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SMALL DEVIATIONS IN A SPACE OF TRAJECTORIES

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Introduction

Consider the sequence

$$\xi_1, \xi_2, \cdots, \xi_n, \cdots \tag{1}$$

of independent identically distributed random variables. If the expectation $\mathbf{E}\xi_1$ exists, we shall take it to be equal to 0. We denote by $F_\alpha(x)$ any strictly stable distribution function (d.f.) with parameter $\alpha \in (0, 2]$[1]. We shall suppose that for some sequence $B(n)$ the weak convergence

$$\mathbf{P}(\xi_1 + \cdots + \xi_n < xB(n)) \Rightarrow F_\alpha(x) \tag{2}$$

holds as $n \to \infty$. This is equivalent to saying that the d.f. $F(x) = \mathbf{P}(\xi_i < x)$ belongs to the domain of attraction of the strictly stable d.f. $F_\alpha(x)$ and, moreover, in the case $\alpha = 1$

$$\beta_n = \int_{-\infty}^{\infty} \sin(tB^{-1}(n))F(dt) = o\left(\frac{1}{n}\right)$$

as $n \to \infty$ (see Theorem 2 of § 5 in Chapter XVII in [1]). The function

$$L^*(u) = u^{\alpha-2} \int_{-u}^{u} t^2 F(dt) \tag{3}$$

changes slowly; the sequence $B(n)$ differs from $n^{1/\alpha}$ by a slowly varying factor, and for some d, $0 < d < \infty$, we have

$$B^*(B(n)) \backsim dn \tag{4}$$

as $n \to \infty$, where $B^*(u) = u^\alpha / L^*(u)$. Without loss of generality we may assume that $d = 1$.

We denote by $D(0, 1)$ the space of real functions $f(t)$, $0 \leqq t \leqq 1$, which have no discontinuities of the second kind, and are right continuous at all points t, $0 \leqq t < 1$, and left continuous at $t = 1$, with the Skorokhod metric $\rho_D(f, g)$ (see [2]).

[1] A stable d.f. $F_\alpha(x)$ is said to be strictly stable if, for $k = 1, 2, \cdots$, $F_\alpha(x) = F_\alpha^{(*k)}(xk^{-1/\alpha})$, where the superscript $(*k)$ denotes the k-fold convolution of the d.f. (see [1], p. 166).

For the sequence (1) and any sequence $\{x(n)\}_{n=1}^{\infty}$ of positive numbers such that $x(n)\to\infty$ as $n\to\infty$, we define a sequence of random functions $s_n\in D(0, 1)$, putting

$$s_n(t)=(\xi_0+\cdots+\xi_k)x^{-1}(n) \quad \text{for } k/n\leqq t<(k+1)n, \tag{5}$$

where $k=0, 1, \cdots, n-1$, and $\xi_0=0$.

We denote by $\xi(t)$, $t\geqq 0$, a homogeneous process with independent increments, for which $\mathbf{P}(\xi(t)<x)=F_\alpha(xt^{-1/\alpha})$ for $t>0$ (we call such processes strictly stable). We assume that the sample trajectories of the process $\xi(t)$ are right continuous with probability 1.

Suppose that the sequence $\{x(n)\}$ in (5) has the form $x(n)=B(n)=n^{1/2}L(n)$ (see (2)). Then, in view of well-known theorems on the convergences of processes (see, e.g., [2] and [3]), for any Borel set $G\subseteq D(0, 1)$ for which $\mathbf{P}(\xi(\cdot)\in\partial G)=0$, where ∂G is the boundary of the set G,

$$\mathbf{P}(s_n(\cdot)\in G)\to\mathbf{P}(\xi(\cdot)\in G) \quad \text{as } n\to\infty. \tag{6}$$

The problem of studying the asymptotics of the sequence

$$\mathbf{P}(s_n(\cdot)\in G) \tag{7}$$

in the domain of large deviations, when $x(n)/B(n)\to\infty$ as $n\to\infty$, has been considered by many authors, but only for sets G of narrow special classes, for example, classes of the form $\{f:\sup_{0\leqq t\leqq 1}f(t)<y\}$ and certain others. The extension in (7) of the class of sets G in question, along with a weakening of the restrictions on the sequence (1), leads to the consideration of a cruder problem, that of studying the asymptotic behavior of the sequence

$$\log\mathbf{P}(s_n(\cdot)\in G). \tag{8}$$

In the domain of large deviations this and analogous problems were considered by A. A. Borovkov in [4], A. D. Vent-tsel' in [5], and the author in [6] (we note that some of the results of the present paper were presented in [6]).

