Exercise 1

We are interested in estimating an unknown signal x , given noisy measurements y. The signal is drawn from a **prior distribution** $P_X(x)$, and the noisy observation process is characterized by the conditional distribution $P_{Y|X}(y \mid x)$ called **Likelihood**. This defines the framework of **Bayesian estimation**.
Indeed using Bayes' theorem, the posterior distribution of x given y, is:

$$P_{X|Y}(x \mid y) = \frac{P_{Y|X}(y \mid x)\, P_X(x)}{P_Y(y)}$$

In the exercise I will consider the case of having gaussian noise with variance Δ and mean 0, so that if x^* is the *true value*, we will have n observations:

$$y_i = x^* + \sqrt{\Delta}\, z_i \quad \text{with} \quad z \sim N(0, \Delta)$$

We want to study the performance of the *Maximum A Posteriori (**MAP**)* estimator:
$\hat{x}_{\text{MAP}} = \arg\max_x P_{X|Y}(x \mid y)$, in terms of the Mean Square Error (**MSE**), for which the Bayes-optimal estimator is: $\hat{x}_{\text{MMSE}} = \mathrm{E}[X \mid Y = y]$.
The notion of Bayes-optimal estimator of a particular error, refer to the estimator that given the posterior $P_{X|Y}(x \mid y)$, gives an estimation of x^*, that on average minimize the error under examination.
In this exercise, we will consider three different priors, associated to:
1. A ***Rademacher*** random variable, $X = \pm 1$, with probability $p = 0.5$
2. A ***Gaussian*** *random Variable* $X \sim N(0, 1)$
3. A ***Gauss-Bernoulli*** *random variable which is 0 with probability 1/2 and Gaussian otherwise. It basically represent the fact that half of the times the signal is missing.*

The posteriors for this model can be computed from the priors, the fact that the likelihood is Gaussian, and imposing the normalization. The probabilities implemented in the code are indeed the expression obtained for them in the textbook.

In[22]:=
```
(*Define the sampling for the different RVs*)
xRad = Sign[Random[] - 0.5];
xGauss = Sqrt[2]*InverseErf[2*Random[]-1];
xGaussBernoulli = If[Random[] < 0.5, 0, Sqrt[2]*InverseErf[2*Random[] - 1]];
```

In[4]:=
```
(* In general the goal will be to generate  noisy observations with mean 0 and variance
Delta = 3;
n = 1000;

yRad = Table[
   xRad + Sqrt[2*Delta]*InverseErf[2*Random[] - 1],
  {i, 1, n}
];
yGauss = Table[
    xGauss + Sqrt[2*Delta]*InverseErf[2*Random[] - 1],
  {i, 1, n}
];

yGaussBernoulli = Table[
  xGaussBernoulli + Sqrt[2*Delta]*InverseErf[2*Random[] - 1],
  {i, 1, n}
];


(* Length of the table (Number of observations) *)
Length[yRad]
```

