1 Introduction and construction

In this section we introduce the main objects of study and construct the East and 1-sided contact processes on the same probability space using the graphical method. This construction gives rise to important properties used later that we discuss.

1.1 Introduction

The East-process is an interacting particle system evolving with a Glauber like dynamics on the state space $\Omega := \{0,1\}^{\mathbb{Z}}$. It belongs to a class of stochastic processes called kinetically constrained spin models (KCMs), with the East-process being the first of these to be studied rigorously. The process evolves as follows: at each site $x \in \mathbb{Z}$ the system tries to update the value of the spin at x to 1 or 0 at rate $p \in (0,1)$ and $q := 1-p$ respectively. The update is accepted only if a local constraint is satisfied, which in the East-process' case is that the occupation variable at site $x-1$ must be equal to 1. Sometimes we will call elements of Ω *configurations* and say a site is *occupied* or *infected* if its spin value is equal to 1.

In the sections to follow we focus on two objects of interest related to the East-process. The first one is the speed of the so-called *front*. Consider an East-process started from the configuration equal to all 0 except at the origin. It is easy to see that the spins on $(-\infty, 0]$ stay frozen for all time, and infection 'spreads' to the right. A natural question to ask then is how fast this spreading of infection happens *if* it happens at all. We define the front to be the rightmost infected site in the configuration at time t. We will show that for large enough p the front of the East-process started from exactly one infection propagates at precisely linear speed. In our study of the speed of the front we will compare the East-process to a second stochastic process called the 1-sided contact process on $\mathbb{Z}$. The 1-sided contact process on $\mathbb{Z}$ has the same state space Ω and evolves as follows: each site infects its neighbour to the right at rate p and *recovers* i.e. sets its own spin to 0 at rate q.

The second object of interest is the mixing time of the East process when restricted to $\{0,1,...,L\}$ for some $L \in \mathbb{N}^+$. We will study the mixing time for the East-process on $\{0,1,...,L\}$ with the occupation of the origin fixed to be 1, so that the evolution at site 1 is unconstrained. We go on to show that for large enough p the mixing time is $\Theta(L)$[1].

1.2 Constructing the basic coupling

Let $\mathscr{P} = (E_{x,k}, B_{x,k})_{x\in\mathbb{Z}, k\in\mathbb{N}^+}$ be a collection of independent random variables with $E_{x,k} \sim \text{Exp}(1)$ and $B_{x,k} \sim \text{Ber}(p = 1-q)$. Define the times

$$T_{x,n} := \sum_{k=1}^{n} E_{x,k}$$

also referred to as *clock rings* and call a clock ring $T_{x,n}$ *legal* if the local constraint of the corresponding process is satisified at site x at time $T_{x,n}$. We can now construct the East-process $(\sigma_t)_{t\geq 0}$ and the 1-sided contact process $(\eta_t)_{t\geq 0}$ using $\mathscr{P}$ as follows.

[1]We say $f = \Theta(g)$ if there exist constants K_1, K_2 such that for all large enough $n \in \mathbb{N}$ it holds that $K_1 g(n) \leq f(n) \leq K_2 g(n)$.

For each site $x \in \mathbb{Z}$ at each time $T_{x,n}$:

- If $B_{x,n} = 1$:
  1. If $\sigma_{T_{x,n}^-}(x-1) = 1$ update σ to 1 at site x.
  2. If $\eta_{T_{x,n}^-}(x-1) = 1$ update η to 1 at site x.
- Else:
  1. If $\sigma_{T_{x,n}^-}(x-1) = 1$ update σ to 0 at site x.
  2. Update η to 0 at site x.

Notation 1.1 (Initial configurations). Suppose we start a stochastic process $(\xi_t)_{t\geq 0}$ with state space Ω from initial configuration $\nu \in \Omega$. The resulting process will be denoted $(\xi_t^\nu)_{t\geq 0}$.