In the present paper we study the asymptotic behavior of the sequence (8) in the case when $x(n)/B(n)\to 0$ as $n\to\infty$. This is the problem of studying the asymptotics of the probabilities of small deviations for $s_n(t)$ in the space of trajectories. In this case, for a certain class of sets $G\subseteq D(0, 1)$, including strips with curvilinear boundaries, the sequence (8) behaves like $CH_\alpha(G)nx^{-\alpha}(n)L^*(x(n))$, where the slowly varying function $L^*(u)$ is defined in (3), the functional $H_\alpha(G)$, $0<H_\alpha(G)<\infty$, is constructed explicitly, and the constant C, $-\infty<C<0$, depends only on the d.f. F_α (Theorem 1). In the case $\alpha=2$, it has been possible to calculate the constant C (Theorem 3). Analogous results are formulated for sequences of processes with independent increments (Theorem 4).

1. Formulation of the Basic Results

Suppose two sequences $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ of positive (negative) numbers are given. We shall write $a_n\gtrsim b_n$, if

$$\limsup_{n\to\infty} a_n^{-1}b_n\leqq 1 \qquad (\liminf_{n\to\infty} a_n^{-1}b_n\geqq 1).$$

If both $a_n \gtrsim b_n$ and $b_n \gtrsim a_n$, we write $a_n \backsim b_n$. Analogously, for positive (negative) functions $a(u), b(u), c(u), 0<u\leqq U$, we write $a(u)\gtrsim b(u)$ or $a(u) \backsim c(u)$ as $u\downarrow 0$, if for any sequence $u(n)\downarrow 0$ as $n\to\infty$ we have respectively $a(u(n))\gtrsim b(u(n))$, $a(u(n)) \backsim c(u(n))$.

We now introduce the classes of sets $G\subseteq D(0, 1)$ for which we shall find the asymptotics of the sequence (8). We denote by $\mathfrak{R}_1$ the class of sets $G\subseteq D(0, 1)$ of the form

$$\{f\in D(0,1): f(0)=0, L_1(t)>f(t)>L_2(t); 0\leqq t\leqq 1\}, \tag{9}$$

where the functions $L_1(t)$, $L_2(t)$ are right-continuous and piecewise constant; moreover, $L_1(t)$ may take on the value $+\infty$ and $L_2(t)$ the value $-\infty$. We denote by $\mathfrak{R}_2$ the class of sets $G\in\mathfrak{R}_1$ such that $G\cap C(0, 1)\neq\varnothing$, where $C(0, 1)$ is the collection of real continuous functions $f(t)$, $0\leqq t\leqq 1$. Obviously a set $G\in\mathfrak{R}_1$ belongs to $\mathfrak{R}_2$ if and only if, for $0\leqq t\leqq 1$,

$$\min\{L_1(t-0), L_1(t+0)\}-\max\{L_2(t-0), L_2(t+0)\}>0.$$

For $G\in\mathfrak{R}_2$ we define the functional

$$H_\alpha(G)=\int_0^1 (L_1(t)-L_2(t))^{-\alpha}\,dt. \tag{10}$$

We denote by $\mathfrak{R}_3$ the class of sets $G\subseteq D(0, 1)$ which, for some $N=1, 2, \cdots$, admit the representation

$$G=\bigcup_{i=1}^{N} G_i; \qquad \min_{1\leqq i\leqq N} H_\alpha(G_i)>0, \tag{11}$$

where $G_i\in\mathfrak{R}_2$. For $G\in\mathfrak{R}_3$, we put

$$H_\alpha(G)=H_\alpha\left(\bigcup_{i=1}^{N} G_i\right)=\min_{1\leqq i\leqq N} H_\alpha(G_i). \tag{12}$$

We shall say that the set $G\subseteq D(0, 1)$ lies in the class $\mathfrak{R}$ if there exist sequences $\{G_N^+\}$, $\{G_n^-\}$ of sets of $\mathfrak{R}_3$ such that

(a) $G_n^+\supseteq G\supseteq G_n^-$ for $n=1, 2, \cdots$;

(b) $\lim\limits_{n\to\infty}[H_\alpha(G_n^-)-H_\alpha(G_n^+)]=0$.

For the class $\mathfrak{R}$ we define a functional $H_\alpha(G)$, writing

$$H_\alpha(G)=\lim_{n\to\infty} H_\alpha(G_n^+)=\lim_{n\to\infty} H_\alpha(G_n^-). \tag{13}$$

The correctness of the definition of the functional $H_\alpha(G)$ on the classes $\mathfrak{R}_3$ and $\mathfrak{R}$ follows from the probabilistic meaning of this functional (see Theorem 1).

