$i = 0$

(3p) 1. Find the coefficients a, b and c of the following quadrature formula:

$$\int_0^1 f(x)dx = af(0) + bf(1) + cf'(1) + R(f)$$

(3p) 2. Consider the plane $\mathcal{P} : x + y + z = 1$ and the line $\ell : \frac{x-1}{2} = y = z + 1$. Using the Gauss partial elimination method, find the coordinates of the intersection point between $\mathcal{P}$ and ℓ.

3. Let $g(x) = -4 + 4x - \frac{1}{2}x^2$. **(2p)** a) Show that $P = 2$ and $P = 4$ are fixed points.

(1p) b) Use the starting value $x_0 = 1$ and compute the approximations x_1 and x_2 of the solution of the equation $g(x) = x$, using succesive approximations method.

1. Find the coefficients a, b and c of the following quadrature formula: $\int_0^1 f(x)dx = af(0) + b\cdot f(1) + c\cdot f'(1) + R(f)$.

Since ~~only~~ $f(0)$, $f(1)$ and $f'(1)$ are available, we will aproximate $f(x)$ using Hermite interpolation formula.

We have $x_0 = 0$ and $x_1 = 0$, so $m = 1$, $r_0 = 0$, $r_1 = 1$ and $n = 1+2-1 = 2$

We check if there exists a solution for the problem (determining f).
We consider $P(x) = a_2x^2 + a_1x + a_0 \in P_2$ and the system

$$\begin{cases} P(0) = f(0) \\ P(1) = f(1) \\ P'(1) = f'(1) \end{cases} \iff \begin{cases} a_0 = f(0) \\ a_2 + a_1 + a_0 = f(1) \\ 2a_2 + a_1 = f'(1) \end{cases}$$

NOT NECESSARY FOR HERMITE, ONLY FOR BIRKHOFF

The determinant of the system is

$$\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{vmatrix} = -1 \neq 0$$

so the problem has a unique solution

The Hermite polynomial is

$(H_2f)(x) = h_{00}(x)f(0) + h_{10}(x)f(1) + h_{11}(x)f'(x_1) \in P_2$

We have $h_{00}(x) = mx^2 + mx + n \in P$ and

$$\begin{cases} h_{00}(0) = 1 \\ h_{00}(1) = 0 \\ h_{00}'(1) = 0 \end{cases} \iff \begin{cases} m\cdot 0^2 + m\cdot 0 + n = 1 \\ m\cdot 1^2 + m\cdot 1 + n = 0 \\ 2m\cdot 1 + m = 0 \end{cases} \iff \begin{cases} n = 1 \\ m + m + n = 0 \\ 2m + m = 0 \end{cases} \iff \begin{cases} n = 1 \\ m = 1 \\ m = -2 \end{cases}$$

so $h_{00}(x) = x^2 - 2x + 1$

We have $h_{10}(x) = g\cdot x^2 + n\cdot x + s \in P$ and

$$\begin{cases} h_{10}(0) = 0 \\ h_{10}(1) = 1 \\ h_{10}'(1) = 0 \end{cases} \iff \begin{cases} g\cdot 0 + n\cdot 0 + s = 0 \\ g\cdot 1 + n\cdot 1 + s = 1 \\ 2g\cdot 1 + n\cdot 1 = 0 \end{cases} \iff \begin{cases} s = 0 \\ g + n + s = 1 \\ 2g + n = 0 \end{cases} \iff \begin{cases} s = 0 \\ g = -1 \\ n = 2 \end{cases}$$

so $h_{10}(x) = -x^2+2x$

We have $h_{11}(x) = t \cdot x^2 + u \cdot x + v$ and

$$\begin{cases} h_{11}(0)=0 \\ h_{11}(1)=0 \\ h_{11}'(1)=0 \end{cases} \Leftrightarrow \begin{cases} t\cdot 0 + u \cdot 0 + v = 0 \\ t \cdot 1 + u \cdot 1 + v = 0 \\ 2t+v=1 \end{cases} \Leftrightarrow \begin{cases} v=0 \\ t+u=0 \\ 2t+u=1 \end{cases} \Leftrightarrow \begin{cases} v=0 \\ t=1 \\ u=-1 \end{cases}$$

so $h_{11}(x) = x^2-x$

With these, $f(x) = (H_2 f)(x) + (R_2 f)(x) = (x^2-2x+1)\cdot f(0) + (-x^2+2x)\cdot f(1) + (x^2-x)\cdot f'(1) +$
$+ (R_2 f)x$

Then, the coefficients a, b, c from the quadrature formula are equal to:

$$a = \int_0^1 (x^2-2x+1)\,dx = \frac{x^3}{3} - x^2 + x \Big|_0^1 = \frac{1}{3}$$

$$b = \int_0^1 (-x^2+2x)\,dx = -\frac{x^3}{3} + x^2 \Big|_0^1 = \frac{2}{3}$$

$$c = \int_0^1 (x^2-x)\,dx = \frac{x^3}{3} - \frac{x^2}{2} \Big|_0^1 = -\frac{1}{6}$$

