1 Basic Coupling

In this section we construct the East-process and the 1-sided contact process on the same probability space using the graphical method.

1.1 Construction

Let $\mathcal{C} = (E_{x,k}, B_{x,k})_{x\in\mathbb{Z},k\in\mathbb{N}^+}$ be a collection of independent random variables with $E_{x,k} \sim \text{Exp}(1)$ and $B_{x,k} \sim \text{Ber}(p = 1-q)$. We construct the East-process $\sigma := (\sigma_t)_{t\geq 0}$ and the 1-sided contact process $\eta := (\eta_t)_{t\geq 0}$ using $\mathcal{C}$ as follows:

For each site $x \in \mathbb{Z}$ at each time $T_{x,n} := \sum_{k=1}^{n} E_{x,k}$:

- If $B_{x,n} = 1$:
  1. If $\sigma_{T_{x,n}}(x-1) = 1$ update σ to 1 at site x
  2. If $\eta_{T_{x,n}}(x-1) = 1$ update η to 1 at site x
- Else:
  1. If $\sigma_{T_{x,n}}(x-1) = 1$ update σ to 0 at site x
  2. Update η to 0 at site x

Remark 1.1 (Time change). In what follows we only consider contact processes with $\frac{p}{q} > \lambda_c$ where λ_c is the critical parameter for the 1-sided contact process on $\mathbb{Z}$. If $\tau^A := \tau(\eta^A_\cdot) := \inf\{t \geq 0 \mid \eta^A_t = \varnothing\}$, then this implies that $\mathbb{P}\left(\tau^{\{0\}} = \infty\right) > 0$. If p satisfies this constraint, we call the correspoding East and contact processes supercritical. (I'm not sure about this remark - might need to use higher value to get access to results from [2])

Suppose we start the processes σ and η from the initial configurations $A, B \in \Omega := \{0,1\}^{\mathbb{Z}}$ respectively. The resulting processes are denoted $(\sigma^A_t)_{t\geq 0}$ and $(\eta^B_t)_{t\geq 0}$. Note that because of the natural bijection between the power set of $\mathbb{Z}$ and Ω, we will consider the values σ_t and η_t as subsets of $\mathbb{Z}$ and elements of Ω interchangeably.

1.2 Monotonicity

Suppose at some time $t \geq 0$ $\eta_t \leq \sigma_t$. Then $\eta_{t+s} \leq \sigma_{t+s}$ $\forall s \geq 0$. To see this note that η updates a particular site to 1 only if σ does too, and σ updates a particular site to 0 only if η does too. In particular, if $X(\xi)$ denotes the position of the rightmost one - also known as the front - of $\xi \in \Omega$ then $X(\eta_{t+s}) \leq X(\sigma_{t+s})$ $\forall s \geq 0$.

2 Front propagation of the East-process

In this section we prove at least linear speed for the front of the East-process by comparing it to the 1-sided contact process, closely following the arguments of [1].

We use the following result without proof:

Lemma 2.1 ([2] Section 4 Theorems 4 & 5) — *If $(\eta_t)_{t\geq 0}$ is a supercritical, 1-sided contact process and τ is as in Remark 1.1 then $\exists\ \alpha > 0$ such that $\forall a < \alpha\ \exists\ \gamma, C > 0$ satisfying*

$$\mathbb{P}\left(X(\eta^{\{0\}}_t) < at \,\middle|\, \tau^{\{0\}} = \infty\right) \leq Ce^{-\gamma t} \qquad \forall t \geq 0 \tag{2.1}$$

Furthermore, if $A \subset \mathbb{Z}$ with $|A| < \infty$ then $\exists\, \gamma, C > 0$ such that

$$\mathbb{P}\left(t < \tau(\eta_\cdot^A) < \infty\right) \leq Ce^{-\gamma t} \qquad \forall t \geq 0 \tag{2.2}$$

2.1 Restart argument

Theorem 2.2 (Coupling East and surviving contact processes) — *There exists a process $(\sigma_t, \eta_t)_{t\geq 0}$ taking values in Ω^2 and a random variable T taking values in $[0, \infty)$ such that*

(i) (σ_t) *is an East-process started from* $\{0\}$

(ii) $\forall t \geq 0$ & $\forall x \in \mathbb{Z}$, $\eta_t(x) \leq \sigma_t(x)$

(iii) $(\eta_{T+t})_{t\geq 0}$ *is a surviving 1-sided contact process started from* $\{0\}$

Furthermore T has exponentially decaying tail probabilities.

