Branching Random Walks with Selection

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Contents

1 Effect of selection on ancestry: an exactly soluble case and its phenomenological generalization

Placeholder text.

2 The space D

This section is based on Sections 3,11 and 12 of [?]. Consider $D(I, \mathbb{R})$ where I is a closed interval. Let Λ be the space of continuous bijections from $[0,1]$ to itself which are zero at zero. The J_1 topology is defined by the metric

$$d_{J_1}(f,g) = \inf_{\lambda \in \Lambda} \{ \|f \circ \lambda\|_\infty \vee \|\lambda - \mathbb{1}\|_\infty \} \tag{2.1}$$

The M_1 topology is defined by the same metric but on the completed graph. These definitions extend to I $[0,\infty)$ by saying that $x_n \in D$ converges to $x \in D$ if convergence happens in $D([0,t])$ for all continuity points t of x. The topology this defines is metrisable (see page 83 of [?]). The topologies they induce are called strong and sometimes written SJ_1 and SM_1. We talk about weak topologies when we consider $D(I, \mathbb{R}^k)$ with $k > 1$ and they are equal to the product topology, but they are not important to our discussion.

The $(S)J_2$ and $(S)M_2$ topologies are the ones induced by applying the Hausdorff metric to the functions graphs and completed graphs respectively.

3 Background

Theorem 3.1 ([?, Theorem 7.4.1.]) — *Let $\{X_{m,n} \mid 0 \le m < n\}$ be a family of random variables satisfying*

(i) $X_{l,n} \le X_{l,m} + X_{m,n}$ *for all* $l < m < n$

(ii) The distribution of $\{X_{m+k,n+k} \mid 0 \le m < n\}$ *does not depend on* $k \in \mathbb{N}$

(iii) $\mathbb{E}\left[X_{0,1}^+\right] < \infty$ *and there exists* $\gamma > -\infty$ *such that* $\mathbb{E}\left[X_{0,n}\right] > \gamma n$ *for all* $n \in \mathbb{N}$.

Then

4 exactly soluble

The model we consider is constructed as follows: The individual at position $x_0 \in \mathbb{R}$ has offspring according to a poisson process with intensity measure that has density $x \mapsto \exp(-(x - x_0))$ with respect to the Lebesgue measure. Out of the offspring of all individuals, the N right-most are selected to form the population of the next generation. By additivity of Poisson processes, the location of the offspring is described by a Poisson process with density $\Psi(x) \sum_{i=1}^{N} \exp(-(x - x_i))$ where x_i denotes the position of the i'th individual in the current generation.

Let us find the joint density of the $N+1$ rightmost particles. Let $x_1, ..., x_{N+1}$ denote them, with $X_{N+1} < X_i$ for $i \in [N]$ but with no particular order on $X_1, ..., X_N$. Then weby the mean value theorem for integrals we have:

$$\mathbb{P}\left(\bigcap_{i=1}^{N+1} \{X_i \in [x_i, x_i + \Delta_i]\} \right) = \frac{1}{N!} \exp\left(- \int_{x_{N+1}}^{\infty} \Psi(u)\, du \right) \prod_{i=1}^{N+1} \{\Delta_i \Psi(x_i)\} (1 + o(1)).$$

Thus, dividing by $\Delta_1 \times ... \times \Delta_{N+1}$ and taking the Δ_i go to zero, we get the density

$$f_{X_1,...,X_{N+1}}(x_1,...,x_{N+1}) = \frac{1}{N!} \exp\left(- \int\limits_{x_{N+1}}^{\infty} \Psi(u)\,du \right) \prod_{i=1}^{N+1} \Psi(x_i).$$

Now we can marginalise to obtain the density of X_{N+1}:

$$f_{X_{N+1}}(x) = \frac{1}{N!} \Psi(x) \left(\int\limits_{x}^{\infty} \Psi(u)\,du \right)^{N} \exp\left(- \int\limits_{x}^{\infty} \Psi(u)\,du \right).$$

From here it is easy to see that conditional on X_{N+1}, the X_i are independent with density

$$f(x) = \frac{\Psi(x)}{\int\limits_{X_{N+1}}^{\infty} \Psi(u)\,du} \mathbb{1}_{\{X_{N+1} < x\}}.$$

5 Speed

Placeholder text.

In generality, the models that we will discuss in this essay can be described as evolving according to two mechanisms: ...

