Hermite interpolation problem with double nodes. Given $m+1$ distinct nodes x_i, $i=\overline{0,m}$ and the values $f(x_i)$, $f'(x_i)$ of an unknown function f and its derivative, find a polynomial $P(x)$ of minimum degree, satisfying the interpolation conditions

1

$$\begin{aligned} P(x_i) &= f(x_i), \\ P'(x_i) &= f'(x_i),\ i=\overline{0,m}. \end{aligned} \tag{1.1}$$

Theorem 1.1. *There is a unique polynomial H_nf of degree at most n, satisfying the interpolation conditions (1.1). This polynomial can be written as*

2

$$H_nf(x) = \sum_{i=0}^{m}\left[h_{i0}(x)f(x_i)+h_{i1}(x)f'(x_i)\right], \tag{1.3}$$

$$l_i(x_j) = \delta_{ij} = \begin{cases} 0, & i\neq j \\ 1, & i=j \end{cases}$$

where

$$\begin{aligned} h_{i0}(x) &= \left[1-2l_i'(x_i)(x-x_i)\right]\left[l_i(x)\right]^2, \\ h_{i1}(x) &= (x-x_i)\left[l_i(x)\right]^2,\ i=0,\dots,m. \end{aligned} \tag{1.4}$$

H_nf is called the ***Hermite interpolation polynomial*** *of f at the double nodes $x_0,x_1,\dots,x_m$. The functions $h_{i0}(x), h_{i1}(x)$, $i=\overline{0,m}$ are called* ***Hermite fundamental polynomials*** *associated with these points.*

$$h_{i0}(x) = \frac{(x-x_i)^2}{j!}\left[\sum_{k=0}^{1}\frac{\tau_{i-i}}{k!}(x-x_i)^k\left[\frac{1}{u_i(x)}\right]^{(k)}_{x=x_i}\right]u_i(x).$$

$$h_{i1}(x) = \frac{(x-x_i)^2}{j!}\left[\sum_{k=0}^{1}\frac{\tau_{i-i}}{k!}(x-x_i)^k\left[\frac{1}{u_i(x)}\right]^{(k)}_{x=x_i}\right]u_i(x).$$

$$\begin{aligned} u(x) &= \prod_{i=0}^{m}(x-x_i)^{r_i+1}, \\ u_i(x) &= \prod_{\substack{j=0\\ j\neq i}}^{m}(x-x_j)^{r_j+1} = \frac{u(x)}{(x-x_i)^{r_i+1}}, \end{aligned}$$

5

Newton's form $N_nf(x) = f(z_0)+f[z_0,z_1](x-z_0)+\dots$
$+\ f[z_0,\dots,z_n](x-z_0)\dots(x-z_{n-1})$

$$\begin{aligned} R_n(x) &= f(x)-N_n(x) = f[z,z_0,\dots,z_n](x-z_0)\dots(x-z_n) \\ &= \frac{u(x)}{(n+1)!}f^{(n+1)}(\xi_x),\ \xi_x\in(a,b). \end{aligned}$$

, $i=\overline{0,m}$, $j=\overline{0,r_i}$, are called **Hermite fundamental polynomials**

6

Newton's divided difference form $z_0,z_1\to x_0,\ \ z_2,z_3\to x_1,\ \dots,\ z_{2i},z_{2i+1}\to x_i,\ \dots\ z_{2m},z_{2m+1}\to x_m$.

3

$$\begin{aligned} N_n(x) &= f(x_0)+f[x_0,x_0](x-x_0)+f[x_0,x_0,x_1](x-x_0)^2 \\ &+\ f[x_0,x_0,x_1,x_1](x-x_0)^2(x-x_1)+\dots \\ &+\ f[x_0,x_0,\dots,x_m,x_m](x-x_0)^2\dots(x-x_{m-1})^2(x-x_m) \end{aligned}$$

and for the remainder,

$$R_n(x) = f(x)-H_nf(x) = [\psi_m(x)]^2\frac{f^{(n+1)}(\xi_x)}{(n+1)!},\ \xi_x\in(a,b).$$

$$f(x)-N_n(x) = f[x,x_0,x_0,\dots,x_m,x_m](x-x_0)^2\dots(x-x_m)^2.$$

Hermite interpolation problem. Given $m+1$ distinct nodes $x_i\in[a,b]$, $i=\overline{0,m}$,

