{Z}^{(n)}_{f,g}(x, y, \delta)\right]\\
&= \overline{\mathbf{E}}\left[\frac{1}{W_n} \sum_{|u|=n} \mathbf{1}_{\left\{V(u)\in[xn^{1/3},yn^{1/3}]\right\}} \mathbf{1}_{\left\{V(u_j)\in I_j^{(n)}, j\leq n\right\}} \mathbf{1}_{\left\{\xi(u_j)\leq\delta n^{1/3}, j\leq n\right\}}\right]\\
&\geq \widehat{\mathbf{E}}\left[e^{-V(w_n)} \mathbf{1}_{\left\{V(w_n)\in[xn^{1/3},(x+\varepsilon)n^{1/3}]\right\}} \mathbf{1}_{\left\{V(w_j)\in I_j^{(n)}, \xi(w_j)\leq\delta n^{1/3}, j\leq n\right\}}\right]\\
&\geq e^{-(x+\varepsilon)n^{1/3}} \widehat{\mathbf{P}}\left[V(w_n) \in [xn^{1/3}, (x + \varepsilon)n^{1/3}], V(w_j) \in I_j^{(n)}, \xi(w_j) \leq \delta n^{1/3}, j \leq n\right].
\end{aligned}$$

Setting $X = \xi(w_1)$, (5.1.3) implies $\widehat{\mathbf{E}}(X^2) < +\infty$, thus $\lim_{z\to+\infty} z^2 \widehat{\mathbf{P}}(X \geq z) = 0$. Applying Corollary 5.2.6, we obtain

$$\liminf_{n\to+\infty} n^{-1/3} \log \mathbf{E}\left[\widetilde{Z}^{(n)}_{f,g}(x, y, \delta)\right] \geq -(x + \varepsilon) - H_1(f, g),$$

and conclude the proof of (5.3.4) by letting $\varepsilon \to 0$.

We now take care of the second moment. Using again Proposition 5.2.1, we have

$$\begin{aligned}
&\mathbf{E}\left[(\widetilde{Z}^{(n)}_{f,g}(x, y, \delta))^2\right]\\
=&\overline{\mathbf{E}}\left[\frac{\widetilde{Z}^{(n)}_{f,g}(x, y, \delta)}{W_n} \sum_{|u|=n} \mathbf{1}_{\left\{V(u)\in[xn^{1/3},yn^{1/3}]\right\}} \mathbf{1}_{\left\{V(u_j)\in I_j^{(n)}, j\leq n\right\}} \mathbf{1}_{\left\{\xi(u_j)\leq\delta n^{1/3}, j\leq n\right\}}\right]\\
\leq&\widehat{\mathbf{E}}\left[e^{-V(w_n)} Z^{(n)}_{f,g}(x, y) \mathbf{1}_{\left\{V(w_n)\in[xn^{1/3},yn^{1/3}]\right\}} \mathbf{1}_{\left\{V(w_j)\in I_j^{(n)}, j\leq n\right\}} \mathbf{1}_{\left\{\xi(w_j)\leq\delta n^{1/3}, j\leq n\right\}}\right]\\
\leq&e^{-xn^{1/3}} \widehat{\mathbf{E}}\left[Z^{(n)}_{f,g}(x, y) \mathbf{1}_{\left\{V(w_n)\in[xn^{1/3},yn^{1/3}]\right\}} \mathbf{1}_{\left\{V(w_j)\in I_j^{(n)}, j\leq n\right\}} \mathbf{1}_{\left\{\xi(w_j)\leq\delta n^{1/3}, j\leq n\right\}}\right].
\end{aligned} \tag{5.3.6}$$

We decompose $Z^{(n)}_{f,g}(x,y)$ according to the generation at which individuals split with the spine, i.e.,

$$Z^{(n)}_{f,g}(x,y) = \mathbf{1}_{\left\{V(w_n)\in[xn^{1/3},yn^{1/3}]\right\}}\mathbf{1}_{\left\{V(w_j)\in I_j^{(n)},j\leq n\right\}} + \sum_{k=1}^n \sum_{u\in\Upsilon_k} \Lambda(u),$$

where $\Lambda(u) = \sum_{|v|=n,v\geq u} \mathbf{1}_{\left\{V(v)\in[xn^{1/3},yn^{1/3}]\right\}}\mathbf{1}_{\left\{V(v_j)\in I_j^{(n)},j\leq n\right\}}$ for $u \in \mathbf{T}$, and $\Upsilon_k = \Upsilon(w_k)$ is the set of children of w_{k-1} which are different from w_k.

By definition of $\widehat{\mathbf{P}}$, conditionally on $\widehat{\mathcal{F}}_k$ the subtree of the descendants of $u \in \Upsilon_k$ is distributed as a branching random walk starting from $V(u)$. For any $k \leq n$ and $u \in \Upsilon_k$, applying Lemma 5.2.2 we have

$$\begin{aligned}
&\mathbf{E}\left[\Lambda(u)\middle|\widehat{\mathcal{F}}_k\right]\\
=&\mathbf{1}_{\left\{V(w_j)\in I_j^{(n)},j\leq k\right\}}\mathbf{E}_{V(u)}\left[\sum_{|v|=n-k}\mathbf{1}_{\left\{V(v)\in[xn^{1/3},yn^{1/3}]\right\}}\mathbf{1}_{\left\{V(v_j)\in I_{k+j}^{(n)},j\leq n-k\right\}}\right]\\
=&\mathbf{1}_{\left\{V(w_j)\in I_j^{(n)},j\leq k\right\}}e^{-V(u)}\mathbf{E}_{V(u)}\left[e^{-S_{n-k}}\mathbf{1}_{\left\{S_{n-k}\in[xn^{1/3},yn^{1/3}]\right\}}\mathbf{1}_{\left\{S_j\in I_{k+j}^{(n)},j\leq n-k\right\}}\right]\\
\leq& e^{V(w_k)-xn^{1/3}}e^{V(u)-V(w_k)}\mathbf{P}_{V(u)}\left[S_j\in I_{k+j}^{(n)}, j\leq n-k\right].
\end{aligned}$$

Thus, by definition of $\xi(w_k)$,

$$\sum_{u\in\Upsilon_k}\mathbf{E}\left[\Lambda(u)\middle|\widehat{\mathcal{F}}_k\right] \leq e^{V(w_k)-xn^{1/3}+\xi(w_k)}\sup_{z\in\mathbb{R}}\mathbf{P}_z\left[S_j\in I_{k+j}^{(n)}, j\leq n-k\right].$$

Let $A \in \mathbb{N}$. For any $a \leq A$ we write $m_a = \lfloor na/A\rfloor$. For any $k \leq m_a$ and $z \in \mathbb{R}$, applying the Markov property at time $m_a - k$ we have

$$\mathbf{P}_z\left[S_j\in I_{k+j}^{(n)}, j\leq n-k\right] \leq \sup_{z'\in\mathbb{R}}\mathbf{P}_{z'}\left[S_j\in I_{m_a+j}^{(n)}, j\leq n-m_a\right].$$

We write $\Psi_a^{(n)} = \sup_{z'\in\mathbb{R}}\mathbf{P}_{z'}\left[S_j\in I_{m_a+j}^{(n)}, j\leq n-m_a\right]$. By Corollary 5.2.4, we have

$$\limsup_{n\to+\infty} n^{-1/3}\log\Psi_a^{(n)} \leq -\left(H_1(f,g) - H_{a/A}(f,g)\right).$$

Moreover, (5.3.6) becomes

$$\begin{aligned}
\mathbf{E}\left[\left(\widetilde{Z}^{(n)}_{f,g}(x,y)\right)^2\right] \leq{}& e^{-xn^{1/3}}\mathbf{P}(S_j\in I_j^{(n)}, j\leq n)\\
&+ e^{-2xn^{1/3}}\sum_{a=0}^{A-1}\Psi_{a+1}^{(n)}\sum_{k=m_a+1}^{m_{a+1}}\mathbf{E}\left[e^{V(w_k)+\xi(w_k)}\mathbf{1}_{\left\{V(w_j)\in I_j^{(n)},\xi(w_j)\leq\delta n^{1/3},j\leq n\right\}}\right].
\end{aligned}$$

We set $\overline{g}_{a,A} = \sup_{s\in[\frac{a}{A},\frac{a+1}{A}]} g_s$, we have

$$\mathbf{E}\left[e^{V(w_k)+\xi(w_k)}\mathbf{1}_{\left\{V(w_j)\in I_j^{(n)},\xi(w_j)\leq\delta n^{1/3},j\leq n\right\}}\right] \leq e^{n^{1/3}(\overline{g}_{a,A}+\delta)}\mathbf{P}(S_j\in I_j^{(n)}, j\leq n).$$

We apply Theorem 5.2.3 to obtain

$$\limsup_{n\to+\infty} n^{-1/3} \log \sum_{k=m_a+1}^{m_{a+1}} \mathbf{E}\left[e^{V(w_k)+\xi(w_k)} \mathbf{1}_{\left\{V(w_j)\in I_j^{(n)}, \xi(w_j)\leq \delta n^{1/3}, j\leq n\right\}} \right] \leq \overline{g}_{a,A} + \delta - H_1(f,g).$$

We conclude that

$$\limsup_{n\to+\infty} n^{-1/3} \log \mathbf{E}\left[(\widetilde{Z}^{(n)}_{f,g}(x,y))^2 \right] \leq -(2x + H_1(f,g)) + \delta + \max_{a<A} \overline{g}_{a,A} + H_{\frac{a+1}{A}}(f,g).$$

Letting $A \to +\infty$ concludes the proof. □

A straightforward consequence of Lemma 5.3.3 is a lower bound on the asymptotic behaviour of the probability for $Z^{(n)}_{f,g}$ to be positive.

Corollary 5.3.4. *Under the assumptions of Lemma 5.3.3, we have*

$$\liminf_{n\to+\infty} n^{-1/3} \log \mathbf{P}\left[Z^{(n)}_{f,g}(x,y) \geq 1 \right] \geq - \sup_{t\in[0,1]} g_t + H_t(f,g).$$

Proof. For any $\delta > 0$, we have $Z^{(n)}_{f,g}(x,y) \geq \widetilde{Z}^{(n)}_{f,g}(x,y,\delta)$. As a consequence,

$$\mathbf{P}\left[Z^{(n)}_{f,g}(x,y) \geq 1 \right] \geq \mathbf{P}\left[\widetilde{Z}^{(n)}_{f,g}(x,y,\delta) \geq 1 \right] \geq \frac{\mathbf{E}\left[\widetilde{Z}^{(n)}_{f,g}(x,y,\delta) \right]^2}{\mathbf{E}\left[\widetilde{Z}^{(n)}_{f,g}(x,y,\delta)^2 \right]}$$

by the Cauchy-Schwarz inequality. Therefore using Lemma 5.3.3 we have

$$\liminf_{n\to+\infty} n^{-1/3} \log \mathbf{P}\left[Z^{(n)}_{f,g}(x,y) \geq 1 \right] \geq - \sup_{t\in[0,1]} g_t + H_t(f,g).$$

□

Another application of Lemma 5.3.3 is a lower bound on the value of the sum of a large number of i.i.d. versions of $Z^{(n)}_{f,g}(x,y)$. This is useful observing that after time k, there exists with large probability at least ϱ^k individuals, each of which starting an independent branching random walk.

Corollary 5.3.5. *Under the assumptions of Lemma 5.3.3, we set* $(Z^{(n),j}_{f,g}(x,y), j \in \mathbb{N})$ *i.i.d. copies of* $Z^{(n)}_{f,g}(x,y)$. *Let* $z > 0$, *we write* $p = \left\lfloor e^{zn^{1/3}} \right\rfloor$. *For any* $\varepsilon > 0$, *we have*

$$\limsup_{n\to+\infty} n^{-1/3} \log \mathbf{P}\left[\sum_{j=1}^{p} Z^{(n),j}_{f,g}(x,y) \leq \exp\left(n^{1/3}(z - x - H_1(f,g) - \varepsilon \right) \right] \leq -z + \sup_{t\in[0,1]} g_t + H_t(f,g).$$

Proof. The proof is based on the following observation. Let $(X_j, j \in \mathbb{N})$ be i.i.d. random variables with finite variance. Using the Bienaymé-Chebychev inequality, we have

$$\mathbf{P}\left(\sum_{j=1}^{p} X_j \leq \frac{1}{2} \mathbf{E}\left(\sum_{j=1}^{p} X_j \right) \right) \leq \mathbf{P}\left(\left| \sum_{j=1}^{p} X_j - p\,\mathbf{E}(X_1) \right| \geq p\,\mathbf{E}(X_1)/2 \right)$$
$$\leq 4 \frac{\mathbf{V}\mathrm{ar}\left(\sum_{j=1}^{p} X_j \right)}{p^2\,\mathbf{E}(X_1)^2} \leq 4 \frac{\mathbf{V}\mathrm{ar}(X_1)}{p\,\mathbf{E}(X_1)} \leq 4 \frac{\mathbf{E}(X_1^2)}{p\,\mathbf{E}(X_1)^2}. \quad (5.3.7)$$

Let $\delta > 0$, as $Z^{(n)}_{f,g}(x,y) \geq \widetilde{Z}^{(n)}_{f,g}(x,y,\delta)$, we have

$$\mathbf{P}\left[\sum_{j=1}^{p} Z^{(n),j}_{f,g}(x,y) \leq \exp\left(n^{1/3}(z - x - H_1(f,g) - \varepsilon)\right)\right] \\ \leq \mathbf{P}\left[\sum_{j=1}^{p} \widetilde{Z}^{(n),j}_{f,g}(x,y,\delta) \leq \exp\left(n^{1/3}(z - x - H_1(f,g) - \varepsilon\right)\right],$$

where $(\widetilde{Z}^{(n),j}_{f,g}(x,y,\delta), j \in \mathbb{N})$ is a sequence of i.i.d. copies of $\widetilde{Z}^{(n),j}_{f,g}(x,y,\delta)$. By Lemma 5.3.3,

$$\liminf_{n\to+\infty} n^{-1/3} \log \mathbf{E}\left(\widetilde{Z}^{(n)}_{f,g}(x,y,\delta)\right) \geq -\left(x + H_1(f,g)\right),$$

thus, for any $\varepsilon > 0$, for any $n \geq 1$ large enough we have

$$\mathbf{E}\left(\widetilde{Z}^{(n)}_{f,g}(x,y,\delta)\right)/2 \geq e^{-n^{1/3}(x+H_1(f,g)+\varepsilon)}.$$

Therefore, using again Lemma 5.3.3 and (5.3.7), we have

$$\limsup_{n\to+\infty} n^{-1/3} \log \mathbf{P}\left[\sum_{j=1}^{p} \widetilde{Z}^{(n),j}_{f,g}(x,y,\delta) \leq \exp\left(n^{1/3}(z - x - H_1(f,g) - \varepsilon\right)\right] \\ \leq -z + \delta + \sup_{t\in[0,1]} g_t + H_t(f,g).$$

Consequently, letting $\delta \to 0$ we have

$$\limsup_{n\to+\infty} n^{-1/3} \log \mathbf{P}\left[\sum_{j=1}^{p} Z^{(n),j}_{f,g}(x,y) \leq \exp\left(n^{1/3}(z - x - H_1(f,g) - \varepsilon\right)\right] \\ \leq -z + \sup_{t\in[0,1]} g_t + H_t(f,g).$$

$\square$

5.3.2 Asymptotic behaviour of the branching random walk with a killing boundary

The results of Section 5.3.1, in particular Lemma 5.3.1 and Corollaries 5.3.4 and 5.3.5, emphasize the importance of the functions g verifying

$$\forall t \in [0,1], g_t = g_0 - H_t(f,g) > f_t \tag{5.3.8}$$

in the study of $\mathbf{T}^{(n)}_f$. For such a function, the estimates of Lemmas 5.3.1, 5.3.2 and 5.3.3 are tight. They enable to precisely study the asymptotic behaviour of $\mathbf{T}^{(n)}_f$.

Theorem 5.3.6. *We consider a branching random walk* $(\mathbf{T}, V)$ *satisfying* (5.1.1), (5.1.2) *and* (5.1.3). *Let* $f \in \mathcal{C}([0,1])$ *be such that* $f_0 < 0$. *If there exists a continuous function* g *such that*

$$g_0 = 0, \quad \forall t \in [0,1], g_t = -\frac{\pi^2\sigma^2}{2}\int_0^t \frac{ds}{(g_s - f_s)^2} \quad \text{and} \quad \forall t \in [0,1], g_t > f_t,$$

then almost surely for $n \geq 1$ large enough, $\{u \in \mathbf{T}_f^{(n)} : |u| = n\} \neq \emptyset$ and

$$\lim_{n\to+\infty} \frac{1}{n^{1/3}} \# \left\{ u \in \mathbf{T}_f^{(n)} : |u| = n \right\} = g_1 - f_1,$$
$$\lim_{n\to+\infty} \frac{1}{n^{1/3}} \min_{u\in\mathbf{T}_f^{(n)}, |u|=n} V(u) = f_1 \quad \text{and} \quad \lim_{n\to+\infty} \frac{1}{n^{1/3}} \max_{u\in\mathbf{T}_f^{(n)}, |u|=n} V(u) = g_1 \quad \text{a.s.} \tag{5.3.9}$$

Otherwise, writing

$$\lambda = \inf \left\{ g_0, g \in \mathcal{C}([0,1]) : \forall t \in [0,1], g_t = g_0 + \frac{\pi^2\sigma^2}{2} \int_0^t \frac{ds}{(g_s - f_s)^2} > f_t \right\}, \tag{5.3.10}$$

then

$$\lim_{n\to+\infty} n^{-1/3} \log \mathbf{P} \left(\left\{ u \in \mathbf{T}_f^{(n)} : |u| = n \right\} \neq \emptyset \right) = -\lambda. \tag{5.3.11}$$

Proof. We study the solutions of the differential equation (5.3.8). As $(t,x) \mapsto -\frac{\pi^2\sigma^2}{2(x-f_t)^2}$ is locally Lipschitz on $\{(t,x) \in [0,1] \times \mathbb{R} : x > f_t\}$, the Cauchy-Lipschitz theorem implies that for any $x > f_0$, there exists a unique continuous function g^x defined on the maximal interval $[0, t_x]$ such that $g_0^x = x$, either $t_x = 1$ or $g_{t_x} = f_{t_x}$, and for any $t < t_x$

$$g_t^x = x - \frac{\pi^2\sigma^2}{2} \int_0^t \frac{ds}{(g_s^x - f_s)^2}.$$

Moreover, we observe that t_x is increasing with respect to x and g_t^x is decreasing in t and increasing in x on $\{(t,x) \in [0,1] \times (f_0, +\infty) : t \leq t_x\}$. With these notations, we have

$$\lambda = \inf \{x > f_0 : t_x = 1\}.$$

As $\lim_{x\to+\infty} \sup_{t\in[0,1]} \frac{\pi^2\sigma^2}{2(x-f_t)^2} = 0$, there exists $x > 0$ large enough such that $t_x = 1$. This implies $\lambda < +\infty$.

We note that for any $x > 0$ such that $g^x > f$ on $[0,1]$, applying Corollary 5.3.4 we obtain

$$\begin{aligned} \liminf_{n\to+\infty} n^{-1/3} \log \mathbf{P} \left[\{u \in \mathbf{T}_f^{(n)} : |u| = n\} \neq \emptyset \right] &\geq \liminf_{n\to+\infty} n^{-1/3} \log \mathbf{P} \left[Z_{f,g^x}^{(n)}(f_1), g_1^x) \geq 1 \right] \\ &\geq -x. \end{aligned}$$

Therefore, we have $\liminf_{n\to+\infty} n^{-1/3} \log \mathbf{P} \left[\{u \in \mathbf{T}_f^{(n)} : |u| = n\} \neq \emptyset \right] \geq -\min(\lambda, 0)$.

If $\lambda \geq 0$, writing $t = t_\lambda$, we use the fact that at some time before t_λ every individual in $\mathbf{T}_f^{(n)}$ crosses $n^{1/3} g_{./n}$ before time tn, thus

$$\mathbf{P} \left(\exists |u| = n : u \in \mathbf{T}_f^{(n)} \right) \leq \mathbf{P} \left(\exists u \in \mathbf{T}_f^{(n)} : V(u) \geq n^{1/3} g_{|u|/n} \right).$$

We set $f_s^{(1)} = f_{st}/t^{1/3}$ and $g_s^{(1)} = g_{st}^\lambda / t^{1/3}$. Applying Lemma 5.3.1, and writing $m = \lfloor tn \rfloor$ we have

$$\limsup_{n\to+\infty} n^{-1/3} \log \mathbf{E} \left(Y_{f^{(1)}, g^{(1)}}^{(m)} \right) \leq -\lambda,$$

which by Markov inequality yields

$$\limsup_{n\to+\infty} n^{-1/3} \log \mathbf{P} \left(u \in \mathbf{T}_f : |u| \leq tn, V(u) \geq n^{1/3} g_{|u|/n} \right) \leq -\lambda,$$

concluding the proof of (5.3.11).

We now assume $\lambda < 0$, or equivalently $g^0 > f$. Applying Lemma 5.3.1, for any $\varepsilon > 0$ we have

$$\limsup_{n\to+\infty} n^{-1/3} \log \mathbf{P}\left(\exists u \in \mathbf{T}_f^{(n)} : V(u) \geq n^{1/3} g^\varepsilon_{|u|/n}\right) \leq - \inf_{t\in[0,1]} g_t^\varepsilon + H_t(f, g^\varepsilon) = -\varepsilon.$$

By the Borel-Cantelli lemma, almost surely for any $n \geq 1$ large enough, we have

$$\left\{u \in \mathbf{T}_f^{(n)} : V(u) \geq n^{1/3} g^\varepsilon_{|u|/n}\right\} = \emptyset. \tag{5.3.12}$$

In particular, letting $\varepsilon \to 0$ we have

$$\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \max_{u\in\mathbf{T}_f^{(n)}, |u|=n} V(u) = g_1 \quad \text{a.s.}$$

Moreover, by Lemma 5.3.2 we have

$$\mathbf{E}\left[Z^{(n)}_{f,g^\varepsilon}(f_1, g_1^\varepsilon)\right] \leq -(f_1 + H_1(f, g^\varepsilon)) = g_1^\varepsilon - f_1 - \varepsilon.$$

Thus, by the Markov inequality and the Borel-Cantelli Lemma

$$\limsup_{n\to+\infty} n^{-1/3} \log Z^{(n)}_{f,g^\varepsilon}(f_1, g_1^\varepsilon) \leq g_1^\varepsilon - f_1.$$

Mixing with (5.3.12) and letting $\varepsilon \to 0$, we conclude

$$\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \#\left\{u \in \mathbf{T}_f^{(n)} : |u| = n\right\} \leq g_1 - f_1.$$

To obtain the other bounds of (5.3.9), we apply Lemma 5.2.8. For any $\varepsilon > 0$ there exists $\varrho > 1$ and $\delta > 0$ such that almost surely for any $n \geq 1$ large enough,

$$\#\left\{u \in \mathbf{T}_f^{(n)} : |u| = \left\lfloor \delta n^{1/3} \right\rfloor \text{ and } V(u) \in [-\varepsilon n^{1/3}, \varepsilon n^{1/3}]\right\} \geq \varrho^{\delta n^{1/3}}.$$

We write S_n this event. On S_n, each of these $\varrho^{\delta n^{1/3}}$ individuals starts an independent branching random walk from some point in $[-\varepsilon n^{1/3}, \varepsilon n^{1/3}]$ with a killing boundary $n^{1/3} f_{./n}$. For ε small enough, we use Corollary 5.3.5 to bound from below the number of descendants that stay between $f + 2\varepsilon$ and $g^{-2\varepsilon} + 2\varepsilon$. We have

$$\begin{aligned}\limsup_{n\to+\infty} n^{-1/3} \log \mathbf{P}&\left[\#\left\{u \in \mathbf{T}_f^{(n)} : |u| = n\right\} \leq e^{n^{1/3}(g_1^{-2\varepsilon} - f_1)} \middle| S_n\right]\\ &\leq -\eta + \sup_{t\in[0,1]} g_t^{-2\varepsilon} + 2\varepsilon + H_t(f + 2\varepsilon, g^{-2\varepsilon} + 2\varepsilon) = -\eta.\end{aligned}$$

Using again the Borel-Cantelli lemma, we obtain

$$\liminf_{n\to+\infty} n^{-1/3} \log \#\left\{u \in \mathbf{T}_f^{(n)} : |u| = n\right\} \geq g_1^{-2\varepsilon} - f_1 \quad \text{a.s.}$$

Consequently, letting $\varepsilon \to 0$ we conclude

$$\lim_{n\to+\infty} n^{-1/3} \log \#\left\{u \in \mathbf{T}_f^{(n)} : |u| = n\right\} = g_1^0 - f_1 \quad \text{a.s.}$$

In particular, almost surely for $n \geq 1$ large enough, $\mathbf{T}^{(n)}_f$ survives until time n, which is enough to prove

$$\liminf_{n \to +\infty} \frac{1}{n^{1/3}} \min_{u \in \mathbf{T}^{(n)}_f, |u|=n} V(u) \geq f_1 \quad \text{a.s.}$$

By Corollary 5.3.4, for any $\varepsilon > 0$ small enough, for any $f_1 + 2\varepsilon < x < y < g_1^{-2\varepsilon} + 2\varepsilon$ we have

$$\liminf_{n \to +\infty} n^{-1/3} \log \mathbf{P}\left(Z^{(n)}_{f+2\varepsilon, g^{-2\varepsilon}+2\varepsilon}(x,y) > 0 \right) \geq 0.$$

Therefore, for any $f_1 < x < y < g_1$, for any $\varepsilon > 0$ small enough we have

$$\mathbf{P}\left(Z^{(n)}_{f,g}(x,y) = 0 \middle| S_n \right) = \left(1 - e^{o(n^{1/3})}\right)^{e^{\eta n^{1/3}}}.$$

We conclude that for any $\zeta > 0$ small enough,

$$\liminf_{n \to +\infty} n^{-1/3} \log\left(-\log \mathbf{P}\left(Z^{(n)}_{f,g}(f_1 + \zeta, f_1 + 2\zeta) = 0 \right) \right) > 0$$

as well as

$$\liminf_{n \to +\infty} n^{-1/3} \log\left(-\log \mathbf{P}\left(Z^{(n)}_{f,g}(g_1 - 2\zeta, g_1 - \zeta) = 0 \right) \right) > 0.$$

Using once again the Borel-Cantelli lemma, we obtain respectively

$$\limsup_{n \to +\infty} \frac{1}{n^{1/3}} \min_{u \in \mathbf{T}^{(n)}_f, |u|=n} V(u) \leq f_1 \quad \text{a.s.}$$

$$\text{and} \quad \liminf_{n \to +\infty} \frac{1}{n^{1/3}} \max_{u \in \mathbf{T}^{(n)}_f, |u|=n} V(u) \geq g_1^0 \quad \text{a.s.}$$

which concludes the proof. □

5.3.3 Applications

Using the results developed in this section, we deduce the asymptotic behaviour of the consistent maximal displacement at time n of the branching random walk.

Theorem 5.3.7 (Consistent maximal displacement of the branching random walk, [FZ10, FHS12])**.** *We consider a branching random walk* $(\mathbf{T}, V)$ *satisfying* (5.1.1)*,* (5.1.2) *and* (5.1.3)*. We have*

$$\lim_{n \to +\infty} \frac{\max_{|u|=n} \min_{k \leq n} V(u_k)}{n^{1/3}} = -\left(\frac{3\pi^2 \sigma^2}{2} \right)^{1/3}.$$

Proof. To prove this result, we only have to show that for any $\delta > 0$, almost surely for $n \geq 1$ large enough we have

$$\left\{ u \in \mathbf{T}^{(n)}_{\left(-\frac{3\pi^2\sigma^2}{2}\right)^{1/3}+\delta} : |u| = n \right\} = \emptyset \quad \text{and} \quad \left\{ u \in \mathbf{T}^{(n)}_{\left(-\frac{3\pi^2\sigma^2}{2}\right)^{1/3}-\delta} : |u| = n \right\} \neq \emptyset.$$

We solve for $x < 0$ the differential equation

$$g_t = -\frac{\pi^2 \sigma^2}{2} \int_0^t \frac{ds}{(g_s - x)^2}.$$

The solution is $g_t = x + \left(-x^3 - \frac{3\pi^2\sigma^2}{2}t\right)^{1/3}$ for $t < \frac{-2x^3}{3\pi^2\sigma^2}$. Applying Theorem 5.3.6, for any $x > -\left(\frac{3\pi^2\sigma^2}{2}\right)^{1/3}$, almost surely for any $n \geq 1$ large enough the tree $\mathbf{T}_x^{(n)}$ gets extinct before time n. For any $x < -\left(\frac{3\pi^2\sigma^2}{2}\right)^{1/3}$, almost surely for $n \geq 1$ large enough the tree $\mathbf{T}_x^{(n)}$ survives until time n. □

Similarly, we provide the asymptotic behaviour, as $\varepsilon \to 0$ of the probability of survival of a branching random walk with a killing boundary of slope $-\varepsilon$.

Theorem 5.3.8 (Survival probability in the killed branching random walk [GHS11])**.** *Let* $(\mathbf{T}, V)$ *be a branching random walk satisfying* (5.1.1), (5.1.2) *and* (5.1.3)*. We have*

$$\lim_{\varepsilon\to 0} \varepsilon^{1/2} \log \mathbf{P}\left(\forall n \in \mathbb{N}, \exists |u| = n : V(u_j) \geq -\varepsilon j, j \leq n\right) = -\frac{\pi\sigma}{2^{1/2}}.$$

Proof. For any $\varepsilon > 0$ and $n \in \mathbb{N}$, we set $\varrho(n,\varepsilon) = \mathbf{P}\left(\exists |u| = n : V(u_j) \geq -\varepsilon j, j \leq n\right)$ and

$$\varrho(\varepsilon) = \lim_{n\to+\infty} \varrho(n,\varepsilon) = \mathbf{P}\left(\forall n \in \mathbb{N}, \exists |u| = n : V(u_j) \geq -\varepsilon j, j \leq n\right).$$

In a first time, we prove that for any $\theta > 0$, we have

$$-\frac{\pi\sigma}{(2\theta)^{1/2}} \leq \liminf_{n\to+\infty} n^{-1/3} \log \varrho\left(n, \theta n^{-2/3}\right) \leq \limsup_{n\to+\infty} n^{-1/3} \log \varrho\left(n, \theta n^{-2/3}\right) \leq \Phi^{-1}(\theta), \tag{5.3.13}$$

where $\Phi : \lambda \mapsto \frac{\pi^2\sigma^2}{2\lambda^2} - \frac{\lambda}{3}$.

Applying Lemma 5.3.1 with functions $f : t \mapsto -\theta t$ and $g : t \mapsto \lambda(1-t)^{1/3} - \theta t$ we prove the upper bound of (5.3.13). Using the fact that an individual staying above $f^{(n)}$ until time n crosses $g^{(n)}$ at some time $k \leq n$, the Markov inequality implies

$$\begin{aligned}
\limsup_{n\to+\infty} n^{-1/3} \log \varrho(n, \theta n^{-2/3}) &\leq \limsup_{n\to+\infty} n^{-1/3} \log \mathbf{E}(Y_{f,g}^{(n)})\\
&\leq -\inf_{t\in[0,1]} g_t + H_t(f,g)\\
&\leq -\inf_{t\in[0,1]} \lambda(1-t)^{1/3} - \theta t + \frac{\pi^2\sigma^2}{2}\int_0^t \frac{ds}{(\lambda(1-s)^{1/3})^2}\\
&\leq -\inf_{t\in[0,1]} \lambda - \theta t + 3\Phi(\lambda)\left[1-(1-t)^{1/3}\right].
\end{aligned}$$

We observe that $t \mapsto 1-(1-t)^{1/3}$ is a convex function on $[0,1]$, with derivative $1/3$ at $t = 0$. Thus, for any $\lambda > 0$ such that $\Phi(\lambda) > 0$, for all $t \in [0,1]$, $3\Phi(\lambda)\left[1-(1-t)^{1/3}\right] \geq \Phi(\lambda)t$. We conclude that for any $\lambda > 0$ such that $\Phi(\lambda) \geq \theta > 0$, we have

$$\limsup_{n\to+\infty} n^{-1/3} \log \varrho(n, \theta n^{-2/3}) \leq -\lambda.$$

With $\lambda = \Phi^{-1}(\theta)$, we conclude the proof of the upper bound of (5.3.13). We now observe that for any $\varepsilon > 0$, we have $\varrho(\varepsilon) \leq \varrho(n,\varepsilon)$. Setting $n = \left\lfloor (\varepsilon/\theta)^{3/2} \right\rfloor$, for any $\theta > 0$ we have

$$\limsup_{\varepsilon\to 0} \varepsilon^{1/2} \log \varrho(\varepsilon) \leq \limsup_{\varepsilon\to 0} \varepsilon^{1/2} \log \varrho(n,\varepsilon) \leq -\theta^{1/2}\Phi^{-1}(\theta).$$

We note that $\lim_{\theta\to+\infty} \theta^{1/2}\Phi^{-1}(\theta) = \lim_{\lambda\to 0} \lambda\Phi(\lambda)^{1/2} = \frac{\pi\sigma}{2^{1/2}}$, which concludes the proof of the upper bound in Theorem 5.3.8.