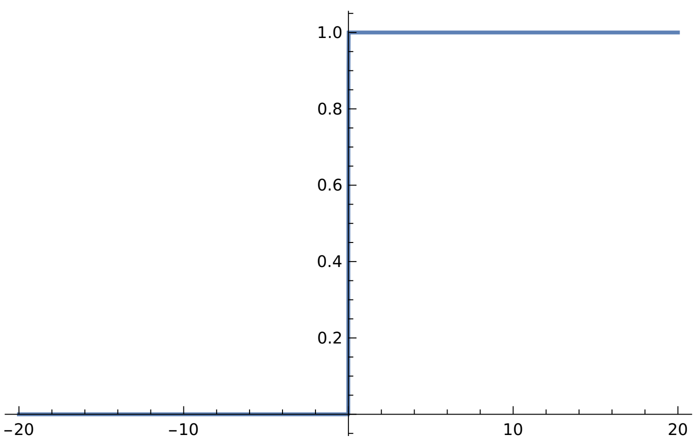
Out[9]= 1000

In[10]:=

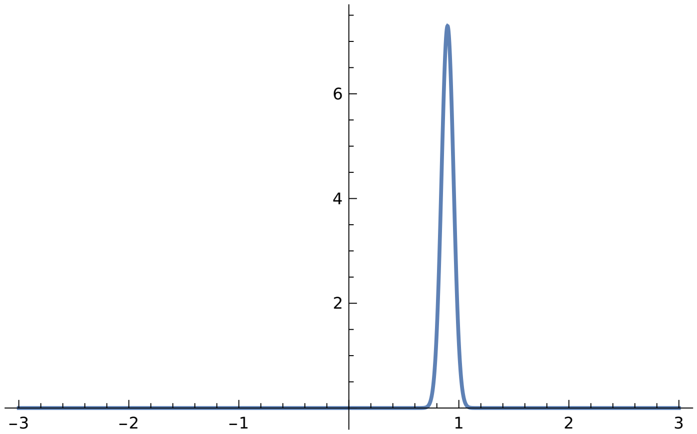
```
(*Now we have our noisy data, we want to find back the true value, giving the best esti
(*Thruogh the Bayes theorem is possible to find the exact form of the posterior probabi
P(x|Y)for the three models*)
Prad[x_,yRad_,Delta_]:=1/(1+Exp[-2*x*Total[yRad]/Delta]);
Pgauss[x_,yGauss_,Delta_,n_]:=PDF[NormalDistribution[Total[yGauss]/(n+Delta),Sqrt[Delta/(n+Del
PgaussBernoulli[x_,yGaussBernoulli_,Delta_,n_]:=Module[{mean,variance,z,weight1,weight2,g
 variance=Delta/(n+Delta);
 z=mean^2/(2*variance);
 weight1=1/(1+(Delta/(n+Delta)) Exp[-z]);
 weight2=1/(1+((n+Delta)/Delta) Exp[-z]);
 gaussian=PDF[NormalDistribution[mean,Sqrt[variance]],x];
 weight1 DiracDelta[x]+weight2 gaussian];

(*Plot the posterior distributions*)
Plot[Prad[x,yRad,Delta],{x,-20,+20}]
Plot[Pgauss[x,yGauss,Delta,n],{x,-3,+3},PlotRange→All]
Plot[PgaussBernoulli[x,yGaussBernoulli,Delta,n],{x,-3,+3},PlotRange→All]
```

Out[13]=

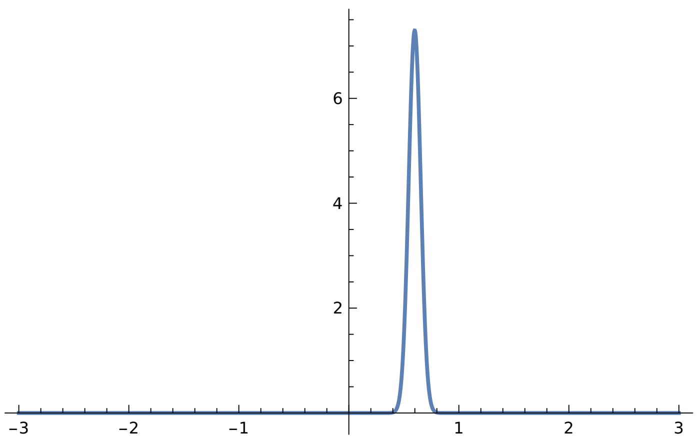


Out[14]=



General: 1. $5.72934496226 \times 10^{-940}$ is too small to represent as a normalized machine number; precision may be lost.

