A summary of Projective Geometric Algebra

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Abstract

Information on Geometric Algebra is scattered and scarce. This is an explanation of how to calculate PGA, but not necessarily why it works - for that you can try sudgylacmoe on youtube. I would also point you to bivector.net, but their dual calculations are whack, the ganja.js ones are presumably correct, since their demos seem to work.

I. Geometric numbers

$$\begin{aligned} 1 + x^2 &= 0 \\ x &= ? \end{aligned}$$

x is not a real number; but if it's not real, why should the other numbers be real?

$$(e_1)^2 + (e_2)^2 = (e_0)^2$$
$$(e_1)^2 = 1;\ (e_2)^2 = -1;\ (e_0)^2 = 0$$

In fact, we can define as many of these as we want, the simplest examples being:

$a + be_1$ `// hyperbolic numbers`

$a + be_2$ `// complex numbers`

$a + be_0$ `// dual numbers`

We can multiply these numbers together using the geometric product:

$$e_i e_i = \{1, -1, 0\}$$
$$e_i e_j = -e_j e_i$$

This product is neither commutative nor anticommutative, but it is distributive and associative:

$$AB \neq BA$$
$$AB \neq -BA$$
$$A(B + C) = AB + AC$$
$$(AB)C = A(BC)$$
$$aB = Ba;\ a \in \mathbb{R}$$

Thus the product of two complex numbers:

$$\begin{aligned} &(A_1 + A_2e_2)(B_1 + B_2e_2) \\ &= A_1B_1 + A_1B_2e_2 + A_2e_2B_1 + A_2e_2B_2e_2 \\ &= A_1B_1 + A_1B_2e_2 + A_2B_1e_2 + A_2B_2 \\ &= (A_1B_1 + A_2B_2) + (A_1B_2 + A_2B_1)e_2 \end{aligned}$$

II. Rotations

A multivector with n basis vectors consists of 2^n blades:

```
- scalar = 0-vector = 1
- vector = 1-vector
- bivector = 2-vector
- trivector = 3-vector
- ...
- (n-1)-vector = pseudovector
- n-vector = pseudoscalar = 𝟙
```

Where a k-vector has $\binom{n}{k}$ blades, for example:

$$\begin{aligned} A = A_1 &\\ &+ A_2e_0 + A_3e_1 + A_4e_2 \\ &+ A_5e_{01} + A_6e_{02} + A_7e_{12} \\ &+ A_8e_{012} \end{aligned}$$

We can abbreviate blades like e_1e_2 as e_{12}.

Multiplying two multivectors gives you another multivector, we can use the taylor series expansion of the exponential function to find a rotation e^A:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\begin{aligned} e^{ae_2} &= 1 + ae_2 - \frac{a^2}{2} - \frac{a^3}{3}e_2 + ... \\ &= (1 - \frac{a^2}{2} + ...) + (a - \frac{a^3}{3} + ...)e_2 \\ &= cos(a) + sin(a)e_2 \end{aligned}$$

Similarly you can find

$$e^{e_i} = \begin{cases} cos(a) + sin(a)e_i & ((e_i)^2 = -1) \\ cosh(a) + sinh(a)e_i & ((e_i)^2 = 1) \\ 1 + e_i & ((e_i)^2 = 0) \end{cases}$$

This gives us rotations, hyperbolic rotations and translations (rotations through infinity) respectively.

Then for a multivector we would have:

$$e^A = e^{A_1} e^{A_2 e_0} e^{A_3 e_1} ... e^{A_n e_I}$$

III. Unary operators

For the i-th blade in a multivector

$$X_i \in \text{k-vector}$$

we can define some operations, like reversing the order of basis vectors in the blade, that amount to flipping some signs:

$$\begin{aligned} \tilde{X}_i &= (-1)^{\lfloor k/2 \rfloor} X_i \ \texttt{// reverse} \\ X_i^{\dagger} &= (-1)^{\lfloor k \rfloor} X_i \ \texttt{// involute} \\ \bar{X}_i &= (-1)^{\lfloor k + k/2 \rfloor} X_i \ \texttt{// conjugate} \\ f(A) &= \sum_i f(X_i) \end{aligned}$$

Poincaré duality states that maps between k-vectors and (n-k)-vectors exist.

$$\begin{aligned} X_i \, dual(X_i) &= \pm \mathbb{1} \\ dual(X_i) &= \pm X_{2^n - i + 1} \\ dual(A) &= \sum_i dual(X_i) \end{aligned}$$

For example:

$$\begin{aligned} \underline{X_i} X_i &= \mathbb{1} \ \texttt{// left complement} \\ X_i \overline{X_i} &= \mathbb{1} \ \texttt{// right complement} \\ X_i X_i^{\star} &= sign(X_i^{ND} \widetilde{X_i^{ND}}) \mathbb{1} \ \texttt{// hodge dual} \end{aligned}$$

Where X_i^{ND} is X_i without degenerate basis vectors, e.g.

$$X_i = e_{012};\ X_i^{ND} = e_{12}$$

Let $\mathbb{G}_{a,b,c}$ be a geometric algebra with a positive, b negative and c zero basis vectors.