To prove the lower bound in (5.3.13), we apply Corollary 5.3.4 to functions $f : t \mapsto -\theta t$ and $g : t \mapsto \lambda - \theta t$. We have

$$\liminf_{n\to+\infty} n^{-1/3} \log \varrho(n, \theta n^{-2/3}) \geq \liminf_{n\to+\infty} n^{-1/3} \log \mathbf{P}\left(Z^{(n)}_{f,g}(f_1, g_1) \geq 1\right)$$
$$\geq - \sup_{t\in[0,1]} \lambda - \theta t + \frac{\pi^2\sigma^2}{2\lambda^2} t.$$

Choosing $\lambda = \frac{\pi\sigma}{(2\theta)^{1/2}}$, we obtain

$$\liminf_{n\to+\infty} n^{-1/3} \log \varrho(n, \theta n^{-2/3}) \geq -\frac{\pi\sigma}{(2\theta)^{1/2}},$$

proving the lower bound of (5.3.13). To extend this lower bound into the lower bound in Theorem 5.3.8 needs more care than the upper bound. First, we observe that this equation implies that for any $\theta > 0$,

$$\liminf_{n\to+\infty} n^{-1/3} \log \varrho(\theta^{3/2} n, n^{-2/3}) \geq -\frac{\pi\sigma}{2^{1/2}}.$$

By (5.1.1), there exist $a > 0$ and $P \in \mathbb{N}$ such that

$$\mathbf{E}\left(\left(\sum_{|u|=1} \mathbf{1}_{\{V(u)\geq -a\}}\right) \wedge P\right) > 1.$$

Consequently, there exist $\varrho > 1$ and a random variable W positive with positive probability such that

$$\liminf_{n\to+\infty} \frac{\#\{|u| = n : \forall j \leq n, V(u_j) \geq -aj\}}{\varrho^n} \geq W \quad \text{a.s.}$$

We conclude there exist $a > 0$, $r > 0$ and $\varrho > 1$ such that

$$\inf_{n\in\mathbb{N}} \mathbf{P}\left(\#\{|u| = n : \forall j \leq n, V(u_j) \geq -aj\} \geq \varrho^n\right) \geq r.$$

With these notations, we observe that for any $\theta > 0$, $\varepsilon > 0$, $\delta > 0$ and $n \in \mathbb{N}$, we have

$$\mathbf{P}\left(\#\left\{|u| = (\theta+\delta)n : \forall j \leq n, V(u_j) \geq -\left(\frac{\theta\varepsilon + \delta a}{\theta + \delta}\right) j\right\} \geq \varrho^{\delta n}\right) \geq r\varrho\left(\theta n, \varepsilon\right).$$

Given $\lambda > \frac{\pi\sigma}{2^{1/2}}$ and $\theta > 0$, we set $\varepsilon > 0$ small enough such that

$$\varepsilon^{1/2} \log \varrho\left(\left\lceil 2\theta^2 \varepsilon^{-3/2}\right\rceil, \varepsilon\right) > -\lambda.$$

We write $\delta = \frac{\theta\varepsilon}{a - 2\varepsilon}$ and $n = \left\lfloor (\theta+\delta)\varepsilon^{-3/2} \right\rfloor$, choosing $\varepsilon > 0$ small enough such that $\delta < \theta$. We have

$$\mathbf{P}\left(\#\{|u| = n : \forall j \leq n, V(u_j) \geq -2\varepsilon j\} \geq \varrho^{\delta n}\right) \geq r e^{-\lambda\varepsilon^{-1/2}},$$

We construct a Galton-Watson process $(G_p(\varepsilon), p \geq 0)$ based on the branching random walk $(\mathbf{T}, V)$ such that

$$G_p(\varepsilon) = \#\{|u| = pn : \forall j \leq pn, V(u_j) \geq -2\varepsilon j\}.$$

We note that $G(\varepsilon)$ dominates the Galton-Watson process $\widetilde{G}(\varepsilon)$, in which individuals make $N_\varepsilon = \left\lfloor \varrho^{\delta n} \right\rfloor$ children with probability $p_\varepsilon = r e^{-\lambda\varepsilon^{-1/2}}$ and none with probability $1 - p$. As $\varepsilon \to 0$ we have

$$\lim_{\varepsilon\to 0} \varepsilon^{1/2} \log(p_\varepsilon N_\varepsilon) = -\lambda + \theta^2 \log \varrho,$$

which is positive choosing some $\theta > 0$ large enough. With this choice of θ, for any $\varepsilon > 0$ small enough $p_\varepsilon N_\varepsilon > 2$. Consequently q_ε the probability of survival of $\widetilde{G}(\varepsilon)$ is positive for any $\varepsilon > 0$ small enough. Moreover, we have $\varrho(2\varepsilon) \geq q_\varepsilon$.

We introduce $f_\varepsilon : s \mapsto \mathbf{E}(s^{\widetilde{G}(\varepsilon)})$ which is a convex function verifying $f_\varepsilon(1) = 1$ and $f_\varepsilon(1 - q_\varepsilon) = 1 - q_\varepsilon$. Moreover, for any $h > 0$, for any $\varepsilon > 0$ small enough

$$f_\varepsilon(1 - hp_\varepsilon) = 1 - p_\varepsilon + p_\varepsilon(1 - hp_\varepsilon)^{N_\varepsilon} \leq 1 - p_\varepsilon + p_\varepsilon \exp(-hp_\varepsilon N_\varepsilon) \leq 1 - p_\varepsilon + p_\varepsilon e^{-2h}.$$

Choosing $h > 0$ small enough, for any $\varepsilon > 0$ small enough we have $f_\varepsilon(1 - hp) < 1 - hp$. This proves that $q_\varepsilon > hp_\varepsilon$, leading to

$$\liminf_{\varepsilon \to 0} \varepsilon^{1/2} \log \varrho(\varepsilon) \geq \liminf_{\varepsilon \to 0} \varepsilon^{1/2} \log p_\varepsilon \geq -\lambda.$$

Letting $\lambda \to -\frac{\pi\sigma}{2^{1/2}}$ concludes the proof. $\square$

5.4 Branching random walk with selection

In this section, we consider a branching random walk on $\mathbb{R}$ in which at each generation only the rightmost individuals live. Given a positive continuous function h, at any time $k \leq n$ only the $\left\lfloor e^{n^{1/3} h_{k/n}} \right\rfloor$ rightmost individuals remain alive. The process is constructed as follows. Let $((\mathbf{T}^p, V^p), p \in \mathbb{N})$ be an i.i.d. sequence of independent branching random walks, for any $n \in \mathbb{N}$ we write $\mathcal{T}_{(n)}$ for the disjoint union of $\mathbf{T}^p$ for $p \leq n$, and introduce the function $V : u \in \mathcal{T}_{(n)} \mapsto V^p(u)$ if $u \in \mathbf{T}^p$. We rank individuals at a given generation according to their position, from highest to lowest, breaking ties uniformly at random. For any $u \in \mathcal{T}_{(n)}$, we write $N_{(n)}(u)$ for the ranking of u in the $|u|^{\text{th}}$ generation.

Let h be a positive continuous function on $[0, 1]$, we write $q = \left\lfloor e^{h_0 n^{1/3}} \right\rfloor$ and

$$\mathbf{T}^h_{(n)} = \left\{ u \in \mathcal{T}_{(q)} : |u| \leq n, \forall j \leq |u|, \log N_{(q)}(u_j) \leq n^{1/3} h_{j/n} \right\}.$$

The process $(\mathbf{T}^h_{(n)}, V)$ is a branching random walk with selection of the $e^{n^{1/3} h_\cdot}$ rightmost individuals. We write

$$M^h_n = \max_{u \in \mathbf{T}^h_{(n)}, |u| = n} V(u) \quad \text{and} \quad m^h_n = \min_{u \in \mathbf{T}^h_{(n)}, |u| = n} V(u).$$

We study $(\mathbf{T}^h_{(n)}, V)$ by comparing it with q independent branching random walks with a killing boundary f, choosing f in a way that

$$\log \# \left\{ u \in \mathbf{T}^{(n)}_f : |u| = \lfloor tn \rfloor \right\} \approx n^{1/3}(h_t - h_0).$$

Using Lemmas 5.3.1 and 5.3.2, we choose functions (f, g) verifying

$$\forall t \in [0, 1], \begin{cases} g_t + \frac{\pi^2\sigma^2}{2} \int_0^t \frac{ds}{(g_s - f_s)^2} = h_0 \\ f_t + \frac{\pi^2\sigma^2}{2} \int_0^t \frac{ds}{(g_s - f_s)^2} = h_0 - h_t. \end{cases}$$

which solution is

$$f : t \in [0, 1] \mapsto h_0 - h_t - \frac{\pi^2\sigma^2}{2} \int_0^t \frac{ds}{h_s^2} \quad \text{and} \quad g : t \in [0, 1] \mapsto h_0 - \frac{\pi^2\sigma^2}{2} \int_0^t \frac{ds}{h_s^2}. \tag{5.4.1}$$

To compare branching random walk with selection and branching random walks with killing boundary, we couple them in a fashion preserving a certain partial order, that we describe now. Let μ, ν be two Radon measures on $\mathbb{R}$, we write

$$\mu \preccurlyeq \nu \iff \forall x \in \mathbb{R}, \mu((x, +\infty)) \leq \nu((x, +\infty)).$$

The relation $\preccurlyeq$ forms a partial order on the set of Radon measures, that can be used to rank populations, representing an individual by a Dirac mass at its position.

A branching-selection process is defined as follows. Given $\varphi : \mathbb{Z}_+ \to \mathbb{N}$ a process adapted to the filtration of $\mathcal{T}(\varphi_0)$, we denote by

$$\mathbf{T}^\varphi = \left\{ u \in \mathcal{T}_{(\varphi_0)} : \forall j \leq |u|, N_{(\varphi_0)}(u_j) \leq \varphi_j \right\}.$$

Let $(x_1, \dots x_{\varphi_0}) \in \mathbb{R}^{\varphi_0}$, we write $V : u \in \mathbf{T}^\varphi \mapsto x_p + V^p(u)$ if $u \in \mathbf{T}^p$. The process $(\mathbf{T}^\varphi, V)$ is a branching-selection process with $\varphi(n)$ individuals at generation n and initial positions $(x_1, \dots x_{\varphi_0})$. Note that both $\mathbf{T}^h_{(n)}$ and $\mathbf{T}^{(n)}_f$ can be described as branching-selection processes. We prove there exists a coupling between branching-selection processes preserving partial order $\preccurlyeq$. Note this lemma is essentially an adaptation of [BG10, Corollary 2].

Lemma 5.4.1. *Let φ and ψ be two adapted processes, on the event*

$$\left\{ \sum_{\substack{u \in \mathbf{T}^\varphi \\ |u|=0}} \delta_{V(u)} \preccurlyeq \sum_{\substack{u \in \mathbf{T}^\psi \\ |u|=0}} \delta_{V(u)} \quad \text{and} \quad \forall j \leq n, \varphi_j \leq \psi_j \right\},$$

we have $\sum_{u \in \mathbf{T}^\varphi, |u|=n} \delta_{V(u)} \preccurlyeq \sum_{u \in \mathbf{T}^\psi, |u|=n} \delta_{V(u)}$.

Proof. The lemma is a direct consequence of the following observation. Given $m \leq n$, $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ such that $\sum_{j=1}^m \delta_{x_j} \preccurlyeq \sum_{j=1}^n \delta_{y_j}$ and $(z_i^j, j \leq n, i \in \mathbb{N})$, we have

$$\sum_{j=1}^m \sum_{i=1}^{+\infty} \delta_{x_j + z_i^j} \preccurlyeq \sum_{j=1}^n \sum_{i=1}^{+\infty} \delta_{y_j + z_i^j}.$$

Consequently, step k of the branching-selection process preserves order $\preccurlyeq$ if $\varphi_k \leq \psi_k$. $\square$

This lemma implies that branching random walks with selection and branching random walk with killing can be coupled in an increasing fashion for the order $\preccurlyeq$, as soon as there are at any time $k \leq n$ more individuals in one process than in the other. The main result of the section is the following estimate on the extremal positions in the branching random walk with selection.

Theorem 5.4.2. *Assuming* (5.1.1), (5.1.2) *and* (5.1.3)*, for any continuous positive function h we have*

$$\lim_{n \to +\infty} \frac{M_n^h}{n^{1/3}} = h_0 - \frac{\pi^2 \sigma^2}{2} \int_0^1 \frac{ds}{h_s^2} \quad \text{and} \quad \lim_{n \to +\infty} \frac{m_n^h}{n^{1/3}} = h_0 - h_1 - \frac{\pi^2 \sigma^2}{2} \int_0^1 \frac{ds}{h_s^2} \quad \text{a.s.}$$

Remark 5.4.3. It is worth noting that choosing h as a constant, Theorem 5.4.2 provides information on the Brunet-Derrida's N-BRW, at the time scale $\frac{(\log N)^3}{h^3}$. If we let $h \to 0$, we study the large times asymptotic behaviour of the N-BRW.

The proof of Theorem 5.4.2 is based on the construction of an increasing coupling existing between $(\mathbf{T}^h_{(n)}, V)$ and approximatively $e^{h_0 n^{1/3}}$ independent branching random walks with a killing boundary $n^{1/3} f_{./n}$. Using Lemma 5.4.1, it is enough to bound the size of the population at any time in the branching random walks with a killing boundary to prove the coupling. In a first time, we bound from below the branching random walk with selection by $e^{(h_0-2\varepsilon)n^{1/3}}$ independent branching random walks with a killing boundary.

Lemma 5.4.4. *We assume that* (5.1.1) *and* (5.1.2) *hold. For any positive continuous function h and $\varepsilon > 0$, there exists a coupling between $(\mathbf{T}^h_{(n)}, V)$ and i.i.d. branching random walks $((\mathbf{T}^j, V^j), j \geq 1)$ such that almost surely for any $n \geq 1$ large enough, we have*

$$\forall k \leq n, \sum_{\substack{u \in \mathbf{T}^h_{(n)} \\ |u|=k}} \delta_{V(u)} \succcurlyeq \sum_{j=1}^{e^{(h_0-2\varepsilon)n^{1/3}}} \sum_{\substack{u \in \mathbf{T}^j \\ |u|=k}} \mathbf{1}_{\left\{V^j(u_i) \geq (f_{i/n}-\varepsilon)n^{1/3}, i \leq k\right\}} \delta_{V^j(u)}. \tag{5.4.2}$$

Proof. Let $n \in \mathbb{N}$ and $\varepsilon > 0$, we denote by $p = \left\lfloor e^{(h_0-2\varepsilon)n^{1/3}} \right\rfloor$ and by $\widetilde{\mathbf{T}}^{(n)}_{f-\varepsilon}$ the disjoint union of $\mathbf{T}^{j\,(n)}_{f-\varepsilon}$ for $j \leq p$. For $u \in \widetilde{\mathbf{T}}^{(n)}_{f-\varepsilon}$, we write $V(u) = V^j(u)$ if $u \in \mathbf{T}^j$. By Lemma 5.4.1, it is enough to prove that almost surely, for any $n \geq 1$ large enough we have

$$\forall k \leq n, \log \# \left\{ u \in \widetilde{\mathbf{T}}^{(n)}_{f-\varepsilon} : |u| = k \right\} \leq n^{1/3} h_{k/n}.$$

We first prove that with high probability, no individual in $\widetilde{\mathbf{T}}^{(n)}_{f-\varepsilon}$ crosses the boundary $(g_{k/n} - \varepsilon)n^{1/3}$ at some time $k \leq n$. By Lemma 5.3.1, we have

$$\begin{aligned}
\limsup_{n \to +\infty} n^{-1/3} \log \mathbf{P}\left(\exists u \in \widetilde{\mathbf{T}}^{(n)}_{f-\varepsilon} : V(u) \geq (g_{|u|/n} - \varepsilon)n^{1/3} \right) \\
\leq \limsup_{n \to +\infty} n^{-1/3} \log \left(p \mathbf{P}\left(\exists u \in \mathbf{T}^{(n)}_{f-\varepsilon} : V(u) \geq (g_{|u|/n} - \varepsilon)n^{1/3} \right) \right) \\
\leq h_0 - 2\varepsilon - \inf_{t \in [0,1]} g_t - \varepsilon + \frac{\pi^2 \sigma^2}{2} \int_0^t \frac{ds}{(g_s - f_s)^2} = -\varepsilon.
\end{aligned}$$

Using the Borel-Cantelli lemma, almost surely for any $n \geq 1$ large enough and $u \in \widetilde{\mathbf{T}}^{(n)}_{f-\varepsilon}$, we have $V(u) \leq (g_{|u|/n} - \varepsilon)n^{1/3}$.

By this result, almost surely, for $n \geq 1$ large enough and for $k \leq n$, the size of the k^{th} generation in $\widetilde{\mathbf{T}}^{(n)}_{f-\varepsilon}$ is given by

$$Z^{(n)}_k = \sum_{u \in \widetilde{T}^{(n)}_{f-\varepsilon}} \mathbf{1}_{\{|u|=k\}} \mathbf{1}_{\left\{V(u_j) \leq (g_{j/n}-\varepsilon)n^{1/3}, j \leq k\right\}}.$$

Using the Markov inequality, we have

$$\mathbf{P}\left(\exists k \leq n : Z^{(n)}_k \geq e^{n^{1/3} h_{k/n}} \right) \leq \sum_{k=1}^n e^{-n^{1/3} h_{k/n}} \mathbf{E}\left[Z^{(n)}_k \right].$$

We now provide an uniform upper bound for $\mathbf{E}(Z^{(n)}_k)$. Applying Lemma 5.2.2, for any $1 \leq k \leq n$ we have

$$\begin{aligned}
\mathbf{E}\left[Z^{(n)}_k \right] &\leq p \, \mathbf{E}\left[e^{-S_k} \mathbf{1}_{\left\{S_j \in \left[(f_{j/n}-\varepsilon)n^{1/3}, (g_{j/n}-\varepsilon)n^{1/3}\right]\right\}} \right] \\
&\leq p e^{-(f_{k/n}-\varepsilon)n^{1/3}} \mathbf{P}\left(S_j \in \left[(f_{j/n} - \varepsilon)n^{1/3}, (g_{j/n} - \varepsilon)n^{1/3} \right], j \leq k \right).
\end{aligned}$$

Let $A \in \mathbb{N}$. For any $a \leq A$ we write $m_a = \lfloor na/A \rfloor$ and $\underline{f}_{a,A} = \inf_{s\in[a/A,(a+1)/A]} f_s$. For any $k \in (m_a, m_{a+1}]$, applying the Markov property at time m_a and Theorem 5.2.3 we have

$$\mathbf{E}\left[Z_k^{(n)}\right] \leq \exp\left[(h_0 - 2\varepsilon)n^{1/3} - n^{1/3}\left(\underline{f}_{a,A} - \varepsilon + \frac{\pi^2\sigma^2}{2}\int_0^{a/A}\frac{ds}{h_s^2}\right)\right]$$

As $h_0 = f_t + h_t + \frac{\pi^2\sigma^2}{2}\int_0^t \frac{ds}{h_s^2}$, letting $A \to +\infty$ we have

$$\limsup_{n\to+\infty} n^{-1/3}\log \mathbf{P}\left(\exists k \leq n : Z_k^{(n)} \geq e^{n^{1/3}h_{k/n}}\right) \leq -\varepsilon.$$

Consequently, applying the Borel-Cantelli lemma again, for any $n \geq 1$ large enough we have

$$\forall k \leq n, \log \#\left\{u \in \widetilde{\mathbf{T}}^{(n)}_{f-\varepsilon} : |u| = k\right\} \leq n^{1/3}h_{k/n}$$

which concludes the proof, by Lemma 5.4.1. □

Similarly, we prove that the branching random walk with selection is bounded from above by $\left\lfloor e^{(h_0+2\varepsilon)n^{1/3}}\right\rfloor$ independent branching random walks with a killing boundary.

Lemma 5.4.5. *We assume* (5.1.1)*,* (5.1.2) *and* (5.1.3) *hold. For any continuous positive function h and $\varepsilon > 0$, there exists a coupling between $(\mathbf{T}^h_{(n)}, V)$ and i.i.d. branching random walks $((\mathbf{T}^j, V^j), j \geq 1)$ such that almost surely for any $n \geq 1$ large enough we have*

$$\forall k \leq n, \sum_{\substack{u\in\mathbf{T}^h_{(n)}\\|u|=k}} \delta_{V(u)} \preccurlyeq \sum_{j=1}^{e^{(h_0+2\varepsilon)n^{1/3}}} \sum_{\substack{u\in\mathbf{T}^j\\|u|=k}} \mathbf{1}_{\left\{V^j(u_i)\geq(f_{i/n}-\varepsilon)n^{1/3}, i\leq k\right\}}\delta_{V^j(u)}. \tag{5.4.3}$$

Proof. Let $n \in \mathbb{N}$ and $\varepsilon > 0$, we denote by $p = \left\lfloor e^{(h_0+2\varepsilon)n^{1/3}}\right\rfloor$ and by $\widetilde{\mathbf{T}}^{(n)}_{f-\varepsilon}$ the disjoint union of ${\mathbf{T}^j}^{(n)}_{f-\varepsilon}$ for $j \leq p$. For $u \in \widetilde{\mathbf{T}}^{(n)}_{f-\varepsilon}$, we write $V(u) = V^j(u)$ if $u \in \mathbf{T}^j$. Similarly to the previous lemma, the key tool is a bound from below of the size of the population at any time in $\widetilde{\mathbf{T}}^{(n)}_{f-\varepsilon}$. For any $1 \leq k \leq n$, we set

$$Z_k^{(n)} = \sum_{u\in\widetilde{\mathbf{T}}^{(n)}_{f-\varepsilon}} \mathbf{1}_{\{|u|=k\}}\mathbf{1}_{\left\{V(u_j)\leq(g_{j/n}-\varepsilon)n^{1/3}, j\leq k\right\}} \quad \text{and}$$

$$\widetilde{Z}_k^{(n)} = \sum_{u\in\widetilde{\mathbf{T}}^{(n)}_{f-\varepsilon}} \mathbf{1}_{\{|u|=k\}}\mathbf{1}_{\left\{V(u)\geq f_1 n^{1/3}\right\}}\mathbf{1}_{\left\{V(u_j)\leq(g_{j/n}-\varepsilon)n^{1/3}, j\leq k\right\}}.$$

For any $t \in (0,1)$, applying Corollary 5.3.5, we have

$$\limsup_{n\to+\infty} n^{-1/3}\log \mathbf{P}\left[\widetilde{Z}^{(n)}_{\lfloor nt\rfloor} \leq e^{(h_t+\varepsilon)n^{1/3}}\right] \leq -3\varepsilon.$$

Let $A \in \mathbb{N}$, for $a \leq A$ we set $m_a = \lfloor na/A \rfloor$. By the Borel-Cantelli lemma, almost surely, for any $n \geq 1$ large enough we have

$$\forall a \leq A, \log \widetilde{Z}^{(n)}_{m_a} \geq n^{1/3}(h_{\frac{a}{A}} + \varepsilon).$$

We extend this result into an uniform one. To do so, we notice that Theorem 5.3.7 implies there exists $r > 0$ small enough and $\lambda > 0$ large enough such that

$$\inf_{n\in\mathbb{N}} \mathbf{P}\left[\exists |u| = n : \forall k \leq n, V(u_k) \geq -\lambda n^{1/3}\right] > r.$$

Consequently, every individual alive at time m_a above $f_{a/A}n^{1/3}$ start an independent branching random walk, which has probability at least r to have a descendant at time m_{a+1} which stayed at any time in $k \in [m_a, m_{a+1}]$ above $(f_{a/A} - \lambda A^{-1/3})n^{1/3}$. Choosing $A > 0$ large enough, conditionally on $\mathcal{F}_{m_a}$, $\inf_{k\in[m_a,m_{a+1}]} Z_k^{(n)}$ is stochastically bounded from below by a binomial variable with parameters $Z_{m_a}^{(n)}$ and r. We conclude from an easy large deviation estimate and the Borel-Cantelli lemma again, that almost surely for $n \geq 1$ large enough we have

$$\forall k \leq n, \log Z_k^{(n)} \geq n^{1/3} h_{k/n}.$$

Applying Lemma 5.4.1, we conclude that

$$\forall k \leq n, \sum_{\substack{u\in\mathbf{T}^h_{(n)}\\|u|=k}} \delta_{V(u)} \preccurlyeq \sum_{\substack{u\in\widetilde{\mathbf{T}}^{(n)}_{f-\varepsilon}\\|u|=k}} \delta_{V(u)}.$$

□

Using Lemmas 5.4.4 and 5.4.5, we easily bound the maximal and the minimal displacement in the branching random walk with selection.

Proof of Theorem 5.4.2. The proof is based on the observation that for any pair of sequences $x_1 \geq x_2 \geq \cdots \geq x_p$ and $y_1 \geq y_2 \geq \cdots \geq y_q$, if $\sum_{j=1}^p \delta_{x_j} \preccurlyeq \sum_{j=1}^q \delta_{y_j}$ then $p \leq q$, $x_1 \leq y_1$ and $x_p \leq y_p$.

Let $n \in \mathbb{N}$ and $\varepsilon > 0$, we denote by $\check{p} = \left\lfloor e^{(h_0-2\varepsilon)n^{1/3}} \right\rfloor$ and by $\widehat{p} = \left\lfloor e^{(h_0+2\varepsilon)n^{1/3}} \right\rfloor$. Given $((\mathbf{T}^j, V^j), j \in \mathbb{N})$ i.i.d. branching random walks, we set $\check{\mathbf{T}}^{(n)}_{f-\varepsilon}$ (respectively $\widehat{\mathbf{T}}^{(n)}_{f-\varepsilon}$) the disjoint union of ${\mathbf{T}^j}^{(n)}_{f-\varepsilon}$ for $j \leq \check{p}$ (resp. $j \leq \widehat{p}$). For $u \in \widehat{\mathbf{T}}^{(n)}_{f-\varepsilon}$, we write $V(u) = V^j(u)$ if $u \in \mathbf{T}^j$. By Lemmas 5.4.4 and 5.4.5, we have

$$\max_{u\in\check{\mathbf{T}}^{(n)}_{f-\varepsilon},|u|=n} V(u) \leq M_n^h \leq \max_{u\in\widehat{\mathbf{T}}^{(n)}_{f-\varepsilon},|u|=n} V(u).$$

For any $\delta > -h_0$, we denote by g^δ the solution of the differential equation

$$g_t^\delta + \frac{\pi^2\sigma^2}{2}\int_0^t \frac{ds}{(g_s^\delta - f_s)^2} = h_0 + \delta.$$

Note that g^δ is well-defined on $[0,1]$ for δ in a neighbourhood of 0, that $g^0 = g$ and that $\delta \mapsto g^\delta$ is continuous with respect to the uniform norm. Moreover

$$\begin{aligned}\mathbf{P}\left(\max_{u\in\widehat{\mathbf{T}}^{(n)}_{f-\varepsilon},|u|=n} V(u) \geq g_1^\delta n^{1/3}\right)&\\ \leq \mathbf{P}\left(\exists u \in \widehat{\mathbf{T}}^{(n)}_{f-\varepsilon} : V(u) \geq g^\delta_{|u|/n} n^{1/3}\right)&\\ \leq \widehat{p}\mathbf{P}\left(\exists |u| \leq n : V(u) \geq g^\delta_{|u|/n} n^{1/3}\right)&.\end{aligned}$$

Consequently, using Lemma 5.3.1, we have

$$\begin{aligned}\limsup_{n\to+\infty} n^{-1/3} \log \mathbf{P}\left(\max_{u\in\widehat{\mathbf{T}}^{(n)}_{f-\varepsilon},|u|=n} V(u) \geq g_1^\delta n^{1/3}\right)&\\ \leq h_0 + 2\varepsilon - \inf_{t\in[0,1]} g_t^\delta + \frac{\pi^2\sigma^2}{2}\int_0^t \frac{ds}{(g_s^\delta - f_s + \varepsilon)^2}&.\end{aligned}$$

For any $\delta > 0$, for any $\varepsilon > 0$ small enough we have

$$\limsup_{n\to+\infty} n^{-1/3} \log \mathbf{P}\left(M_n^h \geq g_1^\delta n^{1/3} \right) < 0.$$

By the Borel-Cantelli lemma, we have $\limsup_{n\to+\infty} \frac{M_n^h}{n^{1/3}} \leq g_1^\delta$ a.s. Letting $\delta \to 0$ concludes the proof of the upper bound of the maximal displacement.

To obtain a lower bound, we notice that

$$\begin{aligned}\mathbf{P}\left(M_n^h \leq (g_1^\delta - 2\varepsilon) n^{1/3} \right) &\leq \mathbf{P}\left(\max_{u\in\check{\mathbf{T}}^{(n)}_{f-\varepsilon}, |u|=n} V(u) \leq (g_1^\delta - 2\varepsilon)n^{1/3} \right)\\ &\leq \mathbf{P}\left(\max_{|u|=n} V(u) \leq (g_1^\delta - 2\varepsilon)n^{1/3} \right)^{\check{p}}.\end{aligned}$$

We only consider individuals u alive at time n that stayed at any time $k \leq n$ between the curves $n^{1/3}(f_{k/n} - \varepsilon)$ and $n^{1/3}(g^{-\delta}_{k/n} - \varepsilon)$, applying Corollary 5.3.4, for any $\delta > 0$ small enough, for any $\varepsilon > 0$ small enough, we have

$$\begin{aligned}\liminf_{n\to+\infty} n^{-1/3} &\log \mathbf{P}\left(\exists |u| = n : V(u) \geq (g_1^{-\delta} - 2\varepsilon)n^{1/3} \right)\\ &\geq - \sup_{t\in[0,1]} g_t^{-\delta} - \varepsilon + \frac{\pi^2\sigma^2}{2} \int_0^t \frac{ds}{(g_s^\delta - f_s)^2} \geq \varepsilon - h_0 + \delta.\end{aligned}$$

As a consequence,

$$\liminf_{n\to+\infty} n^{-1/3} \log\left(-\log \mathbf{P}\left(M_n^h \leq (g_1^\delta - 2\varepsilon)n^{1/3} \right)\right) \geq \delta - \varepsilon.$$

For any $\delta > 0$ small enough, for any $\varepsilon > 0$ small enough, applying the Borel-Cantelli lemma we have

$$\liminf_{n\to+\infty} \frac{M_n^h}{n^{1/3}} \geq g_1^\delta - 2\varepsilon \quad \text{a.s.}$$

Letting $\varepsilon \to 0$ then $\delta \to 0$ concludes the almost sure asymptotic behaviour M_n^h.

