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N-BRANCHING RANDOM WALK WITH α-STABLE SPINE*

B. MALLEIN†

Abstract. We consider a branching-selection particle system on the real line, introduced by Brunet and Derrida in [*Phys. Rev. E*, 56 (1997), pp. 2597–2604]. In this model the size of the population is fixed to a constant N. At each step individuals in the population reproduce independently, making children around their current position. Only the N rightmost children survive to reproduce at the next step. Bérard and Gouéré studied the speed at which the cloud of individuals drifts in [*Comm. Math. Phys.*, 298 (2010), pp. 323–342], assuming the tails of the displacement decays at exponential rate; Bérard and Maillard [*Electron. J. Probab.*, 19 (2014), 22] took interest in the case of heavy tail displacements. We take interest in an intermediate model, considering branching random walks in which the critical "spine" behaves as an α-stable random walk.

Key words. branching random walk, selection, stable distribution

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1. Introduction. Let $\mathscr{L}$ be the law of a random point process on $\mathbf{R}$. Brunet et al. introduced in [3], [4] a discrete-time branching-selection particle system on $\mathbf{R}$ in which the size of the population is limited by some integer N. This process evolves as follows: for any $n \in \mathbf{N}$, every individual alive at the nth generation dies giving birth to children around its current position, according to an independent version of a point process of law $\mathscr{L}$. Only the N children with the largest position are kept alive and form the $(n+1)$st generation of the process. We write $(x_n^N(1), \ldots, x_n^N(N))$ for the positions at time n of individuals in the process, ranked in decreasing order. This process is called the N-*branching random walk*, or N-BRW for short.

In [1], Bérard and Gouéré proved that under some appropriate integrability conditions, the cloud of particles drifts at some deterministic speed

$$v_N := \lim_{n\to+\infty} \frac{x_n^N(1)}{n} = \lim_{n\to+\infty} \frac{x_n^N(N)}{n} \quad \text{a.s.,} \tag{1.1}$$

and obtained the following asymptotic behavior for v_N:

$$v_\infty - v_N \underset{N\to+\infty}{\sim} \frac{C}{(\log N)^2}, \tag{1.2}$$

in which C is an explicit positive constant that depends only on the law $\mathscr{L}$. Their argument is based on a coupling (recalled in section 4.2) between the N-branching random walk and a branching random walk, which we define below.

A *branching random walk* with branching law $\mathscr{L}$ is a process defined as follows. It starts with a unique individual located at position 0 at time 0. At each time $k \in \mathbf{N}$, every individual alive in the process at time k dies giving birth to children. The children are positioned around their parent according to i.i.d. point processes with law $\mathscr{L}$.

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†Laboratoire de Probabilités et Modèles Aléatoires, Université Pierre et Marie Curie (Paris 6), and Département de Mathématiques et Applications, École Normale Supérieure, Paris, France (bastien.mallein@ens.fr).

We write $\mathbf{T}$ for the genealogical tree of the process. For $u \in \mathbf{T}$, we denote by $V(u)$ the position of u, by $|u|$ the time at which u is alive, by πu the parent of u (provided that u is not the root of $\mathbf{T}$), and by u_k the ancestor alive at time k of u. Let $\Omega(u)$ be the set of siblings of u, i.e., the set of individuals $v \in \mathbf{T}$ such that $\pi v = \pi u$ and $v \neq u$. We observe that $\mathbf{T}$ is a (random) Galton–Watson tree with reproduction law $\#\mathscr{L}$.

We list assumptions on the point process law $\mathscr{L}$. Let L be a point process with law $\mathscr{L}$. We first assume that the Galton–Watson tree $\mathbf{T}$ is supercritical and a.s. infinite, i.e.,

$$\mathbf{E}[\#L] > 1 \quad \text{and} \quad \mathbf{P}(\#L = 0) = 0. \tag{1.3}$$

Note that if $\mathbf{P}(\#L = 0) > 0$, while $\mathbf{T}$ might be infinite with positive probability, the N-BRW dies out almost surely. We also suppose the point process law $\mathscr{L}$ to be in the stable boundary case in the following sense:

$$\mathbf{E}\left[\sum_{\ell \in L} e^{\ell}\right] = 1, \tag{1.4}$$

and the random variable X defined by

$$\mathbf{P}(X \leqslant x) = \mathbf{E}\left[\sum_{\ell \in L} \mathbf{1}_{\{\ell \leqslant x\}} e^{\ell}\right] \tag{1.5}$$

is in the domain of attraction of a stable random variable Y verifying $\mathbf{P}(Y \geqslant 0) \in (0,1)$.

Using [5, Chap. XVII], we provide a necessary and sufficient condition for X to be in the domain of attraction of Y. Let $\alpha \in (0,2]$ be such that Y is an α-stable random variable verifying $\mathbf{P}(Y \geqslant 0) \in (0,1)$. We introduce the function

$$L^* \colon x \mapsto x^{\alpha-2} \mathbf{E}[Y^2 \mathbf{1}_{\{|Y| \leqslant x\}}]. \tag{1.6}$$

This function is slowly varying (i.e., for all $\lambda > 0$, $\lim_{t \to +\infty} L^*(\lambda t)/L^*(t) = 1$). We set

$$b_n = \inf\left\{x > 0 \colon \frac{x^\alpha}{L^*(x)} = n\right\}. \tag{1.7}$$

The random variable X is in the domain of attraction of Y if and only if for (S_n) a random walk with step distribution with the same law as X, S_n/b_n converges in law to Y.

Note that if $\mathbf{E}[|X|] < +\infty$, then by the strong law of large numbers we have $S_n/n \to \mathbf{E}[X]$ a.s. Thus (1.5) implies that $\mathbf{E}[X] = 0$. In that case, $\mathscr{L}$ is in the boundary case, as defined in [2]. Up to an affine transformation, several point process laws verify these properties, adapting the discussion in Appendix A of [8] to this setting.

Since Y is an α-stable random variable, there exists an α-stable Lévy process $(Y_t, t \geqslant 0)$ such that Y_1 has the same law as Y. Using Lemma 1 of [13], we define

$$C_* := \lim_{t \to +\infty} \left[-\frac{1}{t} \log \mathbf{P}\left(|Y_s| \leqslant \frac{1}{2},\, s \leqslant t\right)\right] \in (0, +\infty). \tag{1.8}$$

The next integrability assumption on $\mathscr{L}$ ensures that the spine of the branching random walk (see section 2.1) behaves as a typical individual staying close to the boundary of the process:

$$\lim_{x\to+\infty} \frac{x^\alpha}{L^*(x)} \mathbf{E}\Big[\sum_{\ell\in L} e^\ell \mathbf{1}_{\{\log \sum_{\ell'\in L} e^{\ell'} > x+\ell\}}\Big] = 0. \tag{1.9}$$

Finally, we assume that

$$\mathbf{E}\Big[\Big|\max_{\ell\in L} \ell\Big|^2\Big] < +\infty; \tag{1.10}$$

this condition is not expected to be optimal but is sufficient to bound from below in a crude way the minimal position in the N-BRW, which we use when the coupling fails.

THEOREM 1.1. *Under the previous assumptions, for an N-BRW with reproduction law $\mathscr{L}$, the sequence $(v_N, N \geqslant 1)$ defined in* (1.1) *exists and verifies*

$$v_N \underset{N\to+\infty}{\sim} -C_* \frac{L^*(\log N)}{(\log N)^\alpha}. \tag{1.11}$$

We observe that if $\mathscr{L}$ satisfies

$$\mathbf{E}\Big[\sum_{\ell\in L} e^\ell \log\Big(\sum_{\ell'\in L} e^{\ell'-\ell}\Big)^2\Big] + \mathbf{E}\Big[\sum_{\ell\in L} \ell^2 e^\ell\Big] < +\infty, \tag{1.12}$$

then Theorem 1.1 implies that (1.2) holds with $C = (\pi^2/2)\mathbf{E}\big[\sum_{\ell\in L}\ell^2 e^\ell\big]$, which is consistent with the result of Bérard and Gouéré [1].

Examples. We present two point process laws that satisfy the hypotheses of Theorem 1.1. Let X be a random variable on $(0,+\infty)$. We write $\Lambda(\theta)$ for the log-Laplace transform of the law of X. We assume that there exist $\theta^* > 0$ such that $\Lambda(\theta^*) = \log 2$, and $\alpha > 1$ verifying

$$\mathbf{P}(X \geqslant x) \sim e^{-\theta^* x} x^{-\alpha-1}.$$

In this case, there exists $\mu := \mathbf{E}\big[X e^{\theta^* X}\big]/2$ such that the point process $\mathscr{L}$ defined as the law of a pair of independent random variables (Y_1, Y_2) which have the same law as $\theta^*(X-\mu)$ satisfies the hypotheses of Theorem 1.1.

Let ν_α be the law of an α-stable random variable Y such that $\mathbf{P}(Y \geqslant 0) \in (0,1)$. If $\widetilde{\mathscr{L}}$ is the law of a Poisson point process on $\mathbf{R}$ with intensity $\nu(dx)\,e^{-x}$, then $\widetilde{\mathscr{L}}$ satisfies the assumptions of Theorem 1.1, and the spine of such a branching random walk is in the domain of attraction of Y.

The rest of the article is organized as follows. In section 2, we introduce the spinal decomposition that links the computation of additive branching random walk moments with random walks estimates and the Mogul'skii small deviations estimate for random walks. In section 3, these results are used to compute the asymptotic behavior of the survival probability of a branching random walk with a killing line of slope $-\varepsilon$, using the same technique as [7], [12]. This asymptotics is then used in section 4 to prove Theorem 1.1, applying the methods introduced in [1].

2. Spinal decomposition and small deviations estimate.

2.1. The spinal decomposition. The spinal decomposition is a tool introduced by Lyons, Pemantle, and Peres in [11] to study branching processes. It has been extended to branching random walks by Lyons in [10]. It provides two descriptions of a law absolutely continuous with respect to the law $\mathbf{P}_a$ of the branching random walk $(\mathbf{T}, V+a)$. Let $W_n = \sum_{|u|=n} e^{V(u)}$, and let $\mathscr{F}_n = \sigma(u, V(u), |u| \leqslant n)$ be the natural filtration on the set of marked trees. By (1.4), (W_n) is a nonnegative martingale. We define the probability measure $\overline{\mathbf{P}}_a$ on $\mathscr{F}_\infty$ such that for any $n \in \mathbf{N}$,

$$\left.\frac{d\overline{\mathbf{P}}_a}{d\mathbf{P}_a}\right|_{\mathscr{F}_n} = e^{-a} W_n. \tag{2.1}$$

We write $\overline{\mathbf{E}}_a$ for the corresponding expectation.

We construct a second probability measure $\widehat{\mathbf{P}}_a$ on the set of marked trees with spine. For a marked tree $(\mathbf{T}, V)$, we say that $w = (w_n, n \geqslant 0)$ is a spine of $\mathbf{T}$ if for any $n \in \mathbf{N}$

$$|w_n| = n, \quad w_n \in \mathbf{T}, \quad \text{and} \quad (w_n)_{n-1} = w_{n-1}.$$

We introduce

$$\frac{d\widehat{\mathscr{L}}}{d\mathscr{L}} = \sum_{\ell \in L} e^{\ell}, \tag{2.2}$$

another point process law. The probability measure $\widehat{\mathbf{P}}_a$ is the law of the process $(\mathbf{T}, V, w)$ constructed as follows. It starts at time 0 with a unique individual w_0 located at position a. It makes children according to a point process of law $\widehat{\mathscr{L}}$. An individual w_1 is chosen at random among children u of w_0 with probability $e^{V(u)}/W_1$. Similarly, at each $n \in \mathbf{N}$, every individual u in the nth generation dies, giving birth to children according to independent point processes, with law $\widehat{\mathscr{L}}$ if $u = w_n$ or law $\mathscr{L}$ otherwise. Finally w_{n+1} is chosen among children v of w_n with probability proportional to $e^{V(v)}$. To shorten notation we write $\overline{\mathbf{P}} = \overline{\mathbf{P}}_0$ and $\widehat{\mathbf{P}} = \widehat{\mathbf{P}}_0$.

