Maths

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Contents

Chapter 1

Derivatives

The derivative is the rate of change of a function at a specific point:

$$
\begin{aligned}
\text{When } \overbrace{\frac{d}{dx}f(x)}^{\text{the derivative of f(x)}} > 0,\ f(x) \text{ is increasing} \\
\text{When } \frac{d}{dx}f(x) < 0,\ f(x) \text{ is decreasing} \\
\text{When } \frac{d}{dx}f(x) = 0,\ f(x) \text{ is constant}
\end{aligned}
$$

1.1 Notation

If x is a variable, then dx is a small change in x.

dx approaches 0, but is not equal to 0 (because we divide by dx later).

Instead we ask what happens as dx gets closer and closer to 0 :

$$dx \to 0$$

If y is a function of x, then dy and $df(x)$ are the change caused by dx :

$$y = f(x)$$

$$\Rightarrow dy = df(x)$$

$$df(x) = f(x + dx) - f(x)$$

The derivative of $f(x)$ is the ratio between $df(x)$ and dx :

$$\frac{d}{dx}f(x) = \frac{df(x)}{dx}$$

Putting all of these ideas together gives us the equation for a derivative:

$$\left.\begin{array}{l} f(x)' = \frac{d}{dx}f(x) = \frac{df(x)}{dx} \\ df(x) = f(x + dx) - f(x) \\ dx \to 0 \ (dx \neq 0) \end{array}\right\} \frac{d}{dx}f(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

We can also compute $df(x)$:

$$\begin{aligned}
\frac{d}{dx}f(x) &= \frac{df(x)}{dx} \\
df(x) &= \frac{d}{dx}f(x) * dx \\
df(x) &= \lim_{h\to 0} f(x+h) - f(x) \\
df(x) &= f(x+dx) - f(x)
\end{aligned}$$

1.2 Elementary derivatives

1.2.1 By $\frac{d}{dx}f(x)$:

$$\begin{aligned}
a &\in R \\
f(x) &= a \\
\frac{d}{dx}f(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
\frac{d}{dx}a &= \lim_{h\to 0} \frac{a-a}{h} \\
\frac{d}{dx}a &= \lim_{h\to 0} \frac{0}{h} \\
\frac{d}{dx}a &= 0 \ \because \ h \neq 0
\end{aligned}$$

1.2.2 By $df(x)$:

$$
\begin{aligned}
d(x^a) &= (x+dx)^a - x^a \\
&= x^a + \left.\begin{matrix} \overbrace{+(dx+x+x+x+\ldots)}^{x^{a-1}dx} \\ +(x+dx+x+x+\ldots) \\ +(x+x+dx+x+\ldots) \\ +(x+x+x+dx+\ldots) \\ \vdots \end{matrix}\right\} \text{a times} + (\ldots)dx^2 - x^a \\
d(x^a) &= ax^{a-1}dx + (\ldots)dx^2 \\
\frac{d(x^a)}{dx} &= ax^{a-1} + (\ldots)dx \ \because\ dx \neq 0 \\
\frac{d}{dx}x^a &= ax^{a-1} \ \because\ dx \to 0 \\
&\Rightarrow \frac{d}{dx}x = 1
\end{aligned}
$$

1.2.3 By definition :

$$\frac{d}{dx}e^x = e^x$$

1.2.4 By implicit differentiation :

If you have a function of one variable, then take the change in that function with respect to that variable.

$$y = \ln x$$
$$e^y = x$$
$$d(e^y) = d(x)$$
$$e^y * dy = 1 * dx$$
$$\frac{dy}{dx} = \frac{1}{e^y}$$
$$\frac{d}{dx}y = \frac{1}{x}$$
$$\frac{d}{dx}\ln x = \frac{1}{x}$$

If you have a function of many variables, apply section 1.3 Rules to break it down into functions of one variable.

1.2.5 By fancy geometry :

see https://lbry.tv/@3Blue1Brown:b/derivative-formulas-through-geometry

$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d}{dx}\cos x = -\sin x$$

Plus other trigonometric functions which you probably don't need.

