A summary of Projective Geometric Algebra

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Abstract

Information on Geometric Algebra is scattered and scarce. This is an explanation of how to calculate PGA, but not necessarily why it works - for that you can try sudgylacmoe on youtube and Eric Lengyel's cheatsheet for point-based PGA (though he goes mad with power). I would also point you to bivector.net, but their dual calculations are whack, the ganja.js ones are presumably correct, since their demos seem to work.

I. Geometric numbers

$$1 + x^2 = 0$$
$$x = ?$$

x is not a real number; but if it's not real, why should the other numbers be real?

$$(e_1)^2 + (e_2)^2 = (e_0)^2$$
$$(e_1)^2 = 1;\ (e_2)^2 = -1;\ (e_0)^2 = 0$$

In fact, we can define as many of these as we want, the simplest examples being:

$a + be_1$ `// hyperbolic numbers`

$a + be_2$ `// complex numbers`

$a + be_0$ `// dual numbers`

We can multiply these numbers together using the geometric product:

$$e_i e_i = \{1, -1, 0\}$$
$$e_i e_j = -e_j e_i$$

This product is neither commutative nor anticommutative, but it is distributive and associative:

$$AB \neq BA$$
$$AB \neq -BA$$
$$A(B + C) = AB + AC$$
$$(AB)C = A(BC)$$
$$aB = Ba;\ a \in \mathbb{R}$$

Thus the product of two complex numbers:

$$
\begin{aligned}
&(A_1 + A_2 e_2)(B_1 + B_2 e_2) \\
&= A_1 B_1 + A_1 B_2 e_2 + A_2 e_2 B_1 + A_2 e_2 B_2 e_2 \\
&= A_1 B_1 + A_1 B_2 e_2 + A_2 B_1 e_2 + A_2 B_2 \\
&= (A_1 B_1 + A_2 B_2) + (A_1 B_2 + A_2 B_1) e_2
\end{aligned}
$$

II. Rotations

A multivector with n basis vectors consists of 2^n blades:

```
- scalar = 0-vector = 1
- vector = 1-vector
- bivector = 2-vector
- trivector = 3-vector
- ...
- (n-1)-vector = pseudovector
- n-vector = pseudoscalar = 𝟙
```

Where a k-vector has $\binom{n}{k}$ blades, for example:

$$
\begin{aligned}
A = A_1 & \\
& + A_2 e_0 + A_3 e_1 + A_4 e_2 \\
& + A_5 e_{01} + A_6 e_{02} + A_7 e_{12} \\
& + A_8 e_{012}
\end{aligned}
$$

We can abbreviate blades like $e_1 e_2$ as e_{12}.

Multiplying two multivectors gives you another multivector, we can use the taylor series expansion of the exponential function to find a rotation e^A:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$
\begin{aligned}
e^{ae_2} &= 1 + ae_2 - \frac{a^2}{2} - \frac{a^3}{3} e_2 + ... \\
&= (1 - \frac{a^2}{2} + ...) + (a - \frac{a^3}{3} + ...) e_2 \\
&= cos(a) + sin(a) e_2
\end{aligned}
$$

Similarly you can find

$$
e^{e_i} = \begin{cases} cos(a) + sin(a) e_i & ((e_i)^2 = -1) \\ cosh(a) + sinh(a) e_i & ((e_i)^2 = 1) \\ 1 + e_i & ((e_i)^2 = 0) \end{cases}
$$

This gives us rotations, hyperbolic rotations and translations (rotations through infinity) respectively.