PROPOSITION 2.1 (spinal decomposition). *Under assumption* (1.4), *for any* $n \in \mathbf{N}$, *we have*

$$\left.\widehat{\mathbf{P}}_a\right|_{\mathscr{F}_n} = \left.\overline{\mathbf{P}}_a\right|_{\mathscr{F}_n}.$$

Moreover, for any $z \in \mathbf{T}$ *such that* $|z| = n$,

$$\widehat{\mathbf{P}}_a(w_n = z \mid \mathscr{F}_n) = \frac{e^{V(z)}}{W_n},$$

and $(V(w_n), n \geqslant 0)$ *is a random walk starting from* a, *with step distribution defined in* (1.5).

A direct consequence of this proposition is the many-to-one lemma. Introduced by Peyrière [9], [14], this lemma links additive moments of the branching random walks with random walk estimates. Given (X_n), an i.i.d. sequence of random variables with law defined by (1.5), we set $S_n = S_0 + \sum_{j=1}^n X_j$ such that $\mathbf{P}_a(S_0 = a) = 1$.

LEMMA 2.1 (many-to-one lemma). *Under assumption* (1.4), *for any measurable function g and $n \geqslant 1$ we have*

$$\mathbf{E}_a\Big[\sum_{|u|=n} g(V(u_1),\dots,V(u_n))\Big] = \mathbf{E}_a\big[e^{a-S_n} g(S_1,\dots,S_n)\big]. \tag{2.3}$$

Proof. We use Proposition 2.1 to compute

$$\begin{aligned}\mathbf{E}_a\Big[\sum_{|u|=n} g(V(u_1),\dots,V(u_n))\Big] &= \overline{\mathbf{E}}_a\Big[\frac{e^a}{W_n}\sum_{|u|=n} g(V(u_1),\dots,V(u_n))\Big]\\ &= \widehat{\mathbf{E}}_a\Big[e^a\sum_{|u|=n}\widehat{\mathbf{P}}_a(w_n=u \mid \mathscr{F}_n)e^{-V(u)}g(V(u_1),\dots,V(u_n))\Big]\\ &= \widehat{\mathbf{E}}_a\big[e^{a-V(w_n)}g(V(w_1),\dots,V(w_n))\big].\end{aligned}$$

We now observe that $(S_n, n\geqslant 0)$ under $\mathbf{P}_a$ has the same law as $(V(w_n), n\geqslant 0)$ under $\widehat{\mathbf{P}}_a$, which ends the proof.

The many-to-one lemma can be used to bound the maximal displacement in a branching random walk. For example, for all $y \geqslant 0$, we have

$$\begin{aligned}\mathbf{E}\Big[\sum_{u\in\mathbf{T}}\mathbf{1}_{\{V(u)\geqslant y\}}\mathbf{1}_{\{V(u_j)<y,\, j<|u|\}}\Big] &= \sum_{k=1}^{+\infty}\mathbf{E}\Big[\sum_{|u|=k}\mathbf{1}_{\{V(u)\geqslant y\}}\mathbf{1}_{\{V(u_j)<y,\, j<|u|\}}\Big]\\ &= \sum_{k=1}^{+\infty}\mathbf{E}\big[e^{-S_k}\mathbf{1}_{\{S_k\geqslant y\}}\mathbf{1}_{\{S_j<y,\, j<k\}}\big]\\ &\leqslant e^{-y}\sum_{k=1}^{+\infty}\mathbf{P}(S_k\geqslant y,\, S_j<y,\, j<k)\leqslant e^{-y}.\end{aligned}$$

Obviously, this computation leads to

$$\sup_{n\in\mathbf{N}}\mathbf{P}\Big(\max_{|u|=n}V(u)\geqslant y\Big)\leqslant \mathbf{P}\Big(\max_{u\in\mathbf{T}}V(u)\geqslant y\Big)\leqslant e^{-y}. \tag{2.4}$$

Using the spinal decomposition, to compute the number of individuals in a branching random walk who stay in a well-chosen path, it is enough to know the probability for a random walk decorated by additional random variables to follow that path.

2.2. Small deviations estimate and variations. Let S be a random walk in the domain of attraction of an α-stable random variable Y. We recall that

$$L^*(u) = u^{\alpha-2}\mathbf{E}Y^2\mathbf{1}_{\{|Y|\leqslant u\}} \quad\text{and}\quad \frac{b_n^\alpha}{L^*(b_n)} = n.$$

For any $z\in\mathbf{R}$, we define $\mathbf{P}_z$ such that S under law $\mathbf{P}_z$ has the same law as $S+z$ under law $\mathbf{P}$. The Mogul'skii small deviation estimate enables one to compute the probability for S to present fluctuations of order $o(b_n)$.

THEOREM 2.1 (Mogul'skii [13]). *Let $(a_n)\in\mathbf{R}_+^{\mathbf{N}}$ be such that*

$$\lim_{n\to+\infty} a_n = +\infty \quad \textit{and} \quad \lim_{n\to+\infty}\frac{a_n}{b_n} = 0.$$

Next, let $f < g$ be two continuous functions that verify $f(0)<0<g(0)$. If $\mathbf{P}(Y\leqslant 0)\in(0,1)$, then

$$\lim_{n\to+\infty}\frac{a_n^\alpha}{nL^*(a_n)}\log\mathbf{P}\left(\frac{S_j}{a_n}\in\left[f\left(\frac{j}{n}\right),g\left(\frac{j}{n}\right)\right],\leqslant j\leqslant n\right)$$
$$=-C_*\int_0^1\frac{ds}{(g(s)-f(s))^\alpha},$$

where C_* is defined in (1.8).

This result can be seen as a consequence of an α-stable version of the Donsker theorem, obtained by Prokhorov. This result yields the convergence of the normalized trajectory of the random walk S to the trajectory of an α-stable Lévy process $(Y_t, t\in[0,1])$ such that Y_1 has the same law as Y.

THEOREM 2.2 (Prokhorov [15]). *If S_n/b_n converges in law to a stable random variable Y, then $(\lfloor nt\rfloor/b_n,\ t\in[0,1])$ converges in law to $(Y_t,\ t\in[0,1])$ in $\mathscr{D}([0,1])$ equipped with the Skorokhod topology.*

We observe that the Mogul'skii estimate holds uniformly with respect to the starting point.

COROLLARY 2.1. *With the same notation as in Theorem* 2.1*, we have*

$$\lim_{n\to+\infty}\frac{a_n^\alpha}{nL^*(a_n)}\log\sup_{y\in\mathbf{R}}\mathbf{P}_y\left(\frac{S_j}{a_n}\in\left[f\left(\frac{j}{n}\right),g\left(\frac{j}{n}\right)\right],\ 0\leqslant j\leqslant n\right)$$
$$=-C_*\int_0^1\frac{ds}{(g(s)-f(s))^\alpha}.$$

Proof. Observe first that if $y\notin[a_nf(0),a_ng(0)]$, then

$$\mathbf{P}_y\left(\frac{S_j}{a_n}\in\left[f\left(\frac{j}{n}\right),g\left(\frac{j}{n}\right)\right],\ 0\leqslant j\leqslant n\right)=0.$$

We now choose $\delta>0$ and write $K=\lceil(g(0)-f(0))/\delta\rceil$. We have

$$\sup_{y\in\mathbf{R}}\mathbf{P}_y\left(\frac{S_j}{a_n}\in\left[f\left(\frac{j}{n}\right),g\left(\frac{j}{n}\right)\right],\ 0\leqslant j\leqslant n\right)\leqslant\max_{k\leqslant K}\Pi_{f(0)+k\delta,f(0)+(k+1)\delta}(f,g),$$

where

$$\Pi_{x,x'}(f,g)=\sup_{y\in[xa_n,x'a_n]}\mathbf{P}_y\left(\frac{S_j}{a_n}\in\left[f\left(\frac{j}{n}\right),g\left(\frac{j}{n}\right)\right],\ 0\leqslant j\leqslant n\right)$$
$$\leqslant\mathbf{P}\left(\frac{S_j}{a_n}\in\left[f\left(\frac{j}{n}\right)-x',g\left(\frac{j}{n}\right)-x\right],\ 0\leqslant j\leqslant n\right).$$

Therefore, for all $k\leqslant K$,

$$\limsup_{n\to+\infty}\frac{a_n^\alpha}{nL^*(a_n)}\log\Pi_{f(0)+k\delta,f(0)+(k+1)\delta}(f,g)\leqslant-C_*\int_0^1\frac{ds}{(g(s)-f(s)+\delta)^\alpha},$$

which leads to

$$\limsup_{n\to+\infty} \frac{a_n^\alpha}{nL^*(a_n)} \log \sup_{y\in\mathbf{R}} \mathbf{P}\left(\frac{S_j+y}{a_n} \in \left[f\left(\frac{j}{n}\right), g\left(\frac{j}{n}\right)\right], 0 \leqslant j \leqslant n\right)$$
$$\leqslant -C_* \int_0^1 \frac{ds}{(g(s)-f(s)+\delta)^\alpha}.$$

Letting $\delta \to 0$ concludes the proof, as the lower bound is a direct consequence of Theorem 2.1.

Using an adjustment of the original proof of Mogul'skii, one can prove a similar estimate for enriched random walks. Let (X_n, ξ_n) be a sequence of i.i.d. random variables on $\mathbf{R} \times \mathbf{R}_+$, with X_1 in the domain of attraction of the stable random variable Y, such that $\mathbf{P}(Y \geqslant 0) \in (0,1)$. We introduce $S_n = S_0 + X_1 + \cdots + X_n$, which is a random walk in the domain of attraction of Y. The following estimate then holds.

LEMMA 2.2. *Let $(a_n) \in \mathbf{R}_+^{\mathbf{N}}$ be such that $\lim_{n\to+\infty} a_n/b_n = 0$. We set $E_n = \{\xi_j \leqslant n,\ j \leqslant n\}$ and assume that*

$$\lim_{n\to+\infty} \frac{a_n^\alpha}{L^*(a_n)} \mathbf{P}(\xi_1 \geqslant n) = 0. \tag{2.5}$$

There exists $C_* > 0$, given by (1.8), *such that, for any pair (f,g) of continuous functions verifying $f < g$ and for any $f(0) < x < y < g(0)$,*

$$\lim_{n\to+\infty} \frac{a_n^\alpha}{nL^*(a_n)} \log \inf_{z\in[xa_n, ya_n]} \mathbf{P}_z\left(\frac{S_j}{a_n} \in \left[f\left(\frac{j}{n}\right), g\left(\frac{j}{n}\right)\right], j \leqslant n,\ E_n\right)$$
$$= -C_* \int_0^1 \frac{ds}{(g(s)-f(s))^\alpha}.$$

Proof. We first assume that f, g are two constant functions. For $n \geqslant 1$, $f < x < y < g$, and $f < x' < y' < g$, we denote

$$P_{x,y}^{x',y'}(f,g) = \inf_{z\in[x,y]} \mathbf{P}_{za_n}\left(\frac{S_n}{a_n} \in [x',y'],\ \frac{S_j}{a_n} \in [f,g],\ j \leqslant n,\ E_n\right). \tag{2.6}$$

Let $A > 0$ and $r_n = \lfloor A a_n^\alpha / L^*(a_n) \rfloor$. We divide $[0,n]$ into $K = \lfloor n/r_n \rfloor$ intervals of length r_n. For any $k \leqslant K$, we set $m_k = kr_n$ and $m_{K+1} = n$. Applying the Markov property at time $m_K, \ldots, m_1$, and restricting to trajectories which are in $[x'a_n, y'a_n]$ at any time m_k, we have

$$P_{x,y}^{x',y'}(f,g) \geqslant \pi_{x,y}^{x',y'}(f,g) \big(\pi_{x',y'}^{x',y'}(f,g)\big)^K, \tag{2.7}$$

where we set $\pi_{x,y}^{x',y'}(f,g) = \inf_{z\in[x,y]} \mathbf{P}_{za_n}\big(S_{r_n}/a_n \in [x',y'],]\, S_j/a_n \in [f,g], j \leqslant r_n,\ E_{r_n}\big)$.

