Asymptotics for the survival probability in a supercritical branching random walk

by

Nina Gantert, Yueyun Hu and Zhan Shi

Universität Münster, Université Paris XIII & Université Paris VI

This version: November 3, 2008

Summary. Consider a discrete-time one-dimensional supercritical branching random walk. We study the probability that there exists an infinite ray in the branching random walk that always lies above the line of slope $\gamma - \varepsilon$, where γ denotes the asymptotic speed of the right-most position in the branching random walk. Under mild general assumptions upon the distribution of the branching random walk, we prove that when $\varepsilon \to 0$, the probability in question decays like $\exp\{-\frac{\beta+o(1)}{\varepsilon^{1/2}}\}$, where β is a positive constant depending on the distribution of the branching random walk. In the special case of i.i.d. Bernoulli(p) random variables (with $0 < p < \frac{1}{2}$) assigned on a rooted binary tree, this answers an open question of Robin Pemantle [10].

Keywords. Branching random walk, maximal displacement, survival probability.

2000 Mathematics Subject Classification. 60J80.

1 Introduction

We consider a one-dimensional branching random walk in discrete time. Before introducing the model and the problem, we start with an example, borrowed from Pemantle [10], in the study of binary search trees.

Example 1.1 Let $\mathbb{T}_{\mathtt{bs}}$ be a binary tree ("$\mathtt{bs}$" for binary search), rooted at e. Let $(Y(x),\, x \in \mathbb{T}_{\mathtt{bs}})$ be a collection, indexed by the vertices of the tree, of i.i.d. Bernoulli random variables with mean $p \in (0, \frac{1}{2})$. For any vertex $x \in \mathbb{T}_{\mathtt{bs}} \backslash \{e\}$, let $[\![e,\, x]\!]$ denote the shortest path connecting e with x, and let $]\!]e,\, x]\!] := [\![e,\, x]\!] \backslash \{e\}$. We define

$$U_{\mathtt{bs}}(x) := \sum_{v \in]\!]e,\, x]\!]} Y(v), \qquad x \in \mathbb{T}_{\mathtt{bs}} \backslash \{e\},$$

and $U_{\mathtt{bs}}(e) := 0$. Then $(U_{\mathtt{bs}}(x),\, x \in \mathbb{T}_{\mathtt{bs}})$ is a binary branching Bernoulli random walk. It is known (Kingman [7], Hammersley [4], Biggins [2]) that

$$\lim_{n\to\infty} \frac{1}{n} \max_{|x|=n} U_{\mathtt{bs}}(x) = \gamma_{\mathtt{bs}}, \qquad \text{a.s.},$$

where the constant $\gamma_{\mathtt{bs}} = \gamma_{\mathtt{bs}}(p) \in (0,\, 1)$ is the unique solution of

$$\gamma_{\mathtt{bs}} \log \frac{\gamma_{\mathtt{bs}}}{p} + (1-\gamma_{\mathtt{bs}}) \log \frac{1-\gamma_{\mathtt{bs}}}{1-p} - \log 2 = 0. \tag{1.1}$$

For any $\varepsilon > 0$, let $\varrho_{\mathtt{bs}}(\varepsilon, p)$ denote the probability that there exists an infinite ray[1] $\{e =: x_0,\, x_1,\, x_2,\, \cdots\}$ such that $U_{\mathtt{bs}}(x_j) \ge (\gamma_{\mathtt{bs}} - \varepsilon) j$ for any $j \ge 1$. It is conjectured by Pemantle [10] that there exists a constant $\beta_{\mathtt{bs}}(p)$ such that[2]

$$\log \varrho_{\mathtt{bs}}(\varepsilon, p) \;\sim\; -\frac{\beta_{\mathtt{bs}}(p)}{\varepsilon^{1/2}}, \qquad \varepsilon \to 0. \tag{1.2}$$

We prove the conjecture, and give the value of $\beta_{\mathtt{bs}}(p)$. Let $\psi_{\mathtt{bs}}(t) := \log[2(p\mathrm{e}^t + 1 - p)]$, $t > 0$. Let $t^* = t^*(p) > 0$ be the unique solution of $\psi_{\mathtt{bs}}(t^*) = t^* \psi_{\mathtt{bs}}'(t^*)$. [One can then check that the solution of equation (1.1) is $\gamma_{\mathtt{bs}} = \frac{\psi_{\mathtt{bs}}(t^*)}{t^*}$.] Our main result, Theorem 1.2 below, implies that conjecture (1.2) holds, with

$$\beta_{\mathtt{bs}}(p) := \frac{\pi}{2^{1/2}} [t^* \psi_{\mathtt{bs}}''(t^*)]^{1/2}.$$

A particular value of $\beta_{\mathtt{bs}}$ is as follows: if $0 < p_0 < \frac{1}{2}$ is such that $16 p_0 (1-p_0) = 1$ (i.e., if $\gamma_{\mathtt{bs}}(p_0) = \frac{1}{2}$), then

$$\beta_{\mathtt{bs}}(p_0) = \frac{\pi}{4} \Big(\frac{\gamma_{\mathtt{bs}}'(p_0)}{1-2p_0} \Big)^{1/2} \log \frac{1}{4p_0},$$

[1] By an infinite ray, we mean that each x_j is the parent of x_{j+1}.

[2] Throughout the paper, by $a(\varepsilon) \sim b(\varepsilon)$, $\varepsilon \to 0$, we mean $\lim_{\varepsilon\to 0} \frac{a(\varepsilon)}{b(\varepsilon)} = 1$.

where $\gamma'_{\mathsf{bs}}(p_0)$ denotes the derivative of $p \mapsto \gamma_{\mathsf{bs}}(p)$ at p_0. This is, informally, in agreement with the following theorem of Aldous ([1], Theorem 6): if $p \in (p_0, \frac{1}{2})$ is such that $\gamma_{\mathsf{bs}}(p) = \frac{1}{2} + \varepsilon$, then the probability that there exists an infinite ray x with $U_{\mathsf{bs}}(x_i) \geq \frac{1}{2}\, i$, $\forall i \geq 1$, is

$$\exp\Big(- \frac{\pi \log(1/(4p_0))}{4(1-2p_0)^{1/2}} \, \frac{1}{(p-p_0)^{1/2}} + O(1)\Big), \qquad \varepsilon \to 0. \qquad \square$$

As a matter of fact, the main result of this paper (Theorem 1.2 below) is valid for more general branching random walks: the tree $\mathbb{T}_{\mathsf{bs}}$ can be random (Galton–Watson), the random variables assigned on the vertices of the tree are not necessarily Bernoulli, nor necessarily identically distributed, nor necessarily independent if the vertices share a common parent.

Our model is as follows, which is a one-dimensional discrete-time branching random walk. At the beginning, there is a single particle located at position $x = 0$. Its children, who form the first generation, are positioned according to a certain point process. Each of the particles in the first generation gives birth to new particles that are positioned (with respect to their birth places) according to the same point process; they form the second generation. The system goes on according to the same mechanism. We assume that for any n, each particle at generation n produces new particles independently of each other and of everything up to the n-th generation.

