Maths

Patrolin

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Contents

Chapter 1

Derivatives

The derivative is the rate of change at a specific point:

$$
\begin{aligned}
&\text{When } \frac{d}{dx}f(x) > 0,\ f(x) \text{ is increasing}\\
&\text{When } \frac{d}{dx}f(x) < 0,\ f(x) \text{ is decreasing}\\
&\text{When } \frac{d}{dx}f(x) = 0,\ f(x) \text{ is constant}
\end{aligned}
$$

1.1 Notation

If x is a variable, then dx is a small change in x.

dx approaches 0, but is not equal to 0.

Instead we ask what happens as dx gets closer and closer to 0 :

$$dx \to 0$$

If y is a function of x, then dy and $df(x)$ are the change caused by dx :

$$y = f(x)$$

$$\Rightarrow dy = df(x)$$

$$df(x) = f(x + dx) - f(x)$$

The derivative of $f(x)$ is the ratio between $df(x)$ and dx :

$$\frac{d}{dx} f(x) = \frac{df(x)}{dx}$$

Putting all of these ideas together gives us the equation for a derivative:

$$\left.\begin{array}{l} f(x)' = \frac{d}{dx} f(x) = \frac{df(x)}{dx} \\ df(x) = f(x + dx) - f(x) \\ dx \to 0 \ (dx \neq 0) \end{array}\right\} \frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

If we know the derivative, we can also compute $df(x)$:

$$\begin{aligned}\frac{d}{dx}f(x) &= \frac{df(x)}{dx} \\ df(x) &= \frac{d}{dx}f(x) * dx\end{aligned}$$

1.2 Elementary derivatives

1.2.1 By $\frac{d}{dx}f(x)$:

$$a \in R$$

$$\begin{aligned}f(x) &= a \\ \frac{d}{dx}f(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ \frac{d}{dx}a &= \lim_{h\to 0}\frac{a-a}{h} \\ \frac{d}{dx}a &= \lim_{h\to 0}\frac{0}{h} \\ \frac{d}{dx}a &= 0 \ \because \ h \neq 0\end{aligned}$$

1.2.2 By $df(x)$:

$$
\begin{aligned}
d(x^a) &= (x+dx)^a - x^a \\
&= x^a + \left.\begin{matrix} \overbrace{+(dx+x+x+x+\dots)}^{x^{a-1}dx} \\ +(x+dx+x+x+\dots) \\ +(x+x+dx+x+\dots) \\ +(x+x+x+dx+\dots) \\ \vdots \end{matrix}\right\} \text{a times} + (\dots)dx^2 - x^a \\
d(x^a) &= ax^{a-1}dx + (\dots)dx^2 \\
\frac{d(x^a)}{dx} &= ax^{a-1} + (\dots)dx \\
\frac{d}{dx}x^a &= ax^{a-1} \quad \because \; dx \to 0 \\
&\Rightarrow \frac{d}{dx}x = 1
\end{aligned}
$$

1.2.3 By definition :

$$\frac{d}{dx}e^x = e^x$$

1.2.4 By implicit differentiation :

This involves taking the derivatives of each variable separately. The variables may also be on one side of the equation.

$$
\begin{aligned}
y &= \ln x \\
e^y &= x \\
d(e^y) &= d(x) \\
e^y dy &= 1dx \\
\frac{dy}{dx} &= \frac{1}{e^y} \\
\frac{d}{dx}y &= \frac{1}{x} \\
\frac{d}{dx}\ln x &= \frac{1}{x}
\end{aligned}
$$

1.2.5 By fancy geometry :

see https://lbry.tv/@3Blue1Brown:b/derivative-formulas-through-geometry

$$
\begin{aligned}
\frac{d}{dx}\sin x &= \cos x \\
\frac{d}{dx}\cos x &= -\sin x
\end{aligned}
$$

Plus other trigonometric functions which you probably don't need.

