Branching Random Walks

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Outline

Definition

Branching Random Walk (BRW)

- Discrete time, measure-valued Markov chain
- Evolves according to point process $\mathscr{L}$
- Denoted $X = (X_n)_{n \geq 0}$

Branching Random Walk with Selection (N-BRW)

- At each step only $N \geq 2$ rightmost particles survive

Assumptions and notation

Notation

- $\#\mathscr{L}$ — total mass of $\mathscr{L}$
- $\max \mathscr{L}$ and $\min \mathscr{L}$ — right- and leftmost particles
- $\mathbb{T}$ — underlying Galton-Watson tree
- For vertex $x \in \mathbb{T}$
  - $|x|$ — distance from root
  - $root = x_0, x_1, ..., x_{|n|} = x$ — unique path to root
  - $V(x)$ — position of particle x
- $\sum_{l \in \mathscr{L}} [\cdots] \equiv \sum_{|x|=1} [\cdots]$ — sum over particles of $\mathscr{L}$

Basic assumptions

$$\#\mathscr{L} \geq 1 \qquad \text{almost surely, and} \qquad 1 < \mathbb{E}[\#\mathscr{L}] < \infty.$$

Problems of interest

Questions

1. How does $\max X_n/n$ behave as $n \to \infty$?
2. How does $v_N := \lim_{n\to\infty} \max X_n/n$ depend on N?

Answer

Depends on $\mathscr{L}$: light-tails/heavy-tails, continuous/non-continuous...

We now present some cases where these questions have been studied

Light tails

Define the logarithmic moment generating function

$$\psi(t) := \log \mathbb{E} \sum_{|x|=1} e^{tV(x)}.$$

Suppose that there exist $\delta_- < 0 < \delta_+$ such that

$$\psi(\delta_-),\, \psi(\delta_+) < \infty.$$

This implies $\psi \in C^\infty$ near 0.

The special case $\mathscr{L} = \delta_{Y_1} + \delta_{Y_2}$ with Y_1, Y_2 i.i.d. was studied in the seminal paper [2] by Bérard and Gouéré.

A technical condition

Finiteness of ψ near 0 is not enough. We need

$$\psi'(t^*)t^* = \psi(t^*) \qquad \text{for some } t^* > 0,$$

to apply results about killed BRWs ([3]).

Example

If Y_1, Y_2 as on previous slide, then any distribution that is absolutely continuous with finite moment generating function everywhere would do; e.g. Gaussian

Results I

Proposition

Let $d(X_n) = \max X_n - \min X_n$. *Then*

$$\frac{d(X_n)}{n} \xrightarrow[n\to\infty]{a.s.,\, L^1} 0.$$

Proposition

There exists $v_N \in \mathbb{R}$ *such that*

$$\frac{\max X_n}{n} \xrightarrow[n\to\infty]{a.s.,\, L^1} v_N,$$

and $v_N \uparrow v := t^*\psi(t^*)$ *as* $N \to \infty$.

Results II

Theorem ([2, Theorem 1])

As $N \to \infty$,

$$v_N = v - \frac{\pi^2 t^* \psi''(t^*)}{2(\log N)^2} + o((\log N)^{-2}).$$

The slow convergence rate $(\log N)^{-2}$ is called the 'Brunet-Derrida' behaviour.

Example

Y_1, Y_2 i.i.d. $\mathcal{N}(\mu, \sigma^2)$. Then $\psi(t) = \log 2 + \mu t + t^2\sigma^2/2$ and $t^* = \sqrt{\frac{\log 4}{\sigma^2}}$. This gives $v = \mu + \sqrt{\sigma^2 \log 4}$ and the correction term is $\frac{\pi^2\sqrt{\sigma^2 \log 4}}{2(\log N)^2}$.

Connection with $1-d$ random walks

If $\exists\, t>0$ s.t. $\psi(t)<\infty$ transform $\mathscr{L}$ and assume $\psi(1)=0$. Define $1-d$ random walk $(S_n)_{n\geq 0}$ (called *spine*) with step distribution X s.t.

$$\mathbb{P}\left(X\leq x\right)=\mathbb{E}\sum_{|u|=1}\mathbb{1}_{\{V(u)\leq x\}}e^{V(u)}. \tag{1}$$

Lemma (Many-to-One)

Take g measurable and $n\geq 1$. Provided the integrals exist,

$$\mathbb{E}\sum_{|x|=n}g(V(x_1),...,V(x_n))=\mathbb{E}\left[e^{-S_n}g(S_1,...,S_n)\right]. \tag{2}$$

α-stable spine

Previous conditions

$\psi(\delta_-), \psi(\delta_+) < \infty$ and $t^*\psi'(t^*) = \psi(t^*)$. Meaning: Spine $(S_n)_{n\geq 0}$ has all moments and is centered

Mallein's conditions ([4])

$\psi(\delta_+) < \infty$ and X in domain of attraction of α-stable Y.

