# 1 Introduction and construction

In this section we construct the East-process and the 1-sided contact process on the same probability space using the graphical method. This construction gives rise to an important monotonicity property that we use extensively in later sections.

## 1.1 Introduction

The East-process is an interacting particle system evolving with a Glauber like dynamics on the state space $\Omega = \{0,1\}^{\mathbb{Z}}$. It is part of a class of stochastic processes called kinetically constrained spin models (KCMs), with the East-process being the first of these to be studied rigorously. The process evolves as follows: at each site $x \in \mathbb{Z}$ the system tries to update the value of the spin at $x$ to 1 or 0 at rate $p \in (0,1)$ and $q := 1-p$ respectively. The update is accepted only if the local constraint is satisfied, which in the East-processes case is that the spin value at site $x-1$ must be equal to 1. Sometimes we will call elements of $\Omega$ *configurations* and say a site is *occupied* or *infected* if its spin value is equal to 1.

In the sections to follow we focus on two objects of interest related to the East-process. The first one is the speed of the so-called *front*. Consider an East-process started from the configuration equal to all 0 except at the origin. It is easy to see that the spins on $(\infty, 0]$ stay frozen for all time, and infection 'spreads' to the right. A natural question to ask then how fast this spreading of infection happens if it happens at all. The front at time $t$ of this process is defined as the rightmost infected site in the configuration at time $t$. We will show that for large enough $p$ the front propagates at exactly linear speed.

The second object of interest is the mixing time of the East process when restricted to $\{1,2,...,L\}$ for some $L \in \mathbb{N}^+$. We will study the mixing time for the East-process on $\{1,2,...,L\}$ with a 1 fixed at the origin, so that the evolution at site 1 is unconstrained and go on to show that for large enough $p$ the mixing time is $\Omega(L)$.

In our study of the speed of the front we will compare the East-process to a second stochastic process called the 1-sided contact process on $\mathbb{Z}$. The 1-sided contact process on $Z$ has the same state space $\Omega$ and evolves as follows: each site infects its neighbour to the right at rate $p$ and 'recovers' i.e. sets its own spin to 0 at rate $q$.

## 1.2 Constructing the basic coupling

Let $\mathcal{C} = (E_{x,k}, B_{x,k})_{x\in\mathbb{Z}, k\in\mathbb{N}^+}$ be a collection of independent random variables with $E_{x,k} \sim \text{Exp}(1)$ and $B_{x,k} \sim \text{Ber}(p = 1-q)$. Define the times $T_{x,n} := \sum_{k=1}^{n} E_{x,k}$ also referred to as *clock rings* and call a clock ring $T_{x,n}$ legal if the local constraint of the corresponding process is satisified at site $x$ and time $T_{x,n}$. We construct the East-process $\sigma := (\sigma_t)_{t\geq 0}$ and the 1-sided contact process $\eta := (\eta_t)_{t\geq 0}$ using $\mathcal{C}$ as follows:

For each site $x \in \mathbb{Z}$ at each time $T_{x,n}$:

- If $B_{x,n} = 1$:
    1. If $\sigma_{T_{x,n}^-}(x-1) = 1$ update $\sigma$ to 1 at site x
    2. If $\eta_{T_{x,n}^-}(x-1) = 1$ update $\eta$ to 1 at site x

- Else:

    1. If $\sigma_{T^-_{x,n}}(x-1) = 1$ update $\sigma$ to 0 at site x
    2. Update $\eta$ to 0 at site x

*Notation* 1.1 (Initial configurations). Suppose we start a stochastic process $(\xi_t)_{t\geq 0}$ with state space $\Omega$ from initial configuration $\nu \in \Omega$. The resulting process will be denoted $(\xi^\nu_t)_{t\geq 0}$.

*Notation* 1.2 ($\Omega$ and $\mathcal{P}(\mathbb{Z})$). Because of the natural bijection between the power set of $\mathbb{Z}$ and $\Omega$, we will consider configurations as both subsets of $\mathbb{Z}$ and elements of $\Omega$, switching between the two interpretations without explicit mention.

## 1.3 Time change

In what follows we only consider contact processes with $\frac{p}{q} > \lambda_c$ where $\lambda_c$ is the critical parameter for the 1-sided contact process on $\mathbb{Z}$. A 1-sided contact process with rates satisfying this condition is called supercritical. The extinction time $\tau(\eta^{\{0\}}_\cdot) := \inf\{t \geq 0 \mid \eta^{\{0\}}_t = \varnothing\}$ of a supercritical 1-sided contact process satisfies $\mathbb{P}\left(\tau(\eta^{\{0\}}_\cdot) = \infty\right) > 0$ i.e. the process survives forever with positive probability.