5.1 Exponentially decaying tails

5.1.1 Construction

The first variation of the N-branching random walk that we consider is the one studied in [1] by Bérard and Gouéré. Suppose that at timestep $n \geq 0$ there is a particle at $x \in \mathbb{R}$. During the branching step the particle dies giving birth to two children, whose positions independently (from each other and the past) follow a distribution with cumulative distribution function $p(\,\cdot - x)$. Out of all $2N$ children, the population at time $n+1$ is then formed by the N rightmost particles.

Construction. Let $X = (X_n)_{n\geq 0} = (\sum_{i=1}^{N} \delta_{X_n(i)})_{n\geq 0}$ denote the $\mathcal{C}_N$-valued discrete time Markov process defined by the branching-selection procedure detailed above. Note that we suppress the dependence on N in our notation for simplicity. We can construct X easily: Let $\mathcal{E}_N := (\epsilon_{l,i,j})_{l\geq 0,\, i\in[\![1,N]\!],\, j=1,2}$ be an i.i.d. collection of random variables distributed like p. For $n \geq 0$ define $Y_{n+1} := \sum_{i=1}^{N} \sum_{j=1,2} \delta_{X_n(i)+\epsilon_{n,i,j}}$ and take X_{n+1} to be the counting measure supported on the rightmost N atoms of Y_{n+1}. This construction gives rise to an important monotonicity property that we record in the following Lemma:

Lemma 5.1 ([1, Corollary 2]) — *For any $1 \leq N_1 \leq N_2$ and $\mu_i \in \mathcal{C}_{N_i}$ with $i = 1, 2$ such that $\mu_1 \preceq \mu_2$, there exists a coupling $(X_n^{(1)}, X_n^{(2)})_{n\geq 0}$ between two versions of the branching-selection particle system started from μ_1 and μ_2 respectively satisfying $X_n^{(1)} \preceq X_n^{(2)}$ almost surely for all $n \geq 0$.*

Proof. The proof is a straightworward consequence of our construction. The idea is to take an i.i.d. family $\mathcal{E}_{N_2} = (\epsilon_{l,i,j})_{l\geq 0,\, i\in[\![1,N_2]\!],\, j=1,2}$ of random variables with law given by p and to use it to construct both $(X^{(1)})_n$ and $(X_n^{(2)})_{n\geq 0}$. □

5.1.2 Properties of the model

Let us now define the logarithmic moment generation function of p:

$$\Lambda(t) := \log \int_{\mathbb{R}} \exp(tx)dp(x).$$

In their analysis, Bérard and Gouéré impose some conditions on the domain $\mathcal{D}(\Lambda) := \{t \mid \Lambda(t) < \infty\}$ of Λ in order for the results of [2] to be applicable.

Assumption 1. Λ is finite in some neighbourhood of 0.

Assumption 2. There exists $t^* > 0$ in the interior of $\mathcal{D}(\Lambda)$ such that $t^*\Lambda'(t^*) - \Lambda(t^*) = \log 2$.

Assumption 1 is in fact equivalent to the requirement that p have exponentially decaying tails, furthermore it implies that p has finite moments of all orders. The results that follow in this section are conditional upon Assumptions 1 and 2 being satisfied.

Denote by $\max X_n$ and $\min X_n$ the right- and leftmost atom of X_n respectively. It is worth noting that $\min X_n$ and $\max X_n$ are integrable and hence finite by Assumption 1 when started from any fixed $X_0 \in \mathcal{C}_N$. Denote by $d(X_n) := \max X_n - \min X_n$ the diameter of X_n.

PROPOSITION 5.1 ([1, Corollary 1]) — *For any $N \geq 1$ and initial population $X_0 \in \mathcal{C}_N$, we have*

$$\frac{d(X_n)}{n} \xrightarrow[n\to\infty]{a.s.,\, L^1} 0.$$

PROPOSITION 5.2 ([1, Proposition 2]) — *There exists $v_N = v_N(p) \in \mathbb{R}$ such that for any initial population $X_0 \in \mathcal{C}_N$ the following holds almost surely and in L^1:*

$$\lim_{n\to\infty} \frac{\min X_n}{n} = \lim_{n\to\infty} \frac{\max X_n}{n} = v_N. \tag{5.1}$$

