Taylor's polynomial of degree n *of* f *at* x_0 **Lecture 1**

$$p_n(x) = f(x_0) + \frac{(x-x_0)}{1!}f'(x_0) + \dots + \frac{(x-x_0)^n}{n!}f^{(n)}(x_0) \quad (1.2)$$

error (1.3)

$$R_{n+1}(x) = \frac{1}{n!}\int^{x}(x-t)^n f^{(n+1)}(t)dt = \frac{(x-x_0)^{n+1}}{(n+1)!}f^{(n+1)}(\xi),\ \xi \text{ between } x \text{ and } x_0$$

Polynomial Interpolation **Lecture 2**

Lagrange Interpolation

$$l_i(x) = \prod_{\substack{j=0\\ j\neq i}}^{n}\frac{x-x_j}{x_i-x_j} = \frac{u_i(x)}{u_i(x_i)} = \frac{u_i(x)}{u'(x_i)} \quad (1.4)$$

$$L_nf(x) = \sum_{i=0}^{n} l_i(x)f(x_i) \quad (1.3)$$

exists $\xi \in (a,b)$ $\quad (R_nf)(x) = \frac{u(x)}{(n+1)!}f^{(n+1)}(\xi).$ (1.6)

Barycentric interpolation **Lecture 3**

$$w_i = \frac{1}{u'(x_i)} = \frac{1}{\prod_{\substack{j=0\\ j\neq i}}^{n}(x_i-x_j)},\ i=0,1,\dots,n. \quad \textbf{barycentric weights.}\ (1.2)$$

$$\boxed{L_nf(x) = u(x)\sum_{i=0}^{n}\frac{w_i}{x-x_i}f_i}$$

first barycentric formula. (1.3)

second barycentric formula.

$$L_nf(x) = \frac{\sum_{i=0}^{n}\frac{w_i}{x-x_i}f_i}{\sum_{i=0}^{n}\frac{w_i}{x-x_i}}, \quad (1.4)$$

Newton-type methods

Newton's divided difference formula

$$\psi_n(x) = (x-x_0)\dots(x-x_{n-1})(x-x_n) = (x-x_n)\psi_{n-1}(x)$$

$$\psi_n'(x_i) = (x_i-x_n)\psi_{n-1}'(x_i),\ i=0,\dots,n-1$$

$$\psi_n'(x_n) = \psi_{n-1}(x_n). \quad (1.5)$$

$$(1.6)\quad L_nf(x) = \sum_{i=0}^{n}\frac{\psi_n(x)}{(x-x_i)\psi_n'(x_i)}f_i.$$

$N_nf(x)$ **Newton's divided difference form / Newton's interpolation polynomial** (1.12)

$$L_nf(x) = f(x_0) + f[x_0,x_1](x-x_0) + \dots + f[x_0,\dots,x_n](x-x_0)\dots(x-x_{n-1})$$

$$R_nf(x) = \frac{(x-x_0)\dots(x-x_n)}{(n+1)!}f^{(n+1)}(\xi),\ \xi\in(a,b)$$

Newton's forward and backward difference formula $x_i = x_0 + ih, i = 0,1,\dots$

Newton's forward difference formula. $\boxed{s = (x-x_0)/h}$ $\boxed{\binom{s}{k} = \frac{s(s-1)\cdots(s-k+1)}{k!}}$

$$L_nf(x) = f_0 + \binom{s}{1}\Delta f_0 + \binom{s}{2}\Delta^2 f_0 + \dots + \binom{s}{n}\Delta^n f_0 \quad (1.16)$$

$$f(x) - L_nf(x) = h^{n+1}\binom{s}{n+1}f^{(n+1)}(\xi_x), \quad (1.17)$$

Newton's backward difference formula. $\boxed{s = (x-x_n)/h}$

$$L_nf(x) = f_n + \binom{s}{1}\nabla f_n + \binom{s+1}{2}\nabla^2 f_n + \dots + \binom{s+n-1}{n}\nabla^n f_n \quad (1.18)$$

$$f(x) - L_nf(x) = h^{n+1}\binom{s+n}{n+1}f^{(n+1)}(\xi_x). \quad (1.19)$$

Divided Differences **Lecture 2**

first-order divided difference $\quad f[x_0,x_1] = \frac{f(x_1)-f(x_0)}{x_1-x_0} \quad (4.1)$