Definition 1.1 (Supercritical East-process) — As per the previous discussion, we call an East-process supercritical if $\frac{p}{q} > \lambda_c$.

## 1.4 Monotonicity of the basic coupling

The basic coupling has two important properties that follow immediately from its definition. First, it lets us construct both processes started from any initial configuration on the same probability space.

The second property is monotonicity: suppose at some time $t \geq 0$ $\eta_t \leq \sigma_t$. Then $\eta_{t+s} \leq \sigma_{t+s}$ $\forall s \geq 0$. To see this note that $\eta$ updates a particular site to 1 only if $\sigma$ does too, and $\sigma$ updates a particular site to 0 only if $\eta$ does too. In particular, if $X(\xi)$ denotes the position of the front of $\xi \in \Omega$ then $X(\eta_{t+s}) \leq X(\sigma_{t+s})$ $\forall s \geq 0$.

## 2 Front propagation of the East-process

In this section we prove linear speed for the front of the East-process. The upper bound follows from classical results for Poisson point processes, while the lower bound is established by a comparison with the 1-sided contact process, closely following the arguments of [1].

First we formalize the notion of the front that we discussed Section **??**.

Definition 2.1 — For $A \subseteq \mathbb{Z}$ the front of $A$ is defined as $X(A) := \max(A) \in \mathbb{N} \cup \{-\infty, \infty\}$ with $\max(\varnothing) := -\infty$.

The following result will be used without proof:

Lemma 2.1 ([2] Section 4 Theorems 4 & 5) — *If $(\eta_t)_{t\geq 0}$ is a supercritical, 1-sided contact process, $\tau$ is the extinction time as defined in Remark 1.3 and $X(\eta_t)$ denotes the front of $\eta_t$, then $\exists\ \alpha > 0$ such that $\forall a < \alpha\ \exists\ \gamma, C > 0$ satisfying*

$$\mathbb{P}\left( X(\eta_t^{\{0\}}) < at \middle| \tau(\eta_\cdot^{\{0\}}) = \infty \right) \leq Ce^{-\gamma t} \qquad \forall t \geq 0 \tag{2.1}$$

*Furthermore, if $A \subset \mathbb{Z}$ with $|A| < \infty$ then $\exists\ \gamma, C > 0$ such that*

$$\mathbb{P}\left(t < \tau(\eta_\cdot^A) < \infty\right) \leq Ce^{-\gamma t} \qquad \forall t \geq 0 \tag{2.2}$$

### 2.1 Restart argument

Theorem 2.2 (Coupling East and surviving contact processes) — *There exists a process $(\sigma_t, \eta_t)_{t\geq 0}$ taking values in $\Omega^2$ and a random variable $T$ taking values in $[0,\infty)$ such that*

*(i) $(\sigma_t)$ is a supercritical East-process started from $\{0\}$*

*(ii) $\forall t \geq 0$ & $\forall x \in \mathbb{Z},\ \eta_t(x) \leq \sigma_t(x)$*

*(iii) $(\eta_{T+t})_{t\geq 0}$ is a surviving 1-sided contact process started from $\{0\}$*

*Furthermore $T$ has exponentially decaying tail probabilities.*

*Proof.* Let $\{\mathcal{C}^{(i)}\}_{i\in\mathbb{N}^+}$ be independent copies of $\mathcal{C}$. Denote by $\eta_\cdot^{(i)}$ the 1-sided contact process started from $\{0\}$, constructed using $\mathcal{C}^{(i)}$. Furthermore let $U_i = \tau(\eta_\cdot^{(i)})$ be the extinction time of $\eta_\cdot^{(i)}$.
The $U_i$ are iid, and $\mu := \mathbb{P}(U_1 = \infty) > 0$ by Remark **??**. Define $L = \min\{i : U_i = \infty\}$ and note that $L$ has geometric distribution. Finally, let $T = \sum_{i=1}^{L-1} U_i$ with $T = 0$ if $L = 1$.
First we show that $T$ has exponentially decaying tail probabilities. Note that this is equivalent to finiteness of $\mathbb{E}\left[e^{s_i T}\right]$ $i = 1,2$ for some $s_1 > 0$ and $s_2 < 0$. To see the latter holds for $T$ observe that conditional on $L$, $U_1, ..., U_{L-1}$ are iid with distribution equal to that of $U_1$ given $U_1 < \infty$. From (2.1) it follows that $U_1|U_1 < \infty$ has exponentially decaying tail probabilities:

$$\mathbb{P}(U_1 > t|U_1 < \infty) = \frac{\mathbb{P}(t < U_1 < \infty)}{\mathbb{P}(U_1 < \infty)} \leq \frac{Ce^{-\gamma t}}{1-\mu}$$