4

x_0 of multiplicity r_0+1,
x_1 of multiplicity r_1+1,
…
x_i of multiplicity r_i+1,
…
x_m of multiplicity r_m+1,

$$H_nf(x) = \sum_{i=0}^{m}\sum_{j=0}^{r_i}h_{ij}(x)f^{(j)}(x_i).$$

$$n+1 \overset{\text{not}}{=} \sum_{i=0}^{m}(r_i+1)$$

and the values $f^{(j)}(x_i)$, $i=0,1,\dots,m$, $j=0,\dots,r_i$, of an unknown function $f:[a,b]\to\mathbb{R}$

7

$$h_{kj}^{(p)}(x_\nu)=0,\ \text{when } k\neq\nu,\ p=0,\dots,r_\nu$$

$$h_{kj}^{(p)}(x_k)=\delta_{jp},\ \text{when } p=0,\dots,r_k,\ j=0,\dots,r_k,\ \text{and } \nu,k=0,\dots,m$$

$\delta_{jp}=0$ $(j\neq p)$ and 1 $(j=p)$.

does not always have a solution

$$B_nf(x) = \sum_{k=0}^{m}\sum_{j\in I_k}b_{kj}(x)f^{(j)}(x_k).$$

1

$n+1=|I_0|+\dots+|I_m|$.

The terms $b_{kj}(x)$ are called **Birkhoff fundamental polynomials** and they satisfy the relations

$$\begin{aligned} b_{kj}^{(p)}(x_\nu) &= 0,\ \nu\neq k,\ p\in I_\nu, \\ b_{kj}^{(p)}(x_k) &= \delta_{jp},\ p\in I_k,\ \text{for } j\in I_k \text{ and } \nu,k=0,1,\dots,m, \end{aligned}$$

- For $m\in\mathbb{N}$, $\mathbb{P}_m$ denotes the space of all polynomials of degree at most m. Obviously, $\mathbb{P}_m\subset C^\infty[a,b], \forall m\geq 0$.
- The *kernel* of a linear map $L:V\to W$ between two vector spaces V and W, is the set of al vectors in V that are mapped to zero: $\ker L = \{v\in V\mid L(v)=0_W\}$ $\quad 0_W$ is null vector in W

$$H^n[a,b] = \{f:[a,b]\to\mathbb{R}\mid f\in C^{n-1}[a,b], f^{(n-1)} \text{ absolutely continuous on } [a,b]\}.$$

2

positive part of z $z_+ = \begin{cases} z, & z\geq 0 \\ 0, & z<0 \end{cases}$

Any function $f\in H^n[a,b]$ has a Taylor-type representation, with the remainder in integral form

$$f(x) = \sum_{k=0}^{n-1}\frac{(x-a)^k}{k!}f^{(k)}(a)+\int_a^x\frac{(x-t)^{n-1}}{(n-1)!}f^{(n)}(t)\,dt.$$

Let $L:H^n[a,b]\to\mathbb{R}$ be a linear functional that commutes with the definite integral operator. If $\ker L=\mathbb{P}_{n-1}$, then

3

$$Lf = \int_a^b K_n(t)f^{(n)}(t)dt,$$

Peano kernel.

$$K_n(t) = \frac{1}{(n-1)!}L\left[(\cdot-t)_+^{n-1}\right]$$

Corollary 1.7. If the kernel K has constant sign on $[a,b]$ and $f^{(n)}$ is continuous on $[a,b]$, then there exists $\xi\in[a,b]$ such that