divided difference of order n *of* f *at the distinct nodes* $x_0, x_1, \dots, x_n$

$$f[x_0,x_1,\dots,x_n] = \frac{f[x_1,x_2,\dots,x_n]-f[x_0,x_1,\dots,x_{n-1}]}{x_n-x_0} \quad (4.3)$$

$$\begin{array}{llllllll}
x_0 & f[x_0] & \longrightarrow & f[x_0,x_1] & \longrightarrow & f[x_0,x_1,x_2] & \longrightarrow & f[x_0,x_1,x_2,x_3]\\
 & & \nearrow & & \nearrow & & \nearrow & \\
x_1 & f[x_1] & \longrightarrow & f[x_1,x_2] & \longrightarrow & f[x_1,x_2,x_3] & & \\
 & & \nearrow & & \nearrow & & & \\
x_2 & f[x_2] & \longrightarrow & f[x_2,x_3] & & & & \\
 & & \nearrow & & & & & \\
x_3 & f[x_3] & & & & & &
\end{array}$$

$$(4.4)\quad f[x_0,x_0,\dots,x_0] = \frac{f^{(n)}(x_0)}{n!}$$

divided difference of order n at the node x_0, of multiplicity $n+1$

$$f[x_0,x_1,\dots,x_n] = \sum_{i=0}^{n}\frac{f(x_i)}{u'(x_i)} = \sum_{i=0}^{n}\frac{f(x_i)}{u_i(x_i)}, \quad (4.5)$$

$$u(x) = (x-x_0)(x-x_1)\dots(x-x_n) \text{ and } u_i(x) = \frac{u(x)}{x-x_i}.$$

Finite Differences

$x_i = x_0 + ih, i = 0,1,\dots,n,\ h>0.$

k*th-order forward difference of* f *with step* h*, at* x_i. (4.12)

$$\Delta^k f(x_i) = \Delta^{k-1}f(x_{i+1}) - \Delta^{k-1}f(x_i) = \Delta^{k-1}f_{i+1} - \Delta^{k-1}f_i$$

$$\begin{array}{llllllll}
x_0 & f_0 & \longrightarrow & \Delta f_0 & \longrightarrow & \Delta^2 f_0 & \longrightarrow & \Delta^3 f_0\\
 & & \nearrow & & \nearrow & & \nearrow & \\
x_1 & f_1 & \longrightarrow & \Delta f_1 & \longrightarrow & \Delta^2 f_1 & & \\
 & & \nearrow & & \nearrow & & & \\
x_2 & f_2 & \longrightarrow & \Delta f_2 & & & & \\
 & & \nearrow & & & & & \\
x_3 & f_3 & & & & & &
\end{array}$$

$$\Delta^n f(x_0) = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f_k \quad (4.14)$$

backward difference ∇ (4.13)

$$\nabla^k f_i = \nabla^{k-1}f_i - \nabla^{k-1}f_{i-1},$$

Aitken-type methods **Lecture 3**

Neville's method.

$$P_{i,0} := f(x_i),\quad i=\overline{0,n},$$

$$P_{i,j} := \frac{(x-x_{i-j})P_{i,j-1}-(x-x_i)P_{i-1,j-1}}{x_i-x_{i-j}} = \frac{1}{x_i-x_{i-j}}\begin{vmatrix} x-x_{i-j} & P_{i-1,j-1}\\ x-x_i & P_{i,j-1}\end{vmatrix},\quad i\geq j>0. \quad (1.22)$$

$$\begin{array}{lllll}
x_0 & P_0 & & & \\
x_1 & P_1 & P_{01} & & \\
x_2 & P_2 & P_{12} & P_{012} & \\
x_3 & P_3 & P_{23} & P_{123} & P_{0123}
\end{array}$$

$$P_{i,j} = \tilde{P}_{i-j,i-j+1,\dots,i-1,i},\ j=i,i-1,\dots 0,$$

Aitken's method (1.23)

$$P_{i,0} := f(x_i),\quad i=\overline{0,n},$$

$$P_{i,j+1} := \frac{1}{x_i-x_j}\begin{vmatrix} x-x_j & P_{j,j}\\ x-x_i & P_{i,j}\end{vmatrix} = \frac{(x-x_j)P_{i,j}-(x-x_i)P_{j,j}}{x_i-x_j},\quad i>j\geq 0.$$