We denote by $(U(x), \, |x| = n)$ the positions of the particles in the n-th generation, and by $Z_n := \sum_{|x|=n} 1$ the number of particles in the n-th generation. Clearly, $(Z_n, \, n \geq 0)$ forms a Galton–Watson process. [In Example 1.1, $Z_n = 2^n$, whereas $(U(x), \, |x| = 1)$ is a pair of independent Bernoulli(p) random variables.]

We assume that for some $\delta > 0$,

$$\mathbf{E}(Z_1^{1+\delta}) < \infty, \qquad \mathbf{E}(Z_1) > 1; \tag{1.3}$$

in particular, the Galton–Watson process $(Z_n, \, n \geq 0)$ is supercritical. We also assume that there exists $\delta_+ > 0$ such that

$$\mathbf{E}\Big(\sum_{|x|=1} \mathrm{e}^{\delta_+ U(x)} \Big) < \infty. \tag{1.4}$$

An additional assumption is needed (which in Example 1.1 corresponds to the condition $p < \frac{1}{2}$). Let us define the logarithmic generating function for the branching walk:

$$\psi(t) := \log \mathbf{E}\Big(\sum_{|x|=1} \mathrm{e}^{tU(x)} \Big), \qquad t > 0. \tag{1.5}$$

Let $\zeta := \sup\{t : \psi(t) < \infty\}$. Under condition (1.4), we have $0 < \zeta \le \infty$, and ψ is C^∞ on $(0, \zeta)$. We assume that there exists $t^* \in (0, \zeta)$ such that

$$\psi(t^*) = t^* \psi'(t^*). \tag{1.6}$$

For discussions on this condition, see the examples presented after Theorem 1.2 below.

Recall that (Kingman [7], Hammersley [4], Biggins [2]) conditional of the survival of the system,

$$\lim_{n\to\infty} \frac{1}{n} \max_{|x|=n} U(x) = \gamma, \qquad \text{a.s.,} \tag{1.7}$$

where $\gamma := \frac{\psi(t^*)}{t^*}$ is a constant, with t^* and $\psi(\cdot)$ defined in (1.6) and (1.5), respectively.

For $\varepsilon > 0$, let $\varrho_U(\varepsilon)$ denote the probability that there exists an infinite ray $\{e =: x_0, x_1, x_2, \cdots\}$ such that $U(x_j) \ge (\gamma - \varepsilon) j$ for any $j \ge 1$. Our main result is as follows.

Theorem 1.2 *Assume (1.3) and (1.4). If (1.6) holds, then*

$$\log \varrho_U(\varepsilon) \ \sim\ -\frac{\pi}{(2\varepsilon)^{1/2}} \, [t^* \psi''(t^*)]^{1/2}, \qquad \varepsilon \to 0, \tag{1.8}$$

where t^* and ψ are as in (1.6) and (1.5), respectively.

Since $(U(x),\ |x| = 1)$ is not a deterministic set (excluded by the combination of (1.6) and (1.3)), the function ψ is strictly convex on $(0, \zeta)$. In particular, we have $0 < \psi''(t^*) < \infty$.

We now discuss Assumption (1.6) by means of a few examples.

Example 1.1 (continuation). In Example 1.1, conditions (1.3) and (1.4) are obviously satisfied, whereas (1.6) is equivalent to $p < \frac{1}{2}$. In this case, (1.8) becomes (1.2). □

Example 1.3 Consider the example of Bernoulli branching random walk, i.e., such that $U(x) \in \{0, 1\}$ for any $|x| = 1$; to avoid trivial cases, we assume $\mathbf{E}(\sum_{|x|=1} \mathbf{1}_{\{U(x)=1\}}) > 0$ and $\mathbf{E}(\sum_{|x|=1} \mathbf{1}_{\{U(x)=0\}}) > 0$.

Condition (1.4) is automatically satisfied as long as we assume (1.3). Elementary computations show that condition (1.6) is equivalent to $\mathbf{E}(\sum_{|x|=1} \mathbf{1}_{\{U(x)=1\}}) < 1$; in particular, if we assign independent Bernoulli(p) random variables on the edge of a rooted binary tree, this says that $p < \frac{1}{2}$. □

Example 1.4 Assume the distribution of U is bounded from the above, i.e., there exists a constant $C \in \mathbb{R}$ such that $\sup_{|x|=1} U(x) \le C$. Let $s_U := \operatorname{ess\,sup} \sup_{|x|=1} U(x) = \sup\{a \in \mathbb{R} : \mathbf{P}(\sup_{|x|=1} U(x) \ge a) > 0\} < \infty$. Under (1.3) and (1.4), condition (1.6) is satisfied if $\sup_{|x|=1} U(x)$ does not have an atom at s_U, i.e., if $\mathbf{P}\{\sup_{|x|=1} U(x) = s_U\} = 0$. $\square$

The rest of the paper is as follows. In Section 2, we make a linear transformation of our branching random walk so that it will become a boundary case in the sense of Biggins and Kyprianou [3]; the linear transformation is possible due to Assumption (1.6). Section 3 is devoted to the proof of the upper bound in Theorem 1.2, whereas in Section 4 we give the proof of the lower bound.

2 A linear transformation

We define

$$V(x) := -t^* U(x) + \psi(t^*)\, |x|. \tag{2.1}$$

Then

$$\mathbf{E}\Big(\sum_{|x|=1} \mathrm{e}^{-V(x)} \Big) = 1, \qquad \mathbf{E}\Big(\sum_{|x|=1} V(x) \mathrm{e}^{-V(x)} \Big) = 0. \tag{2.2}$$

The new branching random walk $(V(x))$ satisfies $\lim_{n\to\infty} \frac{1}{n} \inf_{|x|=n} V(x) = 0$ a.s. conditional on non-extinction. Let

$$\varrho(\varepsilon) = \varrho(V, \varepsilon) := \mathbf{P}\Big\{ \exists \text{ infinite ray } \{e =: x_0,\, x_1,\, x_2,\, \cdots\}\colon\ V(x_j) \le \varepsilon j,\ \forall j \ge 1 \Big\}. \tag{2.3}$$

Theorem 1.2 will be a consequence of the following estimate: assume (2.2), then

$$\log \varrho(\varepsilon) \ \sim\ -\frac{\pi\sigma}{(2\varepsilon)^{1/2}}, \qquad \varepsilon \to 0, \tag{2.4}$$

where σ is the constant in (2.5) below.

It is (2.4) we are going to prove: an upper bound is proved in Section 3, and a lower bound in Section 4.