We choose a sufficiently small $\delta > 0$ such that $M = \lceil (y-x)/\delta \rceil \geqslant 3$. We observe easily that

$$\pi_{x,y}^{x',y'}(f,g) \geqslant \min_{0\leqslant m\leqslant M} \pi_{x+m\delta, x+(m+1)\delta}^{x',y'}(f,g)$$
$$\geqslant \min_{0\leqslant m\leqslant M} \pi_{x,x}^{x'-(m-1)\delta, y'-(m+1)\delta}(f-(m-1)\delta,\ g-(m+1)\delta). \tag{2.8}$$

Moreover, we have

$$\pi^{x',y'}_{x,x}(f,g) = \mathbf{P}_{xa_n}\left(\frac{S_{r_n}}{a_n} \in [x',y'],\ \frac{S_j}{a_n} \in [f,g],\ j \leqslant r_n,\ E_{r_n}\right)$$
$$\geqslant \mathbf{P}_{xa_n}\left(\frac{S_{r_n}}{a_n} \in [x',y'],\ \frac{S_j}{a_n} \in [f,g],\ j \leqslant r_n\right) - r_n\mathbf{P}(\xi_1 \geqslant n).$$

By (2.5), $\lim_{n\to+\infty} r_n\mathbf{P}(\xi_1 \geqslant n) = 0$. Moreover, $r_n \sim Aa_n^\alpha/L^*(a_n)$, and X_1 is in the domain of attraction of Y. Thus S_{r_n}/a_n converges in law towards $A^{1/\alpha}Y$ as $n \to +\infty$. By Theorem 2.2, the process $\big(S_{\lfloor r_n t/A\rfloor}/a_n,\ t \in [0,A]\big)$ converges as $n \to +\infty$ under law $\mathbf{P}_{xa_n}$ to a stable Lévy process $(x + Y_t,\ t \in [0,A])$ such that Y_A has the same law as $A^{1/\alpha}Y$. In particular,

$$\liminf_{n\to+\infty} \pi^{x',y'}_{x,x}(f,g) \geqslant \mathbf{P}_x\big(Y_A \in (x',y'),\ Y_u \in (f,g),\ u \leqslant A\big).$$

Using (2.8), we have

$$\liminf_{n\to+\infty} \pi^{x',y'}_{x,y}(f,g)$$
$$\geqslant \min_{0\leqslant m\leqslant M} \mathbf{P}_{x+m\delta}(Y_A \in (x'+\delta, y'-\delta),\ Y_u \in (f+\delta, g-\delta),\ u \leqslant A).$$

As a consequence, recalling that $K \sim nL^*(a_n)/(Aa_n^\alpha)$, (2.7) leads to

$$\liminf_{n\to+\infty} \frac{a_n^\alpha}{nL^*(a_n)} \log P^{x',y'}_{x,y}(f,g) \tag{2.9}$$
$$\geqslant \frac{1}{A}\min_{0\leqslant m\leqslant M} \log \mathbf{P}_{x'+m\delta}(Y_A \in (x'+\delta, y'-\delta),\ Y_u \in (f+\delta, g-\delta),\ u \leqslant A).$$

By Lemma 1 of [13] we have

$$\lim_{t\to+\infty} \frac{1}{t} \log \mathbf{P}_x(Y_t \in (x',y'),\ Y_s \in (f,g),\ s \leqslant t) = -\frac{C_*}{(g-f)^\alpha},$$

where C_* is defined by (1.8). Letting $A \to +\infty$ and then $\delta \to 0$, (2.9) yields

$$\liminf_{n\to+\infty} \frac{a_n^\alpha}{nL^*(a_n)} \log P^{x',y'}_{x,y}(f,g) \geqslant -\frac{C_*}{(g-f)^\alpha}, \tag{2.10}$$

which is the expected result when f, g are two constants.

Further, we consider two continuous functions $f < g$. Let $f(0) < x < y < g(0)$. We set h a continuous function such that $f < h < g$ and $h(0) = (x+y)/2$. Let $\varepsilon > 0$ be such that $6\varepsilon \leqslant \inf_{t\in[0,1]} \min(g(t)-h(t), h(t)-f(t))$. We choose $A > 0$ such that

$$\sup_{|t-s|\leqslant 2/A} \big(|f(t)-f(s)| + |g(t)-g(s)| + |h(t)-h(s)|\big) \leqslant \varepsilon,$$

and for $a \leqslant A$, we define $m_a = \lfloor an/A\rfloor$ and $I_{a,A} = [f(a/A)+\varepsilon, g(a/A)-\varepsilon]$. We next set $J_{0,A} = [x,y]$, and for $1 \leqslant a \leqslant A$, $J_{a,A} = [h(a/A)-\varepsilon, h(a/A)+\varepsilon]$. Applying the Markov property at times $m_{A-1},\dots,m_1$, we have

$$\inf_{z\in[xa_n,ya_n]} \mathbf{P}_z\left(\frac{S_j}{a_n} \in \left[f\left(\frac{j}{n}\right), g\left(\frac{j}{n}\right)\right],\ j \leqslant n,\ E_n\right)$$
$$\geqslant \prod_{a=0}^{A-1} \inf_{z\in J_{a,A}} \mathbf{P}_{za_n}\left(\frac{S_{m_{a+1}}}{a_n} \in J_{a+1,A},\ \frac{S_j}{a_n} \in I_{a,A},\ j \leqslant m_{a+1}-m_a,\ E_{m_{a+1}-m_a}\right).$$

Therefore, using (2.10), we have

$$\liminf_{n\to+\infty} \frac{a_n^\alpha}{nL^*(a_n)} \log \inf_{z\in[xa_n,ya_n]} \mathbf{P}_z\left(\frac{S_j}{a_n} \in \left[f\left(\frac{j}{n}\right), g\left(\frac{j}{n}\right)\right], j \leqslant n,\, E_n\right)$$
$$\geqslant -\frac{1}{A}\sum_{a=0}^{A-1} C_* \frac{1}{(g(a/A) - f(a/A) - 2\varepsilon)^\alpha}.$$

Since the upper bound is a straightforward consequence of Theorem 2.1, we let $A \to +\infty$ and $\varepsilon \to 0$ to conclude the proof of Lemma 2.2.

3. Branching random walk with a barrier.

Let $(\mathbf{T}, V)$ be a branching random walk with reproduction law $\mathscr{L}$ satisfying the hypotheses of Theorem 1.1. In this section we study the asymptotic behavior, as $n \to +\infty$ and $\varepsilon \to 0$, of the quantity

$$\varrho(n,\varepsilon) = \mathbf{P}\big(\exists\, |u| = n\colon \forall\, j \leqslant n,\, V(u_j) \geqslant -\varepsilon j\big). \tag{3.1}$$

The asymptotic behavior of $\varrho(\infty,\varepsilon)$ has been studied by Gantert, Hu, and Shi in [7] for a branching random walk with a spine in the domain of attraction of a Gaussian random variable. They examined the asymptotic behavior of $\varrho(n,\varepsilon)$ for $\varepsilon \approx \theta n^{-2/3}$. Using the same arguments, we obtain sharp estimates on the asymptotic behavior of $\varrho(n,\varepsilon)$ for $\varepsilon \approx \theta\Lambda(n)n^{-\alpha/(\alpha+1)}$, where Λ is a well-chosen slowly varying function.

We apply the spinal decomposition and the Mogul'skii estimate to compute the number of individuals that stay at any time $k \leqslant n$ between curves $a_n f(k/n)$ and $a_n g(k/n)$, for an appropriate choice of (a_n), f, and g. We note that

$$\mathbf{E}\Bigg[\sum_{|u|=n} \mathbf{1}_{\{V(u_j)\in[a_n f(j/n),\, a_n g(j/n)],\, j\leqslant n\}}\Bigg] = \mathbf{E}\big[e^{-S_n}\mathbf{1}_{\{S_j\in[a_n f(j/n),\, a_n g(j/n)],\, j\leqslant n\}}\big]$$
$$\approx e^{-a_n g(1)}\mathbf{P}\left(S_j \in \left[a_n f\left(\frac{j}{n}\right), a_n g\left(\frac{j}{n}\right)\right], j \leqslant n\right)$$
$$\approx \exp\left(-a_n g(1) - \frac{nL^*(a_n)}{a_n^\alpha} C_* \int_0^1 \frac{ds}{(g(s)-f(s))^\alpha}\right).$$

This informal computation hints that to obtain tight estimates it is appropriate to choose a sequence (a_n) satisfying $a_n \sim_{n\to+\infty} nL^*(a_n)/a_n^\alpha$ and functions f and g verifying

$$\forall\, t \in [0,1], \quad g(t) + C_* \int_0^t \frac{ds}{(g(s)-f(s))^\alpha} = g(0). \tag{3.2}$$

However, instead of solving explicitly $g'(t) = -C_*(g(t)+\theta t)^{-\alpha}$ as a function of (t,θ), we use approximate solutions for (3.2).

For $n \in \mathbf{N}$, we define

$$a_n = \inf\left\{x \geqslant 0\colon \frac{x^{\alpha+1}}{L^*(x)} = n\right\}, \tag{3.3}$$

and we introduce the function

$$\begin{aligned} \Phi\colon (0,+\infty) &\to \mathbf{R}, \\ \lambda &\mapsto \frac{C_*}{\lambda^\alpha} - \frac{\lambda}{\alpha+1}. \end{aligned} \tag{3.4}$$

Note that Φ is a $\mathscr{C}^\infty$ strictly decreasing function on $(0,+\infty)$ that admits a well-defined inverse Φ^{-1}. The main result of this section is the following.