Proof. Let $\{\mathcal{C}^{(i)}\}_{i\in\mathbb{N}^+}$ be independent copies of $\mathcal{C}$. Denote by $\eta_\cdot^{(i)}$ the 1-sided contact process started from $\{0\}$, constructed using $\mathcal{C}^{(i)}$. Furthermore let $U_i = \tau(\eta_\cdot^{(i)})$ be the extinction time of $\eta_\cdot^{(i)}$.
The U_i are iid, and $\mu := \mathbb{P}(U_1 = \infty) > 0$ by Remark 1.1. Define $L = \min\{i : U_i = \infty\}$ and note that L has geometric distribution. Finally, let $T = \sum_{i=1}^{L-1} U_i$ with $T = 0$ if $L = 1$.
First we show that T has exponentially decaying tail probabilities. Note that this is equivalent to both 1) finiteness of $\mathbb{E}\left[e^{sT}\right]$ for all s in a neighbourhood of zero 2) finiteness of $\mathbb{E}\left[e^{s_i T}\right]$ $i = 1, 2$ for some $s_1 > 0$ and $s_2 < 0$. To see the latter holds for T observe that conditional on L, $U_1, ..., U_{L-1}$ are iid with distribution equal to that of U_1 given $U_1 < \infty$. From 2.1 it follows that $U_1|U_1 < \infty$ has exponentially decaying tail probabilities:

$$\mathbb{P}(U_1 > t | U_1 < \infty) = \frac{\mathbb{P}(t < U_1 < \infty)}{\mathbb{P}(U_1 < \infty)} \leq \frac{Ce^{-\gamma t}}{1 - \mu}$$

Thus for $s > 0$ small enough such that $\mathbb{E}\left[e^{sU_1} \,\middle|\, U_1 < \infty\right] < \infty$, we have

$$\begin{aligned}\mathbb{E}\left[e^{sT}\right] &= \mathbb{E}\left[\mathbb{E}\left[\exp\left(s\sum_{i=1}^{L-1} U_i\right) \middle| L\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[e^{sU_1} \,\middle|\, U_1 < \infty\right]^{L-1}\right] < \infty\end{aligned}$$

Where finiteseness follows as L has geometric distribution. Since $T \geq 0$ the moment generating function is finite for all $s < 0$.
Now we construct the process $(\sigma_t, \eta_t)_{t\geq 0}$:

1. Let $(\sigma_t^{[1]}, \eta_t^{[1]})_{t\geq 0}$ be the basic coupling started from $(\{0\}, \{0\})$, constructed using $\mathcal{C}^{(1)}$.

2. Assuming $(\sigma_t^{[i]}, \eta_t^{[i]})_{t\geq 0}$ has been constructed, define $(\sigma_t^{[i+1]}, \eta_t^{[i+1]})_{t\geq 0}$ as :

   - If $T_i := \sum_{j=1}^{i} U_j = \infty$ then $(\sigma_t^{[i+1]}, \eta_t^{[i+1]})_{t\geq 0} := (\sigma_t^{[i]}, \eta_t^{[i]})_{t\geq 0}$
   - Else, set $(\sigma_t^{[i+1]}, \eta_t^{[i+1]})_{T_i > t\geq 0} := (\sigma_t^{[i]}, \eta_t^{[i]})_{T_i > t\geq 0}$ and let $(\sigma_t^{[i+1]}, \eta_t^{[i+1]})_{t\geq T_i}$ be the basic coupling started from $(\sigma_{T_i}^{[i]}, \{0\})$, constructed using $\mathcal{C}^{(i+1)}$.

