# NUMERICAL CALCULUS

- crash course -

## Course 1

**Notations:**

- Let $x^* \in \mathbb{R}$ be an unknown value of interest. An element $\bar{x} \in \mathbb{R}$ which approximates $x^*$ is called the approximation or the approximant of $x^*$.
- $\Delta x = x^* - \bar{x}$ or $\Delta x = \bar{x} - x^*$ is called the error
- $|\Delta x|$ is called the absolute error
- $\delta x = \frac{|\Delta x|}{|x^*|}$, $x^* \neq 0$ is called the relative error or the percent error
- If $V$ is a $K$-linear space then a real function $p: V \to [0, \infty)$ with the properties:

1) $p(v_1 + v_2) \leq p(v_1) + p(v_2)$ $\forall v_1, v_2 \in V$
2) $p(\alpha v) = |\alpha| p(v)$ $\forall \alpha \in K, v \in V$
3) $p(v) = 0 \Rightarrow v = 0$

is called the norm on $V$

- Let $K$ be a field and $V$ be a given set. We say that $V$ is a $K$ linear space if there exists an internal operation:

"+" : $V \times V \to V$ $(v_1, v_2) \to v_1 + v_2$

and an external operation:

"$\cdot$" : $K \times V \to V$ $(\alpha, v) \to \alpha v$

that satisfy the following conditions:

1) $(V, +)$ is a commutative group
2)
   a) $(\alpha + \beta) v = \alpha v + \beta v$ $\forall \alpha, \beta \in K$, $\forall v, v_1, v_2 \in V$
   b) $\alpha(v_1 + v_2) = \alpha v_1 + \alpha v_2$
   c) $(\alpha \beta) v_1 = \alpha(\beta v_1)$
   d) $1 \cdot v = v$

the elements of $V$ are called vectors and those of $K$ are scalars

- Let $V$ and $V'$ be 2 $K$ linear spaces. A function $f: V \to V'$ is called linear transformation if: (or linear operator)

1) $f(v_1 + v_2) = f(v_1) + f(v_2)$ $\forall v_1, v_2 \in V$ aditivity
2) $f(\alpha v) = \alpha f(v)$ $\forall \alpha \in K$ $\forall v \in V$ homogenity

or shortly

$$f(\alpha v_1 + \beta v_2) = \alpha f(v_1) + \beta f(v_2)$$

- Let $V$ be a linear space on $\mathbb{R}$ or $\mathbb{C}$. A linear operator $P: V \to V$ is called projector if

$$P \circ P = P \quad (\text{shortly } P^2 = P)$$

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1) The identity operator $I: V \to V$ $I(v) = v$ and the null operator $O: V \to V$ $O(v) = 0$ are projectors
2) $P$ is projector $\Rightarrow$ $P^c := I - P$, the complement of $P$ is projector

# Finite, divided differences

## Finite differences

Let $M=\{a_i \mid a_i = a + ih,\ i=\overline{0,m},\ a,h\in\mathbb{R}^*,\ m\in\mathbb{N}^*\}$ and $F=\{f \mid f: M\to\mathbb{R}\}$

- For $f\in F$

$$(\Delta_h f)(a_i) = f(a_{i+1}) - f(a_i) \quad i<m$$

is the **finite difference of the first order** of function $f$ with sthep $h$ at point $a_i$.

- operator $\Delta_h$ is a linear operator with respect to $f$
- Let $0\le i<m$, $k\in\mathbb{N}$ and $1\le k\le m-i$

$$(\Delta_h^k f)(a_i) = (\Delta_h(\Delta_h^{k-1} f))(a_i)$$
$$= (\Delta_h^{k-1} f)(a_{i+1}) - (\Delta_h^{k-1} f)(a_i)$$

$\Delta_h^0 = I$ and $\Delta_h^1=\Delta_h$ is called the **k-th order finite finite difference** of the function $f$ with step $h$ at point $a_i$

- If $0\le i<m$; $k,p\in\mathbb{N}$ and $1\le p+k\le m-i$, then

$$(\Delta_h^p(\Delta_h^k f))(a_i) = \Delta_h^k(\Delta_h^p f)(a_i) = (\Delta_h^{p+k} f)(a_i)$$