THEOREM 3.1. *Under the assumptions of Theorem* 1.1, *for any* $\theta > 0$

$$-\frac{C_*^{1/\alpha}}{\theta^{1/\alpha}} \leqslant \liminf_{n\to+\infty} \frac{1}{a_n} \log \varrho\left(n, \theta\frac{a_n}{n}\right) \leqslant \limsup_{n\to+\infty} \frac{1}{a_n} \log \varrho\left(n, \theta\frac{a_n}{n}\right) \leqslant -\Phi^{-1}(\theta).$$

Remark 3.1. For any $\mu > 0$ we have $a_{\lfloor \mu n\rfloor} \sim_{n\to+\infty} \mu^{1/(\alpha+1)} a_n$, by inversion of regularly varying functions. Consequently, Theorem 3.1 implies that, for any $\theta > 0$,

$$\begin{aligned} -1 &\leqslant \liminf_{n\to+\infty} \frac{1}{a_n} \log \varrho\left(\left\lfloor \left(\frac{\theta}{C_*}\right)^{(\alpha+1)/\alpha} n\right\rfloor, C_*\frac{a_n}{n}\right)\\ &\leqslant \limsup_{n\to+\infty} \frac{1}{a_n} \log \varrho\left(\left\lfloor \left(\frac{\theta}{C_*}\right)^{(\alpha+1)/\alpha} n\right\rfloor, C_*\frac{a_n}{n}\right) \leqslant -\frac{\theta^{1/\alpha}\Phi^{-1}(\theta)}{C_*^{1/\alpha}}. \end{aligned} \tag{3.5}$$

Since $\lim_{\theta\to+\infty} \theta^{1/\alpha}\Phi^{-1}(\theta) = C_*^{1/\alpha}$, this leads to

$$\begin{aligned} &\lim_{h\to+\infty} \liminf_{n\to+\infty} \frac{1}{a_n} \log \varrho\left(\lfloor hn\rfloor, C_*\frac{a_n}{n}\right)\\ &\quad= \lim_{h\to+\infty} \limsup_{n\to+\infty} \frac{1}{a_n} \log \varrho\left(\lfloor hn\rfloor, C_*\frac{a_n}{n}\right) = -1. \end{aligned} \tag{3.6}$$

To prove Theorem 3.1, we prove separately the upper bound in Lemma 3.1 and the lower bound in Lemma 3.2. The upper bound is obtained by computing the number of individuals that stay above the line of slope $-\theta a_n/n$ during n units of time.

LEMMA 3.1. *Under the assumptions of Theorem* 1.1, *for all* $\theta > 0$,

$$\limsup_{n\to+\infty} \frac{1}{a_n} \log \varrho\left(n,\, \theta\frac{a_n}{n}\right) \leqslant -\Phi^{-1}(\theta).$$

Proof. Let $\theta > 0$ and $\lambda > 0$. We set $g\colon t \mapsto -\theta t + \lambda(1-t)^{1/(\alpha+1)}$. For $j \leqslant n$, we introduce the intervals

$$I_j^{(n)} = \left[-\theta a_n \frac{j}{n},\, a_n g\left(\frac{j}{n}\right)\right].$$

Since $I_n^{(n)} = \{g(1)a_n\}$, an individual that stays above the curve of slope $-\theta a_n/n$ crosses at some time $k \leqslant n$ the line $g(\,\cdot\,/n)a_n$, and therefore,

$$\begin{aligned} \varrho\left(n, \theta\frac{a_n}{n}\right) &= \mathbf{P}\left(\exists\, u,\, |u| = n\colon \forall\, j \leqslant n,\, V(u_j) \geqslant -\theta a_n \frac{j}{n}\right)\\ &\leqslant \mathbf{P}\left(\exists\, u,\, |u| \leqslant n\colon V(u) \geqslant a_n g\left(\frac{|u|}{n}\right),\, V(u_j) \in I_j^{(n)},\, j < |u|\right). \end{aligned}$$

Thus, setting

$$Y_n = \sum_{|u|\leqslant n} \mathbf{1}_{\{V(u)\geqslant a_n g(|u|/n)\}} \mathbf{1}_{\{V(u_j)\in I_j^{(n)},\, j<|u|\}},$$

by the Markov inequality we have $\varrho(n, \theta a_n/n) \leqslant \mathbf{E}[Y_n]$. Applying Lemma 2.1 yields

$$\begin{aligned}\mathbf{E}[Y_n] &= \sum_{k=1}^{n} \mathbf{E}\Bigg[\sum_{|u|=k} \mathbf{1}_{\{V(u_j)\in I_j^{(n)},\, j<k\}} \mathbf{1}_{\{V(u)\geqslant a_n g(k/n)\}}\Bigg] \\ &= \sum_{k=1}^{n} \mathbf{E}\Big[e^{-S_k} \mathbf{1}_{\{S_j\in I_j^{(n)},\, j<k\}} \mathbf{1}_{\{S_k\geqslant a_n g(k/n)\}}\Big] \\ &\leqslant \sum_{k=1}^{n} e^{-g(k/n)a_n} \mathbf{P}\big(S_j \in I_j^{(n)},\, j<k\big).\end{aligned}$$

Let $A \in \mathbf{N}$. We set $m_a = \lfloor na/A \rfloor$ and $g_{a,A} = \inf_{s\in[(a-1)/A,(a+2)/A]} g(s)$. Then

$$\begin{aligned}\mathbf{E}[Y_n] &\leqslant \sum_{a=0}^{A-1} \sum_{k=m_a+1}^{m_{a+1}} e^{-g(k/n)a_n} \mathbf{P}\big(S_j \in I_j^{(n)},\, j<k\big) \\ &\leqslant n \sum_{a=0}^{A-1} e^{-g_{a,A}a_n} \mathbf{P}\big(S_j \in I_j^{(n)},\, j\leqslant m_a\big).\end{aligned}$$

Therefore, by Corollary 2.1, we have

$$\begin{aligned}\limsup_{n\to+\infty} \frac{1}{a_n} \log \mathbf{E}[Y_n] &\leqslant \max_{a\leqslant A-1} \left(-g_{a,A} - C_* \int_0^{a/A} \frac{ds}{(g(s)+\theta s)^\alpha}\right) \\ &\leqslant \max_{a\leqslant A-1} \left(-g_{a,A} - \frac{C_*(\alpha+1)}{\lambda^\alpha}\left[1-\left(1-\frac{a}{A}\right)^{1/(\alpha+1)}\right]\right).\end{aligned}$$

Letting $A \to +\infty$, since g is uniformly continuous, we have

$$\begin{aligned}&\limsup_{n\to+\infty} \frac{1}{a_n} \log \varrho\left(n, \theta\frac{a_n}{n}\right) \\ &\quad\leqslant \sup_{t\in[0,1]} \left\{\theta t - \lambda(1-t)^{1/(\alpha+1)} - \frac{C_*(\alpha+1)}{\lambda^\alpha}\big[1-(1-t)^{1/(\alpha+1)}\big]\right\} \\ &\quad\leqslant -\lambda + \sup_{t\in[0,1]} \big\{\theta t - (\alpha+1)\Phi(\lambda)\big[1-(1-t)^{1/(\alpha+1)}\big]\big\}.\end{aligned}$$

Note that $t \mapsto 1-(1-t)^{1/(\alpha+1)}$ is a convex function with slope $1/(\alpha+1)$ at $t=0$. Therefore, if we choose $\lambda = \Phi^{-1}(\theta)$, the function

$$t \mapsto \theta t - (\alpha+1)\Phi(\lambda)\big[1-(1-t)^{1/(\alpha+1)}\big]$$

is concave and decreasing. As a consequence,

$$\limsup_{n\to+\infty} \frac{1}{a_n} \log \varrho\left(n, \theta\frac{a_n}{n}\right) \leqslant -\lambda,$$

which concludes the proof of Lemma 3.1.

To obtain a lower bound, we bound from below the probability for an individual to stay between two given curves, while having not too many children. To do so, we compute the first two moments of the number of such individuals, and apply the Cauchy–Schwarz inequality to conclude.

LEMMA 3.2. *Under the assumptions of Theorem* 3.1, *for all* $\theta > 0$,

$$\liminf_{n\to+\infty} \frac{1}{a_n} \log \varrho\left(n, \theta\frac{a_n}{n}\right) \geqslant -\frac{C_*^{1/\alpha}}{\theta^{1/\alpha}}.$$

Proof. For $u \in \mathbf{T}$, we recall that $\Omega(u) = \{v \in \mathbf{T} \colon \pi v = \pi u \text{ and } v \neq u\}$ is the set of siblings of u. We introduce $\xi(u) = \log \sum_{v\in\Omega(u)} e^{V(v)-V(u)}$. Note that (1.9) implies

$$\lim_{x\to+\infty} \frac{x^\alpha}{L^*(x)} \widehat{\mathbf{P}}(\xi(w_1) \geqslant x) = 0. \tag{3.7}$$

Let $\theta > 0$, $\lambda > 0$, and $\delta > 0$. We set

$$I_j^{(n)} = \left[-a_n\frac{\theta j}{n}, a_n\left(\lambda - \frac{\theta j}{n}\right)\right], \qquad j \leqslant n,$$
$$X_n = \sum_{|u|=n} \mathbf{1}_{\{V(u_j)\in I_j^{(n)},\, j\leqslant n\}} \mathbf{1}_{\{\xi(u_j)\leqslant \delta a_n,\, j\leqslant n\}}.$$

We observe that

$$\begin{aligned}\varrho\left(n, \theta\frac{a_n}{n}\right) &= \mathbf{P}\left(\exists\, u,\, |u| = n \colon V(u_j) \geqslant -a_n\frac{\theta j}{n},\, j \leqslant n\right)\\ &\geqslant \mathbf{P}\big(\exists\, u,\, |u| = n \colon V(u_j) \in I_j^{(n)},\, j \leqslant n\big) \geqslant \mathbf{P}(X_n \geqslant 1).\end{aligned}$$

Thus by the Cauchy–Schwarz inequality, $\varrho(n, \theta a_n/n) \geqslant (\mathbf{E}[X_n])^2/\mathbf{E}[X_n^2]$.

We first bound $\mathbf{E}[X_n]$ from below. Using Proposition 2.1, we obtain

$$\begin{aligned}\mathbf{E}[X_n] &= \overline{\mathbf{E}}\left[\frac{1}{W_n}\sum_{|u|=n} \mathbf{1}_{\{V(u_j)\in I_j^{(n)},\, j\leqslant n\}} \mathbf{1}_{\{\xi(u_j)\leqslant \delta a_n,\, j\leqslant n\}}\right]\\ &= \widehat{\mathbf{E}}\left[\sum_{|u|=n} e^{-V(u)} \widehat{\mathbf{P}}(u = w_n \mid \mathscr{F}_n) \mathbf{1}_{\{V(u_j)\in I_j^{(n)},\, j\leqslant n\}} \mathbf{1}_{\{\xi(u_j)\leqslant \delta a_n,\, j\leqslant n\}}\right]\\ &= \widehat{\mathbf{E}}\left[e^{-V(w_n)} \mathbf{1}_{\{V(w_j)\in I_j^{(n)},\, j\leqslant n\}} \mathbf{1}_{\{\xi(w_j)\leqslant \delta a_n,\, j\leqslant n\}}\right].\end{aligned}$$

Let $\varepsilon \in (0, \lambda)$. Since $I_n^{(n)} = [-\theta a_n, (\lambda - \theta)a_n]$, we have

$$\begin{aligned}\mathbf{E}[X_n] &\geqslant \widehat{\mathbf{E}}\left[e^{-V(w_n)} \mathbf{1}_{\{V(w_n)\leqslant(\varepsilon-\theta)a_n\}} \mathbf{1}_{\{V(w_j)\in I_j^{(n)},\, j\leqslant n\}} \mathbf{1}_{\{\xi(w_j)\leqslant \delta a_n,\, j\leqslant n\}}\right]\\ &\geqslant e^{(\theta-\varepsilon)a_n} \widehat{\mathbf{P}}\big(V(w_n) \leqslant (\varepsilon - \theta)a_n,\, V(w_j) \in I_j^{(n)},\, \xi(w_j) \leqslant \delta a_n,\, j \leqslant n\big).\end{aligned}$$

Let $0 < x < y$ and $A > 0$ be such that $\widehat{\mathbf{P}}(V(w_1) \in [x, y],\, \xi(w_1) \leqslant A) > 0$. Applying the Markov property at time $p = \lfloor \varepsilon a_n \rfloor$, for any $n \geqslant 1$ large enough we have

$$\begin{aligned}&\widehat{\mathbf{P}}\big(V(w_j) \in I_j^{(n)},\, \xi(w_j) \leqslant \delta a_n,\, j \leqslant n\big)\\ &\quad\geqslant \big[\widehat{\mathbf{P}}(V(w_1) \in [x, y],\, \xi(w_1) \leqslant A)\big]^p\\ &\qquad\times \inf_{z\in[x\varepsilon a_n, y\varepsilon a_n]} \widehat{\mathbf{P}}_z\big[V(w_j) \in I_{j+p}^{(n)},\, \xi(w_j) \leqslant \delta a_n,\, j \leqslant n - p\big].\end{aligned}$$