Thus for $s > 0$ such that $\mathbb{E}\left[e^{sU_1}\middle| U_1 < \infty\right] < \infty$, we have

$$\begin{aligned}\mathbb{E}\left[e^{sT}\right] &= \mathbb{E}\left[\mathbb{E}\left[\exp\left(s\sum_{i=1}^{L-1} U_i\right)\middle| L\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[e^{sU_1}\middle| U_1 < \infty\right]^{L-1}\right] < \infty\end{aligned}$$

Where finiteseness follows as $L$ has geometric distribution and so finite moment generating function for all $s \in \mathbb{R}$. Since $T \geq 0$, its moment generating function is finite for all $s < 0$ thus T has exponentially decaying tail probabilities.
Now we construct the process $(\sigma_t, \eta_t)_{t\geq 0}$:

1. Let $(\sigma_t^{[1]}, \eta_t^{[1]})_{t\geq 0}$ be the basic coupling started from $(\{0\}, \{0\})$, constructed using $\mathcal{C}^{(1)}$.
2. Assuming $(\sigma_t^{[i]}, \eta_t^{[i]})_{t\geq 0}$ has been constructed, define $(\sigma_t^{[i+1]}, \eta_t^{[i+1]})_{t\geq 0}$ as :
   - If $T_i := \sum_{j=1}^{i} U_j = \infty$ then $(\sigma_t^{[i+1]}, \eta_t^{[i+1]})_{t\geq 0} := (\sigma_t^{[i]}, \eta_t^{[i]})_{t\geq 0}$
   - Else, set $(\sigma_t^{[i+1]}, \eta_t^{[i+1]})_{T_i > t\geq 0} := (\sigma_t^{[i]}, \eta_t^{[i]})_{T_i > t\geq 0}$ and let $(\sigma_t^{[i+1]}, \eta_t^{[i+1]})_{t\geq T_i}$ be the basic coupling started from $(\sigma_{T_i}^{[i]}, \{0\})$, constructed using $\mathcal{C}^{(i+1)}$.

Since $L$ has a geometric distribution, $L < \infty$ a.s. and we may define $(\sigma_t, \eta_t)t \geq 0 := (\sigma_t^{[L]}, \eta_t^{[L]})_{t\geq 0}$. As the $U_i$ are stopping times, $(\sigma_t)_{t\geq 0}$ is an East-process started from $\{0\}$. It also follows that $(\eta_{T+t})_{t\geq 0}$ is a surviving 1-sided contact process started from $\{0\}$. Noting that an East-process started from $\{0\}$ alwas has a 1 at the origin, it follows that $\eta_t \leq \sigma_t \ \forall t \geq 0$. $\square$

## 2.2 Linear lower bound on propagation

Corollary 2.2.1 — *Let $(\sigma_t)_{t\geq 0}$ be a supercritical East-process and $X(\sigma_t)$ be its front. Then $\exists\ \alpha > 0$ such that $\forall\ a < \alpha\ \exists\ \gamma, C > 0$ satisfying*

$$\mathbb{P}\left(X(\sigma_t) < at\right) \leq Ce^{-\gamma t} \qquad \forall t \geq 0 \tag{2.3}$$

*Remark* 2.1. In the following proof the values of the constants $\gamma$ & $C$ change from line to line, without explicit mention.