We conclude this section with a change-of-probabilities formula, which is the raison d'être of the linear transformation. Let $S_0 := 0$, and let $(S_i - S_{i-1},\ i \ge 0)$ be a sequence of i.i.d. random variables such that for any measurable function $f : \mathbb{R} \to [0, \infty)$,

$$\mathbf{E}[f(S_1)] = \mathbf{E}\Big[\sum_{|x|=1} \mathrm{e}^{-V(x)} f(V(x)) \Big].$$

In particular, $\mathbf{E}(S_1) = 0$ (by (2.2)). In words, (S_n) is a mean-zero random walk. We denote

$$\sigma^2 := \mathbf{E}(S_1^2) = \mathbf{E}\Big[\sum_{|x|=1} V(x)^2 \mathrm{e}^{-V(x)}\Big] = (t^*)^2 \psi''(t^*). \tag{2.5}$$

According to Biggins and Kyprianou [3], under (2.2) and (1.3), we have, for any $n \ge 1$ and any measurable function $F : \mathbb{R}^n \to [0, \infty)$,

$$\mathbf{E}\Big[\sum_{|x|=n} \mathrm{e}^{-V(x)} F(V(x_i),\ 1 \le i \le n)\Big] = \mathbf{E}[F(S_i,\ 1 \le i \le n)], \tag{2.6}$$

where, for any x with $|x| = n$, $\{e =: x_0,\ x_1,\ \ldots, x_n := x\}$ is the shortest path connecting e to x.

We complete this section by recalling a useful result of Mogulskii [9].

Fact 2.1 (Mogulskii [9]) *Let $S_0 := 0$ and let $(S_i - S_{i-1},\ i \ge 0)$ be a sequence of i.i.d. random variables with $\mathbf{E}(S_1) = 0$ and $\sigma^2 := \mathbf{E}(S_1^2) \in (0, \infty)$. Let $g_1 \le g_2$ be continuous functions on $[0, 1]$ with $g_1(0) \le 0 \le g_2(0)$. Let (a_j) be positive numbers with $a_j \to \infty$ and $\frac{j}{a_j^2} \to \infty$. Consider the measurable event*

$$E_j := \left\{ g_1\left(\frac{i}{j}\right) \le \frac{S_i}{a_j} \le g_2\left(\frac{i}{j}\right),\ \forall 1 \le i \le j \right\}.$$

We have

$$\lim_{j\to\infty} \frac{a_j^2}{j} \log \mathbf{P}(E_j) = -\frac{\pi^2\sigma^2}{2} \int_0^1 \frac{\mathrm{d}t}{[g_2(t) - g_1(t)]^2}. \tag{2.7}$$

Moreover, for any $b > 0$,

$$\lim_{j\to\infty} \frac{a_j^2}{j} \log \mathbf{P}\Big\{E_j,\ \frac{S_j}{a_j} \ge g_2(1) - b\Big\} = -\frac{\pi^2\sigma^2}{2} \int_0^1 \frac{\mathrm{d}t}{[g_2(t) - g_1(t)]^2}. \tag{2.8}$$

The change-of-probabilities formula (2.6) and Mogulskii's estimate (2.7) will be used several times in the next sections.

3 Proof of Theorem 1.2: the upper bound

In this section, we prove the upper bound in (2.4): under conditions (2.2) and (1.3), we have

$$\limsup_{\varepsilon\to 0} \varepsilon^{1/2} \log \varrho(\varepsilon) \le -\frac{\pi\sigma}{2^{1/2}}, \tag{3.1}$$

where $\varrho(\varepsilon)$ is defined in (2.3), and σ is the constant in (2.5).

The main idea in this section is borrowed from Kesten [6]. We start with the trivial inequality that for any $n \ge 1$ (an appropriate value for $n = n(\varepsilon)$ will be chosen later on),

$$\varrho(\varepsilon) \le \mathbf{P}\Big\{\exists\, |x| = n :\ V(x_i) \le \varepsilon i,\ \forall i \le n\Big\}.$$

Let $(b_i,\, i \ge 0)$ be a sequence of non-negative real numbers whose value (depending on n) will be given later on. For any x, let $H(x) := \inf\{i \ge 1 :\ V(x_i) \le \varepsilon i - b_i\}$, with $\inf \emptyset := \infty$. Then

$$\varrho(\varepsilon) \le \varrho_1(\varepsilon) + \varrho_2(\varepsilon),$$

where

$$\begin{aligned}
\varrho_1(\varepsilon) &:= \mathbf{P}\Big\{\exists\, |x| = n :\ H(x) = \infty,\ V(x_i) \le \varepsilon i,\ \forall i \le n\Big\},\\
\varrho_2(\varepsilon) &:= \mathbf{P}\Big\{\exists\, |x| = n :\ H(x) \le n,\ V(x_i) \le \varepsilon i,\ \forall i \le n\Big\}.
\end{aligned}$$

We now estimate $\varrho_1(\varepsilon)$ and $\varrho_2(\varepsilon)$ separately.

By definition,

$$\begin{aligned}
\varrho_1(\varepsilon) &= \mathbf{P}\Big\{\exists\, |x| = n :\ \varepsilon i - b_i < V(x_i) \le \varepsilon i,\ \forall i \le n\Big\}\\
&= \mathbf{P}\Big\{\sum_{|x|=n} \mathbf{1}_{\{\varepsilon i - b_i < V(x_i) \le \varepsilon i,\ \forall i \le n\}} \ge 1\Big\}\\
&\le \mathbf{E}\Big(\sum_{|x|=n} \mathbf{1}_{\{\varepsilon i - b_i < V(x_i) \le \varepsilon i,\ \forall i \le n\}}\Big),
\end{aligned}$$

the last inequality being a consequence of Chebyshev's inequality. Applying the change-of-probabilities formula (2.6) to $F(z) := \mathrm{e}^{z_n} \mathbf{1}_{\{\varepsilon i - b_i < z_i \le \varepsilon i,\ \forall i \le n\}}$ for $z := (z_1, \cdots, z_n) \in \mathbb{R}^n$, this yields, in the notation of (2.6),

$$\varrho_1(\varepsilon) \le \mathbf{E}\Big(\mathrm{e}^{S_n} \mathbf{1}_{\{\varepsilon i - b_i < S_i \le \varepsilon i,\ \forall i \le n\}}\Big) \le \mathrm{e}^{\varepsilon n}\, \mathbf{P}\Big\{\varepsilon i - b_i < S_i \le \varepsilon i,\ \forall i \le n\Big\}. \tag{3.2}$$

To estimate $\varrho_2(\varepsilon)$, we observe that

$$\begin{aligned}
\varrho_2(\varepsilon) &\le \sum_{j=1}^{n} \mathbf{P}\Big\{\exists\, |x| = n :\ H(x) = j,\ V(x_i) \le \varepsilon i,\ \forall i \le n\Big\}\\
&\le \sum_{j=1}^{n} \mathbf{P}\Big\{\exists\, |x| = n :\ H(x) = j,\ V(x_i) \le \varepsilon i,\ \forall i \le j\Big\}.
\end{aligned}$$

Since $\{\exists\, |x| = n :\ H(x) = j,\ V(x_i) \le \varepsilon i,\ \forall i \le j\} \subset \{\exists\, |y| = j :\ H(y) = j,\ V(y_i) \le \varepsilon i,\ \forall i \le j\}$, this yields

$$\varrho_2(\varepsilon) \le \sum_{j=1}^{n} \mathbf{P}\Big\{\exists\, |y| = j :\ \varepsilon i - b_i < V(y_i) \le \varepsilon i,\ \forall i < j,\ V(y_j) \le \varepsilon j - b_j\Big\}.$$