Finite difference table

| $a$ | $f$ | $\Delta_h f$ | $\Delta_h^2 f$ ... | $\Delta_h^{m-1} f$ | $\Delta_h^m f$ |
|---|---|---|---|---|---|
| $a_0$ | $f_0$ | $\Delta_h f_0$ | $\Delta_h^2 f_0$ | $\Delta_h^{m-1} f_0$ | $\Delta_h^m f_0$ |
| $a_1$ | $f_1$ | $\Delta_h f_1$ | $\Delta_h^2 f_1$ | $\Delta_h^{m-1} f_1$ | |
| | | | | | |
| $a_{m-3}$ | $f_{m-3}$ | $\Delta_h f_{m-3}$ | $\Delta_h^2 f_{m-3}$ | | |
| $a_{m-2}$ | $f_{m-2}$ | $\Delta_h f_{m-2}$ | $\Delta_h^2 f_{m-2}$ | | |
| $a_{m-1}$ | $f_{m-1}$ | $\Delta_h f_{m-1}$ | | | |
| $a_m$ | $f_m$ | | | | |

Example: $h=0.25$, $a=1$, $a_i = a+ih$ $i=\overline{0,4}$

| $a$ | $f$ | $\Delta_h f$ | $\Delta_h^2 f$ | $\Delta_h^3 f$ | $\Delta_h^4 f$ |
|---|---|---|---|---|---|
| 1 | 0 | 2 | 2 | 2 | -11 |
| 1.25 | 2 | 4 | 4 | -9 | |
| 1.5 | 6 | 8 | -5 | | |
| 1.75 | 14 | 3 | | | |
| 2 | 17 | | | | |

## Divided Differences

Let $X=\{x_i \mid x_i\in\mathbb{R},\ i=\overline{0,m},\ m\in\mathbb{N}^*\}$ and $f:X\to\mathbb{R}$.

- For $r\in\mathbb{N}$, $r<m$,

$$(Df)(x_r) := [x_r, x_{r+1}; f] = \frac{f(x_{r+1})-f(x_r)}{x_{r+1}-x_r}$$ is called

the **first order divided difference** of the function $f$, regarding the points $x_r$ and $x_{r+1}$

- The operator $D$ is linear with respect to $f$
- $(D^k f)(x_r) = \dfrac{(D^{k-1}f)(x_{r+1}) - (D^{k-1}f)(x_r)}{x_{r+k}-x_r}$ is called the **k-th order divided difference** $D^0=I$, $D^1=D$
- The operator $D^k$ is linear with respect $f$

**Example:** $x_0=0,\ x_1=1,\ x_2=2,\ x_3=4$
$f_0=3\quad f_1=4\quad f_2=7\quad f_3=19$

| a | f | Df | $D^2f$ | $D^3f$ |
|---|---|---|---|---|
| 0 | 3 | 1 | 1 | 0 |
| 1 | 4 | 3 | 1 | |
| 2 | 7 | 6 | | |
| 4 | 19 | | | |

**Example** Form div. difference table for: $x_0=2\ x_1=4\ x_2=6\ x_3=8$
$f_0=4\quad f_1=8\quad f_2=20\quad f_3=48$

| a | f | Df | $D^2f$ | $D^3f$ |
|---|---|---|---|---|
| 2 | 4 | 2 | 1 | $\frac{1}{6}$ |
| 4 | 8 | 6 | 2 | |
| 6 | 20 | 14 | | |
| 8 | 48 | | | |

**Example** Form the div difference for:
$x_0=1,\ x_1=2,\ x_2=3,\ x_3=5,\ x_4=7$
$f_0=3,\ f_1=5,\ f_2=9,\ f_3=11,\ f_4=15$

| a | f | Df | $D^2f$ | $D^3f$ | $D^4f$ |
|---|---|---|---|---|---|
| 1 | 3 | 2 | 1 | $-\frac{1}{2}=-0.5$ | $\frac{3}{4}\cdot\frac{1}{6}=\frac{1}{8}$ |
| 2 | 5 | 4 | $-1$ | $\frac{1}{4}$ | |
| 3 | 9 | 1 | 0,25 | | |
| 5 | 11 | 2 | | | |
| 7 | 15 | | | | |