Derivate

$(x^n)' = nx^{n-1}$

$(\cos x)' = -\sin x$

$(\log_a x)' = \frac{1}{x\ln a}$

$(a^x)' = a^x \ln a$

$\left(\frac{f}{g}\right)' = \frac{f'g-fg'}{g^2}$

$(fg)' = f'g + fg'$

Integrale nedefinite

$\int \ln x dx = x\ln x - x + C$

$\int a^x dx = \frac{a^x}{\ln a} + C$

$\int \sin x\, dx = -\cos x + C$

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$

$\int \frac{dx}{x} = \ln|x| + C$

$\int f(x)f'(x)\,dx = \frac{f^2(x)}{2} + C$

$\int f(x)F(x)\,dx = \frac{F^2(x)}{2} + C$

$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$

$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$

$\int \frac{f(x)}{F(x)}dx = \ln|F(x)| + C$

Hermite Interpolation **Lecture 4**

double nodes

$$H_nf(x) = \sum_{i=0}^{m}\left[h_{i0}(x)f(x_i) + h_{i1}(x)f'(x_i)\right] \quad (1.3)$$

$$h_{i0}(x) = \left[1-2l_i'(x_i)(x-x_i)\right]\left[l_i(x)\right]^2, \quad (1.4)$$

$$h_{i1}(x) = (x-x_i)\left[l_i(x)\right]^2,\ i=0,\dots,m.$$

Newton's divided difference form

$$\begin{aligned}
N_n(x) &= f(x_0) + f[x_0,x_0](x-x_0) + f[x_0,x_0,x_1](x-x_0)^2\\
&+ f[x_0,x_0,x_1,x_1](x-x_0)^2(x-x_1) + \dots \\
&+ f[x_0,x_0,\dots,x_m,x_m](x-x_0)^2\dots(x-x_{m-1})^2(x-x_m)
\end{aligned} \quad (1.6)$$

$$R_n(x) = f(x) - H_nf(x) = \left[\psi_m(x)\right]^2\frac{f^{(n+1)}(\xi_x)}{(n+1)!},\ \xi_x\in(a,b). \quad (1.9)$$

$$h_{ij}(x) = \frac{(x-x_i)^j}{j!}\left[\sum_{k=0}^{r_i-j}\frac{(x-x_i)^k}{k!}\left[\frac{1}{u_i(x)}\right]^{(k)}_{x=x_i}\right]u_i(x). \quad (1.13)$$

$$\begin{aligned}
N_nf(x) &= f(z_0) + f[z_0,z_1](x-z_0) + \dots \\
&+ f[z_0,\dots,z_n](x-z_0)\dots(x-z_{n-1}),
\end{aligned} \quad (1.14)$$

$$R_n(x)\ (1.15) = \frac{u(x)}{(n+1)!}f^{(n+1)}(\xi_x),\ \xi_x\in(a,b).$$

Birkhoff Interpolation **Lecture 5**

$$P^{(j)}(x_k) = f^{(j)}(x_k),\ k=\overline{0,m},\ j\in I_k.$$

$$B_nf(x) = \sum_{k=0}^{m}\sum_{j\in I_k}b_{kj}(x)f^{(j)}(x_k) \quad (1.2)$$

$$\delta_{jp} = \begin{cases} 0, & j\neq p\\ 1, & j=p\end{cases}$$

$$\begin{aligned}
b_{kj}^{(p)}(x_\nu) &= 0,\ \nu\neq k,\ p\in I_\nu,\\
b_{kj}^{(p)}(x_k) &= \delta_{jp},\ p\in I_k,\ \text{for } j\in I_k \text{ and } \nu,k = 0,1,\dots,m,
\end{aligned} \quad (1.3)$$

[Peano] *Peano kernel.*

$$Lf = \int_a^b K_n(t)f^{(n)}(t)dt, \qquad K_n(t) = \frac{1}{(n-1)!}L\left[(\cdot - t)_+^{n-1}\right]$$

$$Lf = \frac{1}{n!}f^{(n)}(\xi)Le_n, \quad (1.13)$$

Spline Interpolation **Lecture 6**

Linear Splines

$$s_1(f;x) = f_i + f[x_i,x_{i+1}](x-x_i) = f_i + \frac{f(x_{i+1})-f(x_i)}{x_{i+1}-x_i}(x-x_i).$$

$$|f(x)-s_1(f;x)| \leq \frac{|(x-x_i)(x-x_{i+1})|}{2!}\max_{x\in[x_i,x_{i+1}]}|f''(x)| \leq \frac{h_i^2}{8}\max_{x\in[x_i,x_{i+1}]}|f''(x)|,$$

