Maths

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Contents

Chapter 1

Derivatives

The derivative is the rate of change of a function at a given point:

$$
\begin{aligned}
\text{When } \overbrace{\frac{d}{dx}f(x)}^{\text{the derivative of f(x)}} > 0,\ f(x) \text{ is increasing.}\\
\text{When } \frac{d}{dx}f(x) < 0,\ f(x) \text{ is decreasing.}\\
\text{When } \frac{d}{dx}f(x) = 0,\ f(x) \text{ is constant.}
\end{aligned}
$$

1.1 Notation

If x is a variable, then dx is a small change in x.

dx is close to 0, but not equal to 0. 0 is $\frac{1}{\infty}$, and ∞ is evil!

$$dx \to 0$$

If y is a function of x, then dy and $df(x)$ are the change caused by dx.

$$\begin{aligned} y &= f(x) \\ dy &= df(x) \\ df(x) &= \underbrace{f(x) - f(x - dx)}_{\text{left subderivative}} = \underbrace{f(x + dx) - f(x)}_{\text{right subderivative}} \end{aligned}$$

If the function is discontinuous, then the derivative is undefined.

$$\frac{d}{dx} f(x) = \frac{df(x)}{dx}$$

Putting all of these ideas together gives us the equation for a derivative:

$$\left.\begin{aligned}
&f(x)' = \tfrac{d}{dx}f(x) = \tfrac{df(x)}{dx} \\
&df(x) = f(x+dx) - f(x) \\
&dx \to 0 \ (dx \neq 0)
\end{aligned}\right\} \frac{d}{dx}f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

1.2 Elementary derivatives

1.2.1 $\frac{d}{dx}a = 0$

$$a \in R$$

$$\begin{aligned}
f(x) &= a \\
\frac{d}{dx}f(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
\frac{d}{dx}a &= \lim_{h \to 0} \frac{a - a}{h} \\
\frac{d}{dx}a &= \lim_{h \to 0} \frac{0}{h} \\
\frac{d}{dx}a &= 0 \ \because \ h \neq 0
\end{aligned}$$

1.2.2 $\frac{d}{dx}x^a = ax^{a-1}$

$$d(x^a) = (x+dx)^a - x^a$$

$$d(x^a) = x^a + \underbrace{\left.\begin{array}{c} +(dx+x+x+x+\dots) \\ +(x+dx+x+x+\dots) \\ +(x+x+dx+x+\dots) \\ +(x+x+x+dx+\dots) \\ \vdots \end{array}\right\}}_{x^{a-1}dx} \text{a times} + (\dots)dx^2 - x^a$$

$$d(x^a) = ax^{a-1}dx + (\dots)dx^2$$

$$\frac{d(x^a)}{dx} = ax^{a-1} + (\dots)dx \quad \because \ dx \neq 0$$

$$\frac{d}{dx}x^a = ax^{a-1} \quad \because \ dx \to 0$$

$$\Downarrow$$

$$\frac{d}{dx}x = 1$$

$$\frac{d}{dx}x^2 = 2x$$

$$\frac{d}{dx}x^3 = 3x^2$$

$$\vdots$$

1.2.3 $\frac{d}{dx}e^x = e^x$

This is how the value of e is defined.

$e = 2.718\ldots$

1.2.4 $\frac{d}{dx}\ln x = \frac{1}{x}$

If you have a function of one variable, then take the change in that function with respect to that variable (implicit differentiation).

$$\begin{aligned}
y &= \ln x \\
e^y &= x \\
d(e^y) &= d(x) \\
e^y * dy &= 1 * dx \\
\frac{dy}{dx} &= \frac{1}{e^y} \\
\frac{d}{dx}y &= \frac{1}{x} \\
\frac{d}{dx}\ln x &= \frac{1}{x}
\end{aligned}$$

If you have a function of many variables, apply **?? ??** to break it down into functions of one variable.

1.2.5 Trigonometric functions

see https://lbry.tv/@3Blue1Brown:b/derivative-formulas-through-geometry

$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d}{dx}\cos x = -\sin x$$

Plus others which you probably don't need.

1.3 Rules

1.3.1 The derivative is a linear function

The derivative of a sum is the sum of derivatives.

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) \ + \ \frac{d}{dx}g(x)$$

Constants can be multiplied outside.

$$a \in R$$
$$\frac{d}{dx}(a * x) = a * \frac{d}{dx}x$$

Proof is left as an excercise to the reader.

1.3.2 Product rule

see https://lbry.tv/@3Blue1Brown:b/visualizing-the-chain-rule-and-product

TODO(): put in a picture instead of a link

$$
\begin{aligned}
d(f(x)*g(x)) &= df(x)*g(x)+f(x)*dg(x)+df(x)*dg(x)\\
\frac{d(f(x)*g(x))}{dx} &= \frac{df(x)}{dx}*g(x)+f(x)*\frac{dg(x)}{dx}+\frac{df(x)}{dx}*dg(x)\\
\frac{d}{dx}(f(x)*g(x)) &= \frac{d}{dx}f(x)*g(x)+f(x)*\frac{d}{dx}g(x) \quad \because \; dg(x)\to 0
\end{aligned}
$$

$$
\begin{aligned}
&\frac{d}{dx}(x\ln x) = 1*\ln x+\frac{x}{x} = \ln x+1\\
&\frac{d}{dx}(\sin x*x^2) = \cos x*x^2+\sin x*2x\\
&\frac{d}{dx}\frac{\sin x}{x} = \frac{d}{dx}(\sin x*1/x) = \frac{\cos x}{x}-\frac{\sin x}{x^2}
\end{aligned}
$$

1.3.3 $\frac{d}{dg(x)}f(g(x))$

Derivative of $f(g(x))$ with respect to $g(x)$.