We can now use the same argument as for $\varrho_1(\varepsilon)$, namely, Chebyshev's inequality and then the change-of-probability formula (2.2), to see that

$$\begin{aligned}
\varrho_2(\varepsilon) &\le \sum_{j=1}^{n} \mathbf{E}\Big[\sum_{|y|=j} \mathbf{1}_{\{\varepsilon i - b_i < V(y_i) \le \varepsilon i,\ \forall i<j,\ V(y_j) \le \varepsilon j - b_j\}}\Big] \\
&= \sum_{j=1}^{n} \mathbf{E}\Big[\mathrm{e}^{S_j} \mathbf{1}_{\{\varepsilon i - b_i < S_i \le \varepsilon i,\ \forall i<j,\ S_j \le \varepsilon j - b_j\}}\Big] \\
&\le \sum_{j=1}^{n} \mathrm{e}^{\varepsilon j - b_j} \mathbf{P}\Big\{\varepsilon i - b_i < S_i \le \varepsilon i,\ \forall i < j\Big\}.
\end{aligned}$$

Together with (3.2), and recalling that $\varrho(\varepsilon) \le \varrho_1(\varepsilon) + \varrho_2(\varepsilon)$, this yields

$$\begin{aligned}
\varrho(\varepsilon) &\le \mathrm{e}^{\varepsilon n}\, \mathbf{P}\Big\{\varepsilon i - b_i < S_i \le \varepsilon i,\ \forall i \le n\Big\} + \sum_{j=1}^{n} \mathrm{e}^{\varepsilon j - b_j} \mathbf{P}\Big\{\varepsilon i - b_i < S_i \le \varepsilon i,\ \forall i < j\Big\} \\
&= \mathrm{e}^{\varepsilon n} I(n) + \sum_{j=0}^{n-1} \mathrm{e}^{\varepsilon (j+1) - b_{j+1}} I(j),
\end{aligned}$$

where

$$I(j) := \mathbf{P}\Big\{\varepsilon i - b_i < S_i \le \varepsilon i,\ \forall i \le j\Big\}, \qquad 0 \le j \le n.$$

The idea is now to apply Mogul'skii's estimate (2.7) to $I(j)$ for suitably chosen (b_i). Unfortunately, since ε depends on n, we are not allowed to apply (2.7) simultaneously to all $I(j)$, $0 \le j \le n$. So let us first do a dirty trick, and then apply (2.7) to only a few of $I(j)$.

We assume that (b_i) is non-increasing. Fix an integer $N \ge 2$, and take $n := kN$ for $k \ge 1$. Then

$$\begin{aligned}
\varrho(\varepsilon) &\le \mathrm{e}^{\varepsilon kN} I(kN) + \sum_{j=0}^{k-1} \mathrm{e}^{\varepsilon (j+1) - b_{j+1}} I(j) + \sum_{\ell=1}^{N-1} \sum_{j=\ell k}^{(\ell+1)k-1} \mathrm{e}^{\varepsilon (j+1) - b_{j+1}} I(j) \\
&\le \mathrm{e}^{\varepsilon kN} I(kN) + k \exp\Big(\varepsilon k - b_k\Big) + k \sum_{\ell=1}^{N-1} \exp\Big(\varepsilon (\ell+1) k - b_{(\ell+1)k}\Big) I(\ell k). \qquad (3.3)
\end{aligned}$$

We choose $b_i = b_i(n) := b(n-i)^{1/3} = b(kN-i)^{1/3}$, $0 \le i \le n$, and $\varepsilon := \frac{\theta}{n^{2/3}} = \frac{\theta}{(Nk)^{2/3}}$, where $b>0$ and $\theta>0$ are fixed constants. By definition, for $1 \le \ell \le N$,

$$I(\ell k) = \mathbf{P}\Big\{\theta(\frac{\ell}{N})^{2/3}\frac{i}{\ell k} - b(\frac{N}{\ell} - \frac{i}{\ell k})^{1/3} < \frac{S_i}{(\ell k)^{1/3}} \le \theta(\frac{\ell}{N})^{2/3}\frac{i}{\ell k},\ \forall i \le \ell k\Big\}.$$

Applying (2.7) to $j := \ell k$, $a_i := i^{1/3}$, $g_1(t) := \theta(\frac{\ell}{N})^{2/3}t - b(\frac{N}{\ell} - t)^{1/3}$ and $g_2(t) := \theta(\frac{\ell}{N})^{2/3}t$, we see that, for $1 \le \ell \le N$,

$$\limsup_{k\to\infty} \frac{1}{(\ell k)^{1/3}} \log I(\ell k) \le -\frac{\pi^2\sigma^2}{2b^2}\int_0^1 \frac{\mathrm{d}t}{(\frac{N}{\ell} - t)^{2/3}} = -\frac{3\pi^2\sigma^2}{2b^2}\,\frac{N^{1/3} - (N-\ell)^{1/3}}{\ell^{1/3}},$$

where σ is the constant in (2.5). Going back (3.3), we obtain:

$$\limsup_{k\to\infty} \frac{\theta^{1/2}}{(Nk)^{1/3}} \log \varrho\Big(\frac{\theta}{(Nk)^{2/3}}\Big) \le \theta^{1/2}\alpha_{N,b},$$

where the constant $\alpha_{N,b} = \alpha_{N,b}(\theta)$ is defined by

$$\begin{aligned}\alpha_{N,b} \;\; := \;\; \max_{1\le \ell\le N-1} \Big\{ & \theta - \frac{3\pi^2\sigma^2}{2b^2},\ \frac{\theta}{N} - b(1-\frac{1}{N})^{1/3}, \\ & \frac{\theta(\ell+1)}{N} - b(1 - \frac{\ell+1}{N})^{1/3} - \frac{3\pi^2\sigma^2}{2b^2}\,\frac{N^{1/3} - (N-\ell)^{1/3}}{N^{1/3}}\Big\}.\end{aligned}$$

Since $\varepsilon \mapsto \varrho(\varepsilon)$ is non-decreasing, this yields

$$\limsup_{\varepsilon\to 0} \varepsilon^{1/2} \log \varrho(\varepsilon) \le \theta^{1/2}\alpha_{N,b}.$$

We let $N \to \infty$. By definition,

$$\limsup_{N\to\infty} \alpha_{N,b} \le \max\Big\{\theta - \frac{3\pi^2\sigma^2}{2b^2},\ -b,\ f(\theta,b)\Big\},$$

where $f(\theta,b) := \sup_{t\in(0,\,1]}\{\theta t - b(1-t)^{1/3} - \frac{3\pi^2\sigma^2}{2b^2}[1-(1-t)^{1/3}]\}$.