Example. Suppose that the non-empty set $G\subseteq D(0, 1)$ has the form

$$G=\{f\in D(0,1): f(0)=0, S_1(t)>f(t)>S_2(t); 0\leqq t\leqq 1\}, \tag{14}$$

where $S_1, S_2 \in C(0, 1)$. Obviously, $G \in \mathfrak{N}$ and

$$H_\alpha(G) = \int_0^1 (S_1(t) - S_2(t))^{-\alpha}\, dt.$$

Theorem 1. *Suppose that the strongly stable d.f. $F_\alpha(x)$ in* (2) *satisfies the inequalities*

$$0 < F_\alpha(0) < 1.$$

Then, for sequences $\{x(n)\}$ such that

$$\lim_{n\to\infty} x(n) = \infty, \quad \lim_{n\to\infty} x(n)B^{-1}(n) = 0,$$

and sets $G \in \mathfrak{N}$,

$$\log \mathbf{P}(s_n(\cdot) \in G) \backsim CH_\alpha(G)nx^{-\alpha}(n)L^*(x(n)), \tag{15}$$

where the slowly varying function $L^*(u)$ was defined in (3), *and the constant C, $-\infty < C < 0$, depends only on the d.f. F_α* (see Lemma 1).

Theorem 2. *Suppose that the d.f. $\mathbf{P}(\xi(1) < x) = F_\alpha(x)$ of the strongly stable process $\xi(t)$ satisfies the inequalities*

$$0 < F_\alpha(x) < 1.$$

Then, for $G \in \mathfrak{N}$, as $a \downarrow 0$,

$$\log \mathbf{P}(a^{-1}\xi(\cdot) \in G) \backsim CH_\alpha(G)a^{-\alpha}, \tag{16}$$

where C is the constant in Theorem 1.

REMARK. If the condition $0 < F_\alpha(x) < 1$ is not satisfied[2], then the asymptotic behavior of the sequence $\log \mathbf{P}(s_n(\cdot) \in G)$ can in general be quite different. Suppose for example that $F_\alpha(0) = 0$ and that the set $G \in \mathfrak{N}$ has the form (14) with $S_1(t) = 1 - 2t$, $S_2(t) = -1 - 2t$, $0 \leqq t \leqq 1$. Then, roughly speaking, the principal role in the asymptotics of the sequence

$$\mathbf{P}(s_n(\cdot) \in G) \tag{17}$$

is played by the behavior of $F(x) = \mathbf{P}(\xi_1 < x)$ as $x \to -\infty$; inasmuch as in our case the left tail satisfies only the relation $F(x) = o(1 - F(-x))$ as $x \to -\infty$, for a fixed sequence $\{x(n)\}$ the probability (17) may decrease arbitrarily fast (in fact even be equal to zero).

2. Auxiliary Lemmas

We introduce some notation. For $a > 0$, $b \geqq 0$, $c \leqq 0$, we put

$$J_a = \{f \in D(0, 1) : f(0) = 0, \sup_{0\leqq t\leqq 1} f(t) - \inf_{0\leqq t\leqq 1} f(t) < 2a\};$$

$$I_c^b = \{f \in D(0, 1) : f(0) = 0, b > \sup_{0\leqq t\leqq 1} f(t) \geqq \inf_{0\leqq t\leqq 1} f(t) > c\}.$$

[2] It is well known that the cases $F_\alpha(0) = 0$ and $F_\alpha(0) = 1$ can only occur for $0 < \alpha < 1$. The case $F_\alpha(0) = 0$, for example, corresponding to the case when the sample trajectories of the process $\xi(t)$ are non-decreasing functions.

For any set $G\subseteq D(0,1)$ and numbers $0<a\leqq 1$, $b>c$, we introduce the set

$$Y_c^b(a)G=\{f\in G: b\geqq f(a)>c\}.$$

Suppose that $\xi(t)$, $0\leqq t\leqq 1$, is a stable process with d.f. $\mathbf{P}(\xi(t)<x)=F_\alpha(xt^{-1/\alpha})$. In the lemmas which we present below we study the asymptotics of

$$\log \mathbf{P}(\xi(\cdot)\in aG)$$

as $a\downarrow 0$, where $aG=\{f\in D(0,1): a^{-1}f\in G\}$, for some concrete sets $G\subseteq D(0,1)$.

Lemma 1. *Suppose that* $0<F_\alpha(0)<1$.

I. *As* $a\downarrow 0$,

$$\log \mathbf{P}(\xi(\cdot)\in aI_{-1}^1)\backsim a^{-\alpha}C, \tag{18}$$

where the constant C, $-\infty<C<0$, *depends, of course, only on the d.f.* $F_\alpha(x)$.

II. *Suppose that* $1\geqq b>c>-1$, $|d|<1$. *Then, as* $a\downarrow 0$,

$$\log \mathbf{P}(\xi(\cdot)\in aI_{d-1}^{d+1})\backsim \log \mathbf{P}(\xi(\cdot)\in aJ_1), \tag{19}$$

$$\log \mathbf{P}(\xi(\cdot)\in aI_{d-1}^{d+1})\backsim \log \mathbf{P}(\xi(\cdot)\in aY_{d+c}^{d+b}(1)I_{d-1}^{d+1}). \tag{20}$$

