Algorithms and Complexity: Final

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Professor Bradford

Patrik Sokolowski

Problem 1

Internet Advertisement Impression Auctions sell advertisement space for individuals. These ad-space sellers differentiate their ad impressions by attributes and categories. Different attributes and categories may have different reserve prices. However, the number of attributes and categories may vary greatly, so the key question is how much revenue is lost given fewer reserve prices?
Some defintions,

1. R_ℓ is the total expected revenue given the best choices of upto ℓ reserve prices.
2. m bidders $d_j \in [m]$.
3. n impression-types - each as a single auction item
4. Impression type t_i occurs with probability p_i. That is, $\mathbb{P}[I = t_i] = p_i$.
   So, $1 = p_1 + p_2 + \cdots + p_n$.
5. Impression type t_i gets an evaluation $v(i, j)$ from bidder d_j. This evaluated value is not necessarily a bid.
6. There are ℓ reserve prices $r_1, r_2, \cdots, r_\ell$.
7. Each impression-type t_i is assigned the max reserve price that is lower or equal to $v(i, j)$ for some $j \in [m]$. Or 0 is when no reserve price exists.
   Each impression-type i's highest bid is h_i.
8. Set the reserve prices,

$$r'_i = \begin{cases} \max_{i \in [\ell]}\{v(i, j)\} \\ 0 \end{cases}$$

   for each advertisement impression $i \in [n]$.

The highest bidder is d_h and their bid for impression i is $b(i, h)$. The second highest bidder is d_s with bid $b(i, s)$ for impression i. If there is a single bidder for impression i, then $b(i, s) = 0$.

The Bidding Process works as follows

1. Randomly arriving impression-type t_i is announced.
2. Each bidder j announces their bids $b(i, j)$ for impression-type t_i.
3. Winning bidders are determined by the rules in Table 1.

Condition	**Description**	**Revenue**
If $b(i, h) < r'_i$	No bidder wins	0
If $b(i, s) < r'_i \leq b(i, h)$	Bidder d_h wins paying price r'_i	r'_i
If $r'_i \leq b(i, s)$	Bidder d_h wins paying the second price $b(i, s)$	$b(i, s)$

Table 1: **Tests and their target**

Since this is a second-price auction, selecting the reserve prices $r'_i = h_i$ **prove** the next fact.

Fact 1. Given the probabilites $p_1 + \cdots + p_n = 1$ for impressions 1 though n respectively. At most, it must be that,

$$R_\infty \quad = \quad \sum_{i=1}^{n} h_i \, p_i.$$

We can prove that this Fact is most true. Using the given information:

1) R_∞ = Total revenue gained through given probability
2) p_i = given n probabilities in summation
3) that the "probabilites $p_1 + \cdots + p_n = 1$ for impressions 1 though n respectively" (definition 4)
4) given n impression types, h_n is the total revenue of given impression types.

We can prove $R_\infty = \sum_{i=1}^{n} h_i \, p_i$.

We can safely say using given info number 3 (and definition 4): $R_\infty = \sum_{i=1}^{n} h_i$ (1 (a factor of probability)).

Given info 1 states that R_∞ = Total revenue gained through given probability, and that the summation of h_i (i being n summation types) is the total revenue of bids in an n number of bids, both represent the same value. h_i total revenue being modified by the probabilities p_i (our factor of probability) we get total (sum through summation) revenue gained with probability, which R_∞ is by definition.

Problem 2

Let H_n be the n-th harmonic number. That is,

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Also, R_1 is maximum revenue given the *best selection* of a single reserve price.
Without loss of generality, let the highest bids be ordered as $h_1 \geq h_2 \geq \cdots \geq h_n$. Thus, given a single reserve price $r' = h_i$ for all ad impressions, then this reseve price gives total revenue ih_i.
Prove the next lemma:

Lemma 1. Suppose the impression-types are randomly and uniform distributed, then

$$R_1 \quad \geq \quad R_\infty / H_n.$$

Using the given information:

1) R_1 = the max revenue given best selection of a single reserve price.
2) $\mathrm{R}_\infty = \sum_{i=1}^{n} h_i \, p_i.$ = Expected revenue with probability.
3) $\mathrm{H}_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$

We can already say that $\mathrm{R}_1 \geq R_\infty/H_n.$ will be smaller by some unkown as of yet degree as R_1 represents total best reserve prices as the non-bias reserve price total R_∞ is being divided (making it naturally smaller). This lemma is also assuming $\mathbb{P}[I_i = i] = \frac{1}{n}$ due to all impression types being the same probability. n total impression types: $i \in [1, 2, ..., n]$. We also have the harmonic series being able to = log(n) by constant.

