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UNIVERSITY OF OXFORD

DEPARTMENT OF STATISTICS

The oriented contact process and the East model

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Overview

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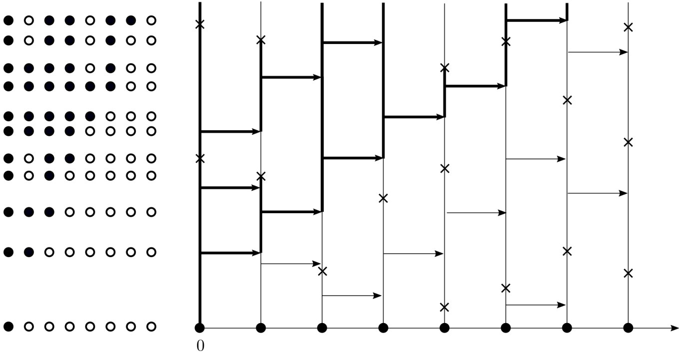

Figure: East process started from $\{0\}$

Front propagation

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Definition (Front)

Let $(\sigma_t)_{t\geq 0}$ be an East process started from $\{0\}$. The front of $(\sigma_t)_{t\geq 0}$ is the rightmost occupied site, written $X(\sigma_t) := \sup \sigma_t$.

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Theorem (Blondel, 2013, Lemma 3.2)

There exist constants $0 < \alpha < v$ *and* $\gamma, C > 0$ *such that*

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Proof.

We propose an alternative proof (under restrictions on p) detailed later. □

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Alternative proof using only front propagation.

Upper bound: Coupling time vs. propagation of the front.
Lower bound: Hitting times of large sets (Peres and Sousi, 2015). □

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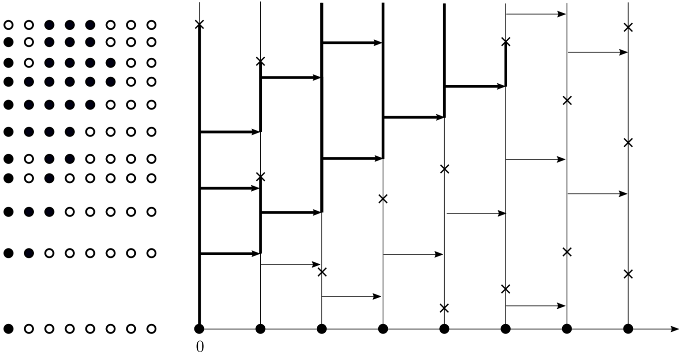

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For this to be useful, we need $p > p_c$ where
$p_c := \sup\{p : |\eta| \to 0\ a.s.\}$. From here on consider only such p.

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Remark

Hinted at in Durrett and Griffeath, 1983 but never proved in detail.

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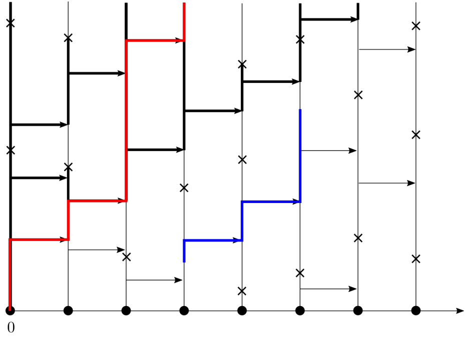

Figure: Red and blue lines are examples of a path.

Percolation construction 2

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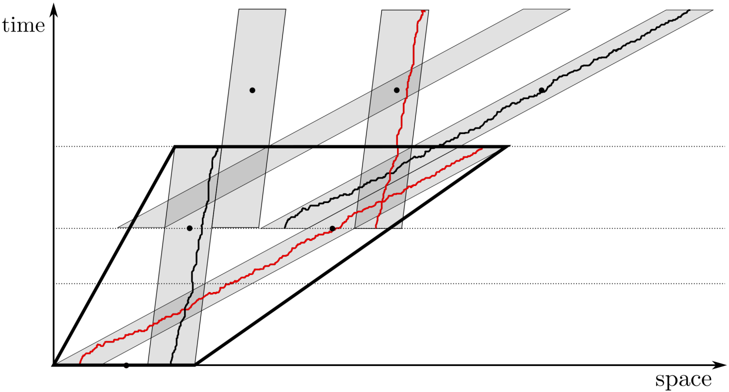

Figure: Red path showing how oriented percolation to infinity occurs.

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