We now bound m_n^h. By Lemma 5.4.5, almost surely for $n \geq 1$ large enough, the $\left\lfloor e^{n^{1/3}h_1} \right\rfloor^{\text{th}}$ rightmost individual at generation n in $\widehat{\mathbf{T}}^{(n)}_{f-\varepsilon}$ is above m_n^h. Therefore for any $x \in \mathbb{R}$, almost surely for $n \geq 1$ large enough,

$$\mathbf{1}_{\left\{ m_n^h \geq x n^{1/3} \right\}} \leq \mathbf{1}_{\left\{ \#\left\{ u\in\widehat{\mathbf{T}}^{(n)}_{f-\varepsilon} : |u|=n, V(u)\geq xn^{1/3} \right\} \geq e^{h_1 n^{1/3}} \right\}}.$$

Let $\delta > 0$. By Lemma 5.3.1, we have

$$\limsup_{n\to+\infty} n^{-1/3} \log \mathbf{P}\left(\exists u \in \widehat{\mathbf{T}}^{(n)}_{f-\varepsilon} : V(u) \geq (g^\delta_{k/n} - \varepsilon)n^{1/3} \right) \leq h_0 - (h_0 + \delta - \varepsilon).$$

Consequently, for any $\delta > 0$, for any $\varepsilon > 0$ small enough, almost surely for $n \geq 1$ large enough the population in $\widehat{\mathbf{T}}^{(n)}_{f-\varepsilon}$ at time k belongs to $I_k^{(n)}$. We write

$$Z^{(n)}(x) = \sum_{u\in\widehat{\mathbf{T}}^{(n)}_{f-\varepsilon}} \mathbf{1}_{\{|u|=n\}} \mathbf{1}_{\left\{ V(u)\geq xn^{1/3} \right\}} \mathbf{1}_{\left\{ V(u_j)\leq (g^\delta_{j/n} - \varepsilon)n^{1/3}, j\leq n \right\}}.$$

By Lemma 5.3.2, we have

$$\begin{aligned}\limsup_{n\to+\infty} n^{-1/3}\log \mathbf{E}\left[Z^{(n)}(x)\right] &\leq h_0 - \left(x + \frac{\pi^2\sigma^2}{2}\int_0^t \frac{ds}{(g_s^\delta - f_s)^2}\right)\\ &\leq g_1^\delta - \delta - x.\end{aligned}$$

Using the Markov inequality and the Borel-Cantelli lemma, for any $\delta > 0$, for any $n \geq 1$ large enough, we have $Z^{(n)}(g_1^\delta - h_1) \leq e^{h_1 n^{1/3}}$, which leads to

$$\limsup_{n\to+\infty} \frac{m_n^h}{n^{1/3}} \leq g_1^\delta - h_1 \quad \text{a.s.}$$

Letting $\delta \to 0$ concludes the proof of the upper bound of m_n^h.

The lower bound is obtained in a similar fashion. For any $\zeta > 0$, we write $k = \left\lfloor \zeta n^{1/3} \right\rfloor$. Almost surely, for $n \geq 1$ large enough we have

$$\sum_{\substack{u\in\check{\mathbf{T}}^{(n)}_{f-\varepsilon}\\ |u|=n-k}} \delta_{V(u)} \preccurlyeq \sum_{\substack{u\in\mathbf{T}^h_{(n)}\\ |u|=n-k}} \delta_{V(u)}.$$

This inequality is not enough to obtain a lower bound on m_n^h, as there are less than $e^{h_1 n^{1/3}}$ individuals alive in $\check{\mathbf{T}}^{(n)}_{f-\varepsilon}$ at generation $n-k$. Therefore, starting from generation $n-k$, we start a modified branching-selection procedure that preserve the order $\preccurlyeq$ and guarantees there are $\left\lfloor e^{h_1 n^{1/3}} \right\rfloor$ individuals alive at generation n.

In a first time, we bound from below the size of the population alive at generation $n-k$. We write, for $\delta > 0$ and $\eta > 0$

$$X^{(n)} = \sum_{u\in\check{\mathbf{T}}^{(n)}_{f-\varepsilon}} \mathbf{1}_{\{|u|=n-k\}} \mathbf{1}_{\left\{V(u_j)\leq (g^{-\delta}_{j/n}-\varepsilon)n^{1/3}, \xi(u_j)\leq e^{\eta n^{1/3}}, j\leq n-k\right\}}.$$

By Lemma 5.3.3, we have

$$\liminf_{n\to+\infty} n^{-1/3}\log \mathbf{E}(X^{(n)}) \geq h_0 - 2\varepsilon - \left((f_1-\varepsilon) + \frac{\pi^2\sigma^2}{2}\int_0^1 \frac{ds}{(g_s^{-\delta}-f_s)^2}\right) = \delta - \varepsilon + (g_1^{-\delta} - f_1).$$

Consequently, using the fact that for $\check{p}$ i.i.d. random variables (X_j), we have

$$\mathbf{P}\left(\sum_{j=1}^{\check{p}} X_j \leq \check{p}\,\mathbf{E}(X_1)/2\right) \leq \frac{4\,\mathbf{E}(X_1^2)}{\check{p}\,\mathbf{E}(X_1)^2},$$

for any $\varepsilon > 0$ and $\delta > 0$ small enough enough, Lemma 5.3.3 leads to

$$\limsup_{n\to+\infty} n^{-1/3}\log \mathbf{P}\left(X^{(n)} \leq e^{((g_1^{-\delta}-f_1)+\delta)n^{1/3}}\right) \leq \eta + h_0 - \delta - \varepsilon - (h_0 - 2\varepsilon).$$

For any $\xi > 0$, choosing $\delta > 0$ small enough, and $\varepsilon > 0$ and $\eta > 0$ small enough, we conclude by the Borel-Cantelli lemma that almost surely, for $n \geq 1$ large enough

$$\#\left\{u \in \check{\mathbf{T}}^{(n)}_{f-\varepsilon} : |u| = n-k\right\} \geq \exp\left(n^{1/3}(h_1 - \xi)\right).$$

In a second time, we observe by (5.1.1) there exists $a > 0$ and $\varrho > 1$ such that

$$\mathbf{E}\left(\sum_{|u|=1} \mathbf{1}_{\{V(u)\geq -a\}}\right) > \varrho.$$

We consider the branching-selection process that starts at time $n-k$ with the population of the $(n-k)^{\text{th}}$ generation of $\check{\mathbf{T}}^{(n)}$, in which individuals reproduce independently according to the law $\mathcal{L}$, with the following selection process: an individual is erased if it belongs to generation $n-k+j$ and is below $n^{1/3} f_{(n-k)/n} - ja$, or if it is not one of the $e^{n^{1/3} h_{(n-k+j)/n}}$ rightmost individuals. By Lemma 5.4.1, this branching-selection process stays at any time $n-k \leq j \leq n$ below $(\mathbf{T}^h_{(n)}, V)$ for the order $\preccurlyeq$. Moreover, by definition, the leftmost individual alive at time n is above $n^{1/3}(f_{(n-k)/n} - \varepsilon - a\zeta)$.

We now bound the size of the population in this process. We write $(X_j, j \in \mathbb{N})$ for a sequence of i.i.d. random variables with the same law as $\sum_{|u|=1} \mathbf{1}_{\{V(u)\geq -a\}}$. By Cramér's theorem, there exists $\lambda > 0$ such that for any $n \in \mathbb{N}$, we have

$$\mathbf{P}\left(\sum_{k=1}^{n} X_j \leq n\varrho\right) \leq e^{-\lambda n}.$$

The probability there exists $j \in [n-k, n]$ such that the size of the population at time j in the branching-selection process is less than $\min\left(\varrho^{k+j-n} e^{(h_{(n-k)/n}-\xi)n^{1/3}}, e^{h_{j/n} n^{1/3}}\right)$ decays exponentially fast with n. Applying the Borel-Cantelli lemma, for any $\zeta > 0$, there exists $\xi > 0$ such that almost surely for $n \geq 1$ large enough, the number of individuals alive at generation n in the bounding branching-selection process is $\left\lfloor e^{h_1 n^{1/3}} \right\rfloor$. On this event, m^h_n is greater than the minimal position in this process. We conclude, letting n grows to $+\infty$ then ε and ζ decrease to 0 that

$$\liminf_{n\to+\infty} \frac{m^h_n}{n^{1/3}} \geq h_0 - h_1 - \frac{\pi^2\sigma^2}{2} \int_0^1 \frac{ds}{h_s^2} \quad \text{a.s.}$$

completing the proof of Theorem 5.4.2. □

An application of Theorem 5.4.2 leads to Theorem 5.1.1.

Proof of Theorem 5.1.1. Let $a > 0$, we denote by $\varphi : n \mapsto \left\lfloor e^{an^{1/3}} \right\rfloor$ and by $(\mathbf{T}^\varphi, V)$ the branching random walk with selection of the $\varphi(n)$ rightmost individuals at generation n. For $n \in \mathbb{N}$ we write

$$M^\varphi_n = \max_{u\in\mathbf{T}^\varphi, |u|=n} V(u) \quad \text{and} \quad m^\varphi_n = \min_{u\in\mathbf{T}^\varphi, |u|=n} V(u).$$

Let $\varepsilon > 0$ and $n \in \mathbb{N}$, we set $k = \lfloor n\varepsilon \rfloor$ and $h : t \mapsto a(t+\varepsilon)^{1/3}$. By Lemma 5.4.1, for any two continuous non-negative functions $h_1 \leq h_2$, and $k \leq n$ we have

$$\sum_{\substack{u\in\mathbf{T}^{h_1}_{(n)}\\ |u|=k}} \delta_{V(u)} \preccurlyeq \sum_{\substack{u\in\mathbf{T}^{h_2}_{(n)}\\ |u|=k}} \delta_{V(u)}.$$

As a consequence, for any $n \in \mathbb{N}$ and $\varepsilon > 0$, we couple the branching random walk with selection $(\mathbf{T}^\varphi, V)$ with two branching random walks with selection $(\mathbf{T}^{h,+}_{(n)}, V)$ and $(\mathbf{T}^{h,-}_{(n)}, V)$ in a way that

$$\sum_{\substack{u\in\mathbf{T}^{h,-}_{(n)}\\ |u|=n-k}} \delta_{V(u)+m^\varphi_k} \preccurlyeq \sum_{\substack{u\in\mathbf{T}^{\varphi}\\ |u|=n}} \delta_{V(u)} \preccurlyeq \sum_{\substack{u\in\mathbf{T}^{h,-}_{(n)}\\ |u|=n-k}} \delta_{V(u)+M^\varphi_k}, \tag{5.4.4}$$

using the fact that the population at time k in $\mathbf{T}^\varphi$ is between m^φ_k and M^φ_k.

Applying Theorem 5.4.2, we have

$$\limsup_{n\to+\infty} \frac{M^\varphi_n - M^\varphi_k}{n^{1/3}} \leq \limsup_{n\to+\infty} \frac{M^h_{n-k}}{n^{1/3}} \leq a\varepsilon^{1/3} - \frac{\pi^2\sigma^2}{2}\int_0^{1-\varepsilon} \frac{ds}{(a(s+\varepsilon)^{1/3})^2} \quad \text{a.s.}$$

as well as

$$\liminf_{n\to+\infty} \frac{m^\varphi_n - m^\varphi_k}{n^{1/3}} \geq \liminf_{n\to+\infty} \frac{m^h_{n-k}}{n^{1/3}} \geq -a - \frac{\pi^2\sigma^2}{2}\int_0^{1-\varepsilon} \frac{ds}{(a(s+\varepsilon)^{1/3})^2} \quad \text{a.s.}$$

As $\lim_{\varepsilon\to 0}\int_0^{1-\varepsilon} \frac{ds}{(a(s+\varepsilon)^{1/3})^2} = \frac{3}{a^2}$, for any $\delta > 0$, for any $\varepsilon > 0$ small enough we have

$$\limsup_{n\to+\infty} \frac{M^\varphi_n - M^\varphi_{\lfloor \varepsilon n\rfloor}}{n^{1/3}} \leq -\frac{3\pi^2\sigma^2}{2a^2} + \delta \quad \text{a.s.}$$

We set $p = \left\lfloor -\frac{\log n}{\log \varepsilon}\right\rfloor$, and observe that

$$\begin{aligned}\frac{M^\varphi_n}{n^{1/3}} &= \frac{1}{n^{1/3}}\sum_{j=0}^{p-2}\left(M^\varphi_{\lfloor \varepsilon^j n\rfloor} - M^\varphi_{\lfloor \varepsilon^{j+1} n\rfloor}\right) + \frac{M^\varphi_{\lfloor \varepsilon^{p-1} n\rfloor}}{n^{1/3}}\\ &\leq \sum_{j=0}^{p-2} \varepsilon^{j/3}\frac{M^\varphi_{\lfloor \varepsilon^j n\rfloor} - M^\varphi_{\lfloor \varepsilon^{j+1} n\rfloor}}{(\varepsilon^j n)^{1/3}} + \frac{\sup_{j\leq \varepsilon^{-2}} M^\varphi_j}{n^{1/3}}.\end{aligned}$$

Using a straightforward adaptation of the Cesàro lemma, we obtain

$$\limsup_{n\to+\infty} \frac{M^\varphi_n}{n^{1/3}} \leq \frac{-\frac{3\pi^2\sigma^2}{2a^2} + \delta}{1-\varepsilon^{1/3}} \quad \text{a.s.}$$

Letting $\varepsilon \to 0$ then $\delta \to 0$ we have

$$\limsup_{n\to+\infty} \frac{M^\varphi_n}{n^{1/3}} \leq -\frac{3\pi^2\sigma^2}{2a^2} \quad \text{a.s.} \tag{5.4.5}$$

Similarly, for any $\delta > 0$, for any $\varepsilon > 0$ small enough we have

$$\liminf_{n\to+\infty} \frac{m^\varphi_n - m^\varphi_{\lfloor \varepsilon n\rfloor}}{n^{1/3}} \geq -a - \frac{3\pi^2\sigma^2}{2a^2} - \delta \quad \text{a.s.}$$

Setting $p = \left\lfloor -\frac{\log n}{\log \varepsilon}\right\rfloor$ and observing that

$$\frac{m^\varphi_n}{n^{1/3}} \geq \sum_{j=0}^{p-2} \varepsilon^{j/3}\frac{m^\varphi_{\lfloor \varepsilon^j n\rfloor} - m^\varphi_{\lfloor \varepsilon^{j+1} n\rfloor}}{(\varepsilon^j n)^{1/3}} + \frac{\inf_{j\leq \varepsilon^{-2}} m^\varphi_j}{n^{1/3}},$$

we use again the Cesàro lemma to obtain, letting ε then δ decrease to 0,

$$\liminf_{n\to+\infty} \frac{m^\varphi_n}{n^{1/3}} \geq -a - \frac{3\pi^2\sigma^2}{2a^2} \quad \text{a.s.} \tag{5.4.6}$$

To obtain the other bounds, we observe that (5.4.4) also leads to

$$\liminf_{n\to+\infty} \frac{M^\varphi_n}{n^{1/3}} \geq \liminf_{n\to+\infty} \frac{M^h_{n-k} + m^\varphi_k}{n^{1/3}} \geq -\frac{\pi^2\sigma^2}{2a^2}\int_0^{1-\varepsilon} \frac{ds}{(s+\varepsilon)^{2/3}} - \left(a + \frac{3\pi^2\sigma^2}{2a^2}\right)\varepsilon^{1/3} \quad \text{a.s.}$$

by Theorem 5.4.2 and (5.4.6). Letting $\varepsilon \to 0$ we have

$$\liminf_{n\to+\infty} \frac{M_n^\varphi}{n^{1/3}} \geq -\frac{3\pi^2\sigma^2}{2a^2} \quad \text{a.s.}$$

Similarly, we have

$$\limsup_{n\to+\infty} \frac{m_n^\varphi}{n^{1/3}} \leq \limsup_{n\to+\infty} \frac{m_{n-k}^h + M_k^\varphi}{n^{1/3}} \leq -a - \frac{\pi^2\sigma^2}{2a^2}\int_0^{1-\varepsilon} \frac{ds}{(s+\varepsilon)^{2/3}} \quad \text{a.s.}$$

using Theorem 5.3.6 and (5.4.5). We let $\varepsilon \to 0$ to obtain

$$\limsup_{n\to+\infty} \frac{m_n^\varphi}{n^{1/3}} \leq -a - \frac{3\pi^2\sigma^2}{2a^2} \quad \text{a.s.}$$

$\square$

The careful reader will notice that, for almost any $a \in \mathbb{R}$ there exist $\overline{a} \neq a$ such that

$$a + \frac{3\pi^2\sigma^2}{2a^2} = \overline{a} + \frac{3\pi^2\sigma^2}{2\overline{a}^2}.$$

With these notation, both the branching random walk with selection of the $e^{an^{1/3}}$ rightmost individuals at generation n and the branching random walk with selection of the $e^{\overline{a}n^{1/3}}$ rightmost ones are coupled, between times εn and n with branching random walks with the same killing barrier

$$f : t \in [\varepsilon, 1] \mapsto \left(a + \frac{3\pi^2\sigma^2}{2a^2}\right) t^{1/3},$$

the difference between the processes being the number of individuals initially alive in the processes, respectively $e^{a(\varepsilon n)^{1/3}}$ and $e^{\overline{a}(\varepsilon n)^{1/3}}$.

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CHAPTER 6

The N-branching random walk with stable spine

"C'est très vilain de faire du mal à un livre, à un arbre ou à une bête."

René Goscinny – Le Petit Nicolas

Abstract

We consider a branching-selection particle system on the real line introduced by Brunet and Derrida in [BD97]. In this model, the size of the population is fixed to some constant N. At each step, these individuals reproduce independently. Only the N rightmost children survive to reproduce on the next generation. Bérard and Gouéré studied the speed at which the cloud of individuals drifts in [BG10], assuming the tails of the displacement decays at exponential rate; Bérard and Maillard [BM14] took interest in the case of heavy tail displacements. We offer here some interpolation between these two models, considering branching random walks in which the critical spine behaves as an α-stable random walk.

NOTA: This chapter is mainly extracted from the article *N-Branching random walk with α-stable spine*, available on arXiv:1503.03762.

6.1 Introduction

Let $\mathcal{L}$ be the law of a random point process on $\mathbb{R}$. Brunet, Derrida et al. introduced in [BD97, BDMM07] a discrete-time branching-selection particle system on $\mathbb{R}$ in which the size of the population is fixed to some integer N. This process evolves as follows: for any $n \in \mathbb{N}$, every particle in the n^{th} generation dies giving birth to children around its current position, according to an independent version of a point process of law $\mathcal{L}$. Only the N new individuals with the largest position are kept alive and form the $(n+1)^{\text{st}}$ generation of the process. We write $(x_n^N(1), \dots X_n^N(N))$ for the positions at time n of particles in the process, ranked in the decreasing order.

In [BG10], Berard and Gouéré proved that under some appropriated integrability conditions, the cloud of particles drifts at some deterministic speed

$$v_N = \lim_{n \to +\infty} \frac{x_n^N(0)}{n} = \lim_{n \to +\infty} \frac{x_n^N(N)}{n} \quad \text{a.s.,} \tag{6.1.1}$$

and obtained the following asymptotic behaviour for v_N as $N \to +\infty$

$$v_\infty - v_N \underset{N\to+\infty}{\sim} \frac{C}{(\log N)^2}, \tag{6.1.2}$$

in which C is an explicit positive constant that depends only on the law $\mathcal{L}$. Their arguments are based on precise computations on a branching random walk, defined below; and a coupling argument recalled in Section 6.4.2.

A *branching random walk* with branching law $\mathcal{L}$ is a process defined as follows. It begins with a unique individual located at position 0 at time 0. At each time $k \in \mathbb{N}$, each individual alive in the process at time k dies giving birth to children. The children are position around their parent according to i.i.d. point processes with law $\mathcal{L}$.

We write $\mathbf{T}$ for the genealogical tree of the process. For $u \in \mathbf{T}$, we denote by $V(u)$ the position of u, by $|u|$ the time at which u is alive, by πu the parent of u (provided that u is not the root of $\mathbf{T}$) and by u_k the ancestor alive at time k of u. We set $\Omega(u)$ the set of siblings of u i.e. the set of individuals $v \in \mathbf{T}$ such that $\pi v = \pi u$ and $v \neq u$. We observe that $\mathbf{T}$ is a (random) Galton-Watson tree with reproduction law $\#\mathcal{L}$, and that $(\mathbf{T}, V)$ is a (plane rooted) marked tree that we refer to as the branching random walk.

The point process $\mathcal{L}$ is supposed to verify some integrability assumptions. We write L for a point process with law $\mathcal{L}$. We assume in this chapter that the Galton-Watson tree $\mathbf{T}$ is supercritical and survives a.s., i.e.

$$\mathbf{E}\left[\#L\right] > 1 \quad \text{and} \quad \mathbf{P}\left(\#L = 0\right) = 0. \tag{6.1.3}$$

We suppose the point process law $\mathcal{L}$ to be in the stable boundary case, in the following sense:

$$\mathbf{E}\left[\sum_{\ell \in L} e^{\ell}\right] = 1, \tag{6.1.4}$$

and the random variable X defined by

$$\mathbf{P}(X \leq x) = \mathbf{E}\left[\sum_{\ell \in L} \mathbf{1}_{\{\ell \leq x\}} e^{\ell}\right] \tag{6.1.5}$$

to be in the domain of attraction of a stable random variable Y verifying $\mathbf{P}(Y \geq 0) \in (0,1)$. Note that if $\mathbf{E}(|X|) < +\infty$, this assumption implies $\mathbf{E}(X) = 0$. In this case, the point process is in the boundary case, as defined in [BK05]. Up to an affine transformation several point processes laws verify these properties, adapting the discussion in [Jaf12, Appendix A] to this setting.

The following result, that gives a necessary and sufficient condition for X to be in the domain of attraction of Y, can be found in [Fel71, Chapter XVII]. Let $\alpha \in (0,2]$ be such that Y is an α-stable random variable verifying $\mathbf{P}(Y \geq 0) \in (0,1)$. We introduce the function

$$L^* : x \mapsto x^{\alpha-2}\, \mathbf{E}\left[Y^2 \mathbf{1}_{\{|Y| \leq x\}}\right]. \tag{6.1.6}$$

This function is slowly varying [1]. We set

$$b_n = \inf\left\{x > 0 : \frac{x^\alpha}{L^*(x)} = n\right\}. \tag{6.1.7}$$

1. i.e. for all $\lambda > 0$, $\lim_{t\to+\infty} \frac{L^*(\lambda t)}{L^*(t)} = 1$.

The random variable X is in the domain of attraction Y if and only if writing (S_n) for a random walk with step distribution with the same law as X, $\frac{S_n}{b_n}$ converges in law to Y.

As Y is an α-stable random variable, there exists an α-stable Lévy process $(Y_t, t \geq 0)$ such that Y_1 has the same law as Y. By [Mog74, Lemma 1], we define

$$C_* = \lim_{t \to +\infty} -\frac{1}{t} \log \mathbf{P}\left(|Y_s| \leq \frac{1}{2}, s \leq t \right) \in (0, +\infty). \tag{6.1.8}$$

We introduce an additional integrability assumption to ensure that the spine in the spinal decomposition –see Section 2.1– has the same behaviour as a typical individual staying close to the boundary. We assume that

$$\lim_{x \to +\infty} \frac{x^\alpha}{L^*(x)} \mathbf{E}\left[\sum_{\ell \in L} e^\ell \mathbf{1}_{\left\{\log\left(\sum_{\ell' \in L} e^{\ell' - \ell}\right) > x\right\}} \right] = 0, \tag{6.1.9}$$

and that

$$\mathbf{E}\left[\left| \max_{\ell \in L} \ell \right|^2 \right] < +\infty. \tag{6.1.10}$$

Condition (6.1.10) is not expected to be optimal. It is used to bound in a crude way from below the minimal position in the branching random walk with selection of the N rightmost individuals.

Theorem 6.1.1. *Under the previous assumptions, the sequence* $(v_N, N \geq 1)$ *described in* (6.1.2) *exists and verifies*

$$v_N \underset{N \to +\infty}{\sim} -C_* \frac{L^*(\log N)}{(\log N)^\alpha}.$$

Examples. We present two point process laws that satisfy the hypotheses of Theorem 6.1.1. Let X be the law of a random variable on $\mathbb{R}$. We write $\Lambda(\theta)$ for the log-Laplace transform of X. We assume there exists $\theta^* > 0$ such that $\Lambda(\theta^*) = \log 2$, and $\alpha > 1$ verifying

$$\mathbf{P}(X \geq x) \sim e^{-\theta^* x} x^{-\alpha-1}.$$

In this case, there exists $\mu := \mathbf{E}\left(X e^{\theta^* X}\right)/2$ such that the point process $\mathcal{L}$ defined as the law of a pair of independent random variables (Y_1, Y_2) which have the same law as $\theta^*(X - \mu)$ satisfies the hypotheses of Theorem 6.1.1.

Let ν_α be the law of an α-stable random variable Y such that $\mathbf{P}(Y \geq 0) \in (0, 1)$. If $\widetilde{\mathcal{L}}$ is the law of a point process on $\mathbb{R}$ with intensity $\nu(dx)e^{-x}$, then $\widetilde{\mathcal{L}}$ satisfies all assumptions of Theorem 6.1.1, and the spine of such a branching random walk is in the domain of attraction of Y.

The rest of the chapter is organised as follows. In Section 6.2, we introduce the spinal decomposition, that links the computation of additive branching random walk moments with random walks estimates; and the Mogul'skiĭ small deviations estimate for random walks. In Section 6.3, we use these two results to compute the asymptotic of the survival of individuals above a killing line of slope $-\varepsilon$, using the same technique as [GHS11]. This asymptotic is then used in Section 6.4 to prove Theorem 6.1.1, applying the methods introduced in [BG10].

6.2 Useful lemmas

6.2.1 The spinal decomposition

The spinal decomposition is a tool introduced by Lyons, Pemantle and Peres in [LPP95] to study branching processes. It has been extended to branching random walks by Lyons in [Lyo97]. It provides two descriptions on a law absolutely continuous with respect to the law $\mathbf{P}_a$, of the branching random walk $(\mathbf{T}, V + a)$. More precisely, we set $W_n = \sum_{|u|=n} e^{V(u)}$ and $\mathcal{F}_n = \sigma(u, V(u), |u| \leq n)$ the natural filtration on the set of marked trees. We observe that (W_n) is a non-negative martingale. We define the probability measure $\overline{\mathbf{P}}_a$ on $\mathcal{F}_\infty$ such that for any $n \in \mathbb{N}$,

$$\left.\frac{d\overline{\mathbf{P}}_a}{d\mathbf{P}_a}\right|_{\mathcal{F}_n} = e^{-a} W_n. \tag{6.2.1}$$

We write $\overline{\mathbf{E}}_a$ for the corresponding expectation.

We construct a second probability measure $\widehat{\mathbf{P}}_a$ on the set of marked trees with spine. For $(\mathbf{T}, V)$ a marked tree, we say that $w = (w_n, n \in \mathbb{N})$ is a spine of $\mathbf{T}$ if for any $n \in \mathbb{N}$, $|w_n| = n$, $w_n \in \mathbf{T}$ and $(w_n)_{n-1} = w_{n-1}$. We introduce

$$\frac{d\widehat{\mathcal{L}}}{d\mathcal{L}} = \sum_{\ell \in \mathcal{L}} e^{\ell}, \tag{6.2.2}$$

another law of point processes. The probability measure $\widehat{\mathbf{P}}_a$ is the law of the process $(\mathbf{T}, V, w)$ constructed as follows. It starts at time 0 with a unique individual w_0 located at position a. It makes children according to a point process of law $\widehat{\mathcal{L}}$. Individual w_1 is chosen at random among children u of w_1 with probability $\frac{e^{V(u)}}{\sum_{|v|=1} e^{V(v)}}$. At each generation $n \in \mathbb{N}$, every individual u in the n^{th} generation dies, giving independently birth to children according to independent point processes, with law $\widehat{\mathcal{L}}$ if $u = w_n$ or law $\mathcal{L}$ otherwise. Finally, w_{n+1} is chosen among children v of w_n with probability proportional to $e^{V(v)}$.

To shorten notations, we write $\overline{\mathbf{P}} = \overline{\mathbf{P}}_0$, $\widehat{\mathbf{P}} = \widehat{\mathbf{P}}_0$. The spinal decomposition links laws $\widehat{\mathbf{P}}$ and $\overline{\mathbf{P}}$.

Proposition 6.2.1 (Spinal decomposition). *Under assumption* (6.1.4)*, for any* $n \in \mathbb{N}$*, we have*

$$\left.\widehat{\mathbf{P}}_a\right|_{\mathcal{F}_n} = \left.\overline{\mathbf{P}}_a\right|_{\mathcal{F}_n}.$$

Moreover, for any $z \in \mathbf{T}$ *such that* $|z| = n$*,*

$$\widehat{\mathbf{P}}_a \left(w_n = z \middle| \mathcal{F}_n \right) = \frac{e^{V(z)}}{W_n},$$

and $(V(w_n), n \geq 0)$ *is a random walk starting from* a*, with step distribution defined in* (6.1.5)*.*

An immediate consequence of Proposition 6.2.1 is the celebrated many-to-one lemma. Introduced by Peyrière in [Pey74], this lemma links an additive moment of the branching random walks with a random walk estimate. Given (X_n) an i.i.d. sequence of random variables with law defined by (6.1.5), we set $S_n = S_0 + \sum_{j=1}^n X_j$ such that $\mathbf{P}_a(S_0 = a) = 1$.

Lemma 6.2.2 (Many-to-one lemma)**.** *Under assumption* (6.1.4)*, for any* $n \geq 1$ *and measurable non-negative function* g*, we have*

$$\mathbf{E}_a\left[\sum_{|u|=n} g(V(u_1), \cdots V(u_n))\right] = \mathbf{E}_a\left[e^{a-S_n} g(S_1, \cdots S_n)\right]. \tag{6.2.3}$$

Proof. We use Proposition 6.2.1 to compute

$$\begin{aligned}
\mathbf{E}_a\left[\sum_{|u|=n} g(V(u_1), \cdots V(u_n))\right] &= \overline{\mathbf{E}}_a\left[\frac{e^a}{W_n}\sum_{|z|=n} g(V(u_1), \cdots V(u_n))\right] \\
&= \widehat{\mathbf{E}}_a\left[e^a \sum_{|u|=n} \widehat{\mathbf{P}}_a\left(w_n = u \middle| \mathcal{F}_n\right) e^{-V(u)} g(V(u_1), \cdots V(u_n))\right] \\
&= \widehat{\mathbf{E}}_a\left[e^{a-V(w_n)} g(V(w_1), \cdots, V(w_n))\right].
\end{aligned}$$

We now observe that $(S_n, n \geq 0)$ under $\mathbf{P}_a$ has the same law as $(V(w_n), n \geq 0)$ under $\widehat{\mathbf{P}}_a$, which ends the proof. □

The many-to-one lemma can be used to bound the maximal displacement in a branching random walk. For example, for all $y \geq 0$, we have

$$\begin{aligned}
\mathbf{E}\left[\sum_{u \in \mathbf{T}} \mathbf{1}_{\{V(u) \geq y\}} \mathbf{1}_{\{V(u_j) < y, j < |u|\}}\right] &= \sum_{k=1}^{+\infty} \mathbf{E}\left[\sum_{|u|=k} \mathbf{1}_{\{V(u) \geq y\}} \mathbf{1}_{\{V(u_j) < y, j < |u|\}}\right] \\
&= \sum_{k=1}^{+\infty} \mathbf{E}\left[e^{-S_k} \mathbf{1}_{\{S_k \geq y\}} \mathbf{1}_{\{S_j < y, j < k\}}\right] \\
&\leq e^{-y} \sum_{k=1}^{+\infty} \mathbf{P}\left(S_k \geq y, S_j < y, j < k\right) \\
&\leq e^{-y}.
\end{aligned}$$

By the Markov inequality, this computation leads to

$$\mathbf{P}\left(\sup_{n \in \mathbb{N}} M_n \geq y\right) \leq \sup_{n \in \mathbb{N}} \mathbf{P}(M_n \geq y) \leq e^{-y}. \tag{6.2.4}$$

Using the spinal decomposition, to compute the number of individuals in a branching random walk who stay in a well-chosen path, it is enough to know the probability for a random walk decorated by additional random variables to follow that path.

6.2.2 Small deviations estimate and variations

Let S be a random walk in the domain of attraction of an α-stable random variable Y. We recall that

$$L^*(u) = u^{\alpha-2}\,\mathbf{E}(Y \mathbf{1}_{\{|Y| \leq u\}}) \quad \text{and} \quad \frac{b_n^\alpha}{L^*(b_n)} = n.$$

For any $z \in \mathbb{R}$, we define $\mathbf{P}_z$ in a way that S under law $\mathbf{P}_z$ has the same law as $S + z$ under law $\mathbf{P}$. The Mogul'skiĭ small deviation estimate enables to compute the probability for S to present typical fluctuations of order $o(b_n)$.

Theorem 6.2.3 (Mogul'skiĭ theorem)**.** *Let $(a_n) \in \mathbb{R}_+^{\mathbb{N}}$ be such that*

$$\lim_{n\to+\infty} a_n = +\infty, \lim_{n\to+\infty} \frac{a_n}{b_n} = 0.$$

Let $f < g$ be two continuous functions such that $f(0) < 0 < g(0)$. If $\mathbf{P}(Y \le 0) \in (0,1)$ then

$$\lim_{n\to+\infty} \frac{a_n^\alpha}{nL^*(a_n)} \log \mathbf{P}\left[\frac{S_j}{a_n} \in \left[f\left(\tfrac{j}{n}\right), g\left(\tfrac{j}{n}\right)\right], 0 \le j \le n\right] = -C_* \int_0^1 \frac{ds}{(g(s)-f(s))^\alpha},$$

where C_* is defined in (6.1.8).