Proof. Recall the definition of $\mathcal{E}_N$ from the construction of X. For each $l \geq 0$ we define the process $(X_n^l)_{n\geq 0}$ by shifting the origin of time by l. More precisely, given the process up to time $n \geq 0$, define X_{n+1}^l to be given by the N rightmost atoms of $\sum_{i=1}^N \sum_{j=1,2} \delta_{X_n^l(i)+\epsilon_{n+l,i,j}}$. It is clear that each $(X_n^l)_{n\geq 0}$ is distributed as the N-branching random walk with offspring law p.
Suppose that for each $l \geq 0$ we start $(X_n^l)_{n\geq 0}$ from $N\delta_0$ and notice that $(X_n^0)_{n\geq 0} = (X_n)_{n\geq 0}$ almost surely, provided that $X_0 = N\delta_0$ also. From Lemma 5.1 it follows easily that

$$\max X_{n+m}^0 \leq \max X_n^0 + \max X_m^n \qquad\qquad \forall\, n, m \geq 0. \tag{5.2}$$

Provided that the conditions of the theorem hold, applying Kingman's Subadditive Ergodic theorem yields $\lim_{n\to\infty} n^{-1} \max X_n = \lim_{n\to\infty} \mathbb{E}\left[n^{-1}\max X_n\right] = \inf_n \mathbb{E}\left[n^{-1}\max X_n\right] = v_N \in \mathbb{R}$ where the first limit is almost sure and also in L^1. Using the fact that p has exponentially decaying tails (Assumption 1) and the independence of $\mathcal{E}_N$, the conditions are easily verified.
From Proposition 5.1 we immediately get $\lim_{n\to\infty} n^{-1}\min X_n = v_N$, so the proof is complete in the case $X_0 = N\delta_0$. By translation invariance of the dynamics of the system the result also follows for initial conditions of the form $N\delta_{x_0}$ for any $x_0 \in \mathbb{R}$. Finally, for arbitrary $X_0 \in \mathcal{C}_N$ note that the result is a consequence of Lemma 5.1 and a sandwiching argument between the initial configurations $N\delta_{\min X_0}$ and $N\delta_{\max X_0}$. $\square$

PROPOSITION 5.3 ([1, Proposition 3]) — *The sequence $(v_N)_{N\geq 1}$ is non-decreasing.*

Proof. This is again a consequence of Lemma 5.1. $\square$

Remark 5.1. From Proposition 5.3 we can deduce that v_N increases to a possibly infinite limit v_∞ as N goes to infinity. Assumption 1 implies that Λ is smooth on the interior of $\mathcal{D}(\Lambda)$ so that both quantities $v := \Lambda'(t^*)$ and $\chi := \frac{\pi^2}{2} t^* \Lambda''(t^*)$ are finite. In Section 5.1.4 we will see that v_∞ is in fact equal to v.

5.1.3 KILLED BRANCHING RANDOM WALKS

Following the notation used in [1], we formally define a Branching Random Walk (BRW) to be a pair $(\mathcal{T}, \Phi)$, where $\mathcal{T}$ is a rooted binary tree and Φ is a random map assigning a random variable $\Phi(u)$ to each vertex $u \in \mathcal{T}$. Φ must be such that $\Phi(\text{root}) = 0$ and $\{\Phi(v) - \Phi(u) \mid u \text{ is the parent of } v\}$ is i.i.d. with common distribution p. We call $\Phi(u)$ the value of the BRW at vertex u and write $\mathcal{T}(n)$ for the set of vertices in $\mathcal{T}$ at distance n from the root. We say a sequence of vertices $u_1, u_2, \ldots$ is a path if u_{i+1} is the parent of u_i for each $i \geq 1$.
Suppose that we have a BRW $(\mathcal{T}, \Phi)$ and take $v \in \mathbb{R}$ and $m \geq 1$. We say that vertex u is (m, v)-good if there exists a path $u = u_0, u_1, \ldots, u_m$ such that $\Phi(u_i) - \Phi(u) \geq vi$ for all $i \in [\![0, m]\!]$. This is

essentially saying that there exists a path started from u that stays to the right of the space-time line through $(u, \Phi(u))$ with slope v, for at least m steps. The definition of an (∞, v)-good vertex is analogous. We now state two results from [2] that we will need to prove Theorem 5.4. Recall the definitions of v and χ from Remark **??**.