So, $\int_0^1 f(x)\,dx = \frac{1}{3} f(0) + \frac{2}{3} f(1) - \frac{1}{6} f'(1) + R_2(f)$

2. Let $P_0(x_0, y_0, z_0)$ be the intersection point

$$P_0 \in P \cap l \Leftrightarrow \begin{cases} x_0+y_0+z_0=1 \\ \frac{x_0-1}{2} = y_0 = z_0 \end{cases} \Leftrightarrow \begin{cases} x_0+y_0+z_0=1 \\ x_0-1 = 2y_0 \\ y_0 = z_0 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x_0+y_0+z_0=1 \\ x_0-2y_0 = 1 \\ y_0-z_0=0 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 0 \\ 0 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

The pivot can be either a_{11} or a_{21}. We choose a_{11}. We have

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -2 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{array}\right) \quad L_2 \leftarrow L_2 - L_1$$

We apply $L_2 \leftarrow L_2 - L_1$ and we have:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -3 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right)$$

The pivot is a_{22}, so the matrix remains the same. We apply $L_3 \leftarrow L_3 + L_2/3$ and we have

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -3 & -1 & 0 \\ 0 & 0 & -\frac{4}{3} & 0 \end{array}\right)$$

Now, we can apply the process of backsubstitution:

$z_0 = 0 : \left(-\frac{4}{3}\right) = 0$

$y_0 = \frac{0 + z_0}{-3} = 0$ So, our point is $P_0(1, 0, 0)$

$x_0 = \frac{1 - y_0 - z_0}{1} = 1$

3. $g(x) = -4 + 4x - \frac{1}{2}x^2$

a) $g(2) = -4 + 4 \cdot 2 - \frac{1}{2} \cdot 2^2 = -4 + 8 - 2 = 2 \Rightarrow P = 2$ is a fixed point for g

$g(4) = -4 + 4 \cdot 4 - \frac{1}{2} \cdot 4^2 = -4 + 16 - 8 = 4 \Rightarrow P = 4$ is a fixed point for g

b) Let $f(x) = g(x) - x$. Then, the solution of $g(x) = x$ is the solution of $f(x) = 0$.

$f(x) = -4 + 3x - \frac{1}{2}x^2$

We will use Newton's method as a successive approximation method

Then,

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{-1 - \frac{1}{2}}{3 - 1} = 1 + \frac{3}{4} = \frac{7}{4} = 1{,}75$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1{,}75 - \frac{-0{,}28125}{1{,}25} = 1{,}975$

1. **(3p)** Find the polynomial that meets the following specifications:

x_k	$f(x_k)$	$f'(x_k)$
0	1	1
1	3	4

This polynomial can be viewed as a switching path between parallel tracks.

2. **(3p)** The solid obtained by rotating the region under the curve $y = f(x)$, where $a \leq x \leq b$, about the $x-$axis has surface area given by $area = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}dx$. Approximate the surface area using the repeated trapezoidal rule with $n = 5$ and $f(x) = x^3$ for $0 \leq x \leq 1$.

3. **(3p)** For the function $f(x) = x^2 - 9x + 18$, find an interval $[a, b]$ so that the bisection method can be applied. Give the first two iterations.

We have $h_{11}(x) = a_4 x^3 + b_4 x^2 + c_4 x + d_4 \in P_3$ and

$$\begin{cases} h_{11}(x_0) = 0 \\ h_{11}'(x_0) = 0 \\ h_{11}(x_0) = 0 \\ h_{11}'(x_0) = 1 \end{cases} \Leftrightarrow \begin{cases} d_4 = 0 \\ c_4 = 0 \\ a_4 + b_4 + c_4 + d_4 = 0 \\ 3a_4 + 2b_4 + c_4 = 1 \end{cases} \Leftrightarrow \begin{cases} d_4 = 0 \\ c_4 = 0 \\ b_4 = -1 \\ a_4 = 1 \end{cases}$$

so, $h_{11}(x) = x^3 - x^2$

Then, $(H_3 f)(x) = h_{00}(x) \cdot f(0) + h_{01}(x) \cdot f'(0) + h_{10}(x) \cdot f(1) + h_{11}(x) \cdot f'(1)$

$= 2x^3 - 3x^2 + 1 + 2x^3 - 3x^2 + x - 2x^3 + 3x^2 + 4x^3 - 4x^2 = 6x^3 - 7x^2 + x + 1$

2. $\text{area} = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2}\, dx$

$f(x) = x^3$; $a = 0$; $b = 1$; $m = 5$; $x_k = a + kh$; $k = \overline{0, m} = \overline{0, 5}$; $h = \frac{b-a}{m} = \frac{1}{5}$

$\text{area} = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2}\, dx = 2\pi \int_0^1 \underbrace{x^3 \cdot \sqrt{1 + 9x^4}}_{g(x)}\, dx = 2\pi \int_0^1 g(x)\, dx$