The proof of Lemma 1 is based on a number of estimates, which we shall formulate in the form of a separate lemma. First of all we introduce a notation: for a set $G\subseteq D(0,1)$ and numbers $0\leqq a<b\leqq 1$ we put

$$X(a,b)G=\{f\in D(0,1): f(t)=g(t),\ a\leqq t\leqq b,\ g\in G\}.$$

Lemma 2. *Suppose that* $0<F_\alpha(0)<1$. *Then,*

(a) *for* $\varepsilon>0$, $|b|<1$, $|c|<1$, *as* $a\downarrow 0$,

$$\log \mathbf{P}(\xi(\cdot)\in aI_{b-1}^{b+1})\lesssim \log \mathbf{P}(\xi(\cdot)\in aI_{c-(1+\varepsilon)}^{c+1+\varepsilon}); \tag{21}$$

(b) *for* $\varepsilon>0$, *as* $a\downarrow 0$,

$$\log \mathbf{P}(\xi(\cdot)\in aI_{-1}^1)\lesssim \log \mathbf{P}(\xi(\cdot)\in aJ_1)\lesssim \log \mathbf{P}(\xi(\cdot)\in aI_{-(1+\varepsilon)}^{1+\varepsilon}); \tag{22}$$

(c) *for* $0<c\leqq 1$, $k=[c^{-1}]$,

$$\mathbf{P}(\xi(\cdot)\in aI_{-1}^1)\leqq \mathbf{P}^k(\xi(\cdot)\in aX(0,c)J_1); \tag{23}$$

(d) *for* $\varepsilon>0$, $0<c\leqq 1$, $k=[c^{-1}]+1$,

$$\mathbf{P}(\xi(\cdot)\in aI_{-(1+4\varepsilon)}^{1+4\varepsilon})\geqq \Big[\min_{-3\leqq i\leqq 3}\mathbf{P}(\xi(\cdot)\in aX(0,c)Y_{(i-1)\varepsilon}^{(i+1)\varepsilon}(c)I_{-1}^1)\Big]^k; \tag{24}$$

(e) *for* $\varepsilon>0$, $-1\leqq c<b\leqq 1$, *as* $a\downarrow 0$,

$$\log \mathbf{P}(\xi(\cdot)\in aY_c^b(1)I_{-(1+\varepsilon)}^{1+\varepsilon})\gtrsim \log \mathbf{P}(\xi(\cdot)\in aI_{-1}^1). \tag{25}$$

PROOF OF LEMMA 2. (a) Consider the inequality

$$\mathbf{P}(\xi(\cdot)\in aI_{x_2}^{x_1})\geqq \mathbf{P}(\xi(\cdot)\in aX(0,l)Y_{y_2}^{y_1}(l)I_{x_2}^{x_1})\mathbf{P}(\xi(\cdot)\in aX(0,1-l)I_{x_2-y_2}^{x_1-y_1}),$$

where $0<l\leqq 1$, $x_1\geqq y_1\geqq y_2\geqq x_2$. Putting $l=a^\alpha$, $x_1=c+1+\varepsilon$, $x_2=c-(1+\varepsilon)$, $y_1=c-b+\varepsilon$, $y_2=c-b-\varepsilon$, we get

$$\mathbf{P}(\xi(\cdot)\in aI_{c-(1+\varepsilon)}^{c+1+\varepsilon})\geqq \mathbf{P}(\xi(\cdot)\in aX(0,a^\alpha)Y_{c-b-\varepsilon}^{c-b+\varepsilon}(a^\alpha)I_{c-(1+\varepsilon)}^{c+1+\varepsilon})$$
$$\times \mathbf{P}(\xi(\cdot)\in aX(0,1-a^\alpha)I_{b-1}^{b+1})\geqq p\mathbf{P}(\xi(\cdot)\in aI_{b-1}^{b+1}),$$

where

$$p=\liminf_{a\downarrow 0}\mathbf{P}(\xi(\cdot)\in aX(0,a^{\alpha})Y_{c-b-\varepsilon}^{c-b+\varepsilon}(a^{\alpha})I_{c-1}^{c+1})=\mathbf{P}(\xi(\cdot)\in Y_{c-b-\varepsilon}^{c-b+\varepsilon}(1)I_{c-1}^{c+1})>0.$$

Relation (21) is proved.

(b) The assertion of this point follows from the easily verified relations

$$I_{-1}^{1}\subseteq J_1\subseteq\bigcup_{i=-k}^{k}\{I_{i/k-(1+1/k)}^{i/k+1+1/k}\}$$

and from assertion (21).

