1. Let $x_0 = 1$, $x_1 = 1.5$, $x_2 = 2$. Because $S(x)$ is a cubic spline, we have

$$S_1(x_1) = S_2(x_1) \quad (i)$$

$$S_1'(x_1) = S_2'(x_1) \quad (ii)$$

$$S_1''(x_1) = S_2''(x_1). \quad (iii)$$

$$\begin{cases} S_1'(x) = 3(x+1)^2 + 2a(x+1) - c \\ S_1''(x) = 6(x+1) + 2a \end{cases}$$

$$\begin{cases} S_2'(x) = -3x^2 + 2cx + b \\ S_2''(x) = -6x + 2c \end{cases}$$

Replacing $x_1 = 0$ gives the system

$$1 + a - c + b = 7 \quad (i)$$

$$3 + 2a - c = b \quad (ii)$$

$$6 + 2a = 2c \quad (iii)$$

The system can be solved using any method. Final solution is $a = 9$, $b = 9$, $c = 12$.

2. Repeated Simpson formula

$$\int_a^b f(x)\,dx = \frac{b-a}{6n}\left[f(a)+f(b)+4\sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right)+2\sum_{k=1}^{n-1} f(x_k)\right]$$

$$+R_n(f).$$

In this case, $a=1$, $b=2$, $f(x)=3x^3+2x+1$, $n=2$, $h=0.5$

$x_0=1$, $x_1=1.5$, $x_2=2$

Replacing gives

$$\int_1^2 (3x^3+2x+1)\,dx \simeq \frac{2-1}{12}\cdot\left[f(1)+f(2)+4\cdot\left[f\left(\frac{1+1.5}{2}\right)+f\left(\frac{1.5+2}{2}\right)\right]+2f(1.5)\right]$$

Just compute $f(1)$, $f(1.25)$, $f(1.5)$, $f(1.75)$ and $f(2)$.

Replacing values should give the approximation 15.23

~~Ad~~ Actual value: $\int_1^2 (3x^3+2x+1)\,dx = x^4+x^2+x\Big|_1^2$

$= 15.$

3. $x_0 = -2, \ x_1 = -1, \ x_2 = 0, \ x_3 = 1, \ x_4 = 2$

$f(x_0) = f_0 = 2, \ f(x_1) = f_1 = 1, \ f(x_2) = f_2 = 0, \ f(x_3) = f_3 = 1,$

$f(x_4) = f_4 = 2.$

$$N_m f(x) = f(x_0) + \sum_{i=1}^{m} (x - x_0) \dots (\underbrace{x - x_{i-1}}_{x_{i-1}}) D^i f(x_0)$$

Construct divided difference table:

x	f	Df	$D^2 f$	$D^3 f$	$D^4 f$
				$\frac{1}{3}$	$-\frac{1}{6}$
-2	2	-1	0		
-1	1	-1	1	$-\frac{1}{3}$	
0	0	1	0		
1	1	1			
2	2				

$$N_m f(x) = \cancel{1-2}1 +$$
$$+ (x+2) \cdot (-1)$$
$$+ \cancel{(x+2)(x+1) \cdot 0}$$
$$+ (x+2)(x+1)(x) \cdot \frac{1}{3}$$
$$+ (x+2)(x+1)(x)(x-1) \cdot -\frac{1}{6}$$

$$= -x + (x)(x+1)(x+2) \cdot \frac{1}{6} \cdot \left(2 - (x-1)\right)$$

$$= +\frac{1}{6}(x)(x+1)(x+2)(3-x) - x.$$