

A summary of Projective Geometric Algebra

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October 10, 2022

Abstract

Information on Geometric Algebra is scattered and scarce. This is an explanation of how to calculate PGA, but not necessarily why it works - for that you can try [sudgylacmoe](#) on youtube and [Eric Lengyel's](#) cheatsheet for point-based PGA (though he goes mad with power). We would also point you to [bivector.net](#) (but their calculator just, doesn't), the [ganja.js](#) ones are presumably correct, since their demos seem to work.

I. GEOMETRIC NUMBERS

associative:

$$1 + x^2 = 0 \\ x = ?$$

x is not a real number; but if it's not real, why should the other numbers be real?

$$(e_1)^2 + (e_2)^2 = (e_0)^2 \\ (e_1)^2 = 1; (e_2)^2 = -1; (e_0)^2 = 0$$

In fact, we can define as many of these as we want with a geometric algebra $G_{a,b,c}$ having a numbers that square to 1, b that square to -1 and c that square to 0 (degenerate basis vectors):

$$(a + be_1) \in G_{1,0,0} \text{ // hyperbolic numbers}$$

$$(a + be_2) \in G_{0,1,0} \text{ // complex numbers}$$

$$(a + be_0) \in G_{0,0,1} \text{ // dual numbers}$$

We can multiply these numbers together using the geometric product:

$$(e_i)^2 = \{1, -1, 0\} \\ e_i e_j = -e_j e_i$$

The result is another number in the same algebra $G_{a,b,c}$

This product is neither commutative nor anticommutative, but it is distributive and as-

$$AB \neq BA$$

$$AB \neq -BA$$

$$A(B + C) = AB + AC$$

$$(AB)C = A(BC)$$

$$aB = Ba; a \in \mathbb{R}$$

Thus the product of two complex numbers:

$$\begin{aligned} (A_1 + A_2 e_2)(B_1 + B_2 e_2) \\ = A_1 B_1 + A_1 B_2 e_2 + A_2 e_2 B_1 + A_2 e_2 B_2 e_2 \\ = A_1 B_1 + A_1 B_2 e_2 + A_2 B_1 e_2 + A_2 B_2 \\ = (A_1 B_1 + A_2 B_2) + (A_1 B_2 + A_2 B_1) e_2 \end{aligned}$$

II. ROTATIONS

A multivector with n basis vectors consists of 2^n blades:

- scalar = 0-vector = 1
- vector = 1-vector
- bivector = 2-vector
- trivector = 3-vector
- ...
- pseudovector = (n-1)-vector
- pseudoscalar = n-vector = 1

Where a k-vector has $\binom{n}{k}$ blades, for example:

$$\begin{aligned} A = & A_1 \\ & + A_2 e_0 + A_3 e_1 + A_4 e_2 \\ & + A_5 e_{01} + A_6 e_{02} + A_7 e_{12} \\ & + A_8 e_{012} \end{aligned}$$

We can abbreviate blades like $e_1 e_2$ as e_{12} .

We can use the exponential function to find a rotation e^A :

$$\begin{aligned} e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \\ e^{ae_2} &= 1 + ae_2 - \frac{a^2}{2} - \frac{a^3}{3} e_2 + \dots \\ &= (1 - \frac{a^2}{2} + \dots) + (a - \frac{a^3}{3} + \dots) e_2 \\ &= \cos(a) + \sin(a) e_2 \end{aligned}$$

Similarly you can find

$$e^{e_i} = \begin{cases} \cosh(a) + \sinh(a) e_i & ((e_i)^2 = 1) \\ \cos(a) + \sin(a) e_i & ((e_i)^2 = -1) \\ 1 + e_i & ((e_i)^2 = 0) \end{cases}$$

This gives us hyperbolic rotations, rotations and translations (rotations through infinity) respectively.

For the i-th blade X_i in a multivector:

$$e^A = \prod_i e^{A_i X_i}$$

III. UNARY OPERATORS

We can define some operations, like reversing the order of basis vectors in the blade, that amount to flipping some signs:

$$\check{X}_i = (-1)^{\lfloor k/2 \rfloor} X_i \quad // \text{ reverse}$$

$$X_i \in \text{k-vector}$$

$$f(A) = \sum_i f(X_i)$$

Poincaré duality states that maps between k-vectors and (n-k)-vectors exist.

$$\begin{aligned} X_i \text{ dual}(X_i) &= \pm 1 \\ \text{dual}(X_i) &= \pm X_{2^n - i + 1} \end{aligned}$$

For example:

$$X_i X_i = 1 \quad // \text{ left complement}$$

$$X_i \overline{X_i} = 1 \quad // \text{ right complement}$$

$$X_i X_i^* = \text{sign}(X_i^{ND} \widetilde{X_i^{ND}}) 1 \quad // \text{ hodge dual}$$

Where X_i^{ND} is X_i without degenerate basis vectors, e.g.