This result, proved in [Mog74], can be seen as a consequence of an α-stable version of the Donsker theorem, obtained by Prokhorov. This result yields the convergence of the trajectory of the random walk S, suitably normalized, to the trajectory of an α-stable Lévy process $(Y_t, t \in [0,1])$ such that Y_1 has the same law as Y.

Theorem 6.2.4 (Prokhorov theorem [Pro56])**.** *If $\frac{S_n}{b_n}$ converges in law to a stable random variable Y, then the process $(\frac{S_{\lfloor nt \rfloor}}{b_n}, t \in [0,1])$ converges in law to $(Y_t, t \in [0,1])$ in $\mathcal{D}([0,1])$ equipped with the Skorokhod topology.*

We first note that the Mogul'skiĭ estimate holds uniformly with respect to the starting point.

Corollary 6.2.5. *With the same notation as Theorem 6.2.3, we have*

$$\lim_{n\to+\infty} \frac{a_n^\alpha}{nL^*(a_n)} \log \sup_{y\in\mathbb{R}} \mathbf{P}_y\left[\frac{S_j}{a_n} \in [f(\tfrac{j}{n}), g(\tfrac{j}{n})], 0 \le j \le n\right] = -C_* \int_0^1 \frac{ds}{(g(s)-f(s))^\alpha}.$$

Proof. Observe in a first time that if $y \notin [a_n f(0), a_n g(0)]$, then

$$\mathbf{P}_y\left[\frac{S_j}{a_n} \in [f(\tfrac{j}{n}), g(\tfrac{j}{n})], 0 \le j \le n\right] = 0.$$

We now choose $\delta > 0$, and write $K = \left\lceil \frac{g(0)-f(0)}{\delta} \right\rceil$, we have

$$\sup_{y\in\mathbb{R}} \mathbf{P}_y\left[\frac{S_j}{a_n} \in [f(\tfrac{j}{n}), g(\tfrac{j}{n})], 0 \le j \le n\right] \le \max_{k\le K} \Pi_{f(0)+k\delta, f(0)+(k+1)\delta}(f,g),$$

where

$$\begin{aligned}\Pi_{x,x'}(f,g) &= \sup_{y\in[xa_n, x'a_n]} \mathbf{P}_y\left[\frac{S_j}{a_n} \in [f(\tfrac{j}{n}), g(\tfrac{j}{n})], 0 \le j \le n\right] \\ &\le \mathbf{P}\left[\frac{S_j}{a_n} \in [f(\tfrac{j}{n}) - x', g(\tfrac{j}{n}) - x], 0 \le j \le n\right].\end{aligned}$$

Therefore, for all $k \le K$, we have

$$\limsup_{n\to+\infty} \frac{a_n^\alpha}{nL^*(a_n)} \log \Pi_{f(0)+k\delta, f(0)+(k+1)\delta}(f,g) \le -C_* \int_0^1 \frac{ds}{(g(s)-f(s)+\delta)^\alpha},$$

which leads to

$$\begin{aligned}\limsup_{n\to+\infty} \frac{a_n^\alpha}{nL^*(a_n)} \log \sup_{y\in\mathbb{R}} \mathbf{P}\left[\frac{S_j+y}{a_n} \in [f(\tfrac{j}{n}), g(\tfrac{j}{n})], 0 \le j \le n\right]& \\ \le -C_* \int_0^1 \frac{ds}{(g(s)-f(s)+\delta)^\alpha}&.\end{aligned}$$

We let $\delta \to 0$, which concludes the proof, as the lower bound is a direct consequence of Theorem 6.2.3. □

Using an adjustment of the original proof of Mogul'skiĭ, one can prove a similar estimate for enriched random walks. We set (X_n, ξ_n) a sequence of i.i.d. random variables on $\mathbb{R} \times \mathbb{R}_+$, with X_1 in the domain of attraction of the stable random variable Y, such that $\mathbf{P}(Y > 0) \in (0, 1)$. We denote by $S_n = S_0 + X_1 + \cdots + X_n$, which is a random walk in the domain of attraction of Y. The following estimate then holds.

Lemma 6.2.6. *Let $(a_n) \in \mathbb{R}_+^{\mathbb{N}}$ be such that $\lim_{n\to+\infty} \frac{a_n}{b_n} = 0$. We set $E_n = \{\xi_j \leq n, j \leq n\}$ and we assume that*

$$\lim_{n\to+\infty} \frac{a_n^\alpha}{L^*(a_n)} \mathbf{P}(\xi_1 \geq n) = 0. \tag{6.2.5}$$

There exists $C_* > 0$, given by (6.1.8), such that for any pair (f, g) of continuous functions verifying $f < g$, for any $f(0) < x < y < g(0)$ we have

$$\lim_{n\to+\infty} \frac{a_n^\alpha}{nL^*(a_n)} \log \inf_{z\in[xa_n, ya_n]} \mathbf{P}_z \left(\frac{S_j}{a_n} \in \left[f_{j/n}, g_{j/n} \right], j \leq n, E_n \right) = -C_* \int_0^1 \frac{ds}{(g(s) - f(s))^\alpha}.$$

Proof. We assume in a first time that f, g are two constant functions. Let $n \geq 1$, $f < x < 0 < y < g$ and $f < x' < y' < g$, we denote by

$$P_{x,y}^{x',y'}(f, g) = \inf_{z\in[x,y]} \mathbf{P}_{za_n} \left(\frac{S_n}{a_n} \in [x', y'], \frac{S_j}{a_n} \in [f, g], j \leq n, E_n \right). \tag{6.2.6}$$

Let $A > 0$ and $r_n = \left\lfloor A \frac{a_n^\alpha}{L^*(a_n)} \right\rfloor$. We divide $[0, n]$ into $K = \left\lfloor \frac{n}{r_n} \right\rfloor$ intervals of length r_n. For any $k \leq K$, we set $m_k = kr_n$ and $m_{K+1} = n$. Applying the Markov property at time $m_K, \ldots m_1$, and restricting to trajectories which are, at any time m_k in $[x'a_n, y'a_n]$, we have

$$P_{x,y}^{x',y'}(f, g) \geq \pi_{x,y}^{x',y'}(f, g) \left(\pi_{x',y'}^{x',y'}(f, g) \right)^K, \tag{6.2.7}$$

where we set

$$\pi_{x,y}^{x',y'}(f, g) = \inf_{z\in[x,y]} \mathbf{P}_{za_n} \left(\frac{S_{r_n}}{a_n} \in [x', y'], \frac{S_j}{a_n} \in [f, g], j \leq r_n, E_{r_n} \right).$$

Let $\delta > 0$ be chosen small enough such that $M = \left\lceil \frac{y-x}{\delta} \right\rceil \geq 3$. We observe easily that

$$\begin{aligned} \pi_{x,y}^{x',y'}(f, g) &\geq \min_{0\leq m\leq M} \pi_{x+m\delta, x+(m+1)\delta}^{x',y'}(f, g) \\ &\geq \min_{0\leq m\leq M} \pi_{x,x}^{x'+(m+1)\delta, y'+(m-1)\delta}(f + (m + 1)\delta, g + (m - 1)\delta). \end{aligned} \tag{6.2.8}$$

Moreover, we have

$$\begin{aligned} \pi_{x,x}^{x',y'}(f, g) &= \mathbf{P}_{xa_n} \left(\frac{S_{r_n}}{a_n} \in [x', y'], \frac{S_j}{a_n} \in [f, g], E_{r_n} \right) \\ &\geq \mathbf{P}_{xa_n} \left(\frac{S_{r_n}}{a_n} \in [x', y'], \frac{S_j}{a_n} \in [f, g] \right) - r_n \mathbf{P}(\xi_1 \geq n). \end{aligned}$$

By (6.2.5), we have $\lim_{n\to+\infty} r_n \mathbf{P}(\xi_1 \geq n) = 0$. Applying Theorem 6.2.4, the random walk $(\frac{S_{\lfloor r_n t \rfloor}}{a_n}, t \in [0, A])$ converges as $n \to +\infty$ under law $\mathbf{P}_{xa_n}$ to a stable Lévy process $(x + Y_t, t \in [0, A])$ such that Y_1 has the same law than Y. In particular

$$\liminf_{n\to+\infty} \pi_{x,x}^{x',y'}(f, g) \geq \mathbf{P}_x(Y_A \in (x', y'), Y_u \in (f, g), u \leq A).$$

Using (6.2.8), we have

$$\liminf_{n\to+\infty} \pi_{x,y}^{x',y'}(f,g) \geq \min_{0\leq m\leq M} \mathbf{P}_{x+m\delta}(Y_A \in (x'+\delta, y'-\delta), Y_u \in (f+\delta, g-\delta), u \leq A).$$

As a consequence, recalling that $K \sim \frac{nL^*(a_n)}{Aa_n^\alpha}$, (6.2.7) leads to

$$\liminf_{n\to+\infty} \frac{a_n^\alpha}{nL^*(a_n)} \log P_{x,y}^{x',y'}(f,g)$$
$$\geq \frac{1}{A} \min_{0\leq m\leq M} \log \mathbf{P}_{x+m\delta}(Y_A \in (x'+\delta, y'-\delta), Y_u \in (f+\delta, g-\delta), u \leq A). \quad (6.2.9)$$

By [Mog74, Lemma 1], we have

$$\lim_{t\to+\infty} \frac{1}{t} \log_x \mathbf{P}(Y_t \in (x', y'), Y_s \in (f,g), s \leq t) = -\frac{C_*}{(g-f)^\alpha},$$

where C_* is defined by (6.1.8). Letting $A \to +\infty$ then $\delta \to 0$, (6.2.9) yields

$$\liminf_{n\to+\infty} \frac{a_n^\alpha}{nL^*(a_n)} \log P_{x,y}^{x',y'}(f,g) \geq -\frac{C_*}{(g-f)^\alpha} \quad (6.2.10)$$

which is the expected result when f, g are two constants.

In a second time, we consider two continuous functions $f < g$. Let $f(0) < x < y < g(0)$. We set h a continuous function such that $f < h < g$ and $h(0) = \frac{x+y}{2}$. Let $\varepsilon > 0$ such that $6\varepsilon \leq \inf_{t\in[0,1]} \min(g_t - h_t, h_t - f_t)$. We choose $A > 0$ such that

$$\sup_{|t-s|\leq \frac{2}{A}} |f_t - f_s| + |g_t - g_s| + |h_t - h_s| \leq \varepsilon.$$

and for $a \leq A$, we write $m_a = \lfloor an/A \rfloor$ and $I_{a,A} = [f_{a/A}+\varepsilon, g_{a/A}-\varepsilon]$. We define $J_{0,A} = [x,y]$, and for $1 \leq a \leq A$, $J_{a,A} = [h_{a/A} - \varepsilon, h_{a/A} + \varepsilon]$. Applying the Markov property at times $m_{A-1}, \ldots, m_1$, we have

$$\inf_{z\in[xa_n, ya_n]} \mathbf{P}_z \left(\frac{S_j}{a_n} \in \left[f_{j/n}, g_{j/n} \right], j \leq n, E_n \right)$$
$$\geq \prod_{a=0}^{A-1} \inf_{x\in J_{a,A}} \mathbf{P}_{xa_n} \left(\frac{S_{m_{a+1}}}{a_n} \in J_{a+1,A}, \frac{S_j}{a_n} \in I_{a,A}, j \leq m_{a+1} - m_a, E_{m_{a+1}-m_a} \right).$$

Therefore, using equation (6.2.10), we have

$$\liminf_{n\to+\infty} \frac{nL^*(a_n)}{a_n^\alpha} \log \mathbf{P} \left(\frac{S_j}{a_n} \in \left[f_{j/n}, g_{j/n} \right], E_n, j \leq n \right) \geq -\frac{1}{A} \sum_{a=0}^{A-1} C_* \frac{1}{(g_{a,A} - f_{a,A} - 2\varepsilon)^\alpha}.$$

As the upper bound is a direct consequence of Theorem 6.2.3, we let $A \to +\infty$ and $\varepsilon \to 0$ to conclude the proof. □

6.3 Branching random walk with a barrier

In this section, we study the asymptotic, as $n \to +\infty$ and $\varepsilon \to 0$ of the quantity

$$\varrho(n,\varepsilon) = \mathbf{P} \left(\exists |u| = n : \forall j \leq n, V(u_j) \geq -\varepsilon j \right). \quad (6.3.1)$$

The asymptotic behaviour of $\varrho(\infty, \varepsilon)$ has been studied by Gantert, Hu and Shi in [GHS11] for a branching random walk with a spine in the domain of attraction of a Gaussian random variable. To do so, they studied the asymptotic behaviour of $\varrho(n, \varepsilon)$ for $\varepsilon \approx \theta n^{-2/3}$. Using the same arguments, we obtain sharp estimates on the asymptotic behaviour of $\varrho(n, \varepsilon)$ for $\varepsilon \approx \theta\Lambda(n)n^{-\frac{\alpha}{\alpha+1}}$, where Λ is a well-chosen slowly varying function.

We apply the spinal decomposition and the Mogul'skiĭ estimate to compute the number of individuals that stay at any time $k \leq n$ between curves $a_n f(k/n)$ and $a_n g(k/n)$, for an appropriate choice of (a_n), f and g. We note that

$$\begin{aligned}\mathbf{E}\left[\sum_{|u|=n} \mathbf{1}_{\{V(u_j)\in[a_n f(j/n), a_n g(j/n)], j\leq n\}}\right] &= \mathbf{E}\left[e^{-S_n}\mathbf{1}_{\{S_j\in[a_n f(j/n), a_n g(j/n), j\leq n\}}\right]\\ &\approx e^{-a_n g(1)}\mathbf{P}\left(S_j \in [a_n f(j/n), a_n g(j/n)], j \leq n\right)\\ &\approx \exp\left(-a_n g(1) - \frac{nL^*(a_n)}{a_n^\alpha} C_* \int_0^1 \frac{ds}{(g_s - f_s)^\alpha}\right).\end{aligned}$$

This informal computation hints that to obtain tight estimates, it is appropriate to choose a sequence (a_n) satisfying $a_n \sim_{n\to+\infty} \frac{nL^*(a_n)}{a_n^\alpha}$, and functions f and g verifying

$$\forall t \in [0, 1], g(t) + C_* \int_0^t \frac{ds}{(g_s - f_s)^\alpha} = g(0), \tag{6.3.2}$$

However, the differential equation $g'_t = -C_*(g_t - \theta t)^\alpha$ being uneasy to solve as a function of t and θ we use approximate solutions for (6.3.2).

We define the sequence

$$\forall n \in \mathbb{N}, a_n = \inf\left\{x \geq 0 : \frac{x^{\alpha+1}}{L^*(x)} = n\right\}. \tag{6.3.3}$$

and we introduce the function

$$\Phi : \begin{array}{ccc} (0, +\infty) & \to & \mathbb{R} \\ \lambda & \mapsto & \frac{C_*}{\lambda^\alpha} - \frac{\lambda}{\alpha+1}. \end{array} \tag{6.3.4}$$

Note that Φ is a $\mathcal{C}^\infty$ strictly decreasing function on $(0, +\infty)$, that admits a well-defined inverse Φ^{-1}. The main result of the section is the following.

Theorem 6.3.1. *Under the assumptions of Theorem 6.1.1, for all $\theta > 0$ we have*

$$-\frac{C_*^{\frac{1}{\alpha}}}{\theta^{\frac{1}{\alpha}}} \leq \liminf_{n\to+\infty} \frac{1}{a_n} \log \varrho\left(n, \theta\frac{a_n}{n}\right) \leq \limsup_{n\to+\infty} \frac{1}{a_n} \log \varrho\left(n, \theta\frac{a_n}{n}\right) \leq -\Phi^{-1}(\theta).$$

Remark 6.3.2. By inversion of regularly varying functions, for all $\mu > 0$ we have

$$a_{\lfloor \mu n\rfloor} \sim_{n\to+\infty} \mu^{\frac{1}{\alpha+1}} a_n.$$

Consequently, Theorem 6.3.1 implies that for any $\theta > 0$, for all $n \geq 1$ large enough,

$$\begin{aligned}-1 &\leq \liminf_{n\to+\infty} \frac{1}{a_n} \log \varrho\left(\left\lfloor (\theta/C_*)^{\frac{\alpha+1}{\alpha}} n\right\rfloor, C_*\frac{a_n}{n}\right)\\ &\leq \limsup_{n\to+\infty} \frac{1}{a_n} \log \varrho\left(\left\lfloor (\theta/C_*)^{\frac{\alpha+1}{\alpha}} n\right\rfloor, C_*\frac{a_n}{n}\right) \leq -\frac{\theta^{\frac{1}{\alpha}}\Phi^{-1}(\theta)}{C_*^{\frac{1}{\alpha}}}.\end{aligned} \tag{6.3.5}$$

We observe that $\lim_{\theta\to+\infty} \theta^{\frac{1}{\alpha}} \Phi^{-1}(\theta) = C_*^{\frac{1}{\alpha}}$, therefore

$$\lim_{h\to+\infty} \liminf_{n\to+\infty} \frac{1}{a_n} \log \varrho\left(\lfloor hn \rfloor, C_* \frac{a_n}{n}\right) = \lim_{h\to+\infty} \limsup_{n\to+\infty} \frac{1}{a_n} \log \varrho\left(\lfloor hn \rfloor, C_* \frac{a_n}{n}\right) = -1. \tag{6.3.6}$$

We prove separately the upper and the lower bound. To prove Theorem 6.3.1, we prove separately an upper bound in Lemma 6.3.3 and the lower bound in Lemma 6.3.4. The upper bound is obtained by computing the mean number of individuals that stay above the line of slope $-\theta \frac{a_n}{n}$ during n units of time.

Lemma 6.3.3. *Under the assumptions of Theorem 6.1.1, for all $\theta > 0$ we have*

$$\limsup_{n\to+\infty} \frac{1}{a_n} \log \varrho\left(n, \theta \frac{a_n}{n}\right) \leq -\Phi^{-1}(\theta).$$

Proof. Let $\theta > 0$ and $\lambda > 0$, we set $g : t \mapsto -\theta t + \lambda(1-t)^{\frac{1}{\alpha+1}}$. For $j \leq n$, we introduce the intervals

$$I_j^{(n)} = \left[-\theta a_n j/n, a_n g(j/n)\right].$$

We observe that

$$\begin{aligned} \varrho\left(n, \theta \frac{a_n}{n}\right) &\leq \mathbf{P}\left(\exists |u| = n : \forall j \leq n, V(u_j) \geq -\theta a_n \frac{j}{n}\right) \\ &\leq \mathbf{P}\left(\exists |u| \leq n : V(u) \geq a_n g(|u|/n), V(u_j) \in I_j^{(n)}, j < |u|\right). \end{aligned}$$

Therefore, defining

$$Y_n = \sum_{|u|\leq n} \mathbf{1}_{\{V(u)\geq a_n g(|u|/n)\}} \mathbf{1}_{\left\{V(u_j)\in I_j^{(n)}, j<|u|\right\}},$$

by the Markov inequality we have $\varrho\left(n, \theta \frac{a_n}{n}\right) \leq \mathbf{E}(Y_n)$. Applying Lemma 6.2.2, we have

$$\begin{aligned} \mathbf{E}(Y_n) &= \sum_{k=1}^n \mathbf{E}\left[\sum_{|u|=k} \mathbf{1}_{\left\{V(u_j)\in I_j^{(n)}, j<k\right\}} \mathbf{1}_{\{V(u)\geq a_n g(k/n)\}}\right] \\ &= \sum_{k=1}^n \mathbf{E}\left[e^{-S_k} \mathbf{1}_{\left\{S_j \in I_j^{(n)}, j<k\right\}} \mathbf{1}_{\{S_k \geq a_n g(k/n)\}}\right] \\ &\leq \sum_{k=1}^n e^{-g(k/n)a_n} \mathbf{P}\left(S_j \in I_j^{(n)}, j < k\right). \end{aligned}$$

Let $A \in \mathbb{N}$, we set $m_a = \lfloor na/A \rfloor$ and $g_{a,A} = \sup_{s\in[\frac{a-1}{A}, \frac{a+2}{A}]} g(s)$, we have

$$\begin{aligned} \mathbf{E}(Y_n) &\leq \sum_{a=0}^{A-1} \sum_{k=m_a+1}^{m_{a+1}} e^{-g(k/n)a_n} \mathbf{P}\left(S_j \in I_j^{(n)}, j < k\right) \\ &\leq n \sum_{a=0}^{A-1} e^{-g_{a,A}a_n} \mathbf{P}\left(S_j \in I_j^{(n)}, j \leq m_a\right), \end{aligned}$$

by the Markov inequality applied at time m_a. Therefore, by Corollary 6.2.5, we have

$$\begin{aligned} \limsup_{n\to+\infty} \frac{1}{a_n} \log \mathbf{E}(Y_n) &\leq \max_{a\leq A-1} \left(-g_{a,A} - C_* \int_0^{\frac{a}{A}} \frac{ds}{(g(s)+\theta s)^\alpha}\right) \\ &\leq \max_{a\leq A-1} \left(-g_{a,A} - C_*(\alpha+1)\left[1 - (1-a/A)^{\frac{1}{\alpha+1}}\right]\right). \end{aligned}$$

Letting $A \to +\infty$, as g is uniformly continuous, we have

$$\begin{aligned}\limsup_{n\to+\infty} \frac{1}{a_n} \log \varrho\left(n, \theta\frac{a_n}{n}\right) &\leq \sup_{t\in[0,1]} \left\{\theta t - \lambda(1-t)^{\frac{1}{\alpha+1}} - C_*(\alpha+1)\left[1-(1-t)^{\frac{1}{\alpha+1}}\right]\right\}\\ &\leq -\lambda + \sup_{t\in[0,1]} \theta t - (\alpha+1)\Phi(\lambda)\left[1-(1-t)^{\frac{1}{\alpha+1}}\right].\end{aligned}$$

Note that $t \mapsto 1-(1-t)^{\frac{1}{\alpha+1}}$ is a convex function with slope $\frac{1}{\alpha+1}$ at $t=0$. Therefore, if we choose $\lambda = \Phi^{-1}(\theta)$, the function $t \mapsto \theta t - (\alpha+1)\Phi(\lambda)\left[1-(1-t)^{\frac{1}{\alpha+1}}\right]$ is concave and decreasing. As a consequence

$$\limsup_{n\to+\infty} \frac{1}{a_n} \log \varrho\left(n, \theta\frac{a_n}{n}\right) \leq -\lambda,$$

which concludes the proof. □

To obtain a lower bound, we bound from below the probability for an individual to stay between two given curves, while having not too many children. To do so, we compute the first two moments of the number of such individuals, and apply the Cauchy-Schwarz inequality to conclude.

Lemma 6.3.4. *Under the assumptions of Theorem 6.3.1, for all $\theta > 0$ we have*

$$\liminf_{n\to+\infty} \frac{1}{a_n} \log \varrho\left(n, \theta\frac{a_n}{n}\right) \geq -\frac{C_*^{\frac{1}{\alpha}}}{\theta^{\frac{1}{\alpha}}}.$$

Proof. For $u \in \mathbf{T}$, we introduce $\xi(u) = \log \sum_{v\in\Omega(u)} e^{V(v)-V(u)}$, where

$$\Omega(u) = \{v \in \mathbf{T} : \pi v = \pi u \quad \text{and} \quad v \neq u\}$$

is the set of siblings of u. Note that (6.1.9) implies

$$\lim_{x\to+\infty} \frac{x^\alpha}{L^*(x)} \widehat{\mathbf{P}}\left(\xi(w_1) \geq x\right) = 0. \tag{6.3.7}$$

Let $\theta > 0$, $\lambda > 0$ and $\delta > 0$. For $j \leq n$, we set $I_j^{(n)} = [-a_n\theta j/n, a_n(\lambda - \theta j/n)]$ and

$$X_n = \sum_{|u|=n} \mathbf{1}_{\left\{V(u_j)\in I_j^{(n)}, j\leq n\right\}} \mathbf{1}_{\{\xi(u_j)\leq \delta a_n, j\leq n\}}.$$

We observe that

$$\begin{aligned}\varrho\left(n, \theta\frac{a_n}{n}\right) &\geq \mathbf{P}\left(\exists |u| = n : V(u_j) \geq -a_n\theta j/n, j \leq n\right)\\ &\geq \mathbf{P}\left(\exists |u| = n : V(u_j) \in I_j^{(n)}, j \leq n\right)\\ &\geq \mathbf{P}\left(X_n \geq 1\right),\end{aligned}$$

thus by the Cauchy-Schwarz inequality, $\varrho\left(n, \theta\frac{a_n}{n}\right) \geq \frac{\mathbf{E}(X_n)^2}{\mathbf{E}(X_n^2)}$.

In a first time, we bound from below $\mathbf{E}(X_n)$. Using Proposition 6.2.1, we have

$$\begin{aligned}\mathbf{E}(X_n) &= \overline{\mathbf{E}}\left[\frac{1}{W_n}\sum_{|u|=n}\mathbf{1}_{\left\{V(u_j)\in I_j^{(n)},j\leq n\right\}}\mathbf{1}_{\{\xi(u_j)\leq\delta a_n,j\leq n\}}\right]\\
&= \widehat{\mathbf{E}}\left[\sum_{|u|=n}e^{-V(u)}\widehat{\mathbf{P}}(u=w_n|\mathcal{F}_n)\mathbf{1}_{\left\{V(u_j)\in I_j^{(n)},j\leq n\right\}}\mathbf{1}_{\{\xi(u_j)\leq\delta a_n,j\leq n\}}\right]\\
&= \widehat{\mathbf{E}}\left[e^{-V(w_n)}\mathbf{1}_{\left\{V(w_j)\in I_j^{(n)},j\leq n\right\}}\mathbf{1}_{\{\xi(w_j)\leq\delta a_n,j\leq n\}}\right].\end{aligned}$$

Let $\varepsilon > 0$, we have

$$\begin{aligned}\mathbf{E}(X_n) &\geq \widehat{\mathbf{E}}\left[e^{-V(w_n)}\mathbf{1}_{\{V(w_n)\leq(-\theta+\varepsilon)a_n\}}\mathbf{1}_{\left\{V(w_j)\in I_j^{(n)},j\leq n\right\}}\mathbf{1}_{\{\xi(w_j)\leq\delta a_n,j\leq n\}}\right]\\
&\geq e^{(\theta-\varepsilon)a_n}\widehat{\mathbf{P}}\left[V(w_n)\leq(-\theta+\varepsilon)a_n, V(w_j)\in I_j^{(n)}, \xi(w_j)\leq\delta a_n, j\leq n\right].\end{aligned}$$

We introduce $0 < x < y$ and $A > 0$ such that $\widehat{\mathbf{P}}(V(w_1)\in[x,y],\xi(w_1)\leq A) > 0$. Applying the Markov property at time $p = \lfloor \varepsilon a_n \rfloor$, for any $n \geq 1$ large enough we have

$$\begin{aligned}&\widehat{\mathbf{P}}\left[V(w_j)\in I_j^{(n)},\xi(w_j)\leq\delta a_n, j\leq n\right]\\
&\geq \widehat{\mathbf{P}}(V(w_1)\in[x,y],\xi(w_1)\leq A)^p \inf_{z\in[x\varepsilon a_n, y\varepsilon a_n]}\widehat{\mathbf{P}}_z\left[V(w_j)\in I_{j+p}^{(n)},\xi(w_j)\leq\delta a_n, j\leq n-p\right]\end{aligned}$$

As (6.3.7) holds, we apply Lemma 6.2.6, we have

$$\liminf_{n\to+\infty}\frac{1}{a_n}\log\mathbf{E}(X_n) \geq \theta-\varepsilon-\frac{C_*}{\lambda^\alpha}+\varepsilon\log\widehat{\mathbf{P}}(V(w_1)\in[x,y],\xi(w_1)\leq A).$$

Letting $\varepsilon \to 0$, we have

$$\liminf_{n\to+\infty}\frac{1}{a_n}\log\mathbf{E}(X_n)\geq\theta-\frac{C_*}{\lambda^\alpha}.$$

We now bound from above the second moment of X_n, using once again the spinal decomposition. Observe that

$$\begin{aligned}\mathbf{E}(X_n^2) &= \overline{\mathbf{E}}\left(\frac{X_n}{W_n}\sum_{|u|=n}\mathbf{1}_{\left\{V(u_j)\in I_j^{(n)},j\leq n\right\}}\mathbf{1}_{\{\xi(u_j)\leq\delta a_n,j\leq n\}}\right)\\
&= \overline{\mathbf{E}}\left(X_n\sum_{|u|=n}e^{-V(u)}\overline{\mathbf{P}}(w_n=u|\mathcal{F}_n)\mathbf{1}_{\left\{V(u_j)\in I_j^{(n)},j\leq n\right\}}\mathbf{1}_{\{\xi(u_j)\leq\delta a_n,j\leq n\}}\right)\\
&= \widehat{\mathbf{E}}\left(e^{-V(w_n)}X_n\mathbf{1}_{\left\{V(w_j)\in I_j^{(n)},j\leq n\right\}}\mathbf{1}_{\{\xi(w_j)\leq\delta a_n,j\leq n\}}\right)\\
&\leq e^{\theta a_n}\widehat{\mathbf{E}}\left[X_n\mathbf{1}_{\left\{V(w_j)\in I_j^{(n)},j\leq n\right\}}\mathbf{1}_{\{\xi(w_j)\leq\delta a_n,j\leq n\}}\right].\end{aligned}$$

We decompose the set of individuals counted in X_n under law $\widehat{\mathbf{P}}$ according to their most recent common ancestor with the spine w, we have

$$X_n = \mathbf{1}_{\left\{V(w_j)\in I_j^{(n)},j\leq n\right\}}\mathbf{1}_{\{\xi(w_j)\leq\delta a_n,j\leq n\}} + \sum_{j=1}^n\sum_{u\in\Omega(w_j)}\Lambda(u),$$

where $u' \geq u$ means u' is a descendant of u and

$$\Lambda(u) = \sum_{|u'|=n, u' \geq u} \mathbf{1}_{\left\{V(u'_j) \in I_j^{(n)}, j \leq n\right\}} \mathbf{1}_{\left\{\xi(u'_j) \leq \delta a_n, j \leq n\right\}}.$$

Let $k \leq n$ and $u \in \Omega(w_k)$. Conditionally on $\mathcal{G}$, the subtree rooted at u with marks V is a branching random walk with law $\mathbf{P}_{V(u)}$, therefore

$$\begin{aligned}
\mathbf{E}\left(\Lambda(u) \middle| \mathcal{G}\right) &\leq \mathbf{E}_V(u) \left(\sum_{|u'|=n-k} \mathbf{1}_{\left\{V(u'_j) \in I_{k+j}^{(n)}, j \leq n-k\right\}} \right) \\
&\leq e^{V(u)} \mathbf{E}_{V(u)} \left(e^{-S_{n-k}} \mathbf{1}_{\left\{S_j \in I_{k+j}^{(n)}, j \leq n-k\right\}} \right] \\
&\leq e^{V(u)} e^{-g^{\lambda,\delta}(1) a_n} \sup_{z \in \mathbb{R}} \mathbf{P}_z \left(S_j \in I_{k+j}^{(n)}, j \leq n-k \right).
\end{aligned}$$

Let $A \in \mathbb{N}$, we set $m_a = \lfloor na/A \rfloor$ and

$$\Psi_{a,A} = \sup_{z \in \mathbb{R}} \mathbf{P}_z \left(S_j \in I_{m_a+j}^{(n)}, j \leq n - m_a \right).$$

For all $k \leq m_{a+1}$ and $u \in \Omega(w_k)$, we have

$$\mathbf{E}\left(\Lambda(u) \middle| \mathcal{G}\right) \leq e^{V(u)} e^{\theta a_n} \Psi_{a+1,A}.$$

Therefore

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{1}_{\left\{V(w_j) \in I_j^{(n)}, j \leq n\right\}} \mathbf{1}_{\left\{\xi(w_j) \leq \delta a_n, j \leq n\right\}} \sum_{k=m_a+1}^{m_{a+1}} \sum_{u \in \Omega(w_k)} \Lambda(u) \right] \\
\leq &\sum_{k=m_a+1}^{m_{a+1}} \mathbf{E}\left[\mathbf{1}_{\left\{V(w_j) \in I_j^{(n)}, j \leq n\right\}} \sum_{u \in \Omega(w_k)} \mathbf{1}_{\left\{\xi(w_k) \leq \delta a_n\right\}} \Lambda(u) \right] \\
\leq &\Psi_{a+1,A} e^{\theta a_n} \sum_{k=m_a+1}^{m_{a+1}} \mathbf{E}\left[\mathbf{1}_{\left\{V(w_j) \in I_j^{(n)}, j \leq n\right\}} e^{\xi(w_k)+V(w_k)} \mathbf{1}_{\left\{\xi(w_k) \leq \delta a_n\right\}} \right] \\
\leq &n \Psi_{a+1,A} \Psi_{0,A} e^{(\lambda + (1-a/A)\theta + \delta) a_n}.
\end{aligned}$$

Consequently, applying Corollary 6.2.5, we have

$$\begin{aligned}
\limsup_{n \to +\infty} \frac{1}{a_n} \log \mathbf{E}(X_n^2) &\leq \max_{a \leq A} \lambda + (2 - a/A)(\theta - \frac{C_*}{\lambda^\alpha}) + \delta \\
&\leq \lambda + 2\theta - 2\frac{C_*}{\lambda^\alpha} + \delta,
\end{aligned}$$

as soon as $\theta \geq \frac{C_*}{\lambda^\alpha}$.