Since (3.7) holds, we apply Lemma 2.2:

$$\liminf_{n\to+\infty} \frac{1}{a_n} \log \mathbf{E}[X_n] \geqslant \theta - \varepsilon - \frac{C_*}{\lambda^\alpha} + \varepsilon \log \widehat{\mathbf{P}}(V(w_1) \in [x,y],\, \xi(w_1) \leqslant A).$$

Letting $\varepsilon \to 0$ yields

$$\liminf_{n\to+\infty} \frac{1}{a_n} \log \mathbf{E}[X_n] \geqslant \theta - \frac{C_*}{\lambda^\alpha}.$$

To bound from above the second moment of X_n, we apply once again the spinal decomposition,

$$\begin{aligned}
\mathbf{E}[X_n^2] &= \overline{\mathbf{E}}\Bigg[\frac{X_n}{W_n} \sum_{|u|=n} \mathbf{1}_{\{V(u_j)\in I_j^{(n)},\, j\leqslant n\}} \mathbf{1}_{\{\xi(u_j)\leqslant \delta a_n,\, j\leqslant n\}}\Bigg] \\
&= \overline{\mathbf{E}}\Bigg[X_n \sum_{|u|=n} e^{-V(u)} \widehat{\mathbf{P}}(w_n = u \mid \mathscr{F}_n) \mathbf{1}_{\{V(u_j)\in I_j^{(n)},\, j\leqslant n\}} \mathbf{1}_{\{\xi(u_j)\leqslant \delta a_n,\, j\leqslant n\}}\Bigg] \\
&= \widehat{\mathbf{E}}\Big[e^{-V(w_n)} X_n \mathbf{1}_{\{V(w_j)\in I_j^{(n)},\, j\leqslant n\}} \mathbf{1}_{\{\xi(w_j)\leqslant \delta a_n,\, j\leqslant n\}}\Big] \\
&\leqslant e^{\theta a_n} \widehat{\mathbf{E}}\Big[X_n \mathbf{1}_{\{V(w_j)\in I_j^{(n)},\, j\leqslant n\}} \mathbf{1}_{\{\xi(w_j)\leqslant \delta a_n,\, j\leqslant n\}}\Big].
\end{aligned}$$

We decompose the set of individuals counted in X_n under law $\widehat{\mathbf{P}}$ according to their most recent common ancestor with the spine w:

$$X_n = \mathbf{1}_{\{V(w_j)\in I_j^{(n)},\, j\leqslant n\}} \mathbf{1}_{\{\xi(w_j)\leqslant \delta a_n,\, j\leqslant n\}} + \sum_{j=1}^{n} \sum_{u\in\Omega(w_j)} \Lambda(u),$$

where

$$\Lambda(u) = \sum_{|u'|=n,\, u'\geqslant u} \mathbf{1}_{\{V(u'_j)\in I_j^{(n)},\, j\leqslant n\}} \mathbf{1}_{\{\xi(u'_j)\leqslant \delta a_n,\, j\leqslant n\}}$$

($u' \geqslant u$ means u' is a descendant of u).

We write

$$\mathscr{G} = \sigma\big(\big(w_k,\, \Omega(w_k),\, V(u),\, u \in \Omega(w_k)\big),\, k \geqslant 0\big)$$

for the sigma-field of the information of the spine. Let $k \leqslant n$ and $u \in \Omega(w_k)$. Conditionally on $\mathscr{G}$, the subtree rooted at u with marks V is a branching random walk with law $\mathbf{P}_{V(u)}$, and therefore,

$$\begin{aligned}
\widehat{\mathbf{E}}[\Lambda(u) \mid \mathscr{G}] &\leqslant \mathbf{E}_{V(u)}\Bigg[\sum_{|u'|=n-k} \mathbf{1}_{\{V(u'_j)\in I_{k+j}^{(n)},\, j\leqslant n-k\}}\Bigg] \\
&\leqslant e^{V(u)} \mathbf{E}_{V(u)}\big[e^{-S_{n-k}} \mathbf{1}_{\{S_j\in I_{k+j}^{(n)},\, j\leqslant n-k\}}\big] \\
&\leqslant e^{V(u)} e^{\theta a_n} \sup_{z\in\mathbf{R}} \mathbf{P}_z\big(S_j \in I_{k+j}^{(n)},\, j \leqslant n-k\big).
\end{aligned}$$

Let $A \in \mathbf{N}$. We set $m_a = \lfloor na/A \rfloor$ and

$$\Psi_{a,A} = \sup_{z\in\mathbf{R}} \mathbf{P}_z\big(S_j \in I_{m_a+j}^{(n)},\, j \leqslant n - m_a\big).$$

For any $k \leqslant m_{a+1}$ and $u \in \Omega(w_k)$, we have $\widehat{\mathbf{E}}[\Lambda(u) \mid \mathscr{G}] \leqslant e^{V(u)} e^{\theta a_n} \Psi_{a+1,A}$, and thus

$$\begin{aligned}
&\widehat{\mathbf{E}}\left[\mathbf{1}_{\{V(w_j)\in I_j^{(n)},\, j\leqslant n\}} \mathbf{1}_{\{\xi(w_j)\leqslant \delta a_n,\, j\leqslant n\}} \sum_{k=m_a+1}^{m_{a+1}} \sum_{u\in\Omega(w_k)} \Lambda(u)\right] \\
&\quad \leqslant \sum_{k=m_a+1}^{m_{a+1}} \widehat{\mathbf{E}}\left[\mathbf{1}_{\{V(w_j)\in I_j^{(n)},\, j\leqslant n\}} \sum_{u\in\Omega(w_k)} \mathbf{1}_{\{\xi(w_k)\leqslant \delta a_n\}} \Lambda(u)\right] \\
&\quad \leqslant \Psi_{a+1,A} e^{\theta a_n} \sum_{k=m_a+1}^{m_{a+1}} \widehat{\mathbf{E}}\left[\mathbf{1}_{\{V(w_j)\in I_j^{(n)},\, j\leqslant n\}} e^{\xi(w_k)+V(w_k)} \mathbf{1}_{\{\xi(w_k)\leqslant \delta a_n\}}\right] \\
&\quad \leqslant n \Psi_{a+1,A} \Psi_{0,A} \exp\left(\lambda + \left(1 - \frac{a}{A}\right)\theta + \delta\right) a_n.
\end{aligned}$$

Consequently, applying Corollary 2.1, as soon as $\theta \geqslant C_*/\lambda^\alpha$ we have

$$\limsup_{n\to+\infty} \frac{1}{a_n} \log \mathbf{E}[X_n^2] \leqslant \max_{a\leqslant A}\left(\lambda + \left(2 - \frac{a}{A}\right)\left(\theta - \frac{C_*}{\lambda^\alpha}\right) + \delta\right) \leqslant \lambda + 2\theta - 2\frac{C_*}{\lambda^\alpha} + \delta.$$

Using the first and second moment estimates of X_n, this gives

$$\liminf_{n\to+\infty} \frac{1}{a_n} \log \varrho\left(n, \theta\frac{a_n}{n}\right) \geqslant -\lambda - \delta.$$

Making $\delta \to 0$ and $\lambda \to (C_*/\theta)^{1/\alpha}$ concludes the proof of Lemma 3.2.

Remark 3.2. If we assume (f^θ, g^θ) are a pair of functions solving the differential equations

$$\begin{cases} f(t) = -\theta t, \\ g(t) = -\theta + C_* \displaystyle\int_t^1 \frac{ds}{(g(s) - f(s))^\alpha}, \end{cases}$$

using the estimates similar to those developed in Lemmas 3.1 and 3.2, we prove that for all $\theta \in \mathbf{R}$

$$\lim_{n\to+\infty} \frac{1}{a_n} \log \varrho\left(n, \theta\frac{a_n}{n}\right) = -g^\theta(0).$$

Theorem 3.1 is used to obtain bounds for $g^\theta(0)$ admitting a closed expression that are precise for large θ. Using similar methods, as applied to different functions, we also obtain estimates on the behavior of $g^\theta(0)$ for small values of θ, namely,

$$\lim_{\theta\to 0} g^\theta(0) = ((\alpha+1)C_*)^{1/(\alpha+1)}.$$

4. Speed of the N-branching random walk.

In [1], to prove that $\lim_{n\to+\infty} (\log N)^2 v_N = C$ for a branching random walk in the usual boundary case, the essential tool was a version of Theorem 3.1, which was found in [7]. The same methods are applied to compute the asymptotic behavior of v_N under the assumptions of Theorem 1.1. Loosely speaking, we compare the N-BRW with N independent branching random walks in which individuals crossing a linear boundary with slope $-\nu_N$ defined by

$$\nu_N := C_* \frac{L^*(\log N)}{(\log N)^\alpha} \tag{4.1}$$

are killed. By (3.6), for any $h > 0$ and $N \geqslant 1$ large enough,

$$\varrho\left(h \frac{(\log N)^{\alpha+1}}{L^*(\log N)}, \nu_N\right) \approx \frac{1}{N}.$$

Thus $(\log N)^{\alpha+1}/L^*(\log N)$ is expected to be the correct time scale for the study of the process.

We start this section with a more precise definition of the branching-selection particle system we consider. We introduce additional notation that enables us to describe it as a measure-valued Markov process. In section 4.2, we introduce an increasing coupling between branching-selection particle systems, and use it to prove the existence of v_N. Finally, we obtain in section 4.3 an upper bound for v_N and in section 4.4 a lower bound that are enough to conclude the proof of Theorem 1.1.

4.1. Definition of the N-branching random walk and notation. The branching-selection models we consider are particle systems on $\mathbf{R}$. It is often convenient to represent the state of a particle system by a counting measure on $\mathbf{R}$ with finite integer-valued mass on every interval of the form $[x, +\infty)$. We let $\mathscr{M}$ denote the set of such measures. A Dirac mass at position $x \in \mathbf{R}$ indicates the presence of an individual alive at position x. With this interpretation, a measure in $\mathscr{M}$ represents a population with a rightmost individual and no accumulation point. For $N \in \mathbf{N}$, we write $\mathscr{M}_N$ for the set of measures in $\mathscr{M}$ with total mass N that represent populations of N individuals. If $\mu \in \mathscr{M}_N$, then there exists $(x_1, \dots, x_N) \in \mathbf{R}^N$ such that $\mu = \sum_{j=1}^N \delta_{x_j}$.