$\text{area} \xlongequal[\text{Trap.}]{\text{Rep.}} 2\pi \left[\frac{b-a}{2m} \left(g(a) + g(b) + 2 \sum_{k=1}^{m-1} g(x_k) \right) + R_m(f) \right] =$

$= 2\pi \left\{ \frac{1}{10} \left[g(0) + g(1) + 2\left(g(\tfrac{1}{5}) + g(\tfrac{2}{5}) + g(\tfrac{3}{5}) + g(\tfrac{4}{5}) \right) \right] + R_5(f) \right\}$

$\cong 2\pi \left\{ \frac{1}{10} \left[0 + \sqrt{10} + 2(0{,}00805 + 0{,}07099 + 0{,}31732 + 1{,}10838) \right] + R_5(f) \right\}$

$\cong 2\pi \left[\frac{1}{10} (3{,}16227 + 3{,}01068) + R_5(f) \right]$

$= 2\pi (0{,}617295 + R_5(f)) \cong 2\pi \cdot 0{,}617295 \cong 3{,}87857$

3. $f(x) = x^2 - 9x + 18$

$\left.\begin{array}{l} f(2) = 4 - 18 + 18 = 4 > 0 \\ f(5) = 25 - 45 + 18 = -2 < 0 \\ f \text{ is continuous} \end{array}\right| \Rightarrow \exists\, x \in (2, 5) \text{ s.t. } f(x) = 0$

1. $x_0=0,\ x_1=1,\ f(0)=1,\ f(1)=3,\ f'(0)=1,\ f'(1)=4$

We will use the Hermite interpolation formula since $f(0), f(1)$ and $f'(0), f'(1)$ are available.

We have x_0, x_1 so $m=1,\ r_0=1,\ r_1=1$ and $n=m+r_0+r_1=3$

The Hermite polynomial is:

$$(H_3 f)(x) = h_{00}(x)\cdot f(x_0) + h_{01}(x)\cdot f'(x_0) + h_{10}(x)\cdot f(x_1) + h_{11}(x)\cdot f'(x_1)$$

We have $h_{00}(x) = a_1x^3 + b_1x^2 + c_1x + d \in P_3$ and

$$\begin{cases} h_{00}(x_0)=1 \\ h_{00}'(x_0)=0 \\ h_{00}(x_1)=0 \\ h_{00}'(x_1)=0 \end{cases} \Leftrightarrow \begin{cases} d_1=1 \\ c_1=0 \\ a_1+b_1+c_1+d_1=0 \\ 3a_1+2b_1+c_1=0 \end{cases} \Leftrightarrow \begin{cases} d_1=1 \\ c_1=0 \\ b_1=-3 \\ a_1=2 \end{cases}$$

so, $h_{00}(x) = 2x^3-3x^2+1$

We have $h_{01}(x) = a_2x^3 + b_2x^2 + c_2x + d_2 \in P_3$ and

$$\begin{cases} h_{01}(x_0)=0 \\ h_{01}'(x_0)=1 \\ h_{01}(x_1)=0 \\ h_{01}(x_1)=0 \end{cases} \Leftrightarrow \begin{cases} d_2=0 \\ c_2=1 \\ a_2+b_2+c_2+d_2=0 \\ 3a_2+2b_2+c_2=0 \end{cases} \Leftrightarrow \begin{cases} d_2=0 \\ c_2=1 \\ b_2=-3 \\ a_2=2 \end{cases}$$

so, $h_{01}(x) = 2\cdot x^3 - 3x^2 + x$

We have $h_{10}(x) = a_3x^3 + b_3x^2 + c_3x + d_3 \in P_3$ and

$$\begin{cases} h_{10}(x_0)=0 \\ h_{10}'(x_0)=0 \\ h_{10}(x_0)=1 \\ h_{10}'(x_0)=0 \end{cases} \Leftrightarrow \begin{cases} d_3=0 \\ c_3=0 \\ a_2+b_2+c_3+d_3=1 \\ 3a_2+2b_2+c_3=0 \end{cases} \Leftrightarrow \begin{cases} d_3=0 \\ c_3=0 \\ b_2=3 \\ a_2=-2 \end{cases}$$

so, $h_{10}(x) = -2x^3+3x^2$

First Iteration

$a_0=2,\ b_0=5,\ c_0=\frac{2+5}{2}=3,5,\ f(c_0)=12,25-31,5+18=-1,25$

$f(a_0)\cdot f(c_0)=4\cdot(-1,25)<0 \Rightarrow a_1=2,\ b_1=c_0=3,5$

Second Iteration

$a_1=2,\ b_1=3,5,\ c_1=\frac{2+3,5}{2}=2,75,\ f(c)=7,5625-24,75+18=$

$=0,8125$

$f(a_1)\cdot f(c_1)=4\cdot 0,8125>0 \Rightarrow a_2=c_1=2,75,\ b_2=3,5$