

A summary of Projective Geometric Algebra

PATROLIN

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Abstract

Information on Geometric Algebra is scattered and scarce. This is an explanation of how to calculate PGA, but not necessarily why it works - for that you can try [sudgylacmoe](#) on youtube and [Eric Lengyel's](#) cheatsheet for point-based PGA (though he goes mad with power). We would also point you to [bivector.net](#) (but their calculator just, doesn't), the [ganja.js](#) math is presumably correct, since their demos seem to work.

I. GEOMETRIC NUMBERS

associative:

$$1 + x^2 = 0 \\ x = ?$$

x is not a real number; but if it's not real, why should the other numbers be real?

$$(e_1)^2 + (e_2)^2 = (e_0)^2 \\ (e_1)^2 = 1; (e_2)^2 = -1; (e_0)^2 = 0$$

In fact, we can define as many of these as we want with a geometric algebra $G_{a,b,c}$ having a numbers that square to 1, b that square to -1 and c that square to 0 (degenerate basis vectors):

$$(a + be_1) \in G_{1,0,0} \text{ // hyperbolic numbers}$$

$$(a + be_2) \in G_{0,1,0} \text{ // complex numbers}$$

$$(a + be_0) \in G_{0,0,1} \text{ // dual numbers}$$

We can multiply these numbers together using the geometric product:

$$(e_i)^2 = \{1, -1, 0\} \\ e_i e_j = -e_j e_i$$

The result is another number in the same algebra $G_{a,b,c}$

This product is neither commutative nor anticommutative, but it is distributive and as-

$$AB \neq BA$$

$$AB \neq -BA$$

$$A(B + C) = AB + AC$$

$$(AB)C = A(BC)$$

$$aB = Ba; a \in \mathbb{R}$$

Thus the product of two complex numbers:

$$\begin{aligned} (A_1 + A_2 e_2)(B_1 + B_2 e_2) \\ = A_1 B_1 + A_1 B_2 e_2 + A_2 e_2 B_1 + A_2 e_2 B_2 e_2 \\ = A_1 B_1 + A_1 B_2 e_2 + A_2 B_1 e_2 + A_2 B_2 \\ = (A_1 B_1 + A_2 B_2) + (A_1 B_2 + A_2 B_1) e_2 \end{aligned}$$

II. ROTATIONS

A multivector with n basis vectors consists of 2^n blades:

- scalar = 0-vector = 1
- vector = 1-vector
- bivector = 2-vector
- trivector = 3-vector
- ...
- pseudovector = (n-1)-vector
- pseudoscalar = n-vector = 1

Where a k-vector has $\binom{n}{k}$ blades, for example:

$$\begin{aligned} A &= A_1 \\ &+ A_2 e_0 + A_3 e_1 + A_4 e_2 \\ &+ A_5 e_{01} + A_6 e_{02} + A_7 e_{12} \\ &+ A_8 e_{012} \end{aligned}$$

We can abbreviate blades like $e_1 e_2$ as e_{12} .

We can use the exponential function to find a rotation e^A :

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\begin{aligned} e^{a e_2} &= 1 + a e_2 - \frac{a^2}{2} - \frac{a^3}{6} e_2 + \dots \\ &= (1 - \frac{a^2}{2} + \dots) + (a - \frac{a^3}{6} + \dots) e_2 \\ &= \cos(a) + \sin(a) e_2 \end{aligned}$$

Similarly you can find

$$e^{e_i} = \begin{cases} \cosh(a) + \sinh(a) e_i & ((e_i)^2 = 1) \\ \cos(a) + \sin(a) e_i & ((e_i)^2 = -1) \\ 1 + e_i & ((e_i)^2 = 0) \end{cases}$$

This gives us hyperbolic rotations, rotations and translations (rotations through infinity) respectively.

With X_i = the i-th blade in a multivector:

$$e^A = \prod_i e^{A_i X_i}$$

This operation depends on the order of the basis blades.
For graphics you would want rotation, then translation.
(e_0 should be last)

III. UNARY OPERATORS

We can define some operations, like reversing the order of basis vectors in the blade, that amount to flipping some signs:

$$\begin{aligned} \tilde{X}_i &= (-1)^{\lfloor k/2 \rfloor} X_i \quad // \text{ reverse} \\ X_i &\in \text{k-vector} \\ f(A) &= \sum_i f(X_i) \end{aligned}$$

Poincaré duality states that maps between k-vectors and (n-k)-vectors exist.

$$\begin{aligned} X_i \text{ dual}(X_i) &= \pm 1 \\ \text{dual}(X_i) &= \pm X_{2^n - i + 1} \end{aligned}$$

For example:

$$\begin{aligned} X_i X_i &= 1 \quad // \text{ left complement} \\ X_i \overline{X_i} &= 1 \quad // \text{ right complement} \\ X_i X_i^* &= \text{sign}(X_i^{ND} \widetilde{X_i^{ND}}) 1 \quad // \text{ hodge dual} \end{aligned}$$