THEOREM 5.2 ([2, Theorem 1.2]) — *Let $\rho(\infty, \epsilon)$ denote the probability that the root of the BRW with offspring distribution p is $(\infty, v-\epsilon)-$ good. Then, as $\epsilon > 0$ goes to zero,*

$$\rho(\infty, \epsilon) \leq \exp\left(-\left(\frac{\chi + o(1)}{\epsilon}\right)^{1/2}\right). \tag{5.3}$$

A similar result can be stated for the probability of observing a $(m, v-\epsilon)$-good root with m finite:

THEOREM 5.3 ([1, Theorem 3]) — *Let $\rho(m, \epsilon)$ denote the probability that the root of the BRW with offspring distribution p is $(m, v-\epsilon)$-good. For any $0 < \beta < \chi$, there exists $\theta > 0$ such that for all large m,*

$$\rho(m, \epsilon) \leq \exp\left(-\left(\frac{\chi - \beta}{\epsilon}\right)^{1/2}\right), \qquad \text{with } \epsilon := \theta m^{-2/3}. \tag{5.4}$$

5.1.4 Brunet-Derrida behaviour

We are now ready to present the main result of Bérard and Gouéré on N-branching random walks with exponentially decaying tails:

THEOREM 5.4 — *As N goes to infinity,*

$$v_\infty - v_N = \frac{\chi}{(\log N)^2} + o((\log N)^{-2}). \tag{5.5}$$

Let us describe the coupling between the N-branching random walk and N independent branching random walks, which will allow us to relate Theorems 5.2 and 5.4 to the N-branching random walk. Let $(\mathrm{BRW}_i)_{i \in [\![1,N]\!]} = ((\mathcal{T}_i, \Phi_i))_{i \in [\![1,N]\!]}$ be a set of N independent copies of the BRW with offspring distribution p. Define $\mathbb{T}_n := \bigsqcup_{i=1}^N \mathcal{T}_i(n)$ to be the disjoint union of vertices at depth n in the N BRWs, and fix an arbitrary (nonrandom) total order on $\mathbb{T}_n$ for each n. We now inductively define a sequence $(G_n)_{n \geq 0}$ of random subsets of $\mathbb{T}_n$, each with exactly N elements. These random subsets will correspond to the particles alive in the coupled N-braching random walk at time n. Define $G_0 = \mathbb{T}_0$ and given G_n, define H_n to be the vertices in $\mathbb{T}_{n+1}$ that descend from vertices in G_n. Finally, set G_{n+1} to be the set of N vertices in H_n with the gratest value, resolving ties via the fixed total order on $\mathbb{T}_{n+1}$. If we now define (with some abuse of notation) $\mathfrak{X}_n = \sum_{u,i:u \in G_n \cap \mathcal{T}_i} \delta_{\Phi_i(u)}$ then $(\mathfrak{X}_n)_{n \geq 0}$ has the same distribution as X started from $N\delta_0$.
Let us record a technical lemma that will be used in the proof of the lower bound in Theorem 5.4.

LEMMA 5.5 ([3, Adapted by Bérard and Gouéré from Lemma 5.2]) — *Let $v_1 < v_2 \in \mathbb{R}$ and $1 \leq m \leq n \in \mathbb{N}$. Suppose $0 =: x_0, ..., x_n$ is a sequence of real numbers such that $\max_{i \in [\![0,n-1]\!]}(x_{i+1} - x_i) \leq K$ for some $K > 0$, and define $I := \{i \in [\![0, n-m]\!] \mid x_{i+j} - x_i \geq jv_1, \quad \forall j \in [\![0, m]\!]\}$. If $x_n \geq v_2 n$, then $|I| \geq \frac{v_2 - v_1}{K - v_1}\frac{m}{n} - \frac{K}{K - v_1}$.*

Proof of lower bound in Theorem 5.4. As before, set $X_0 = N\delta_0$. Our aim is to show $v_N := \lim_{n\to\infty} \mathbb{E}\left[n^{-1} \max X_n\right] \leq v_\infty - \chi/(\log N)^2 + o((\log N)^{-2})$. However, we shall show this with v_∞ replaced by v, which combined with the upper bound also proves that $v_\infty = v$. Let $\beta \in (0, \chi)$ and let $\theta > 0$ be as in Theorem 5.4. Let $\lambda > 0$, and define

$$m := \left\lceil \theta^{3/2} \left(\frac{(1+\lambda)\log N}{(\chi - \beta)^{1/2}}\right)^3 \right\rceil, \tag{5.6}$$

and $\epsilon := \theta\, m^{-2/3}$. The scale of ϵ and m is carefully chosen so that by Theorem 5.4,

$$\rho(m, \epsilon) \leq N^{-(1+\lambda)} \qquad \text{for all large } N. \tag{5.7}$$