We introduce a partial order on $\mathscr{M}$: given $\mu, \nu \in \mathscr{M}$, we write $\mu \preccurlyeq \nu$ if for all $x \in \mathbf{R}$, $\mu([x, +\infty)) \leqslant \nu([x, +\infty))$. Note that if $\mu \preccurlyeq \nu$, then $\mu(\mathbf{R}) \leqslant \nu(\mathbf{R})$. A similar partial order can be defined on the set of laws of point processes. We say that $\mathscr{L} \preccurlyeq \widetilde{\mathscr{L}}$ if there exists a coupling $(L, \widetilde{L})$ of these two laws such that L has law $\mathscr{L}$, $\widetilde{L}$ has law $\widetilde{\mathscr{L}}$, and

$$\sum_{\ell \in L} \delta_\ell \preccurlyeq \sum_{\tilde{\ell} \in \tilde{L}} \delta_{\tilde{\ell}} \quad \text{a.s.}$$

Let $N \in \mathbf{N}$. We introduce a Markov chain $(X_n^N, n \geqslant 0)$ on $\mathscr{M}_N$ called the N-BRW. For any $n \geqslant 0$, we denote by $(x_n^N(1), \dots, x_n^N(N)) \in \mathbf{R}^N$ the random vector that verifies

$$X_n^N = \sum_{j=1}^N \delta_{x_n^N(j)} \quad \text{and} \quad x_n^N(1) \geqslant x_n^N(2) \geqslant \cdots \geqslant x_n^N(N).$$

Conditionally on X_n^N, the point process X_{n+1}^N is constructed as follows. Let $L_n^1, \dots, L_n^N$ be N i.i.d. point processes with law $\mathscr{L}$. Then we set

$$Y_{n+1}^N = \sum_{i=1}^N \sum_{\ell^i \in L_n^i} \delta_{x_n^N(i) + \ell^i} \in \mathscr{M},$$

which is the population after the branching step. We set

$$y = \sup\big\{x \in \mathbf{R} \colon Y_{n+1}^N([x,+\infty)) \geqslant N\big\} \quad \text{and} \quad P = Y_{n+1}^N\big((y,+\infty)\big).$$

We write $X_{n+1}^N = Y_{n+1|(y,+\infty)}^N + (N-P)\delta_y$. The natural filtration associated with the N-BRW is defined, for $n \in \mathbf{N}$, by $\mathscr{F}_n = \sigma(L_j^1, \dots, L_j^N, j \leqslant n)$. Whereas this is not done here, genealogical information can be freely added to this process, breaking ties in any $(\mathscr{F}_n)$-adapted manner to choose which of the individuals at the leftmost position are killed.

4.2. Increasing coupling of branching-selection models. We construct here a coupling between N-BRWs that preserves the order $\preccurlyeq$. This coupling was introduced in [1] in a special case and is a key tool in the study of the branching-selection processes we consider. It is used to bound from above and from below the behavior of the N-BRW by a branching random walk in which individuals that cross a line of slope $-\nu_N$ are killed. First, we couple a single step of the N-BRW.

LEMMA 4.1. *Let $1 \leqslant m \leqslant n$, and let $\mu \in \mathscr{M}_m$, $\widetilde{\mu} \in \mathscr{M}_n$ be such that $\mu \preccurlyeq \widetilde{\mu}$. Next, let $\mathscr{L} \preccurlyeq \widetilde{\mathscr{L}}$ be two laws of point processes. Then, for any $1 \leqslant M \leqslant N$, there exists a coupling of X_1^M (the first step of an M-BRW with reproduction law $\mathscr{L}$ starting from μ) with $\widetilde{X}_1^N$ (the first step of an N-BRW with reproduction law $\widetilde{\mathscr{L}}$ starting from $\widetilde{\mu}$) in a way that $X_1^M \preccurlyeq \widetilde{X}_1^N$ a.s.*

Proof. Let $(L, \widetilde{L})$ be a pair of point processes such that $\sum_{\ell \in L} \delta_\ell \preccurlyeq \sum_{\ell \in \tilde{L}} \delta_\ell$ a.s., L has law $\mathscr{L}$, and $\widetilde{L}$ has law $\widetilde{\mathscr{L}}$. We set $((L_j, \widetilde{L}_j), j \geqslant 0)$ i.i.d. random variables with the same law as $(L, \widetilde{L})$. We write $\mu = \sum_{i=1}^m \delta_{x_i}$ and $\widetilde{\mu} = \sum_{i=1}^n \delta_{y_i}$ in a way that $(x_j, j \leqslant m)$ and $(y_j, j \leqslant n)$ are ranked in decreasing order. We set

$$\mu^1 = \sum_{i=1}^m \sum_{\ell_i \in L_i} \delta_{x_i + \ell_i} \quad \text{and} \quad \widetilde{\mu}^1 = \sum_{i=1}^n \sum_{\ell_i \in \tilde{L}_i} \delta_{y_i + \ell_i}.$$

We observe that $\mu^1 \preccurlyeq \widetilde{\mu}^1$ a.s.

We set X_1^M for the M individuals with highest positions in μ^1 and $\widetilde{X}_1^N$ for the N individuals with highest positions in $\widetilde{\mu}^1$. Once again, we have $X_1^M \preccurlyeq \widetilde{X}_1^N$ a.s., proving Lemma 4.1.

A direct consequence of this lemma is the existence of an increasing coupling between N-BRWs.

COROLLARY 4.1. *Let $\mathscr{L} \preccurlyeq \widetilde{\mathscr{L}}$ be two laws of point processes. For all $1 \leqslant M \leqslant N \leqslant +\infty$, if $X_0^M \preccurlyeq \widetilde{X}_0^N$, then there exists a coupling between the M-BRW (X_n^M) with law $\mathscr{L}$ and the N-BRW $(\widetilde{X}_n^N)$ with law $\widetilde{\mathscr{L}}$ verifying*

$$\forall\, n \in \mathbf{N}, \quad X_n^M \preccurlyeq \widetilde{X}_n^N \quad a.s.$$

Using this increasing coupling, we prove that with high probability the cloud of particles in the N-BRW does not spread.

LEMMA 4.2. *Under the assumptions* (1.3), (1.4), *and* (1.10) *there exists $C > 0$ such that, for all $N \geqslant 2$, $y \geqslant 1$, and $n \geqslant C(\log N + \log y)$,*

$$\mathbf{P}\big(x_n^N(1) - x_n^N(N) \geqslant y\big) \leqslant C\bigg(\frac{N(\log N + \log y)}{y}\bigg)^2.$$

Proof. Given $n \in \mathbf{N}$ and $k \leqslant n$, we will bound $x_n^N(1) - x_{n-k}^N(1)$ from above and $x_n^N(N) - x_{n-k}^N(1)$ from below to estimate the size of the cloud of particles at time n. An appropriate choice of k will conclude the proof of Lemma 4.2.

We first observe that the N-BRW starting from position X_{n-k}^N can be coupled with N i.i.d. branching random walks $((\mathbf{T}^j, V^j), j \leqslant N)$ with $(\mathbf{T}^j, V^j)$ starting from position $x_{n-k}^N(j)$, in a way that

$$X_n^N \preccurlyeq \sum_{j=1}^N \sum_{u \in \mathbf{T}^j,\, |u|=k} \delta_{V^j(u)}.$$

As a consequence, by (2.4), for any $y \in \mathbf{R}$ and $k \leqslant n$

$$\mathbf{P}\big(x_n^N(1) - x_{n-k}^N(1) \geqslant y\big) \leqslant \mathbf{P}\Big(\max_{j \leqslant N} \max_{u \in \mathbf{T}^j,\, |u|=k} V^j(u) \geqslant y\Big) \leqslant N e^{-y}. \tag{4.2}$$

We now bound from below the displacements in the N-BRW. Let L be a point process with law $\mathscr{L}$. By assumption (1.3), there exists $R > 0$ such that

$$\mathbf{E}\Big[\sum_{\ell \in L} \mathbf{1}_{\{\ell \geqslant -R\}}\Big] > 1.$$

We denote by L_R the point process that consists of the maximal point in L as well as any other point that is greater than $-R$. Using Corollary 4.1, we couple $(X_{n-k+m}^N, m \geqslant 0)$ with the N-BRW $(X_m^{N,R}, m \geqslant 0)$ of reproduction law $\mathscr{L}_R$, starting from a unique individual located at $x_{n-k}^N(1)$ at time 0 in an increasing fashion.

Since $X_m^{N,R} \preccurlyeq X_{n-k+m}^N$, if $X_k^{N,R}(\mathbf{R}) = N$, then $x_k^{N,R}(N) \leqslant x_n^N(N)$. Moreover, by definition of L_R the minimal displacement made by one child with respect to its parent is given by $\min(-R, \max L)$. For $n \in \mathbf{N}$, we denote by Q_n a random variable defined as the sum of n i.i.d. copies of $\min(-R, \max L)$. Observe that Q_{kN} is stochastically dominated by $x_k^{N,R}(N) - x_{n-k}^N(1)$. Consequently,

$$\mathbf{P}\big(x_n^N(N) - x_{n-k}^N(1) \leqslant -y\big) \leqslant \mathbf{P}\big(X_k^{N,R}(\mathbf{R}) < N\big) + \mathbf{P}(Q_{kN} \leqslant -y).$$

By (1.10), we have $\mathbf{P}(Q_{kN} \leqslant -y) \leqslant Ck^2N^2/y^2$. Moreover, the process $(X_n^{N,R}(\mathbf{R}), n \geqslant 0)$ is a Galton–Watson process with the reproduction law given by $\#L_R$ that saturates at N. Setting

$$m_R = \mathbf{E}[\#L_R] \quad \text{and} \quad \alpha = -\frac{\log \mathbf{P}(\#L_R = 1)}{\log m_R},$$

we have $\mathbf{P}(X_k^{N,R}(\mathbf{R}) < N) \leqslant C N^\alpha m_R^{-k\alpha}$ by [6]. We conclude that

$$\mathbf{P}\big(x_n^N(N) - x_{n-k}^N(1) \leqslant y\big) \leqslant C\frac{k^2N^2}{y^2} + C\frac{N^\alpha}{m_R^{k\alpha}}. \tag{4.3}$$

Combining (4.2) and (4.3), for all $y \geqslant 1$ and $k \in \mathbf{N}$, this gives

$$\mathbf{P}\big(x_n^N(1) - x_n^N(N) \geqslant 2y\big) \leqslant N e^{-y} + C\frac{k^2N^2}{y^2} + C\frac{N^\alpha}{m_R^{k\alpha}}.$$

Thus, setting $k = \lfloor 3(\log N + \log y)/(\alpha \log m_R) \rfloor$, there exists $C > 0$ such that for any $y \geqslant 1$ and $N \geqslant 1$ large enough, for any $n \geqslant k$,

$$\mathbf{P}\big(x_n^N(1) - x_n^N(N) \geqslant 2y\big) \leqslant C\left(\frac{N(\log N + \log y)}{y}\right)^2.$$

Lemma 4.2 is proved.

Applying Lemma 4.1 and the Borel–Cantelli lemma, for any $N \geqslant 2$ we have

$$\lim_{n\to+\infty} \frac{x_n^N(1) - x_n^N(N)}{n} = 0 \quad \text{a.s. and in } L^1.$$

LEMMA 4.3. *Under assumptions* (1.3), (1.4), *and* (1.10), *for any* $N \geqslant 1$, *there exists* $v_N \in \mathbf{R}$ *such that for all* $j \leqslant N$

$$\lim_{n\to+\infty} \frac{x_n^N(j)}{n} = v_N \quad \textit{a.s. and in } L^1. \tag{4.4}$$

Moreover, if $X_0^N = N\delta_0$, *we have*

$$v_N = \inf_{n\geqslant 1} \frac{\mathbf{E}[x_n^N(1)]}{n} = \sup_{n\geqslant 1} \frac{\mathbf{E}[x_n^N(N)]}{n}. \tag{4.5}$$

Proof. This proof is based on the Kingman subadditive ergodic theorem. We first prove that if $X_0^N = N\delta_0$, then $(x_n^N(1))$ is a subadditive sequence, and $(x_n^N(N))$ is an overadditive one. Thus $x_n^N(1)/n$ and $x_n^N(N)/n$ converge, and $\lim_{n\to+\infty} x_n^N(1)/n = \lim_{n\to+\infty} x_n^N(N)/n$ a.s. by Lemma 4.2. We then treat the case of a generic starting value $X_0^N \in \mathscr{M}_N$, using Corollary 4.1.