Where X_i^{ND} is X_i without degenerate basis vectors, e.g.

$$X_i = e_{012}; X_i^{ND} = e_{12}$$

For $G_{a,0,c}$: $X_i^* = \overline{X_i}$

And applying a dual twice changes the signs, so we also want their inverses:

$$\begin{aligned} (X_i^*)^{\star^{-1}} &= X_i \\ \overline{\overline{X_i}} &= X_i \end{aligned}$$

In 2D and 3D PGA, we can simplify implementation by swapping two basis vectors in some blades such that $\overline{X_i}$ does not flip signs, e.g.

$$\begin{aligned} A &= A_1 + A_2 e_0 + A_3 e_1 + A_4 e_2 \\ &+ A_5 e_{01} + A_6 e_{20} + A_7 e_{12} + A_8 e_{012} \\ \underline{A} = \overline{A} &= A_8 + A_7 e_0 + A_6 e_1 + A_5 e_2 \\ &+ A_4 e_{01} + A_3 e_{20} + A_2 e_{12} + A_1 e_{012} \end{aligned}$$

IV. SHAPES AND SIZES

Let $G_{d,0,1}$ be a d-dimensional PGA:

$$(e_0)^2 = 0; (e_i)^2 = 1$$

Now we have a choice to make:

1. Point-based PGA
 - vectors are points
 - (n-1)-vectors are hyperplanes
2. Plane-based PGA
 - vectors are hyperplanes
 - (n-1)-vectors are points

Both of these are equally valid and many operations make use of computing in the dual algebra via $\overline{A} \text{ op } \overline{B}$.

$$\begin{aligned} point &= e_0 + xe_1 + ye_2 + \dots + we_d \\ hyperplane &= \overline{e_0} + xe_1 + ye_2 + \dots + we_d \end{aligned}$$

$$\begin{aligned} hyperplane &= e_0 + xe_1 + ye_2 + \dots + we_d \\ point &= \overline{e_0} + xe_1 + ye_2 + \dots + we_d \end{aligned}$$

We will denote Plane-based operations inside boxes whenever they differ.

Let $\langle A \rangle_k$ be the grade selection operator:

$$\begin{aligned} \langle X_i \rangle_k &= \begin{cases} X_i & (X_i \in \text{k-vector}) \\ 0 & (X_i \notin \text{k-vector}) \end{cases} \\ \langle A \rangle_k &= \sum_i \langle X_i \rangle_k \end{aligned}$$

Then we can define the wedge \wedge and anti-wedge \vee products:

$$\begin{aligned} A \wedge B &= \sum_{j,k} \langle \langle A \rangle_j; \langle B \rangle_k \rangle_{j+k} = \overline{A \vee B} \\ A \vee B &= \sum_{j,k} \langle \langle A \rangle_j; \langle B \rangle_k \rangle_{n - ((n-j) + (n-k))} \\ &= \overline{A \wedge B} \end{aligned}$$

These operators retain distributivity, associativity and noncommutativity.

We can then join points into lines and lines into planes:

$$\begin{aligned} line &= point_1 \text{ join } point_2 \\ plane &= point_1 \text{ join } point_2 \text{ join } point_3 \\ A \text{ join } B &= A \wedge B \end{aligned}$$

$$A \text{ join } B = A \vee B$$

In fact this works with any two geometric objects, operators that don't have this property aren't really worth your time.

And we can meet two objects:

$$\begin{aligned} point &= line_1 \text{ meet } line_2 \\ point &= plane \text{ meet } line \\ line &= plane_1 \text{ meet } plane_2 \\ A \text{ meet } B &= A \vee B \end{aligned}$$

$$A \text{ meet } B = A \wedge B$$

If you meet two parallel lines, you get a point at infinity = an infinite point:

$$line_1 \text{ meet } line_1 = xe_1 + ye_2 + \dots + we_d$$

$$line_1 \text{ meet } line_1 \approx \overline{xe_1 + ye_2 + \dots + we_d}$$

You probably want to ignore the sign flips from the dual, otherwise you'd be changing the coordinates.

All objects in PGA have an direction A^D and position A^P :

$$\begin{aligned} A &= A^D + A^P \\ A^D &= \sum_i A_i X_i \quad (e_0 \in X_i) \\ A^P &= \sum_i A_i X_i \quad (e_0 \notin X_i) \end{aligned}$$

$$\begin{aligned} A^D &= \sum_i A_i X_i \quad (e_0 \notin X_i) \\ A^P &= \sum_i A_i X_i \quad (e_0 \in X_i) \end{aligned}$$