Cubic Splines

$$\begin{aligned}
s_3|_{[x_i,x_{i+1}]} &= p_i(x)\in\mathbb{P}_3,\ i=1,2,\dots,n-1,\\
s_3(f;x_i) &= f_i,\ i=1,2,\dots,n,\\
s_3'(f;x_i) &= m_i,\ i=1,2,\dots,n.
\end{aligned}$$

$$p_i(x) = f_i + m_i(x-x_i) + \frac{f[x_i,x_{i+1}]-m_i}{h_i}(x-x_i)^2 + \frac{m_{i+1}-2f[x_i,x_{i+1}]+m_i}{h_i^2}(x-x_i)^2(x-x_{i+1}).$$

$$h_im_{i-1} + 2(h_{i-1}+h_i)m_i + h_{i-1}m_{i+1} = b_i,\ i=2,\dots,n-1,$$

Piecewise cubic Hermite interpolation

$$b_i = 3\left(h_if[x_{i-1},x_i] + h_{i-1}f[x_i,x_{i+1}]\right).$$

1. Complete (clamped) splines. $m_1 = f'(a),\ m_n = f'(b).$

2. Endpoint second derivative splines.

$$\begin{aligned} 2m_1 + m_2 &= 3f[x_1,x_2] - \frac{1}{2}f''(a)h_1,\\ m_{n-1} + 2m_n &= 3f[x_{n-1},x_n] - \frac{1}{2}f''(b)h_{n-1}. \end{aligned}$$

3. Natural cubic splines. $s_3''(f;a) = s_3''(f;b) = 0,$

$$\begin{aligned} 2m_1 + m_2 &= 3f[x_1,x_2],\\ m_{n-1} + 2m_n &= 3f[x_{n-1},x_n]. \end{aligned}$$

4. "Not-a-knot" (deBoor) splines.

$$\begin{aligned} h_2^2\, m_1 + (h_2^2 - h_1^2)m_2 - h_1^2\, m_3 &= \beta_1,\\ h_{n-1}^2\, m_{n-2} + (h_{n-1}^2 - h_{n-2}^2)m_{n-1} - h_{n-2}^2\, m_n &= \beta_2,\\ \beta_1 &= 2\Big(h_2^2 f[x_1,x_2] - h_1^2 f[x_2,x_3]\Big),\\ \beta_2 &= 2\Big(h_{n-1}^2 f[x_{n-2},x_{n-1}] - h_{n-2}^2 f[x_{n-1},x_n]\Big). \end{aligned}$$

Finding cubic splines using the second derivatives — Lecture 6

$$\begin{aligned} s_3|_{[x_i,x_{i+1}]} &= p_i(x) \in \mathbb{P}_3,\ i = 1,2,\ldots,n-1,\\ s_3(f;x_i) &= f_i,\ i = 1,2,\ldots,n,\\ s_3''(f;x_i) &= M_i,\ i = 1,2,\ldots,n. \end{aligned}$$

$$\begin{aligned} h_{i-1}M_{i-1} + 2(h_{i-1}+h_i)M_i + h_i M_{i+1} &= 6\left(f[x_i,x_{i+1}] - f[x_{i-1},x_i]\right),\\ 2M_1 + M_2 &= 6\left(f[x_1,x_2] - f_1'\right),\\ M_{n-1} + 2M_n &= 6\left(f_n' - f[x_{n-1},x_n]\right), \end{aligned}$$

Least Squares Approximation

$$\begin{aligned} a\sum_{i=i}^n x_i^2 + b\sum_{i=i}^n x_i &= \sum_{i=i}^n x_i y_i\\ a\sum_{i=i}^n x_i + nb &= \sum_{i=i}^n y_i. \end{aligned}$$

$$\begin{aligned} a\sum_{i=1}^n x_i^4 + b\sum_{i=1}^n x_i^3 + c\sum_{i=1}^n x_i^2 &= \sum_{i=1}^n x_i^2 y_i,\\ a\sum_{i=1}^n x_i^3 + b\sum_{i=1}^n x_i^2 + c\sum_{i=1}^n x_i &= \sum_{i=1}^n x_i y_i,\\ a\sum_{i=1}^n x_i^2 + b\sum_{i=1}^n x_i + nc &= \sum_{i=1}^n y_i, \end{aligned}$$