Notation 1.2 (Ω and $\mathcal{P}(\mathbb{Z})$). Because of the natural bijection between the power set of $\mathbb{Z}$ and Ω, we will consider configurations as both subsets of $\mathbb{Z}$ and elements of Ω, regularly switching between the two interpretations.

1.3 Time change

In what follows we only consider contact processes with $\frac{p}{q} > \lambda_c$ where λ_c is the critical parameter for the 1-sided contact process on $\mathbb{Z}$. A 1-sided contact process with rates satisfying this condition is called supercritical. The extinction time $\tau(\eta_\cdot^{\{0\}}) := \inf\{t \geq 0 \mid \eta_t^{\{0\}} = \varnothing\}$ of a supercritical 1-sided contact process satisfies $\mathbb{P}\left(\tau(\eta_\cdot^{\{0\}}) = \infty\right) > 0$ i.e. the process survives forever with positive probability.

Notation 1.3 (Supercritical East-process). As per the previous discussion, we call an East-process supercritical if $\frac{p}{q} > \lambda_c$.

1.4 Domination and other properties

The basic coupling has two important properties that follow immediately from its definition. First, it lets us construct both processes started from any initial configuration on the same probability space. The second property is domination: if at some time $t \geq 0$ $\eta_t \leq \sigma_t$ then $\eta_{t+s} \leq \sigma_{t+s},\ \forall s \geq 0$. To see this note that under the graphical construction η updates a particular site to 1 only if σ does too, and σ updates a particular site to 0 only if η does too. In particular, if $X(\cdot)$ denotes the position of the front then $X(\eta_{t+s}) \leq X(\sigma_{t+s}),\ \forall s \geq 0$.

Domination is what enables us to bound the East-process from below by the 1-sided contact process. The reason we might want to do this is that contact processes posess desirable qualities that KCMs in general might not. Contact processes are *attractive* in the sense that if $\nu \subseteq \xi \subseteq \mathbb{Z}$ then $\eta_t^\nu \leq \eta_t^\xi$ for all $t \geq 0$ under the basic coupling. Moreover they are also *additive*: if $\nu, \xi \subseteq \mathbb{Z}$ then $\eta_t^{\nu\cup\xi} = \eta_t^\nu \cup \eta_t^\xi$ for all $t \geq 0$ under the basic coupling. These qualities make contact processes more amenable to analysis than KCMs, and there is a breadth of methods and results already established. The East-process lacks both attractivity and additivity, thus the desire to compare it to the 'simpler' 1-sided contact process is justified.

2 Front propagation of the East-process

In this section we prove linear speed for the front of the East-process. The upper bound follows from classical results for Poisson point processes, while the lower bound is established by a comparison with the 1-sided contact process, closely following the arguments of [1].

First we formalize the notion of the front that we discussed Subsection 1.1.

Definition 2.1 — For $A \subseteq \mathbb{Z}$ the front of A is defined as $X(A) := \max(A) \in \mathbb{N} \cup \{-\infty, \infty\}$ with $\max(\varnothing) := -\infty$.

The following result will be used without proof:

Lemma 2.1 ([2, Section 4 Theorems 4 & 5]) — *If $(\eta_t)_{t\geq 0}$ is a supercritical, 1-sided contact process and τ is the extinction time as defined in Subsection 1.3, then $\exists\ \alpha > 0$ such that for all $a < \alpha$ there exist constants $\gamma, C > 0$ satisfying*

$$\mathbb{P}\left(X(\eta_t^{\{0\}}) < at \,\middle|\, \tau(\eta_\cdot^{\{0\}}) = \infty \right) \leq Ce^{-\gamma t} \qquad \forall t \geq 0. \tag{2.1}$$

Furthermore, if $|A| < \infty$ then there exist constants $\gamma, C > 0$ such that

$$\mathbb{P}\left(t < \tau(\eta_\cdot^{A}) < \infty \right) \leq Ce^{-\gamma t} \qquad \forall t \geq 0. \tag{2.2}$$