$$X_i = e_{012}; \quad X_i^{ND} = e_{12}$$

$$\text{For } \mathbb{G}_{n,0,c}: X_i^* = \overline{X_i}$$

And applying a dual twice changes the signs, so we also want their inverses:

$$\begin{aligned} (X_i^*)^{\star^{-1}} &= X_i \\ \overline{\overline{X_i}} &= X_i \end{aligned}$$

In 2D and 3D PGA, we can simplify implementation by swapping two basis vectors in some blades such that $\overline{X_i}$ does not flip signs, e.g.

$$\begin{aligned} A = & A_1 + A_2 e_0 + A_3 e_1 + A_4 e_2 \\ & + A_5 e_{01} + A_6 e_{20} + A_7 e_{12} + A_8 e_{012} \end{aligned}$$

$$\begin{aligned} \underline{A} = \overline{A} = & A_8 + A_7 e_0 + A_6 e_1 + A_5 e_2 \\ & + A_4 e_{01} + A_3 e_{20} + A_2 e_{12} + A_1 e_{012} \end{aligned}$$

IV. SHAPES AND SIZES

Let $\mathbb{G}_{d,0,1}$ be a d-dimensional PGA:

$$(e_0)^2 = 0; (e_i)^2 = 1$$

Now we have a choice to make:

1. Point-based PGA

- vectors are points
- (n-1)-vectors are hyperplanes

2. Plane-based PGA

- vectors are hyperplanes
- (n-1)-vectors are points

Both of these are equally valid and many operations make use of computing in the dual algebra via $\underline{A} \text{ op } \overline{B}$.

$$\text{point} = e_0 + x e_1 + y e_2 + \dots + w e_d$$

$$\text{hyperplane} = \overline{e_0 + x e_1 + y e_2 + \dots + w e_d}$$

$$\boxed{\begin{aligned} \text{hyperplane} &= e_0 + xe_1 + ye_2 + \dots + we_d \\ \text{point} &= \overline{e_0 + xe_1 + ye_2 + \dots + we_d} \end{aligned}}$$

We will denote Plane-based operations inside boxes whenever they differ.

Let $\langle A \rangle_k$ be the grade selection operator:

$$\begin{aligned} \langle X_i \rangle_k &= \begin{cases} X_i & (X_i \in \text{k-vector}) \\ 0 & (X_i \notin \text{k-vector}) \end{cases} \\ \langle A \rangle_k &= \sum_i \langle X_i \rangle_k \end{aligned}$$

Then we can define the wedge \wedge and anti-wedge \vee products:

$$\begin{aligned} A \wedge B &= \sum_{j,k} \langle \langle A \rangle_j \langle B \rangle_k \rangle_{j+k} = \overline{A \vee B} \\ A \vee B &= \sum_{j,k} \langle \langle A \rangle_j \langle B \rangle_k \rangle_{n - ((n-j) + (n-k))} \\ &= \overline{A \wedge B} \end{aligned}$$

These operators retain distributivity and associativity.

We can then join points into lines and lines into planes:

$$\begin{aligned} \text{line} &= \text{point}_1 \text{ join } \text{point}_2 \\ \text{plane} &= \text{point}_1 \text{ join } \text{point}_2 \text{ join } \text{point}_3 \end{aligned}$$

$$A \text{ join } B = A \wedge B$$

$$\boxed{A \text{ join } B = A \vee B}$$

In fact this works with any two geometric objects, operators that don't have this property aren't really worth your time.

And we can meet two objects:

$$\begin{aligned} \text{point} &= \text{line}_1 \text{ meet } \text{line}_2 \\ \text{point} &= \text{plane} \text{ meet } \text{line} \\ \text{line} &= \text{plane}_1 \text{ meet } \text{plane}_2 \end{aligned}$$

$$A \text{ meet } B = A \vee B$$

$$\boxed{A \text{ meet } B = A \wedge B}$$

If you meet two parallel lines, you get a point at infinity = an infinite point:

$$\text{line}_1 \text{ meet } \text{line}_1 = xe_1 + ye_2 + \dots + we_d$$

$$\boxed{\text{line}_1 \text{ meet } \text{line}_1 = \overline{xe_1 + ye_2 + \dots + we_d}}$$

If you flip the direction of one of the lines, you get an infinite point in the other direction

All objects in PGA have an direction A^D and position A^P :

$$\begin{aligned} A &= A^D + A^P \\ A^D &= \sum_i A_i X_i \quad (e_0 \in X_i) \\ A^P &= \sum_i A_i X_i \quad (e_0 \notin X_i) \end{aligned}$$

$$\boxed{\begin{aligned} A^D &= \sum_i A_i X_i \quad (e_0 \notin X_i) \\ A^P &= \sum_i A_i X_i \quad (e_0 \in X_i) \end{aligned}}$$