# Polynomial interpolation
## Taylor Polynomial

- Let $f \in C^n[a,b]$ such that there exists $f^{(n+1)}$ on $[a,b]$ and consider $x_0 \in [a,b]$. The taylor polynomial is

$$T_n(x) = \sum_{k=0}^{n} \frac{(x-x_0)^k}{k!} f^{(k)}(x_0)$$

and we have the approximation formula

$$f(x) = T_n(x) + R_n(x) \quad , R_n \text{ is the error}$$

- For $\forall x \in [a,b]$ there exists a number $\xi$ between $x_0$ and $x$ s that

$$R_n(x) = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

**Example:** $f(x)=\frac{1}{x}$ and $x_0=1$. Approximate the value of $f(3)$ by the first and the second degree Taylor Polynomials

$$T_1 = \sum_{k=0}^{1} \frac{(x-1)^k}{k!} f^k(1) = 1 + (x-1)\cdot(-1) = -x+2$$

$$T_2 = 1 + (x-1)\cdot(-1) + \frac{(x-1)^2}{2}\cdot 2 = -x+2+x^2-2x+1 =$$
$$= x^2-3x+3$$

$$f'(x) = \left(\frac{1}{x}\right)' = (x^{-1})' = -x^{-2} = -\frac{1}{x^2}$$

$$f''(x) = \frac{2}{x^3}$$

$$T_1(3) = -1$$
$$T_2(3) = 9-9+3 = 3$$

# Lagrange Interpolation

- Lagrange interpolation polynomial is given by

$$(L_m f)(x) = \sum_{i=0}^{m} l_i(x) f(x_i)$$

where by $l_i(x)$ we denote **the Lagrange fundamental interpolation polynomials**

## Course 2

- Let $[a,b] \subset \mathbb{R}$, $x_i \in [a,b]$, $i = \overline{0,m}$ such that $x_i \neq x_j$ for $i \neq j$ and consider $f : [a,b] \to \mathbb{R}$
- **Lagrange interpolation problem** consists in determining the polynomial $P$ of smallest degree for which

$$P(x_i) = f(x_i) \quad i = \overline{0,m}$$

- Lagrange interpolation polynomial is given by

$$(L_m f)(x) = \sum_{i=0}^{m} l_i(x) f(x_i)$$

- We have:

$$u(x) = \prod_{j=0}^{m} (x - x_j)$$

$$u_i(x) = \frac{u(x)}{x - x_i} = \prod_{\substack{j=0 \\ j \neq i}}^{m} (x - x_j)$$

$$l_i(x) = \frac{u_i(x)}{u_i(x_i)} = \prod_{\substack{j=0 \\ j \neq i}}^{m} \frac{x - x_j}{x_i - x_j}$$

- The operator $L_m$ is linear

Example: Find the lagrange polynomial that interpolates the data in the following table and find the approximative value of $f(-0.5)$.

| $x$ | $-1$ | $0$ | $3$ |
|---|---|---|---|
| $f(x)$ | $8$ | $-2$ | $4$ |

$x_0 = -1 \quad x_1 = 0 \quad x_2 = 3$

$m = 2$

$$(L_2 f)(x) = \sum_{i=0}^{2} l_i(x) \cdot f(x_i) =$$

$$l_0(x) = \frac{(x - x_1)}{x_0 - x_1} \cdot \frac{x - x_2}{x_0 - x_2} = \frac{x \cdot (x-3)}{(-1)\cdot(-4)} = \frac{x^2 - 3x}{4}$$

$$l_1(x) = \frac{x - x_0}{x_1 - x_0} \cdot \frac{x - x_2}{x_1 - x_2} = \frac{(x+1)(x-3)}{-3} = \frac{x^2 - 2x - 3}{-3}$$

$$l_2(x) = \frac{x - x_0}{x_2 - x_0} \cdot \frac{x - x_1}{x_2 - x_1} = \frac{x^2 + x}{12}$$

$$(L_2 f)(x) = \frac{x^2 - 3x}{4} \cdot 8 + (-2) \cdot \frac{x^2 - 4x + 3}{-3} + 4 \cdot \frac{x^2 + x}{12}$$

$$= 2x^2 - 6x + \frac{2x^2 - 8x + 6}{3} + \frac{x^2 + x}{3} = \frac{9x^2 - 19x + 6}{3} = 3x^2 - 7x + 2$$