Out[15]=



In[25]:=

```
(*Consider the evolution of the error,
avaraged over many observations, either with the sample size n,
or the noise variance Delta, for MAP and MMSE estimator*)
(*Rademcher *)
Prad[x_, yRad_, Delta_] := 1 / (1 + Exp[-2 * x * Total[yRad] / Delta]);
ResultsRadMSE = {};
ResultsRadMAP = {};
Delta = 1;
n = 100;
(*UNCOMMENT JUST ONE OF THE TWO FOR TO ITERATE OVER DELTA OR n*)
For[Delta = 0.1, Delta ≤ 20, Delta = Delta + 0.5,
  (*For[n = 100,n ≤ 10000,n=10*n,*)
  MSE = 0;
  MAPMSE = 0;
  For[instances = 0, instances < 1000, instances++,
       (*Generate different instance at each iteration*)
        xRad = Sign[Random[] - 0.5];
        yRad = Table[xRad + Sqrt[2 * Delta] * InverseErf[2 * Random[] - 1], {i, 1, n}];
       xRadMAP = If[Prad[1, yRad, Delta] > Prad[-1, yRad, Delta], 1, -1];
       xRadMMSE = Prad[1, yRad, Delta] - Prad[-1, yRad, Delta];
        MSE = MSE + (xRadMMSE - xRad) ^ 2; (*Sum to take the avg at the end*)
        MAPMSE = MAPMSE + (xRadMAP - xRad) ^ 2;
  ];
  (*take the avarage of the mean square error among all the observations*)
      AppendTo[ResultsRadMSE, {Delta, MSE / (1000)}] ;
      AppendTo[ResultsRadMAP, {Delta, MAPMSE / (1000)}] ;
 ];

ListPlot[{ResultsRadMSE, ResultsRadMAP}, PlotStyle → {Blue, Red},
 PlotLegends → {"MSE", "MAP"}, AxesLabel → {"Delta", "Value"},
 PlotLabel → "Results for MSE and MAP", Joined → True , PlotRange → All]
```

General: Exp[−2160.9] is too small to represent as a normalized machine number; precision may be lost.

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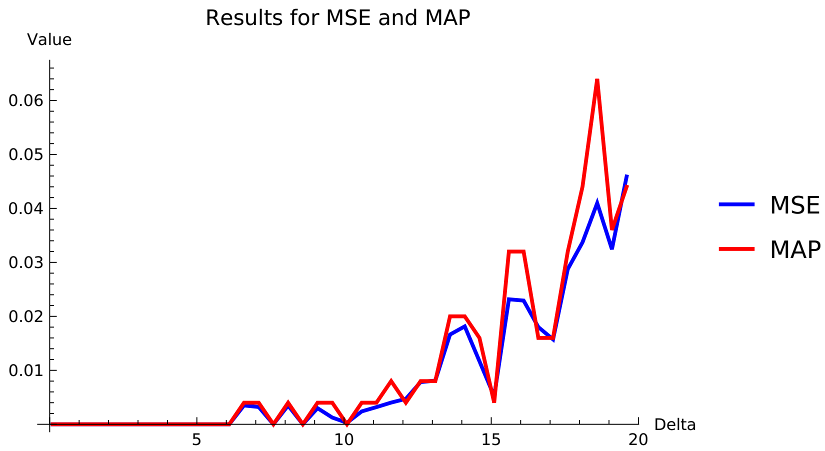
General: Exp[−1938.57] is too small to represent as a normalized machine number; precision may be lost.

General: Further output of General::munfl will be suppressed during this calculation.

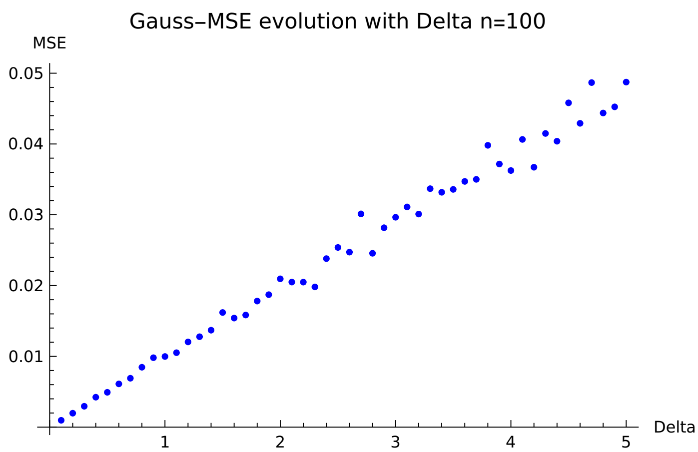
Out[31]=


Results for MSE and MAP
Value
0.06
0.05
0.04
0.03
0.02
0.01
5
10
15
20
Delta
MSE
MAP

