Subject: Numerical Calculus - Exercises (Examples) Data: ....

Lecture 1 Finite Differences

ex. 1 $h=0.25$, $a=1$, $a_i = a+ih$, $i=\overline{0,4}$

$f_0=0,\ f_1=2,\ f_2=6,\ f_3=14,\ f_4=17$

sol:

a	f	$\Delta_h f$	$\Delta_h^2 f$	$\Delta_h^3 f$	$\Delta_h^4 f$
1	0	2	2	2	-11
1.25	2	4	4	-9	
1.5	6	8	-5		
1.75	14	3			
2	17				

ex. 2 $f(x)=e^x \to (\Delta_h^k f)(a_i)$, $a_i = a+ih$, $i\in\mathbb{N}$ $\Rightarrow a_{i+1}=a_i+h$

sol. $(\Delta_h f)(a_i) = f(a_{i+1}) - f(a_i) = e^{a+(i+1)h} - e^{a+ih} =$

$= e^{a+ih}(e^h-1) = e^{a_i}(e^h-1)$

$(\Delta_h^2 f)(a_i) = (\Delta_h f)(a_{i+1}) - (\Delta_h f)(a_i) = e^{a_{i+1}}(e^h-1) - e^{a_i}(e^h-1)$

$= e^{a_i}(e^h-1)^2$

Induction: suppose $P(k)$: $(\Delta_h^k f)(a_i) = e^{a_i}(e^h-1)^k \ (*)$, $\forall k\in\mathbb{N}^*$ (valid)

Prove $P(k+1)$: $(\Delta_h^{k+1} f)(a_i) = e^{a_i}(e^h-1)^{k+1}$

Verif step $P(1)$ above ✓

$P(k+1)$: $(\Delta_h^{k+1} f)(a_i) \overset{D.9}{=} (\Delta_h^k f)(a_{i+1}) - (\Delta_h^k f)(a_i) =$

$\overset{(*)}{=} e^{a_{i+1}}(e^h-1)^k - e^{a_i}(e^h-1)^k = (e^{a_i+h} - e^{a_i})(e^h-1)^k =$

$= e^{a_i}(e^h-1)^{k+1}$

Q.E.D.

$\Rightarrow (\Delta_h^k f)(a_i) = e^{a_i}(e^h-1)^k$

Divided Differences

ex. 6 $x_0=0,\ x_1=1,\ x_2=2,\ x_3=4$, $f_0=3,\ f_1=4,\ f_2=7,\ f_3=19$

x	f	Df	$D^2 f$	$D^3 f$
0	3	$\frac{4-3}{1-0}=1$	$\frac{3-1}{2-0}=1$	$\frac{1-1}{4-0}=0$
1	4	$\frac{7-4}{2-1}=3$	$\frac{6-3}{4-1}=1$	
2	7	$\frac{19-7}{4-2}=6$		
4	19			

Subiect: Data:

ex 17 $x_0=2,\ x_1=4,\ x_2=6,\ x_3=8;\ f_0=4,\ f_1=8,\ f_2=20,\ f_3=48$

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$
2	4	$\frac{8-4}{4-2}=2$	$\frac{6-2}{6-2}=1$	$\frac{2-1}{8-2}=\frac{1}{6}$
4	8	$\frac{20-8}{6-4}=6$	$\frac{14-6}{8-4}=2$	
6	20	$\frac{48-20}{8-6}=14$		
8	48			

ex 18 $x_0=1,\ x_1=2,\ x_2=3,\ x_3=5,\ x_4=7,\ f_0=3,\ f_1=5,\ f_2=9,\ f_3=11,\ f_4=15$

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
1	3	$\frac{5-3}{2-1}=2$	$\frac{4-2}{3-1}=1$	$\frac{-1-1}{5-1}=-\frac{1}{2}$	$\frac{-\frac{1}{4}+\frac{1}{2}}{7-1}=\frac{1}{24}$
2	5	$\frac{9-5}{3-2}=4$	$\frac{1-4}{5-2}=-1$	$\frac{-\frac{1}{4}+1}{7-2}=-\frac{1}{4}$	
3	9	$\frac{11-9}{5-3}=1$	$\frac{2-1}{7-3}=-\frac{1}{4}$		
5	11	$\frac{15-11}{7-5}=2$			
7	15				