- Barycentric form of Lagrange interpolation

$$A_i = \frac{1}{u_i(x_i)}$$

$$(L_m f)(x) = \frac{\sum_{i=0}^{m} \frac{A_i f(x_i)}{x - x_i}}{\sum_{i=0}^{m} \frac{A_i}{x - x_i}}$$

- The Lagrange polynomial generates the Lagrange interpolation formula

$$f = L_m f + R_m f$$

where $R_m f$ denotes the remainder (the error)

- $$(R_m f)(x) = \frac{u(x)}{(m+1)!} f^{(m+1)}(\xi)$$

Let $\alpha = \min\{x, x_0, \dots, x_m\}$ and $\beta = \max\{x, x_0, \dots, x_m\}$. If $f \in C^m[\alpha, \beta]$ and $f^{(m)}$ is derivable on $(\alpha, \beta)$ then $\forall x \in (\alpha, \beta)$ there exists $\xi \in (\alpha, \beta)$ such that we have the error formula from above.

- If $f \in C^{m+1}[a, b]$ then

$$|(R_m f)(x)| \le \frac{|u(x)|}{(m+1)!} \|f^{(m+1)}\|_\infty \quad x \in [a, b]$$

$\|\cdot\|_\infty$ denotes the uniform norm and $\|f\|_\infty = \max_{x \in [a,b]} |f(x)|$

Example: If we know $\lg 2 = 0.301$, $\lg 3 = 0.477$, $\lg 5 = 0.699$, find $\lg 76$

| $x$ | 2 | 3 | 5 |
|---|---|---|---|
| $f(x)$ | 0.301 | 0.477 | 0.699 |

$m = 2$

$$(L_2 f)(x) = \sum_{i=0}^{2} l_i(x) \cdot f(x_i) = l_0(x) \cdot 0.301 + l_1(x) \cdot 0.477 + l_2(x) \cdot 0.699$$

$$= \frac{x^2 - 8x + 15}{3} \cdot 0.301 + \frac{x^2 - 7x + 10}{-2} \cdot 0.477 + \frac{x^2 - 5x + 6}{6} \cdot 0.699$$

$$l_0(x) = \frac{(x - x_1)}{x_0 - x_1} \cdot \frac{(x - x_2)}{x_0 - x_2} = \frac{(x-3)(x-5)}{(-1) \cdot (-3)} = \frac{x^2 - 8x + 15}{3}$$

$$l_1(x) = \frac{(x - x_0)}{x_1 - x_0} \cdot \frac{x - x_2}{x_1 - x_2} = \frac{(x-2)(x-5)}{(1) \cdot (-2)} = \frac{x^2 - 7x + 10}{-2}$$

$$l_2(x) = \frac{x - x_0}{x_2 - x_0} \cdot \frac{x - x_1}{x_2 - x_1} = \frac{(x-2)(x-3)}{3 \cdot 2} = \frac{x^2 - 5x + 6}{6}$$

$$(R_2 f)(x) = \frac{u(x)}{3!} f^{3}(\xi)$$

# Course 3

## Aitken

- Let $[a,b] \subset \mathbb{R}$, $x_i \in [a,b]$, $i=\overline{0,m}$ such that $x_i \neq x_j$ for $i \neq j$ and consider $f: [a,b] \to \mathbb{R}$
- A practical method for computing the Lagrange polynomial is the **Aitken's algorithm**. This consists in generating the table:

| | | | | | | |
|---|---|---|---|---|---|---|
| $x_0$ | $f_{00}$ | | | | | |
| $x_1$ | $f_{10}$ | $f_{11}$ | | | | |
| $x_2$ | $f_{20}$ | $f_{21}$ | $f_{22}$ | | | |
| $x_3$ | $f_{30}$ | $f_{31}$ | $f_{32}$ | $f_{33}$ | | |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | | |
| $x_m$ | $f_{m0}$ | $f_{m1}$ | $f_{m2}$ | $f_{m3}$ | $\dots$ | $f_{mm}$ |

where $f_{i0} = f(x_i)$, $i=\overline{0,m}$
and

$$f_{i,j+1} = \frac{1}{x_i - x_j}\begin{vmatrix} f_{jj} & x_j - x \\ f_{ij} & x_i - x \end{vmatrix} \qquad i=\overline{0,m},\ j=\overline{0,i-1}$$