(c) Inequality (23) is obtained as a consequence of the inclusion

$$I_{-1}^{1}\subseteq\bigcap_{i=0}^{k-1}\{X(ic,(i+1)c)J_1\}.$$

(d) We shall prove inequality (24). For $\varepsilon>0$; $0<c\leqq 1$; $n=1,2,\cdots$; $a_i=i\varepsilon/n$, $-n\leqq i\leqq n$,

$$\begin{aligned}\mathbf{P}(\xi(\cdot)\in aI_{-(1+2\varepsilon)}^{1+2\varepsilon})&\geqq\sum_{i=-n+1}^{n}\mathbf{P}(\xi(\cdot)\in aX(0,1-c)Y_{a_{i-1}}^{a_i}(1-c)I_{-(1+2\varepsilon)}^{1+2\varepsilon})\\&\quad\times\mathbf{P}(\xi(\cdot)\in aX(0,c)I_{-(1+2\varepsilon)-a_{i-1}}^{1+2\varepsilon-a_i})\\&\geqq\mathbf{P}(\xi(\cdot)\in aX(0,1-c)Y_{-\varepsilon}^{\varepsilon}(1-c)I_{-(1+2\varepsilon)}^{1+2\varepsilon})\\&\quad\times\min_{-n+1\leqq i\leqq n}\mathbf{P}(\xi(\cdot)\in aX(0,c)I_{-(1+2\varepsilon)-a_{i-1}}^{1+2\varepsilon-a_i}).\end{aligned}$$

Letting $n\to\infty$, we find, for any $\delta>0$,

$$\begin{aligned}\mathbf{P}(\xi(\cdot)\in aI_{-(1+2\varepsilon)}^{1+2\varepsilon})&\geqq\mathbf{P}(\xi(\cdot)\in aX(0,1-c)Y_{-\varepsilon}^{\varepsilon}(1-c)I_{-(1+2\varepsilon)}^{1+2\varepsilon})\\&\quad\times\inf_{|b|<\varepsilon}\mathbf{P}(\xi(\cdot)\in aX(0,c)I_{b-(1+2\varepsilon-\delta)}^{b+1+2\varepsilon-\delta}).\end{aligned}$$

Using induction on $k=[c^{-1}]+1$, one can prove that, for any $\delta>0$ and $0<c\leqq 1$,

$$\mathbf{P}(\xi(\cdot)\in aI_{-(1+2\varepsilon)}^{1+2\varepsilon})\geqq\left[\inf_{|b|<\varepsilon}\mathbf{P}(\xi(\cdot)\in aX(0,c)Y_{-\varepsilon}^{\varepsilon}(c)I_{b-(1+2\varepsilon-\delta)}^{b+1+2\varepsilon-\delta})\right]^{k}.\tag{26}$$

Put $\delta=\varepsilon$. Note further that, for $|b|\leqq\varepsilon$,

$$\mathbf{P}(\xi(\cdot)\in aX(0,c)Y_{-\varepsilon}^{\varepsilon}(c)I_{b-(1+\varepsilon)}^{b+1+\varepsilon})\geqq\mathbf{P}(\xi(\cdot)\in aX(0,c)Y_{-b-\varepsilon}^{-b+\varepsilon}(c)I_{-1}^{1}).\tag{27}$$

Moreover, for any b with $|b|<\varepsilon$ there exists an integer $i=i(b)$, $-3\leqq i\leqq 3$, such that

$$\mathbf{P}(\xi(\cdot)\in aX(0,c)Y_{-b-\varepsilon}^{-b+\varepsilon}(c)I_{-1}^{1})\geqq\mathbf{P}(\xi(\cdot)\in aX(0,c)Y_{(i-1)\varepsilon/2}^{(i+1)\varepsilon/2}(c)I_{-1}^{1}).\tag{28}$$

Applying (26), (27) and (28), we get (24).

(e) Relation (25) is obtained as a consequence of the inclusion

$$Y_{-1}^{1}(1)I_{-1}^{1}\subseteq\bigcup_{i=-k+1}^{k}\{Y_{(i-1)/k}^{i/k}(1)I_{-1}^{1}\}$$

and the relation

$$\log\mathbf{P}(\xi(\cdot)\in aY_{(j-1)/k}^{j/k}(1)I_{-(1+\varepsilon)}^{1+\varepsilon})\gtrsim\log\mathbf{P}(\xi(\cdot)\in aY_{(i-1)/k}^{i/k}(1)I_{-1}^{1}),$$

where $-k+1 \leqq i, j \leqq k$. This is proved in exactly the same way as the relation (21). Lemma 2 is proved.

PROOF OF LEMMA 1. Suppose that the set $G \subseteq D(0, 1)$, $0 < a \leqq 1$. We denote by $G_a \subseteq D(0, a^\alpha)$ the set

$$G_a = \{f \in D(0, a^\alpha) : a^{-1} f(a^{-\alpha} t) \in G\}.$$

It is not hard to see that for the process $\xi(t)$, $t \geqq 0$, with the strongly stable d.f. $F(\xi(1) < x) = F_\alpha(x)$

$$\mathbf{P}(\xi(\cdot) \in G_a) = \mathbf{P}(\xi(\cdot) \in G). \tag{29}$$

Putting $c = a^\alpha$ in inequalities (23) and (24) and letting $a \downarrow 0$, we find on the basis of (29) that

$$-\infty < C_- = \liminf_{a \downarrow 0} a^\alpha \log \mathbf{P}(\xi(\cdot) \in aI_{-1}^1)$$
$$\leqq \limsup_{a \downarrow 0} a^\alpha \log \mathbf{P}(\xi(\cdot) \in aI_{-1}^1) = C_+ < 0.$$