1.2.6 Bonus section :

Come back after reading section 1.3 Rules.

$$\frac{d}{dx}e^{-x} = e^{-x} * (-1)$$
$$\frac{d}{dx}e^{-x} = -e^{-x}$$

Note that $\ln(a^x)$ is the same as $x \ln a$.

$$a^x = e^{\ln a^x}$$
$$a^x = e^{x \ln a}$$
$$\frac{d}{dx}a^x = e^{x \ln a} * \ln a$$
$$\frac{d}{dx}a^x = a^x \ln a$$

$$
\begin{aligned}
y &= \log_a x \\
a^y &= x \\
d(a^y) &= d(x) \\
a^y \ln a * dy &= 1 * dx \\
\frac{dy}{dx} &= \frac{1}{a^y \ln a} \\
\frac{d}{dx} y &= \frac{1}{a^{\log_a x} \ln a} \\
\frac{d}{dx} \log_a x &= \frac{1}{x \ln a}
\end{aligned}
$$

$$
\begin{aligned}
f(x) &= x^x \\
\ln f(x) &= \ln(x^x) \\
\ln f(x) &= x \ln x \\
f(x) &= e^{x \ln x} \\
\frac{d}{dx} x^x &= e^{x \ln x} * (1 * \ln x \ + \ x * 1/x) \\
\frac{d}{dx} x^x &= e^{x \ln x} * (\ln x + 1)
\end{aligned}
$$

1.3 Rules

1.3.1 Sum rule

see https://lbry.tv/@3Blue1Brown:b/derivative-formulas-through-geometry

$$\frac{d}{dx}(g(x) + h(x)) = \frac{d}{dx}g(x) \ + \ \frac{d}{dx}h(x)$$

$$\frac{d}{dx}(\sin x + x^2) = \frac{d}{dx}\sin x \ + \ \frac{d}{dx}x^2 = \cos x + 2x$$
$$\frac{d}{dx}(x + 1) = 1 + 0 = 1$$
$$\frac{d}{dx}(x^3 + e^x) = 3x^2 + e^x$$

1.3.2 Product rule

see https://lbry.tv/@3Blue1Brown:b/visualizing-the-chain-rule-and-product

$$\begin{aligned} d(g(x)*h(x)) &= dg(x)*h(x) \;+\; g(x)*dh(x) \;+\; dg(x)*dh(x) \\ \frac{d(g(x)*h(x))}{dx} &= \frac{dg(x)}{dx}*h(x) \;+\; g(x)*\frac{dh(x)}{dx} \;+\; \frac{dg(x)}{dx}*dh(x) \\ \frac{d}{dx}(g(x)*h(x)) &= \frac{d}{dx}g(x)*h(x) \;+\; g(x)*\frac{d}{dx}h(x) \;\because\; dh(x)\to 0 \end{aligned}$$

$$\begin{aligned} \frac{d}{dx}(\sin x*x^2) &= \frac{d}{dx}\sin x*x^2 \;+\; \sin x*\frac{d}{dx}x^2 = \cos x*x^2 \;+\; \sin x*2x \\ \frac{d}{dx}(2*x) &= 0*x \;+\; 2*1 = 2 \\ \frac{d}{dx}\frac{\sin x}{x} &= \frac{d}{dx}(\sin x*1/x) = \frac{\cos x}{x} - \frac{\sin x}{x^2} \end{aligned}$$

This also leads to a formula for division :

$$\frac{d}{dx}\frac{g(x)}{h(x)} = \frac{\frac{d}{dx}g(x)*h(x) \;-\; g(x)*\frac{d}{dx}h(x)}{h(x)^2}$$

1.3.3 Derivative with respect to $f(x)$:

You simply treat $f(x)$ as if it was x :

$$\frac{d}{d(x^2)}\sin(x^2) = \cos(x^2)$$
$$\frac{d}{d(\sin x)}(\sin x)^3 = 3(\sin x)^2$$

1.3.4 Chain rule

$$\frac{d}{dx}g(h(x)) = \frac{dg(h(x))}{dx}$$
$$\frac{d}{dx}g(h(x)) = \frac{dg(h(x))}{dh(x)} \ * \ \frac{dh(x)}{dx}$$
$$\frac{d}{dx}g(h(x)) = \frac{d}{dh(x)}g(h(x)) \ * \ \frac{d}{dx}h(x)$$

$$\frac{d}{dx}\sin(x^2) = \cos(x^2) * 2x$$
$$\frac{d}{dx}\frac{1}{\sin x} = \frac{d}{dx}(\sin x)^{-1} = -1 * (\sin x)^{-2} \ * \ \cos x = -\frac{\cos x}{(\sin x)^2}$$

1.3.5 L'Hopital's rule

$$\lim_{x\to a}\frac{g(x)}{h(x)} = \lim_{x\to a}\frac{dg(x)}{dh(x)}$$

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\cos x * dx}{1 * dx}$$
$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\cos x}{1} \quad \because \; dx \neq 0$$
$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

1.3.6 Derivative of inverse function

$g(x)$ being the inverse of $f(x)$:

$$g(f(x)) = x$$
$$\frac{d}{dx}f(x) * \frac{d}{dx}g(x) = 1$$
$$\frac{d}{dx}g(x) = \frac{1}{\frac{d}{dx}f(x)}$$