Then for $\mathbb{G}_{a,0,c}$:

$$\overline{X_i} = X_i^{\star}$$

And if that wasn't confusing enough, applying a dual twice changes the signs, so we also want the inverses of these duals:

$$\begin{aligned} (X_i^{\star})^{\star^{-1}} &= X_i \\ \overset{-\,-}{\overline{X_i}} &= X_i \\ \underset{-\,-}{\underline{X_i}} &= X_i \end{aligned}$$

In 2D and 3D PGA, we can simplify implementation by swapping two basis vectors in some blades such that $\overline{X_i}$ does not flip signs, e.g.

$$\begin{aligned} A &= A_1 + A_2 e_0 + A_3 e_1 + A_4 e_2 \\ &\quad + A_5 e_{01} + A_6 e_{20} + A_7 e_{12} + A_8 e_{012} \\ \overset{-\,-}{A} = \overline{A} &= A_8 + A_7 e_0 + A_6 e_1 + A_5 e_2 \\ &\quad + A_4 e_{01} + A_3 e_{20} + A_2 e_{12} + A_1 e_{012} \end{aligned}$$

IV. Shapes and sizes

Let $\mathbb{G}_{d,0,1}$ be a d-dimensional PGA:

$$(e_0)^2 = 0;\ (e_i)^2 = 1$$

Now we have a choice to make:

```
1. Point-based PGA
     - vectors are points
     - (n-1)-vectors are hyperplanes
2. Plane-based PGA
     - vectors are hyperplanes
     - (n-1)-vectors are points
```

Both of these are equally valid and many operations make use of computing in the dual algebra via $\overline{\overline{A} \text{ op } \overline{B}}$.

$$point = e_0 + xe_1 + ye_2 + ... + we_d$$
$$hyperplane = \overline{e_0 + xe_1 + ye_2 + ... + we_d}$$

$$\boxed{\begin{aligned} hyperplane &= e_0 + xe_1 + ye_2 + ... + we_d \\ point &= \overline{e_0 + xe_1 + ye_2 + ... + we_d} \end{aligned}}$$

We will denote Plane-based operations inside boxes whenever they differ.

Let $\langle A \rangle_k$ be the grade selection operator:

$$\langle A \rangle_k = \sum_i \langle X_i \rangle_k$$
$$\langle X_i \rangle_k = \begin{cases} X_i & (X_i \in \text{k-vector}) \\ 0 & (X_i \notin \text{k-vector}) \end{cases}$$

Then we can start defining binary operators:

$$A \wedge B = \sum_{j,k} \left\langle \langle A \rangle_j \langle B \rangle_k \right\rangle_{j+k} \texttt{ // wedge product}$$
$$A \vee B = \overline{\overline{A} \wedge \overline{B}} \texttt{ // antiwedge product}$$

These operators retain distributivity and associativity.

We can join points into lines, planes, ...

$$line = point_1 \mathit{\,join\,} point_2$$
$$plane = point_1 \mathit{\,join\,} point_2 \mathit{\,join\,} point_3$$

$$A \mathit{\,join\,} B = A \wedge B$$

$$\boxed{A \mathit{\,join\,} B = A \vee B}$$

In fact this works with any two geometric objects, operators that don't have this property aren't really worth your time.

And we can meet two objects:

$$point = line_1 \mathit{\,meet\,} line_2$$
$$point = plane \mathit{\,meet\,} line$$
$$line = plane_1 \mathit{\,meet\,} plane_2$$

$$A \mathit{\,meet\,} B = A \vee B$$

$$\boxed{A \mathit{\,meet\,} B = A \wedge B}$$

If you meet two parallel lines, you get a point at infinity = an infinite point:

$$line_1 \mathit{\,meet\,} line_1 = xe_1 + ye_2 + ... + we_d$$

$$\boxed{line_1 \mathit{\,meet\,} line_1 = \overline{xe_1 + ye_2 + ... + we_d}}$$

All objects in GA have an orientation, so if you flip the direction of one of the lines, you get an infinite point in the other direction

TODO: something about sizes of line segments / volumes

TODO: projection to camera plane + depth buffer

V. Motors

TODO: something about bivector blades being rotations

$$rotor = e^{bivector^{ND}}$$
$$motor = e^{bivector}$$

In 3D, rotors are quaternions.

TODO: applying motors and line forces per https://enki.ws/ganja.js/examples/coffeeshop.html#sUwbwu9vR