① $\int_{\mathbb{R}} e^{-ax^2-bx}\,dx = \sqrt{\frac{\pi}{a}}\; e^{b^2/4a}$

② $\int_{\mathbb{R}} x\, e^{-ax^2-bx}\,dx = \underbrace{-\frac{1}{2a} e^{-ax^2-bx}\Big|_{\mathbb{R}}}_{=0} - \frac{b}{2a}\int e^{-ax^2-bx}\,dx = -\frac{b}{2a^{3/2}}\sqrt{\pi}\, e^{b^2/4a}$

③ $\int_{\mathbb{R}} \underbrace{x^2 e^{-ax^2-bx}}_{=(xe^{-ax^2})(xe^{-bx})} = \underbrace{-\frac{1}{2a} e^{-ax^2-bx}\, x\Big|_{\mathbb{R}}}_{=0} + \frac{1}{2a}\left[-b\int x\, e^{-ax^2-bx} + \int e^{-ax^2-bx}\right]$

$= \frac{\sqrt{\pi}\, b^2}{4a^{5/2}}\, e^{b^2/4a} + \sqrt{\frac{\pi}{a}}\, e^{b^2/4a}$

④

$$m = \mathbb{E}_{x_*,z}\frac{\int \mathrm{d}x\, P_X(x)\, x x^* e^{\frac{-\lambda m}{2}x^2+(\lambda m x_*+\sqrt{\lambda m}z)x}}{\int \mathrm{d}x\, P_X(x) e^{\frac{-\lambda m}{2}x^2+(\lambda m x_*+\sqrt{\lambda m}z)x}}$$

⑤

$$\phi = -\frac{\lambda}{4}m^2 + \mathbb{E}_{x_*,z}\left[\log\left(\int P_X(x)\,\mathrm{d}x\, e^{\frac{-\lambda m}{2}x^2+(\lambda m x_*+\sqrt{\lambda m}z)x}\right)\right]$$

Ex. 2

→ Gaussian $\quad x, x^* \sim \mathcal{N}(0,1) \quad z \sim \mathcal{N}(0,1)$

$\phi = -\frac{1}{4}m^4 + \mathbb{E}_{x_*,z}\left[\log(J_1)\right]$ ★ $\quad \left[\begin{array}{l} a = \frac{1+\overbrace{\lambda m}^{c^2}}{2} \\ b = -(\underbrace{\sqrt{\lambda m}}_{c}\, z + \underbrace{\lambda m}_{c^2}\, x^*) \end{array}\right.$ $\quad c = \sqrt{\lambda m}$

$J_1 = \int dx\, \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\; e^{-\frac{\lambda m}{2}x^2 + (\lambda m x^* + \sqrt{\lambda m}\, z)x}$ $\quad \Downarrow \quad =$

$= \int dx\, \frac{e^{-ax^2-bx}}{\sqrt{2\pi}} = \frac{1}{\sqrt{2a}}\, e^{b^2/4a} = \frac{1}{\sqrt{1+c^2}}\, e^{\frac{(cz+c^2x^*)^2}{2(1+c^2)}}$

$\mathbb{E}_{x^*,z}\left[\log J_1\right] = \mathbb{E}_{x^*,z}\left[-\frac{1}{2}\log(1+c^2) + \frac{(cz+c^2x^*)^2}{2(1+c^2)}\right] =$

$= \frac{1}{2\pi}\int dz\, e^{-\frac{z^2}{2}}\int dx^*\, e^{-\frac{(x^*)^2}{2}}\left[-\frac{1}{2}\log(1+c^2) + \frac{c^2z^2 + c^4 (x^*)^2 + 2c^3 z x^*}{2(1+c^2)}\right]$

i.d. same integral

0 since mean of two $\mathcal{N}(0,1)$

$= \frac{1}{2\pi}\left[-\pi\log(1+c^2) + \frac{\sqrt{2\pi}}{2}\int dx\, e^{-\frac{x^2}{2}}\, x^2\, \frac{c^2(1+c^2)}{1+c^2}\right] =$

$$= -\frac{1}{2}\log(1+\lambda m) + \frac{1}{2}\lambda m$$

$$\left[\Phi_{RS}(\lambda, m) = -\frac{\lambda}{4} m^2 - \frac{1}{2}\log(1+\lambda m) + \frac{1}{2}\lambda m\right]^*$$

$$\frac{\partial \Phi_{RS}}{\partial m} = -\frac{\lambda m}{2} - \frac{\lambda}{2(1+\lambda m)} + \frac{1}{2}\lambda = 0$$

$$\frac{\cancel{1} + m + \lambda m^2 - \lambda m - \cancel{1}}{(1+\lambda m)} = \frac{\lambda m\left(m + \left(\frac{1}{\lambda} - 1\right)\right)}{(1+\lambda m)}$$

