With these notations, we have (for the trapezoidal rule) $h_k = \frac{b-a}{n_k} = \frac{b-a}{2^{k-1}}$

Romberg's Method

$$\int_a^b f(x)dx = \frac{h_k}{2}\left[f(a)+f(b)+2\sum_{i=1}^{2^{k-1}-1} f(a+ih_k)\right] - \frac{b-a}{12}h_k^2 f''(\xi_k),\ \xi_k \in [a,b].$$

$$R_{k,1} = \frac{1}{2}\left[R_{k-1,1} + h_{k-1}\sum_{i=1}^{2^{k-2}} f\left(a+(i-\tfrac{1}{2})h_{k-1}\right)\right],\ k = \overline{2,n}.\ R_{k,j} = \frac{4^{j-1}R_{k,j-1} - R_{k-1,j-1}}{4^{j-1}-1}$$

$$\begin{matrix} R_{1,1} & & & & \\ R_{2,1} & R_{2,2} & & & \\ R_{3,1} & R_{3,2} & R_{3,3} & & \\ \vdots & \vdots & \vdots & \ddots & \\ R_{n,1} & R_{n,2} & R_{n,3} & \dots & R_{n,n} \end{matrix} \quad |R_{n-1,n-1} - R_{n,n}| < \varepsilon$$

$$J_m(w) = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & 0 \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & \alpha_2 & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_{m-1}} \\ 0 & & & \sqrt{\beta_{m-1}} & \alpha_{m-1} \end{bmatrix}$$

Theorem 3.9. *Let $u(x) = (x-x_1)(x-x_2)\dots(x-x_m)$. Then, the quadrature formula (3.15) is exact for all polynomials $p \in \mathbb{P}_{2m-1}$ if and only if u is orthogonal to the set $\mathbb{P}_{m-1}$, $u \perp \mathbb{P}_{m-1}$, with*

$$< f,g >_w = \int_a^b w(x)f(x)g(x)\,dx.$$

$$\pi_{k+1}(t) = (t-\alpha_k)\pi_k(t) - \beta_k\pi_{k-1}(t)$$

$$\pi_{-1}(t) = 0,\ \pi_0(t) = 1.$$

Name	Notation	Polynomial	Weight fn.	Interval	α_k	β_k
Legendre	l_m	$[(x^2-1)^m]^{(m)}$	1	$[-1,1]$	0	$\beta_0 = 2$, $\beta_k = (4-k^{-2})^{-1}, k \geq 1$
Chebyshev 1st	T_m	$\cos(m \arccos x)$	$(1-x^2)^{-\frac{1}{2}}$	$[-1,1]$	0	$\beta_0 = \pi$, $\beta_1 = \frac{1}{2}$, $\beta_k = \frac{1}{4}, k \geq 2$
Chebyshev 2nd	Q_m	$\frac{\sin[(m+1)\arccos x]}{\sqrt{1-x^2}}$	$(1-x^2)^{\frac{1}{2}}$	$[-1,1]$	0	$\beta_0 = \frac{\pi}{2}$, $\beta_k = \frac{1}{4}, k \geq 1$
Laguerre	L_m^a	$x^{-a}e^x\left(x^{m+a}e^{-x}\right)^{(m)}$	$x^a e^{-x},\ a > -1$	$[0,\infty)$	$2k+a+1$	$\beta_0 = \Gamma(1+a)$, $\beta_k = k(k+a), k \geq 1$
Hermite	H_m	$(-1)^m e^{x^2}\left(e^{-x^2}\right)^{(m)}$	e^{-x^2}	$\mathbb{R}$	0	$\beta_0 = \sqrt{\pi}$, $\beta_k = \frac{k}{2}, k \geq 1$

Jacobi iteration method

$$x_1^{(k+1)} = \frac{1}{9}\left[b_1 - x_2^{(k)} - x_3^{(k)}\right]$$
$$x_2^{(k+1)} = \frac{1}{10}\left[b_2 - 2x_1^{(k)} - 3x_3^{(k)}\right]$$
$$x_3^{(k+1)} = \frac{1}{11}\left[b_3 - 3x_1^{(k)} - 4x_2^{(k)}\right]$$
$$\|x - x^{(k)}\| = \max_{1\leq i\leq n}|x_i - x_i^{(k)}|$$