4

$$Lf = \frac{1}{n!}f^{(n)}(\xi)Le_n,$$

$$e_k(x)=x^k,\ k\in\mathbb{N}.$$

Newton's method for multiple roots

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}.$$

$$\lim_{n\to\infty}\frac{x_{n+1}-\alpha}{(x_n-\alpha)^2} = \frac{1}{2}g''(\alpha)$$

$$g(\alpha) = \alpha,\ g'(\alpha) = 1-m\,\varphi(\alpha) = 0,$$

$$f(\alpha) = f'(\alpha) = \dots = f^{(m-1)}(\alpha) = 0.$$

Definition 2.1. *Let Δ be a grid of the interval $[a,b]$,*

$$\Delta:\ a=x_1<x_2<\dots<x_{n-1}<x_n=b.$$

The set

$$\mathbb{S}_m^k(\Delta) = \{s\mid s\in C^k[a,b],\ s|_{[x_i,x_{i+1}]}\in\mathbb{P}_m,\ i=1,2,\dots,n-1\} \tag{2.1}$$

is called the ***the space of polynomial spline functions*** *of degree m and class k on Δ.*

A basis for this space is given by the so-called *B-spline functions*. Taking $x_0=x_1=a$, $x_{n+1}=$ $x_n=b$, for $i=\overline{1,n}$, we define

$$s(x) = \sum_{i=1}^{n}c_iB_i(x).$$

$$B_i(x_j) = \delta_{ij}.$$

$$B_i(x) = \begin{cases} \dfrac{x-x_{i-1}}{x_i-x_{i-1}}, & \text{If } x_{i-1}\leq x\leq x_i \\ \dfrac{x_{i+1}-x}{x_{i+1}-x_i}, & \text{If } x_i\leq x\leq x_{i+1} \\ 0, & \text{in rest.} \end{cases} \tag{2.5}$$

3

Interpolation with cubic splines $s\in\mathbb{S}_3^2(\Delta)$

$$p_{i-1}''(x_i) = p_i''(x_i),\ i=2,\dots,n-1,$$

which, for the Taylor coefficients in (2.8), means $2c_{i-1,2}+6c_{i-1,3}h_{i-1} = 2c_{i,2},\ i=2,\dots,n-1$

$$h_im_{i-1}+2(h_{i-1}+h_i)m_i+h_{i-1}m_{i+1} = b_i,\ i=2,\dots,n-1,$$

$$b_i = 3\Big(h_if[x_{i-1},x_i]+h_{i-1}f[x_i,x_{i+1}]\Big)$$

6

$$x_{n+1} = x_n - \underbrace{\left(J_f(x_n)\right)^{-1}}_{\left[\begin{array}{c|c} \frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_2} \\ \frac{\partial f_2(x)}{\partial x_1} & \frac{\partial f_2(x)}{\partial x_2} \end{array}\right]^{-1}} f(x_n)$$

$$= \qquad\qquad = J_f(x_n)$$

"Not-a-knot" (deBoor) splines.

$$p_1(x)\equiv p_2(x),\ p_{n-2}(x)\equiv p_{n-1}(x).$$

$$\begin{aligned} h_2m_1+(h_2+h_1)m_2 &= \gamma_1, \\ (h_{n-1}-h_{n-2})m_{n-1}+h_{n-2}m_n &= \gamma_2; \end{aligned}$$

10

$$\gamma_1 = \frac{1}{h_2+h_1}\left[f[x_1,x_2]h_2\left(h_1+2(h_1+h_2)\right)+h_1^2f[x_2,x_3]\right],$$

$$\gamma_2 = \frac{1}{h_{n-1}+h_{n-2}}\left[h_{n-1}^2f[x_{n-2},x_{n-1}]+\left(2(h_{n-1}+h_{n-2})+h_{n-1}\right)h_{n-2}f[x_{n-1},x_n]\right]$$

$$\|u\|_\infty = \max_{t\in[a,b]}|u(t)|, \qquad \|u\|_\infty = \max_{i=\overline{1,m}}|u(t_i)|,$$

14

$$\|u\|_2 = \left(\int_a^b|u(t)|^2\,dt\right)^{1/2}, \qquad \|u\|_2 = \left(\sum_{i=1}^{m}|u(t_i)|^2\right)^{1/2},$$

$$\|u\|_{2,w} = \left(\int_a^b w(t)\,|u(t)|^2\,dt\right)^{1/2} \qquad \|u\|_{2,w} = \left(\sum_{i=1}^{m}w_i\,|u(t_i)|^2\right)^{1/2}$$

$$y = f(x) = c_1\pi_1(x)+\dots c_m\pi_m(x).$$

$$\sum_{j=1}^{m}(\pi_i,\pi_j)c_j = (\pi_i,y),\ i=1,\dots,n.$$

normal equations.