2.1 Restart argument

Theorem 2.2 (Coupling East and surviving contact processes) — *There exists a process $(\sigma_t, \eta_t)_{t\geq 0}$ taking values in Ω^2 and a random variable T taking values in $[0,\infty)$ such that*

(i) (σ_t) is a supercritical East-process started from $\{0\}$

(ii) $\forall t \geq 0$ and $\forall x \in \mathbb{Z}$, it holds that $\eta_t(x) \leq \sigma_t(x)$

(iii) $(\eta_{T+t})_{t\geq 0}$ is a surviving 1-sided contact process started from $\{0\}$

Furthermore T has exponentially decaying tail probabilities.

Proof. Let $\{\mathscr{P}^{(i)}\}_{i\in\mathbb{N}^+}$ be independent copies of $\mathscr{P}$. Denote by $\eta_\cdot^{(i)}$ the 1-sided contact process started from $\{0\}$, constructed using $\mathscr{P}^{(i)}$. Furthermore let $U_i := \tau(\eta_\cdot^{(i)})$ be the extinction time of $\eta_\cdot^{(i)}$. Note that the U_i are i.i.d. and $\mu := \mathbb{P}(U_1 = \infty) > 0$ by Subsection 1.3. Define $L := \min\{i : U_i = \infty\}$ and note that L has geometric distribution. Finally, let

$$T := \begin{cases} 0 & \text{if } L = 1 \\ \sum_{i=1}^{L-1} U_i & \text{otherwise} \end{cases}.$$

First we show that T has exponentially decaying tail probabilities. Note that since $T \geq 0$ almost surely, this is equivalent to finiteness of $\mathbb{E}\left[e^{sT}\right]$ for some $s > 0$. To see the latter holds for T observe that conditional on L the random variables $U_1, ..., U_{L-1}$ are i.i.d. with distribution equal to that of U_1 given $U_1 < \infty$. From (2.1) it follows that $U_1|\{U_1 < \infty\}$ has exponentially decaying tail probabilities:

$$\mathbb{P}\left(U_1 > t \,|U_1 < \infty\right) = \frac{\mathbb{P}\left(t < U_1 < \infty\right)}{\mathbb{P}\left(U_1 < \infty\right)} \leq \frac{Ce^{-\gamma t}}{1-\mu}$$

Therefore there exists $s > 0$ such that $\mathbb{E}\left[e^{sU_1}\middle| U_1 < \infty\right] < \infty$, and so

$$\begin{aligned}\mathbb{E}\left[e^{sT}\right] &= \mathbb{E}\left[\mathbb{E}\left[\exp\left(s\sum_{i=1}^{L-1} U_i\right)\middle| L\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[e^{sU_1}\middle| U_1 < \infty\right]^{L-1}\right] < \infty\,,\end{aligned}$$

where finiteseness follows as L has geometric distribution and so finite moment generating function for all $s \in \mathbb{R}$.
Now we construct the process $(\sigma_t, \eta_t)_{t\geq 0}$:

1. Let $(\sigma_t^{[1]}, \eta_t^{[1]})_{t\geq 0}$ be the basic coupling started from $(\{0\}, \{0\})$, constructed using $\mathscr{P}^{(1)}$.
2. Assuming $(\sigma_t^{[i]}, \eta_t^{[i]})_{t\geq 0}$ has been constructed, define $(\sigma_t^{[i+1]}, \eta_t^{[i+1]})_{t\geq 0}$ as :
   - If $T_i := \sum_{j=1}^{i} U_j = \infty$ then $(\sigma_t^{[i+1]}, \eta_t^{[i+1]})_{t\geq 0} := (\sigma_t^{[i]}, \eta_t^{[i]})_{t\geq 0}$
   - Else, set $(\sigma_t^{[i+1]}, \eta_t^{[i+1]})_{T_i > t\geq 0} := (\sigma_t^{[i]}, \eta_t^{[i]})_{T_i > t\geq 0}$ and let $(\sigma_t^{[i+1]}, \eta_t^{[i+1]})_{t\geq T_i}$ be the basic coupling started from $(\sigma_{T_i}^{[i]}, \{0\})$, constructed using $\mathscr{P}^{(i+1)}$.