Since L has a geometric distribution, $L < \infty$ a.s. and we may define $(\sigma_t, \eta_t)t \geq 0 := (\sigma_t^{[L]}, \eta_t^{[L]})_{t\geq 0}$. As the U_i are stopping times, $(\sigma_t)_{t\geq 0}$ is an East-process started from $\{0\}$. It also follows that $(\eta_{T+t})_{t\geq 0}$ is a surviving 1-sided contact process started from $\{0\}$. Noting that an East-process started from $\{0\}$ alwas has a 1 at the origin, it follows that $\eta_t \leq \sigma_t$ $\forall t \geq 0$. $\square$

2.2 Linear lower bound on front propagation

Corollary 2.2.1 — *Let $(\sigma_t)_{t\geq 0}$ be a supercritical East-process. Then $\exists\ \alpha > 0$ such that $\forall\ a < \alpha\ \exists\ \gamma, C > 0$ satisfying*

$$\mathbb{P}\left(X(\sigma_t) < at\right) \leq Ce^{-\gamma t} \qquad\qquad \forall t \geq 0 \tag{2.3}$$

Remark 2.1. In the following proof the values of the constants γ & C change from line to line, without explicit mention.

Proof. Let $(\sigma_t)_{t\geq 0}$, $(\eta_t)_{t\geq 0}$ and T be as in Theorem 2.2. Since $\eta_{T+\cdot}$ survives, by Lemma 2.1 (i) $\exists\ \alpha > 0$ such that $\forall\ a < \alpha\ \exists\ \gamma, C > 0$ satisfying

$$\mathbb{P}\left(X(\eta_{T+t}) < at\right) \leq Ce^{-\gamma t} \qquad\qquad \forall t \geq 0 \tag{2.4}$$

By Theorem 2.2 (ii) we know that $\eta_t \leq \sigma_t$ for all $t \geq 0$, so that for $c \in (0,1)$ we have

$$\begin{aligned}
\mathbb{P}\left(X(\sigma_t) < at,\ 0 \leq T \leq ct\right) &\leq \mathbb{P}\left(X(\eta_t) < at,\ (1-c)t \leq t - T \leq t\right) \\
&= \mathbb{P}\left(X(\eta_{T+(t-T)}) < at,\ t \leq \frac{t-T}{1-c} \leq \frac{t}{1-c}\right) \\
&\leq \mathbb{P}\left(X(\eta_{T+(t-T)}) < \frac{a(t-T)}{1-c},\ t \leq \frac{t-T}{1-c} \leq \frac{t}{1-c}\right) \\
&\leq \sup_{u\in[(1-c)t,t]} \mathbb{P}\left(X(\eta_{T+u}) < \frac{au}{1-c}\right)
\end{aligned}$$

For c small enough such that $\frac{a}{1-c} < \alpha$ it follows by (2.4) that

$$\begin{aligned}
&\leq \sup_{u\in[(1-c)t,t]} Ce^{-\gamma u} \\
&= Ce^{-\gamma(1-c)t} \\
&= Ce^{-\gamma t}
\end{aligned}$$

To get the conclusion observe that

$$\begin{aligned}
\mathbb{P}\left(X(\sigma_t) < at\right) &\leq \mathbb{P}\left(X(\sigma_t) < at,\ 0 \leq T \leq ct\right) + \mathbb{P}\left(T > ct\right) \\
&\leq C_1 e^{-\gamma_1 t} + C_2 e^{-\gamma_2 t} \leq Ce^{-\gamma t}
\end{aligned}$$

$\square$

3 Mixing of the East-process

In this section we prove that the mixing time of the supercritical East-process on $[0, L]$ with a 1 fixed at the origin is $\mathcal{O}(L)$