*Proof.* Let $(\sigma_t)_{t\geq 0}$, $(\eta_t)_{t\geq 0}$ and $T$ be as in Theorem 2.2. Since $\eta_{T+\cdot}$ survives, by Lemma 2.1 part (i) $\exists\ \alpha > 0$ such that $\forall\ a < \alpha\ \exists\ \gamma, C > 0$ satisfying

$$\mathbb{P}\left(X(\eta_{T+t}) < at\right) \leq Ce^{-\gamma t} \qquad \forall t \geq 0 \tag{2.4}$$

By Theorem 2.2 part (ii) we know that $\eta_t \leq \sigma_t$ for all $t \geq 0$, so that for $c \in (0, 1)$ we have

$$\begin{aligned}
\mathbb{P}\left(X(\sigma_t) < at,\ 0 \leq T \leq ct\right) &\leq \mathbb{P}\left(X(\eta_t) < at,\ (1-c)t \leq t - T \leq t\right) \\
&= \mathbb{P}\left(X(\eta_{T+(t-T)}) < at,\ t \leq \frac{t-T}{1-c} \leq \frac{t}{1-c}\right) \\
&\leq \mathbb{P}\left(X(\eta_{T+(t-T)}) < \frac{a(t-T)}{1-c},\ t \leq \frac{t-T}{1-c} \leq \frac{t}{1-c}\right) \\
&\leq \sup_{u\in[(1-c)t,t]} \mathbb{P}\left(X(\eta_{T+u}) < \frac{au}{1-c}\right)
\end{aligned}$$

For $c$ small enough such that $\frac{a}{1-c} < \alpha$ it follows by (2.4) that

$$\begin{aligned}
&\leq \sup_{u\in[(1-c)t,t]} Ce^{-\gamma u} \\
&= Ce^{-\gamma(1-c)t} \\
&= Ce^{-\gamma t}
\end{aligned}$$

To get the conclusion observe that

$$\begin{aligned}\mathbb{P}\left(X(\sigma_t) < at\right) &\le \mathbb{P}\left(X(\sigma_t) < at,\ 0 \le T \le ct\right) + \mathbb{P}\left(T > ct\right) \\ &\le C_1 e^{-\gamma_1 t} + C_2 e^{-\gamma_2 t} \le C e^{-\gamma t}\end{aligned}$$

□

## 2.3 Finite speed of propagation

Definition 2.2 (Hitting times) — Let $(\sigma_t)_{t\ge0}$ be the East-process started from $\{0\}$ and for $l \in \mathbb{N}$ define $\rho_l := \min\{t \ge 0 \mid \sigma_t(l) = 1\}$ to be the hitting time of site $l$.

Note that $0 = \rho_0 \le \rho_1 \le \rho_2 \le ... < \infty$ with finiteness following from at least linear propagation of the front since $\mathbb{P}\left(\rho_l = \infty\right) = \lim_{t\to\infty} \mathbb{P}\left(\rho_l > t\right) \le \lim_{t\to\infty} \mathbb{P}\left(X\left(\sigma_t\right) < l\right) = 0$.

We wish to bound the quantity $\mathbb{P}\left(\rho_l \le t\right)$ to control the speed of the front since $\{X\left(\sigma_t\right) \ge l\} \subset \{\rho_l \le t\}$. To this end, consider the stopping times $T_i$ for $i \in \mathbb{N}$ defined as $T_{i+1} := \min\{T_{i+1,k} \mid T_{i+1,k} > T_i \ \&\ k \in \mathbb{N}^+\}$ with $T_0 = 0$. Consider the process $M_t := |\{i \ge 1 \mid T_i \le t\}|$ i.e. the process counting the number of $T_i$ that have occured up to time t. By repeated application of the strong markov property we get that the process $(M_t)_{t\ge0}$ is in fact a Poisson process of rate 1.

Define the events $F(l,t) := \{\text{there is a succession of rings up to site } l \text{ in the time interval } [0,t]\}$ for $l \in \mathbb{N}^+$ and $t \ge 0$ where by succession of rings up to site $l$ we mean an increasing sequence of rings starting at site 1 and ending at site $l$. It is clear that $\mathbb{P}\left(\rho_l \le t\right) \le \mathbb{P}\left(F(l,t)\right)$ and by the definition of $(M_t)_{t\ge0}$ it follows that $\mathbb{P}\left(F(l,t)\right) = \mathbb{P}\left(M_t \ge l\right)$. The following standard result gives us the desired upper bound.

Lemma 2.3 — *Let $X \sim Po\left(\lambda\right)$ be a Poisson random variable with mean $\lambda > 0$. Then*

$$\mathbb{P}\left(X \ge x\right) \le \frac{e^{-\lambda}(e\lambda)^x}{x^x} \qquad \forall x > \lambda \tag{2.5}$$

*Proof.* The result follows by a Chernoff bound argument. For all $t > 0$ we have

$$\mathbb{P}\left(X \ge x\right) \le \frac{\mathbb{E}\left[e^{tX}\right]}{e^{tx}} = \exp\left(\lambda e^t - \lambda - tx\right)$$

The minimimum occurs at $t = \log\left(\frac{x}{\lambda}\right)$, giving the result. □

We are now ready to prove the linear upper bound on the speed of the East-process.