E.g. $\begin{cases} m_1 = 0 \\ m_2 = 1 - \frac{1}{\lambda} \end{cases}$ E.g. $\lambda = 1.5$. $\left[m_2 = \frac{1}{3}\right]$

obtain a self consistent equation:

$$\frac{\partial \Phi_{RS}}{\partial m} = 0 \Rightarrow \lambda m + \frac{\lambda}{(1+\lambda m)} = \lambda$$

$$\Downarrow$$

$$\left[m = 1 - \frac{1}{1+\lambda m} = \frac{\lambda m}{1+\lambda m}\right]$$

(same result obtained evaluating (E))

All η are the FRS integral in X just changing the exponent coefficients with a, b (Divide also for x^*)

2. Rademacher

$c = \sqrt{\lambda m}$

$$x, x^* = \begin{cases} -1 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{cases} \qquad z \sim \mathcal{N}(0,1)$$

$$m = \mathbb{E}_{x^*,z}\left[\frac{\frac{1}{2}x^*\left(e^{-\frac{c^2}{2}+(c^2x^*+cz)} - e^{-\frac{c^2}{2}-(c^2x^*+cz)}\right)}{\frac{1}{2}\left(e^{-\frac{c^2}{2}+(c^2x^*+cz)} - e^{-\frac{c^2}{2}-(c^2x^*+cz)}\right)}\right] = \mathbb{E}_{x^*,z}\left[x^* \tanh(c^2x^*+cz)\right] =$$

$$= \mathbb{E}_z\left[\frac{1}{2}\left[\tanh\left[c^2+cz\right] - \tanh\left[-c^2+cz\right]\right]\right] =$$

$$= \frac{1}{\sqrt{2\pi}}\int dz\, e^{-\frac{z^2}{2}}\frac{1}{2}\left[\tanh\left[c^2+cz\right] - \underbrace{\tanh\left[-c^2+cz\right]}_{\overset{(*)}{=}\ \tanh\left[c^2-cz\right] =}\right] =$$

$$= \mathbb{E}_z\left[\tanh\left[\lambda m + \sqrt{\lambda m}\, z\right]\right]$$

$$(*)\quad \int_{-\infty}^{\infty} e^{-z^2/2}\tanh\left[c^2-cz\right]dz \underset{z\to -z}{=} -\int_{+\infty}^{-\infty} e^{-z^2/2}\tanh\left[c^2+cz\right]dz = \int_{-\infty}^{+\infty} e^{-z^2/2}\tanh\left[c^2+cz\right]dz$$

2. Rademacher - Bernoulli

$$P(x) \ (= P(x^*)) = \begin{cases} 0 & 1-\rho \\ \pm\frac{1}{2} & \rho/2 \end{cases}$$

$$I_1 = \int dx\, P_x(x)\, x x^* e^{-\frac{c^2}{2}x^2 + (cz+c^2x^*)x} = \rho x^* e^{-\frac{c^2}{2}} \sinh\left[c^2x^*+cz\right]$$

$$I_2 = \int dx\, P_x(x)\, e^{-\frac{c^2}{2}x^2 + (cz+c^2x^*)x} = (1-\rho) + \rho\, e^{-c^2/2}\cosh\left[c^2x^*+cz\right]$$

$$m = \mathbb{E}_{x^*,z}\left[\frac{\rho x^* e^{-\frac{c^2}{2}}\sinh\left[c^2x^*+cz\right]}{(1-\rho)+\rho e^{-c^2/2}\cosh\left[c^2x^*+cz\right]}\right] =$$

$$= \mathbb{E}_z\left[\frac{\rho^2}{2}\frac{e^{-\frac{c^2}{2}}\sinh\left[c^2+cz\right]}{(1-\rho)+\rho\, e^{-c^2/2}\cosh\left[c^2+cz\right]} - \frac{\rho^2}{2}\frac{e^{-c^2/2}\sinh\left[-c^2+cz\right]}{(1-\rho)+\rho\, e^{-c^2/2}\cosh\left[-c^2+cz\right]}\right] =$$

As before

$$-\int_{\mathbb{R}} e^{-\frac{z^2}{2}} \frac{\sinh[-c^2-cz]}{(1-\rho)+\rho e^{-c^2/2}\cosh[-c^2-cz]}\, dz \overset{z \to -z}{=} +\int_{-\infty}^{+\infty} e^{-z^2/2} \frac{\sinh[c^2+cz]}{(1-\rho)+\rho e^{-c^2/2}\cosh[c^2+cz]}\, dz$$

$$\Downarrow$$

$$m = \mathbb{E}_z\left[\frac{\rho^2 \tanh[\lambda m+\sqrt{\lambda m}\, z]}{\frac{(1-\rho)\, e^{-\frac{\lambda m}{2}}}{\cosh[\lambda m+\sqrt{\lambda m}\, z]}+\rho}\right]$$

if $\rho = 1$
$\Rightarrow$ previous case ✓