15

$$f(x)=ax+b, \qquad E(a,b)=\sum_{i=1}^{n}(ax_i+b-y_i)^2$$

$$E_a' = 2\sum_{i=1}^{n}(ax_i+b-y_i)x_i, \qquad \sum_{i=1}^{n}(ax_i+b-y_i)x_i = 0,$$

$$E_b' = 2\sum_{i=1}^{n}(ax_i+b-y_i), \qquad \sum_{i=1}^{n}(ax_i+b-y_i) = 0,$$

$$a\sum_{i=1}^{n}x_i^2+b\sum_{i=1}^{n}x_i = \sum_{i=1}^{n}x_iy_i$$

$$a\sum_{i=1}^{n}x_i+nb = \sum_{i=1}^{n}y_i.$$

$$f(x)=ax^2+bx+c,$$

$$E(a,b,c)=\sum_{i=1}^{n}(ax_i^2+bx_i+c-y_i)^2.$$

$$a\sum_{i=1}^{n}x_i^4+b\sum_{i=1}^{n}x_i^3+c\sum_{i=1}^{n}x_i^2 = \sum_{i=1}^{n}x_i^2y_i$$

$$a\sum_{i=1}^{n}x_i^3+b\sum_{i=1}^{n}x_i^2+c\sum_{i=1}^{n}x_i = \sum_{i=1}^{n}x_iy_i$$

$$a\sum_{i=1}^{n}x_i^2+b\sum_{i=1}^{n}x_i+nc = \sum_{i=1}^{n}y_i,$$

$e_k(x)=x^k$.
$\ker R=\mathbb{P}_d \iff \begin{cases} R(e_k)=0,\ k=0,1,\dots,d, \\ R(e_{d+1})\neq 0, \end{cases}$

numerical differentiation formula.

$$f^{(k)}(\alpha) = \sum_{i=0}^{m}A_if(x_i)+R(f).$$

forward difference numerical derivative.

$$f'(x) \approx \frac{f(x+h)-f(x)}{h} \equiv D_hf(x)$$

$$(RD_hf)(x) = f'(x)-D_hf(x) = -\frac{h}{2}f''(\xi),\ \xi\in(x,x+h)$$

numerical integration (quadrature) formula.

$$\int_a^b f(x)dx = \sum_{k=0}^{m}\sum_{j\in I_k}A_{kj}f^{(j)}(x_k)+R(f).$$

backward difference numerical derivative

$$f'(x) \approx \frac{f(x)-f(x-h)}{h} \equiv \overline{D}_hf(x)$$

$$(R\overline{D}_hf)(x) = f'(x)-\frac{f(x)-f(x-h)}{h} = \frac{h}{2}f''(\xi),\ \xi\in(x-h,x)$$

$x_0=x-h, x_1=x, x_2=x+h$.

$$\begin{aligned} f'(x) &\approx f[x_0,x_1]+f[x_0,x_1,x_2](x-x_1+x-x_0) \\ &= \frac{f(x+h)+f(x-h)}{2h} \equiv \widehat{D}_hf(x), \end{aligned}$$

central difference numerical derivative formula.

$$(R_2f)'(x) = -\frac{h^2}{6}f'''(\xi_x),\ x-h\leq\xi_x\leq x+h.$$

Method of Undetermined Coefficients

$$D_h^{(2)}f(x) = \frac{f(x+h)-2f(x)+f(x-h)}{h^2}.$$

$$(RD_h^{(2)}f)(x) = f''(x)-D_h^{(2)}f(x) \approx -\frac{h^2}{12}f^{(4)}(x$$

interpolatory quadrature

$$\int_a^b f(x)dx = \sum_{k=0}^{m}A_kf(x_k)+R(f),$$

nodes used are equally spaced **Newton-Cotes quadrature**

interpolatory quadrature $d_{\max}=2m+1$.