Taylor

ex 23 1st 6 Taylor polynomials about $x_0=0$ for $f(x)=e^x \Rightarrow f^{(k)}(x)=e^x\ \forall k\in\mathbb{N}$

$$T_1(x)=\frac{(x-x_0)^0}{0!}f(x_0)+\frac{(x-x_0)^1}{1!}\underbrace{f'(x_0)}_{e^{x_0}=e^0=1}=\underbrace{f(x_0)}_{1}+x\cdot 1=e^0+x=x+1$$

$$T_2(x)=T_1(x)+\frac{(x-x_0)^2}{2!}f''(x_0)=\underbrace{e^0+x}_{x+1}+\frac{x^2}{2}$$

$$T_3(x)=T_2(x)+\frac{(x-x_0)^3}{3!}\underbrace{f^{(3)}(x_0)}_{1}=\frac{e^0}{1}+x+\frac{x^2}{2}+\frac{x^3}{3!}$$

$$\ldots \Rightarrow T_n(x)=\sum_{k=0}^{n}\frac{x^k}{k!}$$

ex 24 $f(x)=\frac{1}{x},\ x_0=1 \to$ approx. $f(3)$ by 1st & 2nd deg. Taylor polynomials

$$f'(x)=-\frac{1}{x^2},\quad f''(x)=\frac{2}{x^3}$$

$$T_0(x)=1$$

$$T_1(x)=\underbrace{\frac{(x-x_0)^0}{0!}}_{1}\underbrace{f(x_0)}_{1}+\frac{(x-x_0)^1}{1!}\underbrace{f'(x_0)}_{-\frac{1}{1^2}=-1}=1-(x-1)=2-x$$

$$T_2(x)=T_1(x)+\frac{(x-x_0)^2}{2!}f''(x_0)=2-x+\frac{(x-1)^2}{\cancel{2}}\cdot\frac{\cancel{2}}{1^3}=2-x+x^2-2x+1=$$
$$=x^2-3x+3$$

$$\Rightarrow T_1(3)=2-3=-1$$

$$T_2(3)=\cancel{3^2}-\cancel{3\cdot 3}+3=3$$

Subject: Numerical Calculus - Exercises (2) Data:

Lecture 1 ex. 25 $f(x)=e^x$, $(T_5 f)(x)$ on $|x|\le 1$, x_0 - center of interval

-> establish error of approx.

sol

$$(T_5 f)(x) \overset{ex.23}{=} 1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}=\sum_{k=0}^{5}\frac{x^k}{k!}$$

$$|\Delta x| = |f(x)-(T_5 f)(x)| = \left|e^x-\sum_{k=0}^{5}\frac{x^k}{k!}\right|$$

$$R_5(x)=\frac{(x-x_0)^6}{6!}f^{(6)}(\xi)=\frac{x^6}{6!}e^{\xi}, \quad \xi\in[\overset{=0}{x_0}, x] \Rightarrow |\xi|\le 1$$

$|x|\le 1$

$$\Rightarrow |R_5(x)| \le \frac{|x|^6}{6!}\cdot e^{|\xi|} \le \frac{1}{6!}\cdot e = \frac{e}{6!}$$

Lecture 2 ex. 5 a) x_0, x_1 - nodes; f - to interp.

b)

x	-1	0	3
$f(x)$	8	-2	4

=> Lagrange polynomial; $\sim f(-0.5)$

sol. a) $u(x)=(x-x_0)(x-x_1)$

$u_1(x)=x-x_0$

$u_0(x)=x-x_1$

$\Rightarrow$ $l_1(x)=\frac{x-x_0}{x_1-x_0}$ & $l_0(x)=\frac{x-x_1}{x_0-x_1}$

$(L_m f)(x)=l_0(x)\cdot f(x_0)+l_1(x)\cdot f(x_1)=\frac{x-x_1}{x_0-x_1}f(x_0)+\frac{x-x_0}{x_1-x_0}f(x_1)$

b) $u(x)=(x+1)(x-0)(x-3)$

$u_0(x)=x(x-3)$, $\quad l_0(x)=\frac{x(x-3)}{-1\cdot(-4)}=\frac{x(x-3)}{4}$

$u_1(x)=(x+1)(x-3)$, $\quad l_1(x)=\frac{(x+1)(x-3)}{-3}$

$u_2(x)=(x+1)x$, $\quad l_2(x)=\frac{(x+1)\cdot x}{(3+1)\cdot 3}=\frac{x(x+1)}{12}$

$$(L_m f)(x)=\frac{x(x-3)}{\cancel{4}}\cdot \overset{2}{\cancel{8}}+\frac{(x+1)(x-3)}{-3}\cdot(-2)+\frac{x(x+1)}{\underset{3}{\cancel{12}}}\cdot\cancel{4}$$

$$\Rightarrow (L_m f)(x)=\frac{6x^2-18x+2x^2-4x-\cancel{2}3+x^2+x}{3}=\frac{9x^2-21x-3\cdot 2}{3}$$

$$=3x^2-7x-2$$

$$(L_m f)(-0.5)=3\cdot(-0.5)^2-7\cdot(-0.5)-2=2.25$$

$72 = 2^3 \cdot 3^2$ | $75 = 3 \cdot 5^2$ | $80 = 2^4 \cdot 5^1$ | $81 = 3^4$

Subiect: Data:

Led 2 | ex 10 $\lg 2 = 0.301$, $\lg 3 = 0.477$, $\lg 5 = 0.699 \Rightarrow \lg 76$

x	2	3	5
$f(x)$	0.301	0.477	0.699

$\lg 75 = \lg 3 + 2\lg 5 = 0.477 + 2 \cdot 0.699 = 1.875$

$\lg 72 = 3\lg 2 + 2\lg 3 = 1.857$

$\lg 80 = 4\lg 2 + \lg 5 = 1.903$

$\lg 81 = 4\lg 3 = 1.908$

$u(x) = (x-x_0)(x-x_1)(x-x_2) = (x-2)(x-3)(x-5)$

$u_0(x) = (x-3)(x-5)$ $\quad l_1(x) =$

$u_1(x) = (x-2)(x-5)$

$u_2(x) = (x-2)(x-3)$

→ we need nodes around the required point

x	72	75	80	81
$f(x)$	1.857	1.875	1.903	1.908

$u(x) = (x-72)(x-75)(x-80)(x-81)$

$u_0(x) = (x-75)(x-80)(x-81)$ $\quad l_0(x) = \frac{(x-75)(x-80)(x-81)}{-216}$

$u_1(x) = (x-72)(x-80)(x-81)$ $\quad l_1(x) = \frac{(x-72)(x-80)(x-81)}{90}$

$u_2(x) = (x-72)(x-75)(x-81)$ $\quad l_2(x) = \frac{(x-72)(x-75)(x-81)}{-40}$

$u_3(x) = (x-72)(x-75)(x-80)$ $\quad l_3(x) = \frac{(x-72)(x-75)(x-80)}{54}$

$(L_m f)(x) = \frac{(x-75)(x-80)(x-81)}{-216} \cdot 1.857 + (x-72)(x-80)(x-81) \cdot 0.0208 -$

$= -0.0085$

$-(x-72)(x-75)(x-81) \cdot 0.0475 + (x-72)(x-75)(x-80) \cdot 0.0353 =$

$= (L_m f)(76) = 1.88088888$

ex 11

x	100	121	144
$f(x)$	10	11	12

$(\text{sqrt} \to \sqrt{x})$ $\quad f(x) =$

$f'(x) = \frac{1}{2} x^{-\frac{1}{2}}$

$f''(x) = -\frac{1}{4} \cdot x^{-\frac{3}{2}}$

$f'''(x) = \frac{3}{8} x^{-\frac{5}{2}}$

$l_0(x) f(x_0) = \frac{(x-121)(x-144)}{924} \cdot 10$ $\quad (L_m f)(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2)$

$l_1(x) f(x_1) = \frac{(x-100)(x-144)}{-483} \cdot 11$ $\quad (L_m f)(115) \cong 10.7227$

$l_2(x) f(x_2) = \frac{(x-100)(x-121)}{1012} \cdot 12$

$|(R_m f)(x)| \leq \frac{|u(x)|}{(m+1)!} \underbrace{\|f^{(m+1)}\|_\infty}_{\to f^{(3)}(100) =} = \left|\frac{(x-100)(x-121)(x-144)}{3!}\right| f^{(3)}(100)$

replace

$\to$ 115

Subiect: Numerical Calculus - Exercises (3)

Lecture 3 ex.3 $f(x) = \sin \frac{\pi}{1} x$, $x_0 = 0, x_1 = \frac{1}{6}, x_2 = \frac{1}{2} \Rightarrow L_m f$, $N_m f$ $(m=2)$

x	0	$\frac{1}{6}$	$\frac{1}{2}$
$f(x)$	0	$\frac{1}{2}$	1

$\bullet (L_m f)(x) = l_0(x) \cdot f(x_0) + l_1(x) f(x_1) + l_2(x) \cdot f(x_2) =$

$= \frac{(x-\frac{1}{6})(x-\frac{1}{2})}{(-\frac{1}{2})(-1)} \cdot 0 + \frac{x(x-\frac{1}{2})}{\frac{1}{6}(\frac{1}{6}-\frac{1}{2})} \cdot \frac{1}{2} + \frac{x(x-\frac{1}{6})}{\frac{1}{2}(\frac{1}{2}-\frac{1}{6})} \cdot 1 =$

algebraic calcul.:

$= \frac{x(2x-1)}{\frac{2}{6}\cdot(-\frac{2}{6})} + \frac{x(6x-1)}{\frac{6}{2}\cdot\frac{2}{6}} = -9x(2x-1) + x(6x-1) =$

$= x(-18x + 9 + 6x - 1) = x(-12x+8) = 4x(2-3x)$

$\bullet (N_m f)(x) = f(x_0) + (x-x_0)(D f)(x_0) + (x-x_0)(x-x_1)(D^2 f)(x_0) =$

x	f	Df	$D^2 f$
0	0	$\frac{\frac{1}{2}}{\frac{1}{6}} = 3$	$\frac{\frac{3}{2}-3}{\frac{1}{2}-0} = -3$
$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1-\frac{1}{2}}{\frac{1}{2}-\frac{1}{6}} = \frac{1}{2}\cdot\frac{6}{2} = \frac{3}{2}$	
$\frac{1}{2}$	1		

$= 3x + x(x-\frac{1}{6}) \cdot (-3) =$

$= 3x + \frac{-3x(6x-1)}{6} = \frac{6x - 6x^2 + x}{2} =$

$= \frac{-6x^2+7x}{2}$ ✓

Lecture 4 ex 3 $f(0) = 1, f'(0) = 2, f(1) = -3 \rightarrow$ Hermite interpol

sol: $x_0 = 0, r_0 = 1 \rightarrow r_k$ - the max. der. order given for x_k

$x_1 = 1, r_1 = 0$ $\quad n$ - degree of pol.

$m+1 = 2$ points $\Rightarrow m = 1$, $n = m + r_0 + r_1 = 1+1+0 = 2$

$(H_2 f)(x) = \sum_{k=0}^{1} \sum_{j=0}^{r_k} h_{kj}(x) f^{(j)}(x_k) =$

$= h_{00}(x) f^{(0)}(x_0) + h_{01}(x) f^{(1)}(x_0) + h_{10}(x) f^{(0)}(x_1) =$

$= h_{00}(x) \underbrace{f(0)}_{1} + h_{01}(x) \underbrace{f'(0)}_{2} + h_{10}(x) \underbrace{f(1)}_{-3}$

$n=2 \Rightarrow h_{kj} = a x^2 + bx + c$

$\cdot\ h_{00}(x) = a_1 x^2 + b_1 x + c_1 \Rightarrow h_{00}' = 2a_1 x + b_1$

$\begin{cases} h_{00}(x_0) = \delta_{00} = 1 \\ h_{00}'(x_0) = \delta_{01} = 0 \\ h_{00}(x_1) = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 1 \\ b_1 = 0 \\ a_1 + b_1 + 1 = 0 \Rightarrow a_1 = -1 \end{cases} \Rightarrow h_{00} = -x^2 + 1$

$x_0=0, x_1=1$

Subiect: Data:

- $h_{01}(x) = a_2x^2+b_2x+c_2$

$$\begin{cases} h_{01}(x_0) = \delta_{10} = 0 \\ h'_{01}(x_0) = \delta_{11} = 1 \\ h_{01}(x_1) = 0 \end{cases} \Rightarrow \begin{cases} c_2 = 0 \\ 2a_2 \cdot 0 + b_2 = 1 \Rightarrow b_2 = 1 \\ a_2+b_2+c_2 = 0 \Rightarrow a_2 = -1 \end{cases} \Rightarrow h_{01}(x) = -x^2+x$$

- $h_{10}(x) = a_3x^2+b_3x+c_3$

$$\begin{cases} h_{10}(x_0) = 0 \\ h'_{10}(x_0) = 0 \\ h_{10}(x_1) = \delta_{00} = 1 \end{cases} \Rightarrow \begin{cases} c_3 = 0 \\ b_3 = 0 \\ a_3+b_3+c_3 = 1 \Rightarrow a_3 = 1 \end{cases} \Rightarrow h_{10}(x) = x^2$$

$$\Rightarrow (H_2f)(x) = -x^2+1 -2x^2+2x-3x^2 = -6x^2+2x+1$$

ex. 7 $f(x) = xe^x$, $f(-1) = -0.3679$, $f(0) = 0$, $f'(0) = 1$, $f(1) = 2.7183$

$m=2$, $x_0=-1 \to r_0 = 0$; $x_1 = 0 \to r_1 = 1$; $x_2 = 1 \to r_2 = 0 \Rightarrow n = m+r_0+r_1+r_2$
$n = 3$