- the stoping criterion is $|f_{ii} - f_{i-1,i-1}| < \varepsilon$
- the aproximation is given by $f_{ii}$

Theoretical example:

$$f_{11} = \frac{1}{x_1 - x_0}\begin{vmatrix} f_{00} & x_0 - x \\ f_{10} & x_1 - x \end{vmatrix} = \frac{1}{x_1 - x_0}\left[(x_1 - x) f(x_0) - (x_0 - x) f(x_1)\right]$$

$$= \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1) = (L_1 f)(x)$$

- The elements from the main diagonal are the Lagrange polynomial
- Recommendation is to sort the nodes $x_0, x_1, \dots, x_m$ with respect to the distance to $x$, such that

$$|x_i - x| \leq |x_j - x| \text{ if } i < j, \quad i,j = \overline{1,m}$$

Example:

Approximate $\sqrt{115}$ with precision $\varepsilon = 10^{-3}$ using Aitken's alg.

$x_0 = 121 \qquad x_1 = 100 \qquad x_2 = 144$

$|115 - 121| = 6 \qquad |115 - 100| = 15 \qquad |115 - 144| = -29$

| | | | |
|---|---|---|---|
| $x_0 = 121$ | $f_{00} = 11$ | | |
| $x_1 = 100$ | $f_{10} = 10$ | $f_{11} = (x + 110) : 21 \simeq 10.71$ | |
| $x_2 = 144$ | $f_{20} = 12$ | $f_{21} = (132 + x) : 23 \simeq 10.73$ | $f_{22} =$ |

$$f_{11} = \frac{1}{x_1 - x_0}\begin{vmatrix} f_{00} & x_0 - x \\ f_{10} & x_1 - x \end{vmatrix} = \frac{1}{-21}\left(11x_1 - 11x - 10x_0 + 10x\right)$$

$$= \frac{1}{-21}\left(1100 - x - 1210\right)$$

$$= \frac{110 + x}{21} \simeq 10.71$$

$$f_{21} = \frac{1}{x_2 - x_0} \begin{vmatrix} f_{00} & x_0 - x \\ f_{20} & x_2 - x \end{vmatrix} = \frac{1}{23} \left[ 11 \quad (144-x) - 12\,(121-x) \right]$$

$$= \frac{1}{23} \left( 1584 + \quad x - 1452 \right)$$

$$= \frac{132 + x}{23} \simeq 10.73$$

$$f_{22} = \frac{1}{x_2 - x_1} \begin{vmatrix} f_{11} & x_1 - x \\ f_{21} & x_2 - x \end{vmatrix} = \frac{1}{44} \left( 10.71\,(144-x) - 10.73\,(100-x) \right)$$

$$= \frac{1}{44} \left( 10.71 \cdot 29 - 10.73 \cdot (-15) \right)$$

$$= \frac{310.59 + 160.95}{44} = \frac{471.54}{44} \simeq 10.716$$

# Newton Interpolation Polynomial

- A useful representation of Lagrange interpolation is

$$(L_m f)(x) := (N_m f)(x) = f(x_0) + \sum_{i=1}^{m} (x - x_0) \dots (x - x_{i-1}) (D^i f)(x_0)$$

which is called **Newton Interpolation polynomial** where $(D^i f)(x_0)$ is i-th order divided difference of the function $f$ at $x_0$, given by the table.

- $(D^i f)(x_0) = [x_0, \dots, x_i; f]$
- **Newton interpolation formula** is
$$f = N_m f + R_m f$$

| | $f$ | $Df$ | | $D^{m+1} f$ |
|---|---|---|---|---|
| $x$ | $f(x)$ | $(Df)(x) = [x, x_0; f]$ | $\dots$ | $[x, x_0, \dots, x_m; f]$ |
| $x_0$ | $f(x_0)$ | $(Df)(x_0) = [x_0, x_1; f]$ | $\dots$ | |
| $x_1$ | $f(x_1)$ | $(Df)(x_1) = [x_1, x_2; f]$ | $\dots$ | |
| $\dots$ | $\dots$ | $\dots$ | | |
| $x_{m-1}$ | $f(x_{m-1})$ | $(Df)(x_{m-1}) = [x_{m-1}, x_m; f]$ | | |
| $x_m$ | $f(x_m)$ | | | |