Lecture 7

$$L(f) = \sum_{i=1}^m A_i L_i(f) + R(f),\ f \in X.$$ *A_i are called **coefficients**,* $\ker R = \mathbb{P}_d \iff \begin{cases} R(e_k) = 0,\ k = 0,1,\ldots,d,\\ R(e_{d+1}) \neq 0, \end{cases}$

Numerical Differentiation

$$f^{(k)}(\alpha) = \sum_{i=0}^m A_i f(x_i) + R(f),$$ *numerical differentiation formula.*

$$\int_a^b f(x)dx = \sum_{k=0}^m \sum_{j\in I_k} A_{kj} f^{(j)}(x_k) + R(f),$$ *numerical integration (quadrature) formula.*

forward difference

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} \equiv D_h f(x),$$

$$(RD_h f)(x) = f'(x) - D_h f(x) = -\frac{h}{2}f''(\xi),\ \xi \in (x, x+h).$$

backward difference $$f'(x) \approx \frac{f(x) - f(x-h)}{h} \equiv \widetilde{D}_h f(x),\ (R\widetilde{D}_h f)(x) = f'(x) - \frac{f(x) - f(x-h)}{h} = \frac{h}{2}f''(\xi),\ \xi \in (x-h, x).$$

Method of Undetermined Coefficients $$D_h^{(2)} f(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.\ (RD_h^{(2)} f)(x) = f''(x) - D_h^{(2)} f(x) \approx -\frac{h^2}{12}f^{(4)}(x).$$

Numerical Integration

$$\int_a^b f(x)dx = \sum_{k=0}^m A_k f(x_k) + R(f),$$

interpolatory quadrature. $d_{\max} = 2m+1.$

Trapezoidal Rule

$$\int_a^b f(x)dx = \frac{b-a}{2}\Big(f(a) + f(b)\Big) - \frac{(b-a)^3}{12}f''(\xi),\ \xi \in (a,b),$$

n subintervals $h = \frac{b-a}{n}$. **composite (repeated)**

$$\int_a^b f(x)dx = \frac{h}{2}\Big[f(a) + 2\big(f_1 + \cdots + f_{n-1}\big) + f(b)\Big] - \frac{h^2(b-a)}{12}f''(\xi),\ \xi \in (a,b).$$

Rectangle (Midpoint) Rule

$$\int_a^b f(x)dx = (b-a)f\Big(\frac{a+b}{2}\Big) + R(f),$$

$$R(f) = \frac{f''(\xi)}{2!}\int_a^b \Big(x - \frac{a+b}{2}\Big)^2 dx = \frac{(b-a)^3}{24}f''(\xi),\ \xi \in (a,b).$$

composite (repeated)

$$\int_a^b f(x)dx = h\sum_{i=0}^{n-1} f\Big(a + (i + \frac{1}{2})h\Big) + \frac{h^2(b-a)}{24}f''(\xi),\ \xi \in (a,b),$$

Simpson's Rule

$$\int_a^b f(x)dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880}f^{(4)}(\xi),\ \xi \in (a,b).$$

$\boxed{h = \frac{b-a}{2m},}$ $n = 2m$ **composite (repeated)**

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(a) + 4\sum_{i=1}^m f_{2i-1} + 2\sum_{i=1}^{m-1} f_{2i} + f(b)\right] - \frac{h^4(b-a)}{180}f^{(4)}(\xi),\ \xi \in (a,b).$$

Adaptive Quadratures — Lecture 8

$$I \approx \frac{2^p I_{2n} - I_n}{2^p - 1} = I_{2n} + \frac{I_{2n} - I_n}{2^p - 1} \overset{\text{not}}{=} R_{2n}$$

Richardson's extrapolation formula.