Gauss-Seidel iteration method

$$x_1^{(k+1)} = \frac{1}{9}\left[b_1 - x_2^{(k)} - x_3^{(k)}\right]$$
$$x_2^{(k+1)} = \frac{1}{10}\left[b_2 - 2x_1^{(k+1)} - 3x_3^{(k)}\right]$$
$$x_3^{(k+1)} = \frac{1}{11}\left[b_3 - 3x_1^{(k+1)} - 4x_2^{(k+1)}\right]$$

Jacobi iteration

$M = D,\ N = L+U$, so

$T_J = D^{-1}(L+U),\ c_J = D^{-1}b.$

$x^{(k+1)} = D^{-1}(L+U)x^{(k)} + D^{-1}b,$

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}x_j^{(k)}\right],\ i = \overline{1,n}$$

for any diagonally dominant system, converges

$$|a_{ii}| > \sum_{\substack{j=1\\ j\neq i}}^{n}|a_{ij}|,\ i = \overline{1,n},$$

Gauss-Seidel iteration

$M = D-L,\ N = U$, so

$T_{GS} = (D-L)^{-1}U,\ c_{GS} = (D-L)^{-1}b.$

$x^{(k+1)} = (D-L)^{-1}Ux^{(k)} + (D-L)^{-1}b,$

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1}a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n}a_{ij}x_j^{(k)}\right]$$

optimal value for ω

$$\omega^* = \frac{2}{1+\sqrt{1-(\rho(T_J))^2}}$$

$$f[x_0,x_0,\dots,x_0] = \frac{f^{(n)}(x_0)}{n!}$$

*The polynomial $p(\lambda) = \det(A-\lambda I_n)$ is called the **characteristic polynomial of** A and the equation $p(\lambda) = 0$ the **characteristic equation of** A.*
*The roots of $p(\lambda)$ are called **eigenvalues (e-values) of** A.*
*If $\lambda \in \mathbb{C}$ is an e-value of A, a vector $x \in \mathbb{R}^n, x \neq 0$ satisfying $(A-\lambda I_n)x = 0$ is called an **eigenvector (e-vector) of** A, corresponding to the e-value λ.*
*The set of all e-values of A, denoted by $\lambda(A)$ is called the **spectrum of** A.*
*The value $\rho(A) = \max\{|\lambda| \mid \lambda \in \lambda(A)\}$ is called the **spectral radius of** A.*
*The value $tr(A) = a_{11} + \dots + a_{nn}$ is called the **trace of** A.*

Acceleration methods *acceleration parameter ω*

$M = \frac{D}{\omega} - L,\ N = \left(\frac{1-\omega}{\omega}D + U\right)$, so

$T_\omega = \left(\frac{D}{\omega}-L\right)^{-1}\left(\frac{1-\omega}{\omega}D+U\right),\ c_\omega = \left(\frac{D}{\omega}-L\right)^{-1}b.$

$$x_i^{(k+1)} = \frac{\omega}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1}a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n}a_{ij}x_j^{(k)}\right] + (1-\omega)x_i^{(k)}$$

$\omega < 1$ is called *subrelaxation*;
$\omega = 1$ is the Gauss-Seidel method; $\rho(T_\omega) \geq |\omega - 1|$
$\omega > 1$ is called *overrelaxation*, the **SOR method**

necessary condition for the convergence of the SOR method is $0 < \omega < 2$

$$x^{(k)} = \omega\tilde{x}^{(k)} + (1-\omega)x^{(k-1)}$$

Definition 1.2. *We say that a sequence of iterates $\{x_n\}_{n\in\mathbb{N}}$ converges to α with **order of convergence** $p \geq 1$, if $|x_{n+1}-\alpha| \leq c\,|x_n-\alpha|^p$, for all $n \in \mathbb{N}$, where $c > 0$ is a constant independent of n.*
*If $p = 1$, the method is said to **converge linearly** to α, in which case we also require that $c < 1$.*
*Then the constant c is called the **rate of linear convergence** of x_n to α.*
*For $1 < p < 2$, we say that the convergence is **superlinear**.*

Theorem 2.1. [Intermediate Value Theorem]
If $f : [a,b] \to \mathbb{R}$ is a continuous function, then it takes on any given value between $f(a)$ and $f(b)$ at some point within the interval. As a consequence, if a continuous function has values of opposite sign inside an interval $[a,b]$, then it has at least a root in that interval.