In 2D and 3D:

$$\begin{aligned} point &= e_0 + A^P \\ line &= A^D + A^P \end{aligned}$$

$$point = \overline{e_0} + A^P$$

In 3D:

$$\begin{aligned} plane &= A^D + A_{15}e_{123} \\ &= normal + distance \end{aligned}$$

$$plane = A^D + A_2e_0$$

We can take norms to measure the direction or the position:

$$\|A\|_D = \sqrt{\sum_i (A_i^D)^2}$$

A is finite when $\|A\|_D \neq 0$, therefore infinite objects have no direction.

$$\|A\|_P = \sqrt{\sum_i (A_i^P)^2}$$

$$\|A\| = \sqrt{\sum_i |(A_i X_i)^2|} = \|A\|_P$$

$$\boxed{\|A\| = \|A\|_D}$$

$$A^{-1} = \tilde{A} \frac{1}{\|A\|^2}$$

We can use this to calculate lengths/areas/volumes/...:

$$\text{length}(\text{edge_loop}) = \sum_i \|line_i\|_P$$

$$\text{area}(\text{edge_loop}) = \frac{1}{2} \sum_i \|line_i\|_D$$

$$line_i = p_i \text{ join } p_{i+1}$$

$$\text{area}(\text{triangle_mesh}) = \frac{1}{2} \sum_i \|plane_i\|_P$$

$$\text{volume}(\text{triangle_mesh}) = \frac{1}{3} \sum_i \|plane_i\|_D$$

$$plane_i = p_i \text{ join } p_{i+1} \text{ join } p_{i+2}$$

i. Rasterization

We could just take the x, y components of a d-dimensional point and get an orthographic projection.

But we could also just make a line from the camera (at the origin) to a point and then intersect it with a plane:

$$p_{\text{perspective}} = (p_{\text{camera}} \text{ join } p_{\text{vertex}}) \text{ meet } (xy_plane)$$

$$xy_plane = p_1 \text{ join } p_2 \text{ join } p_3$$

$$\text{depth}(p_{\text{vertex}}) = \|p_{\text{vertex}}\|_P$$

TODO: $\text{project}(A, B): P = (A \cdot B^{-1})B$?
 TODO: $\text{project}(\text{point_at_origin}, B)$ TODO:
 $\text{project}(\text{plane_at_infinity}, B)$

ii. Raytracing

TODO: $line_1 \wedge line_2$
 $= \text{signed_distance}(line_1, line_2) \|line_1^D \wedge line_2^D\|$
 TODO: correlate of signed distance between lines: $line_1 \vee line_2 \rightarrow \text{ray-triangle intersection}$

\rightarrow raytraced game

TODO: distance of line to A \rightarrow raytraced graphing calculator

TODO: ?

$$d(A, B) = \begin{cases} A \wedge \bar{B} & (\text{type}(A) = \text{type}(B)) \\ d(A, \text{project}(A, B)) & (\text{type}(A) \neq \text{type}(B)) \end{cases}$$

TODO: $\cos(A, B) = A \cdot B / (A.\text{norm}() B.\text{norm}()); \sin(A, B) = ?$

iii. Raymarching

TODO: signed distance of point to A \rightarrow ray-marched graphing calculator

iv.

V. MOTORS

Taking e^{bivector} gives us a motor (motion operator), we can apply motors via MAM :

$$\text{motor} = e^{\text{bivector}}$$

$$\text{rotor} = e^{\text{bivector}^D}; \text{rotor} \in \text{motor}$$

$$\text{translator} = e^{\text{bivector}^P}; \text{translator} \in \text{motor}$$

In 3D, rotors are quaternions.

Where bivector blades are rotations. For example $\frac{\theta}{2}e_{12}$ is an xy rotation by θ degrees and $\frac{d}{2}e_{01}$ is a translation by d :

$$e^{\frac{\theta}{2}e_{12}} = \cos \frac{\theta}{2} + \left(\sin \frac{\theta}{2} \right) e_{12}$$

$$e^{\frac{d}{2}e_{01}} = 1 + \frac{d}{2}e_{01}$$

We can interpolate motors with nlerp or slerp , where BA^{-1} is a transformation that brings A to B:

$$\text{nlerp}(t, A, B) = \frac{\text{lerp}(t, A, B)}{\|\text{lerp}(t, A, B)\|}$$

$$\text{lerp}(t, A, B) = (1 - t)A + tB$$

$$\text{slerp}(t, A, B) = (BA^{-1})^t A$$

nlerp does not allow for unnormalized motors, but we will be normalizing them in the physics simulation anyways.

TODO: line forces per <https://enki.ws/ganja.js/examples/coffeeshop.html#sUwbwu9vR>
and <https://bivector.net/PGADYN.html>

VI. BONUS

Useless operators that are just here for completeness:

$$X_i^\dagger = (-1)^{\lfloor k \rfloor} X_i \text{ // involute}$$

$$\bar{X}_i = (-1)^{\lfloor k+k/2 \rfloor} X_i \text{ // conjugate}$$

$$\|A\|_\infty = \|A\|_D$$

$\|A\|_\infty = \|A\|_P$