$$I - I_{2n} \approx \frac{I_{2n} - I_n}{2^p - 1},$$ **error estimate**

Iterated Quadratures

Romberg's Method

$$R_{k,j} = \frac{4^{j-1}R_{k,j-1} - R_{k-1,j-1}}{4^{j-1} - 1},\ k = \overline{2,n},\ j = \overline{2,k}.$$

$$\begin{matrix} R_{1,1} & & & & \\ R_{2,1} & R_{2,2} & & & \\ R_{3,1} & R_{3,2} & R_{3,3} & & \\ \vdots & \vdots & \vdots & \ddots & \\ R_{n,1} & R_{n,2} & R_{n,3} & \ldots & R_{n,n} \end{matrix}$$

stopping $|R_{n-1,n-1} - R_{n,n}| < \varepsilon.$

$S_{k,1} = R_{k,2}.$

Gaussian Quadratures

$$\int_a^b w(x)f(x)\,dx = \sum_{k=1}^m A_k f(x_k) + R_m(f)$$

maximum degree of precision, $d = 2m - 1.$

$$\mu_j = \int_a^b w(x)x^j\,dx$$

Gaussian Elimination Lecture 9

$$x_n = \frac{b_n}{u_{nn}},$$ **backward substitution:** $$x_i = \frac{1}{u_{ii}}\Big(b_i - \sum_{j=i+1}^n u_{ij}x_j\Big),\ i = \overline{n-1,1}$$

$$x_1 = \frac{b_1}{l_{11}},$$ **forward** $$x_i = \frac{1}{l_{ii}}\Big(b_i - \sum_{j=1}^{i-1} l_{ij}x_j\Big),\ i = \overline{2,n}.$$

partial pivoting (maximal pivoting on columns). $|a_{pk}^{(k)}| = \max_{k\le l\le n} |a_{lk}^{(k)}|.$

$(R_i) \longleftrightarrow (R_p)$ **scaled pivoting on columns:**

$$\frac{|a_{pi}|}{s_i} = \max_{1\le j\le n} \frac{|a_{ji}|}{s_j}$$

$$s_i = \max_{j=\overline{1,n}} |a_{ij}| \text{ or } s_i = \sum_{j=1}^n |a_{ij}|,\ i = \overline{1,n}.$$

total (maximal) pivoting.

$$|a_{pq}| = \max\{|a_{ij}|, i,j = \overline{k,n}\}$$

$(R_k) \longleftrightarrow (R_p),\ (C_k) \longleftrightarrow (C_q).$

$x' = [x_3\ x_2\ x_1]^T.$

LU Factorization *no row interchanges are necessary*

$A = LU,$ **Schur complement** $A' - vw^*/a_{11} = L'U'.$ $A = \begin{bmatrix} a_{11} & w^* \\ v & A' \end{bmatrix}$

$$\begin{aligned} Ly &= b\\ Ux &= y. \end{aligned} \quad \begin{bmatrix} 1 & 0 \\ v/a_{11} & I_{n-1} \end{bmatrix} \begin{bmatrix} a_{11} & w^* \\ 0 & A' - vw^*/a_{11} \end{bmatrix}$$

LUP Factorization $PA = LU.$ $\quad Ly = Pb$, $Ux = y.$

QR Factorization Lecture 10

$A = QR$, *Q orthogonal and R upper triangular* $r_{ii} > 0,$

$Rx = Q^T b.$

Cholesky factorization $A = R^T R,$ $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ 0 & & & a_{nn} \end{bmatrix}$

Iterative Methods Lecture 10

Jacobi and Gauss-Seidel Methods

$A = D - L - U,\quad x^{(k+1)} = Tx^{(k)} + c$

Jacobi iteration, $A = M - N.$

$M = D,\ N = L + U,$ so

$T_J = D^{-1}(L + U),\ c_J = D^{-1}b.$

$x^{(k+1)} = D^{-1}(L + U)x^{(k)} + D^{-1}b,\ k \in \mathbb{N},$

Gauss-Seidel iteration,

$M = D - L,\ N = U,$ so

$T_{GS} = (D - L)^{-1}U,\ c_{GS} = (D - L)^{-1}b.$

$x^{(k+1)} = (D - L)^{-1}Ux^{(k)} + (D - L)^{-1}b,$

Acceleration methods; SOR Method

$M = \frac{D}{\omega} - L,\ N = \left(\frac{1-\omega}{\omega}D + U\right),$ so

$T_\omega = \left(\frac{D}{\omega} - L\right)^{-1}\left(\frac{1-\omega}{\omega}D + U\right),\ c_\omega = \left(\frac{D}{\omega} - L\right)^{-1}b.$

$$D = \begin{bmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{bmatrix} \quad -U = \begin{bmatrix} 0 & a_{12} & \ldots & a_{1n} \\ & 0 & \ldots & a_{2n} \\ & & \ddots & \vdots \\ 0 & & & 0 \end{bmatrix}$$

$$-L = \begin{bmatrix} 0 & & & 0 \\ a_{21} & 0 & & \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \ldots & 0 \end{bmatrix}$$