we add $x$

$$[x, x_0; f] = \frac{f(x_0) - f(x)}{x_0 - x}$$

- $(R_m f)(x) = (x - x_0) \dots (x - x_m) [x, x_0, \dots, x_m; f]$
- $(N_i f)(x) = (N_{i-1} f)(x) + (x - x_0)(x - x_1) \cdot \dots \cdot (x - x_{i-1}) [x_0, \dots, x_i; f]$
- $\dfrac{f^{(m+1)}(\xi)}{(m+1)!} = [x, x_0, \dots, x_m; f]$

Example: Find $L_2 f$ for $f(x) = \sin \pi x$ and $x_0 = 0$ $x_1 = \frac{1}{6}$, $x_2 = \frac{1}{2}$

| $x$ | $f(x)$ | $Df$ | $D^2 f$ |
|---|---|---|---|
| $x_0 = 0$ | 0 | 3 | $-3$ |
| $x_1 = \frac{1}{6}$ | $\frac{1}{2}$ | $\frac{3}{2}$ | |
| $x_2 = \frac{1}{2}$ | 1 | | |

$$(L_2 f)(x) = 0 + \sum_{i=1}^{2} (x - x_0) \dots (x - x_{i-1}) (D^i f)(x_0) = \cancel{0 + (x - 0) \cdot (x - \tfrac{1}{6})}$$

$$= 0 + (x-0)\cdot(D^1 f)(x_0) + (x-0)\left(x-\frac{1}{6}\right)\cdot(D^2 f)(x_0)$$
$$= 0 + 3x + \left(x^2 - \frac{1}{6}x\right)\cdot(-3) =$$
$$= 3x - 3x^2 + \frac{1}{2}x = -3x^2 + \frac{7}{2}x$$

Lagrange classical formula:

$$x_0 = 0 \qquad x_1 = \frac{1}{6} \quad x_2 = \frac{1}{2}$$

$$(L_2 f)(x) = \sum_{i=0}^{2} l_i(x)\, f(x_i)$$

$$l_0(x) = \frac{x-x_1}{x_0-x_1}\cdot\frac{x-x_2}{x_0-x_2} = \frac{\left(x-\frac{1}{6}\right)\left(x-\frac{1}{2}\right)}{\frac{1}{12}} = \cancel{12x^2+6x}$$
$$\cancel{= (12x-2)(12x}$$
$$= (12x-2)\left(x-\frac{1}{2}\right)$$
$$= 12x^2 - 8x + 1$$

$$l_1(x) = \frac{x-x_0}{x_1-x_0}\cdot\frac{x-x_2}{x_1-x_2} = \frac{x\left(x-\frac{1}{2}\right)}{-\frac{1}{3}\cdot\frac{1}{6}}$$
$$= \frac{x^2-\frac{x}{2}}{-\frac{1}{18}} = -18x^2 + 9x$$

$$l_2(x) = \frac{x-x_0}{x_2-x_0}\cdot\frac{x-x_1}{x_2-x_1} = \frac{x\cdot\left(x-\frac{1}{6}\right)}{\frac{1}{2}\cdot\frac{1}{3}} = 6x^2 - x$$

$$(L_2 f)(x) = (12x^2-8x+1)\cdot 0 + (-18x^2+9x)\cdot\frac{1}{2} + 6x^2 - x$$
$$= -9x^2 + \frac{9x}{2} + 6x^2 - x = -3x^2 + \frac{7}{2}x$$

# Neville algorithm

- Let $f$ be a function defined at $x_0, x_1, \dots, x_n$ and suppose that $m_1, m_2, \dots, m_k$ are $k$ distinct integers with $0 \leq m_i < n$ for every $i$. The Lagrange polynomial that interpolates $f(x)$ at the $k$ points $x_{m_1}, \dots, x_{m_k}$ is denoted by $P_{m_1,\dots,m_k}(x)$

Example: Consider $x_0=1$, $x_1=2$, $x_2=3$, $x_3=4$, $x_4=6$ and $f(x)=e^x$. Determine $P_{1,2,4}(x)$ and use it to approximate $f(5)$.