Results

Suppose the N-BRW is in the 'stable boundary' case $\psi(1) = 0$. Let

$$L^*(x) := x^{\alpha-2}\mathbb{E}\left[Y^2 \mathbb{1}_{|Y|\leq x}\right].$$

Theorem

There exists $C^* \in (0, \infty)$ such that as $N \to \infty$,

$$v_N \sim -C^* \frac{L^*(\log N)}{(\log N)^\alpha}.$$

α-stable spine - example

Let ν_α be α-stable with $\nu_\alpha([0,\infty)) \in (0,1)$. Let $\mathscr{L} = PPP(\nu_\alpha(dx)e^{-x})$. Then $\mathscr{L}$ satisfies the hypothesis. Indeed:

- By the Slivnyak-Mecke Theorem ([1, Theorem 1.13]) we are in the stable boundary case:

$$\mathbb{E}\sum_{l\in\mathscr{L}} e^l = \int_{\mathbb{R}} e^x e^{-x} \nu_\alpha(dx) = 1.$$

- Mallein states that the spine is in the domain of attraction of ν_α, I haven't been able to prove this yet. However, by the Many-to-One Lemma

$$\mathbb{E}|X|^\beta = \mathbb{E}\sum_{l\in\mathscr{L}} |l|^\beta e^l = \int_{\mathbb{R}} |x|^\beta \nu_\alpha(dx) = \infty$$

  for $\beta > \alpha$, so we're certainly not in the Bérard-Gouéré case.

Proof outline

Proof is based on idea that TFAE

(a) N i.i.d. BRW do not survive killing below the speed $v - \epsilon$

(b) $v_N < v - \epsilon$.

Shown in [3]: $m \propto \epsilon^{-u}$ with $u \in (0, 3/2]$ we have

$$\begin{aligned} \log \rho(m, -\epsilon) &\propto -\epsilon^{-u/3} && \text{as } \epsilon \downarrow 0, \\ \log \rho(\infty, -\epsilon) &\propto -\epsilon^{1/2} && \text{as } \epsilon \downarrow 0. \end{aligned}$$

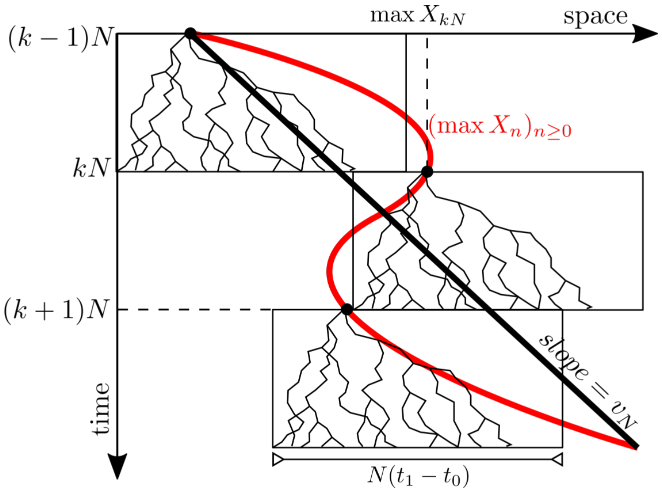
Based on this, expect

$$\rho(\infty, -\epsilon_N) \propto \frac{1}{N},$$

where $\epsilon_N := v - v_N$.

Convergence of diameter

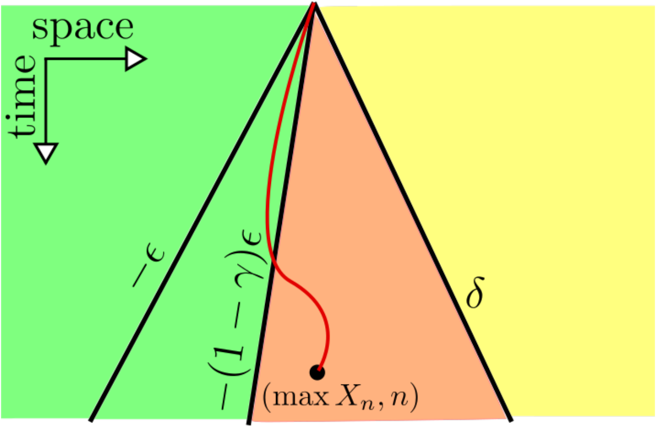
Existence of *a.s.* and L^1 limit of velocity by Subadditive Ergodic Theorem