Since L has a geometric distribution, $L < \infty$ a.s. and we may define $(\sigma_t, \eta_t)_{t\geq 0} := (\sigma_t^{[L]}, \eta_t^{[L]})_{t\geq 0}$. As the U_i are stopping times, $(\sigma_t)_{t\geq 0}$ is an East-process started from $\{0\}$. It also follows that $(\eta_{T+t})_{t\geq 0}$ is a surviving 1-sided contact process started from $\{0\}$. Noting that an East-process started from $\{0\}$ alwas has a 1 at the origin, it follows that $\eta_t \leq \sigma_t\ \forall t \geq 0$. □

2.2 Linear lower bound on propagation

Corollary 2.2.1 — *Let $(\sigma_t)_{t\geq 0}$ be a supercritical East-process and $X(\sigma_t)$ be its front. Then $\exists\ \alpha > 0$ such that $\forall\ a < \alpha$ there exist constants $\gamma, C > 0$ satisfying*

$$\mathbb{P}\left(X(\sigma_t) < at\right) \leq Ce^{-\gamma t} \qquad \forall t \geq 0. \tag{2.3}$$

Remark 2.1. In the following proof the values of the constants γ & C change from line to line, without explicit mention.

Proof. Let $(\sigma_t)_{t\geq 0}$, $(\eta_t)_{t\geq 0}$ and T be as in Theorem 2.2. Since $\eta_{T+\cdot}$ survives, by Lemma 2.1 part (i) $\exists\ \alpha > 0$ such that for all $a < \alpha$ there exist $\gamma, C > 0$ satisfying

$$\mathbb{P}\left(X(\eta_{T+t}) < at\right) \leq Ce^{-\gamma t} \qquad \forall t \geq 0. \tag{2.4}$$

By Theorem 2.2 part (ii) we know that $\eta_t \leq \sigma_t$ for all $t \geq 0$, so that for $c \in (0,1)$ we have

$$\begin{aligned}\mathbb{P}\left(X(\sigma_t) < at,\ 0 \leq T \leq ct\right) &\leq \mathbb{P}\left(X(\eta_t) < at,\ (1-c)t \leq t - T \leq t\right) \\ &= \mathbb{P}\left(X(\eta_{T+(t-T)}) < at,\ t \leq \frac{t-T}{1-c} \leq \frac{t}{1-c}\right) \\ &\leq \mathbb{P}\left(X(\eta_{T+(t-T)}) < \frac{a(t-T)}{1-c},\ t \leq \frac{t-T}{1-c} \leq \frac{t}{1-c}\right) \\ &\leq \sup_{u\in[(1-c)t,t]} \mathbb{P}\left(X(\eta_{T+u}) < \frac{au}{1-c}\right).\end{aligned}$$

For c small enough such that $\frac{a}{1-c} < \alpha$ it follows by (2.4) that

$$\sup_{u\in[(1-c)t,t]} \mathbb{P}\left(X(\eta_{T+u}) < \frac{au}{1-c}\right) \le \sup_{u\in[(1-c)t,t]} Ce^{-\gamma u} = Ce^{-\gamma(1-c)t}$$
$$= Ce^{-\gamma t}.$$

To get the conclusion observe that

$$\mathbb{P}\left(X(\sigma_t) < at\right) \le \mathbb{P}\left(X(\sigma_t) < at,\ 0 \le T \le ct\right) + \mathbb{P}\left(T > ct\right)$$
$$\le C_1 e^{-\gamma_1 t} + C_2 e^{-\gamma_2 t} \le Ce^{-\gamma t}.$$

□

2.3 Finite speed of propagation

Definition 2.2 (Hitting times) — For the East-process $(\sigma_t^{\{0\}})_{t\ge 0}$ define $\rho_l := \min\{t \ge 0 \mid \sigma_t^{\{0\}}(l) = 1\}$ for each $l \in \mathbb{N}$ to be the first time that the front reaches site l.