Orthogonal polynomials and recurrence coefficients

Name	Notation	Polynomial	Weight fn.	Interval	α_k	β_k
Legendre	l_m	$[(x^2-1)^m]^{(m)}$	1	$[-1,1]$	0	$\beta_0 = 2,$ $\beta_k = (4 - k^{-2})^{-1}, k \ge 1$
Chebyshev 1^{st}	T_m	$\cos(m \arccos x)$	$(1-x^2)^{-\frac{1}{2}}$	$[-1,1]$	0	$\beta_0 = \pi,$ $\beta_1 = \frac{1}{2},$ $\beta_k = \frac{1}{4}, k \ge 2$
Chebyshev 2^{nd}	Q_m	$\frac{\sin[(m+1)\arccos x]}{\sqrt{1-x^2}}$	$(1-x^2)^{\frac{1}{2}}$	$[-1,1]$	0	$\beta_0 = \frac{\pi}{2},$ $\beta_k = \frac{1}{4}, k \ge 1$
Laguerre	L_m^a	$x^{-a}e^x\left(x^{m+a}e^{-x}\right)^{(m)}$	$x^a e^{-x},\ a > -1$	$[0,\infty)$	$2k + a + 1$	$\beta_0 = \Gamma(1+a),$ $\beta_k = k(k+a), k \ge 1$
Hermite	H_m	$(-1)^m e^{x^2}\left(e^{-x^2}\right)^{(m)}$	e^{-x^2}	$\mathbb{R}$	0	$\beta_0 = \sqrt{\pi},$ $\beta_k = \frac{k}{2}, k \ge 1$

$f : [a, b] \to \mathbf{R}$ is continuous c in (a, b)
g is an integrable function
that does not change sign on $[a, b]$.

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx$$

$$\operatorname{tr}(A) = \sum_{i=1}^n a_{ii} = \sum_{i=1}^n \lambda_i = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

$$\det(A) = \prod_{i=1}^n \lambda_i = \lambda_1\lambda_2\cdots\lambda_n$$

[Banach] (3.2)
$g \in C[a,b]$, with $g([a,b]) \subseteq [a,b]$
there exists $0 < \lambda < 1$
$|g(x) - g(y)| \le \lambda|x - y|,\ \forall x,y \in [a,b]$
g has a unique fixed point $\alpha \in [a,b]$

Common Rootfinding Methods Lecture 11

Bisection Method
two-point method, $c_n = \frac{a_n + b_n}{2}$

Secant Method *two-point method,*

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})},\ n = 1,2,\ldots,$$

Newton's Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},\ n = 0,1,\ldots.$$ *one-step*

Lecture 12

Newton's Method for Multiple Roots

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}.$$

Newton's Method for Nonlinear Systems

$$\boldsymbol{x_{n+1}} = \boldsymbol{x_n} - \left(\boldsymbol{J_f}(\boldsymbol{x_n})\right)^{-1}\boldsymbol{f}(\boldsymbol{x_n}),\ n \ge 0,$$

$$\boldsymbol{J_f}(\boldsymbol{x_n}) = \begin{bmatrix} \frac{\partial f_1(\boldsymbol{x})}{\partial x_1} & \frac{\partial f_1(\boldsymbol{x})}{\partial x_2} \\ \frac{\partial f_2(\boldsymbol{x})}{\partial x_1} & \frac{\partial f_2(\boldsymbol{x})}{\partial x_2} \end{bmatrix} \quad \begin{aligned} \boldsymbol{J_f}(\boldsymbol{x_n})\boldsymbol{\delta_{n+1}} &= -\boldsymbol{f}(\boldsymbol{x_n}),\\ \boldsymbol{x_{n+1}} &= \boldsymbol{x_n} + \boldsymbol{\delta_{n+1}}. \end{aligned}$$