Take $\gamma \in (0,1)$ and define $v_1 = v - \epsilon$ and $v_2 = v - (1-\gamma)\epsilon$ noting that $v_1 < v_2 < v$. Finally, let $n = \lceil N^\xi \rceil$ for some $0 < \xi < \lambda$ and consider the following inequality with $\delta > 0$:

$$\begin{aligned}\mathbb{E}\left[n^{-1}\max X_n\right] &= \mathbb{E}\left[n^{-1}\max X_n\left[\mathbb{1}_{\{\max X_n < nv_2\}} + \mathbb{1}_{\{nv_2 \le \max X_n < n(v+\delta)n\}} + \mathbb{1}_{\{(v+\delta)n \le \max X_n\}}\right]\right] \\ &\le v_2 + (v+\delta)\underbrace{\mathbb{P}\left(\max X_n \le v_2 n\right)}_{(I)} + \underbrace{\mathbb{E}\left[n^{-1}\max X_n \mathbb{1}_{\{(v+\delta)n \le \max X_n\}}\right]}_{(II)}. \qquad (5.8)\end{aligned}$$

The strategy for the proof is to show that both (I) and (II) are $o((logN)^{-2})$. The result then follows, as $v_2 = v - (1-\gamma)(\chi-\beta)(1+\lambda)^{-2}(\log N)^{-2}$ where γ, β, λ can be taken arbitrarily small.

Let B_n be the number of vertices in $\sqcup_{i=1}^n G_i$ that are (m, v_1)-good with respect to their respective BRWs. Define $K = \kappa \log(2Nn)$ for some $\kappa > 0$ and notice that the quantity $\frac{v_2 - v_1}{K - v_1}\frac{m}{n} - \frac{K}{K - v_1} = \Theta(N^\xi (\log N)^{-4})$ so that for large enough N it is positive. Let $u_0, u_1, ..., u_n$ be a path in $\mathcal{T}_{i_0}$ for some $i_0 \in [\![1, N]\!]$ such that $u_0 = root_{i_0}$ and $u_n \in G_n$ with $\Phi_{i_0}(u_n) = \max X_n$. In other words, let $u_0, ..., u_n$ be the path from the root to the rightmost particle at time n of the coupled N-branching random walk. On the event $E := \{\max X_n \ge v_2 n\}$, we apply Lemma 5.5 to the sequence of real numbers $(\Phi_{i_0}(u_i))_{i \in [\![1,n]\!]}$ to see that either there is an (m, v_1)-good vertex among the u_i or one of the random walk steps along the path is $\ge K$. These events are respectively included in the events that $B_n \ge 1$ and that $M := \max\{\epsilon_{l,i,j} \mid l \in [\![0, n-1]\!],\ i \in [\![1, N]\!],\ j = 1, 2\} \ge K$. We can use this to bound the probability of E:

$$\mathbb{P}(E) \le \mathbb{P}(M \ge K) + \mathbb{P}(B_n \ge 1). \qquad (5.9)$$

Consider a vertex $u \in \mathcal{T}_{i_0}(d)$ for some $i_0 \in [\![1, N]\!]$ at depth $d \in [\![0, n]\!]$. The event $\{u \in G_d\}$ is measurable with respect to the sigma algebra generated by the random variables $\{\Phi_j(v) \mid j \in [\![1, N]\!],\ \mathcal{T}_j \ni v\text{'s depth } \le d\}$. On the other hand, the event $\{u \text{ is } (m, v_1)\text{-good}\}$ is determined by the variables $\{\Phi_{i_0}(v) - \Phi_{i_0}(u) \mid \mathcal{T}_{i_0} \ni v\text{'s depth } > d\}$, so that the two events are independent. We can write B_n as

$$B_n = \sum_{i \in [\![1,N]\!],\ u \in \mathcal{T}_i} \mathbb{1}_{\{u \text{ is } (n,v_1)\text{-good}\}} \mathbb{1}_{\{u \in G_d \text{ for some d} \in [\![0,n]\!]\}}.$$