4) $\mathbb{P}[I_i = i] = \frac{1}{n}$
5) $i \in [1, 2, ..., n]$
6) $\mathrm{H}_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = log(n).$
7)$R\ell$ = total revenue for ℓ reserve prices.

We can restate $\mathrm{R}_1 \geq R_\infty/H_n.$ as $\mathrm{R}_1 \geq R_\infty/log(n).$
Leading to $\mathrm{R}_1 * log(n) \geq R_\infty.$

Being divided by a seriese will make the numerator smaller and smaller so any summation concerning the right side of the lemma will always be smaller. Given no differentiation in probability on the right leads to a greater constant in the end in this case.

In a given scenario:

R_1¡$R_1 \geq R_\infty/H_n.$ for all uniform probability impression types.
for all i : $\frac{ih_i}{n} < R_\infty/H_n.$

Problem 3

The next proof uses the Euler-Mascheroni constant γ where:

$$\begin{aligned}\gamma &= H_n - \ln n - o(1)\\ &= \lim_{n\to\infty}\left(-\ln(n) + \sum_{k-1}^{n}\frac{1}{k}\right)\\ &= \int_1^\infty \left(\frac{1}{\lfloor x\rfloor} - \frac{1}{x}\right)dx.\end{aligned}$$

It is well known that γ is about $0.57721\cdots$.
Prove the next theorem.

Theorem 1. Suppose the impression probabilities are randomly and uniformly distributed. Then for all ℓ : $\ln^{1/2-\epsilon} n \geq \ell \geq 1$ and an arbitrarily small constant $\epsilon > 0$, it must be that,

$$R_\ell \quad \geq \quad (1-o(1))\,(\ell/H_n)\,R_\infty.$$

Also, there is a uniform probability instance where $R_\ell \leq (\ell/H_n)\,R_\infty$.

Impression types here are uniform

First part: Let c = $\ln^{\in} n and dc^\ell \leq n. Say:$

$\sum_{i=1}^{c^\ell}$ $h_i \frac{}{n} \geq R_\infty/c$,

then by Lemma 1, one single reserve price may have revenue of at least:

$\sum_{i=1}^{c^\ell}$ $h_i \frac{}{n}$ / $H_{c^\ell} \geq (1-o(1))\frac{R_\infty}{\ell ln(c)}, Why?$
If we assume uniform probabilities we can constantly assume $R_\ell \geq (1-o(1)) * \ell/H_n * R_\infty$.After this we also use the Euler-Mascheroni constant rule to change the summation into a more 'adjustable' algebriac form.
$\geq (1-o(1))\frac{\ell R_\infty}{(ln^{1-2\in}n)cln(c)}, Why?$
We algebriacly move some of the terms around and further set up for the upcoming steps (now we have the c's in the denominator.
= (1 - o(1)) $\frac{\ell R_\infty}{(ln^{1-\in}n)lnln^{\in}(c)}, Why?$
Here we define c = $\ln^{\in} n$(which helps give us the first step and the further steps on).
= (1 - o(1)) $\frac{ln^{\in}}{lnln^{\in}n} * \frac{\ell}{lnn} * R_\infty, Why?$
There exists an instant with uniform probabilities for $R_\ell \leq \ell/H_n * R_\infty$.
$\geq (1-o(1))\frac{\ell}{H_n} * R_\infty$,Why? Since $\frac{ln^{\in}n}{lnln^{\in}n} \geq 1$.
In the last step, due to the reserve prices always being as good as 1, we 'canceled' out $\frac{ln^{\in}n}{lnln^{\in}n}$and moved $H_n$back in there.

Through each step we can see that the reserve prices are always at least as good as 1.