Using the first and second moment estimates of X_n, we have

$$\liminf_{n \to +\infty} \frac{1}{a_n} \log \varrho \left(n, \theta \frac{a_n}{n} \right) \geq -\lambda - \delta.$$

Letting $\delta \to 0$ and $\lambda \to (\theta/C_*)^{\frac{1}{\alpha}}$ concludes the proof. $\square$

Remark 6.3.5. If we assume (f^θ, g^θ) to be a pair of functions solution of the differential equation

$$\begin{cases} f(t) = -\theta t \\ g(t) = -\theta + C_* \int_t^1 \frac{ds}{(g(s)-f(s))^\alpha}, \end{cases}$$

using similar estimates as the ones developed in Lemmas 6.3.3 and 6.3.4, we prove that for all $\theta \in \mathbb{R}$

$$\lim_{n\to+\infty} \frac{1}{a_n} \log \varrho\left(n, \theta\frac{a_n}{n}\right) = -g^\theta(0).$$

Theorem 6.3.1 is used to obtain closed bounds for $g^\theta(0)$, that are tight for large θ.

6.4 Speed of the N-branching random walk

In [BG10], to prove that $\lim_{n\to+\infty}(\log N)^2 v_N = C$ for a branching random walk in the usual boundary case, the essential tool was a version of Theorem 6.3.1, found in [GHS11]. The same methods are applied to compute the asymptotic behaviour of v_N under the assumptions of Theorem 6.1.1. Loosely speaking, we compare the N-branching random walk with N independent branching random walks in which individuals crossing a linear boundary with slope $-\nu_N$ defined by

$$\nu_N := C_* \frac{L^*(\log N)}{(\log N)^\alpha}. \tag{6.4.1}$$

By (6.3.6), for any $h > 0$ large enough and $N \geq 1$ large enough, $\varrho\left(h\frac{(\log N)^{\alpha+1}}{L^*(\log N)}, \nu_N\right) \approx \frac{1}{N}$. Consequently, $\frac{(\log N)^{\alpha+1}}{L^*(\log N)}$ is expected to be the correct time scale for the study of the process.

In this section, we give in a first time a more precise definition of the branching-selection particle system we consider. We introduce additional notations that enables to describe it as a measure-valued Markov process. In Section 6.4.2, we introduce an increasing coupling between branching-selection particles systems, and use it to prove the existence of v_N. Finally, we obtain in Section 6.4.3 an upper bound for v_N and in Section 6.4.4 a lower bound, that are enough to conclude the proof of Theorem 6.1.1.

6.4.1 Definition of the N-branching random walk and notation

The branching-selection models we consider are particle systems on $\mathbb{R}$. It is often convenient to represent the state of a particle system by a counting measure on $\mathbb{R}$ with finite integer-valued mass on every interval of the form $[x, +\infty)$. The set of such measures is written $\mathcal{M}$. A Dirac mass at position $x \in \mathbb{R}$ indicates the presence of an individual alive at position x. With this interpretation, a measure in $\mathcal{M}$ represents a population with a rightmost individual, and no accumulation point. For $N \in \mathbb{N}$, we write $\mathcal{M}_N$ for the set of measures in $\mathcal{M}$ with total mass N, that represent populations of N individuals. If $\mu \in \mathcal{M}_N$, then there exists $(x_1, \ldots, x_n) \in \mathbb{R}^N$ such that $\mu = \sum_{j=1}^n \delta_{x_j}$.

We introduce a partial order on $\mathcal{M}$: given $\mu, \nu \in \mathcal{M}$, we write $\mu \preccurlyeq \nu$ if for all $x \in \mathbb{R}$, $\mu([x, +\infty)) \leq \nu([x, +\infty))$. Note that if $\mu \preccurlyeq \nu$ then $\mu(\mathbb{R}) \leq \nu(\mathbb{R})$. A similar partial order can be defined on the set of laws point processes. We say that $\mathcal{L} \preccurlyeq \widetilde{\mathcal{L}}$ if there exists a coupling $(L, \widetilde{L})$ of these two laws, such that L has law $\mathcal{L}$, $\widetilde{L}$ has law $\widetilde{\mathcal{L}}$ and

$$\sum_{\ell \in L} \delta_\ell \preccurlyeq \sum_{\widetilde{\ell} \in \widetilde{\mathcal{L}}} \delta_{\widetilde{\ell}} \quad \text{a.s.}$$

Let $N \in \mathbb{N}$. We introduce a Markov chain $(X_n^N, n \geq 1)$ on $\mathcal{M}_N$ that we call the *branching random walk with selection of the N rightmost individuals*, or N-BRW for short. For any $n \in \mathbb{N}$, we denote by $(x_n^N(1), \dots x_n^N(N)) \in \mathbb{R}^{\mathbb{N}}$ the random vector that verifies

$$X_n^N = \sum_{j=1}^{N} \delta_{x_n^N(j)} \quad \text{and} \quad x_n^N(1) \geq x_n^N(2) \geq \cdots \geq x_n^N(N).$$

Conditionally on X_n^N, X_n^{N+1} is constructed as follows. We introduce N i.i.d. point processes $(L_n^1, \dots, L_n^N)$ with law $\mathcal{L}$, and we set

$$Y_{n+1}^N = \sum_{j=1}^{N} \sum_{\ell^i \in L_n^i} \delta_{x_n^N(i)+\ell^i} \in \mathcal{M},$$

which is the population after the branching step. We set $y = \sup\{x \in \mathbb{R} : Y_{n+1}^N([x, +\infty)) \geq N\}$ and $P = Y_{n+1}^N((y, +\infty))$. Then $X_{n+1}^N = Y_{n+1|(y,+\infty)}^N + (N - P)\delta_y$. For $n \in \mathbb{N}$, we set $\mathcal{F}_n = \sigma(X_j^N, j \leq n)$ the natural filtration associated to the N-BRW. Whereas this is not done in this chapter, note that genealogical informations can be added to this process –as long as any ambiguity that might appears, when deciding which of the individuals alive at position y are killed, is settled in a $\mathcal{F}$-adapted manner.

6.4.2 Increasing coupling of branching-selection models

We construct here a coupling between N-BRWs, that preserves the order $\preccurlyeq$. This coupling has been introduced in [BG10], in a special case and is a key tool in the study of the branching-selection processes we consider. It is used to bound from above and from below the behaviour of the N-BRW by a branching random walk in which individuals that cross a line of slope $-\nu_N$ are killed. In a first time, we couple a single step of the N-BRW.

Lemma 6.4.1. *Let $1 \leq m \leq n$ and $\mu \in \mathcal{M}_m, \widetilde{\mu} \in \mathcal{M}_n$ be such that $\mu \preccurlyeq \widetilde{\mu}$. Let $\mathcal{L} \preccurlyeq \widetilde{\mathcal{L}}$ be two laws of point processes. For any $1 \leq M \leq N$, there exists a coupling of X_1^M the first step of a M-BRW with reproduction law $\mathcal{L}$ starting from μ with $\widetilde{X}_1^N$ the first step of a N-BRW with reproduction law $\widetilde{\mathcal{L}}$ starting from $\widetilde{\mu}$, in a way that $X_1^M \preccurlyeq \widetilde{X}_1^N$ a.s.*

Proof. Let $(L, \widetilde{L})$ be a pair of point processes such that $\sum_{\ell \in L} \delta_\ell \preccurlyeq \sum_{\ell \in \widetilde{L}} \delta_\ell$ a.s., L has law $\mathcal{L}$ and $\widetilde{L}$ has law $\widetilde{\mathcal{L}}$. We set $((L_j, \widetilde{L}_j, j \geq 0)$ i.i.d. random variables with the same law as $(L, \widetilde{L})$. We write $\mu = \sum_{i=1}^m \delta_{x_i}$ and $\widetilde{\mu} = \sum_{i=1}^n \delta_{y_i}$ in a way that $(x_j, j \leq m)$ and $(y_j, j \leq n)$ are ranked in the decreasing order. We set

$$\mu^1 = \sum_{i=1}^{m} \sum_{\ell^i \in L_i} \delta_{x_i+\ell_i} \quad \text{and} \quad \widetilde{\mu}^1 = \sum_{i=1}^{n} \sum_{\ell_i \in \widetilde{L}_i} \delta_{y_i+\ell_i}.$$

Note that $\mu^1 \preccurlyeq \nu^1$ a.s.

Moreover, setting X_1^M for the M individuals in the highest position in μ^1 and $\widetilde{X}_1^N$ the N individuals in the highest position in $\widetilde{\mu}^1$. Once again, we have $X_1^M \preccurlyeq \widetilde{X}_1^N$ a.s. □

A direct consequence of this lemma is the existence of an increasing coupling between N-BRWs.

Corollary 6.4.2. *Let $\mathcal{L} \preccurlyeq \widetilde{\mathcal{L}}$ be two laws of point processes. For all $1 \leq M \leq N \leq +\infty$, if $X_0^M \preccurlyeq \widetilde{X}_0^N$, then there exists a coupling between the M-BRW (X_n^M) with law $\mathcal{L}$ and the N-BRW $(\widetilde{X}_n^N)$ with law $\widetilde{\mathcal{L}}$ verifying*

$$\forall n \in \mathbb{N}, X_n^M \preccurlyeq \widetilde{X}_n^N \quad \text{a.s.}$$

Using this increasing coupling, we prove that with high probability, the cloud of particles in the N-BRW does not spread.

Lemma 6.4.3. *Under the assumptions* (6.1.3), (6.1.4) *and* (6.1.10) *there exist* $C > 0$ *and* $\alpha > 0$ *such that for all* $N \geq 2$, $y \geq 1$ *and* $n \geq C(\log N + y)$,

$$\mathbf{P}\left(x_n^N(1) - x_n^N(N) \geq y\right) \leq C\left(\frac{N \log N}{y}\right)^2$$

Proof. Let $n \in \mathbb{N}$ and $k \leq n$, we bound $x_n^N(1) - x_{n-k}^N(1)$ from above and $x_n^N(N) - x_{n-k}^N(1)$ from below to estimate the size of the cloud of particles at time n. In a second time, choosing k appropriately we conclude the proof of Lemma 6.4.3.

We first observe that the N-BRW starting from position X_{n-k}^N can be coupled with N i.i.d. branching random walks $((\mathbf{T}^j, V^j), j \leq N)$ with $(\mathbf{T}^j, V^j)$ starting from position $x_{n-k}^N(j)$, in a way that

$$X_n^N \preccurlyeq \sum_{j=1}^{N} \sum_{u \in \mathbf{T}^j, |u|=k} \delta_{V^j(u)}.$$

As a consequence, by (6.2.4), for any $y \in \mathbb{R}$,

$$\mathbf{P}\left(x_n^N(1) - x_{n-k}^N(1) \geq y\right) \leq \mathbf{P}\left(\max_{j \leq N} \max_{u \in \mathbf{T}^j, |u|=k} V^j(u) \geq y\right) \leq N e^{-y}. \tag{6.4.2}$$

We now bound from below the displacements in the N-BRW. Let L be a point process with law $\mathcal{L}$. By (6.1.3), there exists $R > 0$ such that $\mathbf{E}\left(\sum_{\ell \in L} \mathbf{1}_{\{\ell \geq -R\}}\right) > 1$. We denote by L_R the point process that consists in the maximal point in L as well as any other point that is greater than $-R$. Using Corollary 6.4.2, we couple $(X_{k+n}^N, n \geq 0)$ with the N-BRW $(X_n^{N,R}, n \geq 0)$ of reproduction law $\mathcal{L}_R$, starting from a unique individual located at $x_{n-k}^N(1)$ at time 0 in an increasing fashion.

As $X_n^{N,R} \preccurlyeq X_n^N$, if $X_n^{N,R}(\mathbb{R}) = N$, then $x_n^{N,R}(N) \leq x_n^N(N)$. Moreover by definition of L_R, the minimal displacement made by one child with respect to its parent is given by $\min(-R, \max L)$. For $n \in \mathbb{N}$, we write $Q_{n,N}$ a random variable defined as the sum of n i.i.d. copies of $\min(-R, \max L)$. Observe that Q_{kN} is stochastically dominated by $x_k^{N,R}(N)$. Consequently

$$\mathbf{P}\left(x_n^N(N) - x_{n-k}^N(1) \leq -y\right) \leq \mathbf{P}\left(X_k^{N,R}(\mathbb{R}) < N\right) + \mathbf{P}\left(Q_{kN} \leq -y\right).$$

By (6.1.10), we have $\mathbf{P}(Q_{kN} \leq -y) \leq C\frac{k^2N^2}{y^2}$. Moreover, the process $(X_n^{N,R}(\mathbb{R}), n \geq 0)$ is a Galton-Watson process with reproduction law given by $\#L_R$, that saturates at time N. Consequently, using [FW07] results, setting $m_R = \mathbf{E}(\#L_R)$ and $\alpha = -\frac{\log \mathbf{P}(\#L_R=1)}{\log m_R}$ we have

$$\mathbf{P}\left(X_k^{N,R}(\mathbb{R}) < N\right) \leq C\frac{N^\alpha}{m_R^{k\alpha}}.$$

We conclude that

$$\mathbf{P}\left(x_n^N(N) - x_{n-k}^N(1) \leq y\right) \leq C\frac{k^2N^2}{y^2} + C\frac{N^\alpha}{m_R^{k\alpha}} \tag{6.4.3}$$

Combining (6.4.2) and (6.4.3), for all $y \geq 1$ and $k \in \mathbb{N}$ we have

$$\mathbf{P}\left(x_n^N(1) - x_n^N(N) \geq 2y\right) \leq N e^{-y} + C\frac{k^2N^2}{y^2} + C\frac{N^\alpha}{m_R^{k\alpha}}.$$

Thus, setting $k = \left\lfloor \frac{2\log N + y}{\log m_R} \right\rfloor$, there exists $C > 0$ such that for any $y \geq 1$ and $N \geq 1$ large enough,

$$\mathbf{P}\left(x_n^N(1) - x_n^N(N) \geq 2y\right) \leq Ne^{-y} + C\frac{(\log N)^2 N^2}{y^2} + CN^{-\alpha}e^{-\alpha y \log m_R} \leq C\left(\frac{N\log N}{y}\right)^2.$$

□

Applying Lemma 6.4.3 and the Borel-Cantelli lemma, for all $N \geq 2$ we have

$$\lim_{n\to+\infty} \frac{x_n^N(1) - x_n^N(N)}{n} = 0 \quad \text{a.s. and in } L^1,$$

thus, if we prove that $\frac{x_n^N(1)}{n}$ converges, the same hold for every individual in the cloud. This is done in the next proof, using the Kingman's subadditive ergodic theorem.

Lemma 6.4.4. *Under the assumptions* (6.1.3), (6.1.4) *and* (6.1.10), *for all $N \geq 1$, there exists $v_N \in \mathbb{R}$ such that for all $j \leq N$*

$$\lim_{n\to+\infty} \frac{x_n^N(j)}{n} = v_N \quad \textit{a.s. and in } L^1. \tag{6.4.4}$$

Moreover, if $X_0^N = N\delta_0$, we have

$$v_N = \inf_{n\geq 1} \frac{\mathbf{E}(x_n^N(1))}{n} = \sup_{n\geq 1} \frac{\mathbf{E}(x_n^N(n))}{n}. \tag{6.4.5}$$

Proof. This proof is based on the Kingman's subadditive ergodic theorem. We first prove that if $X_0^N = N\delta_0$, then $(x_n^N(1))$ is a subadditive sequence, and $(x_n^N(N))$ is an overadditive one. Thus $\frac{x_n^N(1)}{n}$ and $\frac{x_n^N(N)}{n}$ converge, and $\lim \frac{x_n^N(1)}{n} = \lim \frac{x_n^N(N)}{n}$ a.s. by Lemma 6.4.3. We treat in a second time the case of a generic starting value $X_0^N \in \mathcal{M}_N$ using Corollary 6.4.2.

Let $N \in \mathbb{N}$, let $(L_{j,n}, j \leq N, n \geq 0)$ be an array of i.i.d. point processes with common law $\mathcal{L}$. We define on the same probability space random measures $(X_{m,n}^N, 0 \leq k \leq n)$ such that for all $m \geq 0$, $(X_{m,m+n}^N, n \geq 0)$ is a N-BRW starting from the initial distribution $N\delta_0$. For ny $m \geq 0$, we set $X_{m,m}^N = N\delta_0$. Let $0 \leq m \leq n$, we assume that $X_{m,n}^N = \sum_{j=1}^N \delta_{x_{m,n}^N(j)}$, with $(x_{m,n}^N(j))$ listed in the decreasing order, is given. We define $(x_{m,n+1}^N(j), j \geq 0)$, again listed in the decreasing order, in a way that

$$\sum_{j=1}^{+\infty} \delta_{x_{m,n+1}^N(j)} = \sum_{j=1}^{N} \sum_{\ell_{j,n}\in L_{j,n}} \delta_{x_{m,n}^N(j)+\ell_{j,n}},$$

and set $X_{m,n+1}^N = \sum_{j=1}^N \delta_{x_{m,n+1}^N}$.

For $x \in \mathbb{R}$, we write φ_x for the shift operator on $\mathcal{M}$, such that $\varphi_x(\mu) = \mu(. - x)$. With this definition, we observe that for all $0 \leq m \leq n$ we have

$$\varphi_{x_{0,n}^N(N)} X_{n,n+m}^N \preccurlyeq X_{0,n+m}^N \preccurlyeq \varphi_{x_{0,n}^N(1)}(X_{n,n+m}^N).$$

As a consequence,

$$x_{0,m+n}^N(1) \leq x_{0,m}^N(1) + x_{m,m+n}^N(1) \quad \text{and} \quad x_{0,m+n}^N(N) \geq x_{0,m}^N(N) + x_{m,m+n}^N(N). \tag{6.4.6}$$

We apply Kingman's subadditive ergodic theorem. Indeed, $(x^N_{m,m+n}(1), n \geq 0)$ is independent of $(x^N_{k,l}(1), 0 \leq k \leq l \leq m)$ and has the same law as $(x^N_{0,n}(1), n \geq 0)$. Moreover, $\mathbf{E}(|x^N_{0,1}(1)|) < +\infty$ by (6.1.10). As a consequence, (6.4.6) implies there exists $v_N \in \mathbb{R}$ such that

$$\lim_{n\to+\infty} \frac{x^N_{0,n}(1)}{n} = v_N \quad \text{a.s. and in } L^1,$$

and $v_N = \inf_{n\in\mathbb{N}} \frac{\mathbf{E}(x^N_{0,n}(1))}{n}$. Similarly, $\lim_{n\to+\infty} \frac{x^N_{0,n}(N)}{n} = \sup_{n\in\mathbb{N}} \frac{\mathbf{E}(x^N_{0,n}(N))}{n}$ a.s. and in L^1, and (6.3.5) is verified. Moreover, by Lemma 6.4.3, these limits are equal.

We now consider the general case. Let $(X^N_n, n \geq 0)$ be a N-BRW. We couple this process with Y^N and Z^N two N-BRWs stating from $N\delta_{x^N(1)}$ and $N\delta_{x^N(N)}$ respectively, such that for all $n \in \mathbb{N}$, $Z^N_n \preccurlyeq X^N_n \preccurlyeq Y^N_n$. We have

$$\forall j \leq N, z^N_n(N) \leq z^N_n(j) \leq x^N_n(1) \leq y^N_n(j) \leq y^N_n(1).$$

We conclude that for all $j \leq N$, $\lim_{n\to+\infty} \frac{x^N_n(j)}{n} = v_N$ a.s. and in L^1. □

We now study the asymptotic behaviour of v_N as $N \to +\infty$. To do so, we couple the N-BRW with a branching random walk in which individuals are killed below the line of slope $-\nu_N$. Applying Theorem 6.3.1, we derive upper and lower bounds for v_N.

6.4.3 An upper bound on the maximal displacement

To obtain an upper bound on the maximal displacement in the N-branching random walk, we link the existence of an individual alive at time n with the event there exists an individual staying above a line of slope $-\nu_N$, during m_N units of time. The following lemma is an easier and less precise version of [BG10, Lemma 2], that is sufficient for our proofs.

Lemma 6.4.5. *Let $v < K$. We set $(x_n, n \geq 0)$ a sequence of real numbers with $x_0 = 0$ such that $\sup_{n\in\mathbb{N}} x_{i+1} - x_i \leq K$. For all $m \leq n$, if $x_n > (n-m)v + Km$, then there exists $i \leq n-m$ such that for all $j \leq m$, $x_{i+j} - x_i \geq vj$.*

Proof. Let (x_n) be a sequence verifying $\sup_{i\in\mathbb{N}} x_{i+1} - x_i \leq K$. We assume that for any $i \leq n-m$, there exists $j_i \leq m$ verifying $x_{i+j_i} - x_i \leq vj_i$. We set $\sigma_0 = 0$ and $\sigma_{k+1} = j_{\sigma_k}$. By definition, we have

$$x_{\sigma_{k+1}} \leq (\sigma_{k+1} - \sigma_k)v + x_{\sigma_k},$$

thus, for all $k \geq 0$, $x_{\sigma_k} \leq \sigma_k v$. Moreover, as (σ_k) is strictly increasing, with steps smaller than m, there exists k_0 such that $\sigma_{k_0} \in [n-m, n]$. We conclude that

$$x_n \leq x_n - x_{\sigma_{k_0}} + x_{\sigma_{k_0}} \leq K(n - \sigma_{k_0}) + v\sigma_{k_0} \leq Km + (n-m)v.$$

□

The previous lemma can be used to extend the estimate obtained thanks to Theorem 6.3.1 from times of order $(\log N)^{\alpha+1}$ to times of order N^ε.

Lemma 6.4.6. *Under the assumptions of Theorem 6.1.1, let X^N be a N-BRW with reproduction law $\mathcal{L}$ starting from $N\delta_0$. For any $\varepsilon \in (0,1)$, there exists $\delta > 0$ such that for any $N \geq 1$ large enough, we have*

$$\mathbf{P}\left(\frac{x^N_{\lfloor N^\delta \rfloor}(1)}{N^\delta} \geq -(1-\varepsilon)\nu_N \right) \leq N^{-\delta}.$$

Proof. Let $\varepsilon \in (0,1)$ and $\theta > 0$. By (6.3.5), for all $n \geq 1$ large enough we have

$$\limsup_{n\to+\infty} \frac{1}{a_n} \log \varrho\left(\left\lfloor \left(\frac{\theta}{(1-\varepsilon)C_*} \right)^{\frac{\alpha+1}{\alpha}} n \right\rfloor, C_*(1-\varepsilon)\frac{a_n}{n} \right) \leq -\frac{\theta^{\frac{1}{\alpha}}\Phi^{-1}(\theta)}{C_*^{\frac{1}{\alpha}}(1-\varepsilon)^{\frac{1}{\alpha}}}.$$

We set

$$m_N = \left\lfloor \left(\frac{\theta}{(1-\varepsilon)C_*} \right)^{\frac{\alpha+1}{\alpha}} \frac{(\log N)^{\alpha+1}}{L^*(\log N)} \right\rfloor,$$

For any $\varepsilon > 0$ small enough, there exists $\delta \in (0, \varepsilon^{1/2})$ such that for all $\theta > 0$ large enough we have we have $\varrho(m_N, (1-\varepsilon)\nu_N) \leq N^{-(1+2\delta)}$ for all $N \geq 1$ large enough.

We set $n = \left\lfloor N^\delta \right\rfloor$. Observe the N-BRW of length n is built with nN independent point processes of law $\mathcal{L}$ satisfying (6.1.4). If L is a point process with law $\mathcal{L}$, we have

$$\mathbf{P}(\max L \geq x) \leq \mathbf{P}\left(\sum_{\ell \in L} e^\ell \geq e^x \right) \leq e^{-x}.$$

Therefore, setting $K = (1+2\delta)\log N$, the probability there exists one individual in the N-BRW alive before time n that made a step larger than K is bounded from above by $1 - (1 - N^{-(1+2\delta)})^{nN} \leq N^{-\delta}$.

We now consider the path of length n that links an individual alive at time n at position $x_n^N(1)$ with is ancestor alive at time 0. We write $y_n^N(k)$ for the position of the ancestor at time k of this individual. With probability $1-N^{-\delta}$, this is a path with no step greater than K. As for $N \geq 1$ large enough, we have $-(1-\varepsilon-\delta)\nu_N n > -(n-m_N)(1-\varepsilon)\nu_N + K m_N$. By Lemma 6.4.5, for any $N \geq 1$ large enough we have

$$\begin{aligned}\left\{ \forall k < n, y_n^N(k+1) - y_n^N(k) \leq K \right\} \cup \left\{ x_n^N(1) \geq -(1-\delta-\varepsilon)\nu_N n \right\} \\ \subset \left\{ \exists j \leq n - m_N : \forall k \leq m_N, y_n^N(j+k) - y_n^N(j) \geq -(1-\varepsilon)\nu_N k \right\}.\end{aligned}$$

Consequently if $x_n^N(1) \geq -(1-\delta-\varepsilon)\nu_N n$, there exists an individual in the N-BRW that has a sequence of descendants of length m_N staying above the line of slope $-(1-\delta)\nu_N$. This happens with probability at most $nN\varrho(m_N, (1-\varepsilon)\nu_N)$. We conclude from these observations that for any $\varepsilon > 0$ and $N \geq 1$ large enough

$$\mathbf{P}\left(x_n^N(1) \geq -\nu_N(1-\delta-\varepsilon)n \right) \leq CN^{-\delta}.$$

□

Proof of the upper bound of Theorem 6.1.1. We first observe that the maximal displacement at time n in the N-BRW is bounded from above by the maximum of N independent branching random walks. By (6.2.4), for all $y \geq 0$ and $n \in \mathbb{N}$ we have $\mathbf{P}(x_n^N(1) \geq y) \leq Ne^{-y}$.

Moreover, as $(x_n^N(1))$ is a subadditive sequence, for all $p \geq 1$ we have

$$\limsup_{n\to+\infty} \frac{x_n^N(1)}{n} \leq \mathbf{E}\left[\frac{x_p^N(1)}{p} \right] \quad \text{a.s.}$$

For any $\varepsilon > 0$ and $y > 0$, there exists $\delta > 0$ such that setting $p = \left\lfloor N^\delta \right\rfloor$ we have

$$v_N \leq \mathbf{E}\left[\frac{x_p^N(1)}{p}\mathbf{1}_{\left\{x_p^N(1)\geq py\right\}}\right] + \mathbf{E}\left[\frac{x_p^N(1)}{p}\mathbf{1}_{\left\{\frac{x_p^N(1)}{p}\in[-\nu_N(1-\varepsilon),y]\right\}}\right] + \mathbf{E}\left[\frac{x_p^N(1)}{p}\mathbf{1}_{\left\{x_p^N(1)\leq -p(1-\varepsilon)\nu_N\right\}}\right],$$

therefore

$$\begin{aligned} v_N \leq & \int_y^{+\infty} \mathbf{P}\left(x_p^N(1) \geq pz\right) dz + y\mathbf{P}\left(x_p^N(1) \geq -p(1-\varepsilon)\nu_N\right) - (1-\varepsilon)\nu_N \\ \leq & \frac{N}{p}e^{-py} + yN^{-\delta} - (1-\varepsilon)\nu_N. \end{aligned}$$

Letting $N \to +\infty$ then $\varepsilon \to 0$, we conclude that

$$\limsup_{N\to+\infty} \frac{v_N(\log N)^\alpha}{L^*(\log N)} \leq -C^*.$$

□

6.4.4 The lower bound

To bound from below the position of the leftmost individual in the N-BRW, we prove that with high probability, there exists a time $k \leq n_N$ such that $x_k^N(N) \geq -k\nu_N$. We use these events as renewal times for a particle process that stays below the N-BRW.

Lemma 6.4.7. *For any $\lambda > 0$ and any $\varepsilon > 0$ small enough, there exists $\delta > 0$ such that for all $N \geq 1$ large enough,*

$$\mathbf{P}\left(\forall n \leq \lambda\frac{(\log N)^{\alpha+1}}{L^*(\log N)}, x_n^N(N) \leq -n(1+\varepsilon)\nu_N\right) \leq \exp\left(-N^\delta\right).$$

Proof. For $N \in \mathbb{N}$ and $\lambda > 0$, we set $m_N = \left\lfloor \lambda\frac{(\log N)^{\alpha+1}}{L^*(\log N)}\right\rfloor$. Let $\varepsilon > 0$, by (6.3.5), we have

$$\liminf_{N\to+\infty} \frac{1}{\log N} \log \varrho\left(m_N, (1+\varepsilon)\nu_N\right) \geq -(1+\varepsilon)^{-\frac{1}{\alpha}}.$$

Consequently for any $\varepsilon > 0$ small enough, there exists $\delta \in (0, \varepsilon^{1/2})$ such that for all $N \geq 1$ large enough we have $\varrho(m_N, (1+\delta)\nu_N) \geq \frac{1}{N^{1-\delta}}$.

Let L be a point process with law $\mathcal{L}$. Using (6.1.3), there exists $R > 0$ large enough such that $\mathbf{E}\left(\#\left\{\ell \in L : \ell \geq -R\right\}\right) > 1$. We consider the branching random walk in which individuals that cross the line of slope $-R$ are killed. By standard Galton-Watson processes theory [2], there exists $r > 0$ and $\alpha > 0$ such that for all $N \geq 1$ large enough the probability there exists at least N individuals alive at time $\lfloor \alpha \log N \rfloor$ in this process is bounded from below by r. Thus for all $N \geq 1$ large enough, the probability there exists at least $N+1$ individuals alive at time $m_N + \lfloor \alpha \log N \rfloor$ in a branching random walk in which individuals that cross the line of slope $-\nu_N(1+2\varepsilon)$ are killed is bounded from below by $r\varrho(m_N, (1+\varepsilon)\nu_N)$.

2. see, e.g. [FW07].

We set $\mathcal{B}_N = \left\{\forall n \leq m_N + \lfloor \alpha \log N \rfloor, x_n^N \leq -n\nu_N(1+2\varepsilon)\right\}$. By Corollary 6.4.2, the N-BRW can be coupled with N independent branching random walks starting from 0, in which individuals below the line of slope $-\nu_N(1+2\varepsilon)$ are killed, in a way that on $\mathcal{B}_N$, X^N is above the branching random walks for the order $\preccurlyeq$. The probability that at least one of the branching random walks has at least $N+1$ individuals at time $m_N + \lfloor \alpha \log N \rfloor$ is bounded from below by

$$1 - (1 - r\varrho(m_N, (1+\delta)\nu_N))^N \geq 1 - \exp(-N^{\delta/2}),$$

for any $N \geq 1$ large enough. On this event, the coupling is impossible as X^N has no more that N individuals alive at time N, thus $\mathcal{B}_N$ is not satisfied. We conclude that $\mathbf{P}(\mathcal{B}_N) \leq e^{-N^{\delta/2}}$. □

Lower bound of Theorem 6.1.1. The proof is based on a coupling of the N-BRW X^N with another particle system Y^N, in a way that for any $n \in \mathbb{N}$, $Y^N \preccurlyeq X^N$. Let $(L_{j,n}, j \leq N, n \geq 0)$ be an array of i.i.d. point processes with law $\mathcal{L}$. We construct X^N such that $L_{j,n}$ represents the set of children of the individual $x_n^N(j)$, with $X_0^N = N\delta_0$. By Lemma 6.4.7, for any $\varepsilon > 0$ small enough, there exists $\delta > 0$ such that setting $m_N = \left\lfloor \frac{(\log N)^{\alpha+1}}{L^*(\log N)} \right\rfloor$, for any $N \geq 1$ large enough we have

$$\mathbf{P}\left(\forall n \leq m_N, x_n^N(N) \leq -n(1+\varepsilon)\nu_N\right) \leq \exp\left(-N^\delta\right).$$

We introduce $T_0 = 0$ and $Y_0^N = N\delta_0$. The process Y^N behaves as a N-BRW, using the same point processes $(L_{j,n})$ as X until time

$$T_1 = \min\left(m_N, \inf\left\{j \geq 0 : y_j^N(N) > -j\nu_N(1+\varepsilon)\right\}\right).$$

We then write $Y_{T_1+}^N = N\delta_{y_k^N(N)}$, i.e. just after time T_1, the process Y^N starts over at time T_1+ from its leftmost individual. Then for any $k \in \mathbb{N}$, the process behaves as a N-BRW between time T_k+ and T_{k+1}, defined by

$$T_{k+1} = T_k + \min\left(m_N, \inf\left\{j \geq 0 : y_{T_k+j}^N(N) - y_{T_k}^N(N) > -j\nu_N(1+\varepsilon)\right\}\right).$$

We observe easily that for all $k \in \mathbb{N}$, we have $Y^N \preccurlyeq X^N$ a.s. and in particular $y_k^N(N) \leq x_k^N(N)$.