Note that $0 = \rho_0 \le \rho_1 \le \rho_2 \le \rho_l < \infty$ for all $l \in \mathbb{N}$ with finiteness following from at least linear propagation of the front since $\mathbb{P}\left(\rho_l = \infty\right) = \lim_{t\to\infty} \mathbb{P}\left(\rho_l > t\right) \le \lim_{t\to\infty} \mathbb{P}\left(X\left(\sigma_t\right) < l\right) = 0$.

We wish to bound the quantity $\mathbb{P}\left(\rho_l \le t\right)$ to control the speed of the front since $\{X\left(\sigma_t\right) \ge l\} \subset \{\rho_l \le t\}$. To this end, consider the stopping times τ_x for each site $x \in \mathbb{N}$ defined as $\tau_{x+1} := \min\{T_{x+1,k} \mid T_{x+1,k} > \tau_x \text{ and } k \in \mathbb{N}^+\}$ with $\tau_0 = 0$. Consider the process $M_t := |\{x \ge 1 \mid \tau_x \le t\}|$ i.e. the process counting the number of τ_x that have occured up to time t. By repeated application of the strong Markov property we get that the process $(M_t)_{t\ge 0}$ is in fact a Poisson process of rate 1.

For all $t \ge 0$ and $l \in \mathbb{N}$ define $F(l,t)$ to be the event that there is an increasing sequence of rings starting at site 1 and ending at site l in the time interval $[0,t]$. It is clear that $\mathbb{P}\left(\rho_l \le t\right) \le \mathbb{P}\left(F(l,t)\right)$ and by the definition of $(M_t)_{t\ge 0}$ it follows that $\mathbb{P}\left(F(l,t)\right) = \mathbb{P}\left(M_t \ge l\right)$. The following standard result gives us the desired upper bound.

Lemma 2.3 — *Let $X \sim Po\left(\lambda\right)$ be a Poisson random variable with mean $\lambda > 0$. Then*

$$\mathbb{P}\left(X \ge x\right) \le \frac{e^{-\lambda}(e\lambda)^x}{x^x} \qquad \forall x > \lambda. \tag{2.5}$$

Proof. The result follows by a Chernoff bound argument. For all $t > 0$ we have

$$\mathbb{P}\left(X \ge x\right) \le \frac{\mathbb{E}\left[e^{tX}\right]}{e^{tx}} = \exp\left(\lambda e^t - \lambda - tx\right).$$

The minimum occurs at $t = \log\left(\frac{x}{\lambda}\right)$, giving the result. □

We are now ready to prove the linear upper bound on the speed of the East-process.

Theorem 2.4 — *Let $(\sigma_t^{\{0\}})_{t\ge 0}$ be an East-process and $X(\sigma_t^{\{0\}})$ be its front. Then there exist constant $v, \gamma > 0$ such that*

$$\mathbb{P}\left(X(\sigma_t^{\{0\}}) > vt\right) \le e^{-\gamma t} \qquad \forall t \ge 0. \tag{2.6}$$

Remark 2.2. In the following proof we omit the use of the floor function $\lfloor\cdot\rfloor$ for clarity of notation, treating non-integer values as integers. It is however clear that the proof could be adapted to be precise.