Elementary computations show that as long as $b < \frac{3\pi^2\sigma^2}{2b^2} \le b + 3\theta$, we have $f(\theta,b) = \theta - \frac{3\pi^2\sigma^2}{2b^2} + \frac{2}{3(3\theta)^{1/2}}(\frac{3\pi^2\sigma^2}{2b^2} - b)^{3/2}$. Thus $\max\{\theta - \frac{3\pi^2\sigma^2}{2b^2},\ -b,\ f(\theta,b)\} = \max\{f(\theta,b),\ -b\}$, which equals $-b$ if $\theta = \frac{\pi^2\sigma^2}{2b^2} - \frac{b}{3}$. As a consequence, for any $b>0$ satisfying $b < \frac{3\pi^2\sigma^2}{2b^2}$,

$$\limsup_{\varepsilon\to 0} \varepsilon^{1/2}\log\varrho(\varepsilon) \le -b\sqrt{\frac{\pi^2\sigma^2}{2b^2} - \frac{b}{3}} = -\sqrt{\frac{\pi^2\sigma^2}{2} - \frac{b^3}{3}}.$$

Letting $b \to 0$, this yields (3.1) and completes the proof of the upper bound in Theorem 1.2. $\square$

4 Proof of Theorem 1.2: the lower bound

Before proceeding to the proof of the lower bound in Theorem 1.2, we recall two inequalities: the first gives a useful lower tail estimate for the number of individuals in a super-critical Galton–Watson process conditional on survival, whereas the second concerns an elementary property of the conditional distribution of sum of independent random variables. Let us recall that Z_n is the number of particles in the n-th generation.

Fact 4.1 (McDiarmid [8]) *There exists $\vartheta > 1$ such that*

$$\mathbf{P}\{Z_n \le \vartheta^n \,|\, Z_n > 0\} \le \vartheta^{-n}, \qquad \forall n \ge 1. \tag{4.1}$$

Fact 4.2 ([5]) *If $X_1,\, X_2,\, \cdots,\, X_N$ are independent non-negative random variables, and if $F : (0,\, \infty) \to \mathbb{R}_+$ is non-increasing, then*

$$\mathbf{E}\Big\{F(\sum_{i=1}^N X_i) \,\Big|\, \sum_{i=1}^N X_i > 0\Big\} \le \max_{1\le i\le N} \mathbf{E}\{F(X_i) \,|\, X_i > 0\}.$$

This section is devoted to the proof of the lower bound in (2.4): under conditions (2.2) and (1.3),

$$\liminf_{\varepsilon\to 0} \varepsilon^{1/2} \log \varrho(\varepsilon) \ge -\frac{\pi\sigma}{2^{1/2}}\,, \tag{4.2}$$

where $\varrho(\varepsilon)$ and σ are as in (2.3) and (2.5), respectively.

The basic idea consists in constructing a new Galton–Watson tree $\mathbb{G} = \mathbb{G}(\varepsilon)$ within the branching random walk, and obtaining a lower bound for $\varrho(\varepsilon)$ in terms of $\mathbb{G}$.

Recall from (1.7) that conditional on survival, $\frac{1}{j}\max_{|z|\le j} V(z)$ converges almost surely, when $j \to \infty$, to a finite constant. Since the system survives with (strictly) positive probability, we can fix a sufficiently large constant ν such that

$$\inf_{j\ge 0} \mathbf{P}\Big(\max_{|x|\le j} V(x) \le \nu j\Big) \ge \frac{1}{2}, \qquad \kappa := \inf_{j\ge 0} \mathbf{P}\Big(Z_j > 0,\, \max_{|x|\le j} V(x) \le \nu j\Big) > 0, \tag{4.3}$$

where, as before, $Z_j := \#\{|x| = j\}$.

Fix a constant $0 < \alpha < 1$. For any integers $n > L \ge 1$ with $(1-\alpha)\varepsilon L \ge \nu(n-L)$, we consider the set $G_{n,\varepsilon} = G_{n,\varepsilon}(L)$ defined by

$$G_{n,\varepsilon} := \{|x| = n :\, V(x_i) \le \alpha\varepsilon i,\, \forall 1 \le i \le L;\, V(x_j) - V(x_L) \le (1-\alpha)\varepsilon L,\, \forall L < j \le n\}.$$

By definition, for any $x \in G_{n,\varepsilon}$, we have $V(x_i) \le \varepsilon i$, $\forall 1 \le i \le n$.

If $G_{n,\varepsilon} \neq \emptyset$, the elements of $G_{n,\varepsilon}$ form the first generation of the new Galton–Watson tree $\mathbb{G}_{n,\varepsilon}$, and we construct $\mathbb{G}_{n,\varepsilon}$ by iterating the same procedure: for example, the second generation in $\mathbb{G}_{n,\varepsilon}$ consists of y with $|y| = 2n$ being a descendant of some $x \in G_{n,\varepsilon}$ such that $V(y_{n+i}) - V(x) \le \alpha\varepsilon i$, $\forall 1 \le i \le L$ and that $V(y_{n+j}) - V(y_{n+L}) \le (1-\alpha)\varepsilon L$, $\forall L < j \le n$.

Let $q_{n,\varepsilon}$ denote the probability of extinction of the Galton–Watson tree $\mathbb{G}_{n,\varepsilon}$. It is clear that

$$\varrho(\varepsilon) \ge 1 - q_{n,\varepsilon}\,;$$

so we only need to find a lower bound for $1 - q_{n,\varepsilon}$. In order to do so, we introduce, for $b \in \mathbb{R}$ and $n \ge 1$,

$$\varrho(b,\, n) := \mathbf{P}\Big(\exists |x| = n:\ V(x_i) \le bi,\ \forall 1 \le i \le n\Big). \tag{4.4}$$

Let us first prove some preliminary results.

Lemma 4.3 *Let $0 < \alpha < 1$ and $\varepsilon > 0$. Let $n > L \ge 1$ be such that $(1-\alpha)\varepsilon L \ge \nu(n-L)$. Then*

$$\mathbf{P}\{G_{n,\varepsilon} \neq \emptyset\} \ge \frac{1}{2}\varrho(\alpha\varepsilon,\, n).$$

Proof. By definition,

$$\mathbf{P}\{G_{n,\varepsilon} \neq \emptyset\} = \mathbf{E}\Big\{\mathbf{1}_{\{\exists |y|=L:\, V(y_i)\le \alpha\varepsilon i,\, \forall i \le L\}} \mathbf{P}\Big(\max_{|z|\le n-L} V(z) \le (1-\alpha)\varepsilon L\Big)\Big\}.$$

Since $(1-\alpha)\varepsilon L \ge \nu(n-L)$, it follows from (4.3) that

$$\mathbf{P}\{G_{n,\varepsilon} \neq \emptyset\} \ge \frac{1}{2}\mathbf{P}\{\exists |y| = L:\ V(y_i) \le \alpha\varepsilon i,\ \forall i \le L\},$$

which is no smaller than $\frac{1}{2}\varrho(\alpha\varepsilon,\, n)$. □

Lemma 4.4 *Let $0 < \alpha < 1$ and $\varepsilon > 0$. Let $n > L \ge 1$ be such that $(1-\alpha)\varepsilon L \ge \nu(n-L)$. We have,*

$$\mathbf{P}\{1 \le \# G_{n,\varepsilon} \le \vartheta^{n-L}\} \le \frac{1}{\kappa\, \vartheta^{n-L}},$$

where $\kappa > 0$ and $\vartheta > 1$ are the constants in (4.3) and (4.1), respectively.