Then for a multivector we would have:

$$e^A = e^{A_1} e^{A_2 e_0} e^{A_3 e_1} ... e^{A_n e_I}$$

III. Unary operators

For the i-th blade in a multivector

$$X_i \in \text{k-vector}$$

we can define some operations, like reversing the order of basis vectors in the blade, that amount to flipping some signs:

$$
\begin{aligned}
\tilde{X}_i &= (-1)^{\lfloor k/2 \rfloor} X_i \quad \texttt{// reverse} \\
X_i^{\dagger} &= (-1)^{\lfloor k \rfloor} X_i \quad \texttt{// involute} \\
\bar{X}_i &= (-1)^{\lfloor k + k/2 \rfloor} X_i \quad \texttt{// conjugate} \\
f(A) &= \sum_i f(X_i)
\end{aligned}
$$

Poincaré duality states that maps between k-vectors and (n-k)-vectors exist.

$$
\begin{aligned}
X_i \, dual(X_i) &= \pm \mathbb{1} \\
dual(X_i) &= \pm X_{2^n - i + 1} \\
dual(A) &= \sum_i dual(X_i)
\end{aligned}
$$

For example:

$$
\begin{aligned}
\underline{X_i} X_i &= \mathbb{1} \quad \texttt{// left complement} \\
X_i \overline{X_i} &= \mathbb{1} \quad \texttt{// right complement} \\
X_i X_i^{\star} &= sign(X_i^{ND} \widetilde{X_i^{ND}}) \, \mathbb{1} \quad \texttt{// hodge dual}
\end{aligned}
$$

Where X_i^{ND} is X_i without degenerate basis vectors, e.g.

$$X_i = e_{012};\; X_i^{ND} = e_{12}$$

Let $\mathbb{G}_{a,b,c}$ be a geometric algebra with a positive, b negative and c zero basis vectors.

Then for $\mathbb{G}_{a,0,c}$:

$$\overline{X_i} = X_i^{\star}$$

And if that wasn't confusing enough, applying a dual twice changes the signs, so we also want the inverses of these duals:

$$
\begin{aligned}
(X_i^{\star})^{\star^{-1}} &= X_i \\
\overline{\overline{X_i}} &= X_i \\
\underline{\underline{X_i}} &= X_i
\end{aligned}
$$

In 2D and 3D PGA, we can simplify implementation by swapping two basis vectors in some blades such that $\overline{X_i}$ does not flip signs, e.g.

$$
\begin{aligned}
A &= A_1 + A_2 e_0 + A_3 e_1 + A_4 e_2 \\
&\quad + A_5 e_{01} + A_6 e_{20} + A_7 e_{12} + A_8 e_{012} \\
\overline{\overline{A}} = \overline{A} &= A_8 + A_7 e_0 + A_6 e_1 + A_5 e_2 \\
&\quad + A_4 e_{01} + A_3 e_{20} + A_2 e_{12} + A_1 e_{012}
\end{aligned}
$$

IV. Shapes and sizes

Let $\mathbb{G}_{d,0,1}$ be a d-dimensional PGA:

$$(e_0)^2 = 0;\ (e_i)^2 = 1$$

Now we have a choice to make:

1. `Point-based PGA`
   - `- vectors are points`
   - `- (n-1)-vectors are hyperplanes`
2. `Plane-based PGA`
   - `- vectors are hyperplanes`
   - `- (n-1)-vectors are points`

Both of these are equally valid and many operations make use of computing in the dual algebra via $\overline{\overline{A}\ \mathrm{op}\ \overline{B}}$.

$$
\begin{aligned}
point &= e_0 + xe_1 + ye_2 + \ldots + we_d \\
hyperplane &= \overline{e_0 + xe_1 + ye_2 + \ldots + we_d}
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
hyperplane &= e_0 + xe_1 + ye_2 + \ldots + we_d \\
point &= \overline{e_0 + xe_1 + ye_2 + \ldots + we_d}
\end{aligned}
}
$$

We will denote Plane-based operations inside boxes whenever they differ.

Let $\langle A \rangle_k$ be the grade selection operator:

$$\langle A \rangle_k = \sum_i \langle X_i \rangle_k$$

$$\langle X_i \rangle_k = \begin{cases} X_i & (X_i \in \text{k-vector}) \\ 0 & (X_i \notin \text{k-vector}) \end{cases}$$

Then we can start defining binary operators:

$A \wedge B = \sum_{j,k} \left\langle \langle A \rangle_j \langle B \rangle_k \right\rangle_{j+k}$ `// wedge product`

$A \vee B = \overline{\overline{A} \wedge \overline{B}}$ `// antiwedge product`

These operators retain distributivity and associativity.