$$|(R_n f)(x)| \le \frac{|u(x)|}{(n+1)!} \| f^{(n+1)} \|_\infty =$$

$f'(x) = xe^x+e^x = (x+1)e^x$

$f''(x) = (x+2)e^x$

$f^{(k)}(x) = (x+k)e^x$

$$= \frac{|(x-x_0)^{r_0+1}(x-x_1)^{r_1+1}(x-x_2)^{r_2+1}|}{4!} \cdot \|(x+4)e^x\|_\infty$$

$$= \frac{|(x+1)\cdot x^2 \cdot (x-1)|}{4!} \cdot 5e \quad \underset{x=\frac{1}{2}}{\text{repl.}} \quad \frac{\left|\frac{1}{4}\cdot\left(\frac{1}{2}+1\right)\left(\frac{1}{2}-1\right)\right|}{4!} \cdot 5e = \frac{\left|\frac{1}{4}\cdot\frac{3}{2}\cdot\left(-\frac{1}{2}\right)\right|}{1\cdot 2\cdot 3\cdot 4} \cdot 5e$$

$$= \frac{5e}{2^7}$$

Subiect: Numerical Calculus - Exercises (4)   Data:..............................

Lecture 5 - Ex: Hermite interp. w. double nodes

$x_0 = -1,\ f(-1) = -3,\ f'(-1) = 10$

$x_1 = 1,\ f(1) = 1,\ f'(1) = 2$

	z	f	Δf	$\Delta^2 f$	$\Delta^3 f$
0	-1	-3	10	$\frac{2-10}{1-(-1)} = -4$	$\frac{0-(-4)}{1-(-1)} = 2$
1	-1	-3	$\frac{1-(-3)}{1-(-1)} = 2$	$\frac{2-2}{1-(-1)} = 0$	
2	1	1	2		
3	1	1			

$$\Rightarrow (H_3 f)(x) = f(x_0) + \cancel{\sum_{i=1}^{3}} (x-z_0)(\Delta f)(z_0) + (x-z_0)(x-z_1)(\Delta^2 f)(z_0) + (x-z_0)(x-z_1)(x-z_2)(\Delta^3 f)(z_0) =$$

$$= -3 + (x+1)\cdot 10 + (x+1)^2 \cdot (-4) + (x+1)^2 (x-1) \cdot 2 \ \ldots$$

Subiect: Numerical Calculus - Exercises (5) Data:...........

Lecture 5 Birkhoff

Ex. 7 $f \in C^2[0,1]$, $x_0 = 0$, $x_1 = 1$, $f(0) = 1$, $f'(1) = \frac{1}{2}$ $\Rightarrow I_0 = \{0\} \to |I_0| = 1$, $I_1 = \{1\} \to |I_1| = 1$

$n = 1+1-1 = 1$, $m = 1$

We check if $\exists$ sol. of the problem

Consider $P(x) = a_1 x + a_0 \in \mathbb{P}_1 \Rightarrow P'(x) = a_1$

$\begin{cases} P(0) = f(0) \\ P'(1) = f'(1) \end{cases} \Leftrightarrow \begin{cases} a_0 = f(0) \\ a_1 = f'(1) \end{cases} \Rightarrow$ Det. of sys. $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$

$\Rightarrow \exists$ unique sol.

Birkhoff pol. is $(B_1 f)(x) = b_{00}(x) f(x_0) + b_{11}(x) f'(x_1) =$

$= b_{00}(x) + \frac{1}{2} b_{11}(x)$

- $b_{00}(x) = a_1 x + t_1$

$\begin{cases} b_{00}(x_0) = \delta_{00} = 1 \\ b_{00}'(x_1) = 0 \end{cases} \Rightarrow \begin{cases} t_1 = 1 \\ a_1 = 0 \end{cases} \Rightarrow b_{00}(x) = 1$

- $b_{11}(x) = a_2 x + t_2$

$\begin{cases} b_{11}(x_0) = 0 \\ b_{11}'(x_1) = \delta_{11} = 1 \end{cases} \Rightarrow \begin{cases} t_2 = 0 \\ a_2 + t_2 = 1 \to a_2 = 1 \end{cases} \Rightarrow b_{11}(x) = x$

$\Rightarrow (B_1 f)(x) = 1 + \frac{x}{2}$

Ex. 8 $f'(0) = 1$, $f(1) = 2$, $f'(2) = 1$ $\to$ $I_0 = \{1\}$, $I_1 = \{0\}$, $I_2 = \{1\}$ $\Rightarrow f(\frac{1}{2}) \approx ?$