Proof. By definition,

$$\#G_{n,\varepsilon} = \sum_{|x|=L} \eta_x \mathbf{1}_{\{\exists |x|=L:\, V(x_i)\le \alpha\varepsilon i,\ \forall i\le L\}},$$

where[3]

$$\eta_x := \#\{y > x:\ |y| = n\}\, \mathbf{1}_{\{\max_{\{z>x:\, |z|\le n\}}[V(z)-V(x)]\le (1-\alpha)\varepsilon L\}}.$$

By Fact 4.2, for any $\ell \ge 1$,

$$\mathbf{P}\Big(\#G_{n,\varepsilon} \le \ell \,\Big|\, \#G_{n,\varepsilon} > 0\Big) \le \mathbf{P}\Big(Z_{n-L} \le \ell \,\Big|\, Z_{n-L} > 0,\ \max_{|z|\le n-L} V(z) \le (1-\alpha)\varepsilon L\Big),$$

where, as before, $Z_{n-L} := \#\{|x| = n-L\}$. Since $(1-\alpha)\varepsilon L \ge \nu(n-L)$, it follows from (4.3) that $\mathbf{P}(Z_{n-L} > 0,\ \max_{|z|\le n-L} V(z) \le (1-\alpha)\varepsilon L) \ge \kappa > 0$. Therefore,

$$\mathbf{P}(1 \le \#G_{n,\varepsilon} \le \ell) \le \frac{1}{\kappa}\mathbf{P}\Big(Z_{n-L} \le \ell \,\Big|\, Z_{n-L} > 0\Big).$$

This implies Lemma 4.4 by means of Fact 4.1. $\square$

Lemma 4.5 *For any $\theta > 0$,*

$$\liminf_{n\to\infty} \frac{\log \varrho(\theta n^{-2/3},\, n)}{n^{1/3}} \ge -\frac{\pi\sigma}{(2\theta)^{1/2}},$$

where $\sigma > 0$ is the constant in (2.5).

Proof. We use a second moment argument by means of the Paley–Zygmund inequality. Let $0 < \lambda < \frac{\pi\sigma}{(2\theta)^{1/2}}$. For brevity, we write $I_{i,n} := [\frac{\theta i}{n^{2/3}} - \lambda n^{1/3},\ \frac{\theta i}{n^{2/3}}]$ for all $1 \le i \le n$, and consider

$$Y_n := \sum_{|x|=n} \mathbf{1}_{\{V(x_i)\in I_{i,n}\,,\ \forall 1\le i\le n\}}.$$

By the change-of-probabilities formula (2.6) and in its notation,

$$\mathbf{E}(Y_n) = \mathbf{E}\Big(\mathrm{e}^{S_n}\, \mathbf{1}_{\{S_i\in I_{i,n}\,,\ \forall 1\le i\le n\}}\Big),$$

which, for any $\chi > 0$, is

$$\begin{aligned} &\ge\ \mathrm{e}^{(\theta-\chi)n^{1/3}}\, \mathbf{P}\{S_i \in I_{i,n}\,,\ \forall 1\le i\le n,\ S_n \ge (\theta-\chi)n^{1/3}\} \\ &=\ \mathrm{e}^{(\theta-\chi)n^{1/3}}\, \mathbf{P}\Big\{\theta\frac{i}{n} - \lambda \le \frac{S_i}{n^{1/3}} \le \theta\frac{i}{n},\ \forall 1\le i\le n,\ \frac{S_n}{n^{1/3}} \ge \theta-\chi\Big\}. \end{aligned}$$

[3] We write $y > x$ if x is an ancestor of y.

Applying (2.8) to $j := n$, $a_i := i^{1/3}$, $g_1(t) := \theta t - \lambda$ and $g_2(t) := \theta t$, we see that for any $\lambda_1 \in (0, \lambda)$ and all sufficiently large n,

$$\mathbf{E}(Y_n) \geq \mathrm{e}^{(\theta-\chi)n^{1/3}} \exp\Big(- \frac{\pi^2\sigma^2}{2\lambda_1^2}\, n^{1/3}\Big). \tag{4.5}$$

We now estimate the second moment of Y_n. By definition,

$$\begin{aligned}\mathbf{E}(Y_n^2) &= \mathbf{E}\Big\{ \sum_{|x|=n} \sum_{|y|=n} \mathbf{1}_{\{V(x_i)\in I_{i,n}\,,\, V(y_i)\in I_{i,n}\,,\, \forall 1\le i\le n\}} \Big\} \\ &= \mathbf{E}\Big\{ \sum_{j=0}^{n} \sum_{|z|=j} \mathbf{1}_{\{V(z_i)\in I_{i,n}\,,\, \forall i\le j\}} \sum_{(x,y)} \mathbf{1}_{\{V(x_i)\in I_{i,n}\,,\, V(y_i)\in I_{i,n}\,,\, \forall j\le i\le n\}} \Big\},\end{aligned}$$

where the double sum $\sum_{(x,y)}$ is over pairs (x, y) with $|x| = |y| = n$ such that $z < x$ and $z < y$ and that $x_{j+1} \neq y_{j+1}$. Therefore,

$$\mathbf{E}(Y_n^2) \leq \mathbf{E}\Big\{ \sum_{j=0}^{n} \sum_{|z|=j} \mathbf{1}_{\{V(z_i)\in I_{i,n}\,,\, \forall i\le j\}} \sum_{x>z,\, |x|=n} \mathbf{1}_{\{V(x_i)\in I_{i,n}\,,\, \forall j\le i\le n\}}\, h_{j,n} \Big\},$$

where

$$h_{j,n} := \sup_{u\in I_{j,n}} \mathbf{E}\Big(\sum_{|y|=n-j} \mathbf{1}_{\{V(y_\ell)\in [\frac{\theta(\ell+j)}{n^{2/3}} - \lambda n^{1/3} - u,\ \frac{\theta(\ell+j)}{n^{2/3}} - u],\ \forall \ell\le n-j\}} \Big).$$

Accordingly,

$$\mathbf{E}(Y_n^2) \leq \sum_{j=0}^{n} \mathbf{E}(Y_n) h_{j,n} = \mathbf{E}(Y_n) \sum_{j=0}^{n} h_{j,n}.$$

Taking (4.5) into account, we obtain:

$$\frac{\mathbf{E}(Y_n^2)}{[\mathbf{E}(Y_n)]^2} \leq \exp\Big[(-\theta + \chi + \frac{\pi^2\sigma^2}{2\lambda_1^2}) n^{1/3} \Big] \sum_{j=0}^{n} h_{j,n}. \tag{4.6}$$