Theorem 2.2. [Rolle's Theorem]
If a function f is continuous on $[a,b]$ and differentiable on (a,b), with $f(a) = f(b)$, then there exists a point $c \in (a,b)$ such that $f'(c) = 0$. As a consequence, between any two distinct real roots of f, there is a root of the derivative.

Bisection Method

$f(a)f(b) < 0.$

exists $\alpha \in (a,b)$ such that $f(\alpha) = 0$.

Define $c = (a+b)/2$.
If $b-c \leq \varepsilon$, then $\alpha = c$ and exit.
If $\text{sign}(f(b)) \cdot \text{sign}(f(c)) \leq 0$, then $a = c$; otherwise, $b = c$.
Return to step 1.

$$n \geq \log_2\left(\frac{b-a}{\varepsilon}\right) \qquad |x_n - \alpha| \leq b_n - a_n = \frac{b-a}{2^n}, \qquad |x_{n+1}-\alpha| \leq \frac{1}{2}|x_n-\alpha|$$

Theorem 1.1. [Taylor's Theorem] *Let $f : [a,b] \to \mathbb{R}$ be a function with $n+1$ continuous derivatives on $[a,b]$, for some $n \geq 0$ and let $x, x_0 \in [a,b]$. Then*

$$f(x) = p_n(x) + R_{n+1}(x),$$

where

$$p_n(x) = f(x_0) + \frac{(x-x_0)}{1!}f'(x_0) + \dots + \frac{(x-x_0)^n}{n!}f^{(n)}(x_0)$$

*is **Taylor's polynomial of degree n of f at x_0** and*

$$R_{n+1}(x) = \frac{1}{n!}\int_{x_0}^{x}(x-t)^n f^{(n+1)}(t)dt = \frac{(x-x_0)^{n+1}}{(n+1)!}f^{(n+1)}(\xi),\ \xi \text{ between } x \text{ and } x_0, \quad (1.3)$$

*is **the error of the Taylor approximation**.*

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!}e^{\xi_x},$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n\frac{x^{2n}}{(2n)!} + (-1)^{n+1}\frac{x^{2n+2}}{(2n+2)!}\cos\xi_x,$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + (-1)^n\frac{x^{2n+1}}{(2n+1)!}\sin\xi_x,$$
$$(1+x)^a = 1 + \binom{a}{1}x + \binom{a}{2}x^2 + \dots + \binom{a}{n}x^n + \binom{a}{n+1}\frac{x^{n+1}}{(1+\xi_x)^{n+1-a}},$$

Taylor's formula in two dimensions

Theorem 1.4. *Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a function which is $n+1$ times continuously differentiable on D, for some $n \geq 0$ and let $(x,y), (x_0,y_0) \in D$. Then*

$$f(x,y) = p_n(x,y) + R_{n+1}(x,y), \quad (1.4)$$

$$p_n(x,y) = f(x_0,y_0) + \frac{x-x_0}{1!}f'_x(x_0,y_0) + \frac{y-y_0}{1!}f'_y(x_0,y_0)$$
$$+ \frac{1}{2!}\left[(x-x_0)^2f''_{x^2}(x_0,y_0) + 2(x-x_0)(y-y_0)f''_{xy}(x_0,y_0) + (y-y_0)^2f''_{y^2}(x_0,y_0)\right]$$
$$+ \sum_{j=1}^{n}\frac{1}{j!}\left[(x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y}\right]^j f(x,y)\Big|_{\substack{x=x_0\\ y=y_0}}$$

Newton's (tangent) method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},\ n = 0,1,\dots$$

secant method.