1.2.6 By giving up :

$$\frac{d}{dx}x^x = x^x(\ln x + 1)$$
$$\frac{d}{dx}e^{-x} = -e^{-x}$$
$$\frac{d}{dx}a^x = a^x * \ln a$$
$$\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$$

1.3 Rules

1.3.1 Sum rule

see https://lbry.tv/@3Blue1Brown:b/derivative-formulas-through-geometry

$$\frac{d}{dx}(g(x)+h(x)) = \frac{d}{dx}g(x) + \frac{d}{dx}h(x)$$

$$\begin{aligned}
\frac{d}{dx}(\sin x + x^2) &= \frac{d}{dx}\sin x + \frac{d}{dx}x^2 = \cos x + 2x \\
\frac{d}{dx}(x+1) &= 1 + 0 = 1 \\
\frac{d}{dx}(x^3 + e^x) &= 3x^2 + e^x
\end{aligned}$$

1.3.2 Product rule

see https://lbry.tv/@3Blue1Brown:b/visualizing-the-chain-rule-and-product

$$
\begin{aligned}
d(g(x) * h(x)) &= dg(x) * h(x) + g(x) * dh(x) + dg(x) * dh(x) \\
\frac{d(g(x) * h(x))}{dx} &= \frac{dg(x)}{dx} * h(x) + g(x) * \frac{dh(x)}{dx} + \frac{dg(x)}{dx} * h(x) * dh(x) \\
\frac{d}{dx}(g(x) * h(x)) &= \frac{d}{dx}g(x) * h(x) + g(x) * \frac{d}{dx}h(x) \quad \because \; dh(x) \to 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dx}(\sin x * x^2) &= \frac{d}{dx}\sin x * x^2 + \sin x * \frac{d}{dx}x^2 = \cos x * x^2 + \sin x * 2x \\
\frac{d}{dx}(2 * x) &= 0 * x + 2 * 1 = 2 \\
\frac{d}{dx}\frac{\sin x}{x} &= \frac{d}{dx}(\sin x * 1/x) = \frac{\cos x}{x} - \frac{\sin x}{x^2}
\end{aligned}
$$

This also leads to a formula for division :

$$
\frac{d}{dx}\frac{g(x)}{h(x)} = \frac{\frac{d}{dx}g(x) * h(x) - g(x) * \frac{d}{dx}h(x)}{h(x)^2}
$$

1.3.3 Derivative with respect to $f(x)$:

You simply treat $f(x)$ as if it was x :

$$\frac{d}{d(x^2)}\sin(x^2) = \cos(x^2)$$
$$\frac{d}{d(\sin x)}(\sin x)^3 = 3(\sin x)^2$$

1.3.4 Chain rule

$$\frac{d}{dx}g(h(x)) = \frac{dg(h(x))}{dx}$$
$$\frac{d}{dx}g(h(x)) = \frac{dg(h(x))}{dh(x)} * \frac{dh(x)}{dx}$$
$$\frac{d}{dx}g(h(x)) = \frac{d}{dh(x)}g(h(x)) * \frac{d}{dx}h(x)$$

$$\frac{d}{dx}\sin(x^2) = \cos(x^2) * 2x$$
$$\frac{d}{dx}\frac{1}{(\sin x)} = \frac{-1}{(\sin x)^2} * \cos x = -\frac{\cos x}{(\sin x)^2}$$

1.3.5 L'Hopital's rule

$$\lim_{x\to a}\frac{g(x)}{h(x)}=\lim_{x\to a}\frac{dg(x)}{dh(x)}$$

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\cos x*dx}{1*dx}$$
$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\cos x}{1}\quad\because\ dx\neq 0$$
$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

1.3.6 Derivative of inverse function

Given $g(x)$, the inverse of $f(x)$:

$$g(f(x))=x$$
$$\frac{d}{dx}f(x)*\frac{d}{dx}g(x)=1$$
$$\frac{d}{dx}g(x)=\frac{1}{\frac{d}{dx}f(x)}$$