As $(T_k - T_{k-1}, k \geq 1)$ is a sequence of i.i.d. random variables, Lemma 6.4.4 leads to

$$\lim_{n\to+\infty} \frac{x_{T_k}^N}{k} = \mathbf{E}(T_1)v_N \quad \text{a.s.}$$

Moreover, as $(y_{T_k}^N(N) - y_{T_{k-1}(N)}^N, k \geq 1)$ is another sequence of i.i.d. random variables, by law of large numbers we have

$$\lim_{n\to+\infty} \frac{y_{T_k}^N(N)}{k} = \mathbf{E}(y_{T_1}^N(N)) \quad \text{a.s.}$$

Combining these two estimates, we have

$$v_N \geq \frac{\mathbf{E}(y_{T_1}^N(N))}{\mathbf{E}(T_1)}.$$

We now compute

$$\begin{aligned}\mathbf{E}(y^N_{T_1}(N)) &= \mathbf{E}\left(y^N_{T_1}(N)\mathbf{1}_{\{T_1<m_N\}}\right) + \mathbf{E}\left(y^N_{T_1}(N)\mathbf{1}_{\{T_1=m_N\}}\right)\\ &= \mathbf{E}\left(-\nu_N(1+\varepsilon)T_1\mathbf{1}_{\{T_1<m_N\}}\right) + \mathbf{E}\left(y^N_{T_1}(N)\mathbf{1}_{\{T_1=m_N\}}\right)\\ &\geq -\nu_N(1+\varepsilon)\,\mathbf{E}(T_1) + \mathbf{E}\left(y^N_{T_1}(N)\mathbf{1}_{\{T_1=m_N\}}\right)\end{aligned}$$

Note that for all $j \leq T_1$, we have $Y^N_j = X^N_j$. Moreover, by Corollary 6.4.2, we couple X^N with a N-BRW $\widetilde{X}$ in which every individual makes only one child, with a displacement of law $\max L$. Consequently, we have

$$y^N_{T_1}(N) \geq \left(\sum_{n=1}^{T_1} \min_{j\leq N} \max L_{j,n}\right) \quad \text{a.s.}$$

which leads to

$$v_N \geq -\nu_N(1+\varepsilon) + \mathbf{E}\left[\left(\sum_{n=1}^{m_N} \min_{j\leq N} \max L_{j,n}\right)\mathbf{1}_{\{T_1=m_N\}}\right].$$

Using the Cauchy-Schwarz inequality and (6.1.10), we have

$$\mathbf{E}\left[\left(\sum_{n=1}^{m_N} \min_{j\leq N} \max L_{j,n}\right)\mathbf{1}_{\{T_1=m_N\}}\right] \geq -CNm_N\mathbf{P}\left(T_1=m_N\right)^{1/2}.$$

We apply Lemma 6.4.7 and let $N\to+\infty$ then $\delta\to 0$ to prove that

$$\liminf_{N\to+\infty} \frac{v_N(\log N)^\alpha}{L^*(\log N)} \geq -C_*.$$

□

Part III

Simple proofs of classical branching random walks estimates

CHAPTER 7

Maximal displacement in a branching random walk

"J'ai une mémoire admirable, j'oublie tout."

Alphonse Allais

Abstract

This chapter is devoted to the study of the maximal displacement in the branching random walk. We prove here that the asymptotic this quantity is composed of a first ballistic order, plus a logarithmic correction and stochastically bounded fluctuations. This result, proved in [HS09] and [ABR09] under some additional integrability conditions, is given here under close-to-optimal conditions. Borrowing ideas of [AS10], we obtain a simple proof for the two first order of the asymptotic of the maximal displacement in a branching random walk.

7.1 Introduction

A *branching random walk* on $\mathbb{R}$ is a process which starts with one individual located at the origin at time 0, and evolves as follows: at each time k, every individual currently in the process dies, giving birth to a certain number of children, which are positioned around the position of their parent according to independent and identically distributed versions of a point process.

Under some mild integrability conditions, the asymptotic behaviour of the maximal displacement is fully known. Hammersley [Ham74], Kingman [Kin75] and then Biggins [Big76] proved this maximal value grows at linear speed. In 2009, Hu and Shi [HS09] exhibited a logarithmic correction in probability, with almost sure fluctuations; and Addario-Berry and Reed [ABR09] showed the tightness of the maximal displacement, shifted around its median. More recently, Aidékon [Aïd13] proved the fluctuations converge in law to some random shift of a Gumbel variable.

Aidékon and Shi gave in [AS10] a simple way to obtain the asymptotic behaviour of the maximal displacement up to an $o(\log n)$ order. The aim of this chapter is to expose a slight refinement of their methods to prove the asymptotic behaviour up to terms of order 1. Moreover we work here with more general integrability conditions, similar to the ones used in [Aïd13].

The upper bound of the asymptotic behaviour is obtained by "bending the boundary" of the branching random walk. The idea follows from an heuristic bootstrap argument, which is detailed in Section 7.4.1. The close-to-optimal integrability conditions arise naturally when using the *spinal decomposition* of the branching random walk, introduced by Lyons [Lyo97] and recalled in Section 7.2. We introduce some ideas that are used in the further chapters, in a more complicate setting. Thus this chapter can be seen as a sandbox for later computations.

We recall that in this thesis, c, C are two positive constants, respectively small enough and large enough, which may change from line to line, and depend only on the law of the random variables we consider. For a given sequence of random variables $(X_n, n \geq 1)$, we write $X_n = O_{\mathbf{P}}(1)$ if the sequence is tensed, i.e.

$$\lim_{K \to +\infty} \sup_{n \geq 1} \mathbf{P}(|X_n| \geq K) = 0.$$

Moreover, we always assume the convention $\max \emptyset = -\infty$ and $\min \emptyset = +\infty$, and for $u \in \mathbb{R}$, we write $u_+ = \max(u, 0)$, and $\log_+(u) = (\log u)_+$. Finally, $\mathcal{C}_b$ is the set of continuous bounded functions on $\mathbb{R}$.

We consider a point process L on $\mathbb{R}$, of law $\mathcal{L}$. We write κ for the log-Laplace transform $\theta \mapsto \log \mathbf{E}\left[\sum_{\ell \in L} e^{\theta \ell}\right]$ of $\mathcal{L}$. We assume there exists $\theta^* > 0$ such that $\kappa(\theta^*) < +\infty$ and

$$\theta^* \mathbf{E}\left[\sum_{\ell \in L} \ell e^{\theta^* \ell - \kappa(\theta^*)}\right] - \kappa(\theta^*) = 0. \tag{7.1.1}$$

We write $v = \frac{\kappa(\theta^*)}{\theta^*} = \mathbf{E}\left[\sum_{\ell \in L} \ell e^{\theta^* \ell - \kappa(\theta^*)}\right]$ for the *speed* of the branching random walk. We introduce the assumption of finite variance for the spine

$$\mathbf{E}\left[\sum_{\ell \in L} \ell^2 e^{\theta^* \ell}\right] < +\infty, \tag{7.1.2}$$

and the additional integrability condition

$$\mathbf{E}\left[\sum_{\ell \in L} e^{\theta^* \ell} \log_+\left(\sum_{\ell' \in L} (1 + (\ell' - \ell)_+) e^{\theta^*(\ell' - \ell)}\right)^2\right] < +\infty. \tag{7.1.3}$$

Let $(\mathbf{T}, V)$ be a branching random walk with reproduction law $\mathcal{L}$. We denote by $M_n = \max_{|u|=n} V(u)$ the maximal displacement in this branching random walk and by $S = \{\mathbf{T} \text{ is infinite}\}$ the survival event. By definition, $M_n = -\infty$ on S^c for all n large enough. The following result holds.

Theorem 7.1.1. *Under the assumptions* (7.1.1), (7.1.2) *and* (7.1.3), *we have*

$$M_n = nv - \frac{3}{2\theta^*} \log n + O_{\mathbf{P}}(1) \quad \text{on } S.$$

The main tool used to prove this theorem is the following estimate on the right tail of the maximal displacement M_n.

Theorem 7.1.2. *Under the assumptions* (7.1.1), (7.1.2) *and* (7.1.3), *there exists $c, C > 0$ such that for all $n \geq 1$ and $y \in [0, n^{1/2}]$,*

$$c(1+y)e^{-\theta^* y} \leq \mathbf{P}\left[M_n \geq nv - \frac{3}{2\theta^*} \log n + y\right] \leq C(1+y)e^{-\theta^* y}.$$

The rest of this chapter is organised as follows. In Section 7.2, we introduced the so-called spinal decomposition of the branching random walk, which links additive moments of the branching random walk with random walk estimates. In Section 7.3, we recall the random walk estimates obtained in Chapter 1, and their extensions to random walks enriched by additional random variables, which are only correlated to the last step of the walk. Section 7.4 is devoted to the proof of Theorem 7.1.2. This theorem is used in Section 7.5 to prove Theorem 7.1.1, using a coupling between the branching random walk and a Galton-Watson process.

7.2 Spinal decomposition of the branching random walk

We introduce in this section the well-known spinal decomposition of the branching random walk. This result consists in two ways of describing a size-biased version of the law of the branching random walk. Spinal decomposition of a branching process has been introduced for the first time to study Galton-Watson processes in [LPP95]. In [Lyo97], this technique is adapted to the study of branching random walks.

7.2.1 The size-biaised law of the branching random walk

Let $(\mathbf{T}, V)$ be a branching random walk with reproduction law $\mathcal{L}$. For all $x \in \mathbb{R}$, we write $\mathbf{P}_x$ the law of $(\mathbf{T}, V + x)$ and $\mathbf{E}_x$ the corresponding expectation. For all $n \geq 1$, we set

$$W_n = \sum_{|u|=n} \exp\left(\theta^* V(u) - n\kappa(\theta^*)\right).$$

Writing $\mathcal{F}_n = \sigma(u, V(u), u \leq n)$, we observe that (W_n) is a non-negative $(\mathcal{F}_n)$-martingale. We define the law

$$\left.\overline{\mathbf{P}}_x\right|_{\mathcal{F}_n} = e^{-x} W_n \cdot \left.\mathbf{P}_x\right|_{\mathcal{F}_n}. \tag{7.2.1}$$

The spinal decomposition consists in an alternative construction of the law $\overline{\mathbf{P}}_a$, as the projection of a law on the set of planar rooted marked trees with spine, which we define below.

7.2.2 A law on plane rooted marked trees with spine

Let $(\mathbf{T}, V)$ be a marked tree with infinite height. Let $w \in \mathbb{N}^{\mathbb{N}}$ be a sequence of integers, we write $w_n = (w(1), \ldots w(n))$ and we say that w is a spine for $\mathbf{T}$ if for all $n \in \mathbb{N}$, $w_n \in \mathbf{T}$. The triplet $(\mathbf{T}, V, w)$ is called a (plane rooted) marked tree with spine, and the set of such objects is written $\widehat{\mathcal{T}}$. We define the three following filtrations on this set

$$\mathcal{F}_n = \sigma\left(u, V(u) : u \in \mathbf{T}, |u| \leq n\right), \ \widehat{\mathcal{F}}_n = \mathcal{F}_n \vee \sigma\left(w_k, k \leq n\right)$$
$$\text{and } \mathcal{G}_n = \sigma\left(w_k, V(w_k) : k \leq n\right) \vee \sigma\left(u, V(u), u \in \Omega(w_k), k < n\right). \tag{7.2.2}$$

The filtration $(\widehat{\mathcal{F}}_n)$ is the filtration of the knowledge of the marked tree with spine up to height n, $(\mathcal{F}_n)$ has only the informations of the marked tree when forgetting about the spine, and $(\mathcal{G}_n)$ has only the knowledge of the spine and its children.

We introduce

$$\widehat{\mathcal{L}} = \left(\sum_{\ell \in L} e^{\theta^* \ell - \kappa(\theta^*)}\right) \cdot \mathcal{L},$$

a law of a point process with Radon-Nikodým derivative with respect to $\mathcal{L}$, and we write $(\widehat{L}_n, n \in \mathbb{N})$ i.i.d. point processes with law $\widehat{\mathcal{L}}$. Conditionally on this sequence, we choose, for every $n \in \mathbb{N}$, $w(n) \in \mathbb{N}$ independently at random, such that, writing $\widehat{L}_n = (\ell_1, \ldots \ell_{N_k})$, we have

$$\forall h \in \mathbb{N},\ \mathbf{P}\left(w(k) = h \middle| (\widehat{L}_n, n \in \mathbb{N})\right) = \mathbf{1}_{\{h \leq N_k\}} \frac{e^{\theta^* \ell_k(h)}}{\sum_{j \leq N_k} e^{\theta^* \ell_k(j)}}.$$

We write w the sequence $(w(n), n \in \mathbb{N})$.

We now define a random variable $(\mathbf{T}, V, w)$ on $\widehat{\mathcal{T}}$. To do so, we introduce a family of independent point processes $\{L^u, u \in \mathcal{U}\}$ such that $L^{w_k} = \widehat{L}_{k+1}$, and if $u \neq w_{|u|}$, then L^u has law $\mathcal{L}$. For any $u \in \mathcal{U}$ such that $|u| \leq n$, we write $L^u = (\ell^u_1, \ldots \ell^u_{N(u)})$. We construct the random tree

$$\mathbf{T} = \{u \in \mathcal{U} : |u| \leq n, \forall 1 \leq k \leq |u|, u(k) \leq N(u_{k-1})\},$$

and the the function

$$V : \begin{array}{ccc} \mathbf{T} & \to & \mathbb{R} \\ u & \mapsto & \sum_{k=1}^{|u|} \ell^{u_{k-1}}_{u(k)}. \end{array}$$

For all $x \in \mathbb{R}$, the law of $(\mathbf{T}, x+V, w) \in \widehat{\mathcal{T}}_n$ is written $\widehat{\mathbf{P}}_x$, and the corresponding expectation is $\widehat{\mathbf{E}}_x$. This law is called the law of the branching random walk with spine.

We can describe the branching random walk with spine as a process in the following manner. It starts with a unique individual positioned at $x \in \mathbb{R}$ at time 0, which is the ancestral spine w_0. Then, at each time $n \in \mathbb{N}$, every individual alive at generation n dies. Each of these individuals gives birth to children, which are positioned around their parent according to an independent point process. If the parent is w_n, the law of this point process is $\widehat{\mathcal{L}}$, otherwise the law is $\mathcal{L}$. The individual w_{n+1} is then chosen at random among the children u of w_n, with probability proportional to $e^{\theta^* V(u)}$.

7.2.3 The spinal decomposition

The following result, which links the laws $\widehat{\mathbf{P}}_x$ and $\overline{\mathbf{P}}_x$, is the spinal decomposition, proved in [Lyo97].

Proposition X (Spinal decomposition). *For all $x \in \mathbb{R}$, we have*

$$\overline{\mathbf{P}}_x\Big|_{\mathcal{F}_n} = \widehat{\mathbf{P}}_x\Big|_{\mathcal{F}_n}. \tag{7.2.3}$$

Moreover, for any $n \in \mathbb{N}$ and $|u| = n$, we have

$$\widehat{\mathbf{P}}_x(w_n = u|\mathcal{F}) = \frac{\exp\left(\theta^* V(u) - n\kappa(\theta^*)\right)}{W_n}. \tag{7.2.4}$$

Note that a time-inhomogeneous version of this result has been proved in Chapter 1. An immediate consequence of this result, which can also be proved directly by recurrence, is the well-known many-to-one lemma. This equation, known at least from the early works of Peyrière [Pey74] has been used in many forms over the last decades. We denote by μ a probability measure on $\mathbb{R}$ defined by,

$$\mu((-\infty, x]) = \mathbf{E}\left[\sum_{\ell \in L} \mathbf{1}_{\{\ell \leq x\}} e^{\theta^* \ell - \kappa(\theta^*)}\right].$$

Lemma XI (Many-to-one)**.** *Let (X_n) be an i.i.d. sequence of random variables with law μ, we write $S_n = S_0 + \sum_{k=1}^n X_k$ for $n \in \mathbb{N}$, where $\mathbf{P}_x(S_0 = x) = 1$. For all $x \in \mathbb{R}, n \in \mathbb{N}$ and continuous bounded function f, we have*

$$\mathbf{E}_x\left[\sum_{|u|=n} f(V(u_1), \ldots V(u_n))\right] = e^{\theta^* x}\, \mathbf{E}_x\left[e^{-\theta^* S_n + n\kappa(\theta^*)} f(S_1, \ldots S_n)\right]. \tag{7.2.5}$$

Remark 7.2.1. Under the assumption (7.1.1), we observe that the mean S_1 is

$$\mathbf{E}(S_1) = \mathbf{E}\left[\sum_{\ell \in L} \ell e^{\theta^* \ell - \kappa(\theta^*)}\right] = v.$$

Moreover, assumption (7.1.2) leads to $\mathbf{E}(S_1^2) = \mathbf{E}\left[\sum_{\ell \in L} \ell e^{\theta^* \ell - \kappa(\theta^*)}\right] < +\infty$, so (S_n) is a random walk with mean v and finite variance.

Proof. Let f be a continuous bounded function and $x \in \mathbb{R}$, we have, by Proposition X

$$\begin{aligned}
&\mathbf{E}_x\left[\sum_{|u|=n} f(V(u_1), \ldots V(u_n))\right] \\
&\quad = \overline{\mathbf{E}}_x\left[\frac{e^{\theta^* x}}{W_n} \sum_{|u|=n} f(V(u_1), \ldots V(u_n))\right] \\
&\quad = e^{\theta^* x}\overline{\mathbf{E}}_x\left[\sum_{|u|=n} e^{-\theta^* V(u) + n\kappa(\theta^*)} f(V(u_1), \ldots V(u_n))\widehat{\mathbf{P}}(w_n = u | \mathcal{F}_n)\right] \\
&\quad = e^{\theta^* x}\widehat{\mathbf{E}}_x\left[\sum_{|u|=n} e^{-\theta^* V(u) + n\kappa(\theta^*)} f(V(u_1), \ldots V(u_n))\mathbf{1}_{\{w_n = u\}}\right] \\
&\quad = e^{\theta^* x}\widehat{\mathbf{E}}_x\left[e^{-\theta^* V(w_n) + n\kappa(\theta^*)} f(V(w_1), \cdots V(w_n))\right].
\end{aligned}$$

Moreover, by definition of $\widehat{\mathbf{P}}_x$, we observe that the law of $(V(w_1), \ldots, V(w_n))$ is the same as the law of $(S_1, \ldots S_n)$ under $\mathbf{P}_x$, which ends the proof. □

The many-to-one lemma and the spinal decomposition enable to compute moments of any additive functional of the branching random walk, by using random walk estimates, which are obtained in the next section.

7.3 Some random walk estimates

We collect first a series of well-known random walk estimates, such as local limit and ballot theorems, and extend these results to bound the probability for a random walk to make an excursion above a given curve. In a second section, we extend these results to random walks enriched with additional random variables which are correlated with the last step of the random variable.

7.3.1 Classical random walk estimates

We recall in this section the random walk estimates obtained in Chapter 1, that we use to prove Theorem 7.1.2. We denote by $(T_n, n \geq 0)$ a one-dimensional centred random walk, with finite variance σ^2. We begin with Stone's local limit theorem [Sto65]. There exists $C > 0$ such that for all $a \geq 0$ and $h \geq 0$, we have

$$\limsup_{n\to+\infty} n^{1/2} \sup_{|y|\geq an^{1/2}} \mathbf{P}(T_n \in [y, y+h]) \leq C(1+h)e^{-\frac{a^2}{2\sigma^2}}. \tag{7.3.1}$$

Moreover, there exists $H > 0$ such that for all $a < b \in \mathbb{R}$

$$\liminf_{n\to+\infty} n^{1/2} \inf_{y\in[an^{1/2}, bn^{1/2}]} \mathbf{P}(T_n \in [y, y+H]) > 0. \tag{7.3.2}$$

We continue with Caravenna–Chaumont's local limit theorem [CC13]. Let $(r_n, n \geq 0)$ be a positive sequence such that $r_n = O(n^{1/2})$. There exists $C > 0$ such that for all $a \geq 0$ and $h \geq 0$,

$$\limsup_{n\to+\infty} n^{1/2} \sup_{y\in[0,r_n]} \sup_{x\geq an^{1/2}} \mathbf{P}(T_n \in [x, x+h] | T_j \geq -y, j \leq n) \leq C(1+h)ae^{-\frac{a^2}{2\sigma^2}}. \tag{7.3.3}$$

Moreover, there exists $H > 0$ such that for all $a < b \in \mathbb{R}_+$,

$$\liminf_{n\to+\infty} n^{1/2} \inf_{y\in[0,r_n]} \inf_{x\in[an^{1/2}, bn^{1/2}]} \mathbf{P}(T_n \in [x, x+H] | T_j \geq -y, j \leq n) > 0. \tag{7.3.4}$$

Up to a transformation $T \mapsto T/(2H)$, which shrinks the space by a factor $\frac{1}{2H}$, we may and will assume in the rest of this chapter once again that all the random walks we consider are such that (7.3.2) and (7.3.4) hold with $H = 1$.

We next recall the consequence of Kozlov's [Koz76] and Pemantle–Peres' [PP95] ballot theorems, for all $A \geq 0$ and $\alpha \in [0, 1/2)$, there exists $C > 0$ such that for all $n \geq 1$ and $y \geq 0$,

$$\mathbf{P}(T_j \geq -y - Aj^\alpha, j \leq n) \leq C(1+y)n^{-1/2}, \tag{7.3.5}$$

moreover, there exists $c > 0$ such that for all $n \geq 1$ and $y \in [0, n^{1/2}]$

$$\mathbf{P}(T_j \geq -y, j \leq n) \geq c(1+y)n^{-1/2}. \tag{7.3.6}$$

We also obtained in Chapter 1 the following bounds for the probability for a random walk to make an excursion. There exists $C > 0$ such that for any $x, h \geq 0$ and $y \in \mathbb{R}$ we have

$$\begin{aligned}\mathbf{P}(T_{p+q} \in [y+h, y+h+1], T_j \geq -x + y\mathbf{1}_{\{j>p\}}, j \leq n)\\ \leq C\frac{(1+x)\wedge p^{1/2}}{p^{1/2}} \frac{1}{\max(p,q)^{1/2}} \frac{(1+h)\wedge q^{1/2}}{q^{1/2}}.\end{aligned} \tag{7.3.7}$$

Moreover, there exists $c > 0$ such that for all $n \geq 1$ large enough, $x \in [0, n^{1/2}]$ and $y \in [-n^{1/2}, n^{1/2}]$ we have

$$\mathbf{P}_x(T_n \leq y+1, T_j \geq y\mathbf{1}_{\{j>n/2\}}, j \leq n) \geq c\frac{(1+x)}{n^{3/2}}. \tag{7.3.8}$$

This result also holds for excursions above bended curves. For all $A \geq 0$ there exists $C > 0$ such that for all $n \in \mathbb{N}$ and $y, h \geq 0$

$$\mathbf{P}(T_n + A\log n \in [h-y, h-y+1], T_j \geq -A\log \tfrac{n}{n-j+1} - y, j \leq n) \\ \leq C\frac{((1+y)\wedge n^{1/2})((1+h)\wedge n^{1/2})}{n^{3/2}} \quad (7.3.9)$$

We also need the results obtained for random walks enriched with random variables that only depend on its last step. Let (X_n, ξ_n) be a sequence of i.i.d. random vectors in $\mathbb{R}^2$, such that $\mathbf{E}(X_1) = 0$, $\mathbf{E}(X_1^2) < +\infty$ and $\mathbf{E}((\xi_1)_+^2) < +\infty$. We write $T_n = \sum_{j=1}^n X_j$. There exists $C > 0$ that does not depend on the law of ξ_1 such that for all $n \geq 1$, $x, h \geq 0$ and $y \in \mathbb{R}$, we have

$$\mathbf{P}\left[T_j \geq -x, j \leq n, \exists k \leq n : T_k \leq \xi_k - x\right] \leq C\frac{1+x}{n^{1/2}}\left[\mathbf{P}(\xi_1 \geq 0) + \mathbf{E}((\xi_1)_+^2)\right], \quad (7.3.10)$$

as well as

$$\mathbf{P}\left[T_n - x - y - h \in [0,1], T_j \geq -x + y\mathbf{1}_{\{j>tn\}}, j \leq n, \exists k \leq n : T_k \leq \xi_k + y\mathbf{1}_{\{k>tn\}} - x\right] \\ \leq C\frac{(1+x)(1+h)}{n^{3/2}}\left[\mathbf{P}(\xi_1 \geq 0) + \mathbf{E}((\xi_1)_+^2)\right]. \quad (7.3.11)$$

We use these two bounds to control at the same time the position of the spine and the number of its children.

7.4 Bounding the tail of the maximal displacement

Let $(\mathbf{T}, V)$ be a branching random walk, and M_n its maximal displacement at time n. We write $m_n = nv - \frac{3}{2\theta^*}\log n$, the main goal of this section is to prove Theorem 7.1.2. We first give an upper bound for the tail of M_n, by observing there exists a boundary such that, with high probability, no individual in the branching random walk crosses. The lower bound is obtained by bounding from below the probability there exists an individual which is at time n close to $m_n := nv - \frac{3}{2\theta^*}\log n$, such that all its ancestors were below this boundary.

7.4.1 The boundary of the branching random walk

A natural way to compute an upper bound for $\mathbf{P}(M_n \geq m_n + y)$ would be a direct application of the Markov inequality. We have

$$\mathbf{P}(M_n \geq m_n + y) \leq \mathbf{E}\left[\sum_{|u|=n} \mathbf{1}_{\{V(u)\geq m_n+y\}}\right] \leq \mathbf{E}\left[e^{\theta^* S_n - n\kappa(\theta^*)}\mathbf{1}_{\{S_n \geq m_n+y\}}\right]$$

by Lemma XI. Therefore, as $\mathbf{E}(S_1) = v = \frac{\kappa(\theta^*)}{\theta^*}$, we have

$$\mathbf{P}(M_n \geq m_n + y) \leq n^{3/2}e^{-\theta^* y}\sum_{h=0}^{+\infty} e^{-\theta^* h}\mathbf{P}(S_n - m_n - y \in [h, h+1]) \leq Cne^{-\theta^* y}$$

by (7.3.1). Note this computation is not precise enough to yield Theorem 7.1.2.

To obtain a better bound, a natural idea is to compute the number of individuals alive at generation k who would have with high probability a descendant above $m_n + y$ at generation n. If we assume Theorem 7.1.1 to be true, then an individual alive at generation k may have children above m_n with high probability if it is to the right of

$$f_k^{(n)} = kv - \frac{3}{2\theta^*}\left(\log n - \log(n-k+1)\right).$$

We prove that with high probability, no individual crosses the boundary $f_.^{(n)} + y$ before time n.

Lemma 7.4.1. *Under the assumptions* (7.1.1) *and* (7.1.2), *there exists* $C > 0$ *such that for all* $y \geq 0$

$$\mathbf{P}(\exists |u| \leq n : V(u) \geq f_{|u|}^{(n)} + y) \leq C(1+y)e^{-\theta^* y}.$$

Proof. For all $k \leq n$, we write $Z_k^{(n)}(y) = \sum_{|u|=k} \mathbf{1}_{\left\{V(u) \geq f_k^{(n)} + y\right\}} \mathbf{1}_{\left\{V(u_j) \leq f_j^{(n)} + y, j<k\right\}}$ the number of individuals which cross for the first time curve $f^{(n)}$ at time k. By many-to-one lemma, we have

$$\begin{aligned}
\mathbf{E}\left(Z_k^{(n)}(y)\right) &= \mathbf{E}\left[e^{-\theta^* S_k + k\kappa(\theta^*)} \mathbf{1}_{\left\{S_k \geq f_k^{(n)} + y\right\}} \mathbf{1}_{\left\{S_j \leq f_j^{(n)} + y, j<k\right\}}\right] \\
&\leq \frac{n^{3/2}}{(n-k+1)^{3/2}} e^{-\theta^* y} \mathbf{P}\left(S_k \geq f_k^{(n)} + y, S_j \leq f_j^{(n)} + y, j < k\right).
\end{aligned}$$

We condition this probability with respect to the last step $S_k - S_{k-1}$ to obtain

$$\mathbf{P}\left(S_k \geq f_k^{(n)} + y, S_j \leq f_j^{(n)} + y, j < k\right) = \mathbf{E}(\varphi_{k-1}(S_k - S_{k-1})).$$

where $\varphi_k(x) = \mathbf{P}\left(S_k \geq f_{k+1}^{(n)} + y - x, S_j \leq f_j^{(n)} + y, j \leq k\right)$. Applying (7.3.9), there exists $C > 0$ such that for all $k \leq n$ and $x \in \mathbb{R}$

$$\varphi_k(x) \leq C \mathbf{1}_{\{x \geq 0\}} \frac{(1+y)(1+x)^2}{(k+1)^{3/2}}.$$

As a consequence,

$$\begin{aligned}
\mathbf{P}\left(\exists |u| \leq n : V(u) \geq f_{|u|}^{(n)} + y\right) &\leq \sum_{k=1}^n \mathbf{P}(Z_k^{(n)}(y) \geq 1) \leq \sum_{k=1}^n \mathbf{E}(Z_k^{(n)}(y)) \\
&\leq C(1+y)e^{-\theta^* y} \sum_{k=1}^n \frac{n^{3/2}}{k^{3/2}(n-k+1)^{3/2}} \mathbf{E}\left((S_k - S_{k-1})_+^2 + 1\right).
\end{aligned}$$

By decomposition of this sum into $k \leq n/2$ and $k \geq n/2$, we conclude

$$\mathbf{P}(\exists |u| \leq n : V(u) \geq f_{|u|}^{(n)} + y) \leq C(1+y)e^{-\theta^* y}.$$

□

This lemma directly implies the upper bound in Theorem 7.1.2.

Proof of the upper bound in Theorem 7.1.2. As $f_n^{(n)} = m_n$, we observe easily that

$$\mathbf{P}(M_n \geq m_n + y) \leq \mathbf{P}\left(\exists |u| \leq n : V(u) \geq f_{|u|}^{(n)} + y\right),$$

applying Lemma 7.4.1 ends the proof. □

7.4.2 A lower bound through second moment computations

For all $n \in \mathbb{N}$ and $k \leq n$, we write $g_k^{(n)} = kv - \mathbf{1}_{\{k>n/2\}} \frac{3}{2\theta^*} \log n + 1$ a boundary which is close to $f^{(n)}$ but simpler to use. We prove in this section that the set

$$\mathcal{A}_n(y) = \left\{ u \in \mathbf{T}, |u| = n : V(u) \geq m_n + y, V(u_j) \leq g_j^{(n)} + y, j \leq n \right\}$$

is non-empty with positive probability. To do so, we wish to bound the first two moments of the number of individuals in $\mathcal{A}_n(y)$. However, to obtain a good upper bound for a second moment in branching processes, we need to control the reproduction of the individuals we consider. For $u \in \mathbf{T}$, we write

$$\xi(u) = \sum_{u' \in \Upsilon(u)} (1 + (V(u) - V(u'))_+) e^{-\theta^*(V(u') - V(u))}$$

where $\Upsilon(u) = \{v \in \mathbf{T} : \pi v = \pi u, v \neq u\}$ is the set of siblings of u. For any $z > 0$, we set

$$\mathcal{B}_n(z) = \left\{ u \in \mathbf{T}, |u| = n : \xi(u_j) \leq z e^{\frac{\theta^*}{2}(V(u_j) - g_j^{(n)})}, j < n \right\}.$$

We compute the first two moments of $Y_n(y, z) = \sum_{|u|=n} \mathbf{1}_{\{u \in G_n(y,z)\}}$ where $G_n(y, z) = \mathcal{A}_n(y) \cap \mathcal{B}_n(z)$.