In 2D and 3D:

$$\begin{aligned} \text{point} &= e_0 + A^P \\ \text{line} &= A^D + A^P \end{aligned}$$

$$\boxed{\text{point} = \overline{e_0} + A^P}$$

In 3D:

$$\begin{aligned} \text{plane} &= A^D + A_{15}e_{123} \\ &= \text{normal} + \text{distance} \end{aligned}$$

$$\boxed{\text{plane} = A^D + A_2e_0}$$

We can take norms to measure the direction or the position:

$$\begin{aligned} \|A\|_D &= \sqrt{\sum_i (A_i^D)^2} \\ \|A\|_P &= \sqrt{\sum_i (A_i^P)^2} \\ \|A\| &= \sqrt{\sum_i |(A_i X_i)^2|} = \|A\|_P \end{aligned}$$

$$\boxed{\|A\| = \|A\|_D}$$

$$A^{-1} = \tilde{A} \frac{1}{\|A\|^2}$$

A is finite if $\|A\|_D \neq 0$, therefore infinite objects have no direction.

We can use this to calculate lengths/areas/volumes/...:

$$length(edge_loop) = \sum_i \|line_i\|_P$$

$$area(edge_loop) = \frac{1}{2} \sum_i \|line_i\|_D$$

$$line_i = p_i \text{ join } p_{i+1}$$

$$area(triangle_mesh) = \frac{1}{2} \sum_i \|plane_i\|_P$$

$$volume(triangle_mesh) = \frac{1}{3} \sum_i \|plane_i\|_D$$

$$plane_i = p_i \text{ join } p_{i+1} \text{ join } p_{i+2}$$

i. Rasterization

We could just take the x, y components of a d-dimensional point and get an orthographic projection.

But we could also just make a line from the camera (at the origin) to a point and then intersect it with a plane:

$$p_{perspective} = (p_{camera} \text{ join } p_{vertex}) \text{ meet } (xy_plane)$$

$$xy_plane = p_1 \text{ join } p_2 \text{ join } p_3$$

$$depth(p_{vertex}) = \|p_{vertex}\|_P$$

TODO: project(A, B): $P = (A \cdot B^{-1})B$?

TODO: project(point_at_origin, B) TODO: project(plane_at_infinity, B)

ii. Raytracing

TODO: $line_1 \wedge line_2$

$$= signed_distance(line_1, line_2) \|line_1^D \wedge line_2^D\|$$

TODO: correlate of signed distance between lines: $line_1 \vee line_2 \rightarrow$ ray-triangle intersection \rightarrow raytraced game

TODO: distance of line to A \rightarrow raytraced graphing calculator

TODO: ?

$$d(A, B) = \begin{cases} \frac{A \wedge \bar{B}}{d(A, project(A, B))} & (type(A) = type(B)) \\ d(A, project(A, B)) & (type(A) \neq type(B)) \end{cases}$$

TODO: $\cos(A, B) = A \cdot A / (A \cdot norm())$
 $\sin(A, B) = ?$

iii. Raymarching

TODO: signed distance of point to A \rightarrow raymarched graphing calculator

iv.

V. MOTORS

Taking $e^{bivector}$ gives us a motor (motion operator), we can apply motors via $M\tilde{M}$:

$$motor = e^{bivector}$$

$$rotor = e^{bivector^D}; rotor \in motor$$

$$translator = e^{bivector^P}; translator \in motor$$

In 3D, rotors are quaternions.

Where bivector blades are rotations. For example $\frac{\theta}{2}e_{12}$ is an xy rotation by θ degrees and $\frac{d}{2}e_{01}$ is a translation by d :

$$e^{\frac{\theta}{2}e_{12}} = \cos \frac{\theta}{2} + \left(\sin \frac{\theta}{2} \right) e_{12}$$

$$e^{\frac{d}{2}e_{01}} = 1 + \frac{d}{2}e_{01}$$

We can interpolate motors with $nlerp$ or $slerp$, where BA^{-1} is a transformation that brings A to B:

$$nlerp(t, A, B) = \frac{lerp(t, A, B)}{\|lerp(t, A, B)\|}$$

$$lerp(t, A, B) = (1 - t)A + tB$$

$$slerp(t, A, B) = (BA^{-1})^t A$$

$nlerp$ does not allow for unnormalized motors, but we will be normalizing them in the physics simulation anyways.

TODO: line forces per <https://enki.ws/ganja.js/examples/coffee> and <https://bivector.net/PGADYN.html>

VI. BONUS

Useless operators that are just here for completeness:

$$X_i^\dagger = (-1)^{[k]} X_i \text{ // involute}$$

$$\bar{X}_i = (-1)^{[k+k/2]} X_i \text{ // conjugate}$$

$$\|A\|_{\infty} = \|A\|_D$$

$$\boxed{\|A\|_{\infty} = \|A\|_p}$$