$$\int_a^b f(x)dx = \sum_{k=0}^{m}A_kf(x_k)+R(f),$$

nodes used are equally spaced **Newton-Cotes quadrature**

rectangle rule.

$$\int_a^b f(x)dx = (b-a)f\left(\frac{a+b}{2}\right)+R(f),$$

a single *double* node, $x_0=\dfrac{a+b}{2}$ $\qquad d=1.$

$$R(f) = \frac{f''(\xi)}{2!}\int_a^b\left(x-\frac{a+b}{2}\right)^2dx = \frac{(b-a)^3}{24}f''(\xi),\ \xi\in(a,b$$

composite (repeated) rectangle (midpoint) formula.

$$\int_a^b f(x)dx = h\sum_{i=0}^{n-1}f\left(a+\left(i+\frac{1}{2}\right)h\right)+\frac{h^2(b-a)}{24}f''(\xi),\ \xi\in(a,b)$$

trapezoidal rule. 2 nodes, $x_0=a, x_1=b$. $\qquad d=1.$

$$\int_a^b f(x)dx = \frac{b-a}{2}\Big(f(a)+f(b)\Big)-\frac{(b-a)^3}{12}f''(\xi),\ \xi\in(a,b)$$

$$\int_a^b f(x)dx = \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{(b-a)^5}{2880}f^{(4)}(\xi),\ \xi\in(a,b)$$

$x_0=a, x_1=\frac{a+b}{2}$, *double* and $x_2=b$. **(Cavalieri-) Simpson rule** $d=3$.

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(a)+4\sum_{i=1}^{m}f_{2i-1}+2\sum_{i=1}^{m-1}f_{2i}+f(b)\right]-\frac{h^4(b-a)}{180}f^{(4)}(\xi),\ \xi\in(a,b).$$

composite (repeated) Simpson's rule

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(a)+2(f_1+\dots+f_{n-1})+f(b)\right]-\frac{h^2(b-a)}{12}f''(\xi),\ \xi\in(a,b)$$

composite (repeated) trapezoidal rule.

$$\int_a^b f(x)dx$$

Alternatively, we can write it in Taylor's form around x_i. Considering that $x-x_{i+1}=x_i-x_i$, for $x\in[x_i,x_{i+1}]$, we get

4

$$\begin{aligned} p_i(x) &= c_{i,0}+c_{i,1}(x-x_i)+c_{i,2}(x-x_i)^2+c_{i,3}(x-x_i)^3 \\ c_{i,0} &= f_i, \\ c_{i,1} &= m_i, \\ c_{i,2} &= \frac{f[x_i,x_{i+1}]-m_i}{h_i}-c_{i,3}h_i = \frac{3f[x_i,x_{i+1}]-2m_i-m_{i+1}}{h_i} \\ c_{i,3} &= \frac{m_{i+1}-2f[x_i,x_{i+1}]+m_i}{h_i^2}. \end{aligned}$$

Finding cubic splines using the second derivatives

$$\begin{aligned} s_3|_{[x_i,x_{i+1}]} &= p_i(x)\in\mathbb{P}_3,\ i=1,2,\dots,n-1, \\ s_3(f,x_i) &= f_i,\ i=1,2,\dots,n, \\ s_3''(f,x_i) &= M_i,\ i=1,2,\dots,n. \end{aligned}$$

$$\begin{cases} ax_i+b = M_i \\ ax_{i+1}+b = M_{i+1} \end{cases}$$

$$\begin{aligned} s_3''(f;x) &= ax+b,\ s_3''(f;x_i)=M_i,\ s_3''(f;x_{i+1})=M_{i+1},\ i=1,2,\dots,n-1. \\ h_{i-1}M_{i-1}+2(h_{i-1}+h_i)M_i+h_iM_{i+1} &= 6\left(f[x_i,x_{i+1}]-f[x_{i-1},x_i]\right), \\ 2M_1+M_2 &= 6(f[x_1,x_2]-f_1), \\ M_{n-1}+2M_n &= 6(f_n-f[x_{n-1},x_n]) \end{aligned}$$

11

Complete (clamped) splines. $m_1=f'(a)$, $m_n=f'(b)$.
For this type of spline, it can be shown that, if $f\in C^4[a,b]$, then

$$\|f^{(r)}(\cdot)-s_3^{(r)}(f,\cdot)\|_\infty \leq C_r|\Delta|^{4-r}\|f^{(4)}\|_\infty,\ r=0,1,2,3$$

where

$$C_0 = \frac{5}{384},\ C_1 = \frac{1}{24},\ C_2 = \frac{3}{8},$$

and C_3 depends on the ratio $|\Delta|/\min_i h_i$.