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})},\ n = 1,2,\dots$$

$$f[x_0,x_1,\dots,x_n] = \frac{f[x_1,x_2,\dots,x_n] - f[x_0,x_1,\dots,x_{n-1}]}{x_n - x_0}$$

$$f[x_0,x_1,\dots,x_n] = \sum_{i=0}^{n}\frac{f(x_i)}{u'(x_i)} = \sum_{i=0}^{n}\frac{f(x_i)}{u_i(x_i)},$$

where $u(x) = (x-x_0)(x-x_1)\dots(x-x_n)$ and $u_i(x) = \frac{u(x)}{x-x_i}$

the mean-value formula for divided differences

$$f[x_0,x_1,\dots,x_n] = \frac{1}{n!}f^{(n)}(\xi_n),$$

$$\Delta^k f(x_i) = \Delta^{k-1}f(x_{i+1}) - \Delta^{k-1}f(x_i)$$
$$\nabla^k f_i = \nabla^{k-1}f_i - \nabla^{k-1}f_{i-1}$$

Definition 3.1. *A number r written in base b (an even number) has the **floating-point representation** as*

$$r = \pm\, r_0 r_1 \dots r_p \cdot r_{p+1}\dots r_k \times b^c,$$

where $r_0, r_1, \dots, r_k$ form the mantissa (significand), c is the exponent and b is the base.

*approximates r **correctly with n significant digits** if*

$$|\Delta r^*| \leq \frac{1}{2} \times 10^{k-n+1}$$

Definition 3.2. *The **significant digits** of a floating-point number written in base b are any of the digits $1, 2, \dots, b-1$ in its representation that are non-zero, or located between non-zero digits, or preceded by at least one non-zero digit.*

$$\delta x_i \simeq \frac{|\delta f|}{n\left|x_i\frac{d}{dx_i}\ln f\right|}$$

$$\delta f = \frac{\Delta f}{f} \simeq \sum_{i=1}^{n}\Delta x_i\frac{f'_{x_i}()}{f} = \sum_{i=1}^{n}\Delta x_i\frac{d}{dx_i}\ln f() = \sum_{i=1}^{n}x_i\delta x_i\frac{d}{dx_i}\ln f() = \sum_{i=1}^{n}x_i\frac{d}{dx_i}\ln f()\delta x_i.$$

Theorem 1.7. *If no row interchanges are necessary in the Gaussian elimination process for solving the system $Ax = b$, then A can be factored as $A = LU$.* **LU Factorization**

$l_{ii} = 1, i = \overline{1,n}$, we have *Doolittle* factorization and if we impose $u_{ii} = 1, i = \overline{1,n}$, *Crout* factorization

3. The matrix $U = [u_{ij}]$ in the Doolittle factorization is the upper triangular matrix obtained by Gaussian elimination, $u_{ij} = a_{ij}^{(i)}, \ i \leq j$ while $L = [l_{ij}]$ is the matrix of the *multipliers*

$$l_{ij} = m_{ij} = \frac{a_{ij}^{(j)}}{a_{jj}^{(j)}},\ i \geq j$$

$$A = \begin{bmatrix} a_{11} & w^* \\ v & A' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ v/a_{11} & I_{n-1} \end{bmatrix}\begin{bmatrix} a_{11} & w^* \\ 0 & A'-vw^*/a_{11} \end{bmatrix}$$

$A' - vw^*/a_{11}$ is called the **Schur complement** of A with respect to a_{11}.

We take the upper triangular part (*including* the main diagonal) for U and the lower triangular part (*without* the main diagonal) for L (which will have all 1's on the main diagonal).

$$Ly = b$$
$$Ux = y$$

Definition 1.1. *A real square matrix Q is called **orthogonal** if $Q \cdot Q^t = Q^t \cdot Q = I$*

Let A be a real square matrix. Then there exist unique matrices Q and R such that $A = QR$ with Q orthogonal and R upper triangular with positive elements on the main diagonal, r_{ii}

*The pair (Q,R) is called the **QR factorization** of A.*

$$Ax = b \Leftrightarrow Rx = Q^t b$$

LUP Factorization

$PA = LU.$

P a permutation matrix.

