Algorithms and Complexity: Homework #1

Due on February 1, 2019 at 3:10pm

Professor Bradford

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Problem 1

Page 22, Exercise 1.

Solution

Explanation or Counterexample

TrueorFalse? In every instance of the Stable Matching Problem, There is a stable matching containing a pair (m, w) such that m is ranked first on the preference list of w and w is ranked first on the preference list of m.

False!

Consider:

Female preference lists.PNG

M

m1	1	2
m2	1	2

W

w1	1	2
w2	1	2

Picture Provided by Professor Bradford's Power Point

m1 prefers w1 and m2 prefers w1 w1 prefers m1 and w2 prefers m1

It is impossible to have all four people have their number 1 preferences if two on each side prefer one on the other side.
Having m1 and m2 prefer w1 will make it impossible to satisfy this unless a polygamous rule is put in. Both m's cannot be made into a perfect match.
Having w1 and w2 prefer m1 wil make it impossible to satisfy this unless a polygamous rule is put in. Both w's cannot be made into a perfect match.

Problem 2

Page 22, Exercise 2.

Solution

Explanation or Counterexample

$True or False$? Consider an instance of the Stable Matching Problem in which there exists a man m and a woman w such that m is ranked first on the preference list of w and w is ranked first in the preference list of m. Then, in every stable matching S for this intance, the pair (m, w) belongs to S.

$True$!

Consider the pairs:
(m, w^1)$and(m^1, w)$
We know m prefers w and w prefers m. If given any different match-ups such as the ones above we get instabilities which are not perfect pairs. This would not hold in this instance, only showing (m, w) belongs in S. We wouldn't be able to satisfy the instance which wants (m,w) belonging to S, so we need (m, w) matched.

Problem 3

Give an inductive proof of

$$\sum_{i=0}^{n-1}(2i+1) \quad = \quad n^2.$$

Solution

Proof. Put your proof here. □

Proof:

$$\begin{aligned} Basis: n=1 & & \\ Leftside: & \sum_{i=0}^{1-1}(2(0)+1) & =1 \\ Rightside: & (1)^2 & =1 \end{aligned}$$

Both sides are equal, making it true for n=1

$$InductionStep: assuming\,(n=1)\,is\,true\,for\,(n=c)$$

$$\begin{aligned} \sum_{i=0}^{c-1+1}(2i+1) & = & \sum_{i=0}^{c-1}(2i+1)+(2(c+1)-1) \\ = & c^2+2(c+1)-1 & (inductionhypothesis) \\ = & c^2+2c+1 & =(c+1)^2(rightside) \end{aligned}$$

So this must hold for all n$\leq$ 2

Problem 4

Let

$$f(n) \quad = \quad \sum_{i=0}^{n-1} x^i, \tag{1}$$

Do each of the following,

Part A

Prove $f(n) = f(n-1) + x^{n-1}$

Proof. $Basis: f(2) = f(1)$
$f(1) = \sum_{i=0}^{1-1} x^i = 1$
$so \quad f(2) = 1 + x$
$and \quad \sum_{i=0}^{n-1} x^i = 1 + x \quad for \quad n = 2$

Inductive hypothesis:
say for n$\leq k, \ some \ k$
$f(n) = \sum_{i=0}^{k-1} x^i$
$Suppose, FSOC(For\ Sake\ Of\ Contradiction), k\ is\ the\ smallest\ integer\ where\ \ f(k) \neq \sum_{i=0}^{k-1} x^i$
$by\ definition,\ f(k) = f(k-1) + x^{k-1}$
$However\ f(k-1) = \sum_{i=0}^{k-2} x^i$
$and\ f(k) = f(k-1) + x^{k-1}$
$This\ means\ there\ can\ be\ no\ minimal\ k\ where$
$f(k) \neq \sum_{i=0}^{k-1} x^i \quad \square$

Part B

Prove $f(n) = xf(n-1) + 1$

Proof. $Basis: n = 2$
$f(2) = xf(1) + 1$
$\sum_{i=0}^{1-1} x^i + 1$
$f(2) = x + 1$

$Inductive\ hypothesis: f(n) = xf(n-1) + 1 \ \ \forall \ \ n \leq C$
$Inductive\ Step: f(n)\ fails\ first\ for\ n$
$f(n) = xf(n-1) + 1,\ by\ I.H:$
$\sum_{i=0}^{n-2} x^i + 1 \qquad \sum_{i=0}^{n-2} x^{i+1}$
$f(n) = \sum_{i=1}^{n-1} x^i + 1 \qquad \sum_{i=1}^{n-1} x^{i+1}$

$\square$

Problem 4 continued on next page. . .

Part C

Prove $f(n) = (x^n - 1)/(x - 1)$

Proof. $Basis : f(2) = f(1)$
$f(1) = \sum_{i=0}^{1-1} x^i = 1$
$so \quad f(2) = 1 + x$
$and \quad \sum_{i=0}^{n-1} x^i = 1 + x \quad for \quad n = 2$

Inductive hypothesis:
say for n$\leq k, \ some \ k$
$f(n) = \sum_{i=0}^{k-1} x^i$
$Suppose, FSOC(For\ Sake\ Of\ Contradiction), k\ is\ the\ smallest\ integer\ where\ \ f(k) \neq \sum_{i=0}^{k-1} x^i$
$by\ definition,\ f(k) = (x^n - 1)/(x - 1)$
$However\ f(2) = 1 + x \neq (x^2 - 1)/(x - 1)$
$and\ f(k) = (x^2 - 1)/(x - 1)$
$This\ means\ there\ can\ be\ no\ minimal\ k\ where$
$f(k) \neq \sum_{i=0}^{k-1} x^i \quad \square$

Part D

What happens when $x = 1$, how about for Equation 1?
f(n) $= \sum_{i=0}^{n-1} 1^i$
$f(1) = \sum_{i=0}^{1-1} 1^i = 1 \quad = \quad 1$
$f(2) = \sum_{i=0}^{2-1} 1^i = 1 + 1 \quad = \quad 2$
$f(3) = \sum_{i=0}^{3-1} 1^i = 1 + 1 + 1 = \quad 3$
$f(n) = \sum_{i=0}^{n} n$

Part A:
f(n) $=$ f(n-1)x^{n-1}
$f(n) = f(n-1)1^{n-1}$
$Basis : f(1) = 1\ and\ f(2) = 2$
$Suppose, FSOC(For\ Sake\ Of\ Contradiction), k\ is\ the\ smallest\ integer\ where\ \ f(k) \neq \sum_{i=0}^{k-1} x^i$
$by\ definition,\ f(k) = f(k-1)1^{k-1}$
$However, f(1) = 0 \neq \sum_{i=0}^{1-1} 1^i = 1$
$This\ means\ there\ can\ be\ no\ minimal\ k\ where$
$f(k) \neq \sum_{i=0}^{k-1} x^i$

Part B:
Basis: n = 1
f(n) $= \sum_{i=0}^{1-1} 1^i = (1)f(1-1) + 1$
$Inductive hypothesis :$
$\sum_{i=0}^{n-1} 1^i = (1)f(n-1) + 1 \ \forall \ n \ \leq \ C$
$InductiveStep :$
$(1)f(n-1) + 1\ always\ equals\ \ \sum_{i=0}^{n-1} 1^i$

Problem 4 continued on next page. . .

$f(n-1)+1=f(n)$
$by\ definition\ will = \sum_{i=0}^{n-1} 1^i$

Part C:
f(n) = $(\mathrm{x}^2-1)/(x-1)$
$f(n)=(1^2-1)/(1-1)$
$f(n)=0$