# A summary of Projective Geometric Algebra

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**Abstract**

*This is an explanation of how to calculate PGA, but not necessarily why it works - for that you can try sudgylacmoe on youtube. I would also point you to bivector.net, but their dual calculations are whack, the ganja.js ones are presumably correct, since their demos work.*

## I. Geometric numbers

$$1 + x^2 = 0$$
$$x = ?$$

x is not a real number; but if it's not real, why should the other numbers be real?

$$(e_1)^2 + (e_2)^2 = (e_0)^2$$
$$(e_1)^2 = 1;\ (e_2)^2 = -1;\ (e_0)^2 = 0$$

In fact, we can define as many of these as we want, the simplest examples being:

$a + be_1$ `// hyperbolic numbers`

$a + be_2$ `// complex numbers`

$a + be_0$ `// dual numbers`

We can multiply these numbers together using the geometric product:

$$e_i e_i = \{1, -1, 0\}$$
$$e_i e_j = -e_j e_i$$

This product is neither commutative nor anticommutative, but it is distributive and associative:

$$AB \neq BA$$
$$AB \neq -BA$$
$$A(B + C) = AB + AC$$
$$(AB)C = A(BC)$$
$$aB = Ba;\ a \in \mathbb{R}$$

Thus the product of two complex numbers:

$$\begin{aligned}
&(A_1 + A_2e_2)(B_1 + B_2e_2) \\
&= A_1B_1 + A_1B_2e_2 + A_2e_2B_1 + A_2e_2B_2e_2 \\
&= A_1B_1 + A_1B_2e_2 + A_2B_1e_2 + A_2B_2 \\
&= (A_1B_1 + A_2B_2) + (A_1B_2 + A_2B_1)e_2
\end{aligned}$$

## II. Rotations

A multivector with n basis vectors consists of $2^n$ blades:

```
- scalar = 0-vector = 1
- vector = 1-vector
- bivector = 2-vector
- trivector = 3-vector
- ...
- (n-1)-vector = pseudovector
- n-vector = pseudoscalar = 𝟙
```

Where a k-vector has $\binom{n}{k}$ blades, for example:

$$\begin{aligned}
A = A_1 & \\
&+ A_2e_0 + A_3e_1 + A_4e_2 \\
&+ A_5e_{01} + A_6e_{02} + A_7e_{12} \\
&+ A_8e_{012}
\end{aligned}$$

We can abbreviate blades like $e_1e_2$ as $e_{12}$.

Multiplying two multivectors gives you another multivector, we can use the taylor series expansion of the exponential function to find a rotation $e^A$:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\begin{aligned} e^{ae_2} &= 1 + ae_2 - \frac{a^2}{2} - \frac{a^3}{3}e_2 + ... \\ &= (1 - \frac{a^2}{2} + ...) + (a - \frac{a^3}{3} + ...)e_2 \\ &= cos(a) + sin(a)e_2 \end{aligned}$$

Similarly you can find

$$e^{e_i} = \begin{cases} cos(a) + sin(a)e_i & ((e_i)^2 = -1) \\ cosh(a) + sinh(a)e_i & ((e_i)^2 = 1) \\ 1 + e_i & ((e_i)^2 = 0) \end{cases}$$

This gives us rotations, hyperbolic rotations and translations (rotations through infinity) respectively.

Then for a multivector we would have:

$$e^A = e^{A_1} e^{A_2 e_0} e^{A_3 e_1} ... e^{A_n e_I}$$

## III. Duals and complements

For a blade $X_i$ in a k-vector we can define some operations, like reversing the order of basis vectors in the blade, that just amount to some sign flips:

$$\tilde{X}_i = (-1)^{\lfloor k/2 \rfloor} X_i \texttt{ // reverse}$$
$$X_i^\dagger = (-1)^{\lfloor k \rfloor} X_i \texttt{ // involute}$$
$$\bar{X}_i = (-1)^{\lfloor k+k/2 \rfloor} X_i \texttt{ // conjugate}$$
$$complement(A) = \sum_i complement(X_i)$$