Lemma 7.4.2. *Under the assumptions* (7.1.1) *and* (7.1.2)*, there exists $C > 0$ such that for all $y \geq 0$ and $z \geq 1$ we have*

$$\mathbf{E}(Y_n(y, z)^2) \leq Cz(1 + y)e^{-\theta^* y}.$$

Proof. Applying Proposition X, we have

$$\begin{aligned} \mathbf{E}(Y_n(y, z)^2) = \overline{\mathbf{E}}\left[\frac{1}{W_n} Y_n(y, z)^2\right] &= \widehat{\mathbf{E}}\left[\frac{1}{W_n} \sum_{|u|=n} \mathbf{1}_{\{u \in G_n(y,z)\}} Y_n(y, z)\right] \\ &= \widehat{\mathbf{E}}\left[e^{-\theta^* V(w_n) + n\kappa(\theta^*)} \mathbf{1}_{\{w_n \in G_n(y,z)\}} Y_n(y, z)\right]. \end{aligned}$$

Using the fact that $w_n \in \mathcal{A}_n(y) \subset G_n(y, z)$, we have

$$\mathbf{E}(Y_n(y, z)^2) \leq C n^{3/2} e^{-\theta^* y} \widehat{\mathbf{E}}\left[Y_n(y, z) \mathbf{1}_{\{w_n \in G_n(y,z)\}}\right].$$

We decompose $Y_n(y, z)$ along the spine, to obtain

$$Y_n(y, z) \leq \mathbf{1}_{\{w_n \in G_n(y,z)\}} + \sum_{k=1}^{n} \sum_{u \in \Upsilon(w_k)} Y_n(u, y), \tag{7.4.1}$$

where, for $u \in \mathbf{T}$ and $y \geq 0$, we write $Y_n(u, y) = \sum_{|u'|=n, u' \geq u} \mathbf{1}_{\{u' \in \mathcal{A}_n(y)\}}$. We recall that under law $\widehat{\mathbf{P}}$, for any $k \leq n$, the branching random walk of the children of an individual $u \in \Upsilon(w_k)$ has law $\mathbf{P}_{V(u)}$. As a consequence, for $y \geq 0$, $k \leq n$ and $u \in \Upsilon(w_k)$,

$$\widehat{\mathbf{E}}\left[Y_n(u, y) | \mathcal{G}_n\right] = \mathbf{E}_{V(u)}\left[\sum_{|u'|=n-k} \mathbf{1}_{\{V(u') \geq m_n + y\}} \mathbf{1}_{\left\{V(u'_j) \leq g_{k+j}^{(n)} + y, j \leq n-k\right\}}\right].$$

As a consequence, by Lemma XI, we have

$$\begin{aligned}
&\widehat{\mathbf{E}}\left[Y_n(u,y)|\mathcal{G}_n\right]\\
\leq &Cn^{3/2}e^{-\theta^* y}e^{\theta^* V(u)-k\kappa(\theta^*)}\mathbf{P}_{V(u)}\left(S_{n-k}\geq m_n+y, S_j\leq g^{(n)}_{j+k}+y, j\leq n-k\right)\\
\leq &Ce^{-\theta^* y}\frac{n^{3/2}}{(n-k+1)^{3/2}}e^{\theta^* V(u)-k\kappa(\theta^*)}(1+(g^{(n)}_k+y-V(u))_+)
\end{aligned}$$

by (7.3.7).

For all $k\leq n$, we now compute the quantity

$$h_k := \widehat{\mathbf{E}}\left[\mathbf{1}_{\{w_n\in G_n(y,z)\}}\sum_{u\in\Upsilon(w_k)}(1+(g^{(n)}_k+y-V(u))_+)e^{\theta^*(V(u)-g^{(n)}_k)}\right],$$

using the definition of $\xi(w_k)$ and the fact that $x\mapsto x_+$ is Lipschitz, we have

$$\begin{aligned}
h_k &\leq C\widehat{\mathbf{E}}\left[\mathbf{1}_{\{w_n\in G_n(y,z)\}}(1+(g^{(n)}_k+y-V(w_k))_+)e^{\theta^* V(w_k)-g^{(n)}_k}\xi(w_k)\right]\\
&\leq Cz\widehat{\mathbf{E}}\left[\mathbf{1}_{\{w_n\in \mathcal{A}_n(y)\}}(1+(g^{(n)}_k+y-V(w_k))_+)e^{\frac{\theta^*}{2}(V(w_k)-g^{(n)}_k)}\right].
\end{aligned}$$

Decomposing this expectation with respect to the value taken by $V(w_k)$, we have

$$\begin{aligned}
h_k \leq Cze^{\theta^* y}\sum_{h=0}^{+\infty}(1+h)e^{-\theta^* h/2}&\\
&\times\mathbf{P}\left[\begin{array}{l} S_n\geq m_n+y, S_k-g^{(n)}_k-y\in[-h-1,-h]\\ S_j\leq g^{(n)}_j+y, j\leq n\end{array}\right].
\end{aligned}$$

We apply the Markov property at time k to obtain

$$\begin{aligned}
\mathbf{P}&\left[S_n\geq m_n+y, S_k-g^{(n)}_k-y\in[-h-1,-h], S_j\leq g^{(n)}_j+y, j\leq n\right]\\
&\leq \mathbf{P}\left[S_k-g^{(n)}_k-y\in[-h-1,-h], S_j\leq g^{(n)}_j+y, j\leq k\right]\\
&\qquad\times\inf_{z\in[-h-1,-h]}\mathbf{P}_z\left[S_{n-k}\geq m_n-g^{(n)}_k, S_j\leq g^{(n)}_{k+j}-g^{(n)}_k, j\leq n-k\right],
\end{aligned}$$

thus, applying again (7.3.7),

$$\begin{aligned}
h_k &\leq Cze^{\theta^* y}\sum_{h=0}^{+\infty}(1+h)e^{-\theta^* h/2}\frac{(1+h)(1+y)}{(k+1)^{3/2}}\frac{1+h}{(n-k+1)^{3/2}}\\
&\leq Cz\frac{(1+y)e^{\theta^* y}}{(k+1)^{3/2}(n-k+1)^{3/2}}.
\end{aligned}$$

As a consequence,

$$\begin{aligned}
\mathbf{E}&\left[\mathbf{1}_{\{w_n\in G_n(y,z)\}}\sum_{u\in\Upsilon(w_k)}Y_n(u,y)\right]\\
&\leq Ce^{-\theta^* y}\frac{n^{3/2}}{(n-k+1)^{3/2}}h_k e^{\theta^* g^{(n)}_k-k\kappa(\theta^*)}\\
&\leq Cz(1+y)\frac{n^{3/2}}{(k+1)^{3/2}(n-k+1)^3}\frac{1}{1+n^{3/2}\mathbf{1}_{\{k>n/2\}}}.
\end{aligned}$$

We now apply this bound to (7.4.1), we obtain

$$\begin{aligned}
\mathbf{E}(Y_n(y,z)^2) &\leq C n^{3/2} e^{-\theta^* y}\, \mathbf{E}\left[\mathbf{1}_{\{w_n \in G_n(y,z)\}} Y_n(y,z)\right]\\
&\leq C n^{3/2} e^{-\theta^* y} \left[\mathbf{P}\left(w_n \in \mathcal{A}_n(y)\right) + \sum_{k=1}^n \widehat{\mathbf{E}}\left(\mathbf{1}_{\{w_n \in G_n(y,z)\}} \sum_{u \in \Upsilon(w_k)} Y_n(u,y)\right)\right]\\
&\leq C(1+y) e^{-\theta^* y}\left(1 + z \sum_{k=1}^n \frac{1}{(k+1)^{3/2}(n-k+1)^3} \frac{n^3}{1+n^{3/2}\mathbf{1}_{\{k>n/2\}}}\right)\\
&\leq C z (1+y) e^{-\theta^* y},
\end{aligned}$$

using (7.3.7) to bound $\mathbf{P}(w_n \in \mathcal{A}_n(y))$. □

In a second time, we bound $\mathbf{E}(Y_n(y,z))$ from below.

Lemma 7.4.3. *Under the assumptions* (7.1.1), (7.1.2) *and* (7.1.3), *there exist* $c > 0$ *and* $z \geq 1$ *such that for all* $y \in [0, n^{1/2}]$ *and* $n \in \mathbb{N}$

$$\mathbf{E}(Y_n(y,z)) \geq c(1+y)e^{-\theta^* y}.$$

Proof. Let $n \in \mathbb{N}$, $y \in [0, n^{1/2}]$ and $z \geq 1$. By the spinal decomposition, we have

$$\begin{aligned}
\mathbf{E}(Y_n(y,z)) &\geq \widehat{\mathbf{E}}\left[\frac{1}{W_n} \sum_{|u|=n} \mathbf{1}_{\{u \in G_n(y,z)\}}\right] \geq \widehat{\mathbf{E}}\left[e^{V(w_n) - n\kappa(\theta^*)} \mathbf{1}_{\{w_n \in G_n(y,z)\}}\right]\\
&\geq n^{3/2} e^{-\theta^*(y+1)} \widehat{\mathbf{P}}(w_n \in G_n(y,z)).
\end{aligned}$$

To bound this probability, we observe first that

$$\widehat{\mathbf{P}}(w_n \in \mathcal{A}_n(y) \cap \mathcal{B}_n(z)) = \widehat{\mathbf{P}}(w_n \in \mathcal{A}_n(y)) - \widehat{\mathbf{P}}(w_n \in \mathcal{A}_n(y) \cap \mathcal{B}_n(z)^c),$$

and $\widehat{\mathbf{P}}(w_n \in \mathcal{A}_n(y)) \geq c(1+y)n^{-3/2}$ by (7.3.8). Introducing

$$\tau = \inf\left\{k \geq 1 : \frac{\theta^*}{2}\left(V(w_k) - g_k^{(n)}\right) \geq \log \xi(w_{k-1}) - \log z\right\},$$

we rewrite

$$\widehat{\mathbf{P}}(w_n \in \mathcal{A}_n(y) \cap \mathcal{B}_n(z)^c) \leq \mathbf{P}\left(V(w_n) \geq m_n + y, V(w_j) \leq g_j^{(n)} + y, \tau \leq n\right).$$

Therefore, we can apply (7.3.11), there exists $C > 0$ such that

$$\begin{aligned}
&\widehat{\mathbf{P}}(w_n \in \mathcal{A}_n(y) \cap \mathcal{B}_n(z)^c)\\
&\qquad\qquad \leq C \frac{1+y}{n^{3/2}} \left(\mathbf{P}(\log \xi(w_1) \geq \log z) + \widehat{\mathbf{E}}((\log \xi(w_1) - \log z)_+)^2\right).
\end{aligned}$$

By (7.1.3), we have $\widehat{\mathbf{E}}((\log \xi(w_1))_+^2)$, therefore by dominated convergence, we have

$$\lim_{z \to +\infty} \sup_{n \in \mathbb{N}, y \geq 0} \frac{n^{3/2}}{1+y} \widehat{\mathbf{P}}(w_n \in \mathcal{A}_n(y) \cap \mathcal{B}_n(z)^c) = 0,$$

thus we can find $z > 1$ large enough such that $\mathbf{P}(w_n \in \mathcal{A}_n(y) \cap \mathcal{B}_n(z)) \geq c(1+y)n^{-3/2}$, which ends the proof. □

Using these two lemmas, we obtain a lower bound for the right tail of the random variable M_n.

Lower bound in Theorem 7.1.2. By the Cauchy-Schwarz inequality, we have

$$\mathbf{P}(Y_n(y,z) \geq 1) \geq \frac{\mathbf{E}(Y_n(y,z))^2}{\mathbf{E}(Y_n(y,z)^2)}.$$

Using Lemmas 7.4.2 and 7.4.3, there exists $z \geq 1$ such that

$$\begin{aligned} \mathbf{P}(Y_n(y,z) \geq 1) &\geq \frac{\left(c(1+y)e^{-\theta^* y}\right)^2}{Cz(1+y)e^{-\theta^* y}} \\ &\geq c(1+y)e^{-\theta^* y}. \end{aligned}$$

As a consequence, we conclude $\mathbf{P}(M_n \geq m_n + y) \geq \mathbf{P}(G_n(y,z) \neq \emptyset) \geq c(1+y)e^{-\theta^* y}$. □

7.5 Concentration estimates for the maximal displacement

The aim of this section is to prove Theorem 7.1.1 using Theorem 7.1.2. We use the fact that on the survival event S of the branching random walk, the size of the population alive at time k grows at exponential rate, as in a Galton-Watson process. Moreover, each one of the individuals alive at time k have positive probability to make a child to the right of m_n at time $n \gg k$, which is enough to obtain the tension of $M_n - m_n$. We recall the Galton-Watson estimate obtained in Chapter 1.

Lemma 7.5.1. *Let $(Z_n, n \geq 0)$ be a Galton-Watson process with reproduction law μ. We write $b = \min\{k \in \mathbb{Z}_+ : \mu(k) > 0\}$, $m = \mathbf{E}(Z_1) \in (1, +\infty)$ and q the smallest solution of the equation $\mathbf{E}(q^{Z_1}) = q$. There exists $C > 0$ such that for all $z \in (0,1)$ and $n \in \mathbb{N}$ we have*

$$\mathbf{P}(Z_n \leq zm^n) \leq \begin{cases} q + Cz^{\frac{\alpha}{\alpha+1}} & \text{if } b = 0 \\ Cz^\alpha & \text{if } b = 1 \\ \exp\left[-Cz^{-\frac{\log b}{\log m - \log b}}\right] & \text{if } b \geq 2. \end{cases}$$

Proof of Theorem 7.1.1. To prove Theorem 7.1.1, we have to prove that

$$\lim_{y \to +\infty} \limsup_{n \to +\infty} \mathbf{P}(|M_n - m_n| \geq y, \mathbf{T} \text{ is infinite}) = 0.$$

Using the upper bound of Theorem 7.1.2, we have

$$\limsup_{n \to +\infty} \mathbf{P}(M_n \geq m_n + y) \leq C(1+y)e^{-\theta^* y} \underset{y \to +\infty}{\longrightarrow} 0.$$

To complete the proof, we have to strengthen the lower bound of Theorem 7.1.2, given by

$$\exists c > 0, \forall n \in \mathbb{N}, \forall y \in [0, n^{1/2}], \mathbf{P}(M_n \geq m_n + y) \geq c(1+y)e^{-\theta^* y}.$$

To do so, we observe that with high probability, there is a large number of individuals alive at time k and above some given position. For all $h \geq 0$, we write $N_h = \sum_{|u|=1} \mathbf{1}_{\{V(u) \geq -h\}}$ and μ_h the law of N_h the number of children of a given individual which makes a displacement greater than $-h$. We write

$$f_h = \mathbf{E}(s^{N_h}) \quad \text{and} \quad f = \mathbf{E}\left[s^N\right],$$

where $N = \sum_{|u|=1} 1$ is the total progeny of an individual in the branching random walk. By monotone convergence, we have $f_h(s) \underset{h\to+\infty}{\longrightarrow} f(s)$ for all $s \in [0,1]$. In particular, q_h the smallest solution of $f_h(q_h) = q_h$ converge, as $h \to +\infty$ to q the smallest solution of the equation $f(q) = q$. Moreover, it is well-known that $1 - q = \mathbf{P}(\mathbf{T}$ is infinite). Observe that by (7.1.1), we have necessary $\mathbf{E}(N) > 1$, therefore, we can choose h to be large enough such that $\mathbf{E}(N_h) > \varrho^2 > 1$.

We can easily couple the branching random walk with a Galton-Watson process Z^h with reproduction law μ_h in such a way that $Z_n^h \leq \sum_{|u|=n} \mathbf{1}_{\{V(u)\geq -nh\}}$. Applying Lemma 7.5.1, we have

$$\mathbf{P}\left(\sum_{|u|=k} \mathbf{1}_{\{V(u)\geq -kh\}} \leq \varrho^k\right) \leq q_h + C\beta^k$$

for some $\beta < 1$. As a consequence, for all $\varepsilon > 0$, there exists h large enough and k large enough such that

$$\mathbf{P}\left(\sum_{|u|=k} \mathbf{1}_{\{V(u)\geq -kh\}} \leq \varrho^k\right) \leq q + 2\varepsilon.$$

By Theorem 7.1.2, there exists $\eta > 0$ such that for all $n \in \mathbb{N}$, $\mathbf{P}(M_n \geq m_n) \geq \eta$. But there are with high probability at least ϱ^k individuals alive to the right of $-kh$ at time k, each of which starting an independent branching random walk. Therefore

$$\mathbf{P}\left(M_{n+k} \leq m_n - kh\right) \leq (1-\eta)^{\varrho^k} + \mathbf{P}\left(\sum_{|u|=k} \mathbf{1}_{\{V(u)\geq -kh\}} \leq \varrho^k\right) \leq q + 3\varepsilon,$$

as long as k is chosen large enough.

As a consequence, for all $\varepsilon > 0$, there exists $k \in \mathbb{N}$ and $C > 0$ such that for all $n \geq k$, writing $y = -kh - kv + \frac{3}{2\theta^*}\log 2$ we have $\mathbf{P}(M_{n+k} \geq m_{n+k} - y) \geq 1 - q - \varepsilon$. Therefore

$$\lim_{y\to+\infty} \liminf_{n\to+\infty} \mathbf{P}(M_n \leq m_n - y, \mathbf{T} \text{ is infinite}) = 0,$$

which ends the proof. $\square$

CHAPTER 8

Consistent maximal displacement of the branching random walk

"If all you have is a hammer, everything looks like a nail."

Abraham Maslow – The Psychology of Science

Abstract

We obtain in this chapter the asymptotic behaviour of the consistent maximal displacement of the branching random walk. This quantity is the maximal distance between the boundary of the process, and the individual which stayed the closest to it at any time. This result has been obtained by Fang and Zeitouni [FZ10], Fauraud, Hu and Shi [FHS12] under some stronger integrability assumptions. Roberts [Rob12] computed the second order of the asymptotic for the branching Brownian motion. We provide here only the main asymptotic behaviour, but under particularly light integrability assumptions, using a spinal version of the Mogul'skiĭ estimate.

8.1 Introduction

We consider a branching random walk $(\mathbf{T}, V)$ on $\mathbb{R}$. For $\theta > 0$, we write $\kappa(\theta)$ for the log-Laplace transform of the point process used in the branching random walk for reproduction. As in Chapter 1, we assume the following integrability assumption: there exists $\theta^* > 0$ such that $\kappa(\theta^*) < +\infty$ and

$$\theta^* \mathbf{E}\left[\sum_{|u|=1} V(u) e^{\theta^* V(u) - \kappa(\theta^*)}\right] - \kappa(\theta^*) = 0. \tag{8.1.1}$$

Under this assumption, writing $v = \frac{\kappa(\theta^*)}{\theta^*}$, it has been proved in [Big76] that the maximal displacement at time n in the branching random walk increases at ballistic speed v. We introduce

$$\sigma^2 := \mathbf{E}\left[\sum_{|u|=1} (V(u) - v)^2 e^{\theta^* V(u) - \kappa(\theta^*)}\right] < +\infty, \tag{8.1.2}$$

which is the variance of the spine obtained in the spinal decomposition.

The *consistent maximal displacement* of the branching random walk is the quantity defined as

$$L_n = \min_{|u|=n} \max_{k \leq n} \{kv - V(u_k)\}. \tag{8.1.3}$$

It correspond to the maximal distance between the boundary of the process –the line of slope v– and the individual that stayed as close as possible to this highest position [1]. In order to obtain the a.s. asymptotic behaviour of L_n as $n \to +\infty$, we introduce the following integrability assumption

$$\lim_{x\to+\infty} x^2 \mathbf{E}\left[\sum_{\ell \in L} e^{\theta^*\ell} \mathbf{1}_{\left\{\log\left(\sum_{\ell' \in L} e^{\theta^*(\ell'-\ell)}\right)\geq x\right\}}\right] = 0. \tag{8.1.4}$$

This condition ensure that in the spinal decomposition, the spine at time $n+1$ is one of the rightmost children of the spine at time n [2], thus the behaviour of the size-biased process is similar to the behaviour of the original process. In some sense, it is similar to the $\mathbf{E}(N \log N) < +\infty$ condition for the Galton-Watson process. We note that (8.1.4) is implied by the usual integrability condition:

$$\mathbf{E}\left[\left(\sum_{\ell\in L} e^{\theta^*\ell}\right) \log\left(\sum_{\ell \in L} e^{\theta^*\ell}\right)^2\right] < +\infty. \tag{8.1.5}$$

The following theorem is the main result of the chapter.

Theorem 8.1.1. *Assuming* (8.1.1), (8.1.2) *and* (8.1.4) *hold, we have*

$$\lim_{n\to+\infty} \frac{L_n}{n^{1/3}} = \left(\frac{3\pi^2\sigma^2}{2\theta^*}\right)^{1/3} \quad \text{a.s. on } \{\mathbf{T} \text{ is infinite}\}.$$

The rest of the chapter is organised as follows. In Section 8.2, we introduce the Mogul'skiĭ estimate of the probability for a random walk to have smaller than usual fluctuations around its mean. We then extend this result to random walks with spine. Section 8.3 is devoted to the study of the right tail of L_n, using the spinal decomposition of the branching random walk and the previous random walk estimates. We prove Theorem 8.1.1 in Section 8.4, using the previous estimate and a coupling between the branching random walk and a Galton-Watson process.

8.2 A small deviations estimate

Let $(S_n, n \geq 0)$ be a centred random walk with finite variance $\sigma^2 := \mathbf{E}(S_1^2)$. The Mogul'skiĭ estimate gives the rate of decay of the probability for the random walk of length n to have fluctuations of order a_n, where $a_n = o(n^{1/2})$.

Theorem 8.2.1 (Mogul'skiĭ [Mog74])**.** *Let (a_n) be a sequence of real non-negative numbers such that $\lim_{n\to+\infty} a_n = +\infty$ and $\lim_{n\to+\infty} \frac{a_n^2}{n} = 0$. For any pair of continuous functions f, g such that $f < g$ and $f(0) < 0 < g(0)$, we have*

$$\lim_{n\to+\infty} \frac{a_n^2}{n} \log \mathbf{P}\left(\frac{S_j}{a_n} \in \left[f_{j/n}, g_{j/n}\right], j \leq n\right) = -\frac{\pi^2\sigma^2}{2} \int_0^1 \frac{ds}{(g_s - f_s)^2}.$$

1. The “Talleyrand” of the branching random walk.
2. *With great power there must also come -- great responsibility!* – Amazing Fantasy #15.

This theorem, which admit a number of modifications and extension. For example, it holds with a level of uniformity on the starting point, or on the length of the random walk. We prove here that this theorem also work for random walks enriched by random variables correlated to the last step. We denote by (X_j, ξ_j) i.i.d. random variables taking values in $\mathbb{R}^2$. We assume that

$$\mathbf{E}(X_1) = 0, \quad \sigma^2 := \mathbf{E}(X_1^2) < +\infty. \tag{8.2.1}$$

For $n \in \mathbb{N}$, we set $S_n = \sum_{j=1}^n X_j$. The process we consider is $(S_n, \xi_n, n \in \mathbb{N})$.

Theorem 8.2.2 (Spinal Mogul'skiĭ estimate)**.** *Let (a_n) be a sequence of real numbers, such that $\lim_{n\to+\infty} a_n = +\infty$ and $\lim_{n\to 0} \frac{a_n^2}{n} = 0$. If $\lim_{n\to+\infty} a_n^2 \mathbf{P}(\xi_1 \geq n) = 0$, then for any pair of continuous functions (f, g) such that $f < g$ and $f(0) < 0 < g(0)$, and for all $f_1 \leq x < x' \leq g_1$, we have*

$$\lim_{n\to+\infty} \frac{a_n^2}{n} \log \mathbf{P}\left(S_n \in [xa_n, x'a_n], \frac{S_j}{a_n} \in \left[f_{j/n}, g_{j/n}\right], \xi_j \leq n, j \leq n \right)$$
$$= -\frac{\pi^2\sigma^2}{2} \int_0^1 \frac{ds}{(g_s - f_s)^2}.$$

Proof. Note first that the upper bound of this result is a direct consequence of Theorem 8.2.1. Indeed, we have

$$\limsup_{n\to+\infty} \frac{a_n^2}{n} \log \mathbf{P}\left(S_n \in [xa_n, x'a_n], \frac{S_j}{a_n} \in \left[f_{j/n}, g_{j/n}\right], \xi_j \leq n, j \leq n \right)$$
$$\leq \limsup_{n\to+\infty} \frac{a_n^2}{n} \log \mathbf{P}\left(\frac{S_j}{a_n} \in \left[f_{j/n}, g_{j/n}\right], j \leq n \right) \leq -\frac{\pi^2\sigma^2}{2} \int_0^1 \frac{ds}{(g_s - f_s)^2}.$$

The proof of the lower bound is a modification of the original proof of Mogul'skiĭ. It consists in decomposing the time interval $[0, n]$ into subintervals of length n/A, for a given $A > 0$. On these subintervals, the continuous functions can be approached by constants. The intervals are then truncated in smaller ones, of length $\widetilde{A} a_n^2$. On these small intervals, the random walk can be approached by a Brownian motion, as $n \to +\infty$. Finally, we let $\widetilde{A}$ then A grow to $+\infty$ to conclude the proof.

We choose a continuous function h such that $h(0) = 0$, $h(1) = \frac{x+x'}{2}$ and for all $t \in [0, 1]$, $f_t < h_t < g_t$. We set $\delta \in (0, \frac{x'-x}{8})$ such that for all $t \in [0, 1]$, $f_t + 8\delta < h_t < g_t - 8\delta$. We then choose $A > 0$ such that

$$\sup_{|t-s|\leq \frac{2}{A}} |g_t - g_s| + |f_t - f_s| + |h_t - h_s| \leq \delta. \tag{8.2.2}$$

For $a \leq A$, we set $m_a = \lfloor na/A \rfloor$ and $I_a^{(n)} = \left[(h_{a/A} - 3\delta)a_n, (h_{a/A} + 3\delta)a_n\right]$. Considering only random walks paths that are in intervals $I_a^{(n)}$ at times m_a, and applying the Markov property at times $m_{A-1}, \ldots, m_1$, we obtain

$$\mathbf{P}\left(S_n \in [xa_n, x'a_n], \frac{S_j}{a_n} \in \left[f_{j/n}, g_{j/n}\right], \xi_j \leq n, j \leq n \right)$$
$$\geq \prod_{a=0}^{A-1} \inf_{x \in I_a^{(n)}} \mathbf{P}_x \left[\frac{S_{m_{a+1}-m_a}}{a_n} \in I_{a+1}^{(n)}, \frac{S_j}{a_n} \in \left[f_{a/A} + \delta, g_{a/A} - \delta\right], \xi_j \leq n, j \leq m_{a+1} - m_a \right].$$

By (8.2.2) $\left[(h_{a/A}-2\delta)a_n, h_{a/1}+2\delta)a_n\right] \subset I^{(n)}_{a+1}$ for all $a<A$. Therefore, for $a<A$ we have

$$\inf_{y\in I^{(n)}_a} \mathbf{P}_y\left[S_{m_{a+1}-m_a} \in I^{(n)}_{a+1}, \frac{S_j}{a_n} \in \left[f_{a/A}+\delta, g_{a/A}-\delta\right], \xi_j \leq n, j \leq m_{a+1}-m_a\right]$$
$$\geq R_{n,A,\delta}(f_{a/A}-h_{a/A}+\delta, g_{a/A}-h_{a/A}-\delta), \quad (8.2.3)$$

where, for $\alpha < -6\delta$ and $6\delta < \beta$, we denote by

$$R_{n,A,\delta}(\alpha,\beta) := \inf_{|y|\leq 3\delta a_n} \mathbf{P}_y\left[\frac{S_{m_{a+1}-m_a}}{a_n} \in [-2\delta, 2\delta], \frac{S_j}{a_n} \in [\alpha,\beta], \xi_j \leq n, j \leq m_{a+1}-m_a\right].$$

Let $a<A$. We choose $\widetilde{A}>0$, we set $r_n = \left\lfloor \widetilde{A}a_n^2 \right\rfloor$ and $K = \left\lfloor \frac{n}{Ar_n} \right\rfloor$. We denote by $\tau_k = kr_n$, and by $\Delta_a = m_{a+1}-m_a-\tau_K$. We apply the Markov property at times $\tau_K, \ldots \tau_1$ to obtain

$$R_{n,A,\delta}(\alpha,\beta) \geq \left(\inf_{|y|\leq 3\delta a_n} \mathbf{P}_y\left[\frac{S_{r_n}}{a_n} \in [-\delta,\delta], \frac{S_j}{a_n} \in [\alpha+2\delta, \beta-2\delta], \xi_j \leq n, j \leq r_n\right]\right)^K$$
$$\times \inf_{|y|\leq \delta a_n} \mathbf{P}_y\left[\frac{S_{\Delta_a}}{a_n} \in [-2\delta, 2\delta], \xi_j \leq n, j \leq \Delta_a\right]. \quad (8.2.4)$$

As $\Delta_a \in [0, r_n]$, the probability on the second line can be bounded from below by

$$\inf_{|y|\leq \delta a_n} \mathbf{P}_y\left[\frac{S_j}{a_n} \in [-2\delta, 2\delta], \xi_j \leq n, j \leq r_n\right],$$

which is bounded in a similar way than the other quantities present in (8.2.4).

Indeed, we observe that

$$\inf_{|y|\leq 3\delta a_n} \mathbf{P}_y\left[\frac{S_{r_n}}{a_n} \in [-\delta,\delta], \frac{S_j}{a_n} \in [\alpha+2\delta, \beta-2\delta], \xi_j \leq n, j \leq r_n\right]$$
$$\geq \inf_{|y|\leq 3\delta a_n} \mathbf{P}_y\left[\frac{S_{r_n}}{a_n} \in [-\delta,\delta], \frac{S_j}{a_n} \in [\alpha+2\delta, \beta-2\delta], j \leq r_n\right] - r_n\mathbf{P}(\xi_1 \geq n).$$

As $\lim_{n\to+\infty} r_n\mathbf{P}(\xi_1 \geq n) = 0$ and $r_n \sim_{n\to+\infty} \widetilde{A}a_n^2$, by the Donsker theorem, we have

$$\lim_{n\to+\infty} \inf_{|y|\leq 3\delta a_n} \mathbf{P}_y\left[\frac{S_{r_n}}{a_n} \in [-\delta,\delta], \frac{S_j}{a_n} \in [\alpha+2\delta, \beta-2\delta], j \leq r_n\right]$$
$$= \inf_{|y|\leq 3\delta} \mathbf{P}_y\left[\sigma B_{\widetilde{A}} \in [-\delta,\delta], \sigma B_s \in [\alpha+2\delta, \beta-2\delta], s \leq \widetilde{A}\right],$$

We observe that $K \sim_{n\to+\infty} \frac{n}{A\widetilde{A}a_n^2}$, thus for all $a<A$, (8.2.4) leads to

$$\liminf_{n\to+\infty} \frac{a_n^2}{n} \log R_{n,\delta,A}(\alpha,\beta) \geq \frac{1}{A\widetilde{A}} \inf_{|y|\leq 3\delta} \mathbf{P}_y\left[\sigma B_{\widetilde{A}} \in [-\delta,\delta], \sigma B_s \in [\alpha+2\delta, \beta-2\delta], s \leq \widetilde{A}\right].$$

The probability for a Bronwian motion to stay in a strip admit a closed expression (see e.g. [IM74]), for all $\alpha<\beta$ and $\delta < \min(-\alpha,\beta)/3$,

$$\inf_{|y|\leq 3\delta} \mathbf{P}_y\left[\sigma B_{\widetilde{A}} \in [-\delta,\delta], \sigma B_s \in [\alpha,\beta], s \leq \widetilde{A}\right]$$
$$= \inf_{|y|\leq 3\delta} \int_{-\delta}^{\delta} \frac{2}{\beta-\alpha} \sum_{n=1}^{+\infty} e^{-n^2\frac{\pi^2\sigma^2}{2(\beta-\alpha)^2}\widetilde{A}} \sin\left(n\pi\frac{y-\alpha}{\beta-\alpha}\right) \sin\left(n\pi\frac{z-\alpha}{\beta-\alpha}\right) dz.$$

In particular, letting $\widetilde{A} \to +\infty$, we have

$$\liminf_{n\to+\infty} \frac{a_n^2}{n} \log R_{n,\delta,A}(\alpha,\beta) \geq -\frac{\pi^2\sigma^2}{2A} \frac{1}{(\beta-\alpha-4\delta)}.$$

Using this last inequality, (8.2.3) leads to

$$\liminf_{n\to+\infty} \frac{a_n^2}{n} \log \mathbf{P}\left(S_n \in [xa_n, x'a_n], \frac{S_j}{a_n} \in \left[f_{j/n}, g_{j/n}\right], \xi_j \leq n, j \leq n\right) \\ \geq -\frac{\pi^2\sigma^2}{2A} \sum_{a=0}^{A-1} \frac{1}{(g_{a/A} - f_{a/A} - 6\delta)^2}.$$

Letting $A \to +\infty$ then $\delta \to 0$ concludes the proof. □

8.3 Left tail of the consistent maximal displacement

In this section, we use Theorem 8.2.2 as well as the spinal decomposition of the branching random walk to compute first and second moment estimates. These are used to bound the probability there exists an individual in the branching random walk that stay in a given path. We observe that the $n^{1/3}$ order of the consistent maximal displacement is the correct rate such that the exponential term e^{S_n} in the many-to-one lemma exactly balances the probability $\mathbf{P}(|S_j| \leq n^{1/3}, j \leq n)$. The main result of the section is the computation of the left tail of the consistent maximal displacement. We set

$$\lambda^* := \left(\frac{3\pi^2\sigma^2}{2\theta^*}\right)^{1/3}. \tag{8.3.1}$$

Theorem 8.3.1. *We assume that* (8.1.1), (8.1.2) *and* (8.1.4) *hold. For all* $\lambda < \lambda^*$, *we have*

$$\lim_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\left(L_n - \lambda^* n^{1/3} \leq -\lambda n^{1/3}\right) = -\theta^*\lambda.$$

8.3.1 An upper bound of the right tail of the consistent maximal displacement

We start with a boundary estimate, that gives an upper bound of the right tail of the consistent maximal displacement. We prove that with high probability, every individual that stays above $kv - \lambda n^{1/3}$ at any time $k \leq n$ also stay below $kv - g_{k/n} n^{1/3}$, for a well-chosen function g.