Taking expectations gives

$$\mathbb{E}[B_n] \le N(n+1)\rho(m,\epsilon) = \mathcal{O}(N^{\xi-\lambda}) \qquad \text{as } N \text{ goes to infinity,}$$

where we used that G_n has N elements for all n. Recall that the distribution p has exponentially decaying tails, so that there exist $C, \gamma > 0$ such that $\mathbb{P}_{X\sim p}(X > t) \le C\exp(-\gamma t)$ for all large t. This gives $\mathbb{P}(M \ge K) \le 1 - (1 - \exp(-\gamma\kappa\log(2Nn)))^N = \mathcal{O}(N(2Nn)^{-\gamma\kappa}) = \mathcal{O}(N^{1-(\xi+1)\gamma\kappa})$. Thus we can take κ large enough so that we get

$$\mathbb{P}(E) \le \mathcal{O}(N^{\xi-\lambda}), \qquad \text{which proves } (I) = o((\log N)^{-2}). \qquad (5.10)$$

To show that $(II) = o((\log N)^{-2})$ first consider the obvious inequality $\exp(t\max X_n) \le \sum_{i\in[\![1,N]\!],\ u \in \mathcal{T}_i(n)} \exp(t\Phi_i(u))$. Taking expectations gives $\mathbb{E}[\exp(t\max X_n)] \le N2^n\exp(n\Lambda(t))$, where we used a telescoping sum along the path connecting the root and u and the fact that $\#\mathcal{T}_i(n) = 2^n$ for each i. Recalling from Assumption 2 and Remark 5.1 that $\Lambda(t^*) = vt^* - \log 2$, we obtain

$$\mathbb{E}[\exp(t^*\max X_n)] \le N\exp(vnt^*). \qquad (5.11)$$

Lemma 5.6 — *Let $b > 0$. Then for all large enough a,*

$$x\mathbb{1}_{\{x \ge a\}} \le \exp\left(b\left(x - \frac{a}{2}\right)\right), \qquad \forall x \in \mathbb{R}. \qquad (5.12)$$

Proof. Differentiate the map $f : x \mapsto \exp(b(x - a/2)) - x$ to find that for large enough a, f is increasing on $[a, \infty)$. Noting that $f(a) \ge 0$ for all large a concludes the proof. $\square$

Apply Lemma 5.6 with $X = \max X_n - vn$, $a = \delta n, b = t^*$ and take expectations to get

$$\mathbb{E}\left[(\max X_n - vn)\mathbb{1}_{\{\max X_n \geq (v+\delta)n\}}\right] \leq \mathbb{E}\left[\exp(t^*(X_n - vn - \delta n/2))\right],$$

which combined with 5.11 and a Chernoff bound gives

$$(II) = \mathbb{E}\left[\max X_n \mathbb{1}_{\{\max X_n \geq (v+\delta)n\}}\right] \leq N\exp(-\delta n/2)(1 + |v|n) = o((logN)^{-2}).$$

We have shown that for any choice of $\gamma \in (0,1)$, $\beta \in (0,\chi)$ and $\lambda > \xi > 0$, for all N large enough

$$\mathbb{E}\left[\lceil N^{\xi}\rceil^{-1} \max X_{\lceil N^{\xi}\rceil}\right] \leq v - (1-\gamma)\frac{\chi - \beta}{(1+\lambda)^2(\log N)^2} + o((\log N)^{-2}). \tag{5.13}$$

Recall from the proof of Proposition 5.2 that $v_N = \inf_n n^{-1}\mathbb{E}\left[\max X_n\right]$, so the left hand side in 5.13 can be replaced by v_N. Taking γ, β, λ and ξ to zero gives the desired result. $\square$

5.2 Generalised Bernoulli

Placeholder text.

DEfine the *censored* Galton Watson process $(X_n)_{n\geq 0}$ with offspring distribution $\mathcal{X}$ to be the discrete time $\mathbb{N}$ valued stochastic process satisfying

(i) $X_0 = N$

(ii) $X_{n+1} = \min\left\{N, \sum_{i=1}^{X_n} \mathcal{X}_{i,n+1}\right\}$ for all $n \geq 0$

where $(\mathcal{X}_{i,n})_{i\geq 0,\, n\geq 0}$ is an i.i.d. collection of random variables with distribution $\mathcal{X}$. Complete at some later point, since this is really easy material.

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