

A summary of Projective Geometric Algebra

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Abstract

Information on Geometric Algebra is scattered and scarce. This is an explanation of how to calculate PGA, but not necessarily why it works - for that you can try [sudgylacmoe](#) on youtube and [Eric Lengyel's](#) cheatsheet for point-based PGA (though he goes mad with power). We would also point you to [bivector.net](#), but their dual calculations are whack, the [ganja.js](#) ones are presumably correct, since their demos seem to work.

I. GEOMETRIC NUMBERS

$$1 + x^2 = 0 \\ x = ?$$

x is not a real number; but if it's not real, why should the other numbers be real?

$$(e_1)^2 + (e_2)^2 = (e_0)^2 \\ (e_1)^2 = 1; (e_2)^2 = -1; (e_0)^2 = 0$$

In fact, we can define as many of these as we want, the simplest examples being:

$$a + be_1 \text{ // hyperbolic numbers} \\ a + be_2 \text{ // complex numbers} \\ a + be_0 \text{ // dual numbers}$$

We can multiply these numbers together using the geometric product:

$$(e_i)^2 = \{1, -1, 0\} \\ e_i e_j = -e_j e_i$$

This product is neither commutative nor anticommutative, but it is distributive and associative:

$$AB \neq BA \\ AB \neq -BA \\ A(B + C) = AB + AC \\ (AB)C = A(BC) \\ aB = Ba; a \in \mathbb{R}$$

Thus the product of two complex numbers:

$$(A_1 + A_2 e_2)(B_1 + B_2 e_2) \\ = A_1 B_1 + A_1 B_2 e_2 + A_2 e_2 B_1 + A_2 e_2 B_2 e_2 \\ = A_1 B_1 + A_1 B_2 e_2 + A_2 B_1 e_2 + A_2 B_2 \\ = (A_1 B_1 + A_2 B_2) + (A_1 B_2 + A_2 B_1) e_2$$

II. ROTATIONS

A multivector with n basis vectors consists of 2^n blades:

- scalar = 0-vector = 1
- vector = 1-vector
- bivector = 2-vector
- trivector = 3-vector
- ...
- pseudovector = (n-1)-vector
- pseudoscalar = n-vector = 1

Where a k-vector has $\binom{n}{k}$ blades, for example:

$$A = A_1 \\ + A_2 e_0 + A_3 e_1 + A_4 e_2 \\ + A_5 e_{01} + A_6 e_{02} + A_7 e_{12} \\ + A_8 e_{012}$$

We can abbreviate blades like $e_1 e_2$ as e_{12} .

Multiplying two multivectors gives you another multivector, we can use the Taylor series expansion of the exponential function to find a rotation e^A :

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$e^{ae_2} = 1 + ae_2 - \frac{a^2}{2} - \frac{a^3}{3}e_2 + \dots$$

$$= (1 - \frac{a^2}{2} + \dots) + (a - \frac{a^3}{3} + \dots)e_2$$

$$= \cos(a) + \sin(a)e_2$$

Similarly you can find

$$e^{e_i} = \begin{cases} \cos(a) + \sin(a)e_i & ((e_i)^2 = -1) \\ \cosh(a) + \sinh(a)e_i & ((e_i)^2 = 1) \\ 1 + e_i & ((e_i)^2 = 0) \end{cases}$$

This gives us rotations, hyperbolic rotations and translations (rotations through infinity) respectively.

Then for a multivector we would have:

$$e^A = e^{A_1} e^{A_2 e_0} e^{A_3 e_1} \dots e^{A_n e_I}$$

III. UNARY OPERATORS

For the i -th blade in a multivector

$$X_i \in \text{k-vector}$$

we can define some operations, like reversing the order of basis vectors in the blade, that amount to flipping some signs:

$$\tilde{X}_i = (-1)^{\lfloor k/2 \rfloor} X_i \text{ // reverse}$$

$$X_i^\dagger = (-1)^{\lfloor k \rfloor} X_i \text{ // involute}$$

$$\bar{X}_i = (-1)^{\lfloor k+k/2 \rfloor} X_i \text{ // conjugate}$$

$$f(A) = \sum_i f(X_i)$$

Poincaré duality states that maps between k -vectors and $(n-k)$ -vectors exist.

$$X_i \text{ dual}(X_i) = \pm 1$$

$$\text{dual}(X_i) = \pm X_{2^n - i + 1}$$

$$\text{dual}(A) = \sum_i \text{dual}(X_i)$$

For example:

$$\underline{X}_i X_i = 1 \text{ // left complement}$$

$$X_i \overline{X}_i = 1 \text{ // right complement}$$

$$X_i X_i^* = \text{sign}(X_i^{ND} \widetilde{X_i^{ND}}) 1 \text{ // hodge dual}$$

Where X_i^{ND} is X_i without degenerate basis vectors, e.g.

$$X_i = e_{012}; X_i^{ND} = e_{12}$$

Let $\mathbb{G}_{a,b,c}$ be a geometric algebra with a positive, b negative and c zero basis vectors.