$$Ly = Pb$$
$$Ux = y.$$

Theorem 1.5. *Let A be a symmetric positive definite matrix. Then A has a unique **Cholesky factorization** $A = R^t R$. A real square matrix A is called **positive definite**, if*

$$\det(A - \lambda I) = 0$$

eigenvalues are positive and real

$$x^t Ax = \sum_{i,j=1}^{n}a_{ij}x_i x_j > 0,\ \forall x \in \mathbb{R}^n,\ x \neq 0 \quad (1.5)$$

R is an upper triangular matrix with positive elements on the main diagonal, $r_{ii} > 0,\ \forall i$.

$$A = \begin{bmatrix} a_{11} & w^* \\ w & A' \end{bmatrix} = \begin{bmatrix} \alpha & w^*/\alpha \\ w/\alpha & A' \end{bmatrix} \qquad \alpha = \sqrt{a_{11}}$$
$$= \begin{bmatrix} \alpha & 0 \\ w/\alpha & I_{n-1} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & A' - ww^*/a_{11} \end{bmatrix}\begin{bmatrix} \alpha & w^*/\alpha \\ 0 & I_{n-1} \end{bmatrix} = R_1^t A_1 R_1$$

total (maximal) pivoting. At each step k, we find

$$|a_{pq}| = \max\{|a_{ij}|, i,j = \overline{k,n}\}$$
$$(R_k) \longleftrightarrow (R_p),\ (C_k) \longleftrightarrow (C_q)$$

This procedure is repeated until

$$A = \underbrace{R_1^t R_2^t \dots R_w^t}_{R^t} R_w R_{w-1} \dots R_1 = R^t R$$

The peak of the Gaussian formula (3.15) are the eigenvalues of J_m

$$J_m v_k = x_k v_k,\ v_k^T v_k = 1,\ k = 1,\dots,m, A_k = \beta_0 v_{k1}^2.$$

Remark 1.2. The determinant of an upper or lower triangular matrix is equal to the product of its diagonal elements $\det(A) = a_{11}a_{22}\dots a_{nn}$.

We can choose the pivot to be the largest element (in absolute value) in that column, below the main diagonal, i.e. **partial pivoting (maximal pivoting on columns)**.

scaled pivoting on columns:
If there exists an i such that $s_i = 0$, then the matrix is singular. For a nonsingular matrix, we use the scaling factor to choose the pivot. At each step i, we find the smallest $p,\ i \leq p \leq n$ such that and then interchange rows $(R_i) \longleftrightarrow (R_p)$ so the pivot is a_{pi}

$$s_i = \max_{j=\overline{1,n}}|a_{ij}| \text{ or } s_i = \sum_{j=1}^{n}|a_{ij}|,\ i = \overline{1,n} \qquad \frac{|a_{pi}|}{s_p} = \max_{1\leq j\leq n}\frac{|a_{ji}|}{s_j}$$

Theorem 3.3. [Banach] *Let $g \in C[a,b]$, with $g([a,b]) \subseteq [a,b]$. Assume that there exists $0 < \lambda < 1$ such that $|g(x) - g(y)| \leq \lambda|x-y|,\ \forall x,y \in [a,b]$ (i.e., g is a **contraction**)*
Then g has a unique fixed point $\alpha \in [a,b]$. Furthermore, the iterates $x_{n+1} = g(x_n)$ converge to α,

$$|g(x) - g(y)| \leq \lambda|x-y|,\ \forall x,y \in [a,b] \text{ (i.e., } g \text{ is a } \textbf{contraction}).$$

$$|x_n - \alpha| \leq \lambda|x_{n-1} - \alpha|,\ n \geq 1,$$
$$|x_n - \alpha| \leq \frac{\lambda^n}{1-\lambda}|x_1 - x_0|.$$

$$|x_{n+1} - x_n| \leq \frac{1-\lambda}{\lambda}\varepsilon$$

$$\lim_{n\to\infty}\frac{x_{n+1}-\alpha}{(x_n-\alpha)^p} = \frac{1}{p!}g^{(p)}(\alpha)$$

derivative at alpha > 1 does not converge, =-1 does not converge, <1 converges, aprox 0 best rate (linear)

2. Again, as before, Newton's method *does not* always converge, but when it does, it does so faster ($p = 2$) than the bisection method ($p = 1$) and the secant method ($p = (1+\sqrt{5})/2 \approx 1.618$).