We can join points into lines, planes, ...

$$
\begin{aligned}
line &= point_1 \mathbin{\mathit{join}} point_2 \\
plane &= point_1 \mathbin{\mathit{join}} point_2 \mathbin{\mathit{join}} point_3
\end{aligned}
$$

$$A \mathbin{\mathit{join}} B = A \wedge B$$

$$\boxed{A \mathbin{\mathit{join}} B = A \vee B}$$

In fact this works with any two geometric objects, operators that don't have this property aren't really worth your time.

And we can meet two objects:

$$
\begin{aligned}
point &= line_1 \mathbin{\mathit{meet}} line_2 \\
point &= plane \mathbin{\mathit{meet}} line \\
line &= plane_1 \mathbin{\mathit{meet}} plane_2
\end{aligned}
$$

$$A \mathbin{\mathit{meet}} B = A \vee B$$

$$\boxed{A \mathbin{\mathit{meet}} B = A \wedge B}$$

If you meet two parallel lines, you get a point at infinity = an infinite point:

$$line_1 \mathbin{\mathit{meet}} line_1 = xe_1 + ye_2 + \ldots + we_d$$

$$\boxed{line_1 \mathbin{\mathit{meet}} line_1 = \overline{xe_1 + ye_2 + \ldots + we_d}}$$

All objects in GA have an orientation, so if you flip the direction of one of the lines, you get an infinite point in the other direction

TODO: distance of line to A -> raytraced graphing calculator distance of point to A -> raymarched graphing calculator

TODO: something about sizes of line segments / volumes

TODO: projection to camera plane + depth buffer Projection: $P = (A \cdot B^{-1})B$?

V. Motors

Taking $e^{bivector}$ gives us a motor (motion operator), we can apply motors via $MA\tilde{M}$:

$$
\begin{aligned}
rotor &= e^{bivector^{ND}} \\
motor &= e^{bivector} \\
rotor &\in motor
\end{aligned}
$$

In 3D, rotors are quaternions.

Where bivector blades are rotations. For example $\frac{\theta}{2}e_{12}$ is an xy rotation by θ degrees and $\frac{d}{2}e_{01}$ is a translation by d:

$$
\begin{aligned}
e^{\frac{\theta}{2}e_{12}} &= cos\,\frac{\theta}{2} + \left(sin\,\frac{\theta}{2}\right)e_{12} \\
e^{\frac{d}{2}e_{01}} &= 1 + \frac{d}{2}e_{01}
\end{aligned}
$$

We can interpolate motors with nlerp or slerp, where BA^{-1} is a transformation that brings A to B:

$$
\begin{aligned}
nlerp(t, A, B) &= \frac{lerp(t, A, B)}{\|lerp(t, A, B)\|} \\
lerp(t, A, B) &= (1-t)A + tB \\
slerp(t, A, B) &= (BA^{-1})^t A \\
A^{-1} &= \frac{A}{\|A\|^2} \\
\|A\| &= \sqrt{\sum_i |\,(A_i X_i)^2\,|} = \sqrt{\sum_i \left(A_i^{ND}\right)^2}
\end{aligned}
$$

nlerp does not allow for unnormalized motors, but we will be normalizing them in the physics simulation anyways.

While we're at it we might as well define other norms:

$$
\begin{aligned}
\|A\|_D &= \sqrt{\sum_i \left(A_i^D\right)^2} \\
\|A\|_F &= \sqrt{\sum_i \left(A_i\right)^2}
\end{aligned}
$$

Where A^D is A with only degenerate basis vectors.

TODO: line forces per https://enki.ws/ganja.js/examples/coffeeshop.html#sUwbwu9vR and https://bivector.net/PGADYN.html