

Maths

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Chapter 1

Derivatives

The derivative is the rate of change of a function at a specific point:

the derivative of $f(x)$
 $\overbrace{\frac{d}{dx}}$

When $\frac{d}{dx}f(x) > 0$, $f(x)$ is increasing

When $\frac{d}{dx}f(x) < 0$, $f(x)$ is decreasing

When $\frac{d}{dx}f(x) = 0$, $f(x)$ is constant

1.1 Notation

If x is a variable, then dx is a small change in x .

dx approaches 0, but is not equal to 0 (because we divide by dx later).

Instead we ask what happens as dx gets closer and closer to 0 :

$$dx \rightarrow 0$$

If y is a function of x , then dy and $df(x)$ are the change caused by dx :

$$y = f(x)$$

$$\Rightarrow dy = df(x)$$

$$df(x) = f(x + dx) - f(x)$$

The derivative of $f(x)$ is the ratio between $df(x)$ and dx :

$$\frac{d}{dx}f(x) = \frac{df(x)}{dx}$$

Putting all of these ideas together gives us the equation for a derivative:

$$\left. \begin{array}{l} f(x)' = \frac{d}{dx}f(x) = \frac{df(x)}{dx} \\ df(x) = f(x + dx) - f(x) \\ dx \rightarrow 0 \ (dx \neq 0) \end{array} \right\} \frac{d}{dx}f(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

We can also compute $df(x)$:

$$\begin{aligned}\frac{d}{dx}f(x) &= \frac{df(x)}{dx} \\ df(x) &= \frac{d}{dx}f(x) * dx \\ df(x) &= \lim_{h \rightarrow 0} f(x+h) - f(x) \\ df(x) &= f(x+dx) - f(x)\end{aligned}$$

1.2 Elementary derivatives

1.2.1 By $\frac{d}{dx}f(x)$:

$$a \in R$$

$$f(x) = a$$

$$\frac{d}{dx}f(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{d}{dx}a = \lim_{h \rightarrow 0} \frac{a - a}{h}$$

$$\frac{d}{dx}a = \lim_{h \rightarrow 0} \frac{0}{h}$$

$$\frac{d}{dx}a = 0 \quad \because \quad h \neq 0$$

1.2.2 By $df(x)$:

$$\begin{aligned}
 d(x^a) &= (x + dx)^a - x^a \\
 &= x^a + \overbrace{\left. \begin{aligned} &+(dx + x + x + x + \dots) \\ &+(x + dx + x + x + \dots) \\ &+(x + x + dx + x + \dots) \\ &+(x + x + x + dx + \dots) \\ &\vdots \end{aligned} \right\}}^{x^{a-1}dx} \text{ a times } + (\dots)dx^2 - x^a
 \end{aligned}$$

$$\begin{aligned}
 d(x^a) &= ax^{a-1}dx + (\dots)dx^2 \\
 \frac{d(x^a)}{dx} &= ax^{a-1} + (\dots)dx \quad \because dx \neq 0 \\
 \frac{d}{dx}x^a &= ax^{a-1} \quad \because dx \rightarrow 0 \\
 &\Rightarrow \frac{d}{dx}x = 1
 \end{aligned}$$

1.2.3 By definition :

$$\frac{d}{dx}e^x = e^x$$

1.2.4 By implicit differentiation :

If you have a function of one variable, then take the change in that function with respect to that variable.

$$y = \ln x$$

$$e^y = x$$

$$d(e^y) = d(x)$$

$$e^y * dy = 1 * dx$$

$$\frac{dy}{dx} = \frac{1}{e^y}$$

$$\frac{d}{dx} y = \frac{1}{x}$$

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

If you have a function of many variables, apply [section 1.3 Rules](#) to break it down into functions of one variable.

1.2.5 By fancy geometry :

see <https://lbry.tv/@3Blue1Brown:b/derivative-formulas-through-geometry>

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

Plus other [trigonometric functions](#) which you probably don't need.

1.2.6 Bonus section :

Come back after reading [section 1.3 Rules](#).

$$\begin{aligned}\frac{d}{dx}e^{-x} &= e^{-x} * (-1) \\ \frac{d}{dx}e^{-x} &= -e^{-x}\end{aligned}$$

Note that $\ln(a^x)$ is the same as $x \ln a$.