To estimate $\sum_{j=0}^n h_{j,n}$, we observe that by the change-of-probabilities formula (2.6),

$$\begin{aligned} h_{j,n} &= \sup_{u\in I_{j,n}} \mathbf{E}\Big(\mathrm{e}^{S_{n-j}} \mathbf{1}_{\{S_i \in [\frac{\theta(i+j)}{n^{2/3}} - \lambda n^{1/3} - u,\ \frac{\theta(i+j)}{n^{2/3}} - u],\ \forall i\le n-j\}} \Big) \\ &= \sup_{v\in [0,\, \lambda n^{1/3}]} \mathbf{E}\Big(\mathrm{e}^{S_{n-j}} \mathbf{1}_{\{S_i \in [\frac{\theta i}{n^{2/3}} - \lambda n^{1/3} + v,\ \frac{\theta i}{n^{2/3}} + v],\ \forall i\le n-j\}} \Big) \\ &\leq \mathrm{e}^{\theta(n-j)n^{-2/3} + \lambda n^{1/3}} \sup_{v\in [0,\, \lambda n^{1/3}]} \mathbf{P}\Big\{ \frac{\theta i}{n^{2/3}} - \lambda n^{1/3} + v \leq S_i \leq \frac{\theta i}{n^{2/3}} + v,\ \forall i \leq n-j \Big\}. \end{aligned}$$

We now use the same dirty trick as in the proof of the upper bound in Theorem 1.2 by sending n to infinity along a subsequence. Fix an integer $N \geq 1$. Let $n := Nk$, with $k \geq 1$. For any $j \in [(\ell-1)k+1,\, \ell k] \cap \mathbb{Z}$ (with $1 \leq \ell \leq N$), we have

$$h_{j,n} \leq \mathrm{e}^{\theta(N-\ell+1)kn^{-2/3}+\lambda n^{1/3}} \sup_{v\in[0,\, \lambda n^{1/3}]} \mathbf{P}\{v-\lambda n^{1/3} \leq S_i - \frac{\theta i}{n^{2/3}} \leq v,\ \forall i \leq (N-\ell)k\}.$$

Unfortunately, the interval $[0,\, \lambda n^{1/3}]$ in $\sup_{v\in[0,\, \lambda n^{1/3}]} \mathbf{P}\{\cdots\}$ is very large, so we split it into smaller ones of type $[\frac{(m-1)\lambda n^{1/3}}{N},\ \frac{m\lambda n^{1/3}}{N}]$ (for $1 \leq m \leq N$), to see that the $\sup_{v\in[0,\, \lambda n^{1/3}]} \mathbf{P}\{\cdots\}$ expression is

$$\begin{aligned} &\leq \max_{1\leq m\leq N} \mathbf{P}\{\frac{(m-1)\lambda n^{1/3}}{N} - \lambda n^{1/3} \leq S_i - \frac{\theta i}{n^{2/3}} \leq \frac{m\lambda n^{1/3}}{N},\ \forall i \leq (N-\ell)k\} \\ &= \max_{1\leq m\leq N} \mathbf{P}\{-\frac{(N-m+1)\lambda}{N^{2/3}} \leq \frac{S_i}{k^{1/3}} - \frac{\theta}{N^{2/3}}\frac{i}{k} \leq \frac{m\lambda}{N^{2/3}},\ \forall i \leq (N-\ell)k\}. \end{aligned}$$

We are now entitled to apply (2.7) to $j := (N-\ell)k$, $a_i := i^{1/3}$, $g_1(t) := \frac{\theta}{(N-\ell)^{1/3}N^{2/3}}t - \frac{(N-m+1)\lambda}{(N-\ell)^{1/3}N^{2/3}}$ and $g_2(t) := \frac{\theta}{(N-\ell)^{1/3}N^{2/3}}t + \frac{m\lambda}{(N-\ell)^{1/3}N^{2/3}}$, to see that for any $1 \leq \ell \leq N$ and uniformly in $j \in [(\ell-1)k+1,\, \ell k] \cap \mathbb{Z}$ (and in $j=0$, which formally corresponds to $\ell = 0$),

$$\limsup_{k\to\infty} \frac{1}{N^{1/3}k^{1/3}} \log h_{j,n} \leq \frac{\theta(N-\ell+1)}{N} + \lambda - \frac{\pi^2\sigma^2}{2}\frac{(N-\ell)N}{(N+1)^2\lambda^2},$$

which is bounded by $\frac{\theta(N+1)}{N} + \lambda - \frac{\pi^2\sigma^2}{2}\frac{N^2}{(N+1)^2\lambda^2}$ (recalling that $\theta > \frac{\pi^2\sigma^2}{2\lambda^2}$). As a consequence,

$$\limsup_{k\to\infty} \frac{1}{N^{1/3}k^{1/3}} \log \sum_{j=0}^{n} h_{j,n} \leq \frac{\theta(N+1)}{N} + \lambda - \frac{\pi^2\sigma^2}{2}\frac{N^2}{(N+1)^2\lambda^2} =: c(\theta, N, \lambda).$$

Going back to (4.6), we get

$$\limsup_{k\to\infty} \frac{1}{N^{1/3}k^{1/3}} \log \frac{\mathbf{E}(Y_{Nk}^2)}{[\mathbf{E}(Y_{Nk})]^2} \leq -\theta + \chi + \frac{\pi^2\sigma^2}{2\lambda_1^2} + c(\theta, N, \lambda).$$

By the Paley-Zygmund inequality, $\mathbf{P}\{Y_n \geq 1\} \geq \frac{[\mathbf{E}(Y_n)^2]}{\mathbf{E}(Y_n^2)}$. Since $\varrho(\theta n^{-2/3},\, n) \geq \mathbf{P}\{Y_n \geq 1\}$, this yields

$$\liminf_{k\to\infty} \frac{\log \varrho(\theta N^{-2/3}k^{-2/3},\, Nk)}{N^{1/3}k^{1/3}} \geq \theta - \chi - \frac{\pi^2\sigma^2}{2\lambda_1^2} - c(\theta, N, \lambda).$$

By the monotonicity of $n \mapsto \varrho(\theta n^{-2/3},\, n)$, we obtain:

$$\liminf_{n\to\infty} \frac{\log \varrho(\theta n^{-2/3},\, n)}{n^{1/3}} \geq \theta - \chi - \frac{\pi^2\sigma^2}{2\lambda_1^2} - c(\theta, N, \lambda).$$

Sending $N \to \infty$, $\chi \to 0$, $\lambda \to \frac{\pi\sigma}{(2\theta)^{1/2}}$ and $\lambda_1 \to \frac{\pi\sigma}{(2\theta)^{1/2}}$ (in this order) completes the proof of Lemma 4.5. $\square$

We now have all the ingredients for the proof of the lower bound in Theorem 1.2.

Proof of Theorem 1.2: the lower bound. Fix constants $0 < \alpha < 1$ and $b > \max\{\frac{\nu}{1-\alpha}, \frac{(3\pi\sigma)^2}{\alpha(\log\vartheta)^2}\}$. Let $n > 1$. Let

$$\varepsilon = \varepsilon(n) := \frac{b}{n^{2/3}}, \qquad L = L(n) := n - \lfloor n^{1/3} \rfloor.$$

Then $(1-\alpha)\varepsilon L \geq \nu(n-L)$ for all sufficiently large n, say[4] $n \geq n_0$.