$$P_{1,2,4}(x) = \frac{(x-x_2)(x-x_4)}{(x_1-x_2)(x_1-x_4)} f(x_1) + \frac{(x-x_1)(x-x_4)}{(x_2-x_1)(x_2-x_4)} f(x_2) + \frac{(x-x_1)(x-x_2)}{(x_4-x_1)(x_4-x_2)} f(x_4)$$

$$= \frac{(x-3)(x-6)}{(2-3)(2-6)} f(2) + \frac{(x-2)(x-6)}{(3-2)(3-4)} f(3) + \frac{(x-2)(x-3)}{(6-2)(6-3)} f(6)$$

$$f(5) \approx P_{1,2,4}(5) = \frac{(5-3)(5-6)}{4} e^2 + \frac{(5-2)(5-6)}{-1} e^3 + \frac{(5-2)(5-3)}{12} e^6$$

- Let $f$ be a function defined at the points $x_0, x_1, \ldots, x_k$ and let $x_i$ and $x_j$ be two distinct points in this set. Then

$$P(x) = \frac{(x-x_j)P_{0,1,\ldots,j-1,j+1,\ldots,k}(x) - (x-x_i)P_{0,1,\ldots,i-1,i+1,\ldots,k}(x)}{x_i - x_j}$$

is the $k$ th lagrange polynomial that approximates $f$ at the $k+1$ nodes $x_0, x_1, \ldots, x_k$.

- $P_j(x) = f(x_j)$
- the interpolation polynomials can be generated recursively

$$P_{0,1} = \frac{1}{x_1 - x_0}\left[(x-x_0)P_1 - (x-x_1)P_0\right]$$

$$P_{1,2} = \frac{1}{x_2 - x_1}\left[(x-x_1)P_2 - (x-x_2)P_1\right]$$

$$P_{0,1,2} = \frac{1}{x_2 - x_0}\left[(x-x_0)P_{1,2} - (x-x_2)P_{0,1}\right]$$

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| | | | | | |
|---|---|---|---|---|---|
| $x_0$ | $P_0$ | | | | |
| $x_1$ | $P_1$ | $P_{0,1}$ | | | |
| $x_2$ | $P_2$ | $P_{1,2}$ | $P_{0,1,2}$ | | |
| $x_3$ | $P_3$ | $P_{2,3}$ | $P_{1,2,3}$ | $P_{0,1,2,3}$ | |
| $x_4$ | $P_4$ | $P_{3,4}$ | $P_{2,3,4}$ | $P_{1,2,3,4}$ | $P_{0,1,2,3,4}$ |

- $Q_{ij}(x) = P_{i-j,i-j+1,\ldots,i-1,i}(x)$ with $0 \le j \le i$
- the table becomes

| | | | | | |
|---|---|---|---|---|---|
| $x_0$ | $P_0 = Q_{00}$ | | | | |
| $x_1$ | $P_1 = Q_{10}$ | $P_{0,1} = Q_{11}$ | | | |
| $x_2$ | $P_2 = Q_{20}$ | $P_{1,2} = Q_{21}$ | $P_{0,1,2} = Q_{22}$ | | |
| $x_3$ | $P_3 = Q_{30}$ | $P_{2,3} = Q_{31}$ | $P_{1,2,3} = Q_{32}$ | $P_{0,1,2,3} = Q_{33}$ | |
| $x_4$ | $P_4 = Q_{40}$ | $P_{3,4} = Q_{41}$ | $P_{2,3,4} = Q_{4,2}$ | $P_{1,2,3,4} = Q_{43}$ | $P_{0,1,2,3,4} = Q_{44}$ |

## Neville's Iterated Interpolation Algorithm

**Input:** nodes $x_0, x_1, \ldots, x_n$ the values of the function $f(x_0), \ldots, f(x_n)$ as the first column of $Q$ $(Q_{0,0}, Q_{1,0}, \ldots, Q_{n,0})$

**Step 1:** for $i = 1, 2, \ldots, n$
for $j = 1, 2, \ldots, i$

$$Q_{i,j} = \frac{(x-x_{i-j})Q_{i,j-1} - (x-x_i)Q_{i-1,j-1}}{x_i - x_{i-j}}$$

**Step 2:** Stopping criterion

$$|Q_{i,j} - Q_{i-1,j-1}| < \varepsilon$$