3.1 A property of basic coupling

As before, let $(\sigma_t^\xi)_{t\geq 0}$ denote the East-process on $\mathbb{Z}$ started from initial configuration $\xi \in \Omega$, constructed using $\mathcal{C} = (E_{x,k}, B_{x,k})_{x\in\mathbb{Z},\ k\in\mathbb{N}^+}$. For $l \in \mathbb{N}$, let $\rho_l := \min\{t \geq 0 \mid \sigma^{\{0\}}(l) = 1\}$ be the hitting time of l by the East-process started from $\{0\}$. Note that $0 = \rho_0 \leq \rho_1 \leq \rho_2 \leq ... < \infty$ with finiteness following from at least linear propagation of the front. Finally, let $A_l := \{0,\ 1,\ \dots\ l\}$.

Lemma 3.1 — *For each $l \in \mathbb{N}$, and $\{0\} \subseteq \xi \subseteq \mathbb{N}$, it holds that*

$$\sigma^{\{0\}}_{\rho_l+t} \cap A_l = \sigma^{\xi}_{\rho_l+t} \cap A_l \qquad\qquad \forall t \geq 0 \tag{3.1}$$

Proof. We proceed by induction. The claim clearly holds for $l = 0$ since every such ξ has a 1 at the origin forever. For the induction step suppose that the claim holds up to $l = n \geq 1$. Let K be such that $T_{n+1,K} := \sum_{i=1}^{K} E_{n+1,i} = \rho_{n+1}$. Then $B_{n+1,K} = 1$, and since the ring is legal, $\sigma^{\{0\}}_{\rho_{n+1}}(n) = 1$, furthermore by the induction hypothesis also $\sigma^{\xi}_{\rho_{n+1}}(n) = 1$ i.e. the ring is also legal for $\sigma^{\xi}_{\cdot}$. Thus both processes update to 1 at time ρ_{n+1}. Now, since the ρ_i are stopping times, $(\sigma^{\{0\}}_{\rho_{n+1}+t})_{t\geq 0}$ & $(\sigma^{\xi}_{\rho_{n+1}+t})_{t\geq 0}$ are two East-processes with $\sigma^{\{0\}}_{\rho_{n+1}} \cap A_{n+1} = \sigma^{\xi}_{\rho_{n+1}} \cap A_{n+1}$. The conclusion follows by basic coupling. □

3.2 Linear bound on mixing

Theorem 3.2 — *For each supercritical East-process with a 1 fixed at the origin, there exists a constant $K > 0$ such that for all $L \in \mathbb{N}$, the mixing time T^L_{mix} on $\{1, 2, ...L\}$ satisfies $T^L_{mix} \leq KL$.*

Proof. By [3] Corollary 5.3, for any coupling of the East-process it holds that

$$d(t) \leq \max_{\nu,\xi\in\Omega} \mathbb{P}_{\nu,\xi}(\tau_{couple} > t) \tag{3.2}$$

Lemma 3.1 gives $\max_{\nu,\xi\in\Omega} \mathbb{P}_{\nu,\xi}(\tau_{couple} > t) \leq \mathbb{P}(\rho_L > t)$. Borrowing notation from Corollary 2.2.1 we get

$$\begin{aligned}\mathbb{P}(\rho_L > KL) &\leq \mathbb{P}\left(X\left(\sigma^{\{0\}}_{KL}\right) < L\right) \\ &= \mathbb{P}\left(X\left(\sigma^{\{0\}}_{KL}\right) < \frac{KL}{K}\right)\end{aligned}$$

Fix K large enough such that $1/K < \alpha$ to get

$$\leq Ce^{-\gamma KL}$$

Thus for say $L \geq L'$, $\mathbb{P}\left(X\left(\sigma^{\{0\}}_{KL}\right) < L\right) \leq 1/4$. This implies $d(KL) \leq 1/4$ i.e. $T^L_{mix} \leq KL\ \forall L \geq L'$. Take $K' = \max\{K, T^1_{mix}, T^2_{mix}, ...T^{L'-1}_{mix}\}$. Then $T^L_{mix} \leq K'L$ holds for all $L \in \mathbb{N}$ as required. □

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