Now select two sequences $\{a(n)\}_{n=1}^\infty$, $\{b(n)\}_{n=1}^\infty$, such that $a(n) \downarrow 0$, $b(n) \downarrow 0$, $c = c(n) = (a(n)/b(n))^\alpha \downarrow 0$ as $n \to \infty$ and

$$\lim_{n \to \infty} a^\alpha(n) \log \mathbf{P}(\xi(\cdot) \in a(n) I_{-1}^1) = C_+,$$

$$\lim_{n \to \infty} (b(n)(1+\varepsilon))^\alpha \log \mathbf{P}(\xi(\cdot) \in b(n) I_{-(1+\varepsilon)}^{1+\varepsilon}) = C_-.$$

Using inequality (23) and the relation (22), we find that, for any $\varepsilon > 0$,

$$C_+ \leqq (1+\varepsilon)^{-\alpha} C_-.$$

From this it follows that the following limit exists:

$$\lim_{a \downarrow 0} a^\alpha \log \mathbf{P}(\xi(\cdot) \in aI_{-1}^1) = C.$$

The proof of point II follows from the assertions of points (b) and (e) of Lemma 2 and the assertion of point I proved above.

Lemma 1 is proved.

We recall that we have denoted by $s_n \in D(0, 1)$ a random step-function, constructed for the sequences (1) and $x = \{x(n)\}_{n=1}^\infty$. The following assertion is proved in exactly the same way as Lemma 2.

Lemma 3. *Suppose that the conditions of Theorem 1 are satisfied. Then,*

(a) *for* $\varepsilon > 0$, $|b| < 1$, $|c| < 1$, *as* $n \to \infty$,

$$\log \mathbf{P}(s_n(\cdot) \in I_{b-1}^{b+1}) \lesssim \log \mathbf{P}(s_n(\cdot) \in I_{c-(1+\varepsilon)}^{c+1+\varepsilon}); \tag{30}$$

(b) *for* $\varepsilon > 0$, *as* $n \to \infty$,

$$\log \mathbf{P}(s_n(\cdot) \in I_{-1}^1) \lesssim \log \mathbf{P}(s_n(\cdot) \in J_1) \lesssim \log \mathbf{P}(s_n(\cdot) \in I_{-(1+\varepsilon)}^{1+\varepsilon}); \tag{31}$$

(c) *for* $n = 1, 2, \cdots; 1 \leqq m \leqq n$; $c = m/n$, $k = [c^{-1}]$,

$$\mathbf{P}(s_n(\cdot) \in I_{-1}^1) \leqq \mathbf{P}^k(s_n(\cdot) \in X(0, c) J_1); \tag{32}$$

(d) *for* $n=1,2,\cdots;\ 1\leqq m\leqq n,\ c=m/n,\ k=[c^{-1}]+1$,

$$\mathbf{P}(s_n(\cdot)\in I^{1+4\varepsilon}_{-(1+4\varepsilon)})\geqq\left[\min_{-3\leqq i\leqq 3}\mathbf{P}(s_n(\cdot)\in X(0,c)\,Y^{(i+1)\varepsilon}_{(i-1)\varepsilon}(c)I^{1}_{-1})\right]^k; \tag{33}$$

(e) *for* $\varepsilon>0,\ -1\leqq c<b\leqq 1$, *as* $n\to\infty$,

$$\log\mathbf{P}(s_n(\cdot)\in Y^b_c(1)I^{1+\varepsilon}_{-(1+\varepsilon)})\lesssim\log\mathbf{P}(s_n(\cdot)\in I^1_{-1}). \tag{34}$$

Using Lemma 3, one proves the following:

Lemma 4. *Suppose that the conditions of Theorem* 1 *are satisfied.*
I.

$$\log\mathbf{P}(s_n(\cdot)\in I^1_{-1})\backsim nx^{-\alpha}(n)L^*(x(n))C, \tag{35}$$

where the function $L^*(u)$ *is defined in* (3), *and* C *is the constant in Lemma* 1.
II. *Suppose* $\varepsilon>0,\ |b|\leqq 1,\ |c|\leqq 1$. *Then,*

$$\log\mathbf{P}(s_n(\cdot)\in I^{c+1}_{c-1})\backsim\log\mathbf{P}(s_n(\cdot)\in J_1); \tag{36}$$

$$\log\mathbf{P}(s_n(\cdot)\in I^{c+1}_{c-1})\backsim\log\mathbf{P}(s_n(\cdot)\in Y^{c+b+\varepsilon}_{c+b-\varepsilon}(1)I^{c+1}_{c-1}). \tag{37}$$