$x_0 = 0$, $x_1 = 1$, $x_2 = 2$

sol: $m = 2$, $n = 1+1+1-1 = 2$

We check if $\exists$ sol. of the problem, $n=2$ $\Rightarrow P'(x) = 2a_2 x + a_1$

Consider $P(x) = a_2 x^2 + a_1 x + a_0 \in \mathbb{P}_2$

$\begin{cases} P'(0) = f'(0) \\ P(1) = f(1) \\ P'(2) = f'(2) \end{cases} \Leftrightarrow \begin{cases} a_1 = 1 \\ a_2 + a_1 + a_0 = 2 \\ 4a_2 + a_1 = 1 \end{cases} \Rightarrow$ Det. of sys. is $\begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 4 & 1 & 0 \end{vmatrix} = 4 \neq 0 \Rightarrow \exists$ unique sol.

$(B_2 f)(x) = b_{01}(x) f'(x_0) + b_{10}(x) f(x_1) + b_{21}(x) f'(x_2)$

Subiect: Data:

Birkhoff polynomial is $(B_2 f)(x) = b_{01}(x) f'(x_0) + b_{10}(x) f(x_1) + b_{21}(x) f''(x_2)$

- $b_{01}(x) = a_1 x^2 + b_1 x + c_1 \Rightarrow b'_{01}(x) = 2a_1 x + b_1$

$$\begin{cases} b'_{01}(x_0) = 1 \\ b_{01}(x_1) = 0 \\ b''_{01}(x_2) = 0 \end{cases} \Rightarrow \begin{cases} b_1 = 1 \\ a_1 + b_1 + c_1 = 0 \Rightarrow c_1 = -\frac{3}{4} \Rightarrow b_{01}(x) = -\frac{1}{4}x^2 + x - \frac{3}{4} \\ 4a_1 + b_1 = 0 \Rightarrow a_1 = -\frac{1}{4} \end{cases}$$

- $b_{10}(x) = a_2 x^2 + b_2 x + c_2$

$$\begin{cases} b'_{10}(x_0) = 0 \\ b_{10}(x_1) = 1 \\ b''_{10}(x_2) = 0 \end{cases} \Rightarrow \begin{cases} b_2 = 0 \\ a_2 + b_2 + c_2 = 1 \Rightarrow c_2 = 1 \Rightarrow b_{10}(x) = 1 \\ 4a_2 + b_2 = 0 \Rightarrow a_2 = 0 \end{cases}$$

- $b_{21}(x) = a_3 x^2 + b_3 x + c_3$

$$\begin{cases} b'_{21}(x_0) = 0 \\ b_{21}(x_1) = 0 \\ b''_{21}(x_2) = 1 \end{cases} \Rightarrow \begin{cases} b_3 = 0 \\ a_3 + b_3 + c_3 = 0 \Rightarrow c_3 = \frac{1}{4} \Rightarrow b_{21}(x) = +\frac{1}{4}x^2 - \frac{1}{4} \\ 4a_3 + b_3 = 1 \Rightarrow a_3 = +\frac{1}{4} \end{cases}$$

$\Rightarrow (B_2 f)(x) = \cancel{-\frac{1}{4}x^2} + x - \frac{3}{4} + 2 \cdot 1 + \cancel{\frac{1}{4}x^2} - \frac{1}{4} = x + 1$

$\Rightarrow (B_2 f)\left(\frac{1}{2}\right) = \frac{1}{2} + 1 = \frac{3}{2} = 1.5$

Subject: Numerical Calculus (Exercises) - (6) Data:..............................

Lecture 6 - Least Square Approximation

Ex3

x	-3	-1	2
$f(x)$	-4	-2	3

$\sum x_i f(x_i) = 12+2+6 = 20$

$\sum x_i = -2$ $\quad \sum x_i^2 = 14$

$\sum f(x_i) = -3$

a) line $\to \varphi(x) = ax+b$

$$\sum_{i=0}^{2} \left[f(x_i) - \varphi(x_i)\right]^2 \ min$$

$$E(a,b) = \sum_{i=0}^{2} \left[f(x_i) - (ax_i+b)\right]^2 \to min$$

$$\begin{cases} \dfrac{\partial E(a,b)}{\partial a} = 0 \\ \dfrac{\partial E(a,b)}{\partial b} = 0 \end{cases} \Rightarrow \begin{cases} \sum_{i=0}^{2} (-2x_i)\left[f(x_i) - (ax_i+b)\right] = 0 \quad \Big| :(-2) \\ \sum_{i=0}^{2} (-2)\left[f(x_i) - (ax_i+b)\right] = 0 \quad \Big| :(-2) \end{cases}$$