In[16]:=

```
ResultsGauss ={};
Pgauss[x_,yGauss_,Delta_,n_]:=PDF[NormalDistribution[Total[yGauss]/(n+Delta),Sqrt[Delta/(n+Del
For[Delta = 0.1,Delta ≤ 5,Delta = Delta + 0.1,
(*For[n = 100,n ≤ 10000,n=10*n,*)
n=100;
MSE=0; (*Map and MMSE coincide for a Gaussian*)
For[instances = 0,instances <600, instances++,
        xGauss = Sqrt[2]*InverseErf[2*Random[]-1];
        yGauss = Table[xGauss + Sqrt[2*Delta]*InverseErf[2*Random[] - 1],{i, 1, n}];
          xGaussMAP = ArgMax[Pgauss[x,yGauss,Delta,n],x];
       xGaussMMSE = NIntegrate[Pgauss[x,yGauss,Delta,n]*x,{x,-Infinity,+Infinity}];
        MSE = MSE + (xGaussMMSE-xGauss)^2; (*Sum to take the avg at the end*)
];
    AppendTo[ResultsGauss,{Delta,MSE/(600)}] ;
(*take the avarage of the mean square error among all the observations*)
];
ResultsGauss;
plot = ListPlot[ResultsGauss,PlotStyle→Blue,AxesLabel→{"Delta","MSE"},PlotLabel→"Gauss-M
Export["plotImage.png",plot]
```

Out[20]=



Out[21]=

plotImage.png

In[38]:=

```
PgaussBernoulli[x_, yGaussBernoulli_, Delta_, n_] :=
 Module[{mean, variance, z, weight1, weight2, gaussian},
  mean = Total[yGaussBernoulli] / (n + Delta);
  variance = Delta / (n + Delta);
  z = mean ^ 2 / (2 * variance);
  weight1 = 1 / (1 + (Delta / (n + Delta)) Exp[-z]);
  weight2 = 1 / (1 + ((n + Delta) / Delta) Exp[-z]);
  gaussian = PDF[NormalDistribution[mean, Sqrt[variance]], x];
  weight1 DiracDelta[x] + weight2 gaussian]

ResultsGaussBernoulliMMSE = {};
ResultsGaussBernoulliMAP = {};
For[Delta = 0.1, Delta ≤ 3, Delta = Delta + 0.2,

  n = 100;
  MSE = 0;
  MAPMSE = 0;
  For[instances = 0, instances < 1000, instances++,
        xGaussBernoulli = If[Random[] < 0.5, 0, Sqrt[2] * InverseErf[2 * Random[] - 1]];
        yGaussBernoulli =
    Table[xGaussBernoulli + Sqrt[2 * Delta] * InverseErf[2 * Random[] - 1], {i, 1, n}];
        xGaussBernoulliMAP = ArgMax[PgaussBernoulli[x, yGaussBernoulli, Delta, n], x];
        xGaussBernoulliMMSE = NIntegrate[
     PgaussBernoulli[x, yGaussBernoulli, Delta, n] * x, {x, -Infinity, +Infinity}];
        MSE = MSE + (xGaussBernoulliMMSE - xGaussBernoulli) ^ 2;
   (*Sum to take the avg at the end*)
        MAPMSE = MAPMSE + (xGaussBernoulliMAP - xGaussBernoulli) ^ 2;
  ];
     (*take the avarage of the mean square error among all the observations*)
     AppendTo[ResultsGaussBernoulliMMSE, {Delta, MSE / (1000)}] ;
     AppendTo[ResultsGaussBernoulliMAP, {Delta, MSE / (1000)}] ;

 ];
ResultsGauss;
ListPlot[{ResultsGaussBernoulliMMSE, ResultsGaussBernoulliMAP},
 PlotStyle → {Blue, Red}, PlotLegends → {"MSE", "MAP"}, AxesLabel → {"Delta", "Value"},
 PlotLabel → "Results for MSE and MAP", Joined → True ]
```

General : Exp[−919.264] is too small to represent as a normalized machine number; precision may be lost.

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Out[43]=