$\boxed{2.3}$ Let $f(x) = x + 2x^2 + 3x^3 + \dots + 100x^{100}$.

What is the sum of digits of the integer $f(1.01)$?

Solution Multiply by x on both sides of the definition of $f(x)$ to get

$$x f(x) = x^2 + 2x^3 + \dots + 100x^{101}. \qquad (1)$$

Subtracting equation (1) from the definition of $f(x)$, we get

$$f(x)(1-x) = \underbrace{x + x^2 + x^3 + \dots + x^{100}}_{\text{geometric series}} - 100x^{101}.$$

The geometric series $x + x^2 + \dots + x^{100}$ is equal to $x \cdot \frac{x^{100}-1}{x-1}$.

Plug this in to get

$$f(x)(1-x) = x\left(\frac{x^{100}-1}{x-1}\right) - 100x^{101}$$

$$= x\left(\frac{x^{100}-1-100x^{101}+100x^{100}}{x-1}\right)$$

$$= x\left(\frac{101x^{100}-1-100x^{101}}{x-1}\right).$$

Plugging in $x = 1.01$ we have

$$f(1.01)\times(-0.01) = 1.01 \times \frac{101(1.01)^{100} - 1 - \overbrace{100\times(1.01)}^{=101}(1.01)^{100}}{0.01}$$

$$= -101.$$

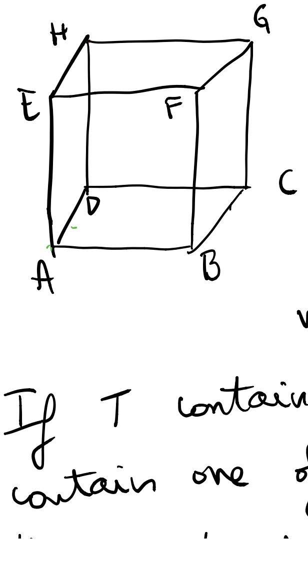
This gives us

$$f(1.01) = 10100.$$

The sum of digits of $f(1.01)$ is 2.

$\boxed{2.4}$ 3 distinct vertices of a cube are chosen at random. What is the probability that they form an equilateral triangle?

<u>Solution</u>: Let ABCDEFGH be a unit cube as shown below. Any two vertices are connected by either 1) an edge 2) a diagonal or 3) a space diagonal with length 1, $\sqrt{2}$ and $\sqrt{3}$ respectively.



Let T be an equilateral triangle made of vertices of the cube.

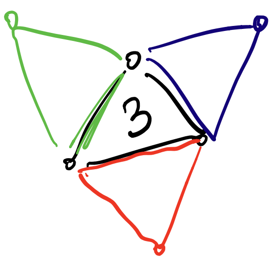
- If T contains an edge, say AB, then it will also contain one of BC, BF, AE or AD. In either case [illegible] be a diagonal. ↯

the third edge will be a diagonal. ↯ ← means contradiction.

- If T contains a space diagonal, say AG, then any other edge will need to be a diagonal or an edge. ↯

This implies all edges of T must be diagonals of the cube.

Let T contain say AF. Then T must be AFH or AFC. Therefore, the number of equilateral triangles made from vertices of the cube is



# of diagonals of the cube

# of equilateral Δ's with given diagonal

$$\frac{12 \cdot 2}{3} = 8$$

triple counting all triangles.

The total number of Δ's is

$$\binom{8}{3} = \frac{8!}{3!\,5!} = \frac{8 \cdot 7 \cdot \cancel{6}}{\cancel{6}} = 56.$$

So the probability is

$$\frac{8}{56} = \frac{1}{7}.$$