The oriented contact process and the East model

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Overview

Let me first give you an overview of what my project was about. It started out primarily concerned with the East-process. The idea was to apply a recent comparison based method that yielded new results for a different process. The corresponding results for the East process are already known, but an alternative, shorter proof could give indication that this comparison method could be useful for other KCMs of oriented nature, the class of processes that the East process belongs to. Upon reviewing the literature it became apparent that the necessary results for the oriented contact process were never precisely stated or proven even though the machinery had been developed. In the past 6 weeks we created a self contained proof of the necessary results and apply them to prove a weaker version of the known mixing time results for the East process.

East process

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3. Its state space is the set of configurations of 0s and 1s on the Integer lattice. Talk about powerset.
4. Talk about graphical representation. Point out cross that doesn't kill spin.

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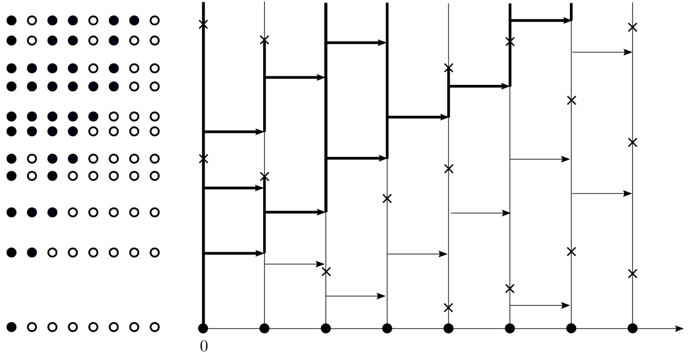

Figure: East process started from $\{0\}$

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Front propagation

1. The first step towards the proof of the linearity of the mixing time is a result about the front of the East process
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3. Read theorem. What this says is ... .
4. We give an alternative proof but for now, assuming this result, we can go on to show the desired linearity of the mixing time.

Front propagation

Definition (Front)

Let $(\sigma_t)_{t\geq 0}$ be an East process started from $\{0\}$. The front of $(\sigma_t)_{t\geq 0}$ is the rightmost occupied site, written $X(\sigma_t) := \sup \sigma_t$.

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Theorem (Blondel, 2013, Lemma 3.2)

There exist constants $0 < \alpha < v$ and $\gamma, C > 0$ such that

$$\mathbb{P}\left(X(\sigma_t) \in (\alpha t, vt)\right) \geq 1 - Ce^{-\gamma t} \qquad \forall t \geq 0.$$

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Proof.

We propose an alternative proof (under restrictions on p) detailed later. $\square$

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Mixing time

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Alternative proof using only front propagation.

Upper bound: Coupling time vs. propagation of the front.
Lower bound: Hitting times of large sets (Peres and Sousi, 2015). $\square$

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One-sided contact process

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DEPARTMENT OF STATISTICS

1. So I explained how the front propagation result can be used to prove the mixing time. But how do we prove the front propagation result? This is where the comparison technique comes in. Instead of studying the spectral gap of the East process, we compare it to a somewhat simpler process - the one-sided/oriented contact process.
2. Explain how basic and oriented contact process works.
3. Point out that cross always implies death.

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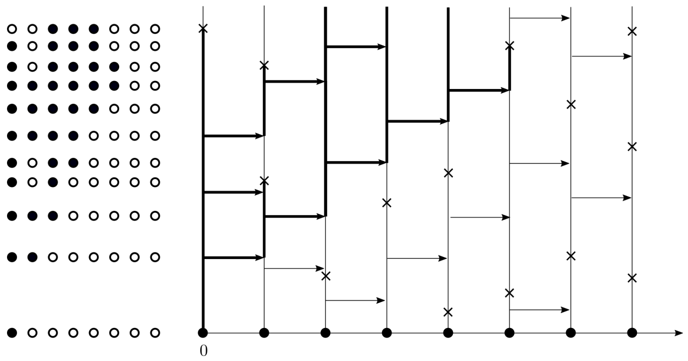

Figure: One-sided contact process started from $\{0\}$

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3. Partial order and how its preserved
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Let $(\eta_t)_{t \geq 0}$ be the one-sided contact process started from $\{0\}$. Then under the basic coupling

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For this to be useful, we need $p > p_c$ where $p_c := \sup\{p : |\eta| \to 0 \; a.s.\}$. From here on consider only such p.

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Results for one-sided contact process

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3. Read theorem and explain
4. These results were never written down or proven in detail before.

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Let $\tau(\eta_\cdot)$ be the extinction time. Then

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If $p > p_c$, there exists $\alpha > 0$ such that for all $a < \alpha$ there exist constants $\gamma, C > 0$ such that

$$\mathbb{P}\left(X(\eta_t) < at \,|\, \tau(\eta_\cdot) = \infty\right) \leq Ce^{-\gamma t} \qquad \forall t \geq 0.$$

Furthermore there exist constants $\gamma, C > 0$ such that

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Remark

Hinted at in Durrett and Griffeath, 1983 but never proved in detail.

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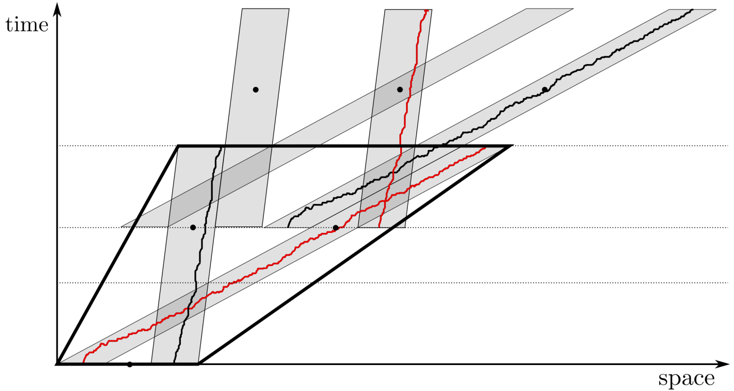

Figure: Red path showing how oriented percolation to infinity occurs.

References

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