Proof. In view of well-known theorems on the convergence of processes (see the Introduction), for $a>0$,

$$\mathbf{P}(s_n(\cdot)\in a^{-1}B(n)x^{-1}(n)I^1_{-1})\backsim\mathbf{P}(\xi(\cdot)\in a^{-1}I^1_{-1}) \tag{38}$$

as $n\to\infty$. For an arbitrary sequence $\{y(n)\}_{n=1}^{\infty}$, $0<y(n)\leqq n$, $y(n)\to\infty$ as $n\to\infty$,

$$\begin{aligned}&\mathbf{P}(s_n(\cdot)\in a^{-1}B(n)x^{-1}(n)I^1_{-1}\\ &\quad\backsim\mathbf{P}(s_n(\cdot)\in a^{-1}B([y(n)])x^{-1}(n)X(0,n^{-1}[y(n)])I^1_{-1}).\end{aligned} \tag{39}$$

Put $y(n)=B^*(ax(n))$, where $B^*(u)=u^\alpha/L^*(u)$; $L^*(u)$ has been defined in (3). From relation (4) (where $d=1$) it is not hard to deduce (using, for example, the representation for a slowly varying function in [1], p. 273) that, as $u\to\infty$,

$$B([B^*(u)])=u(1+o(1)).$$

From this we find that

$$a^{-1}B([y(n)])x^{-1}(n)\backsim 1. \tag{40}$$

Further, it follows from (38)–(40) that, for all $a>0$,

$$\log\mathbf{P}(s_n(\cdot)\in X(0,n^{-1}[B^*(ax(n))])I^1_{-1})\backsim\log\mathbf{P}(\xi(\cdot)\in a^{-1}I^1_{-1}) \tag{41}$$

as $n\to\infty$. Therefore, there exists an arbitrarily slowly increasing sequence $a(n)\uparrow\infty$ such that,

$$\log\mathbf{P}(s_n(\cdot)\in X(0,n^{-1}[B^*(a(n)x(n))])I^1_{-1})\backsim\log\mathbf{P}(\xi(\cdot)\in a^{-1}(n)I^1_{-1}) \tag{42}$$

as $n\to\infty$. We note that in view of Lemma 1

$$\log\mathbf{P}(\xi(\cdot)\in a^{-1}(n)I^1_{-1})\backsim a^\alpha(n)C.$$

In view of the regularity of the function $B^*(u)$ for a sufficiently slowly increasing sequence $a(n)\uparrow\infty$,

$$[B^*(a(n)x(n))]\backsim a^\alpha(n)B^*(x(n)). \tag{43}$$

Thus we have proved that for every sequence $\{x(n)\}_{n=1}^{\infty}$ such that $x(n)\to\infty$, $x(n)/B(n)\to 0$ as $n\to\infty$, there exists a sequence $a(n)\uparrow\infty$ such that

$$\log \mathbf{P}(s_n(\cdot)\in X(0, a^{\alpha}(n)n^{-1}[B^*(x(n))])I_{-1}^{1}) \backsim a^{\alpha}(n)C, \tag{44}$$

where $C=C(F_\alpha)$ from Lemma 1. Further, in order to obtain the upper estimate in (36) we must use relations (32) and (31) of Lemma 3, and for the lower estimate relations (33) and (34) of the same Lemma.

The assertion of point II is proved using relations (31) and (34) of Lemma 3 and the relation (36) just proved.

3. Proof of Theorems 1 and 2

Suppose that $G\in\mathfrak{R}_2$ and that the points $0<t_1<\cdots<t_N<1$ are points of discontinuity of the functions L_1, L_2 defining the set G in (9). On the intervals (t_i, t_{i+1}), $i=0,1,\cdots,N$; $t_0=0$, $t_{N+1}=1$, the set G is a strip with rectilinear boundaries, that is,

$$X(t_i, t_{i+1}-0)G=X(t_i, t_{i+1}-0)I_{b_i}^{a_i},$$

where $a_i=L_1(t)$, $b_i=L_2(t)$ for $t_i<t<t_{i+1}$.

We obtain the estimate from above from the inequality

$$\mathbf{P}(s_n(\cdot)\in G)\leqq\prod_{i=0}^{N}\mathbf{P}(s_n(\cdot)\in X(t_i, t_{i+1})J_{c_i}),$$

where $c_i=(a_i-b_i)/2$, and Lemma 4. The estimate from below follows from the inequality

$$\mathbf{P}(s_n(\cdot)\in G)\geqq\prod_{i=0}^{N}\mathbf{P}(s_n(\cdot)\in(1-\varepsilon)Y_{d_i-\delta}^{d_i+\delta}(t_{i+1}-t_i)X(0, t_{i+1}-t_i)I_{b_i}^{a_i}),$$

where $d_i=(\min\{a_{i-1}, a_i\}+\max\{a_{i-1}, a_i\})/2$, $\delta>0$ is sufficiently small, and Lemma 4. Then assertion (15) is extended in the obvious way to the classes $\mathfrak{R}_3$ and $\mathfrak{R}$.