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(a)+2(f_1+\dots+f_{n-1})+f(b)\right]$$

$$\Delta' :\ a=x_0=x_1<x_2<\dots<x_{n-1}<x_n=x_{n+1}=b.$$

$$\int_a^b|s_{compl}''(f;x)|^2dx \leq \int_a^b|g''(x)|^2dx,$$

equality if and only if $g(\cdot)=s_{compl}(f;\cdot)$.

13

$$\int_a^b|s_{nat}''(f;x)|^2dx \leq \int_a^b|s_{compl}''(f;x)|^2dx.$$

Natural cubic splines. $s_3''(f;a)=s_3''(f;b)=0$, $f''(a)=f''(b)=0$:

$$\begin{aligned} 2m_1+m_2 &= 3f[x_1,x_2], \\ m_{n-1}+2m_n &= 3f[x_{n-1},x_n]. \end{aligned}$$

9

Minimality properties of cubic spline interpolants

Theorem 2.3. *Let $g\in C^2[a,b]$ be any function that interpolates f on Δ. Then with equality if and only if $g(\cdot)=s_{nat}(f;\cdot)$.*

$$\int_a^b|s_{nat}''(f;x)|^2dx \leq \int_a^b|g''(x)|^2dx.$$

12

Let $g\in C^2[a,b]$ be any function that interpolates f on Δ'. Then

$$\Delta' :\ a=x_0=x_1<x_2<\dots<x_{n-1}<x_n=x_{n+1}=b.$$

$$\int_a^b|s_{compl}''(f;x)|^2dx \leq \int_a^b|g''(x)|^2dx,$$

equality if and only if $g(\cdot)=s_{compl}(f;\cdot)$.

13

Endpoint second derivative splines. $s_3''(f,a)=f''(a)$, $s_3''(f,b)=f''(b)$.

8

$$\begin{aligned} 2m_1+m_2 &= 3f[x_1,x_2]-\frac{1}{2}f''(a)h_1, \\ m_{n-1}+2m_n &= 3f[x_{n-1},x_n]-\frac{1}{2}f''(b)h_{n-1} \end{aligned}$$

Piecewise cubic Hermite interpolation

For equally spaced nodes, we have $|\Delta| = (b-a)/(n-1)$

5

$$\|f(\cdot)-s_3(f,\cdot)\|_\infty \leq \frac{1}{384}|\Delta|^4\,\|f^{(4)}\|_\infty.$$

Assuming that the derivatives $f'(x_i)$, $i=1,\dots,n$, are known, we choose $m_i=f'(x_i)$

7

Algorithm 3.1. [Adaptive quadrature]

```
function I = adquad(f,a,b,tol,m)
  I1 = mel(f,a,b,m);
  I2 = mel(f,a,b,2m);
  if |I1 - I2| < ε % success
    I = I2;
    return
  else % recursive subdivision
    I = adquad(f,a,a+b/2,ε,mel,m) + adquad(f,a+b/2,b,ε,mel,m)
  end
```

Richardson's extrapolation formula

$$I \approx \frac{2^pI_{2n}-I_n}{2^p-1} = I_{2n}+\frac{I_{2n}-I_n}{2^p-1}$$

$$I-I_{2n} \approx \frac{I_{2n}-I_n}{2^p-1}$$

$$I-I_n \approx \frac{C}{n^p}$$

$$p \approx \log_2\left(\frac{I_{2n}-I_n}{I_{4n}-I_{2n}}\right) = \frac{1}{\ln 2}\ln\left(\frac{I_{2n}-I_n}{I_{4n}-I_{2n}}\right) \overset{\text{not}}{=} R_{2n}$$