Then for $\mathbb{G}_{a,0,c}$:

$$\overline{X}_i = X_i^*$$

And applying a dual twice changes the signs, so we also want the inverses of these duals:

$$(X_i^*)^{\star^{-1}} = X_i$$

$$\overline{\overline{X}_i} = X_i$$

In 2D and 3D PGA, we can simplify implementation by swapping two basis vectors in some blades such that \overline{X}_i does not flip signs, e.g.

$$A = A_1 + A_2 e_0 + A_3 e_1 + A_4 e_2$$

$$+ A_5 e_{01} + A_6 e_{20} + A_7 e_{12} + A_8 e_{012}$$

$$\underline{A} = \overline{A} = A_8 + A_7 e_0 + A_6 e_1 + A_5 e_2$$

$$+ A_4 e_{01} + A_3 e_{20} + A_2 e_{12} + A_1 e_{012}$$

IV. SHAPES AND SIZES

Let $\mathbb{G}_{d,0,1}$ be a d -dimensional PGA:

$$(e_0)^2 = 0; (e_i)^2 = 1$$

Now we have a choice to make:

1. Point-based PGA

- vectors are points
- $(n-1)$ -vectors are hyperplanes

2. Plane-based PGA

- vectors are hyperplanes
- $(n-1)$ -vectors are points

Both of these are equally valid and many operations make use of computing in the dual algebra via $\overline{A} \text{ op } \overline{B}$.

$$\begin{aligned} point &= e_0 + xe_1 + ye_2 + \dots + we_d \\ hyperplane &= \overline{e_0 + xe_1 + ye_2 + \dots + we_d} \end{aligned}$$

$$\begin{aligned} hyperplane &= e_0 + xe_1 + ye_2 + \dots + we_d \\ point &= \overline{e_0 + xe_1 + ye_2 + \dots + we_d} \end{aligned}$$

We will denote Plane-based operations inside boxes whenever they differ.

Let $\langle A \rangle_k$ be the grade selection operator:

$$\begin{aligned} \langle X_i \rangle_k &= \begin{cases} X_i & (X_i \in \text{k-vector}) \\ 0 & (X_i \notin \text{k-vector}) \end{cases} \\ \langle A \rangle_k &= \sum_i \langle X_i \rangle_k \end{aligned}$$

Then we can define the wedge \wedge and anti-wedge \vee products:

$$\begin{aligned} A \wedge B &= \sum_{j,k} \langle \langle A \rangle_j \langle B \rangle_k \rangle_{j+k} = \overline{A \vee B} \\ A \vee B &= \sum_{j,k} \langle \langle A \rangle_j \langle B \rangle_k \rangle_{n - ((n-j) + (n-k))} \\ &= \overline{A \wedge B} \end{aligned}$$

These operators retain distributivity and associativity.

We can then join points into lines and lines into planes:

$$\begin{aligned} line &= point_1 \text{ join } point_2 \\ plane &= point_1 \text{ join } point_2 \text{ join } point_3 \\ A \text{ join } B &= A \wedge B \end{aligned}$$

$$A \text{ join } B = A \vee B$$

In fact this works with any two geometric objects, operators that don't have this property aren't really worth your time.