You simply treat $g(x)$ as if it was x.

$$\frac{d}{d(x^2)}(\sin x^2) = \frac{d}{dy}(\sin y) = \cos y = \cos x^2$$
$$\frac{d}{d(\sin x)}(\sin x)^3 = 3(\sin x)^2$$
$$\frac{d}{d(\ln x)}(\ln x)^{-1} = -1(\ln x)^{-2}$$

1.3.4 Chain rule

$$\frac{d}{dx}f(g(x)) = \frac{df(g(x))}{dx} = \frac{df(g(x))}{g(x)} \ * \ \frac{dg(x)}{dx}$$
$$\frac{d}{dx}f(g(x)) = \frac{d}{dg(x)}f(g(x)) \ * \ \frac{d}{dx}g(x)$$

$$\frac{d}{dx}(\sin x^2) = \cos x^2 * 2x$$
$$\frac{d}{dx}\frac{1}{\sin x} = \frac{d}{dx}(\sin x)^{-1} = -1 * (\sin x)^{-2} \ * \ \cos x = -\frac{\cos x}{(\sin x)^2}$$
$$\frac{d}{dx}\ln e^x = \frac{1}{e^x} * e^x = 1$$

This also leads to a formula for division.

$$
\begin{aligned}
\frac{d}{dx}\frac{1}{g(x)} &= -1 * g(x)^{-2} * \frac{d}{dx}g(x) = -\frac{\frac{d}{dx}g(x)}{g(x)^2} \\
\frac{d}{dx}\frac{f(x)}{g(x)} &= \frac{\frac{d}{dx}f(x)}{g(x)} + f(x) * \frac{d}{dx}g(x) \\
\frac{d}{dx}\frac{f(x)}{g(x)} &= \frac{\frac{d}{dx}f(x)}{g(x)} - f(x) * \frac{\frac{d}{dx}g(x)}{g(x)^2} \\
\frac{d}{dx}\frac{f(x)}{g(x)} &= \frac{\frac{d}{dx}f(x) * g(x) \ \ - \ \ f(x) * \frac{d}{dx}g(x)}{g(x)^2}
\end{aligned}
$$

1.3.5 L'Hopital's rule

Relies on geometric proof!

$$\lim_{x \to a} \frac{g(x)}{h(x)} = \lim_{x \to a} \frac{dg(x)}{dh(x)}$$

$$
\begin{aligned}
&\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x * dx}{1 * dx} \\
&\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} \quad \because \ dx \neq 0 \\
&\lim_{x \to 0} \frac{\sin x}{x} = 1
\end{aligned}
$$

1.3.6 Derivative of inverse function

With $g(x)$ being the inverse of $f(x)$:

$$
\begin{aligned}
g(f(x)) &= x \\
\frac{d}{dx}f(x) * \frac{d}{dx}g(x) &= 1 \\
\frac{d}{dx}g(x) &= \frac{1}{\frac{d}{dx}f(x)}
\end{aligned}
$$

1.4 Composite derivatives

1.4.1 $\frac{d}{dx}e^{-x} = -e^{-x}$

$$
\begin{aligned}
\frac{d}{dx}e^{-x} &= e^{-x} * (-1) \\
\frac{d}{dx}e^{-x} &= -e^{-x}
\end{aligned}
$$

1.4.2 $\frac{d}{dx}a^x = a^x \ln a$

Note that $\ln a^x = x \ln a$.

$$\begin{aligned}
a^x &= e^{\ln a^x} \\
a^x &= e^{x \ln a} \\
\frac{d}{dx}a^x &= e^{x \ln a} * \ln a \\
\frac{d}{dx}a^x &= a^x \ln a
\end{aligned}$$

1.4.3 $\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$

$$\begin{aligned}
y &= \log_a x \\
a^y &= x \\
d(a^y) &= d(x) \\
a^y \ln a * dy &= 1 * dx \\
\frac{dy}{dx} &= \frac{1}{a^y \ln a} \\
\frac{d}{dx}y &= \frac{1}{a^{\log_a x} \ln a} \\
\frac{d}{dx}\log_a x &= \frac{1}{x \ln a}
\end{aligned}$$

1.4.4 $\frac{d}{dx}x^x = x^x * (\ln x + 1)$

$$f(x) = x^x$$

$$\ln f(x) = \ln(x^x)$$

$$\ln f(x) = x \ln x$$

$$f(x) = e^{x \ln x}$$

$$\frac{d}{dx}x^x = e^{x \ln x} * (\ln x + 1)$$

$$\frac{d}{dx}x^x = x^x * (\ln x + 1)$$