Let $N \in \mathbf{N}$, and let $(L_n^j,\, j \leqslant N,\, n \geqslant 0)$ be an array of i.i.d. point processes with common law $\mathscr{L}$. We define on the same probability space random measures $(X_{m,n}^N,\, 0 \leqslant m \leqslant n)$ such that for all $m \geqslant 0$, $(X_{m,m+n}^N,\, n \geqslant 0)$ is an N-BRW starting from the initial distribution $N\delta_0$. For any $m \geqslant 0$, we set $X_{m,m}^N = N\delta_0$. Letting $0 \leqslant m \leqslant n$, we assume that $X_{m,n}^N = \sum_{j=1}^N \delta_{x_{m,n}^N(j)}$, with $(x_{m,n}^N(j))$ listed in decreasing order, is given. We define $(x_{m,n+1}^N(j),\, j \geqslant 0)$, again listed in decreasing order, in a way that

$$\sum_{j=1}^{+\infty} \delta_{x_{m,n+1}^N(j)} = \sum_{j=1}^N \sum_{\ell_n^j \in L_n^j} \delta_{x_{m,n}^N(j) + \ell_n^j},$$

and set $X_{m,n+1}^N = \sum_{j=1}^N \delta_{x_{m,n+1}^N(j)}$.

For $x \in \mathbf{R}$, we write ϕ_x for the shift operator on $\mathscr{M}$ such that $\phi_x(\mu) = \mu(\cdot - x)$. With this definition, we observe that for any $0 \leqslant m \leqslant n$ we have

$$\phi_{x_{0,n}^N(N)}(X_{n,n+m}^N) \preccurlyeq X_{0,n+m}^N \preccurlyeq \phi_{x_{0,n}^N(1)}(X_{n,n+m}^N).$$

As a consequence,

$$x_{0,n+m}^N(1) \leqslant x_{0,n}^N(1) + x_{n,n+m}^N(1), \quad x_{0,n+m}^N(N) \geqslant x_{0,n}^N(N) + x_{n,n+m}^N(N). \tag{4.6}$$

We apply the Kingman subadditive ergodic theorem. Indeed, for any $n \geqslant 0$, $(x_{n,n+m}^N(1),\, m \geqslant 0)$ is independent of $(x_{k,l}^N(1),\, 0 \leqslant k \leqslant l \leqslant n)$ and has the same law as

$(x^N_{0,m}(1), m \geqslant 0)$. Moreover, $\mathbf{E}[|x^N_{0,1}(1)|] < +\infty$ by (1.10). As a consequence, (4.6) implies that there exists $v_N \in \mathbf{R}$ verifying

$$\lim_{n\to+\infty} \frac{x^N_{0,n}(1)}{n} = v_N \quad \text{a.s. and in } L^1,$$

and $v_N = \inf_{n\in\mathbf{N}} \mathbf{E}[x^N_{0,n}(1)]/n$. Similarly,

$$\lim_{n\to+\infty} \frac{x^N_{0,n}(N)}{n} = \sup_{n\in\mathbf{N}} \frac{\mathbf{E}(x^N_{0,n}(N))}{n} \quad \text{a.s. and in } L^1,$$

which proves that (4.5) is verified. Moreover, by Lemma 4.2, these limits are equal.

We now consider the general case. Let $(X^N_n, n \geqslant 0)$ be an N-BRW. We couple this process with two N-BRWs Y^N and Z^N starting from $N\delta_{x^N_0(1)}$ and $N\delta_{x^N_0(N)}$, respectively, such that for all $n \in \mathbf{N}$, $Z^N_n \preccurlyeq X^N_n \preccurlyeq Y^N_n$. We have

$$\forall\, j \leqslant N, \quad z^N_n(N) \leqslant x^N_n(N) \leqslant x^N_n(j) \leqslant x^N_n(1) \leqslant y^N_n(1).$$

Therefore, for any $j \leqslant N$,

$$\begin{aligned} v_N = \liminf_{n\to+\infty} \frac{z^N_n(N)}{n} &\leqslant \liminf_{n\to+\infty} \frac{x^N_n(j)}{n} \\ &\leqslant \limsup_{n\to+\infty} \frac{x^N_n(j)}{n} \leqslant \limsup_{n\to+\infty} \frac{y^N_n(1)}{n} = v_N \quad \text{a.s.,} \end{aligned}$$

which yields $\lim_{n\to+\infty} x^N_n(j)/n = v_N$ a.s. Similarly, we have

$$\begin{aligned} \mathbf{E}\left[\left|\frac{x^N_n(j)}{n} - v_N\right|\right] &\leqslant \mathbf{E}\left[\left(\frac{y^N_n(1)}{n} - v_N\right)\mathbf{1}_{\{x^N_n(j) \geqslant n v_N\}}\right] \\ &\quad + \mathbf{E}\left[\left(v_N - \frac{z^N_n(N)}{n}\right)\mathbf{1}_{\{x^N_n(j) \leqslant n v_N\}}\right] \\ &\leqslant \mathbf{E}\left[\left|\frac{y^N_n(1)}{n} - v_N\right|\right] + \mathbf{E}\left[\left|\frac{z^N_n(N)}{n} - v_N\right|\right]. \end{aligned}$$

We conclude that $x^N_n(j)/n$ also converges to v_N in L^1. Lemma 4.3 is proved.

Remark 4.1. Lemma 4.3 proves that the limit in (4.4) does not depend on the starting position of the N-BRW. To prove Theorem 1.1, we now study the asymptotic behavior of v_N. This can be done considering only N-BRW starting from the initial condition $N\delta_0$.

To study the asymptotic behavior of v_N as $N \to +\infty$, we couple the N-BRW with a branching random walk in which individuals are killed below the line of slope $-\nu_N$. Applying Theorem 3.1, we derive upper and lower bounds for v_N.

4.3. An upper bound on the maximal displacement. To obtain an upper bound on the maximal displacement in the N-branching random walk, we link the existence of an individual alive at time n that made a large displacement with the event that there exists an individual staying above a line of slope $-\nu_N$ during m_N units of time in a branching random walk. The following lemma is an easier and less precise version of Lemma 2 of [1] that is sufficient for our proofs.

LEMMA 4.4. *Let $v < K$, and let $(x_n, n \geqslant 0)$ be a sequence of real numbers with $x_0 = 0$ such that $\sup_{i \in \mathbf{N}}(x_{i+1} - x_i) \leqslant K$. For all $m \leqslant n$, if $x_n > (n-m)v + Km$, then there exists $i \leqslant n - m$ such that for all $j \leqslant m$, $x_{i+j} - x_i \geqslant vj$.*

Proof. Let (x_n) be a sequence verifying $\sup_{i \in \mathbf{N}}(x_{i+1} - x_i) \leqslant K$. We assume that, for any $i \leqslant n - m$, there exists $j_i \leqslant m$ such that $x_{i+j_i} - x_i \leqslant vj_i$. We set $\sigma_0 = 0$ and $\sigma_{k+1} = \sigma_k + j_{\sigma_k}$. By definition, we have

$$x_{\sigma_{k+1}} \leqslant (\sigma_{k+1} - \sigma_k)v + x_{\sigma_k},$$

and thus, for all $k \geqslant 0$, $x_{\sigma_k} \leqslant \sigma_k v$. Moreover, as (σ_k) is strictly increasing, with steps smaller than m, there exists k_0 such that $\sigma_{k_0} \in [n-m, n]$. Hence

$$\begin{aligned} x_n = (x_n - x_{\sigma_{k_0}}) + x_{\sigma_{k_0}} &\leqslant K(n - \sigma_{k_0}) + v\sigma_{k_0} = Kn - (K-v)\sigma_{k_0} \\ &\leqslant Kn - (K-v)(n-m) \leqslant Km + v(n-m), \end{aligned}$$

which concludes the proof.

Using Lemma 4.4 and Theorem 3.1, we bound from above the maximal position at time N^ε.

LEMMA 4.5. *With the same assumptions as in Theorem* 1.1, *let X^N be an N-BRW with reproduction law $\mathscr{L}$ starting from $N\delta_0$. For any $\varepsilon > 0$ small enough, for any $N \geqslant 1$ large enough, we have*

$$\mathbf{P}\left(x^N_{\lfloor N^\varepsilon \rfloor}(1) \geqslant -(1-2\varepsilon)\nu_N N^\varepsilon\right) \leqslant N^{-\varepsilon}.$$

Proof. Let $\varepsilon \in (0,1)$ and $\theta > 0$. By (3.5),

$$\limsup_{n \to +\infty} \frac{1}{a_n} \log \varrho\left(\left\lfloor \left(\frac{\theta}{(1-\varepsilon)C_*} \right)^{(\alpha+1)/\alpha} n \right\rfloor, C_*(1-\varepsilon)\frac{a_n}{n} \right) \leqslant -\frac{\theta^{1/\alpha}\Phi^{-1}(\theta)}{C_*^{1/\alpha}}.$$

We set

$$m_N = \left\lfloor \left(\frac{\theta}{(1-\varepsilon)C_*} \right)^{(\alpha+1)/\alpha} \frac{(\log N)^{\alpha+1}}{L^*(\log N)} \right\rfloor.$$

Since $a_{(\log N)^{\alpha+1}/L^*(\log N)} \sim_{N \to +\infty} \log N$, we have

$$\limsup_{N \to +\infty} \frac{1}{\log N} \log \varrho(m_N, (1-\varepsilon)\nu_N) \leqslant -\frac{\theta^{1/\alpha}\Phi^{-1}(\theta)}{C_*^{1/\alpha}}.$$

Observe that there exists $C > 0$ such that $\theta^{1/\alpha}\Phi^{-1}(\theta) - C_*^{1/\alpha} \sim_{\theta \to +\infty} -C/\theta$ by definition of Φ. Therefore, for any $\varepsilon > 0$ small enough, there exists $\theta > 0$ such that $\varrho(m_N, (1-\varepsilon)\nu_N) \leqslant N^{-(1+2\varepsilon)}$ for any $N \geqslant 1$ large enough.

We set $n = \lfloor N^\varepsilon \rfloor$. Observe the N-BRW of length n is built with nN independent point processes of law $\mathscr{L}$ satisfying (1.4). If L is a point process with law $\mathscr{L}$, we have

$$\mathbf{P}(\max L \geqslant x) \leqslant \mathbf{P}\left(\sum_{\ell \in L} e^\ell \geqslant e^x \right) \leqslant e^{-x}.$$

Setting $K = (1+2\varepsilon)\log N$, the probability that there exists one individual in the N-BRW alive before time n that made a step larger than K is bounded from above by $1 - (1 - N^{-(1+2\varepsilon)})^{nN} \leqslant N^{-\varepsilon}$.

We now consider the path of length n that links an individual alive at time n at position $x_n^N(1)$ with its ancestor alive at time 0. We write $y_n^N(k)$ for the position of the ancestor at time k of this individual. With probability $1 - N^{-\varepsilon}$, this is a path with no step greater than K. For $N \geqslant 1$ large enough, we have $-(1-2\varepsilon)\nu_N n > -(n-m_N)(1-\varepsilon)\nu_N + Km_N$. Applying Lemma 4.4, for any $N \geqslant 1$ large enough we have

$$\begin{aligned}\left\{\forall\, k < n,\, y_n^N(k+1) - y_n^N(k) \leqslant K\right\} &\cap \left\{x_n^N(1) \geqslant -(1-2\varepsilon)\nu_N n\right\} \\ &\subset \left\{\exists\, j \leqslant n - m_N : \forall\, k \leqslant m_N,\, y_n^N(j+k) - y_n^N(j) \geqslant -(1-\varepsilon)\nu_N k\right\}.\end{aligned}$$

Consequently, if $x_n^N(1) \geqslant -(1-2\varepsilon)\nu_N n$, then there exists an individual in the N-BRW that has a sequence of descendants of length m_N staying above the line of slope $-(1-\varepsilon)\nu_N$. This happens with probability at most $nN\varrho(m_N,(1-\varepsilon)\nu_N)$. We conclude from these observations that for any $\varepsilon > 0$ and $N \geqslant 1$ large enough

$$\mathbf{P}\big(x_n^N(1) \geqslant -\nu_N(1-2\varepsilon)n\big) \leqslant CN^{-\varepsilon}.$$

Lemma 4.5 is proved.