Consider the moment generating function of the reproduction distribution in the Galton–Watson tree $\mathbb{G}_{n,\varepsilon}$:

$$f(s) := \mathbf{E}(s^{\# G_{n,\varepsilon}}), \qquad s \in [0, 1].$$

It is well-known that $q_{n,\varepsilon}$, being the extinction probability of $\mathbb{G}_{n,\varepsilon}$, satisfies $q_{n,\varepsilon} = f(q_{n,\varepsilon})$. Therefore, for any $0 < r < \min\{q_{n,\varepsilon}\,, \frac13\}$,

$$q_{n,\varepsilon} = f(0) + \int_0^{q_{n,\varepsilon}} f'(s)\,\mathrm{d}s = f(0) + \int_0^{q_{n,\varepsilon}-r} f'(s)\,\mathrm{d}s + \int_{q_{n,\varepsilon}-r}^{q_{n,\varepsilon}} f'(s)\,\mathrm{d}s.$$

Since $s \mapsto f'(s)$ is non-decreasing on $[0, 1]$, we have $\int_0^{q_{n,\varepsilon}-r} f'(s)\,\mathrm{d}s \leq f'(1-r)$. On the other hand, since $f'(s) \leq f'(q_{n,\varepsilon}) \leq 1$ for $s \in [0, q_{n,\varepsilon}]$, we have $\int_{q_{n,\varepsilon}-r}^{q_{n,\varepsilon}} f'(s)\,\mathrm{d}s \leq r$. Therefore,

$$q_{n,\varepsilon} \leq f(0) + f'(1-r) + r.$$

Of course, $f(0) = \mathbf{P}\{G_{n,\varepsilon} = \emptyset\}$, whereas $f'(1-r) = \mathbf{E}[(\# G_{n,\varepsilon})(1-r)^{\# G_{n,\varepsilon}-1}]$, which is bounded by $\frac{1}{1-r}\mathbf{E}[(\# G_{n,\varepsilon})\mathrm{e}^{-r\# G_{n,\varepsilon}}]$ (using the elementary inequality $1-u \leq \mathrm{e}^{-u}$ for $u \geq 0$). This leads to (recalling that $r < \frac13 < \frac12$):

$$1 - q_{n,\varepsilon} \geq \mathbf{P}\{G_{n,\varepsilon} \neq \emptyset\} - 2\mathbf{E}[(\# G_{n,\varepsilon})\mathrm{e}^{-r\# G_{n,\varepsilon}}] - r.$$

Since $u \mapsto u\mathrm{e}^{-ru}$ is decreasing on $[\frac1r, \infty)$, it is seen that $\mathbf{E}[(\# G_{n,\varepsilon})\mathrm{e}^{-r\# G_{n,\varepsilon}}]$ is bounded by $\mathbf{E}[(\# G_{n,\varepsilon})\mathrm{e}^{-r\# G_{n,\varepsilon}}\,\mathbf{1}_{\{\# G_{n,\varepsilon} \leq r^{-2}\}}] + r^{-2}\mathrm{e}^{-1/r} \leq r^{-2}\mathbf{P}(1 \leq \# G_{n,\varepsilon} \leq r^{-2}) + r^{-2}\mathrm{e}^{-1/r}$. Accordingly,

$$\begin{aligned} 1 - q_{n,\varepsilon} &\geq \mathbf{P}\{G_{n,\varepsilon} \neq \emptyset\} - \frac{2}{r^2}\,\mathbf{P}(1 \leq \# G_{n,\varepsilon} \leq r^{-2}) - \frac{2\mathrm{e}^{-1/r}}{r^2} - r \\ &\geq \frac12\,\varrho(\alpha\varepsilon, n) - \frac{2}{r^2}\,\mathbf{P}(1 \leq \# G_{n,\varepsilon} \leq r^{-2}) - 2r, \end{aligned}$$

[4]Without further mention, the value of n_0 can change from line to line when other conditions are to be satisfied.

the last inequality following from Lemma 4.3 and the fact that $\sup_{\{0<r\le \frac13\}} \frac{1}{r^3}\mathrm{e}^{-1/r} = \mathrm{e}^{-1} < \frac12$.

We choose $r := \frac{1}{16}\,\varrho(\alpha\varepsilon,\, n)$. [Since $\varrho(\varepsilon) \ge 1 - q_{n,\varepsilon}$, whereas $\lim_{\varepsilon\to 0}\varrho(\varepsilon) = 0$ (proved in Section 3), we have $q_{n,\varepsilon} \to 1$ for $n\to\infty$, and thus the requirement $0 < r < \min\{q_{n,\varepsilon}\,,\, \frac13\}$ is satisfied for all sufficiently large n.]

By Lemma 4.5, $r^{-2} \le \vartheta^{n-L}$ for all $n \ge n_0$ (because $\frac{2\pi\sigma}{(\alpha b)^{1/2}} < \log\vartheta$ by our choice of b). Therefore, an application of Lemma 4.4 tells us that for $n \ge n_0$, $\mathbf{P}(1 \le \# G_{n,\varepsilon} \le r^{-2}) \le \frac{1}{\kappa\,\vartheta^{n-L}}$, which, by Lemma 4.4 again, is bounded by r^3 (because $\frac{3\pi\sigma}{(\alpha b)^{1/2}} < \log\vartheta$). Consequently, for all $n \ge n_0$,

$$1 - q_{n,\varepsilon} \ge \frac12\,\varrho(\alpha\varepsilon,\, n) - 2r - 2r = \frac14\,\varrho(\alpha\varepsilon,\, n).$$

Recall that $\varrho(\varepsilon) \ge 1 - q_{n,\varepsilon}$. Therefore,

$$\liminf_{n\to\infty} \frac{1}{n^{1/3}} \log \varrho(\frac{b_1}{n^{2/3}}) \ge -\frac{\pi\sigma}{(2\alpha b_1)^{1/2}}.$$

Since $\varepsilon \mapsto \varrho(\varepsilon)$ is non-decreasing, we obtain:

$$\liminf_{\varepsilon\to 0} \varepsilon^{1/2} \log \varrho(\varepsilon) \ge -\frac{\pi\sigma}{(2\alpha)^{1/2}}.$$

Sending $\alpha \to 1$ yields (4.2), and thus proves the lower bound in Theorem 1.2. □

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Nina Gantert
Fachbereich Mathematik und Informatik
Universität Münster
Einsteinstrasse 62
D-48149 Münster
Germany
`gantert@math.uni-muenster.de`

Yueyun Hu
Département de Mathématiques
Université Paris XIII
99 avenue J-B Clément
F-93430 Villetaneuse
France
`yueyun@math.univ-paris13.fr`

Zhan Shi
Laboratoire de Probabilités UMR 7599
Université Paris VI
4 place Jussieu
F-75252 Paris Cedex 05
France
`zhan.shi@upmc.fr`