$$\begin{aligned}a^x &= e^{\ln a^x} \\ a^x &= e^{x \ln a} \\ \frac{d}{dx}a^x &= e^{x \ln a} * \ln a \\ \frac{d}{dx}a^x &= a^x \ln a\end{aligned}$$

$$y = \log_a x$$

$$a^y = x$$

$$d(a^y) = d(x)$$

$$a^y \ln a * dy = 1 * dx$$

$$\frac{dy}{dx} = \frac{1}{a^y \ln a}$$

$$\frac{d}{dx} y = \frac{1}{a^{\log_a x} \ln a}$$

$$\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$$

$$f(x) = x^x$$

$$\ln f(x) = \ln(x^x)$$

$$\ln f(x) = x \ln x$$

$$f(x) = e^{x \ln x}$$

$$\frac{d}{dx} x^x = e^{x \ln x} * (1 * \ln x + x * 1/x)$$

$$\frac{d}{dx} x^x = x^x * (\ln x + 1)$$

1.3 Rules

1.3.1 Sum rule

see <https://lbry.tv/@3Blue1Brown:b/derivative-formulas-through-geometry>

$$\frac{d}{dx}(g(x) + h(x)) = \frac{d}{dx}g(x) + \frac{d}{dx}h(x)$$

$$\frac{d}{dx}(\sin x + x^2) = \frac{d}{dx}\sin x + \frac{d}{dx}x^2 = \cos x + 2x$$

$$\frac{d}{dx}(x + 1) = 1 + 0 = 1$$

$$\frac{d}{dx}(x^3 + e^x) = 3x^2 + e^x$$

1.3.2 Product rule

see <https://lbry.tv/@3Blue1Brown:b/visualizing-the-chain-rule-and-product>

$$\begin{aligned}
 d(g(x) * h(x)) &= dg(x) * h(x) + g(x) * dh(x) + dg(x) * dh(x) \\
 \frac{d(g(x) * h(x))}{dx} &= \frac{dg(x)}{dx} * h(x) + g(x) * \frac{dh(x)}{dx} + \frac{dg(x)}{dx} * dh(x) \\
 \frac{d}{dx}(g(x) * h(x)) &= \frac{d}{dx}g(x) * h(x) + g(x) * \frac{d}{dx}h(x) \quad \because dh(x) \rightarrow 0
 \end{aligned}$$

$$\begin{aligned}
 \frac{d}{dx}(\sin x * x^2) &= \frac{d}{dx} \sin x * x^2 + \sin x * \frac{d}{dx}x^2 = \cos x * x^2 + \sin x * 2x \\
 \frac{d}{dx}(2 * x) &= 0 * x + 2 * 1 = 2 \\
 \frac{d}{dx} \frac{\sin x}{x} &= \frac{d}{dx}(\sin x * 1/x) = \frac{\cos x}{x} - \frac{\sin x}{x^2}
 \end{aligned}$$

This also leads to a formula for division :

$$\frac{\frac{d}{dx} g(x)}{h(x)} = \frac{\frac{d}{dx}g(x) * h(x) - g(x) * \frac{d}{dx}h(x)}{h(x)^2}$$

1.3.3 Derivative with respect to $f(x)$:

You simply treat $f(x)$ as if it was x :

$$\begin{aligned}\frac{d}{d(x^2)} \sin(x^2) &= \cos(x^2) \\ \frac{d}{d(\sin x)} (\sin x)^3 &= 3(\sin x)^2\end{aligned}$$

1.3.4 Chain rule

$$\begin{aligned}\frac{d}{dx} g(h(x)) &= \frac{dg(h(x))}{dx} \\ \frac{d}{dx} g(h(x)) &= \frac{dg(h(x))}{dh(x)} * \frac{dh(x)}{dx} \\ \frac{d}{dx} g(h(x)) &= \frac{d}{dh(x)} g(h(x)) * \frac{d}{dx} h(x)\end{aligned}$$

$$\begin{aligned}\frac{d}{dx} \sin(x^2) &= \cos(x^2) * 2x \\ \frac{d}{dx} \frac{1}{\sin x} &= \frac{d}{dx} (\sin x)^{-1} = -1 * (\sin x)^{-2} * \cos x = -\frac{\cos x}{(\sin x)^2}\end{aligned}$$

1.3.5 L'Hopital's rule

$$\lim_{x \rightarrow a} \frac{g(x)}{h(x)} = \lim_{x \rightarrow a} \frac{dg(x)}{dh(x)}$$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin x}{x} &= \lim_{x \rightarrow 0} \frac{\cos x * dx}{1 * dx} \\ \lim_{x \rightarrow 0} \frac{\sin x}{x} &= \lim_{x \rightarrow 0} \frac{\cos x}{1} \quad \because dx \neq 0 \\ \lim_{x \rightarrow 0} \frac{\sin x}{x} &= 1 \end{aligned}$$

1.3.6 Derivative of inverse function

$g(x)$ being the inverse of $f(x)$:

$$\begin{aligned} g(f(x)) &= x \\ \frac{d}{dx} f(x) * \frac{d}{dx} g(x) &= 1 \\ \frac{d}{dx} g(x) &= \frac{1}{\frac{d}{dx} f(x)} \end{aligned}$$