

A summary of Projective Geometric Algebra

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Abstract

I. GEOMETRIC NUMBERS

$$1 + x^2 = 0 \\ x = ?$$

x is not a real number; but if it's not real, why should the other numbers be real?

$$(e_1)^2 + (e_2)^2 = (e_0)^2 \\ (e_1)^2 = 1; (e_2)^2 = -1; (e_0)^2 = 0$$

In fact, we can define as many of these as we want, the simplest examples being:

$$a + be_1 \text{ // hyperbolic numbers} \\ a + be_2 \text{ // complex numbers} \\ a + be_0 \text{ // dual numbers}$$

We can multiply these numbers together using the geometric product:

$$e_i e_i = \{1, -1, 0\} \\ e_i e_j = -e_j e_i$$

This product is neither commutative nor anticommutative, but it is distributive and associative:

$$AB \neq BA \\ AB \neq -BA \\ A(B + C) = AB + AC \\ (AB)C = A(BC) \\ aB = Ba; a \in \mathbb{R}$$

Thus the product of two complex numbers:

$$(A_1 + A_2 e_2)(B_1 + B_2 e_2) \\ = A_1 B_1 + A_1 B_2 e_2 + A_2 e_2 B_1 + A_2 e_2 B_2 e_2 \\ = A_1 B_1 + A_1 B_2 e_2 + A_2 B_1 e_2 + A_2 B_2 \\ = (A_1 B_1 + A_2 B_2) + (A_1 B_2 + A_2 B_1) e_2$$

II. ROTATIONS

A multivector with n basis vectors consists of 2^n blades:

- scalar = 0-vector = 1
- vector = 1-vector
- bivector = 2-vector
- trivector = 3-vector
- ...
- (n-1)-vector = pseudovector
- n-vector = pseudoscalar = 1

Where a k-vector has $\binom{n}{k}$ blades, for example:

$$A = A_1 \\ + A_2 e_0 + A_3 e_1 + A_4 e_2 \\ + A_5 e_{01} + A_6 e_{02} + A_7 e_{12} \\ + A_8 e_{012}$$

We can abbreviate blades like $e_1 e_2$ as e_{12} .

Multiplying two multivectors gives you another multivector, we can use the Taylor series expansion of the exponential function to find a rotation e^A :

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$e^{ae_2} = 1 + ae_2 - \frac{a^2}{2} - \frac{a^3}{3}e_2 + \dots$$

// by the sum being convergent

$$= (1 - \frac{a^2}{2} + \dots) + (a - \frac{a^3}{3} + \dots)e_2$$

$$= \cos(a) + \sin(a)e_2$$

Similarly you can find

$$e^{e_i} = \begin{cases} \cos(a) + \sin(a)e_i & ((e_i)^2 = -1) \\ \cosh(a) + \sinh(a)e_i & ((e_i)^2 = 1) \\ 1 + e_i & ((e_i)^2 = 0) \end{cases}$$

This gives us rotations, hyperbolic rotations and translations (rotations through infinity) respectively.

Then for a multivector we would have:

$$e^A = e^{A_1} e^{A_2 e_0} e^{A_3 e_1} \dots e^{A_n e_I}$$

III. DUALITY

For a blade X in a k -vector we can define some operations, like reversing the order of basis vectors in the blade, that just amount to some sign flips:

$$\tilde{X} = (-1)^{\lfloor k/2 \rfloor} X \text{ // reverse}$$

$$X^\dagger = (-1)^{\lfloor k \rfloor} X \text{ // involute}$$

$$\bar{X} = (-1)^{\lfloor k+k/2 \rfloor} X \text{ // conjugate}$$

$$\text{complement}(A) = \sum \text{complement}(X)$$

Poincaré duality states that maps between k -vectors and $(n-k)$ -vectors exist.

$$X \text{ dual}(X) = \pm 1$$

For example:

$$\underline{X}X = 1 \text{ // left complement}$$

$$X\bar{X} = 1 \text{ // right complement}$$

$$XX^\star = \text{sign}(X_{ND} \widetilde{X_{ND}}) 1 \text{ // hodge dual}$$

Where X_{ND} is X without degenerate basis vectors, e.g.

$$X = e_{012}; X_{ND} = e_{12}$$

Let $\mathbb{G}_{a,b,c}$ be a geometric algebra with a positive, b negative and c zero basis vectors.

Then for $\mathbb{G}_{a,0,c}$:

$$\overline{X} = X^\star$$

And if all that wasn't confusing enough, applying a dual twice changes the signs, so we also want the inverses of these duals:

$$(X^\star)^{\star^{-1}} = X$$

$$\overline{\overline{X}} = X$$

$$\underline{\underline{X}} = X$$

IV. SHAPES AND SIZES

TODO