And we can meet two objects:

$$\begin{aligned} point &= line_1 \text{ meet } line_2 \\ point &= plane \text{ meet } line \\ line &= plane_1 \text{ meet } plane_2 \\ A \text{ meet } B &= A \vee B \end{aligned}$$

$$A \text{ meet } B = A \wedge B$$

If you meet two parallel lines, you get a point at infinity = an infinite point:

$$line_1 \text{ meet } line_1 = xe_1 + ye_2 + \dots + we_d$$

$$line_1 \text{ meet } line_1 = \overline{xe_1 + ye_2 + \dots + we_d}$$

If you flip the direction of one of the lines, you get an infinite point in the other direction

All objects in PGA have an direction A^D and position A^P :

$$\begin{aligned} A^D &= \sum_i A_i X_i \quad (e_0 \in X_i) \\ A^P &= \sum_i A_i X_i \quad (e_0 \notin X_i) \end{aligned}$$

In 2D and 3D:

$$\begin{aligned} point &= e_0 + A^P \\ line &= A^D + A^P \end{aligned}$$

In 3D:

$$\begin{aligned} plane &= A^D + A_{15}e_{123} \\ &= normal + distance \end{aligned}$$

We can take norms to measure the direction or the position:

$$\begin{aligned} \|A\|_D &= \sqrt{\sum_i (A_i^D)^2} \\ \|A\|_P &= \sqrt{\sum_i (A_i^P)^2} \\ &= \sqrt{\sum_i |(A_i X_i)^2|} = \|A\| \\ A^{-1} &= \tilde{A} \frac{1}{\|A\|^2} \end{aligned}$$

A is finite if $\|A\|_D \neq 0$, therefore infinite objects have no direction.

We can use this to calculate lengths/areas/volumes/...:

$$length(edge_loop) = \sum_i \|line_i\|_P$$

$$area(edge_loop) = \frac{1}{2} \sum_i \|line_i\|_D$$

$$line_i = p_i \text{ join } p_{i+1}$$

$$area(triangle_mesh) = \frac{1}{2} \sum_i \|plane_i\|_P$$

$$volume(triangle_mesh) = \frac{1}{3} \sum_i \|plane_i\|_D$$

$$plane_i = p_i \text{ join } p_{i+1} \text{ join } p_{i+2}$$

i. Rasterization

We could just take the x, y components of a d-dimensional point and get an orthographic projection.

But we could also just make a line from the camera (at the origin) to a point and then intersect it with a plane:

$$p_{perspective} = (p_{camera} \text{ join } p_{vertex}) \text{ meet } (xy_plane)$$

$$xy_plane = p_1 \text{ join } p_2 \text{ join } p_3$$

$$depth(p_{vertex}) = \|p_{vertex}\|_P$$

TODO: project(A, B): $P = (A \cdot B^{-1})B$?

TODO: project(point_at_origin, B) TODO: project(plane_at_infinity, B)

ii. Raytracing

TODO: $line_1 \wedge line_2$

$$= signed_distance(line_1, line_2) \|line_1^D \wedge line_2^D\|$$

TODO: correlate of signed distance between lines: $line_1 \vee line_2 \rightarrow$ ray-triangle intersection

\rightarrow raytraced game

TODO: distance of line to A \rightarrow raytraced graphing calculator

TODO: ?

$$distance(A, B) = \begin{cases} \underline{A \wedge \overline{B}} & (type(A) = type(B)) \\ distance(A, project(A, B)) & (type(A) \neq type(B)) \end{cases}$$

TODO: $\cos(A, B)$; $\sin(A, B)$

iii. Raymarching

TODO: signed distance of point to A \rightarrow ray-marched graphing calculator

iv.

V. MOTORS

Taking $e^{bivector}$ gives us a motor (motion operator), we can apply motors via MAM :

$$motor = e^{bivector}$$

$$rotor = e^{bivector^D}; rotor \in motor$$

$$translator = e^{bivector^P}; translator \in motor$$

In 3D, rotors are quaternions.

Where bivector blades are rotations. For example $\frac{\theta}{2}e_{12}$ is an xy rotation by θ degrees and $\frac{d}{2}e_{01}$ is a translation by d :

$$e^{\frac{\theta}{2}e_{12}} = \cos \frac{\theta}{2} + \left(\sin \frac{\theta}{2} \right) e_{12}$$

$$e^{\frac{d}{2}e_{01}} = 1 + \frac{d}{2}e_{01}$$

We can interpolate motors with nlerp or slerp, where BA^{-1} is a transformation that brings A to B:

$$nlerp(t, A, B) = \frac{lerp(t, A, B)}{\|lerp(t, A, B)\|}$$

$$lerp(t, A, B) = (1 - t)A + tB$$

$$slerp(t, A, B) = (BA^{-1})^t A$$

nlerp does not allow for unnormalized motors, but we will be normalizing them in the physics simulation anyways.

TODO: line forces per <https://enki.ws/ganja.js/examples/coffee> and <https://bivector.net/PGADYN.html>