Theorem 2.4 — *Let $(\sigma_t)_{t\ge0}$ be an East-process started from $\{0\}$ and $X(\sigma_t)$ be its front. Then there exist $v, \gamma > 0$ such that*

$$\mathbb{P}\left(X(\sigma_t) > vt\right) \le e^{-\gamma t} \qquad \forall t \ge 0 \tag{2.6}$$

*Remark* 2.2. In the following proof we omit the use of the floor function $\lfloor\cdot\rfloor$ for clarity of notation, treating non-integer values as integers. It is however clear that the proof could be adapted to be precise.

*Proof.* Let $v > 1$. Using Lemma 2.3 and the discussion before it the calculation of an upper bound becomes straightforward:

$$\begin{aligned}\mathbb{P}\left(X(\sigma_t) > vt\right) &\le \mathbb{P}\left(\rho_{vt} \le t\right) \le \mathbb{P}\left(F(vt,t)\right) \\ &= \mathbb{P}\left(M_t \ge vt\right) \le \frac{e^{-t}(et)^{vt}}{(vt)^{vt}} \\ &= \exp\left(-t + vt(log\left({}^{t}\!/\!_{vt}\right) + 1)\right) \\ &\le \exp\left(-t + vt(1 - log(v))\right) \le \exp(vt(1 - log(v)))\end{aligned}$$

The result follows by taking $v > e$ and $\gamma = -v(1 - log(v))$. $\square$

# 3 Mixing of the East-process

In this section we prove that the mixing time of the supercritical East-process on $[0, L]$ with a 1 fixed at the origin is $\Omega(L)$

In this section we will study the evolution of the East-process restricted to the segment [0, L] where $L \in \mathbb{N}$. In doing so we will assume that there is a 1 fixed at the origin. However, because of the local constraint of the East-process, when studying the East-process restricted to $\{0, 1, ..., L\}$ it doesn't matter whether we 1) fix a 1 at the origin or 2) only consider East-processes started from $\{0\} \subseteq \xi \subseteq \mathbb{N}$; the results of the analysis will be the same. Motivated by this we make two definitions:

Definition 3.1 — Define $\widetilde{\Omega} := \{A \subseteq \mathbb{N} \mid 0 \in A\}$ to be the set of configurations that are 0 on the negatives and 1 at the origin. Similarly, for $L \in \mathbb{N}$ define $\Omega_L := \widetilde{\Omega} \cap [0, L]$ to be the state space of the East-process on $\{0, 1, ..., L\}$ with a fixed 1 at the origin.

Another important property of the East-process is that the evolution at some site $x \in \mathbb{N}$ is not influenced by how the process evolves at sites to the right of $x$. More precisely, if $(\sigma_t)_{t\geq 0}$ is an East-process then for any $x \in \mathbb{N}$ and $t \geq 0$, $\sigma_t(x)$ is independent of the sigma algebra $\sigma\left((E_{n,k}, B_{n,k})_{n>x,k>0}\right)$. It is because of this that the East-process with a fixed 1 at the origin is a continuous time Markov chain when restricted to $\{0, 1, ..., L\}$.

## 3.1 Coupling time for basic coupling

As before, let $(\sigma_t^\xi)_{t\geq 0}$ denote the East-process on $\mathbb{Z}$ started from initial configuration $\xi \in \Omega$, constructed using $\mathcal{C} = (E_{x,k}, B_{x,k})_{x\in\mathbb{Z},\ k\in\mathbb{N}^+}$. Recall the definition of the hitting times $(\rho_i)_{i\in\mathbb{N}}$ from Definition 2.2.