Proof. Let $v > 1$. Using Lemma 2.3 and the discussion before it the calculation of an upper bound becomes straightforward:

$$\begin{aligned}\mathbb{P}\left(X(\sigma_t) > vt\right) &\leq \mathbb{P}\left(\rho_{vt} \leq t\right) \leq \mathbb{P}\left(F(vt,t)\right) = \mathbb{P}\left(M_t \geq vt\right) \\ &\leq \frac{e^{-t}(et)^{vt}}{(vt)^{vt}} = \exp\left(-t + vt(\log\left({}^{t}/_{vt}\right) + 1)\right) \\ &= \exp\left(-t + vt(1 - \log(v))\right) \leq \exp(vt(1 - \log(v)))\end{aligned}$$

To conclude take $v > e$ and $\gamma = -v(1 - \log(v))$. $\square$

3 Mixing of the East-process

In this section we prove that the mixing time of the supercritical East-process on $[0, L]$ with a 1 fixed at the origin is $\Theta(L)$

In this section we will study the evolution of the East-process restricted to the segment [0, L] where $L \in \mathbb{N}$. In doing so we will assume that there is a 1 fixed at the origin. However, because of the local constraint of the East-process, when studying the East-process restricted to $\{0, 1, ..., L\}$ it doesn't matter whether we 1) fix a 1 at the origin or 2) only consider East-processes started from $\{0\} \subseteq \xi \subseteq \mathbb{N}$; the results of the analysis will be the same. Motivated by this we make two definitions:

Definition 3.1 — Define $\widetilde{\Omega} := \{A \subseteq \mathbb{N} \mid 0 \in A\}$ to be the set of configurations that are 0 on the negatives and 1 at the origin. Similarly, for $L \in \mathbb{N}$ define $\Omega_L := \{A \cap [0, L] \mid A \in \widetilde{\Omega}\}$ to be the state space of the East-process on $\{0, 1, ..., L\}$ with a fixed 1 at the origin.

Another important property of the East-process is that the evolution at some site $x \in \mathbb{N}$ is not influenced by how the process evolves at sites to the right of x. More precisely, if $(\sigma_t)_{t \geq 0}$ is an East-process then for any $x \in \mathbb{N}$ and $t \geq 0$, $\sigma_t(x)$ is independent of the sigma algebra $\sigma\left((E_{n,k}, B_{n,k})_{n>x,k>0}\right)$. Furthermore, if we fix a 1 at the origin (or equivalently start from a configuration in $\widetilde{\Omega}$) then an even stronger independence holds in that the evolution to the right of the origin is independent of all the clock rings and coin tosses left of the origin. Therefore the East-process with a fixed 1 at the origin is a continuous time Markov chain when restricted to $\{0, 1, ..., L\}$.

3.1 Coupling time for basic coupling

As before, let $(\sigma_t^\xi)_{t \geq 0}$ denote the East-process on $\mathbb{Z}$ started from initial configuration $\xi \in \Omega$, constructed using $\mathscr{P} = (E_{x,k}, B_{x,k})_{x \in \mathbb{Z},\ k \in \mathbb{N}^+}$. Recall the definition of the hitting times $(\rho_i)_{i \in \mathbb{N}}$ from Definition 2.2.

Proposition 3.1 — *For each $l \in \mathbb{N}$ and for all $\xi, \nu \in \widetilde{\Omega}$ it holds that*

$$\sigma^\xi_{\rho_l + t} \cap [0, l] = \sigma^\nu_{\rho_l + t} \cap [0, l] \qquad \forall t \geq 0. \tag{3.1}$$