Lagrange interpolation polynomial

$$L_n f(x) = \sum_{i=0}^{n}l_i(x)f(x_i),$$

$$l_i(x) = \prod_{\substack{j=0\\ j\neq i}}^{n}\frac{x-x_j}{x_i-x_j} = \frac{u_i(x)}{u_i(x_i)} = \frac{u_i(x)}{u'(x_i)}$$

$$l_i(x_j) = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

$$(R_n f)(x) = \frac{u(x)}{(n+1)!}f^{(n+1)}(\xi)$$

$$|(R_n f)(x)| \leq \frac{|u(x)|}{(n+1)!}M_{n+1}(f)$$

$$M_{n+1}(f) = \sup_{t\in[a,b]}|f^{(n+1)}(t)|.$$

second barycentric formula.

$$L_n f(x) = \frac{\sum_{i=0}^{n}\frac{w_i}{x-x_i}f_i}{\sum_{i=0}^{n}\frac{w_i}{x-x_i}},$$

first barycentric formula.

$$L_n f(x) = u(x)\sum_{i=0}^{n}\frac{w_i}{x-x_i}f_i. \qquad w_i = \frac{1}{u'(x_i)} = \frac{1}{\prod_{\substack{j=0\\ j\neq i}}^{n}(x_i-x_j)}$$

$$u(x) = \prod_{j=0}^{n}(x-x_j)$$

$$u_j(x) = \frac{u(x)}{x-x_j},\ j = 0,1,\dots,n$$

$$\begin{matrix} \dots & \dots & \dots & \dots \\ \psi_n(x) & = & (x-x_{n-1})\psi_{n-1}(x) & + & (x-x_0)\psi_{n-1}(x) \\ \psi_{n-1}(x) & = & (x-x_{n-2})\psi_{n-2}(x) & + & (x-x_0)\psi'_{n-2}(x) \\ \psi_{n-2}(x) & = & (x-x_{n-3})\psi_{n-3}(x) & + & (x-x_0)\psi'_{n-3}(x) \\ \dots & \dots & \dots & \dots \end{matrix}$$

Newton's divided difference form

$$L_n f(x) = f(x_0) + f[x_0,x_1](x-x_0) + \dots + f[x_0,\dots,x_n](x-x_0)\dots(x-x_{n-1}).$$

Assume the interval is $[-1,1]$ (then, for a general interval $[a,b]$, we use the linear change of variables $x = \frac{b-a}{2}t + \frac{b+a}{2},\ t \in [-1,1],\ x \in [a,b]$).

An optimal choice of nodes are the roots of the **Chebyshev polynomial of the first kind**:

$T_m(x) = \cos(m \arccos x)$. With the change of variables $x = \cos t, t \in [0,\pi]$, we get

$$T_m(x) = \cos(mt) = \frac{1}{2}\left(e^{imt} + e^{-imt}\right) = \frac{1}{2}[(\cos t + i\sin t)^m + (\cos t - i\sin t)^m] \quad (1.8)$$

$$R_n f(x) = \frac{(x-x_0)\dots(x-x_n)}{(n+1)!}f^{(n+1)}(\xi)$$

x_0	P_{00}		
x_1	P_{10}	P_{11}	
x_2	P_{20}	P_{21}	P_{22}
x_3	P_{30}	P_{31}	P_{32} P_{33}

Newton's forward difference formula. $s = (x-x_0)/h$.

$$L_n f(x) = f_0 + \binom{s}{1}\Delta f_0 + \binom{s}{2}\Delta^2 f_0 + \dots + \binom{s}{n}\Delta^n f_0,$$

$$\binom{s}{k} = \frac{s(s-1)\cdots(s-k+1)}{k!} \qquad f(x) - L_n f(x) = h^{n+1}\binom{s}{n+1}f^{(n+1)}(\xi_x)$$

Newton's backward difference formula.

$$L_n f(x) = f_n + \frac{s}{1!}\nabla f_n + \frac{s(s+1)}{2!}\nabla^2 f_n + \dots + \frac{s(s+1)\dots(s+n-1)}{n!}\nabla^n f_n$$

$$f(x) - L_n f(x) = h^{n+1}\binom{s+n}{n+1}f^{(n+1)}(\xi_x)$$

Neville's method.