Theorem 1 is proved.

Theorem 2 is proved in exactly the same way using Lemma 1.

4. Calculation of the Constant C in the Case $\alpha=2$

Suppose that in (1) the random variables ξ_i are distributed as follows: $\mathbf{P}(\xi_i=\pm 1)=\frac{1}{2}$. We suppose that the sequence $\{x(n)\}$ in (5) consists of integers with the same parity as n (so that $\mathbf{P}(s_n=x(n))>0$ if $|x(n)|\leqq n$). Write

$$u_n^z=\mathbf{P}(-x(n)<\min_{1\leqq k\leqq n-1}s_k\leqq\max_{1\leqq k\leqq n-1}s_k<x(n),\ s_n=z),\qquad u_n=u_n^{-x(n)}.$$

In view of the obvious inequalities

$$\max_{|k|<x(n)}u_{n-2x(n)}^{k}\leqq 2^{2x(n)}u_n,$$

$$\mathbf{P}(s_{n+1}(\cdot)\in I_{-1}^{1})\leqq\sum_{k=-x(n)}^{x(n)}u_n^k,$$

$$\mathbf{P}(s_{n+1}(\cdot)\in J_{-1}^{1})\geqq u_n$$

we find, for sequences $\{x(n)\}$ such that $\lim_{n\to\infty} x(n)=\infty$, $\lim_{n\to\infty} x(n)n^{-1/3}=0$, the following relation:

$$\log \mathbf{P}(s_n(\cdot)\in I_{-1}^1) \backsim \log u_n.$$

Further we make use of the explicit formulas for u_n given in [7], p. 322.

$$u_n=(2x(n))^{-1}\sum_{k=1}^{2x(n)-1}\cos^{n-1}\frac{\pi k}{2x(n)}\cdot\sin\frac{\pi k}{2x(n)}\cdot\sin\frac{\pi k}{2}.$$

The principal roles in the asymptotics of this series are played by the first and last terms, so that

$$\log u_n \backsim \log\left(2(2x(n))^{-1}\cos^{n-1}\frac{\pi}{2x(n)}\cdot\sin\frac{\pi}{2x(n)}\right) \backsim -\frac{\pi^2}{8}nx^{-2}(n).$$

Inasmuch as $\mathbf{E}\xi_i^2=\frac{1}{4}$, we find that the constant C corresponding to the normal (0, 1) distribution is equal to

$$C=-\frac{\pi^2}{2}.$$

Theorem 3. *Suppose that* $\mathbf{E}\xi_i=0$, $\mathbf{E}\xi_i^2=1$ *and that the sequence* $\{x(n)\}$ *is such that*

$$\lim_{n\to\infty} x(n)=0, \qquad \lim_{n\to\infty} x(n)n^{-1/2}=0.$$

Then the relation

$$\log \mathbf{P}(s_n(\cdot)\in G) \backsim -\frac{\pi^2}{2}H_2(G)nx^{-2}(n)$$

holds for sets $G\in\mathfrak{N}$.

5. Small Deviations for Processes

Consider a homogeneous process with independent increments given by $\xi(t)$, $t\geqq 0$. We assume that the sample trajectories of the process are right continuous with probability 1. The expectation $\mathbf{E}\xi(t)$, when it exists, will be taken to be equal to zero.

Suppose that, for some sequence $B(n)$,

$$\mathbf{P}(\xi(n)<xB(n))\Rightarrow F_\alpha(x) \tag{45}$$

as $n\to\infty$, where $F_\alpha(x)$ is a strongly stable d.f. with parameter α, $0<\alpha\leqq 2$. We recall that necessary and sufficient conditions for (45) were presented in the Introduction. As earlier, we suppose that

$$B^*(B(n)) \backsim n,$$

where $B^*(u)=u^\alpha/L^*(u)$, $L^*(u)=u^{\alpha-2}\int_{-u}^{u} t^2\, d\mathbf{P}(\xi(1)<t)$.

Now consider the sequence of processes

$$\{\xi_n(t);\quad 0\leqq t\leqq 1\}_{n=1}^\infty=\{\xi(nt)x^{-1}(n);\quad 0\leqq t\leqq 1\}_{n=1}^\infty,$$

where $x(n)\to\infty$ as $n\to\infty$.

Theorem 4. *Assume that* $0 < F_\alpha(x) < 1$ *and* $x(n)/B(n) \to 0$ *as* $n \to \infty$. *Then for sets* $G \in \mathfrak{R}$

$$\log \mathbf{P}(\xi_n(\cdot) \in G) \backsim CH_\alpha(G) n x^{-\alpha}(n) L^*(x(n)),$$

where C, $-\infty < C < 0$, *is the constant in Theorem* 1.

The proof of Theorem 4 is almost a verbatim repetition of the proof of Theorem 1.

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