$$\Rightarrow \begin{cases} \sum_{i=0}^{2} x_i f(x_i) = a\sum_{i=0}^{2} x_i^2 + b\sum_{i=0}^{2} x_i \\ \sum_{i=0}^{2} f(x_i) = a\sum_{i=0}^{2} x_i + b\sum_{i=0}^{m} 1 \end{cases}$$

$$\Rightarrow \begin{cases} a = \dfrac{(m+1)\sum_{i=0}^{m} x_i f(x_i) - \sum_{i=0}^{m} x_i \sum_{i=0}^{m} f(x_i)}{(m+1)\sum_{i=0}^{m} x_i^2 - \left(\sum_{i=0}^{m} x_i\right)^2} \\ b = \dfrac{\sum_{i=0}^{m} x_i^2 \sum_{i=0}^{m} f(x_i) - \sum_{i=0}^{m} x_i f(x_i) \sum_{i=0}^{m} x_i}{(m+1)\sum_{i=0}^{m} x_i^2 - \left(\sum_{i=0}^{m} x_i\right)^2} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} a = \dfrac{3\cdot 20 - (-2)(-3)}{3\cdot 14 - 4} = \dfrac{\cancel{54}^{27}}{\cancel{38}_{19}} = \dfrac{27}{19} \\ b = \dfrac{14\cdot(-3) - 20\cdot(-2)}{3\cdot 14 - 4} = \dfrac{\cancel{82}^{41}}{\cancel{38}_{19}} = \dfrac{41}{19} \end{cases} \Rightarrow \varphi(x) = \dfrac{27x+41}{19}$$

$\left(\frac{f}{g}\right)' = \frac{f'\cdot g - f\cdot g'}{g^2}$

Subiect: Numerical Calculus - Exercises (7) $\rightarrow$ Data: $x_0=1, x_1=2, x_2=3$

Lecture 7 - ex 12   1. $A=\int_1^3 (2x+1)\,dx$ - repeated trapz. for $m=2$

$$A=\frac{3-1}{2\cdot 2}\Big[f(1)+f(3)+2f(\underset{2}{x_1})\Big]=\frac{1}{2}(3+7+2\cdot 5)=\frac{20}{2}=10$$

2. Approx $\frac{\pi}{4}$ with repeated trapezium's formula, $\varepsilon=10^{-2}$

sol: $\frac{\pi}{4}=\operatorname{arctg}(1)=\int_0^1\frac{dx}{1+x^2} \rightarrow f(x)=\frac{1}{1+x^2} \rightarrow f'(x)=-\frac{2x}{(1+x^2)^2}$

$$f''(x)=-\frac{2(x^2+1)^2-2x\cdot 2\cdot 2x(x^2+1)}{(1+x^2)^4}=\frac{2(x^2+1)(4x^2-x^2-1)}{(x^2+1)^{\cancel{4}\,3}}=\frac{2(3x^2-1)}{(x^2+1)^3}$$

We have to solve $\frac{(1-0)^3}{\cancel{12}\cdot m^2}\cdot\frac{\cancel{2}(3\cdot 1^2-1)}{(1^2+1)^3}<\varepsilon$

$$\frac{1}{2^3\cdot 3\cdot m^2}<\frac{1}{2^2\cdot 5^2}$$

$$m^2>\frac{2^2\cdot 5^2}{2^3\cdot 3}=\frac{25}{6} \qquad 24<25<30 \;\Big|\cdot\frac{1}{6}$$

$$4<\frac{25}{6}<5 \Rightarrow \text{min. } m=5$$

$$\frac{1}{1+\left(\frac{k}{5}\right)^2}=\frac{25}{k^2+25}$$

$$\Rightarrow A=\int_0^1\frac{dx}{1+x^2}=\frac{1-0}{2\cdot 5}\Big[f(0)+f(1)+2\Big(f(\tfrac{1}{5})+f(\tfrac{2}{5})+f(\tfrac{3}{5})+f(\tfrac{4}{5})\Big)\Big]=$$

$$=\frac{1}{10}\left(1+\frac{1}{2}+2\cdot 25\left(\frac{1}{26}+\frac{1}{29}+\frac{1}{34}+\frac{1}{41}\right)\right)=0.7837$$

Subiect: Numerical Calculus - Exercises (8) Data:..................