Proposition 3.1 — *For each $l \in \mathbb{N}$ and $\xi \in \widetilde{\Omega}$ it holds that*

$$\sigma^{\{0\}}_{\rho_l+t} \cap [0, l] = \sigma^{\xi}_{\rho_l+t} \cap [0, l] \qquad \forall t \geq 0 \tag{3.1}$$

*Furthermore for all $\xi, \nu \in \widetilde{\Omega}$ it holds that*

$$\sigma^{\xi}_{\rho_l+t} \cap [0, l] = \sigma^{\nu}_{\rho_l+t} \cap [0, l] \qquad \forall t \geq 0 \tag{3.2}$$

*Proof.* We proceed by induction. The claim clearly holds for $l = 0$ since every such $\xi$ has a 1 at the origin forever. For the induction step suppose that the claim holds up to $l = n \geq 1$. Let $K$ be such that $T_{n+1,K} := \sum_{i=1}^{K} E_{n+1,i} = \rho_{n+1}$. Then $B_{n+1,K} = 1$, and since the ring is legal, $\sigma^{\{0\}}_{\rho_{n+1}}(n) = 1$, furthermore by the induction hypothesis also $\sigma^{\xi}_{\rho_{n+1}}(n) = 1$ i.e. the ring is also legal for $\sigma^{\xi}_{\cdot}$. Thus both processes update to 1 at time $\rho_{n+1}$. Now, since the $\rho_i$ are stopping times, $(\sigma^{\{0\}}_{\rho_{n+1}+t})_{t\geq 0}$ & $(\sigma^{\xi}_{\rho_{n+1}+t})_{t\geq 0}$ are two East-processes with $\sigma^{\{0\}}_{\rho_{n+1}} \cap [0, n+1] = \sigma^{\xi}_{\rho_{n+1}} \cap [0, n+1]$. The conclusion follows by basic coupling and the second claim is immediate. $\square$

## 3.2 Results for continuous Markov chains

We present some standard results for continuous time Markov chains that we will use throughout this section.

Definition 3.2 (Total variation distance) — Let $(\Omega, \mathcal{F})$ be a measurable space and $\nu, \mu$ be two probability measures on it. The total variation distance of $\nu$ and $\mu$ is defined as $\|\nu(\cdot) - \mu(\cdot)\|_{TV} := \sup_{A\in\mathcal{F}} |\nu(A) - \mu(A)|$

Definition 3.3 (Distance from equilibrium) — Let $M := (M_t)_{t\geq 0}$ be a continuous time, irreducible Markov chain taking values in a finite state space $\Omega$ and let $\pi$ be its equilibrium distribution. For $t \geq 0$ define $d(t) := \max_{x\in\Omega} \|\mathbb{P}\left(M_t \in \cdot \mid M_0 = x\right) - \pi(\cdot)\|_{TV}$ to be the worst case total variation distance from equilibrium.

Theorem 3.2 — *Let $M := (M_t)_{t\geq 0}$ be a continuous time, irreducible Markov chain taking values in a finite state space $\Omega$ and let $\pi$ be its equilibrium distribution. Suppose that for each pair of states $x, y \in \Omega$ there is a coupling $(X_t, Y_t)_{t\geq 0}$ of $M$ that is started from $(x, y) \in \Omega^2$. For each of these couplings, let $\tau^{x,y}_{couple} := \min\{t \geq 0 \mid X_t = Y_t\}$ be the first time the chains meet. Then it holds that*

$$d(t) \leq \max_{x,y\in\Omega} \mathbb{P}\left(\tau^{x,y}_{couple} > t\right) \tag{3.3}$$

*Proof.* The result can be found in [3] Corollary 5.3 for discrete time Markov chains, however all the necessary proofs work in the continuous time case without modification. □

## 3.3 Linear upper bound on mixing

Definition 3.4 (Mixing time) — In the setting of Definition 3.3, the mixing time of $M$ is defined as $t_{mix} := \inf\{t \geq 0 \mid d(t) \leq 1/4\}$.

*Remark* 3.1 (Mixing time of East-process on $\{0, 1, ...L\}$). When restricted to $\{0, 1, ...L\}$, the East-process with a fixed 1 at the origin is a finite, irreducible, continuous time Markov chain with equilibrium measure $\pi_L := \delta_1 \times \mathrm{Ber}\,(p)^L$. Therefore the 'mixing time of the East-process on $\{0, 1, ...L\}$' refers to the mixing time of this restricted, finite chain.