Proof of the upper bound in Theorem 1.1. Let X^N be an N-BRW starting from $N\delta_0$. We note that the maximal displacement at time n in the N-BRW is bounded from above by the maximum of N independent branching random walks starting from 0. By inequality (2.4), for any $y \geqslant 0$ and $n \in \mathbf{N}$ we have $\mathbf{P}(x_n^N(1) \geqslant y) \leqslant Ne^{-y}$.

Moreover, by Lemma 4.3 we have $\limsup_{n\to+\infty} x_n^N(1)/n \leqslant \mathbf{E}[x_p^N(1)/p]$ a.s. for all $p \geqslant 1$. Let $\varepsilon > 0$ small enough such that Lemma 4.5 applies and $y > 0$. Setting $p = \lfloor N^{\varepsilon/2}\rfloor$ we have

$$\begin{aligned}v_N \leqslant \mathbf{E}\left[\frac{x_p^N(1)}{p}\mathbf{1}_{\{x_p^N(1)\geqslant py\}}\right] + \mathbf{E}\left[\frac{x_p^N(1)}{p}\mathbf{1}_{\{x_p^N(1)\in[-p\nu_N(1-\varepsilon),\,py]\}}\right]\\ + \mathbf{E}\left[\frac{x_p^N(1)}{p}\mathbf{1}_{\{x_p^N(1)\leqslant -p(1-\varepsilon)\nu_N\}}\right],\end{aligned}$$

and therefore,

$$\begin{aligned}v_N &\leqslant \int_y^{+\infty} \mathbf{P}\big(x_p^N(1) \geqslant pz\big)\,dz + y\mathbf{P}\big(x_p^N(1) \geqslant -p(1-\varepsilon)\nu_N\big) - (1-\varepsilon)\nu_N\\ &\leqslant Ne^{-N^{\varepsilon/2}y} + yN^{-\varepsilon/2} - (1-\varepsilon)\nu_N.\end{aligned}$$

Letting $N \to +\infty$ and then $\varepsilon \to 0$, we conclude that

$$\limsup_{N\to+\infty} \frac{v_N(\log N)^\alpha}{L^*(\log N)} \leqslant -C^*.$$

4.4. The lower bound. To bound from below the position of the leftmost individual in the N-BRW, we prove that with high probability there exists a time $k \leqslant m_N$ such that $x_k^N(N) \geqslant -k\nu_N$. We use these events as renewal times for a particle process that stays below the N-BRW.

LEMMA 4.6. *With the same assumptions as in Theorem* 1.1, *let X^N be an N-BRW with reproduction law $\mathscr{L}$ starting from $N\delta_0$. Then, for any $\lambda > 0$ and any $\varepsilon > 0$ small enough, there exists $\delta > 0$ such that for all $N \geqslant 1$ large enough,*

$$\mathbf{P}\left(\forall\, n \leqslant \lambda\frac{(\log N)^{\alpha+1}}{L^*(\log N)},\, x_n^N(N) \leqslant -n(1+\varepsilon)\nu_N\right) \leqslant \exp\big(-N^\delta\big).$$

Proof. For $N \in \mathbf{N}$ and $\lambda > 0$, we set $m_N = \lfloor \lambda (\log N)^{\alpha+1} / L^*(\log N) \rfloor$. Let $\varepsilon > 0$. By (3.5), we have

$$\liminf_{N \to +\infty} \frac{1}{\log N} \log \varrho(m_N,\, (1+\varepsilon)\nu_N) \geqslant -(1+\varepsilon)^{-1/\alpha} > -1.$$

Consequently, for any $\varepsilon > 0$, for any $\delta > 0$ small enough we have

$$\varrho(m_N, (1+\varepsilon)\nu_N) \geqslant 1/N^{1-\delta}$$

for any $N \geqslant 1$ large enough.

Let L be a point process with law $\mathscr{L}$. There exists $R > 0$ such that

$$\mathbf{E}\bigl(\#\bigl\{\ell \in L\colon \ell \geqslant -R\bigr\}\bigr) > 1.$$

We consider the branching random walk in which individuals that cross the line of slope $-R$ are killed. By standard Galton–Watson processes theory (see, e.g., [6]), there exist $r > 0$ and $\alpha > 0$ such that for any $N \geqslant 1$ large enough the probability that there exist more than N individuals alive at time $\lfloor \alpha \log N \rfloor$ in this process is bounded from below by r. Thus for all $N \geqslant 1$ large enough, the probability that there exist at least $N+1$ individuals alive at time $m_N + \lfloor \alpha \log N \rfloor$ in a branching random walk in which individuals that cross the line of slope $-\nu_N(1+2\varepsilon)$ are killed is bounded from below by $r\varrho(m_N, (1+\varepsilon)\nu_N)$.

We set $\mathscr{B}_N = \bigl\{\forall n \leqslant m_N + \lfloor \alpha \log N \rfloor,\, x_n^N(N) \leqslant -n\nu_N(1+2\varepsilon)\bigr\}$. By Corollary 4.1, the N-BRW can be coupled with N independent branching random walks starting from 0, in which individuals below the line of slope $-\nu_N(1+2\varepsilon)$ are killed, in a way that on $\mathscr{B}_N$, X^N is above the branching random walks for the order $\preccurlyeq$. The probability that at least one of the branching random walks has at least $N+1$ individuals at time $m_N + \lfloor \alpha \log N \rfloor$ is bounded from below by

$$1 - (1 - r\varrho(m_N, (1+\varepsilon)\nu_N))^N \geqslant 1 - \exp(-N^{\delta/2})$$

for any $N \geqslant 1$ large enough. On this event, the coupling is impossible as X^N has no more than N individuals alive at time N, and thus $\mathscr{B}_N$ is not satisfied. We conclude that $\mathbf{P}(\mathscr{B}_N) \leqslant e^{-N^{\delta/2}}$. Lemma 4.6 is proved.

Proof of the lower bound in Theorem 1.1. The proof is based on a coupling of the N-BRW X^N with another particle system Y^N, in a way that for any $n \in \mathbf{N}$, $Y_n^N \preccurlyeq X_n^N$. Let $(L_n^j,\, j \leqslant N,\, n \geqslant 0)$ be an array of i.i.d. point processes with law $\mathscr{L}$. We construct X^N such that L_n^j represents the set of children of the individual $x_n^N(j)$, with $X_0^N = N\delta_0$. By Lemma 4.6, for any $\varepsilon > 0$ small enough, there exists $\delta > 0$ such that with $m_N = \lfloor (\log N)^{\alpha+1} / L^*(\log N) \rfloor$, we have, for any $N \geqslant 1$ large enough,

$$\mathbf{P}\bigl(\forall n \leqslant m_N,\, x_n^N(N) \leqslant -n(1+\varepsilon)\nu_N\bigr) \leqslant \exp(-N^\delta).$$

We introduce $T_0 = 0$ and $Y_0^N = N\delta_0$. The process Y^N behaves as an N-BRW, using the same point processes (L_n^j) as used for X^N until the time

$$T_1 = \min\bigl(m_N,\, \inf\bigl\{j \geqslant 0\colon y_j^N(N) > -j\nu_N(1+\varepsilon)\bigr\}\bigr).$$

We then write $Y_{T_1^+}^N = N\delta_{y_{T_1}^N(N)}$; i.e., just after the time T_1, the process Y^N starts over at the time T_1^+ from its leftmost individual. For any $k \in \mathbf{N}$, the process behaves as an N-BRW between the times T_k^+ and T_{k+1}, defined by

$$T_{k+1} = T_k + \min\bigl(m_N,\, \inf\bigl\{j \geqslant 0\colon y_{T_k+j}^N(N) - y_{T_k}^N(N) > -j\nu_N(1+\varepsilon)\bigr\}\bigr).$$

By construction, for any $k \in \mathbf{N}$ we have $Y_k^N \preccurlyeq X_k^N$ a.s. and, in particular, $y_k^N(N) \leqslant x_k^N(N)$.

Since $(T_k - T_{k-1}, k \geqslant 1)$ is a sequence of i.i.d. random variables, Lemma 4.3 leads to

$$\lim_{k \to +\infty} \frac{x_{T_k}^N(N)}{k} = \mathbf{E}[T_1] v_N \quad \text{a.s.}$$

Moreover, since $(y_{T_k}^N(N) - y_{T_{k-1}}^N(N),\, k \geqslant 1)$ is another sequence of i.i.d. random variables, we have

$$\lim_{k \to +\infty} \frac{y_{T_k}^N(N)}{k} = \mathbf{E}[y_{T_1}^N(N)] \quad \text{a.s.}$$

by the law of large numbers. Combining these two estimates yields

$$v_N \geqslant \frac{\mathbf{E}[y_{T_1}^N(N)]}{\mathbf{E}[T_1]}.$$

We now compute

$$\begin{aligned}
\mathbf{E}[y_{T_1}^N(N)] &= \mathbf{E}\big[y_{T_1}^N(N)\mathbf{1}_{\{T_1 < m_N\}}\big] + \mathbf{E}\big[y_{T_1}^N(N)\mathbf{1}_{\{T_1 = m_N\}}\big] \\
&\geqslant \mathbf{E}\big[-\nu_N(1+\varepsilon)T_1\mathbf{1}_{\{T_1 < m_N\}}\big] + \mathbf{E}\big[y_{T_1}^N(N)\mathbf{1}_{\{T_1 = m_N\}}\big] \\
&\geqslant -\nu_N(1+\varepsilon)\mathbf{E}[T_1] + \mathbf{E}\big[y_{T_1}^N(N)\mathbf{1}_{\{T_1 = m_N\}}\big].
\end{aligned}$$

Note that for all $j \leqslant T_1$, we have $Y_j^N = X_j^N$. Moreover, by Corollary 4.1, we can couple X^N with an N-BRW $\widetilde{X}$ in which individuals make only one child, with a displacement of law $\max L$. Consequently, we have

$$y_{T_1}^N(N) \geqslant \sum_{n=1}^{T_1} \min_{j \leqslant N} \max L_{j,n} \quad \text{a.s.}$$

which leads to

$$v_N \geqslant -\nu_N(1+\varepsilon) + \frac{1}{\mathbf{E}[T_1]}\mathbf{E}\left[\left(\sum_{n=1}^{m_N} \min_{j \leqslant N} \max L_{j,n}\right)\mathbf{1}_{\{T_1 = m_N\}}\right].$$

Using the Cauchy–Schwarz inequality and (1.10), we have

$$\mathbf{E}\left[\left(\sum_{n=1}^{m_N} \min_{j \leqslant N} \max L_{j,n}\right)\mathbf{1}_{\{T_1 = m_N\}}\right] \geqslant -C(Nm_N)^{1/2}\big[\mathbf{P}(T_1 = m_N)\big]^{1/2}.$$

We apply Lemma 4.6 and let $N \to +\infty$ and then $\varepsilon \to 0$ to prove that

$$\liminf_{N \to +\infty} \frac{v_N(\log N)^\alpha}{L^*(\log N)} \geqslant -C_*.$$

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