Proof. We only prove (3.1) for $\nu = \{0\}$ from which the result follows easily for arbitrary ν. We proceed by induction. The claim clearly holds for $l = 0$ since every such ξ has a 1 at the origin forever. For the induction step suppose that the claim holds up to $l = n \geq 0$. If K is such that $T_{n+1,K} = \rho_{n+1}$ then $B_{n+1,K} = 1$, and since the ring is legal, $\sigma^{\{0\}}_{\rho_{n+1}}(n) = 1$. By the induction hypothesis also $\sigma^\xi_{\rho_{n+1}}(n) = 1$ i.e. the ring is also legal for $\sigma^\xi_\cdot$. Thus both processes update to 1 at time ρ_{n+1}. Now, since the ρ_i are stopping times, $(\sigma^{\{0\}}_{\rho_{n+1}+t})_{t \geq 0}$ and $(\sigma^\xi_{\rho_{n+1}+t})_{t \geq 0}$ are two East-processes with $\sigma^{\{0\}}_{\rho_{n+1}} \cap [0, n+1] = \sigma^\xi_{\rho_{n+1}} \cap [0, n+1]$. The conclusion follows by basic coupling. $\square$

3.2 Results for continuous Markov chains

We present some standard results for continuous time Markov chains that we will use throughout this section.

Definition 3.2 (Total variation distance) — Let $(\Omega, \mathcal{F})$ be a measurable space and ν, μ be two probability measures on it. The total variation distance of ν and μ is defined as $\|\nu(\cdot) - \mu(\cdot)\|_{TV} := \sup\limits_{A \in \mathcal{F}} |\nu(A) - \mu(A)|$

Definition 3.3 (Distance from equilibrium) — Let $N := (N_t)_{t\geq 0}$ be a continuous time, irreducible Markov chain taking values in a finite state space Ω and let π be its equilibrium distribution. For $t \geq 0$ define $d(t) := \max_{x\in\Omega} \|\mathbb{P}(N_t \in \cdot \mid N_0 = x) - \pi(\cdot)\|_{TV}$ to be the worst case total variation distance from equilibrium.

Theorem 3.2 — *Let $N := (N_t)_{t\geq 0}$ be a continuous time, irreducible Markov chain taking values in a finite state space Ω and let π be its equilibrium distribution. Suppose that for each pair of states $x, y \in \Omega$ there is a coupling $(X_t, Y_t)_{t\geq 0}$ of N that is started from $(x, y) \in \Omega^2$. For each of these couplings, let $\tau^{x,y}_{couple} := \min\{t \geq 0 \mid X_t = Y_t\}$ be the first time the chains meet. Then it holds that*

$$d(t) \leq \max_{x,y\in\Omega} \mathbb{P}\left(\tau^{x,y}_{couple} > t\right) \tag{3.2}$$

Proof. The result can be found in [3, Corollary 5.3] for discrete time Markov chains, however all the necessary proofs work in the continuous time case without modification. □

3.3 Linear upper bound on mixing

When restricted to $\{0, 1, ...L\}$, the East-process with a fixed 1 at the origin is a finite, irreducible, continuous time Markov chain with equilibrium measure $\pi_L := \delta_1 \times \mathrm{Ber}(p)^L$. Hence we may make the following definition:

Definition 3.4 (Mixing time of East-process) — The mixing time of the East-process with a fixed 1 at the origin is defined as $T^L_{mix} := \inf\{t \geq 0 \mid d(t) \leq 1/4\}$.

Theorem 3.3 — *For each supercritical East-process there exists a constant $K > 0$ such that $T^L_{mix} \leq KL$ for all $L \in \mathbb{N}$.*

Proof. By Theorem 3.2 it holds that

$$d(t) \leq \max_{\nu,\xi\in\tilde{\Omega}} \mathbb{P}\left(\tau^{\nu,\xi}_{couple} > t\right) \tag{3.3}$$

Where $\tau^{\nu,\xi}_{couple}$ is the first time that the East-process started from ξ and ν (and thus with a fixed 1 at the origin) coincide on $\{0, 1...L\}$ under the basic coupling. Lemma 3.1 gives $\max_{\nu,\xi\in\tilde{\Omega}} \mathbb{P}\left(\tau^{\nu,\xi}_{couple} > t\right) \leq \mathbb{P}(\rho_L > t)$. Borrowing notation from Corollary 2.2.1 we get

$$\begin{aligned}\mathbb{P}(\rho_L > KL) &\leq \mathbb{P}\left(X\left(\sigma^{\{0\}}_{KL}\right) < L\right)\\ &= \mathbb{P}\left(X\left(\sigma^{\{0\}}_{KL}\right) < \frac{KL}{K}\right)\end{aligned}$$