Theorem 3.3 — *For each supercritical East-process with a 1 fixed at the origin, there exists a constant $K > 0$ such that for all $L \in \mathbb{N}$, the mixing time $T^L_{mix}$ on $\{0, 1...L\}$ satisfies $T^L_{mix} \leq KL$.*

*Proof.* By Theorem 3.2 it holds that

$$d(t) \leq \max_{\nu,\xi\in\tilde{\Omega}} \mathbb{P}\left(\tau^{\nu,\xi}_{couple} > t\right) \tag{3.4}$$

Where $\tau^{\nu,\xi}_{couple}$ is the first time that the East-process started from $\xi$ and $\nu$ (and thus with a fixed 1 at the origin) coincide on $\{0, 1...L\}$ under the basic coupling. Lemma 3.1 gives $\max_{\nu,\xi\in\tilde{\Omega}} \mathbb{P}\left(\tau^{\nu,\xi}_{couple} > t\right) \leq \mathbb{P}\left(\rho_L > t\right)$. Borrowing notation from Corollary 2.2.1 we get

$$\begin{aligned}\mathbb{P}\left(\rho_L > KL\right) &\leq \mathbb{P}\left(X\left(\sigma^{\{0\}}_{KL}\right) < L\right)\\ &= \mathbb{P}\left(X\left(\sigma^{\{0\}}_{KL}\right) < \frac{KL}{K}\right)\end{aligned}$$

Fix $K$ large enough such that $1/K < \alpha$ to get

$$\leq Ce^{-\gamma KL}$$

Thus for say $L \geq L'$, $\mathbb{P}\left(X\left(\sigma^{\{0\}}_{KL}\right) < L\right) \leq 1/4$. This implies $d(KL) \leq 1/4$ i.e. $T^L_{mix} \leq KL\ \forall L \geq L'$. □

## 3.4 Linear lower bound on mixing

Theorem 3.4 — *For each East-process with a 1 fixed at the origin, there exists a constant $K > 0$ such that for all $L \in \mathbb{N}$, the mixing time $T^L_{mix}$ on $\{0, 1, ...L\}$ satisfies $T^L_{mix} \geq KL$.*

*Proof.* From the proof of Theorem 2.4 we know that $\exists\ \gamma, v > 0$ such that

$$\mathbb{P}\left(\rho_{tv} \leq t\right) \leq e^{-\gamma t} \qquad \forall t \geq 0 \tag{3.5}$$

Thus we get

$$\mathbb{P}\left(\rho_{L/2} \leq L/2v\right) \leq e^{-\gamma \frac{L}{2v}} \qquad \forall L \in \mathbb{N} \tag{3.6}$$

Define the set $A_L := \{\omega \in \Omega \mid \{L/2, L/2+1, ..., L\} \cap \omega \neq \varnothing\}$ to be the set of configurations with at least one occupied site between $L/2$ and $L$. For our proof it is crucial to note that $\mathbb{P}\left(\sigma^{\{0\}}_t \in A_L \middle| \rho_{L/2} > t\right) = 0$: since $\sigma^{\{0\}}$ is started from $\{0\}$, it can be occupied at some site $x \in \mathbb{N}^+$ only if site $x$ has been hit i.e. it can only happen on $\{\rho_x \leq t\}$. Let $\pi_L := \delta_1 \times \mathrm{Ber}\left(p\right)^L$ be the product Bernoulli measure on $\Omega_L$. We have

$$\begin{aligned}
d\left(\frac{L}{2v}\right) &= \max_{\xi \in \widetilde{\Omega}} \left\| \mathbb{P}\left(\sigma^{\xi}_{L/2v} \cap [0, L] \in \cdot\right) - \pi_L(\cdot) \right\|_{TV} \\
&= \max_{\xi \in \Omega} \max_{A \in \Omega_L} \left| \mathbb{P}\left(\sigma^{\xi}_{L/2v} \cap [0, L] \in A\right) - \pi_L(A) \right| \\
&\geq \left| \mathbb{P}\left(\sigma_{L/2v} \in A_L\right) - \pi_L(A_L) \right| \\
&= \left| \mathbb{P}\left(\rho_{L/2} \leq L/2v\right) \mathbb{P}\left(\sigma_{L/2v} \in A_L \middle| \rho_{L/2} \leq L/2v\right) - \left(1 - q^L\right) \right| \\
&\geq \min\left\{ \left| 1 - q^L - e^{-\gamma \frac{L}{2v}} \right|, 1 - q^L \right\} \xrightarrow{L \to \infty} 1
\end{aligned}$$

By the above, for say $L \geq L'$ it holds that $d\left(\frac{L}{2v}\right) > 1/4$ i.e. $\frac{L}{2v} < T^L_{mix}$. Taking $K = \frac{1}{2v}$ the conclusion follows. $\square$

# References


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