Poincaré duality states that maps between k-vectors and (n-k)-vectors exist.

$$X_i \, dual(X_i) = \pm \mathbb{1}$$

For example:

$$\underline{X_i} X_i = \mathbb{1} \texttt{ // left complement}$$
$$X_i \overline{X_i} = \mathbb{1} \texttt{ // right complement}$$
$$X_i X_i^\star = sign(X_i^{ND} \widetilde{X_i^{ND}}) \, \mathbb{1} \texttt{ // hodge dual}$$

Where $X_i^{ND}$ is $X_i$ without degenerate basis vectors, e.g.

$$X_i = e_{012};\ X_i^{ND} = e_{12}$$

Let $\mathbb{G}_{a,b,c}$ be a geometric algebra with $a$ positive, $b$ negative and $c$ zero basis vectors.

Then for $\mathbb{G}_{a,0,c}$:

$$\overline{X_i} = X_i^\star$$

And if all that wasn't confusing enough, applying a dual twice changes the signs, so we also want the inverses of these duals:

$$(X_i^\star)^{\star^{-1}} = X_i$$
$$\overline{\overline{X_i}} = X_i$$
$$\underline{\underline{X_i}} = X_i$$

## IV. Shapes and sizes

Let $\mathbb{G}_{d+1,0,1}$ be a d-dimensional PGA with $(e_0)^2 = 0$ and $(e_i)^2 = 1$.

It turns out, in 2D and 3D, we can simplify implementations by swapping two basis vectors in some blades such that $\overline{X_i}$ does not flip signs, e.g.

$$\begin{aligned} A &= A_1 + A_2 e_0 + A_3 e_1 + A_4 e_2 \\ &\quad + A_5 e_{01} + A_6 e_{20} + A_7 e_{12} + A_8 e_{012} \\ \overline{A} &= A_8 + A_7 e_0 + A_6 e_1 + A_5 e_2 \\ &\quad + A_4 e_{01} + A_3 e_{20} + A_2 e_{12} + A_1 e_{012} \end{aligned}$$

Points in PGA are represented by (n-1)-vectors:

$$\overline{e_0 + xe_1 + ye_2 + ...}$$

This is a kind of a lie, you can make an equivalent dual algebra by letting points be vectors, in fact many operations make use of computing in this dual algebra via $\overline{\overline{A}} \text{ op } \overline{\overline{B}}$

Then we can start defining operators:

$$A \cdot B = \sum_{j,k} \left\langle \langle A \rangle_j \langle B \rangle_k \right\rangle_0 \texttt{ // dot product}$$

$$\langle A \rangle_k = \sum_i \langle X_i \rangle_k$$

$$\langle X_i \rangle_k = \begin{cases} X_i & (X_i \in \text{k-vector}) \\ 0 & (X_i \notin \text{k-vector}) \end{cases}$$

$$A \wedge B = \sum_{j,k} \left\langle \langle A \rangle_j \langle B \rangle_k \right\rangle_{j+k} \texttt{ // wedge product}$$

In other words it's equal to the geometric product if the grades add together: j-vector $\wedge$ k-vector = (j+k)-vector, otherwise it's 0

$$A \vee B = \overline{\underline{\overline{A}} \wedge \underline{\overline{B}}} \quad \texttt{// antiwedge product}$$

Since all of these are based of the geometric product they retain distributivity and associativity

The antiwedge product allows you to join points into lines, planes, ...

$$line = point_1 \vee point_2$$
$$plane = point_1 \vee point_2 \vee point_3$$

In fact this works with any two geometric objects, operators that don't have this property aren't really worth your time.

The wedge product allows you to meet two objects:

$$point = line_1 \wedge line_2$$
$$line = plane_1 \wedge plane_2$$

If you meet two parallel lines, you get an infinite point = a point at infinity:

$$line_1 \wedge line_1 = \overline{xe_1 + ye_2 + ...}$$

All objects in GA have an orientation, so if you flip the direction of one of the lines, you would get an infinite point in the other direction

TODO: something about sizes of line segments / volumes

TODO: something about projections

## V. Motors