A summary of Projective Geometric Algebra

Patrolin

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Abstract

I. Geometric numbers

$$1 + x^2 = 0$$
$$x = ?$$

x is not a real number; but if it's not real, why should the other numbers be real?

$$(e_1)^2 + (e_2)^2 = (e_0)^2$$
$$(e_1)^2 = 1;\ (e_2)^2 = -1;\ (e_0)^2 = 0$$

In fact, we can define as many of these as we want, the simplest examples being:

$a + be_1$ `// hyperbolic numbers`

$a + be_2$ `// complex numbers`

$a + be_0$ `// dual numbers`

We can multiply these numbers together using the geometric product:

$$e_i e_i = \{1, -1, 0\}$$
$$e_i e_j = -e_j e_i$$

This product is neither commutative nor anticommutative, but it is distributive and associative:

$$AB \neq BA$$
$$AB \neq -BA$$
$$A(B + C) = AB + AC$$
$$(AB)C = A(BC)$$
$$aB = Ba;\ a \in \mathbb{R}$$

Thus the product of two complex numbers:

$$\begin{aligned}
&(A_1 + A_2e_2)(B_1 + B_2e_2)\\
&= A_1B_1 + A_1B_2e_2 + A_2e_2B_1 + A_2e_2B_2e_2\\
&= A_1B_1 + A_1B_2e_2 + A_2B_1e_2 + A_2B_2\\
&= (A_1B_1 + A_2B_2) + (A_1B_2 + A_2B_1)e_2
\end{aligned}$$

II. Rotations

A multivector with n basis vectors consists of 2^n blades:

```
- scalar = 0-vector = 1
- vector = 1-vector
- bivector = 2-vector
- trivector = 3-vector
- ...
- (n-1)-vector = pseudovector
- n-vector = pseudoscalar = 𝟙
```

Where a k-vector has $\binom{n}{k}$ blades, for example:

$$\begin{aligned}
A = A_1 &\\
&+ A_2e_0 + A_3e_1 + A_4e_2\\
&+ A_5e_{01} + A_6e_{02} + A_7e_{12}\\
&+ A_8e_{012}
\end{aligned}$$

We can abbreviate blades like e_1e_2 as e_{12}.

Multiplying two multivectors gives you another multivector, we can use the taylor series expansion of the exponential function to find a rotation e^A:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$e^{ae_2} = 1 + ae_2 - \frac{a^2}{2} - \frac{a^3}{3}e_2 + ...$$

`// by the sum being convergent`

$$= (1 - \frac{a^2}{2} + ...) + (a - \frac{a^3}{3} + ...)e_2$$

$$= cos(a) + sin(a)e_2$$

Similarly you can find

$$e^{e_i} = \begin{cases} cos(a) + sin(a)e_i & ((e_i)^2 = -1) \\ cosh(a) + sinh(a)e_i & ((e_i)^2 = 1) \\ 1 + e_i & ((e_i)^2 = 0) \end{cases}$$

This gives us rotations, hyperbolic rotations and translations (rotations through infinity) respectively.

Then for a multivector we would have:

$$e^A = e^{A_1} e^{A_2 e_0} e^{A_3 e_1} ... e^{A_n e_I}$$

III. Duality

For a blade X in a k-vector we can define some operations, like reversing the order of basis vectors in the blade, that just amount to some sign flips:

$$\tilde{X} = (-1)^{\lfloor k/2 \rfloor} X \text{ // reverse}$$
$$X^\dagger = (-1)^{\lfloor k \rfloor} X \text{ // involute}$$
$$\bar{X} = (-1)^{\lfloor k+k/2 \rfloor} X \text{ // conjugate}$$

Poincaré duality states that maps between k-vectors and (n-k)-vectors exist.

$$X\, dual(X) = \pm \mathbb{1}$$

For example:

$$\underline{X}X = \mathbb{1} \text{ // left complement}$$
$$X\overline{X} = \mathbb{1} \text{ // right complement}$$
$$XX^\star = sign(X_{ND}\widetilde{X_{ND}})\, \mathbb{1} \text{ // hodge dual}$$

Where X_{ND} is X without degenerate basis vectors, e.g.

$$X = e_{012};\ X_{ND} = e_{12}$$

Let $\mathbb{G}_{a,b,c}$ be a geometric algebra with a positive, b negative and c zero basis vectors.

Then for $\mathbb{G}_{a,0,c}$:

$$\overline{X} = X^\star$$

And if that wasn't confusing enough applying a dual twice changes the signs, so we also want the inverse of these duals:

$$(X^\star)^{\star^{-1}} = X$$

IV. Shapes and sizes

TODO