Lemma 8.3.2. *We assume that* (8.1.1) *and* (8.1.2) *hold. For any pair of functions* f, g *such that* $f < g$ *and* $f(0) < 0 < g(0)$, *we set*

$$Z_n(f,g) = \sum_{|u|\leq n} \mathbf{1}_{\left\{\frac{V(u)-|u|v}{n^{1/3}} \geq g_{|u|/n}\right\}} \mathbf{1}_{\left\{\frac{V(u_j)-jv}{n^{1/3}} \in \left[f_{j/n}, g_{j/n}\right], j<|u|\right\}}.$$

We have

$$\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{E}(Z_n(f,g)) \leq -\inf_{t\in[0,1]} \left\{\theta^* g_t + \frac{\pi^2\sigma^2}{2} \int_0^t \frac{ds}{(g_s-f_s)^2}\right\}.$$

Proof. To prove this estimate, we need Theorem 8.2.1 with a kind of uniform control in the length of the random walk. Indeed, applying the many-to-one lemma, we have

$$\begin{aligned}
\mathbf{E}(Z_n(f,g)) &= \sum_{k=1}^n \mathbf{E}\left[\sum_{|u|=k} \mathbf{1}_{\left\{\frac{V(u)-kv}{n^{1/3}} \geq g_{k/n}\right\}} \mathbf{1}_{\left\{\frac{V(u_j)-jv}{n^{1/3}} \in \left[f_{j/n}, g_{j/n}\right], j<k\right\}}\right] \\
&= \sum_{k=1}^n \mathbf{E}\left[e^{-\theta^* S_k + k\kappa(\theta^*)} \mathbf{1}_{\left\{\frac{S_k-kv}{n^{1/3}} \geq g_{k/n}\right\}} \mathbf{1}_{\left\{\frac{S_j-jv}{n^{1/3}} \in \left[f_{j/n}, g_{j/n}\right], j<k\right\}}\right] \\
&\leq \sum_{k=1}^n e^{-\theta^* g_{k/n} n^{1/3}} \mathbf{P}\left(\frac{S_j - jv}{n^{1/3}} \in \left[f_{j/n}, g_{j/n}\right], j < k\right).
\end{aligned}$$

We observe that $(S_n - nv, n \geq 0)$ is a centred random walk, thanks to (8.1.1), with finite variance by (8.1.2).

To obtain an uniform bound, we choose $A \in \mathbb{N}$, and divide $[0, n]$ into A intervals of length $\frac{n}{A}$. For $a \leq A$, we set $m_a = \lfloor \frac{na}{A} \rfloor$ and

$$\overline{g}_{a,A} = \sup_{s \in \left[\frac{a-1}{A}, \frac{a+2}{A}\right]} g_s.$$

We observe that for all $k \in (m_a, m_{a+1}]$,

$$\begin{aligned}
&e^{-\theta^* g_{k/n} n^{1/3}} \mathbf{P}\left(\frac{S_j - jv}{n^{1/3}} \in \left[f_{j/n}, g_{j/n}\right], j < k\right) \\
&\qquad\qquad \leq e^{-\theta^* \overline{g}_{a,A} n^{1/3}} \mathbf{P}\left(\frac{S_j - jv}{n^{1/3}} \in \left[f_{j/n}, g_{j/n}\right], j \leq m_a\right),
\end{aligned}$$

therefore

$$\mathbf{E}(Z_n(f,g)) \leq \frac{2n}{A} \sum_{a=0}^{A-1} e^{-\theta^* g_{k/n} n^{1/3}} \mathbf{P}\left(\frac{S_j - jv}{n^{1/3}} \in \left[f_{j/n}, g_{j/n}\right], j \leq m_a\right).$$

We now apply A times Theorem 8.2.1, which leads to

$$\limsup_{n \to +\infty} \frac{1}{n^{1/3}} \log \mathbf{E}(Z_n(f,g)) \leq \max_{a<A} \left\{-\theta^* \overline{g}_{a,A} - \frac{\pi^2 \sigma^2}{2} \int_0^{\frac{a}{A}} \frac{ds}{(g_s - f_s)^2}\right\}.$$

Letting $A \to +\infty$ concludes the proof. □

We now compute the expected number of individuals that stayed at all times between the curves $kv + n^{1/3} f_{k/n}$ and $kv + n^{1/3} g_{k/n}$.

Lemma 8.3.3. *We assume that* (8.1.1) *and* (8.1.2) *hold. For any pair of functions f, g such that $f < g$ and $f(0) < 0 < g(0)$ and for all $x \in [f_1, g_1)$, we set*

$$Y_n(f,g) = \sum_{|u|=n} \mathbf{1}_{\left\{\frac{V(u)-nv}{n^{1/3}} \geq x\right\}} \mathbf{1}_{\left\{\frac{V(u_j)-jv}{n^{1/3}} \in \left[f_{j/n}, g_{j/n}\right], j \leq n\right\}}.$$

We have

$$\lim_{n \to +\infty} \frac{1}{n^{1/3}} \log \mathbf{E}(Y_n(f,g,x)) = -\theta^* x - \frac{\pi^2 \sigma^2}{2} \int_0^1 \frac{ds}{(g_s - f_s)^2}.$$

Proof. This lemma is a direct consequence of the many-to-one lemma and Theorem 8.2.2 when choosing $\xi_n = 0$. Indeed, we have

$$\mathbf{E}\left(Y_n(f,g,x)\right) = \mathbf{E}\left[e^{-\theta^* S_n + n\kappa(\theta^*)} \mathbf{1}_{\left\{ \frac{S_n - nv}{n^{1/3}} \geq x \right\}} \mathbf{1}_{\left\{ \frac{S_j - jv}{n^{1/3}} \in \left[f_{j/n}, g_{j/n} \right], j \leq n \right\}} \right].$$

Consequently, we obtain easily the following upper bound

$$\begin{aligned} \limsup_{n\to+\infty} \frac{1}{n^{1/3}} &\log \mathbf{E}\left(Y_n(f,g,x)\right) \\ &\leq -\theta^* x + \limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\left(\frac{S_j - jv}{n^{1/3}} \in \left[f_{j/n}, g_{j/n} \right], j \leq n \right) \\ &\leq -\theta^* x - \frac{\pi^2\sigma^2}{2} \int_0^1 \frac{ds}{(g_s - f_s)^2}. \end{aligned}$$

The lower bound is derived in a similar way. We set $\varepsilon > 0$ such that $x + \varepsilon < g_1$. We have

$$\begin{aligned} \mathbf{E}\left(Y_n(f,g,x)\right) &\geq \mathbf{E}\left[e^{-\theta^* S_n + n\kappa(\theta^*)} \mathbf{1}_{\left\{ \frac{S_n - nv}{n^{1/3}} \in [x, x+\varepsilon] \right\}} \mathbf{1}_{\left\{ \frac{S_j - jv}{n^{1/3}} \in \left[f_{j/n}, g_{j/n} \right], j \leq n \right\}} \right] \\ &\geq e^{-\theta^*(x+\varepsilon)n^{1/3}} \mathbf{P}\left(\frac{S_n - nv}{n^{1/3}} \in [x, x+\varepsilon], \frac{S_j - jv}{n^{1/3}} \in \left[f_{j/n}, g_{j/n} \right], j \leq n \right). \end{aligned}$$

We apply Theorem 8.2.2, that leads to

$$\liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{E}\left(Y_n(f,g,x)\right) \geq -\theta^*(x+\varepsilon) - \frac{\pi^2\sigma^2}{2} \int_0^1 \frac{ds}{(g_s - f_s)^2}.$$

Letting $\varepsilon \to 0$ concludes the proof. □

These two lemmas can be used to obtain the upper bound of Theorem 8.3.1.

Upper bound of Theorem 8.3.1. Let $\lambda > 0$. For all $n \in \mathbb{N}$, we have

$$\begin{aligned} \mathbf{P}\left(L_n \leq \lambda n^{1/3} \right) &= \mathbf{P}\left(\exists |u| = n : \forall j \leq n, V(u_j) \geq jv - \lambda n^{1/3} \right) \\ &\leq \mathbf{P}\left(Z_n(-\lambda, g) > 0 \right) + \mathbf{P}\left(Y_n(-\lambda, g, -\lambda) > 0 \right), \end{aligned}$$

for any continuous function g such that $g(0) > 0$ and $g > -\lambda$. Applying the Markov inequality and Lemma 8.3.2, we have

$$\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\left(Z_n(-\lambda, g) > 0 \right) \leq - \inf_{t\in[0,1]} \left\{ \theta^* g_t + \frac{\pi^2\sigma^2}{2} \int_0^t \frac{ds}{(g_s + \lambda)^2} \right\}. \tag{8.3.2}$$

Similarly, we have

$$\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\left(Y_n(-\lambda, g, -\lambda) > 0 \right) \leq \theta^* \lambda - \frac{\pi^2\sigma^2}{2} \int_0^1 \frac{ds}{(g_s + \lambda)^2}. \tag{8.3.3}$$

Given $\lambda > 0$, we now choose a convenient function g, that satisfies

$$\forall t \in [0,1], -\theta^* g_t - \frac{\pi^2\sigma^2}{2} \int_0^t \frac{ds}{(g_s + \lambda)^2} = -\theta^* g_0. \tag{8.3.4}$$

We observe that this differential solution admit the following solution:

$$g_t = \left((g_0+\lambda)^3 - (\lambda^*)^3\, t\right)^{1/3} - \lambda.$$

Let $\varepsilon > 0$, we define $g_t = \left((1-t)\,(\lambda^*)^3 + \varepsilon^3\right)^{1/3} - \lambda$, which is a solution of (8.3.4) with the condition $g_1 = \varepsilon - \lambda$. With this definition, (8.3.2) becomes

$$\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\left(Z_n(-\lambda, g) > 0\right) \leq -\theta^* \left[\left((\lambda^*)^3 + \varepsilon^3\right)^{1/3} - \lambda\right],$$

and (8.3.3) gives

$$\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\left(Y_n(-\lambda, g, -\lambda) > 0\right) \leq \theta^*\varepsilon - \theta^* \left[\left((\lambda^*)^3 + \varepsilon^3\right)^{1/3} - \lambda\right].$$

Letting $\varepsilon \to 0$, we obtain

$$\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\left(L_n \leq \lambda n^{1/3}\right) \leq -\theta^* \left[\lambda^* - \lambda\right].$$

We conclude that

$$\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\left(L_n \leq (\lambda^* - \lambda) n^{1/3}\right) \leq -\theta^* \lambda.$$

□

8.3.2 A lower bound by a second moment method

To obtain the lower bound of Theorem 8.3.1, we bound from below the probability there exists an individual that is above $kv - \lambda n^{1/3}$ at any time $k \leq n$. Computing the first two moments of the number of such individuals and applying the Cauchy-Schwarz inequality concludes the proof. Let (f, g) be a pair of continuous functions such that $f < g$ and $f(0) < 0 < g(0)$. We write

$$\mathcal{A}_n(f,g) = \left\{|u| = n : \forall j \leq n, V(u_j) - jv \in \left[f_{j/n} n^{1/3}, g_{j/n} n^{1/3}\right]\right\}.$$

However, a direct second moment method would not lead to a convenient lower bound. To obtain a more precise upper bound, we have to control the reproduction of the individuals we consider. For $u \in \mathbf{T}$, we write $\Upsilon(u)$ the set of siblings of u and

$$\xi(u) = \sum_{u' \in \Upsilon(u)} e^{\theta^*(V(u') - V(u))}, \tag{8.3.5}$$

that is used to control the reproduction along the spine. Note that (8.1.4) is equivalent to

$$\lim_{x\to+\infty} x^2 \widehat{\mathbf{P}}\left(\log\left(\xi(w_1)\right) \geq x\right) = 0. \tag{8.3.6}$$

For all $\delta > 0$, we denote by

$$\mathcal{B}_n^\delta = \left\{|u| = n : \forall j \leq n, \log \xi(u_j) \leq \delta n^{1/3}\right\}.$$

Lemma 8.3.4. *We assume (8.1.1), (8.1.2) and (8.1.4) hold. For any pair of functions f, g such that $f < g$ and $f(0) < 0 < g(0)$, for all $x \in [f_1, g_1]$ and $\delta > 0$, we set*

$$X_n^\delta(f,g,x) = \sum_{|u|=n} \mathbf{1}_{\left\{u \in \mathcal{A}_n(f,g) \cap \mathcal{B}_n^\delta\right\}} \mathbf{1}_{\left\{\frac{V(u)-nv}{n^{1/3}} \geq x\right\}}.$$

We have

$$\liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{E}\left[X_n^\delta(f,g,x)\right] \geq -\theta^* x - \frac{\pi^2\sigma^2}{2} \int_0^1 \frac{ds}{(g_s - f_s)^2},$$

as well as

$$\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{E}\left[\left(X_n^\delta(f,g,x)\right)^2\right] \leq \theta^*\delta - 2\left(\theta^* x + \frac{\pi^2\sigma^2}{2}\int_0^1 \frac{ds}{(g_s-f_s)^2}\right) + \sup_{t\in[0,1]}\left\{\theta^* g_t + \frac{\pi^2\sigma^2}{2}\int_0^t \frac{ds}{(g_s-f_s)^2}\right\}.$$

Proof. We first bound from below the mean of X_n^δ. We set $W_n = \sum_{|u|=n} e^{\theta^* V(u) - n\kappa(\theta^*)}$. Applying the spinal decomposition, we have

$$\begin{aligned}
\mathbf{E}\left[X_n^\delta(f,g,x)\right] &= \overline{\mathbf{E}}\left[\frac{1}{W_n}\sum_{|u|=n} \mathbf{1}_{\left\{u \in \mathcal{A}_n(f,g)\cap\mathcal{B}_n^\delta\right\}} \mathbf{1}_{\left\{\frac{V(u)-nv}{n^{1/3}} \geq x\right\}}\right]\\
&= \widehat{\mathbf{E}}\left[e^{-\theta^* V(w_n) + n\kappa(\theta^*)} \mathbf{1}_{\left\{w_n \in \mathcal{A}_n(f,g)\cap\mathcal{B}_n^\delta\right\}} \mathbf{1}_{\left\{\frac{V(w_n)-nv}{n^{1/3}} \geq x\right\}}\right].
\end{aligned}$$

As a consequence, for all $\varepsilon > 0$ small enough, we have

$$\mathbf{E}\left[X_n^\delta(f,g,x)\right] \geq e^{-\theta^*\varepsilon n^{1/3}} \widehat{\mathbf{P}}\left[\frac{V(w_n)-nv}{n^{1/3}} \in [x, x+\varepsilon], w_n \in \mathcal{A}_n(f,g) \cap \mathcal{B}_n^\delta\right].$$

By (8.1.1) and (8.1.2), $(V(w_n), \xi(w_n), n \in \mathbb{N})$ satisfies (8.2.1). Moreover, by (8.3.6), we have

$$\lim_{n\to+\infty} n^{2/3}\widehat{\mathbf{P}}\left[\log \xi(w_0) \geq \delta n^{1/3}\right] = 0.$$

Therefore, applying Theorem 8.2.2, we have

$$\liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{E}\left[X_n^\delta(f,g,x)\right] \geq -\theta^* x - \frac{\pi^2\sigma^2}{2}\int_0^1 \frac{ds}{(g_s-f_s)^2}.$$

In a second time, we bound from above the second moment of X_n^δ. Applying once again the spinal decomposition, we have

$$\begin{aligned}
\mathbf{E}\left[\left(X_n^\delta(f,g,x)\right)^2\right] =&\overline{\mathbf{E}}\left[\frac{X_n^\delta(f,g,x)}{W_n}\sum_{|u|=n} \mathbf{1}_{\left\{u\in\mathcal{A}_n(f,g)\cap\mathcal{B}_n^\delta\right\}} \mathbf{1}_{\left\{\frac{V(u)-nv}{n^{1/3}}\geq x\right\}}\right]\\
=&\widehat{\mathbf{E}}\left[X_n^\delta(f,g,x) e^{-\theta^* V(w_n)+n\kappa(\theta^*)} \mathbf{1}_{\left\{w_n\in\mathcal{A}_n(f,g)\cap\mathcal{B}_n^\delta\right\}} \mathbf{1}_{\left\{\frac{V(w_n)-nv}{n^{1/3}}\geq x\right\}}\right]\\
\leq& e^{-\theta^* x n^{1/3}} \widehat{\mathbf{E}}\left[X_n^\delta(f,g,x) \mathbf{1}_{\left\{w_n\in\mathcal{A}_n(f,g)\cap\mathcal{B}_n^\delta\right\}} \mathbf{1}_{\left\{\frac{V(w_n)-nv}{n^{1/3}}\geq x\right\}}\right]
\end{aligned}$$

Under the law $\widehat{\mathbf{P}}$, we decompose $X_n^\delta(f,g,x)$ along the spine, we have

$$X_n^\delta(f,g,x) \leq 1 + \sum_{k=1}^n \sum_{u \in \Upsilon(w_k)} \Lambda_k(u),$$

setting, for $k \leq n$ and $u \in \Upsilon(w_k)$,

$$\Lambda_k(u) = \sum_{|u'|=n, u' \geq u} \mathbf{1}_{\{u' \in \mathcal{A}_n(f,g)\}} \mathbf{1}_{\left\{\frac{V(u')-nv}{n^{1/3}} \geq x\right\}}.$$

We set $\mathcal{G} = \sigma(V(w_n), \Omega(w_n), V(u), u \in \Omega(w_n), n \geq 0)$ and $I_k^{(n)} = \left[f_{k/n} n^{1/3}, g_{k/n} n^{1/3}\right]$. We compute, conditionally on $\mathcal{G}$ the value of $\Lambda_k(u)$. Applying the many-to-one lemma, for all $k \leq n$ and $u \in \Upsilon(w_k)$, we have

$$\begin{aligned}
&\mathbf{E}\left[\Lambda_k(u) \middle| \mathcal{G}\right] \\
=&e^{\theta^* V(u) - k\kappa(\theta^*)} \mathbf{E}_{V(u)}\left[e^{-\theta^* S_{n-k} - n\kappa(\theta^*)} \mathbf{1}_{\left\{\frac{S_{n-k}-nv}{n^{1/3}} \geq x\right\}} \mathbf{1}_{\left\{S_j - (k+j)v \in I_{k+j}^{(n)}, j \leq n-k\right\}}\right] \\
\leq& e^{\theta^* x n^{1/3}} e^{\theta^* V(u) - k\kappa(\theta^*)} \mathbf{P}_{V(u)}\left[\frac{S_{n-k}-nv}{n^{1/3}} \geq x, S_j - (k+j)v \in I_{k+j}^{(n)}, j \leq n-k\right].
\end{aligned}$$

We set $A \in \mathbb{N}$, and for $a \leq A$, $m_a = \lfloor na/A \rfloor$. We also introduce

$$\Phi_{a,A}^{\text{end}} = \sup_{x \in I_{m_a}^{(n)}} \mathbf{P}_x\left(S_j - jv \in I_{m_a+j}^{(n)}, j \leq n - m_a\right).$$

Applying the Markov property at time m_{a+1}, for all $k \leq m_{a+1}$ and $u \in \Upsilon(w_k)$, we have

$$\mathbf{E}\left[\Lambda_k(u) \middle| \mathcal{G}\right] \leq e^{-\theta^* x n^{1/3}} e^{\theta^* V(u) - k\kappa(\theta^*)} \Phi_{a+1,A}^{\text{end}}.$$

For $a \leq A$, we now compute the asymptotic behaviour as $n \to +\infty$ of

$$R_{a,A}^{(n)} = \sup_{k \in [m_a, m_{a+1})} \widehat{\mathbf{E}}\left[\left(\sum_{u \in \Upsilon(w_k)} \Lambda_k(u)\right) \mathbf{1}_{\left\{\frac{V(w_n)-nv}{n^{1/3}} \geq x\right\}} \mathbf{1}_{\left\{w_n \in \mathcal{A}_n(f,g) \cap \mathcal{B}_n^\delta\right\}}\right].$$

We have

$$\begin{aligned}
R_{a,A}^{(n)} \leq& e^{-\theta^* x n^{1/3}} \Phi_{a+1,A}^{\text{end}} \widehat{\mathbf{E}}\left[e^{\theta^* V(w_{k-1}) - k\kappa(\theta^*)} \mathbf{1}_{\{w_n \in \mathcal{A}_n(f,g)\}} \xi(w_k) \mathbf{1}_{\left\{\log \xi(w_k) \leq \delta n^{1/3}\right\}}\right] \\
\leq& C e^{(-\theta^* x + \theta^* g_{(k-1)/n} + \delta) n^{1/3}} \Phi_{a+1,A}^{\text{end}} \widehat{\mathbf{P}}\left[w_n \in \mathcal{A}_n(f,g)\right].
\end{aligned}$$

We set $\overline{g}_{a,A} = \sup_{s \in [\frac{a-1}{A}, \frac{a+2}{A}]} g_s$ for all $k \in [m_a, m_{a+1})$, we have

$$R_{a,A}^{(n)} \leq e^{(-\theta^* x + \theta^* g_{k/n} + \delta) n^{1/3}} \Phi_{a+1,A}^{\text{end}} \mathbf{P}\left[S_j - jv \in I_j^{(n)}, j \leq n\right].$$

Applying Theorem 8.2.1, we obtain

$$\begin{aligned}
&\limsup_{n \to +\infty} \frac{1}{n^{1/3}} \sup_{k \in [m_a, m_{a+1})} \log R_{a,A}^{(n)} \\
&\qquad \leq -\theta^* x + \theta^* \overline{g}_{a,A} + \delta - \frac{\pi^2 \sigma^2}{2}\left(\int_0^1 \frac{ds}{(g_s - f_s)^2} + \int_{\frac{a+1}{A}}^1 \frac{ds}{(g_s - f_s)^2}\right).
\end{aligned}$$

We finally observe that

$$\mathbf{E}\left[\left(X_n^\delta(f,g,x)\right)^2\right] \leq \mathbf{E}\left[Y_n(f,g,x)\right] + \frac{n}{A}e^{-\theta^* x n^{1/3}} \sum_{a=0}^{A-1} R_{a,A}^{(n)}.$$

Therefore, we apply Lemma 8.3.3, then let $A \to +\infty$, which concludes the proof. □

We use this lemma to prove the lower bound of Theorem 8.3.1.

Lower bound of Theorem 8.3.1. We keep the notation of Lemma 8.3.4. Let f, g be a pair of continuous functions with $f < g$ and $f_0 < 0 < g_0$, and $x \in [f_1, g_1)$ Applying the Cauchy-Schwarz inequality, for all $\delta > 0$, we have

$$\mathbf{P}\left(X_n^\delta(f,g,x) \geq 1\right) \geq \frac{\mathbf{E}\left(X_n^\delta(f,g,x)\right)^2}{\mathbf{E}\left[X_n^\delta(f,g,x)^2\right]}.$$

Therefore, by Lemma 8.3.4, we have

$$\liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\left(X_n^\delta(f,g,x) \geq 1\right) \geq -\delta - \sup_{t\in[0,1]} \theta^* g_t - \frac{\pi^2\sigma^2}{2} \int_0^t \frac{ds}{(g_s - f_s)^2}.$$

Let $\lambda < \lambda^*$ and $\varepsilon > 0$. For $t \in [0,1]$, we set $f_t = -\lambda$ and $g_t = \left((1-t)\frac{3\pi^2\sigma^2}{2\theta^*} + \varepsilon^3\right)^{1/3} - \lambda$, by the same computations as in the proof of the upper bound, for all $\delta > 0$, we have

$$\begin{aligned}
&\liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\left(L_n \leq \lambda n^{1/3}\right) \\
&\qquad \geq \liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\left(\exists |u| = n : \forall k \leq n, V(u_k) - kv \geq -\lambda n^{1/3}\right) \\
&\qquad \geq \liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\left(X_n^\delta(f,g,-\lambda) \geq 1\right) \\
&\qquad \geq -\delta - \theta^* \left((\lambda^*)^3 + \varepsilon^3\right)^{1/3} - \lambda.
\end{aligned}$$

Letting $\varepsilon \to 0$ and $\delta \to 0$, for all $\lambda < \lambda^*$, we have

$$\liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\left(L_n \leq (\lambda^* - \lambda) n^{1/3}\right) \geq -\theta^* \lambda.$$

□

8.4 Proof of Theorem 8.1.1

This section is very similar to Section 7.5 of Chapter 7. Indeed, we obtained in the previous section the asymptotic behaviour of the tail distribution of L_n, and we offer to strengthen it into a concentration estimate. In order to do so, we use the fact that the tree $\mathbf{T}$ is a supercritical Galton-Watson tree. Therefore, at time $\delta n^{1/3}$, there are at least $e^{\varepsilon n^{1/3}}$ individuals alive, each of which starting an independent branching random walk. We then conclude with the lower bound of Theorem 8.3.1.

Proof of Theorem 8.1.1. We observe first that for all $\lambda < \lambda^*$, by Theorem 8.3.1, we have

$$\sum_{n\in\mathbb{N}} \mathbf{P}\left(L_n \leq (\lambda^* - \lambda) n^{1/3}\right) < +\infty.$$

Thus, by the Borel-Cantelli lemma, we have

$$\liminf_{n\to+\infty} \frac{L_n}{n^{1/3}} \geq \lambda^* \quad \text{a.s.}$$

We now prove the lower bound holds a.s. on the survival event of the branching random walk. We note first that by Theorem 8.3.1, for all $\delta > 0$,

$$\liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbf{P}\left(L_n \leq \lambda^* n^{1/3}\right) \geq -\theta^*\delta. \tag{8.4.1}$$

Let $h > 0$. We write $N = \sum_{|u|=1} 1$ and $N_h = \sum_{|u|=1} \mathbf{1}_{\{V(u)\geq -h\}}$, which are respectively the number of children and the number of children to the right of $-h$ of the ancestor of the branching random walk. We set

$$f_h = \mathbf{E}\left(s^{N_h}\right) \quad \text{and} \quad f = \mathbf{E}\left(s^{N}\right).$$

We set q_h the smallest solution of the equation $f_h(s) = s$ and q the smallest solution of the equation $f(s) = s$. Note that by (8.1.1), $\mathbf{E}(N) > 1$, thus $q < 1$. Moreover, for all h large enough, by monotone convergence, we have $\mathbf{E}(N_h) > 1$, and $q_h < 1$.

We observe that $\mathbf{T}$ is a Galton-Watson tree with reproduction law N. By standard Galton-Watson process theory, we have $\mathbf{P}(\mathbf{T} \text{ is infinite}) = 1 - q$. For $h > 0$, we introduce the tree

$$\mathbf{T}^{(h)} = \{\emptyset\} \cup \left\{u \in \mathbf{T} : u \neq \emptyset, \pi u \in \mathbf{T}^{(h)}, V(u) - V(\pi u) \geq -h\right\}.$$

We observe that $\mathbf{T}^{(h)}$ is the random tree of the individuals in $\mathbf{T}$ in which there is no jump smaller than $-h$. Note that $\mathbf{T}^{(h)}$ is a Galton-Watson tree with reproduction law N_h. Therefore, $\mathbf{P}(\mathbf{T}^{(h)} \text{ is infinite}) = 1 - q_h$. Moreover, by monotone convergence, as $h \to +\infty$, we have $f_h \to f$, thus $q_h \to q$.

We set $S_h = \{\mathbf{T}^{(h)} \text{ is infinite}\}$ and $S = \{\mathbf{T} \text{ is infinite}\}$, we observe that for all $h < h'$, we have

$$S_h \subset S_{h'} \subset S,$$

as $\lim_{h\to+\infty} \mathbf{P}(S_h) = \mathbf{P}(S)$, $S = \cup_{h\geq 0} S_h$ up to a negligible event.

We now apply Lemma 7.5.1 to bound from below the number of individuals alive in $\mathbf{T}^{(h)}$ conditionally to the survival of the process. We choose $\varrho > 1$ such that $\mathbf{E}(N) > \varrho^2 > 1$. For all $h > 0$ large enough, we have $\mathbf{E}(N^h) > \varrho^2$. Therefore, there exists $\beta_h < 1$ and $C_h > 0$ such that for all $k \in \mathbb{N}$

$$\mathbf{P}\left(\#\left\{u \in \mathbf{T}^{(h)} : |u| = k\right\} \leq \varrho^k\right) \leq q_h + C_h \beta_h^k.$$

We set $h > 0$ chosen large enough and bound from above $\mathbf{P}(L_n \leq \lambda n^{1/3}, S_h)$. Note that every individual alive at time k in $\mathbf{T}^{(h)}$ is above $-kh$. For all $\lambda > \lambda^*$, applying the Markov property at time k leads to

$$\begin{aligned}\mathbf{P}\left(L_n \geq \lambda n^{1/3}, S_h\right) &\leq \mathbf{P}\left(\#\left\{u \in \mathbf{T}^{(h)} : |u| = k\right\} \leq \varrho^k, S_h\right) + \mathbf{P}\left(L_{n-k} \geq \lambda n^{1/3} - kh\right)^{\varrho^k}\\ &\leq C_h \beta_h^k + \left(1 - \mathbf{P}\left(L_{n-k} \leq \lambda n^{1/3} - kh\right)\right)^{\varrho^k}.\end{aligned}$$

Let $\varepsilon > 0$, we set $k = \left\lfloor \varepsilon n^{1/3} \right\rfloor$, we have

$$\mathbf{P}\left(L_n \geq \lambda n^{1/3}, S_h\right) \leq C_h \beta_h^{\varepsilon n^{1/3}} + \left(1 - \mathbf{P}\left(L_{n-\varepsilon n^{1/3}} \leq (\lambda - \varepsilon) n^{1/3}\right)\right)^{\varrho^{\varepsilon n^{1/3}}}.$$

For all $\varepsilon > 0$, we set $\delta = \frac{\varepsilon \log \varrho}{2\theta^*}$ and $\lambda = \lambda^* + \varepsilon$. By (8.4.1), we have

$$\log\left[-\log\left(\left(1 - \mathbf{P}\left(L_{n-\varepsilon n^{1/3}} \leq (\lambda - \varepsilon)n^{1/3}\right)\right)^{\varrho^{\varepsilon n^{1/3}}}\right)\right] \sim_{n\to+\infty} \frac{\varepsilon \log \varrho}{2} n^{1/3}.$$

We conclude that $\sum_{n\in\mathbb{N}} \mathbf{P}\left(L_n \geq (\lambda^* + \varepsilon)n^{1/3}\right) < +\infty$, thus by Borel-Cantelli lemma again, for all $h > 0$ large enough and $\varepsilon > 0$,

$$\limsup_{n\to+\infty} \frac{L_n}{n^{1/3}} \leq (\lambda^* + \varepsilon) \quad \text{a.s. on } S_h.$$

As $S = \cup_{h>0} S_h$, letting $\varepsilon \to 0$ and $h \to +\infty$, we conclude that

$$\limsup_{n\to+\infty} \frac{L_n}{n^{1/3}} \leq \lambda^* \quad \text{a.s. on } S.$$

which ends the proof. □

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