Gauss Ex. 8 Partial pivoting

Step 1) $L_1 \leftrightarrow L_4$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 5 \\ -1 & 1 & -5 & 3 \\ 3 & 1 & 7 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 10 \\ 31 \\ -2 \\ 18 \end{pmatrix}$$

pivot 3

$m_{21} = \frac{a_{21}}{a_{11}} = \frac{2}{3}$, $m_{31} = \frac{a_{31}}{a_{11}} = -\frac{1}{3}$, $m_{41} = \frac{a_{41}}{a_{11}} = \frac{1}{3}$

$$\begin{pmatrix} 3 & 1 & 7 & -2 \\ 2 & 3 & 1 & 5 \\ -1 & 1 & -5 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 18 \\ 31 \\ -2 \\ 10 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 3 & 1 & 7 & -2 & 18 \\ 0 & 2.33 & 1-\frac{2}{3}\cdot 7 = -3.66 & 6.33 & 19 \\ 0 & 1+\frac{1}{3} = 1.33 & -2.66 & 2.33 & 4 \\ 0 & 0.66 & -1.33 & 1.66 & 4 \end{array}\right)$$

$m_{32} = \frac{1.33}{2.33} = 0.57$

$m_{42} = \frac{0.66}{2.33} = 0.28$

$$\left(\begin{array}{cccc|c} 3 & 1 & 7 & -2 & 18 \\ 0 & 2.33 & -3.66 & 6.33 & 19 \\ 0 & 0 & -0.57 & -1.27 & -6.83 \\ 0 & 0 & -0.3 & -0.11 & -1.32 \end{array}\right)$$

$m_{43} = \frac{-0.3}{-0.57} = 0.52$

$$\left(\begin{array}{cccc|c} 3 & 1 & 7 & -2 & 18 \\ 0 & 2.33 & -3.66 & 6.33 & 19 \\ 0 & 0 & -0.57 & -1.27 & -6.83 \\ 0 & 0 & 0 & 0.55 & 2.23 \end{array}\right)$$

Upper triangular system

Step 2) Back substitution

$x_4 = \frac{2.23}{0.55} = 4.05$

$x_3 = \frac{-6.83 + 1.27 \cdot x_4}{-0.57} = 2.96$

$x_2 = \frac{19 - 6.33 \cdot x_4 + 3.66 \cdot x_3}{2.33} = 1.8$

$x_1 = \frac{18 + 2x_4 - 7x_3 - x_2}{3} = 1.19$

Ex. 9

$$\begin{cases} 2x + y = 3 \\ 3x - 2y = 1 \end{cases} \Rightarrow \left(\begin{array}{cc|c} 2 & 1 & 3 \\ 3 & -2 & 1 \end{array}\right) \to \left(\begin{array}{cc|c} 3 & -2 & 1 \\ 2 & 1 & 3 \end{array}\right)$$

$m_{21} = \frac{2}{3}$

$$\left(\begin{array}{cc|c} 3 & -2 & 1 \\ 0 & 2.33 & 2.33 \end{array}\right) \Rightarrow$$

$x_2 = \frac{2.33}{2.33} = 1$

$x_1 = \frac{1 + 2x_2}{3} = \frac{1+2}{3} = 1$

Ex. 10 Gauss elimination method

$$\begin{cases} x_1 + x_2 + x_3 = 4 \\ 2x_1 - 2x_2 + 3x_3 = 5 \\ x_1 - x_2 + 4x_3 = 5 \end{cases} \xrightarrow{L_1 \leftrightarrow L_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & -2 & 3 & 5 \\ 1 & -1 & 4 & 5 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 2 & -2 & 3 & 5 \\ 1 & 1 & 1 & 4 \\ 1 & -1 & 4 & 5 \end{array}\right)$$

$m_{21} = \frac{1}{2} = m_{31}$

$$\left(\begin{array}{ccc|c} 2 & -2 & 3 & 5 \\ 0 & 2 & -0.5 & 1.5 \\ 0 & 0 & 2.5 & 2.5 \end{array}\right)$$

$m_{32} = 0$

$$\left(\begin{array}{ccc|c} 2 & -2 & 3 & 5 \\ 0 & 2 & -0.5 & 1.5 \\ 0 & 0 & 2.5 & 2.5 \end{array}\right)$$

$m_{23} = -\frac{1}{5}$, $m_{13} = \frac{6}{5}$

$$\Rightarrow \left(\begin{array}{ccc|c} 2 & -2 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2.5 & 2.5 \end{array}\right)$$

$m_{12} = -\frac{2}{2} = -1$

$$\Rightarrow \left(\begin{array}{ccc|c} 2 & 0 & 0 & 4 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2.5 & 2.5 \end{array}\right) \Rightarrow$$

$x_1 = \frac{4}{2} = 2$

$x_2 = \frac{2}{2} = 1$

$x_3 = \frac{2.5}{2.5} = 1$