$$P_{01\dots k}(x) = \frac{(x-x_0)P_{12\dots k}(x) - (x-x_k)P_{01\dots k-1}(x)}{x_k - x_0}$$

x_0	P_0		
x_1	P_1	P_{01}	
x_2	P_2	P_{12}	P_{012}
x_3	P_3	P_{23}	P_{123}

$P_{i,0} := f(x_i), \quad i = \overline{0,n},$
$P_{i,j} := \frac{(x-x_{i-j})P_{i,j-1} - (x-x_i)P_{i-1,j-1}}{x_i - x_{i-j}}.$
$|P_{ii} - P_{i-1,i-1}| < \varepsilon$

x_0	P_{00}		
x_1	P_{10}	P_{11}	
x_2	P_{20}	P_{21}	P_{22}
x_3	P_{30}	P_{31}	P_{32} P_{33}

Aitken's method

$$P_{i,j+1} := \frac{1}{x_i - x_j}\begin{vmatrix} x-x_j & P_{j,j} \\ x-x_i & P_{ij} \end{vmatrix} = \frac{(x-x_j)P_{i,j} - (x-x_i)P_{j,j}}{x_i - x_j}$$

$$x^* = (1+\delta x)x,$$

$$R_{n+1}(x,y) = \frac{1}{(n+1)!}\left[(x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y}\right]^{n+1}f(x,y)\Big|_{\substack{x=\xi\\ y=\eta}},$$

with (ξ,η) a point on the line segment determined by the points (x,y) and (x_0,y_0).

conditioning matrix.

$$\Gamma(x) = [\gamma_{ij}] = \begin{bmatrix} x_1\frac{\partial f_1}{\partial x_1}\frac{1}{f_1(x)} & \dots & x_m\frac{\partial f_1}{\partial x_m}\frac{1}{f_1(x)} \\ \vdots & & \\ x_1\frac{\partial f_n}{\partial x_1}\frac{1}{f_n(x)} & \dots & x_m\frac{\partial f_n}{\partial x_m}\frac{1}{f_n(x)} \end{bmatrix}$$

condition number of f at x

$$(\text{cond } f)(x) = \|\Gamma(x)\|$$

$$\text{cond}(A) = \|A\|\,\|A^{-1}\|$$

$$(\text{cond } f)(x) = \frac{\|x\|\left\|\frac{\partial f}{\partial x}\right\|}{\|f(x)\|}$$

If the matrix A is singular, by convention, $\text{cond}(A) = \infty$

If the condition number of a problem is large ($(\text{cond } f)(x) \gg 1$), then even for small (relative) errors in the input data, we can expect large errors in the output data. *ill-conditioned* problems.

natural (subordinate, induced) matrix norm

$$\|A\| = \sup_{v\neq 0}\frac{\|Av\|}{\|v\|} = \sup_{\|v\|\leq 1}\|Av\| = \sup_{\|v\|=1}\|Av\|$$

(i) $\|A\| \geq 0,\ \|A\| = 0 \Leftrightarrow A = 0_n$
(ii) $\|\alpha A\| = |\alpha| \cdot \|A\|$.
(iii) $\|A+B\| \leq \|A\| + \|B\|$.
(iv) $\|AB\| \leq \|A\| \cdot \|B\|$.

$$\|A\|_1 = \sup_{v\neq 0}\frac{\|Av\|_1}{\|v\|_1} = \max_{1\leq j\leq n}\sum_{i=1}^{n}|a_{ij}| \text{ (the Minkovski norm).}$$

$$\|A\|_\infty = \sup_{v\neq 0}\frac{\|Av\|_\infty}{\|v\|_\infty} = \max_{1\leq i\leq n}\sum_{j=1}^{n}|a_{ij}| \text{ (the Chebyshev norm)}$$

$$\|A\|_2 = \sup_{v\neq 0}\frac{\|Av\|_2}{\|v\|_2} = \sqrt{\rho(A^TA)} \text{ (the Euclidean norm).}$$

$$\|A\|_F = \left[\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\right]^{1/2} = \sqrt{tr(A^TA)} \quad \textit{Frobenius norm}$$