Upper bound on velocity

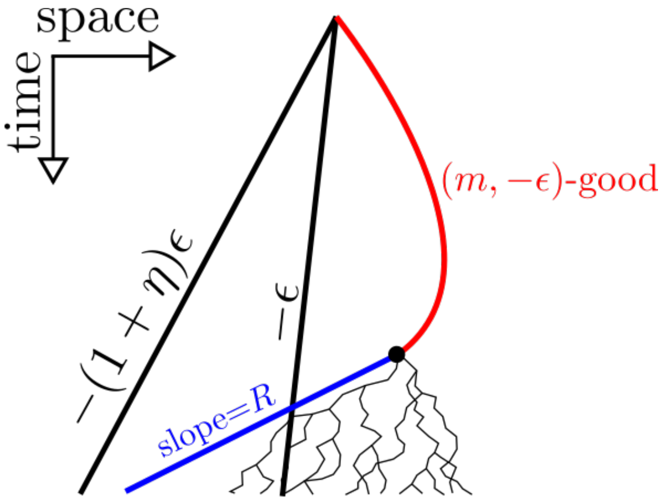
- By Subadditivity $v_N \leq \frac{\mathbb{E}\max X_n}{n}$ for all $n \geq 1$.
- Decompose $\mathbb{E}\max X_n$ and prove upper bound on it:

$$\mathbb{E}\max X_n/n \leq -(1-\gamma)\epsilon + \mathbb{E}\max X_n \mathbb{1}_{\max X_n \geq \delta n} + \delta\,\mathbb{P}\left(\max X_n \geq -(1-\gamma)\epsilon n\right)$$

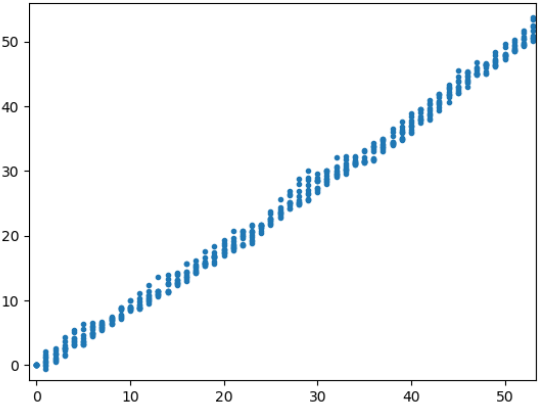


Lower bound on velocity

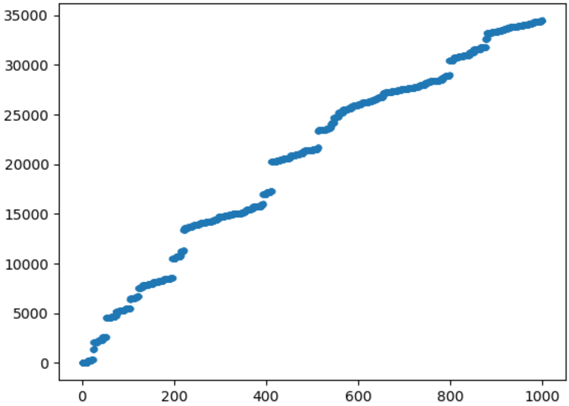
Comes down to showing $\mathbb{P}\left(\min X_k < -(1+\gamma)n \,\forall\, k \in [n]\right)$ is small



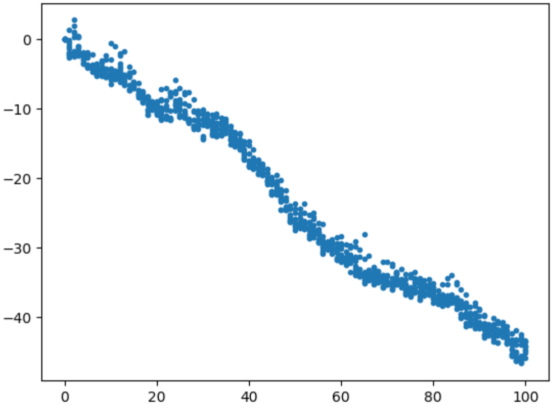
Binary branching, standard normals



Binary branching, Cauchy



$PPP(e^{-x}\nu_\alpha)$



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