Fix K large enough such that $1/K < \alpha$ to get

$$\mathbb{P}\left(X\left(\sigma^{\{0\}}_{KL}\right) < \frac{KL}{K}\right) \leq Ce^{-\gamma KL}.$$

Thus there exists $L' \in \mathbb{N}$ such that for all $L \geq L'$ it holds that $\mathbb{P}\left(X\left(\sigma^{\{0\}}_{KL}\right) < L\right) \leq 1/4$. This implies that $d(KL) \leq 1/4$ in other words that $T^L_{mix} \leq KL$ for all $L \geq L'$. □

3.4 Linear lower bound on mixing

Theorem 3.4 — *For each supercritical East-process there exists a constant $K' > 0$ such that $K'L \le T_{mix}^L$ for all $L \in \mathbb{N}$.*

Proof. From the proof of Theorem 2.4 we know that $\exists\, \gamma, v > 0$ such that $\mathbb{P}\left(\rho_{tv} \le t\right) \le e^{-\gamma t}$ for all $t \ge 0$. Thus we get

$$\mathbb{P}\left(\rho_{L/2} \le L/2v\right) \le e^{-\gamma \frac{L}{2v}} \qquad \forall L \in \mathbb{N} \tag{3.4}$$

Define the set $A_L := \{\omega \in \Omega \mid \{L/2, L/2+1, ..., L\} \cap \omega \neq \varnothing\}$ to be the set of configurations with at least one occupied site between $L/2$ and L. For our proof it is crucial to note that $\mathbb{P}\left(\sigma_t^{\{0\}} \in A_L \middle| \rho_{L/2} > t\right) = 0$: since $(\sigma_t^{\{0\}})_{t\ge 0}$ is started from $\{0\}$, it is infected at some site $x \in \mathbb{N}^+$ only if site x has been hit by the front i.e. it can only happen on $\{\rho_x \le t\}$. Let $\pi_L := \delta_1 \times \mathrm{Ber}\,(p)^L$ be the product Bernoulli measure on Ω_L. We have

$$\begin{aligned} d\left(\frac{L}{2v}\right) &= \max_{\xi \in \widetilde{\Omega}} \left\| \mathbb{P}\left(\sigma^{\xi}_{L/2v} \cap [0,L] \in \cdot\right) - \pi_L(\cdot) \right\|_{TV} \\ &= \max_{\xi \in \widetilde{\Omega}} \max_{A \in \Omega_L} \left| \mathbb{P}\left(\sigma^{\xi}_{L/2v} \cap [0,L] \in A\right) - \pi_L(A) \right| \\ &\ge \left| \mathbb{P}\left(\sigma^{\{0\}}_{L/2v} \in A_L\right) - \pi_L(A_L) \right| \\ &= \left| \mathbb{P}\left(\rho_{L/2} \le L/2v\right) \mathbb{P}\left(\sigma^{\{0\}}_{L/2v} \in A_L \middle| \rho_{L/2} \le L/2v\right) - \left(1 - q^{L/2}\right) \right| \end{aligned}$$

Applying the bound (3.4) gives

$$d\left(\frac{L}{2v}\right) \ge \min\left\{ \left| 1 - q^{L/2} - e^{-\gamma \frac{L}{2v}} \right|, 1 - q^L \right\} \xrightarrow{L \to \infty} 1.$$

By the above, there exists $L' \in \mathbb{N}$ such that for all $L \ge L'$ it holds that $d\left(\frac{L}{2v}\right) > 1/4$, or in other words $\frac{L}{2v} < T_{mix}^L$ for all $L \ge L'$. $\square$

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